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CONLEY THEORY FOR GUTIERREZ-SOTOMAYOR FIELDS

H. MONTÚFAR AND K. A. DE REZENDE

Abstract. In [6], a characterization and genericity theorem for C^1-structurally stable vector fields tangent to a 2-dimensional compact subset M of $\mathbb{R}^k$ are established. Also in [6], new types of structurally stable singularities and periodic orbits are presented. In this work we study the continuous flows associated to these vector fields, which we refer to as the Gutierrez-Sotomayor flows on manifolds M with simple singularities, GS flows, by using Conley Index Theory. The Conley indices of all simple singularities are computed and an Euler characteristic formula is obtained. By considering a stratification of M which decomposes it into a union of its regular and singular strata, certain Euler type formulas which relate the topology of M and the dynamics on the strata are obtained. The existence of a Lyapunov function for GS flows without periodic orbits and singular cycles is established. Using long exact sequence analysis of index pairs we determine necessary and sufficient conditions for a GS flow to be defined on an isolating block. We organize this information combinatorially with the aid of Lyapunov graphs and using a Poincaré-Hopf equality we construct isolating blocks for all simple singularities.

1. Introduction

In [6], C. Gutierrez and J. Sotomayor generalize characterization and genericity theorems obtained by M. Peixoto [8] for structurally stable vector fields tangent to smooth compact two-manifolds. The following definitions 1.1, 1.2 and 1.3 were introduced in [6] and the reader is referred to the original paper for more details.

Definition 1.1. *A subset $M \subset \mathbb{R}^l$ is called a* two-dimensional manifold with simple singularities *if for every point $p \in M$ there is a neighborhood V_p of p in M and a C^∞-diffeomorphism $\Psi : V_p \to \mathcal{G}$ such that $\Psi(p) = 0$, where $\mathcal{G}$ is one of the following subsets of $\mathbb{R}^3$:*

$\mathcal{R} = \{(x, y, z); z = 0\}$, *plane;*

$\mathcal{C} = \{(x, y, z); z^2 - y^2 - x^2 = 0\}$, *cone;*

$\mathcal{D} = \{(x, y, z); xy = 0\}$, *double crossing;*

$\mathcal{W} = \{(x, y, z); zx^2 - y^2 = 0\}$, *Whitney's umbrella;*

$\mathcal{T} = \{(x, y, z); xyz = 0\}$, *triple crossing.*

Ψ is called a local chart at p.

These local charts are essential in order to define the stratified set M in the sense of Thom [10], endowed with the partition $\{M(\mathcal{G}), \mathcal{G}\}$ where $\mathcal{G} = \mathcal{R}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{W}$ or $\mathcal{T}$ and $M(\mathcal{G})$ is the set of points $p \in M$ such that $\Psi(p) = 0$ for $\Psi : V_p \to \mathcal{G}$. Note that $M(\mathcal{R})$ is a smooth two-dimensional manifold called the regular part of M, $M(\mathcal{D})$ is a one-dimensional smooth manifold, while $M(\mathcal{C})$, $M(\mathcal{W})$ and $M(\mathcal{T})$ are discrete sets.

Definition 1.2. *A vector field X of class C^r on $\mathbb{R}^l$ is said to be* tangent *to a manifold $M \subset \mathbb{R}^l$ with simple singularities if it is tangent to the smooth submanifolds $M(\mathcal{G})$, for all $\mathcal{G}$. The space of such vector fields is denoted by $\mathfrak{X}^r(M)$.*

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In [6], C. Gutierrez and J. Sotomayor determine conditions of stability for fixed points, periodic orbits and singular cycles.

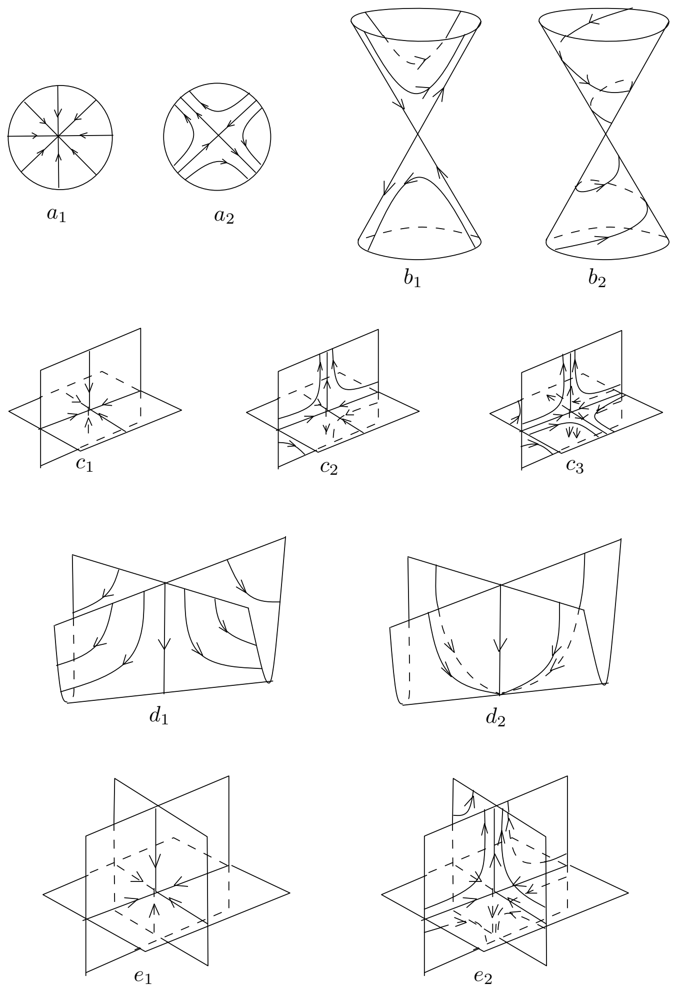

FIGURE 1. Local types of hyperbolic fixed points

Definition 1.3. *Let $M \subset \mathbb{R}^l$ be a two-manifold with simple singularities. We call $\Sigma^r(M)$ the set of vector fields $X \in \mathfrak{X}^r(M)$ such that:*

(1) *X has finitely many fixed points and periodic orbits, all hyperbolic.*
(2) *The singular limit cycles of X are simple and X has no saddle connection.*
(3) *The $\alpha-$limit set and $\omega-$limit set of every trajectory of X is either a fixed point, a periodic orbit or a singular cycle.*

In this work, we refer to the flow X_t associated with the field $X \in \Sigma^r(M)$ as the *Gutierrez-Sotomayor flow*.

In [6], C. Gutierrez and J. Sotomayor proved the following formidable theorem.

Theorem 1.4. *Under either of the following hypotheses on $\mathfrak{X}^r(M)$:*

- *$r = 1$, or*
- *$r = 2, 3, \ldots, \infty$ and each connected component of $M(\mathcal{R})$ is either an orientable two-manifold or an open subset of $P^2 \cup K^2 \cup (T^2 \sharp P^2)$,*

we have that:

1. *$\Sigma^r(M)$ is open and dense in $\mathfrak{X}^r(M)$, and*
2. *$X \in \mathfrak{X}^r(M)$ is structurally stable if and only if $X \in \Sigma^r(M)$.*

In this article, we study Gutierrez-Sotomayor flows from a topological perspective, using Conley index theory. In Section 2 we define a Lyapunov function and in this Gutierrez-Sotomayor context we show its existence for flows without periodic orbits and singular cycles. In proving the existence of Lyapunov functions, we also prove that there is a neighborhood, N of p, in M and a function f on N such that f is continuous and decreases along the orbits of X_t on $N - p$.

In Section 3, we develop the classical Conley theory. In Theorem 3.2, the homotopical index of singularities of a Gutierrez-Sotomayor flow, X_t, on M are obtained. Therefore, by calculating the ranks of the homology of the Conley index of a singularity $p \in M$, denoted by (h_0, h_1, h_2), we present several Euler characteristic type formulas in Section 3.2 which relate the topology of M to the dynamics of the flow X_t.

In Section 4, a more general handle theory is introduced in order to establish a procedure for constructing special isolating neighborhoods of simple singularities of a Gutierrez-Sotomayor flow. GS handles are defined. In Theorem 4.2, a Poincaré-Hopf equality is presented, which relates the first Betti number of the branched one-manifolds which makeup the boundary of the isolating block (N_1, N_0) of the singularity $p \in M$ with the number of boundary components in N_0 and the numerical Conley index (h_0, h_1, h_2) of p. This theorem will guide our constructions of isolating blocks.

In Section 5 we adopt a combinatorial approach, by associating a Lyapunov graph L to a GS flow X_t and a Lyapunov function f, by identifying to a point each connected component of a level set of f.

In Theorem 5.3, through a long exact homological sequence analysis of index pairs we determine properties that a Lyapunov graph must satisfy in order to be associated to a GS-flow. The main results herein generalize results of K. de Rezende and R. Franzosa [3] where Morse-Smale flows and more generally continuous flows are classified on smooth surfaces.

2. Lyapunov function

A Lyapunov function on M is a collection of Lyapunov functions on the strata of $M \subset \mathbf{R}^l$. Note, however, that we do not require the function to be smooth, only continuous.

Definition 2.1. Let M be a two-manifold with simple singularities. If X_t is a Gutierrez-Sotomayor flow on M then a function $f : M \to \mathbb{R}$ is called a *Lyapunov function* if:

1. For each stratum $M(\mathcal{G})$ of M:
   - (a) $f\,|_{M(\mathcal{G})}$ is a smooth function and f is continuous on M.
   - (b) The critical points of $f\,|_{M(\mathcal{G})}$ are nondegenerate and coincide with the singularities of X_t.
   - (c) $\dfrac{d}{dt}(f\,|_{M(\mathcal{G})}\,(X_t x)) < 0$, if x is not a singularity of X_t.
2. If p and q are singularities of X_t, then $f(p) \neq f(q)$.

In Section 2.1, we will construct a Lyapunov function f locally on a neighborhood of a GS singularity. In Section 2.2 we extend this construction to isolating blocks and subsequently to GS two-manifolds.

2.1. **Local Construction.** Throughout this work, for simplicity, a two dimensional disk will be referred to as disk D and a one dimensional disk as a segment I.

Theorem 2.2. *Let M be a 2-dimensional manifold with simple singularities. If $p \in M$ is a singularity of a Gutierrez-Sotomayor flow X_t on M then there exists a neighborhood, N of p on M, sufficiently small, and a function f on N such that f is a Lyapunov function on N.*

Proof.

Case 1:: If $p \in M(\mathcal{R})$ then a neighborhood N of p on M is a disk. Without loss of generality, we can assume the disk N as in Figure 1 (a_1) and (a_2). If p is of type (a_1) then in local coordinates its dynamics in $\mathbb{R}^2$ is given by:

$$\begin{cases} \dot{x} = -2x \\ \dot{y} = -2y \end{cases}$$

Define a function f on N given by $f(x, y) = x^2 + y^2$. Since $\frac{df}{dt} = -4(x^2 + y^2) < 0$ then f is a Lyapunov function on N. If p is as in (a_2) then in these local coordinates its dynamics are given by:

$$\begin{cases} \dot{x} = -y \\ \dot{y} = -x \end{cases}$$

Define a function f on N given by $f(x, y) = xy$. Since $\frac{df}{dt} = -(y^2 + x^2) < 0$ then f is a Lyapunov function on N. If p is as in (a_1) with the reverse dynamics then consider $-f$.

In any case, we can summarize this by writing X in local coordinates as $\dot{x} = Ax + \phi(x)$ where $\phi(0) = d\phi(0) = 0$ and the eigenvalues of A have real part different from zero. This condition is equivalent to the existence of symmetric matrices Q and C with C positive definite and Q non-singular such that the Lyapunov equation $A^T Q + QA = -C$ holds, where the superscript T denotes the transpose of the matrix. Define a function f given by $f(x) = x^T Qx$. Since

$$\frac{df}{dt} = \dot{x}^T Qx + x^T Q\dot{x}$$

$$\frac{df}{dt} = (Ax + \phi(x))^T Qx + x^T Q(Ax + \phi(x))$$

$$\frac{df}{dt} = x^T (A^T Q + QA)x + 2x^T Q\phi(x)$$

$$\frac{df}{dt} = -x^T Cx + 2x^T Q\phi(x)$$

where $2x^T Q\phi(x)$ has higher order terms. For N sufficiently small, f is a Lyapunov function on N.

Case 2:: If $p \in M(\mathcal{C})$ then a neighborhood N of p in M is formed by two disks D_1 and D_2 identified at the singularity p, see Figure 1 (b_1) and (b_2). We can assume without loss of generality that the disks D_i, $i = 1, 2$, in $\mathbb{R}^2$ are as in Figure 2.

If the disks are as in (a) and (b) then we are in the previous case. If D_i is as in (c) then in local coordinates its dynamics are given by:

$$\begin{cases} \dot{x} = 0 \\ \dot{y} = -x^2 - y^2 \end{cases}$$

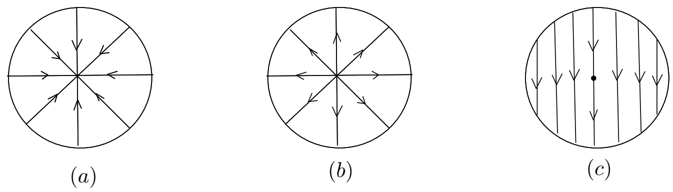

FIGURE 2. Disks D_i in N

Let f_i be a function on D_i given by $f_i(x,y) = y$. As $\frac{df_i}{dt} = -x^2 - y^2 < 0$ then f_i decreases along orbits of X_t on D_i. Define the function f on N:

$$f(x) = \begin{cases} f_1(x) \text{ if } x \in D_1 \\ f_2(x) \text{ if } x \in D_2 \end{cases}$$

Then $f|_{N\setminus\{p\}}$ is a Lyapunov function, hence f is a Lyapunov function on N.

Case 3:: If $p \in M(\mathcal{D})$ then a neighborhood N of p in M is formed by two disks D_i, $i = 1, 2$, that intersect transversally along diameters d_1 and d_2 on D_1 and D_2 respectively, see Figure 1 (c_1), (c_2) and (c_3). Let $d = D_1 \cap D_2$. On each disk D_i the dynamics are the same as defined for $p \in M(\mathcal{R})$, hence a Lyapunov function f_i is defined as in Case 1. By adding appropriate constants we can assume $f_1(p) = f_2(p)$.

Let γ be the orbit on d. By using a diffeomorphism $h : f_1(\gamma) \to f_2(\gamma)$, redefine $f_1 := h \circ f_1$ so that $f_1(x) = f_2(x)$ for $x \in d$. Thus, the transversal intersection of the disks D_i is attained via homeomorphisms on the orbit γ on d given by $x \to (f_2|_\gamma^{-1} \circ f_1|_\gamma)(x)$ we have that for $x \in D_1 \cap D_2$ then $f_1(x) = f_2(x)$. Hence, $f : N \to \mathbb{R}$ is given by

$$f(x) = \begin{cases} f_1(x) \text{ if } x \in D_1 \\ f_2(x) \text{ if } x \in D_2 \end{cases}$$

is a Lyapunov function on N. Indeed for each stratum $M(\mathcal{G}) \subset N$, with $\mathcal{G} = \mathcal{R}$ or $\mathcal{D}$, we have that $f\,|_{M(\mathcal{G})}$ is a Lyapunov function on $M(\mathcal{G})$.

Case 4:: If $p \in M(\mathcal{W})$ then a neighborhood N of p in M can be formed by identifying two distinct rays r_1 and r_2 on a disk D. See Figure 1 (d_1) and (d_2). On the disk D the dynamics are defined as in the case $p \in M(\mathcal{R})$, hence, a Lyapunov function f is defined. Define $\overline{f}$ on $N = D/\sim$ where $\sim$ is given by:

$$x \sim y \Leftrightarrow x = y \text{ or } f(x) = f(y) \text{ with } x \in r_1 \subset W^s(p),\ y \in r_2 \subset W^s(p).$$

Hence, $\overline{f} : N \to \mathbb{R}$ given by $\overline{f}(\overline{x}) = f(x)$ is a Lyapunov function on N. Indeed for each stratum $M(\mathcal{G}) \subset N$ with $\mathcal{G} = \mathcal{R}$ or $\mathcal{D}$, we have that $\overline{f}\,|_{M(\mathcal{G})}$ is a Lyapunov function on $M(\mathcal{G})$. Similarly, when considering the reverse flow the equivalence relation $\sim$ is taken in $W^u(p)$.

Case 5:: If $p \in M(\mathcal{T})$ then a neighborhood N of p in M is formed by three disks D_i, $i = 1, 2, 3$, that intersect transversally in pairwise distinct diameters that intersect at the point p. See Figure 1 (e_1) and (e_2). On the disks D_i the dynamics are as in $p \in M(\mathcal{R})$, hence a Lyapunov function f_i is defined on each disk D_i. If $\widetilde{N}$ is formed by disks D_i and D_j, intersecting transversally, with p a double crossing in $\widetilde{N}$ then define $\widetilde{f}$ on $\widetilde{N}$, decreasing along the orbits of X_t, as in $p \in M(\mathcal{D})$.

Denote by $d_{ki} \subset D_k$ and $d_{kj} \subset D_k$ the lines where $\widetilde{N}$ and D_k intersect transversally. By adding appropriate constants we assume $\widetilde{f}(p) = f_k(p)$ and by using a diffeomorphism $h : f_k(\gamma) \to \widetilde{f}(\gamma)$, redefine $f_k := h \circ f_k$ on the orbits γ of $d_{ki} \cup d_{kj}$ such that $\widetilde{f}(x) = f_k(x)$.

Thus, the transversal intersection of the disk D_k with $\widetilde{N}$ is obtained and via the homeomorphisms defined on the orbits γ on $d_{ki} \cup d_{kj}$

given by $x \to (\widetilde{f}|_{\gamma}^{-1} \circ f_k|_{\gamma})(x)$ we obtain:

$$\begin{cases} \text{if } x \in D_1 \cap D_2 \text{ then } f_1(x) = f_2(x) \\ \text{if } x \in D_1 \cap D_3 \text{ then } f_1(x) = f_3(x) \\ \text{if } x \in D_2 \cap D_3 \text{ then } f_2(x) = f_3(x) \end{cases}$$

since $\widetilde{f}|_{D_i} = f_i$ and $\widetilde{f}|_{D_j} = f_j$. Thus $f : N \to \mathbb{R}$ given by

$$f(x) = \begin{cases} f_1(x) \text{ if } x \in D_1 \\ f_2(x) \text{ if } x \in D_2 \\ f_3(x) \text{ if } x \in D_3 \end{cases}$$

is a Lyapunov function on N. Indeed, for each stratum $M(\mathcal{G}) \subset N$, with $\mathcal{G} = \mathcal{R}$ or $\mathcal{D}$, we have that $f\,|_{M(\mathcal{G})}$ is a Lyapunov function on $M(\mathcal{G})$.

□

We now prove the existence of a continuous real valued function on a neighborhood of a saddle cone singularity, see Proposition 2.3, as well as, in a neighborhood of a periodic orbit or cycle, see Theorem 2.4 that decreases along orbits of the local flow defined on that neighborhood.

Proposition 2.3. *Let M be a two-manifold with simple singularities. If $p \in M(\mathcal{C})$ is a saddle cone type singularity of a Gutierrez-Sotomayor flow X_t on M then there exists a sufficiently small neighborhood N of p, in M, and a function f on N such that f is continuous and decreases along orbits of X_t on $N - \{p\}$.*

Proof. If $p \in M(\mathcal{C})$ is a saddle cone type singularity in M then a neighborhood N of p in M is formed by a union of two discs D_1 and D_2 identified at the singularity p, $D_1 \vee_p D_2$, see Figure 1.1 (b_1). We can assume, without loss of generality, that via a homeomorphism, the discs D_i, $i = 1, 2$, are on the plane $\mathbb{R}^2$, see Figure 2 (c). In these local coordinates, the dynamics are given by:

$$\begin{cases} \dot{x} = 0 \\ \dot{y} = -x^2 - y^2 \end{cases}$$

Let f_i be the function on D_i given by $f_i(x, y) = y$. Since $\frac{df_i}{dt} = -x^2 - y^2 < 0$ then f_i decreases along the orbits of X_t on D_i. Now let the function f on N be such that:

$$f(x) = \begin{cases} f_1(x) \text{ se } x \in D_1 \\ f_2(x) \text{ se } x \in D_2 \end{cases}$$

Then $f\,|_{N \setminus \{p\}}$ is a continuous function that decreases along the orbits of X_t on $N - \{p\}$. □

Theorem 2.4. *Let M be a two manifold with simple singularities. If $\gamma \subset M(\mathcal{R})$ is a periodic orbit of a Gutierrez-Sotomayor flow X_t on M then there exists a neighborhood, sufficiently small, N of γ, on M, and a function f on N such that f decreases along orbits of X_t on $N \setminus \gamma$ and is constant on γ.*

Proof. If $\gamma \subset M(\mathcal{R})$ then a neighborhood N of γ in M is an annulus.

In local coordinates, the dynamics are given by:

$$\begin{cases} \dot{x} = x - y - x(x^2 + y^2) \\ \dot{y} = x + y - y(x^2 + y^2) \end{cases}$$

Define a function f on N by $f(x,y) = \frac{1}{4}\ln^2(x^2+y^2)$. Since

$$\begin{aligned}\frac{df}{dt} &= \frac{x}{x^2+y^2}(\ln(x^2+y^2))(x-y-x(x^2+y^2)) + \frac{y}{x^2+y^2}(\ln(x^2+y^2))(x+y-y(x^2+y^2))\\ &= (\ln(x^2+y^2))(1-(x^2+y^2)) < 0,\end{aligned}$$

we have that f decreases along orbits of X_t on $N\backslash\gamma$ and is constant on γ. □

2.2. **Lyapunov functions - global construction.**

In this section, we study Gutierrez-Sotomayor flows, X_t, with no periodic orbits and no singular cycles on a compact two-manifold with boundary ∂M (which maybe empty). We assume X_t has only GS simple singularities and is transversal to ∂M. Denote by ∂M^- the boundary on which the flow exits and $\partial M^+ = \partial M\backslash\partial M^-$ the boundary on which the flow enters. In general, ∂M is not connected, however there are some attractors as well as repellers defined on manifolds M with boundary where ∂M connected.

If a point p is on the stratum S of M then the tangent space T_pS is well defined. But if M is singular on S then there are possibly infinitely many "tangent spaces" on M at p and we denote them by generalized tangent spaces. Formally, a *generalized tangent space* at $p \in S$ is any plane Q_p of the form $Q_p = \lim_{p_i\to p} T_{p_i}S'$ where p_i is a sequence of points in a stratum S' whose limit is p. See [5] for more details. The *generalized tangent bundle* Q of M is the set of all pairs (x,v) such that $x \in M$ and $v \in Q_p$. Given a Riemannian metric on $\mathbb{R}^l$, for each $p \in S$, the inner product on the space Q_p splits it in a direct sum $Q_p = T_pS \oplus (T_pS)^\perp$ where $(T_pS)^\perp$ is the orthogonal complement of T_pS in Q_p. This means that, locally, the part of the generalized tangent bundle Q that projects on S splits in a tangent bundle TS and a *generalized normal bundle* $TS^\perp$.

Lemma 2.5. *Let M be a two-manifold with simple singularities. If X_t is a Gutierrez-Sotomayor flow on M then there exists a collection of disjoint branched one-submanifolds B_i of M, $i = 0,1,\dots,m$, with the following properties:*

(1) $B_0 = \partial M^-$, $B_m = \partial M^+$
(2) *the flow X_t is transversal to each B_i*
(3) *each B_k, $k \neq 0,m$, splits M in two regions whose closures are denoted by G_k and H_k with $G_k \supset G_{k-1}$, $H_k \supset H_{k+1}$ and G_k contains exactly k singularities. Define $G_0 = B_0$, $H_0 = M$, $G_m = M$ and $H_m = B_m$. Hence, for $i = 0,\dots,m$, $G_i \cap H_i = B_i$ and $G_i \cup H_i = M$.*
(4) *B_k is the entering boundary of the flow X_t on G_k.*

Proof. By induction on k, let $B_0 = \partial M^-$ and assume we constructed B_{k-1} with

$$M = G_{k-1} \cup H_{k-1}, \quad G_{k-1} \cap H_{k-1} = B_{k-1},$$

G_{k-1} contains $k-1$ singularities and the entering boundary of the flow X_t in G_{k-1} is B_{k-1}. Now we will construct B_k.

Let $B_{k-1}\times[-1,1]$ be a product neighborhood[1] of B_{k-1} (in the case $k=1$ consider $B_{k-1}\times[0,1]$) with $B_{k-1} = B_{k-1}\times 0$, $B_{k-1}\times[0,1] \subset H_{k-1}$ and the flow X_t is transversal to $B_{k-1}\times t$ for each t.

(1) Let $p \in M(\mathcal{G})$, with $\mathcal{G} = \mathcal{R}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{W}$ or $\mathcal{T}$, be an attracting simple singularity of X_t.

By Theorem 2.2 we can choose a neighborhood N of p such that X_t is transversal to the boundary. Consider the disjoint union of N with G'_k to obtain G_k where G'_k is obtained by gluing to G_{k-1} the collar of B_{k-1} (contained in H_{k-1}), see Figure 3.

[1]Bicollar of B_{k-1} and a collar in the case $k=1$.

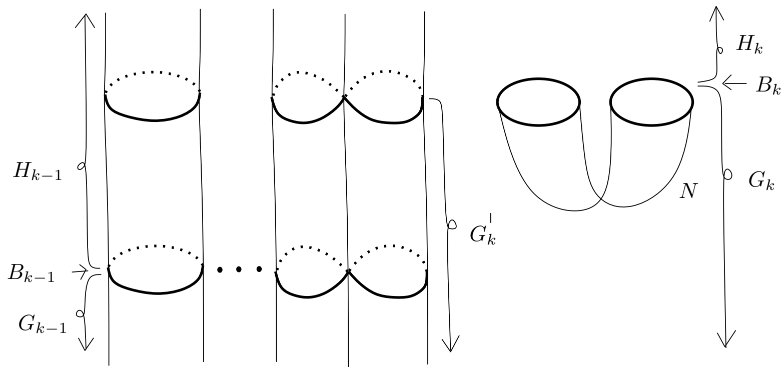

FIGURE 3. Construction of B_k

Hence, $B_k = \partial G_k$ is a disjoint union of branched one-manifolds with one more component than B_{k-1} if $p \in M(\mathcal{G})$ where $\mathcal{G} = \mathcal{R}$, $\mathcal{D}$, $\mathcal{W}$ or $\mathcal{T}$, and with two more components than B_{k-1} if $p \in M(\mathcal{C})$.

(2) Let $p \in M(\mathcal{G})$, with $\mathcal{G} = \mathcal{R}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{W}$ or $\mathcal{T}$, be a singularity of X_t which is not an attractor or repeller. We first construct S_ϵ for each singularity $p \in M(\mathcal{G})$ as the image of the exponential map $Exp : U \subset TM \to M$.

(a) If $p \in M(\mathcal{R}) \cup M(\mathcal{C})$ then by Theorem 2.2 we can choose a neighborhood N of p, a real valued function f on N and $\delta > 0$ such that the disk bounded by

$$f^{-1}(\delta) \cap W^s(p) = \widetilde{W}$$

is contained in N. Let E_ϵ be the normal bundle of $W^s(p)\backslash\{p\}$ in $M(\mathcal{R})$ restricted to $\widetilde{W}$ with vectors of magnitude $\leq \epsilon$. Denote by S_ϵ the image of E_ϵ under the exponential map.

(b) If $p \in M(\mathcal{W})$ then by Theorem 2.2 we can choose a neighborhood N of p, a real valued function f on N and $\delta > 0$ such that the disks bounded by

$$f^{-1}(\delta) \cap W^s(p) = \widetilde{W}$$

are in N. Let E_ϵ be the generalized normal bundle of $W^s(p)\backslash\{p\}$ in M restricted to $\widetilde{W}$ with vectors of magnitude $\leq \epsilon$. Denote by S_ϵ the image of E_ϵ under the exponential map restricted to each Q_p.

(c) If $p \in M(\mathcal{D})$ then by Theorem 2.2 we can choose a neighborhood N of p, a real valued function f on N and $\delta > 0$ such that the disks bounded by

$$f^{-1}(\delta) \cap W^s(p) \cap \overline{N\backslash W^s(p)} = \widetilde{W}$$

are in N. Let E_ϵ be the generalized normal bundle of $W^s(p) \cap \overline{N\backslash W^s(p)}$ in $\overline{N\backslash W^s(p)}$ restricted to $\widetilde{W}$ with vectors of magnitude $\leq \epsilon$. Denote by S_ϵ the image of E_ϵ under the exponential map restricted to each Q_p.

(d) if $p \in M(\mathcal{T})$ then by Theorem 2.2 we can choose a neighborhood N of p, a real valued function f on N and $\delta > 0$ such that the disks bounded by

$$f^{-1}(\delta) \cap W^s(p) \cap \overline{N\backslash W^s(p)} = \widetilde{W}$$

are in N. Let E_ϵ be the generalized normal bundle of $W^s(p) \cap \overline{N\backslash W^s(p)}$ in $\overline{N\backslash W^s(p)}$ restricted to $\widetilde{W}$ with vectors of magnitude $\leq \epsilon$. Denote by S_ϵ the image of E_ϵ under the exponential map restricted to each Q_p.

Choose ϵ sufficiently small such that S_ϵ is transversal to X. By the continuity of the flow X_t of X we can define $T : S_\epsilon \backslash \widetilde{W} \to \widetilde{V}$ which maps $x \in S_\epsilon \backslash \widetilde{W}$ to the point on the orbit of x which intersects $\widetilde{V}$.

Now, define a C^∞ embedding $F : \partial S_\epsilon \times [-1, 1] \to M$ by $F(x, -1) = x$, $F(x, 1) = T(x)$ and $F(x, t)$ is on the orbit that joins x to $T(x)$ and the distance from x to $F(x, t)$ is proportional to t. Extend F to a C^∞ embedding of $\partial S_\epsilon \times [-2, 2]$ that sends $x \times [-2, 2]$ to a regular orbit, for each x.Fix a Riemannian metric on $M(\mathcal{R})$ and let $v(p, t)$ be the unit normal vector field on the image of F with orientation given by (induced by) the vectors on ∂S_ϵ pointing outwards on $\widetilde{W}$. Let $\eta > 0$, be a small constant and $F_\eta(p, t)$ be the point at a distance ηt of $F(p, t)$ along the geodesic determined by $v(p, t)$, see Figure 4.

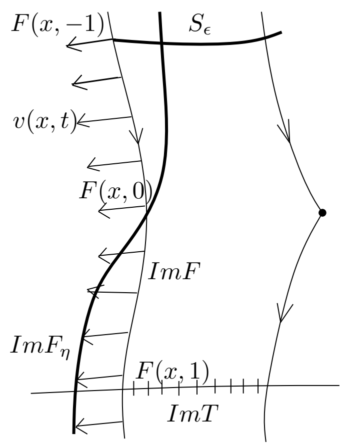

FIGURE 4. Construction of F_η

Choose small η such that the image of F_η, imF_η, is disjoint from the image of T, imT. Also, we have that X_t is transversal to imF_η, and $imF_\eta \cap S_\epsilon$, $imF_\eta \cap \widetilde{V}$ are diffeomorphic to $imF \cap S_\epsilon$, $imF \cap \widetilde{V}$, respectively.

In this way, we obtain a one-dimensional singular submanifold B'_k of M made up of:

- the part of S_ϵ bounded by $imF_\eta \cap S_\epsilon$;
- $\widetilde{V}$ except for regions bounded by $imF_\eta \cap \widetilde{V}$ that contains $W^u(p) \cap \widetilde{V}$;
- the part of imF_η bounded by $imF_\eta \cap S_\epsilon$ and $imF_\eta \cap \widetilde{V}$.

Thus, we have that X_t is transversal to B'_k. We verify that $M \backslash B'_k = G'_k \cup H'_k$ with G'_k containing G_{k-1} and the singular point p. Moreover, G'_k differs from G_k since $B'_k = \partial G'_k$ is not a differentiable submanifold, *i.e.,* differentiability fails along $imF_\eta \cap S_\epsilon$ and $imF_\eta \cap \widetilde{V}$. This can be smoothened easily in order to obtain the desired G_k and B_k. See Figure 5.

(3) Finally, if $p \in M(\mathcal{G})$, with $\mathcal{G} = \mathcal{R}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{W}$ or $\mathcal{T}$, is a repeller singularity of X_t then by Theorem 2.2 choose a neighborhood N of p whose boundary is transversal to X_t. Thus, $G_k = G_{k-1} \cup B_{k-1} \times [0, 1] \cup N$.

□

Lemma 2.6. *Let M be a two-manifold with simple singularities. If X_t is a Gutierrez-Sotomayor flow with only one singularity p then there is a Lyapunov function f on M such that f has value $c - \frac{1}{2}$ on ∂M^-, $c + \frac{1}{2}$ on ∂M^+ and $f(p) = c$.*

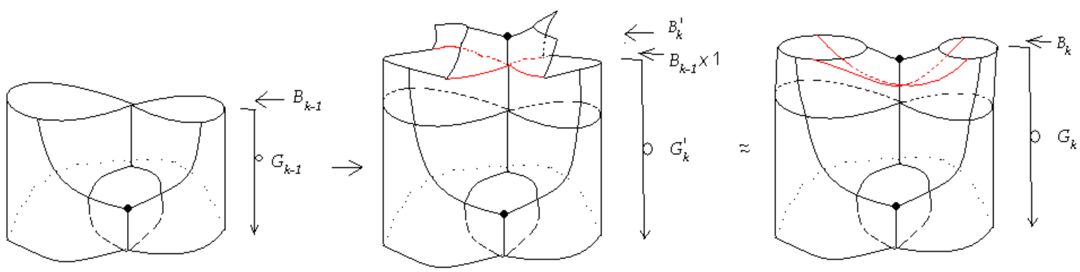

FIGURE 5. Smoothening B'_k to obtain B_k on a Whitney block

Proof. First define a function in a neighborhood of $W^s(p) \cup W^u(p)$. Let N be a neighborhood of p and f a function on N as in Theorem 2.2 and assume $f(p) = c$ by adding appropriate constants. Then let $f^{-1}(c+\delta) \cap N = R^+$, $f^{-1}(c-\delta) \cap N = R^-$, with δ chosen as in the previous lemma[2], $R^+_\epsilon = \{(u,v) \in R; \|v\| \leq \epsilon\}$ and $R^-_\epsilon = \{(u,v) \in R^-; \|u\| \leq \epsilon\}$.

Fix a Riemannian metric on $\mathbb{R}^l$ and take $\epsilon = \frac{1}{10}$. For $x \in R^+_\epsilon$ redefine f on $X_t(x)$, $t \leq 0$, such that $f(X_0(x)) = c + \delta$, $f(y) = c + \frac{1}{2}$ where y is the point of $X_t(x)$ that intersects ∂M^+. Define f proportional to the arclength of the points on the orbit that connect $X_0(x)$ and y. In this way, we obtain a function f in a neighborhood of $W^s(p)$ satisfying the required conditions on the boundary, although non-differentiable on $f^{-1}(c+\delta)$. We can smoothen f, see [7], so that it is C^∞ on $f^{-1}(c+\delta)$.

In a similar fashion, using R^-_ϵ, we obtain a real-valued function f defined in a neighborhood Q of $W^u(p)$ as well as in a neighborhood of $W^s(p)$, satisfying $f(Q \cap \partial M^-) = c - \frac{1}{2}$. Hence, we obtain the desired function f in an open neighborhood P of $W^s(p) \cup W^u(p)$. Without loss of generality, we can assume that if $x \in P$ then $X_t(x) \in P$, $\forall t$.

Now extend f to M. Choose $U \subset \partial M^- \cap P$ a compact neighborhood of $W^u(p) \cap \partial M^-$. Let λ be a C^∞ real valued function on ∂M^- satisfying $0 \leq \lambda \leq 1$ with $\lambda = 1$ on U and $\lambda = 0$ on $\partial M^- \backslash P \cap \partial M^-$. For $x \in M \backslash (W^s(p) \cup W^u(p))$ let $l(x)$ be the length of the orbit passing through x, $v(x)$ arclength of the orbit joining $\{X_t(x)\} \cap \partial M^-$ to x and $g(x) = c - \frac{1}{2} + \frac{v(x)}{l(x)}$. Hence, the function $\overline{\lambda} f + (1 - \overline{\lambda})g$ on M is the desired function where $\overline{\lambda}(x) = \lambda(X_t(x) \cap \partial M^-)$ or equals one if $X_t(x)$ does not intersect ∂M^-. □

Theorem 2.7. *Let M be a compact two-manifold with simple singularities. If X_t is a Gutierrez-Sotomayor flow on M then there exists a Lyapunov function f on M.*

Proof. Consider $G_k - G_{k-1}$, $\forall k$, defined in Lemma 2.5. Let f_k be the function in Lemma 2.6 defined on the closure of $G_k - G_{k-1}$. Juxtaposing the f_k we obtain a function f well defined on M and smooth [3] in a neighborhood of $B_1, \ldots, B_{m-1}$. Therefore, the desired Lyapunov function is obtained. □

3. The Conley Index

In this section, we compute the Conley homotopy index and homology index of simple singularities of a Gutierrez-Sotomayor flow X_t on M. We also prove a result relating the singularities on the regular and singular parts of X_t with the homology of M.

[2]The disk bounded by $f^{-1}(c+\delta) \cap W^s(p)$ is in N.
[3]As in the proof of Lemma 2.6.

A compact set $N \subset M$ is an *isolating neighborhood* if

$$Inv(N) := \{x \in N; X_t(x) \subset N,\ \forall t\} \subset int(N),$$

where $int(N)$ denotes the interior of N. Λ is an *isolated invariant set* if $\Lambda = Inv(N)$ for some isolating neighborhood N.

If Λ is an isolated invariant set, a topological pair of spaces[4] (N, L) is an *index pair* for Λ if:

(1) $\Lambda = Inv(cl(N \backslash L))$ and $N \backslash L$ is an isolating neighborhood for Λ.

(2) L is *positively invariant* in N, *i.e.,* given $x \in L$ and $X_t(x) \subset N$ for $t \in [0, t_0]$ then $X_t(x) \subset L$ for $t \in [0, t_0]$.

(3) L is an *exit set* for N; *i.e.,* given $x \in N$ and $t_1 > 0$ such that $X_{t_1}(x) \notin N$ then there exists $t_0 \in [0, t_1]$ such that $X_t(x) \subset N$, for $t \in [0, t_0]$, and $X_{t_0}(x) \in L$.

In [2], Conley proves the existence of an index pair (N, L) for an isolated invariant set Λ. Furthermore, if (N, L) and (N', L') are index pairs for an isolated invariant set Λ then $(N/L, [L])$ has the same homotopy type as $(N'/L', [L'])$.

In what follows we define, the homotopy index as the homotopy type of the pointed space $(N/L, [L])$. Since homology is an invariant of homotopic spaces thus the homology index is well defined.

Definition 3.1. *We define:*

(1) *The Conley homotopic index of* Λ*,* $h(\Lambda)$*, is the homotopy type of the pointed space* $(N/L, [L])$ *where* (N, L) *is an index pair for* Λ*.*

(2) *The Conley homology index of* Λ *is defined by* $CH_*(\Lambda) := H_*(h(\Lambda))$ *where* H_* *denotes the homology on* $\mathbb{Z}$*.*

(3) *The numerical Conley indices of* Λ *are defined as the ranks of the Conley homology indices of* Λ*,* $h_* = rank CH_*(\Lambda)$*.*

In order to compute the Conley homology indices we make use of the isomorphism:

$$\widetilde{H}_n(X \vee Y) \approx \widetilde{H}_n(X) \oplus \widetilde{H}_n(Y)$$

if the base points of X and Y which are identified in $X \vee Y$ are deformation retracts of neighborhoods $U \subset X$ and $V \subset Y$.

If $p \in M$ is a singularity of a Gutierrez-Sotomayor flow X_t and N a sufficiently small neighborhood, as in the proof of Lemma 2.2. We say that p is of:

- type **a** if $p \in M(\mathcal{G})$, where $\mathcal{G} = \mathcal{R}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{W}$ or $\mathcal{T}$, is an attracting singularity.
- type **s** if $p \in M(\mathcal{R}) \cup M(\mathcal{C})$ is neither an attracting or repelling singularity.
- type **r** if $p \in M(\mathcal{G})$, where $\mathcal{G} = \mathcal{R}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{W}$ or $\mathcal{T}$, is a repelling singularity.
- type $\mathbf{s_u}$ if $p \in M(\mathcal{W})$ is a saddle singularity on a bidimensional disc with the unstable manifold identified to the fold.
- type $\mathbf{s_s}$ if $p \in M(\mathcal{W})$ is a saddle singularity on a bidimensional disc with the stable manifold identified to the fold.
- type **sa** if $p \in M(\mathcal{D})$ and N is formed by a sink and a saddle.
- type **sr** if $p \in M(\mathcal{D})$ and N is formed by a source and a saddle identified at the fold.
- type $\mathbf{ss_u}$ if $p \in M(\mathcal{D})$ and N is formed by two saddles with their unstable manifolds identified to the fold.
- type $\mathbf{ss_s}$ if $p \in M(\mathcal{D})$ and N is formed by two saddles with their stable manifolds identified to the fold.

[4]A *topological pair of spaces* is an ordered pair (N, L) of spaces such that L is a closed subspace of N.

- type **ssa** if $p \in M(\mathcal{T})$ and N is formed by a sink and two saddles.
- type **ssr** if $p \in M(\mathcal{T})$ and N is formed by a source and two saddles.

3.1. **Conley index of GS Singularities.** In the next theorem we compute the Conley homotopy index, as well as, the ranks of the homology indices.

Theorem 3.2. *Let M be a two-manifold with simple singularities and X_t a Gutierrez-Sotomayor flow on M. Let p be a singularity of X_t with type specified in the table below. Then, the numerical Conley index of each type of singularity is as given in the table.*

Type	$p \in M(\mathcal{R})$	$p \in M(\mathcal{C})$	$p \in M(\mathcal{W})$	$p \in M(\mathcal{D})$	$p \in M(\mathcal{T})$
a	S^0	S^0	S^0	S^0	S^0
	$(1,0,0)_\mathcal{R}$	$(1,0,0)_\mathcal{C}$	$(1,0,0)_\mathcal{W}$	$(1,0,0)_\mathcal{D}$	$(1,0,0)_\mathcal{T}$
s	S^1	S^1	—	—	—
	$(1,0,0)_\mathcal{R}$	$(0,1,0)_\mathcal{C}$	—	—	—
$\mathbf{s_u}$	—	—	$\bar{0}$	—	—
	—	—	$(0,0,0)_\mathcal{W}$	—	—
$\mathbf{s_s}$	—	—	S^1	—	—
	—	—	$(0,1,0)_\mathcal{W}$	—	—
sa	—	—	—	S^1	—
	—	—	—	$(0,1,0)_\mathcal{D}$	—
sr	—	—	—	S^2	—
	—	—	—	$(0,0,1)_\mathcal{D}$	—
$\mathbf{ss_u}$	—	—	—	S^1	—
	—	—	—	$(0,1,0)_\mathcal{D}$	—
$\mathbf{ss_s}$	—	—	—	$\vee_{i=1}^3 S^1$	—
	—	—	—	$(0,3,0)_\mathcal{D}$	—
ssa	—	—	—	—	S^1
	—	—	—	—	$(0,1,0)_\mathcal{T}$
ssr	—	—	—	—	S^2
	—	—	—	—	$(0,0,1)_\mathcal{T}$
r	S^2	$S^2 \vee S^2 \vee S^1$	$S^2 \vee S^2$	$\vee_{i=1}^3 S^2$	$\vee_{i=1}^7 S^2$
	$(0,0,1)_\mathcal{R}$	$(0,1,2)_\mathcal{C}$	$(0,0,2)_\mathcal{W}$	$(0,0,3)_\mathcal{D}$	$(0,0,7)_\mathcal{T}$

Proof. If p is a singularity of X_t, we choose an index pair (N, L) for p in M and calculate the Conley homotopic index $h(p)$. The homology $CH_i(p)$ has a factor $\mathbb{Z}$ for each S^i of the homotopical index, thus the Conley numerical index (h_0, h_1, h_2) in each case of Theorem 3.2 is obtained. See Figures 6 through 22.

(1) If $p \in M(\mathcal{R})$, let N be a closed disk and $L = \partial N^-$ the exiting set of N. Thus, the Conley homotopy index of p is S^0 (S^1 or S^2) if p is an attractor (saddle or repeller) singularity.

(2) If $p \in M(\mathcal{C})$, a neighborhood N of p in M is formed by two disks D_1 and D_2 centered at p such that $D_1 \cap D_2 = \{p\}$.
   (a) If p is of type **a** then $L = \emptyset$ and thus is identified to a point. On the other hand, it is easy to see that the double cone, when retracted along the stable manifold of N, has the homotopy type of a point. Hence, $h(p) = S^0$. See Figure 6.
   (b) If p is of type **s** then $w^u(p) \cap \partial N = \{x_1, x_2\}$ where $x_i \in \partial D_i$, $i = 1, 2$. Let $C_i \subset \partial D_i$, $i = 1, 2$, be the two arcs from which the flow exits, then $x_i \in C_i$, $i = 1, 2$ and $L = C_1 \cup C_2$ is the exit set for N. Collapsing L to a point and retracting along

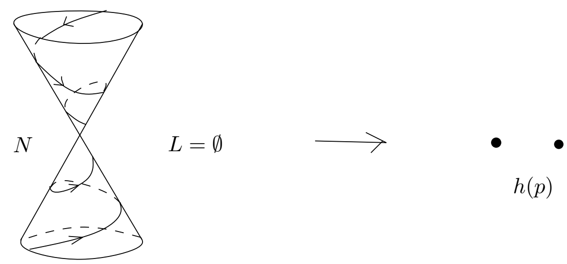

FIGURE 6. Conley index of a singularity of type **a** in $M(\mathcal{C})$

the stable manifold of N we conclude that N/L has the homotopy type of S^1, *i.e.,* $h(p) = S^1$. See Figure 7.

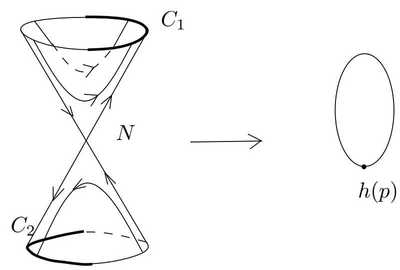

FIGURE 7. Conley index of a singularity of type **s** in $M(\mathcal{C})$

(c) If p is of type **r** then $L = \partial N = \partial D_1 \cup \partial D_2$. Collapsing L to a point we conclude that N/L has the homotopy type of $S^2 \vee S^2 \vee S^1$, *i.e.,* $h(p) = S^2 \vee S^2 \vee S^1$. See Figure 8.

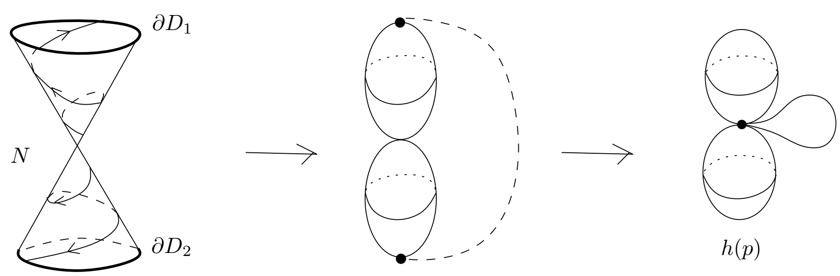

FIGURE 8. Conley index of a singularity of type **r** in $M(\mathcal{C})$

(3) If $p \in M(\mathcal{D})$, a neighborhood N of p in M is formed by two disks D_i, $i = 1, 2$, that intersect transversally along two diameters d_1 and d_2 in D_1 and D_2 respectively.
   (a) If p is of type **a** then $L = \emptyset$ and hence is identified to a point. On the other hand, it is easy to see that by retracting the stable manifold on N it has the homotopy type of a point, hence, $h(p) = S^0$. See Figure 9.
   (b) If p is of type **r** then $L = \partial N = \partial D_1 \cup \partial D_2$ where ∂D_1 and ∂D_2 intersect transversally at two points. Collapsing L to a point we conclude that N/L has the homotopy type of $S^2 \vee S^2 \vee S^2$, *i.e.,* $h(p) = S^2 \vee S^2 \vee S^2$. See Figure 10.

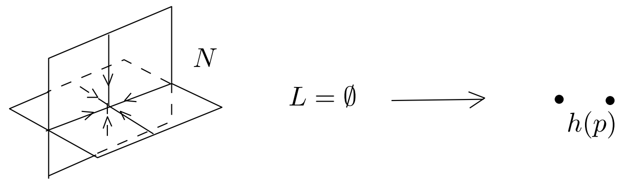

FIGURE 9. Conley index of a singularity of type **a** in $M(\mathcal{D})$

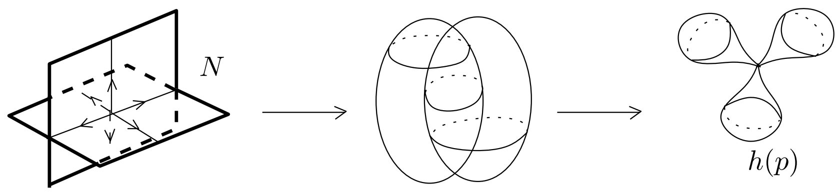

FIGURE 10. Conley index of a singularity of type **r** in $M(\mathcal{D})$

(c) If p is of type **sa** then $w^u(p) \cap \partial N = \{x_1, x_2\}$ where $x_1, x_2 \in \partial D_i$ and D_i is the disk that contains the saddle. Let $C_1, C_2 \subset \partial D_i$ be the two arcs from which the flow exits N hence $x_i \in C_i$, $i = 1, 2$ and $L = C_1 \cup C_2$ is the exit set for N. Collapsing L to a point and retracting along the stable manifold of N we conclude that N/L has the homotopy type of S^1, *i.e.*, $h(p) = S^1$. See Figure 11.

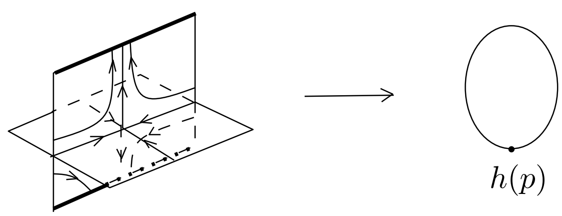

FIGURE 11. Conley index of a singularity of type **sa** in $M(\mathcal{D})$

(d) If p is of type **sr** then $w^u(p) \cap \partial N = \partial D_i$ where D_i is the disk that contains the repeller. Let $C_1, C_2 \subset \partial D_j$, $j \neq i$, be the two transversal arcs to ∂D_i from where the flow exits hence $L = \partial D_i \cup C_1 \cup C_2$ is the exit set for N. Collapsing L to a point and retracting along the stable manifold of N we conclude that N/L has the homotopy type of S^2, *i.e.*, $h(p) = S^2$. See Figure 12.

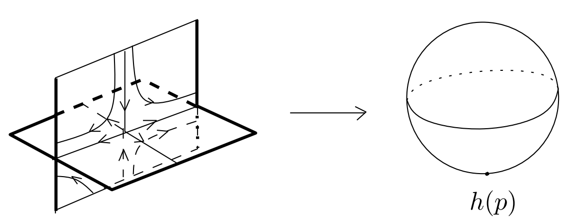

FIGURE 12. Conley index of a singularity of type **sr** in $M(\mathcal{D})$

(e) If p is of type $\mathbf{ss_u}$ then

$$w^u(p) \cap \partial N = \{x_1, x_2\},$$

where $x_1, x_2 \in \partial D_1$ and $x_1, x_2 \in \partial D_2$. Let $B_1, B_2 \subset \partial D_1$ and $C_1, C_2 \subset \partial D_2$ be the arcs from where the flow exits, $B_i \pitchfork C_i = \{x_i\}$ and $L = (B_1 \cup C_1) \sqcup (B_2 \cup C_2)$ is the exit set for N. Collapsing L to a point and retracting along the stable manifold of N we conclude that N/L has the homotopy type of S^1, *i.e.*, $h(p) = S^1$. See Figure 13.

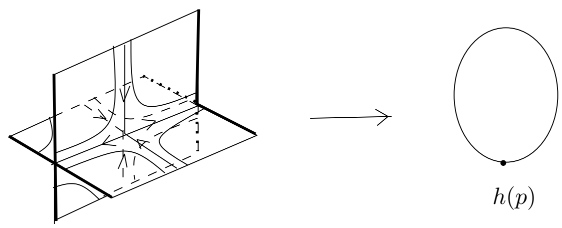

FIGURE 13. Conley index of a singularity of type $\mathbf{ss_u}$ in $M(\mathcal{D})$

(f) If p is of the type $\mathbf{ss_s}$ then $w^u(p) \cap \partial N = \{x_1, x_2, y_1, y_2\}$ where $x_1, x_2 \in \partial D_1$ and $y_1, y_2 \in \partial D_2$. Let $B_1, B_2 \subset \partial D_1$ and $C_1, C_2 \subset \partial D_2$ be the arcs from where the flow exits, $x_i \in B_i$, $i = 1, 2$, $y_i \in C_i$, $i = 1, 2$, and $L = B_1 \sqcup B_2 \sqcup C_1 \sqcup C_2$ is the exit set for N. Collapsing L to a point and retracting along the stable manifold of N we conclude that N/L has the homotopy type of $S^1 \vee S^1 \vee S^1$, *i.e.*, $h(p) = \bigvee_{i=1}^{3} S^1$. See Figure 14.

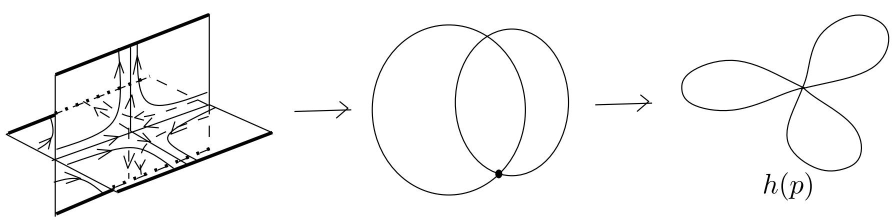

FIGURE 14. Conley index of a singularity of type $\mathbf{ss_s}$ in $M(\mathcal{D})$

(4) If $p \in M(\mathcal{W})$, a neighborhood N of p in M is a disk D with two distinct rays r_1 and r_2 identified.
   (a) If p is of type $\mathbf{a}$ then $L = \emptyset$ and hence is identified to a point. On the other hand, it is easy to see that by retracting the stable manifold of N, it has the homotopy type of a point, hence, $h(p) = S^0$.
   (b) If p is of type $\mathbf{r}$ then $L = \partial N$ is homeomorphic to a figure "eight". Collapsing L to a point we conclude that N/L has the homotopy type of $S^2 \vee S^2$, *i.e.*, $h(p) = S^2 \vee S^2$. See Figure 16.
   (c) If p is of type $\mathbf{s_u}$ then $w^u(p) \cap \partial N = \{x\}$. Let $C_1, C_2 \subset \partial N$ be the arcs from where the flow exits, hence, $C_1 \pitchfork C_2 = \{x\}$ and $L = C_1 \cup C_2$ is the exit set for N. Collapsing L to a point and retracting along the stable manifold of N we conclude that N/L has the homotopy type of a point, *i.e.*, $h(p) = \overline{0}$. See Figure 17.

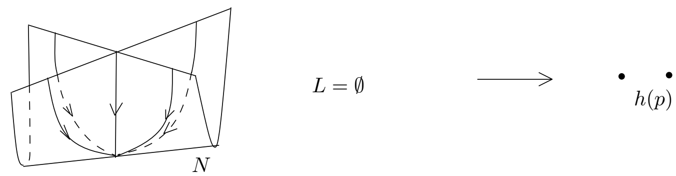

FIGURE 15. Conley index of a singularity of type **a** in $M(\mathcal{W})$

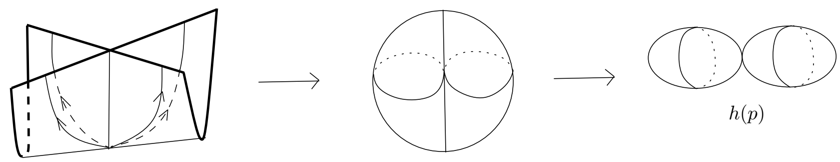

FIGURE 16. Conley index of a singularity of type **r** in $M(\mathcal{W})$

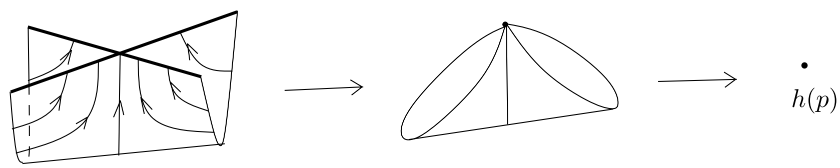

FIGURE 17. Conley index of a singularity of type $\mathbf{s_u}$ in $M(\mathcal{W})$

(d) If p is of type $\mathbf{s_s}$ then $w^u(p) \cap \partial N = \{x_1, x_2\}$. Let $C_i \subset \partial N$, $i = 1, 2$, be the arcs from where the flow exits, hence, $x_i \in C_i$, $i = 1, 2$ and $L = C_1 \cup C_2$ is the exit set for N. Collapsing L to a point and retracting along the stable manifold of N we conclude that N/L has the homotopy type of S^1, *i.e.,* $h(p) = S^1$. See Figure 18.

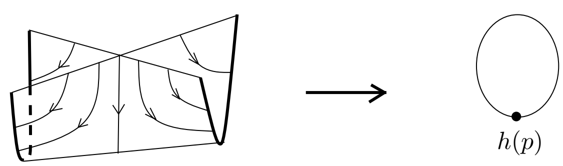

FIGURE 18. Conley index of a singularity of type $\mathbf{s_s}$ in $M(\mathcal{W})$

(5) If $p \in M(\mathcal{T})$, a neighborhood N of p in M is formed by three disks D_i, $i = 1, 2, 3$, that intersect transversally in pairwise disjoint diagonals that go through the point p.
  (a) If p is of the type **a** then $L = \emptyset$ and thus is identified to a point. On the other hand,it is easy to see that by retracting the stable manifold of N, it has the homotopy type of a point, hence, $h(p) = S^0$.
  (b) If p is of type **r** then $L = \partial N = \partial D_1 \cup \partial D_2 \cup \partial D_3$ where ∂D_1, ∂D_2 and ∂D_3 intersect transversally pairwise at two points. Collapsing L to a point we conclude that N/L has the homotopy type of $\vee_{i=1}^{7} S^2$, *i.e.,* $h(p) = \vee_{i=1}^{7} S^2$. See Figure 20.
  (c) If p is of type **ssa** then $w^u(p) \cap \partial N = \{x_1, x_2\}$ where $x_1, x_2 \in \partial D_2$ and $x_1, x_2 \in \partial D_3$. Let $B_1, B_2 \subset \partial D_2$ and $C_1, C_2 \subset \partial D_3$ be the arcs from where the flow exits, hence, $B_i \pitchfork C_i = \{x_i\}$ and $L = (B_1 \cup C_1) \sqcup (B_2 \cup C_2)$ is the exit set for N. Collapsing L

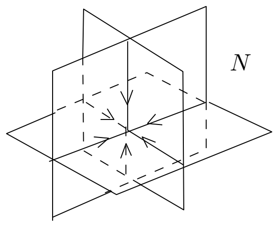

FIGURE 19. Conley index of a singularity of type **a** in $M(\mathcal{T})$

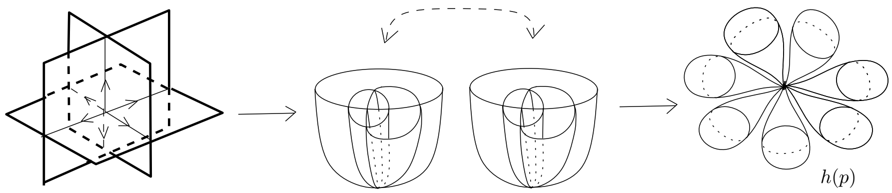

FIGURE 20. Conley index of a singularity of type **r** in $M(\mathcal{T})$

to a point and retracting along the stable manifold of N we conclude that N/L has the homotopy type of S^1, *i.e.,* $h(p) = S^1$. See Figure 21.

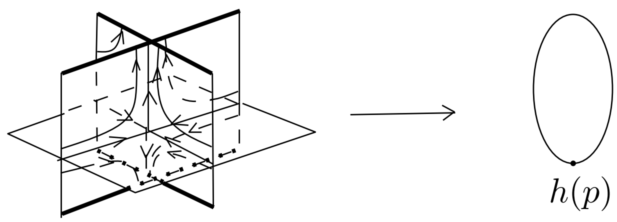

FIGURE 21. Conley index of a singularity of type **ssa** in $M(\mathcal{T})$

(d) If p is of type **ssr** then $w^u(p) \cap \partial N = \partial D_1$ where D_1 is the disk which contains the repeller. Let $B_1, B_2 \subset \partial D_2$ and $C_1, C_2 \subset \partial D_3$ transversal arcs to ∂D_1 from where the flow exits, hence, $L = \partial D_1 \cup B_1 \cup B_2 \cup C_1 \cup C_2$ is the exit set for N. Collapsing L to a point and retracting along the stable manifold of N we conclude that N/L has the homotopy type of S^2, *i.e.,* $h(p) = S^2$. See Figure 22.

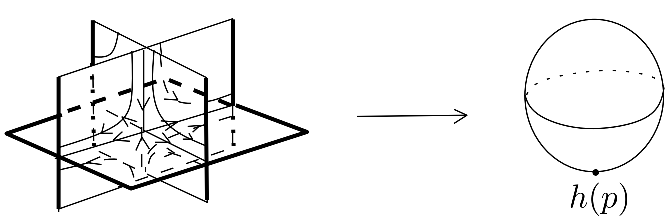

FIGURE 22. Conley index of a singularity of type **ssr** in $M(\mathcal{T})$

It is straightforward to compute the homology of the Conley indices $CH_*(\Lambda)$ and its ranks $h_* = \text{rank } CH_*(\Lambda)$ in each case of $\Lambda = \{p\}$ within this proof. Thus, this numerical Conley index appears in the table as

$$(h_0, h_1, h_2) = (\text{rank } CH_0(\Lambda),\ \text{rank } CH_1(\Lambda),\ \text{rank } CH_2(\Lambda)).$$

□

3.2. **Euler type Characteristic Formulas for GS manifolds.** Let $X = |K|$ be a topological space of dimension n. Define α_j as the number of j-simplices of K. The Poincaré Theorem asserts that the sum $\sum_{j=0}^{n}(-1)^j\alpha_j$ is independent of the simplicial complex K, such that $X = |K|$. This number is the *Euler-Poincaré Characteristic* and is denoted by $\chi(X)$. Also, Poincaré asserts the equality

$$\chi(X) = \sum_{j=0}^{n}(-1)^j\beta_j,$$

where β_j where the rank of $H_j(K)$ is the j-th Betti number of K.

For example, $\chi(S^2) = 2$, $\chi(\text{pinched sphere}) = 3$, $\chi(\text{pinched torus}) = 1$,

$$\chi(\text{sine torus or torus with a fold}) = 1$$

and $\chi(\text{crosscap}) = 2$. See Figure 24 and Figure 25.

We next present the Morse-Conley inequalities for manifolds with simple singularities. We make use of the ranks of the homology indices computed in Theorem 3.2.

Proposition 3.3. *Let M be a two-manifold with simple singularities and X_t a Gutierrez-Sotomayor flow on M with limit set $\mathfrak{L} = \bigcup_{i=1}^{m} L_i$. If (h_0^i, h_1^i, h_2^i) is the numerical Conley index of L_i then*

$$\sum_{i=1}^{m}(h_0^i - h_1^i + h_2^i) = \chi(M) \tag{1}$$

where $\chi(M)$ is the Euler characteristic of M.

Proof. Let f be a Lyapunov function associated to X_t and $G_k \subset M$ as in the proof of Theorem 2.7. Hence, $G_0 \subset G_1 \subset \cdots \subset G_m$ such that (G_i, G_{i-1}) is an index pair for L_i. Consider the long exact sequence of the pair (G_i, G_{i-1})

$$\cdots \overset{p_j}{\to} H_j\left(G_i, G_{i-1}\right) \overset{\partial_j}{\to} H_{j-1}\left(G_{i-1}\right) \overset{i_*}{\to} H_{j-1}\left(G_i\right) \overset{p_{j-1}}{\to} H_{j-1}\left(G_i, G_{i-1}\right) \overset{\partial_{j-1}}{\to} \cdots$$

By exactness,

$$\begin{aligned}\dim \text{ im } (p_j) &= \dim \ker\ (\partial_j) = \dim\ H_j\left(G_i, G_{i-1}\right) - \dim \text{ im } (\partial_j)\\ &= \dim\ H_j\left(G_i, G_{i-1}\right) - \dim \ker\ (i_*)\end{aligned}$$

$$\begin{aligned}\dim \text{ im } (p_{j-1}) &= -\dim \ker\ (p_{j-1}) + \dim\ H_{j-1}(G_i)\\ &= -\dim \text{ im } (i_*) + \dim\ H_{j-1}(G_i).\end{aligned}$$

Thus,

$$\begin{aligned}&\dim \text{ im } (p_j) + \dim \text{ im } (p_{j-1}) =\\ &\dim\ H_j\left(G_i, G_{i-1}\right) - \dim \ker\ (i_*) - \dim \text{ im } (i_*) + \dim\ H_{j-1}(G_i) =\\ &\dim\ H_j\left(G_i, G_{i-1}\right) - \dim\ H_{j-1}(G_{i-1}) + \dim\ H_{j-1}(G_i).\end{aligned}$$

Since $CH_*(L_i) \cong H_*(G_i, G_{i-1})$, then $h_j(L_i) = \dim H_j(G_i, G_{i-1})$. Thus,

$$\dim \text{im } (p_j) + \dim \text{im } (p_{j-1}) = h_j(L_i) - \beta_{j-1}(G_{i-1}) + \beta_{j-1}(G_i).$$

For fixed i, consider the alternated sum over j:

$$\sum_{j=0}^{2}(-1)^j h_j(L_i) + \sum_{j=0}^{3}(-1)^j(\beta_{j-1}(G_i) - \beta_{j-1}(G_{i-1})) = 0.$$

Now, consider the sum of the above expression for $i = 1, \ldots, m$

$$\sum_{i,j}(-1)^j h_j(L_i) + \sum_{j=0}^{3}(-1)^j(\beta_{j-1}(G_m)) = 0.$$

Since $G_m = M$, we obtain the desired result $\chi(M) = \sum_{i,j}(-1)^j h_j^i$, for $i = 1, \ldots, m$ and $j = 0, 1, 2$. □

3.3. **Conley Index restricted to the Strata.** The calculations in the previous section were realized considering isolating neighborhoods of a simple singularity in M. However, one may also compute the Conley indices of the simple singularities of X_t with respect to subspaces of M. In particular, with respect to the singular part of a stratification of M.

A two-manifold with simple singularities M equipped with a partition $\{M(\mathcal{G}), \mathcal{G}\}$ is a stratified manifold. One can define a partition, by distinguishing the regular part from the singular part, as follows:

- $\mathcal{R}$ is the union of the strata of dimension 2.
- $\mathcal{S} = M \backslash \mathcal{R}$ is the union of the strata of dimension 0 and 1.

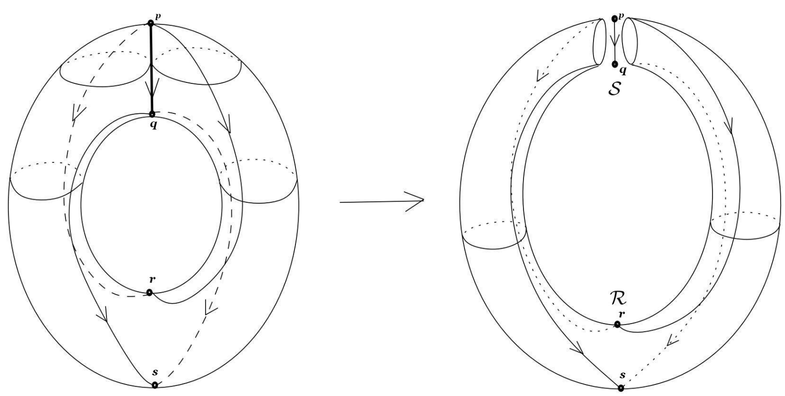

Figure 23. Stratification of the sine torus.

A stratification for $M = \mathcal{R} \sqcup \mathcal{S}$, where $\sqcup$ is a disjoint union. Hence, all points in $\mathcal{S}$ are singular points of the stratification. Observe that $p \in \mathcal{S}$ is not necessarily a singular point of the manifold nor of the flow. In the same way, a singular point of the manifold is not necessarily a singular point of the flow.

Consider the example in Figure 23. The points p, q, r, s are singularities of the flow. All points in $\mathcal{S}$ are singular points of the stratification as well as singular points of the manifold. In the example in Figure 24 (left), $\mathcal{S}$ is the figure "eight" and on it there are 5 singularities of the flow and on $\mathcal{R}$ there are an additional 4 singularities of the flow. All points on the figure "eight" are singular points of the stratification but only the cone point is a singular point of the manifold.

Consider the polar flow on S^2, one repeller and one attractor. Define the singular part, $\mathcal{S}$, to be a great circle C that contains these two singularities. The flow has two singularities, north and south pole. All points in C are singular points of the stratification and there are no singular points of the manifold.

Note that in this last example, a neighborhood U of $\mathcal{S}$, contains orbits of the flow that are both entering and exiting U. We will not consider this type of stratification. We will require that a neighborhood U of $\mathcal{S}$ is either an attracting or repelling basin.

Definition 3.4. *Let $\mathcal{E}$ be a stratification of M and $U_{\mathcal{S}}$ a tubular neighborhood of $\mathcal{S}$, the singular part of the stratification $\mathcal{E}$, of M. We define the distinguished class $\Sigma_{\mathcal{E}}$ of the stable vector fields, as*

$$\Sigma_{\mathcal{E}} = \{X \in \Sigma(M)\colon X \text{ either points inward i.e., } \partial U_{\mathcal{S}} \text{ is the incoming set, or points outward i.e., } \partial U_{\mathcal{S}} \text{ is the exit set, but not both}\}$$

The pair $(X, \mathcal{E})$ is called a distinguished field on M if $X \in \Sigma_{\mathcal{E}}$ and in the case of a flow $(X_t, \mathcal{E})$ is called a distinguished flow.

In what follows we will compute the Conley indices of a GS flow with respect to the stratification $\mathcal{E}$ of $M = \mathcal{R} \sqcup \mathcal{S}$, *i.e.,* if $p \in \mathcal{R}$ is a singularity of X_t the Conley index will be computed with respect to $\mathcal{R}$ and if $p \in \mathcal{S}$ it will be computed with respect to $\mathcal{S}$. In order to compute the Conley index relative to the singular strata, choose an index pair (N, L) in $\mathcal{S}$. Then the Conley numerical index

$$(s_0, s_1) = (\text{rank } H_0(N/L), \text{rank } H_1(N/L)),$$

of $p \in \mathcal{S}$.

We establish the following notation:

- $\mathcal{R}_0 = \sum_{p\in\mathcal{R}} h_0(p)$, $\mathcal{R}_1 = \sum_{p\in\mathcal{R}} h_1(p)$ and $\mathcal{R}_2 = \sum_{p\in\mathcal{R}} h_2(p)$, where $h_i(p)$ is the i-th Conley numerical index of p.
- $\mathcal{S}_0 = \sum_{p\in\mathcal{S}} s_0(p)$ and $\mathcal{S}_1 = \sum_{p\in\mathcal{S}} s_1(p)$.

Note that in Proposition 3.3, we did not take into account a stratification on M. Hence, if we do not take into account a stratification, the above notation implies that equation (1) in Proposition 3.3 can be rewritten as:

$$\mathcal{R}_2 - \mathcal{R}_1 + \mathcal{R}_0 = \chi(M). \tag{2}$$

Let us consider an example of this calculation restricted to the strata.

Example 3.5. *Consider the pinched torus in Figure 24, where the singular part is a circle. The two dimensional stratum is the complement of this circle, a disk and is the regular part. Although the circle itself is not singular, the cone singularity on that circle is a singular point of the manifold and of the flow. This cone singularity is a zero-dimensional stratum and its complement on the circle is the one-dimensional stratum. This flow has three singularities, a repeller in the regular part and two singularities in the singular part.*

Hence, the Conley index of this repeller is $h(p) = S^2$ and its homology index,

$$CH_i(p) = \begin{cases} \mathbb{Z} & \text{if } i = 2 \\ 0 & \text{otherwise.} \end{cases}$$

Hence, the Conley numerical index of the regular part is $(h_0, h_1, h_2) = (0, 0, 1)$.

The singularities of the singular part $\mathcal{S}$, are in S a repeller and an attractor. The repeller in $\mathcal{S}$ has Conley index $h(p) = S^1$ and homological index:

$$CH_i(p) = \begin{cases} \mathbb{Z} & \text{if } i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Hence, the numerical Conley index is $(s_0, s_1) = (0, 1)$. *The attractor in* $\mathcal{S}$ *has Conley index* $h(p) = S^0$ *and homology index:*

$$CH_i(p) = \begin{cases} \mathbb{Z} & se \ \ i = 0 \\ 0 & otherwise. \end{cases}$$

Hence, the numerical Conley index is $(s_0, s_1) = (1, 0)$.

Theorem 3.7 relates the Euler characteristic of the regular part and the Euler characteristic of the singular part of Gutierrez-Sotomayor flows X_t on M, both expressed in terms of the numerical Conley indices to the Euler characteristic of M.

We first prove a lemma that shows that the numerical Conley indices of the singular part $\mathcal{S}$ of M is the same if computed with respect to M or with respect to $\mathcal{S}$.

Lemma 3.6. *Let* M *be a two-manifold with simple singularities and* X_t *the Gutierrez-Sotomayor flow on* M. *If* M *admits a stratification* $\mathcal{E}$ *such that* $(X_t, \mathcal{E})$ *is a distinguished flow then for the singularities* $\{p_1, p_2, \dots, p_n\} \subset \mathcal{S}$ *the following holds:*

$$\mathcal{R}_0 - \mathcal{R}_1 + \mathcal{R}_2 = \mathcal{S}_0 - \mathcal{S}_1 \tag{3}$$

Proof.

$$\mathcal{R}_2 - \mathcal{R}_1 + \mathcal{R}_0 = \chi(\overline{U_{\mathcal{S}}}) = \chi(\mathcal{S}) = -\mathcal{S}_1 + \mathcal{S}_0$$

The first equality follows from Proposition 3.3, the second equality follows from the fact that $\overline{U_{\mathcal{S}}}$ is a deformation retract of $\mathcal{S}$. Finally the third equality follows from Proposition 3.3 adjusted to the one dimensional setting. □

Theorem 3.7. *Let* M *be a two-manifold with simple singularities and* X_t *a Gutierrez-Sotomayor flow on* M. *If* M *admits a stratification* $\mathcal{E}$ *such that* $(X_t, \mathcal{E})$ *is a distinguished flow then*

$$(\mathcal{R}_2 - \mathcal{R}_1 + \mathcal{R}_0)|_{M \setminus \mathcal{S}} = \mathcal{S}_1 - \mathcal{S}_0 + \chi(M) \tag{4}$$

Proof. Consider a sufficiently small tubular neighborhood, $U_{\mathcal{S}}$ of the singular part $\mathcal{S}$ of M which contains no other singularities apart from the ones in $\mathcal{S}$. Suppose that on $\partial U_{\mathcal{S}}$, X points inward to $U_{\mathcal{S}}$ and denote by $\tilde{M} = M - U_{\mathcal{S}}$. Then by Proposition 3.3 we have that:

$$(\mathcal{R}_2 - \mathcal{R}_1 + \mathcal{R}_0)|_{\tilde{M}} = \chi(\tilde{M}, \partial\tilde{M}^-) \tag{5}$$

On the other hand, M is a CW-complex formed by the union of subcomplexes $\tilde{M}$ and $\overline{U_S}$ hence, $\chi(M) = \chi(\tilde{M}) + \chi(\overline{U_S}) - \chi(\partial\tilde{M}^-)$ since $\tilde{M} \cap \overline{U_S} = \partial\tilde{M} = \partial\tilde{M}^-$. Using the exact sequence of the pair $(\tilde{M}, \partial\tilde{M}^-)$ we have that $\chi(\tilde{M}, \partial\tilde{M}^-) = \chi(\tilde{M}) - \chi(\partial\tilde{M}^-)$. Thus,

$$\chi(\tilde{M}) + \chi(\overline{U_S}) - \chi(\partial\tilde{M}^-) = \chi(M)$$
$$\chi(\tilde{M}, \partial\tilde{M}^-) + \chi(\overline{U_S}) = \chi(M)$$
$$(\mathcal{R}_2 - \mathcal{R}_1 + \mathcal{R}_0)|_{\tilde{M}} + \chi(S) = \chi(M)$$

Since $X \in \Sigma_{\mathcal{E}}$ in $U_{\mathcal{S}} \setminus \mathcal{S}$ has no fixed points then $(\mathcal{R}_2 - \mathcal{R}_1 + \mathcal{R}_0)|_{U_{\mathcal{S}} \setminus \mathcal{S}} = 0$, thus from the above equality we have that:

$$(\mathcal{R}_2 - \mathcal{R}_1 + \mathcal{R}_0)|_{M \setminus \mathcal{S}} + \mathcal{S}_0 - \mathcal{S}_1 = \chi(M)$$
$$(\mathcal{R}_2 - \mathcal{R}_1 + \mathcal{R}_0)|_{M \setminus \mathcal{S}} = \mathcal{S}_1 - \mathcal{S}_0 + \chi(M)$$

□

Corollary 3.8. *Let* M *be a two-manifold with simple singularities and* X_t *a Gutierrez-Sotomayor flow on* M. *If* M *admits a stratification* $\mathcal{E}$ *such that* $(X_t, \mathcal{E})$ *is a distinguished flow then*

$$(\mathcal{R}_2 - \mathcal{R}_1 + \mathcal{R}_0)|_{M \setminus \mathcal{S}} = \chi(M) - \chi(S) \tag{6}$$

Proof. Follows directly from Theorem 3.7. □

3.3.1. *Examples.* Fix a stratification $\mathcal{E}$ on a a two-manifold with simple singularities of X_t, a Gutierrez-Sotomayor flow on M. Let $(X_t, \mathcal{E})$ be a distinguished flow.

Example 3.9. *Let M be a two manifold with cone singularities (e.g. a pinched sphere or a pinched torus) with stratification $\mathcal{E}$.*

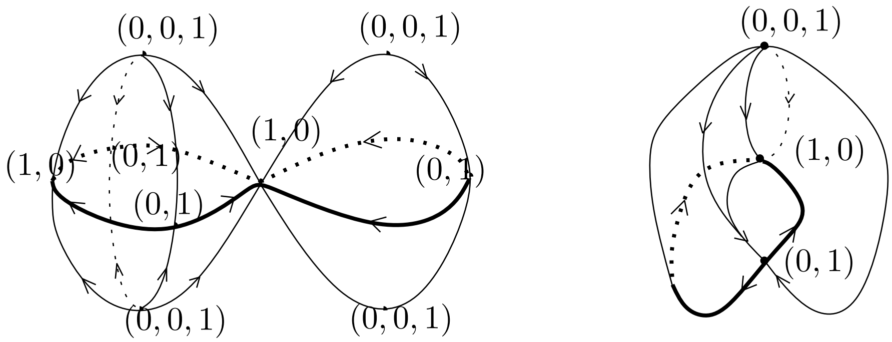

FIGURE 24. Flows on the pinched sphere and the pinched torus

(1) *Let X_t be Gutierrez-Sotomayor flow on the pinched sphere with 9 singularities: two attractors, three saddles and four repellers on M, see Figure 24(left).*

With respect to the stratification $\mathcal{E}$, R has four components homeomorphic to disks with one repelling singularity in the center of each disk. Hence, each singularity has numerical Conley indices equal to $(h_0, h_1, h_2) = (0, 0, 1)$ and thus,

$$R_0 = 0 + 0 + 0 + 0 = 0, \quad R_1 = 0 + 0 + 0 + 0 = 0 \quad e \quad R_2 = 1 + 1 + 1 + 1 = 4.$$

In the singular part of M one has 5 singularities of X_t two of which are attractors and three repellers. Hence, the numerical Conley indices are (s_0, s_1) equal to $(1, 0)$ and $(0, 1)$ respectively. Hence,

$$S_0 = 1 + 1 + 0 + 0 + 0 = 2 \quad and \quad S_1 = 0 + 0 + 1 + 1 + 1 = 3.$$

Substituting these values in equation (4): $4 - 0 + 0 = 3 - 2 + \chi(M)$. Thus, $\chi(M) = 3$.

(2) *Let X_t be Gutierrez-Sotomayor flow on the pinched torus with 3 singularities: one attractor, one saddle and one repeller on M, see Figure 24(right). In Example 3.5 we computed on the regular part*

$$R_0 = 0, \quad R_1 = 0 \quad and \quad R_2 = 1.$$

and on the singular part

$$S_0 = 1 + 0 = 1 \quad and \quad S_1 = 0 + 1 = 1.$$

Substituting these values in equation (4): $1 - 0 + 0 = 1 - 1 + \chi(M)$. Thus, $\chi(M) = 1$.

Example 3.10. *Let M be a manifold with Whitney umbrella singularity (e.g. a crosscap or a torus with a fold) with stratification $\mathcal{E}$.*

(1) *Let X_t be a Gutierrez-Sotomayor flow on a crosscap with 3 singularities: one attractor, one saddle and one repeller on M, see Figure 25(a).*

With respect to the stratification $\mathcal{E}$, R has one component homeomorphic to a disk with an attracting singularity at its center. Hence, $(h_0, h_1, h_2) = (1, 0, 0)$ and thus,

$$R_0 = 1, \quad R_1 = 0 \quad e \quad R_2 = 0.$$

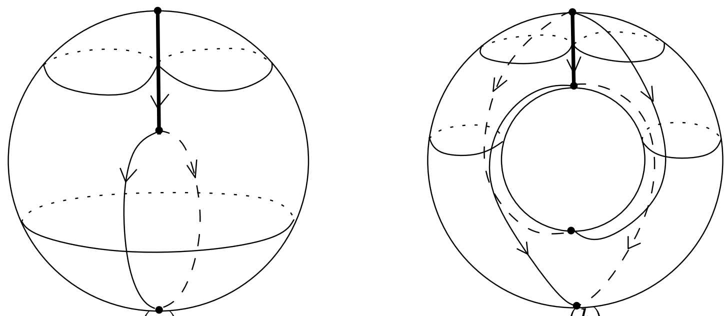

FIGURE 25. Flows on a crosscap and on a torus with a fold.

In the singular part of M there are two singularities of X_t one of which is an attractor and the other a repeller. Hence the numerical Conley indices (s_0, s_1) are equal to $(1,0)$ and $(0,0)$ respectively. Thus,

$$S_0 = 1 + 0 = 1 \quad \textit{and} \quad S_1 = 0 + 0 = 0.$$

Substituting these values in equation (4):

$$0 - 0 + 1 = 0 - 1 + \chi(M).$$

Thus, $\chi(M) = 2$.

(2) *Let X_t be a Gutierrez-Sotomayor flow on a torus with a fold with 4 singularities: an attractor, two saddles and one repeller on M, see Figure 25(b).*

With respect to the stratification $\mathcal{E}$, R has one component homeomorphic to a cylinder with two singularities in its interior a saddle and an attractor. The numerical Conley indices are $(h_0, h_1, h_2) = (0,1,0)$ for the saddle and $(h_0, h_1, h_2) = (1,0,0)$ for the attractor. Hence,

$$R_0 = 0 + 1 = 1, \quad R_1 = 1 + 0 = 1 \quad \textit{and} \quad R_2 = 0 + 0 = 0.$$

On the singular part of M there are two singularities of X_t one of which is an attractor and the other a repeller with numerical Conley indices equal to $(s_0, s_1) = (1,0)$ for the attractor and $(s_0, s_1) = (0,0)$ for the repeller. Hence,

$$S_0 = 0 + 1 = 1 \quad \textit{and} \quad S_1 = 0 + 0 = 0.$$

Substituting these values in equation (4):

$$0 - 1 + 1 = 0 - 1 + \chi(M).$$

Thus, $\chi(M) = 1$.

4. Isolating Blocks

In this section we will develop a theory of generalized handles to present a procedure of constructing special isolating neighborhoods for a simple singularity of a Gutierrez-Sotomayor flow. These isolating neighborhoods have the property that the flow is transversal to their boundary. Furthermore, we require that:

Definition 4.1. An *isolating block* is an isolating neighborhood N for an isolated invariant set Λ of the flow φ such that

$$N^- = \{x \in N | \varphi([0,T), x) \nsubseteq N, \forall T > 0\}$$

is closed.

A similar condition is required for the entering boundary N^+ for $T < 0$.

The existence of isolating blocks is an immediate consequence of the existence of Lyapunov functions f for Gutierrez-Sotomayor flows with simple singularities. If p is a singular point with $f(p) = c$ and $\epsilon > 0$ such that in $[c-\epsilon, c+\epsilon]$ there are no critical values then define an isolating block, N, for p as the connected component $f^{-1}([c-\epsilon, c+\epsilon])$ that contains p and $N^- = f^{-1}(c-\epsilon) \cap N$. Moreover, (N, N^-) is an index pair for $Inv(N) = \{p\}$.

4.1. **The Poincaré-Hopf Condition.** The following theorem establishes a relation between the first Betti number of the branched one-manifolds which make up the boundary N_0 of an isolating block N_1 for the singularity p, the number of boundary components of N_0 and the numerical Conley indices of p, (h_0, h_1, h_2).

Theorem 4.2. *Let (N_1, N_0) be an index pair where N_1 is an isolating block for a singularity p in a two dimensional manifold with simple singularities M. Let $X \in \Sigma^r(M)$ and (h_0, h_1, h_2) be the numerical Conley indices for p. Then*

$$(h_2 - h_1 + h_0) - (h_2 - h_1 + h_0)^* = e^+ - \mathcal{B}^+ - e^- + \mathcal{B}^- \tag{7}$$

where * indicates the index of the time-reversed flow, $e^+(e^-)$ is the number of entering (exiting) boundary components of N_1 and $\mathcal{B}^+ = \sum_{k=1}^{e^+} b_k^+$ ($\mathcal{B}^- = \sum_{k=1}^{e^-} b_k^-$) where $b_k^+(b_k^-)$ is the first Betti number of the kth entering (exiting) boundary components of N_1.

Proof. Proposition 3.3 asserts that $h_2 - h_1 + h_0 = \chi(N_1, N_0)$. By the long exact sequence of the pair (N_1, N_0) we have that $\chi(N_1, N_0) = \chi(N_1) - \chi(N_0)$. But $N_0 = \partial N_1^-$ hence,

$$h_2 - h_1 + h_0 + \chi(\partial N_1^-) = \chi(N_1)$$

Using the same arguments for the reverse flow, we obtain

$$(h_2 - h_1 + h_0)^* + \chi(\partial N_1^+) = \chi(N_1).$$

Subtracting these two equations, one concludes that

$$(h_2 - h_1 + h_0) - (h_2 - h_1 + h_0)^* = \chi(\partial N_1^+) - \chi(\partial N_1^-)$$

$$(h_2 - h_1 + h_0) - (h_2 - h_1 + h_0)^* = \sum_{k=1}^{e^+}(1 - b_k^+) - \sum_{k=1}^{e^-}(1 - b_k^-)$$

$$(h_2 - h_1 + h_0) - (h_2 - h_1 + h_0)^* = e^+ - \mathcal{B}^+ - e^- + \mathcal{B}^-$$

□

4.2. **The Gutierrez-Sotomayor Handle Theory.** In this section we will define a notion of generalized handles and specify their attaching regions. As in classical handle theory, the attaching regions produce different topological spaces depending on how the handle is glued.

Since the fixed points of $X \in \Sigma^r(M)$ are in $M(\mathcal{G})$, with $\mathcal{G} = \mathcal{R}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{W}$ or $\mathcal{T}$, one must consider different types of handles which we refer to as two dimensional Gutierrez-Sotomayor handles, GS handles for short.

A GS handle $\mathcal{H}_x^{\mathcal{G}}$ is a subspace of $\mathbb{R}^3$ with well defined dynamics where a fixed point is on $M(\mathcal{G})$, *i.e.,* it may be on the regular part, on the cone, on the Whitney fold, on double crossings or triple crossings. Hence, we will denote them by regular handles, cone handles, Whitney handles, double handles or triple handles respectively.

In order to specify the dynamics on the handles we consider the following vector fields defined on disks in $\mathbb{R}^2$:

(*a*) $X(x, y) = (-2x, -2y)$ (*b*) $X(x, y) = (x, -y)$

(c) $X(x, y) = (2x, 2y)$ (d) $X(x, y) = (0, -x^2 - y^2)$

Equivalently, one can consider the respective singularities on the handles and their local flow. We will now proceed to define the handles in each case above.

Definition 4.3. *A regular handle is formed by a disk D centered at p with a flow defined as in the cases below: See Figure 26.*

(1) *A regular handle $\mathcal{H}_a^{\mathcal{R}}$ has a flow defined by the vector field in (a). The attaching region of the handle is the empty set.*
(2) *A regular handle $\mathcal{H}_s^{\mathcal{R}}$ has a flow defined by the vector field in (b). The attaching region of the handle is homeomorphic to two disjoint segments from where the flow exits.*
(3) *A regular handle $\mathcal{H}_r^{\mathcal{R}}$ has a flow defined by the vector field in (c). The attaching region of the handle is homeomorphic to a circle from where the flow exits.*

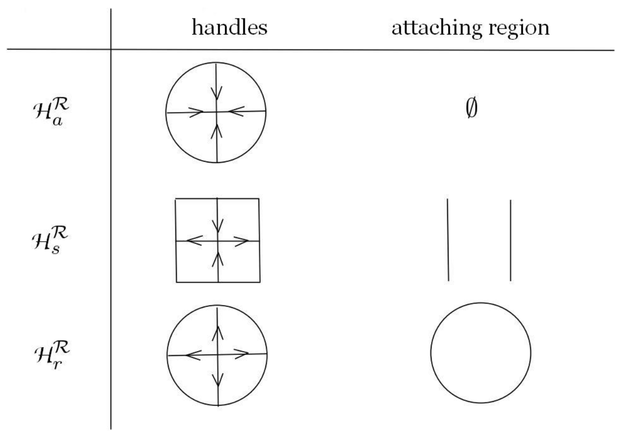

FIGURE 26. Regular handles $\mathcal{H}_a^{\mathcal{R}}$, $\mathcal{H}_s^{\mathcal{R}}$ and $\mathcal{H}_r^{\mathcal{R}}$.

Definition 4.4. *A cone handle is formed by two disks D_1 and D_2 centered at p such that $D_1 \cap D_2 = \{p\}$ with a flow defined as in the cases below. See Figure 27.*

(1) *A cone handle $\mathcal{H}_a^{\mathcal{C}}$ has a flow defined on both disks by the vector field in (a). The attaching region of the handle is the empty set.*
(2) *A cone handle $\mathcal{H}_s^{\mathcal{C}}$ has a flow defined on both disks by the vector field in (d). The attaching region of the handle is the disjoint union of two arcs in ∂D_1 and ∂D_2 respectively, from where the flow exits.*
(3) *A cone handle $\mathcal{H}_r^{\mathcal{C}}$ has a flow defined on both disks by the vector field in (c). The attaching region of the handle are the two circles that correspond to ∂D_i, from where the flow exits.*

Definition 4.5. *A Whitney handle is formed by a disk D with two regular orbits identified as in the cases below. See Figure 28.*

(1) *A Whitney handle $\mathcal{H}_a^{\mathcal{W}}$ has a flow defined on D by the vector field in (a), with two regular orbits identified to a ray of D. The attaching region of the handle is the empty set.*

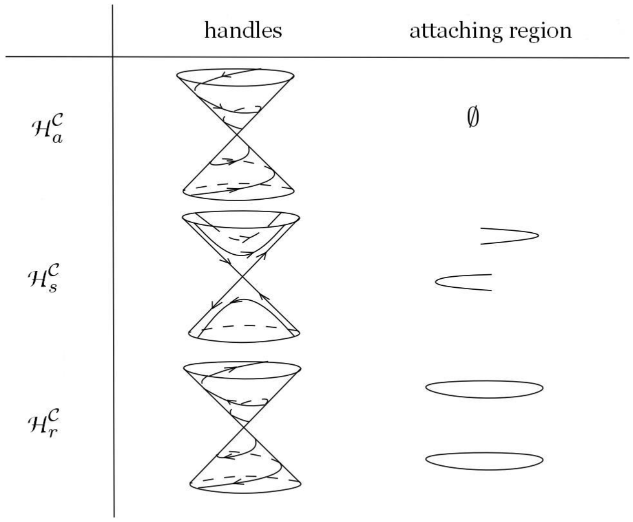

FIGURE 27. Cone handles $\mathcal{H}_a^{\mathcal{C}}$, $\mathcal{H}_s^{\mathcal{C}}$ and $\mathcal{H}_r^{\mathcal{C}}$.

(2) *A Whitney handle $\mathcal{H}_{s_s}^{\mathcal{W}}$ has a flow defined on D by the vector field in* (*b*), *with two regular orbits on the stable manifold identified to a ray of D. The attaching region of the handle is the disjoint union of two arcs in ∂D, from where the flow exits.*

(3) *A Whitney handle $\mathcal{H}_{s_u}^{\mathcal{W}}$ has a flow defined on D by the vector field in* (*b*), *with two regular orbits on the unstable manifold identified to a ray of D. The attaching region of the handle is a transversal intersection of two arcs from where the flow exits.*

(4) *A Whitney handle $\mathcal{H}_r^{\mathcal{W}}$ has a flow defined on D by the vector field in* (*c*), *with two regular orbits identified to a ray of D. The attaching region of the handle is the boundary ∂D which after the identification is homeomorphic to a figure "eight" from where the flow exits.*

Definition 4.6. *A double handle is formed by two disks D_1 and D_2 centered at p and intersecting transversally along diameters d_1 and d_2 of D_1 and D_2 respectively. These diameters are formed by a union of orbits as described below. See Figure 29.*

(1) *A double handle $\mathcal{H}_a^{\mathcal{D}}$ has a flow defined on D_1 and D_2 by the vector field in* (*a*). *The attaching region of the handle is the empty set.*

(2) *A double handle $\mathcal{H}_{sa}^{\mathcal{D}}$ has a flow defined on D_1 by the vector field in* (*a*) *and defined on D_2 by the vector field in* (*b*) *where d_2 is the stable manifold in D_2. The attaching region of the handle is homeomorphic to two disjoint segments from where the flow exits.*

(3) *A double handle $\mathcal{H}_{ss_u}^{\mathcal{D}}$ has a flow defined on D_1 and D_2 by the vector field in* (*b*) *where d_1 and d_2 are the unstable manifolds on the respective disks. The attaching region of the handle is homeomorphic to two copies of two segments that intersect transversally and from where the flow exits.*

(4) *A double handle $\mathcal{H}_{ss_s}^{\mathcal{D}}$ has a flow defined on D_1 and D_2 by the vector field in* (*b*) *where d_1 and d_2 are the stable manifolds on the respective disks. The attaching region of the handle is homeomorphic to two copies of two segments from where the flow exits.*

(5) *A double handle $\mathcal{H}_{sr}^{\mathcal{D}}$ has a flow defined on D_1 by the vector field in* (*c*) *and has a flow defined on D_2 by the vector field in* (*b*) *where d_2 is the unstable manifold in D_2. The*

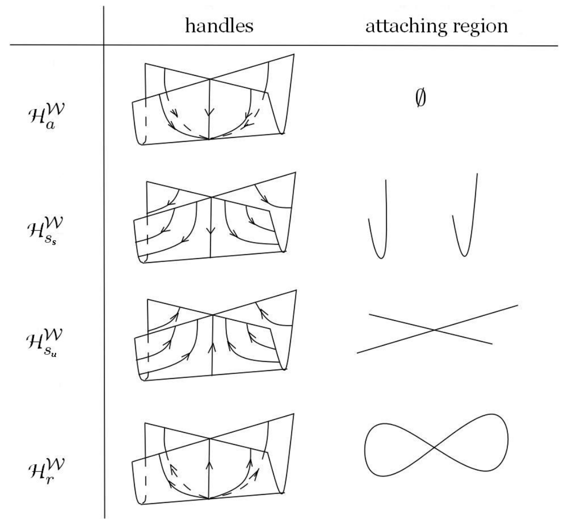

FIGURE 28. Whitney handles $\mathcal{H}_a^{\mathcal{W}}$, $\mathcal{H}_{s_s}^{\mathcal{W}}$, $\mathcal{H}_{s_u}^{\mathcal{W}}$ and $\mathcal{H}_r^{\mathcal{W}}$.

attaching region of the handle is homeomorphic to ∂D_2 on which two segments intersect transversally and from where the flow exits.

(6) *A double handle $\mathcal{H}_r^{\mathcal{D}}$ has a flow defined on D_1 and D_2 by the vector field in (c). The attaching region of the handle is homeomorphic to ∂D_1 and ∂D_2 intersecting transversally at two distinct points and from where the flow exits.*

Definition 4.7. *A triple handle is formed by three disks D_1, D_2 and D_3 centered at p with diameters $d_1 \subset D_1$, $d_2 \subset D_2$ and $d_3 \subset D_3$ intersecting transversally at p and pairwise disjoint. These diameters are formed by a union of orbits as described below. See Figure 30.*

(1) *A triple handle $\mathcal{H}_a^{\mathcal{T}}$ has a flow defined on D_1, D_2 and D_3 by the vector field in (a). The attaching region of the handle is the empty set.*

(2) *A triple handle $\mathcal{H}_{ssa}^{\mathcal{T}}$ has a flow defined on D_1 by the vector field in (a) and has a flow defined on D_2 and D_3 by the vector field in (b) where d_2 and d_3 are stable manifolds of D_2 and D_3 respectively. The attaching region of the handle is homeomorphic to two copies of two segments that intersect transversally from where the flow exits.*

(3) *A triple handle $\mathcal{H}_{ssr}^{\mathcal{T}}$ has a flow defined on D_1 by the vector field in (c) and has a flow defined on D_2 and D_3 by the vector field in (b) where d_2 and d_3 are unstable manifolds of D_2 and D_3 respectively. The attaching region of the handle is homeomorphic to ∂D_2 from where the flow exits with four segments intersecting ∂D_2 transversally and also from where the flow exits.*

(4) *A triple handle $\mathcal{H}_r^{\mathcal{T}}$ has a flow defined on D_1, D_2 and D_3 by the vector field in (c). The attaching region of the handle is homeomorphic to three circles, all from which the flow exits and that pairwise intersect transversally at two points.*

4.3. **Constructing Isolating Blocks.** In this section, we construct an isolating block by gluing a GS handle $\mathcal{H}_x^{\mathcal{G}}$ to a collar of a distinguished branched one manifold $N^- \times [0,1]$.

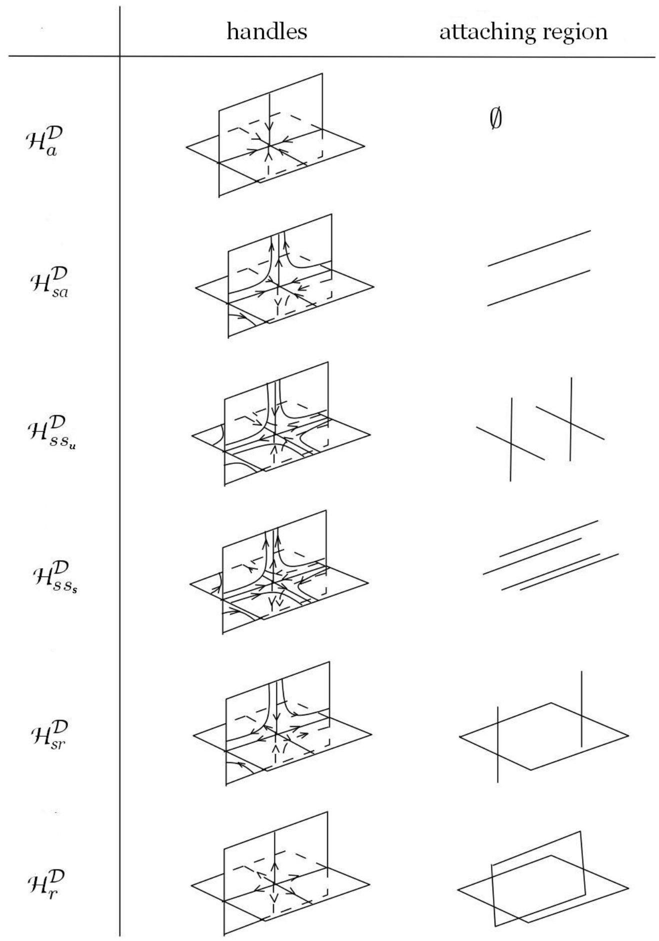

FIGURE 29. Double handles $\mathcal{H}_a^{\mathcal{D}}$, $\mathcal{H}_{sa}^{\mathcal{D}}$, $\mathcal{H}_{ss_u}^{\mathcal{D}}$, $\mathcal{H}_{ss_s}^{\mathcal{D}}$, $\mathcal{H}_{sr}^{\mathcal{D}}$ and $\mathcal{H}_r^{\mathcal{D}}$.

Definition 4.8. *A distinguished branched one manifold is a topological space, having at most four connected components, locally constructed from a finite number of branched charts. Each branched chart is the transversal intersection of two arcs in the plane.*

In Figure 31, we present examples of distinguished branched 1-manifolds.

It is interesting to note that the different attachments of a given GS handle $\mathcal{H}_x^{\mathcal{G}}$ produces non-homeomorphic isolating blocks (N, N^-). However, all isolating blocks have the same Conley index, *i.e.,* the homotopy type of N/N^- is the same and independent of the block.

Theorem 4.9. *Let p be a simple singularity of a Gutierrez-Sotomayor flow X_t on M. Suppose that p satisfies the Poincaré-Hopf condition for the positive numbers e^+ , e^-, $\{b_k^+\}_{k=1}^{e^+}$ and $\{b_k^-\}_{k=1}^{e^-}$. Then there exists an isolating block N for p with $\partial N = \partial N^+ \cup \partial N^-$ such that the following holds:*

(1) *e^+ (respectively e^-) is the number of connected components of ∂N^+ (respectively ∂N^-), corresponding to the entering (respectively exiting) boundary components of the flow. In*

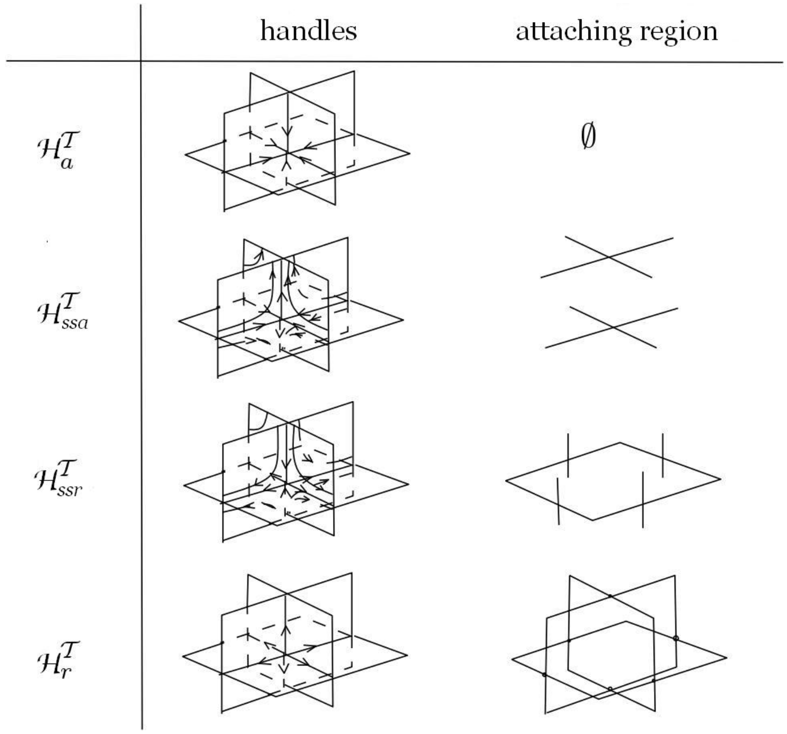

FIGURE 30. Triple handles $\mathcal{H}_a^{\mathcal{T}}$, $\mathcal{H}_{ssa}^{\mathcal{T}}$, $\mathcal{H}_{ssr}^{\mathcal{T}}$ e $\mathcal{H}_r^{\mathcal{T}}$.

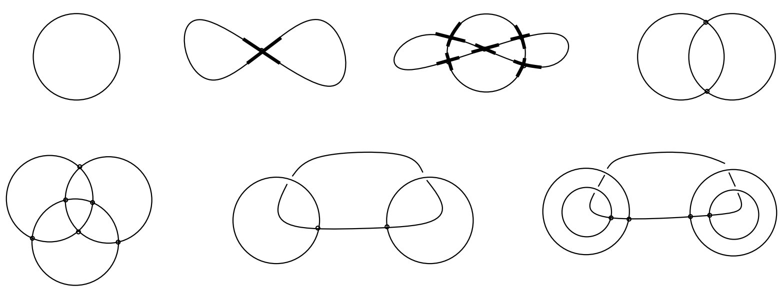

FIGURE 31. Distinguished branched one-manifolds.

other words, we have a disjoint union

$$\partial N^+ = \bigcup_{k=1}^{e^+} \partial N_k^+ \quad \left(\text{respectively } \partial N^- = \bigcup_{k=1}^{e^-} \partial N_k^-\right).$$

(2) *the rank* $H_1(\partial N_k^+) = b_k^+$ *with* $k = 1, ..., e^+$ *and the rank*$H_1(\partial N_k^-) = b_k^-$ *with* $k = 1, ..., e^-$.

(3) *the rank* $H_*(N/\partial N^-) = h_*$ *where* (h_0, h_1, h_2) *is the numerical Conley index of* p.

Proof. For each attractor and repeller, the GS handle $\mathcal{H}_a^{\mathcal{G}}$ where $\mathcal{G} = \mathcal{R}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{W}$ or $\mathcal{T}$ is always an isolating block. For saddle handles there are different topological types of isolating blocks depending on the distinguished branched one manifolds and the attaching maps to their collars. Consider a distinguished branched one manifold $N^- = \bigcup_{k=1}^{e^-} N_k^-$ with e^- components and each N_k^- with b_k^- as its first Betti number. Let $\mathcal{H}_x^{\mathcal{G}}$ be a GS handle with attaching region A_k and the collar $\bigcup_{k=1}^{e^-}(N_k^- \times I)$ of N_k^-. Attach the handle to the distinguished branched one manifold via an embedding

$$f : A_k \to \bigcup_{k=1}^{e^-}(N_k^- \times 1).$$

□

See Figures 32, 33, 34, 35, 36 and 37, where we present constructions for specific cases of saddle type isolating blocks for a simple singularity of a Gutierrez-Sotomayor flow X_t.

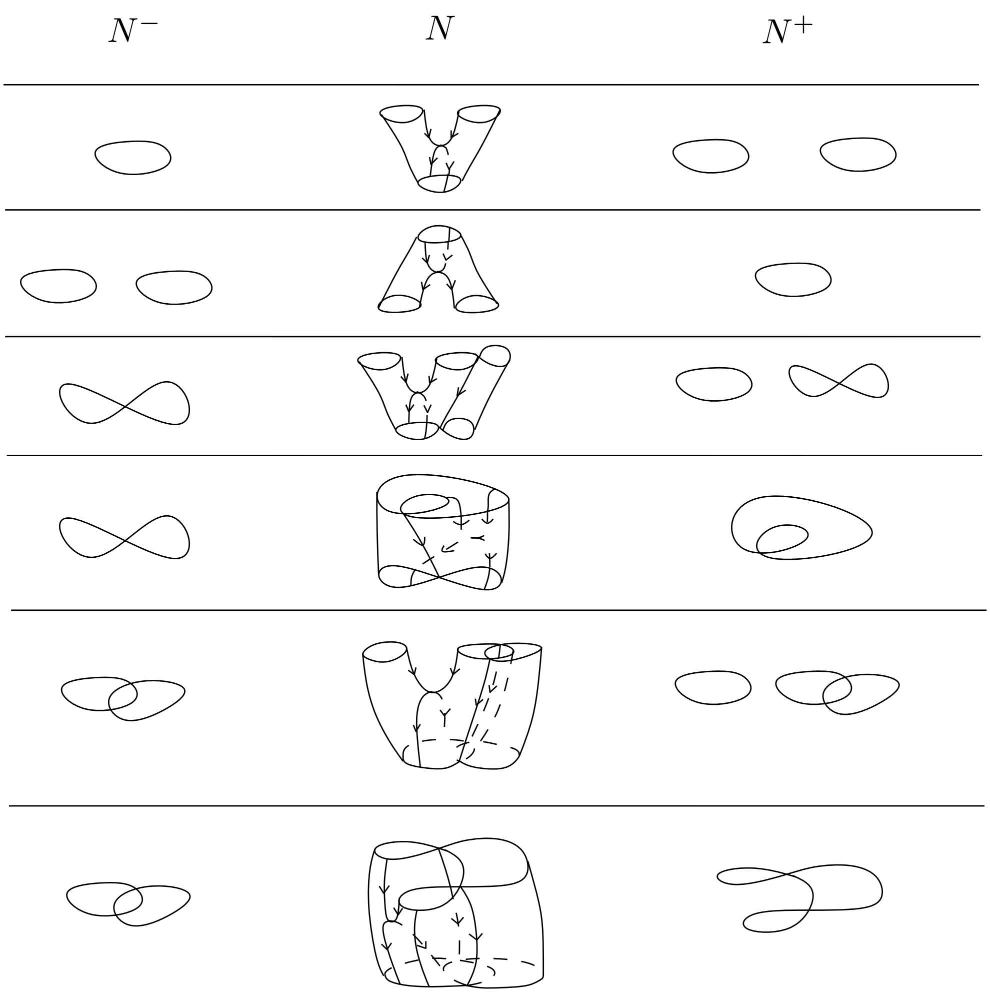

FIGURE 32. Isolating blocks containing a regular handle $\mathcal{H}_s^{\mathcal{R}}$.

Other blocks can be constructed from these by adding cylinders where the flow is trivial. See Figure 38.

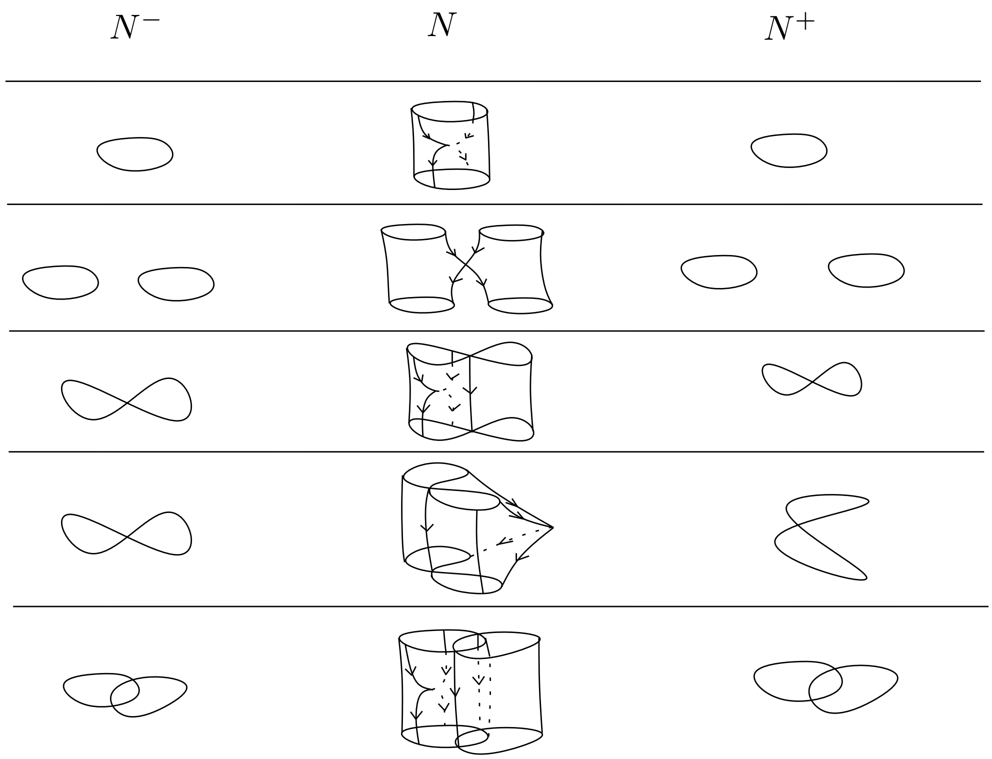

FIGURE 33. Isolating blocks containing a cone handle $\mathcal{H}_s^{\mathcal{C}}$.

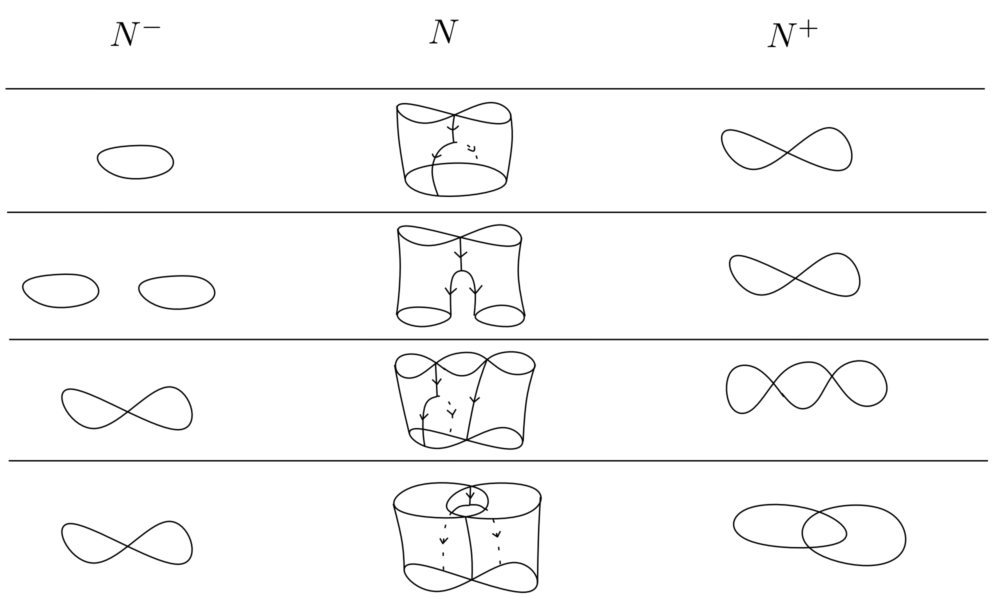

FIGURE 34. Isolating blocks containing a Whitney handle $\mathcal{H}_{s_s}^{\mathcal{W}}$.

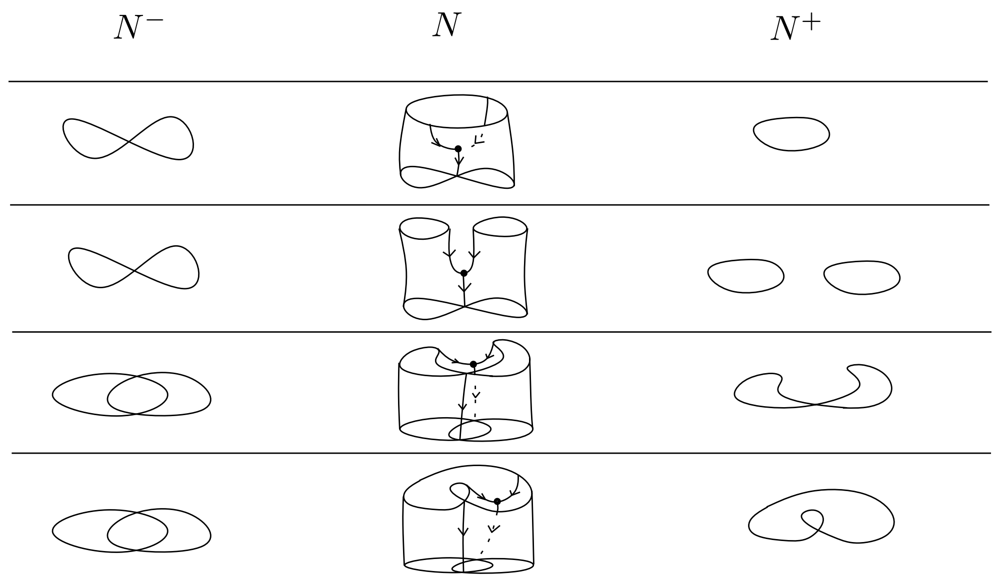

FIGURE 35. Isolating blocks containing a Whitney handle $\mathcal{H}^{\mathcal{W}}_{s_u}$.

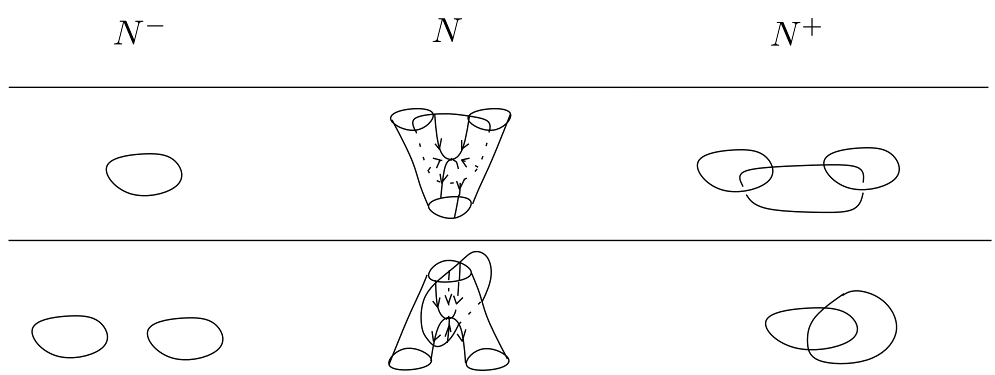

FIGURE 36. Isolating blocks containing a double handle $\mathcal{H}^{\mathcal{D}}_{sa}$.

5. Lyapunov Graphs

Let f be a Lyapunov function associated to the Gutierrez-Sotomayor flow X_t on the two-manifold M with simple singularities. We define the following equivalence relation on M: $x \sim_f y \Leftrightarrow x$ and y belong to the same connected component of a level set of f.

We call $M/\sim_f$ the Lyapunov graph associated to X_t and f.

On $M/\sim_f$ each connected component of a level set $f^{-1}(c)$ collapses to a point, thus $f^{-1}(c)/\sim_f$ is a finite set of distinct points on $M/\sim_f$. A point on $M/\sim_f$ is a vertex if by the equivalence relation it corresponds to a component of a level set containing a unique singularity. All other points are edge points. The vertices v of $M/\sim_f$ can be labelled with the type of singularity and we denote by e_v^+ the number of positively incident edges and e_v^- the number of negatively incident edges to v.

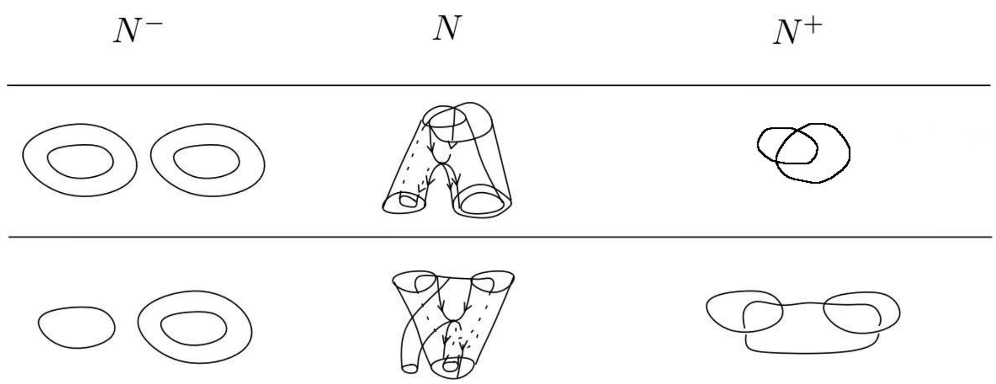

FIGURE 37. Isolating blocks containing a double handle $\mathcal{H}^{\mathcal{D}}_{ss_s}$.

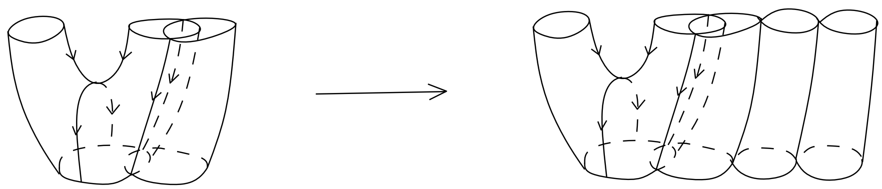

FIGURE 38. Isolating blocks containing a regular handle.

Theorem 5.1. *Supose that $X_t : M \to M$ is a Gutierrez-Sotomayor flow with Lyapunov function $f : M \to \boldsymbol{R}$. Let $L = M/\sim_f$, then L is a finite directed graph without oriented cycles.*

Proof. By the definition of a Lyapunov function we have that the critical points of f correspond to the singularities of X_t. Since X_t has a finite number of singularities then there exists a finite number of critical values of f, $c_1, c_2, \ldots, c_n$. Thus, $f^{-1}(c_i, c_{i+1})$ is diffeomorphic to $N \times (0, 1)$ where $N = f^{-1}(c)$ with $c \in (c_i, c_{i+1})$. Hence by Lemma 2.5, N is a branched one manifold with a finite number of components.

Also, $f^{-1}(c_i)$ has a finite number of components since if this were not the case $f^{-1}(c_i + \epsilon)$ would have infinite components for any $\epsilon > 0$. Only one of these components, denoted by X_i, contains the critical point of f since by definition a Lyapunov function f separates critical points.

Now if $N_0 \subset f^{-1}(c_i)$ does not contain critical points of f then the component of $f^{-1}(c_{i-1}, c_{i+1})$ that contains N_0 is diffeomorphic to $N_0 \times (0, 1)$. Indeed, $M - \bigcup_i X_i$ is diffeomorphic to the disjoint union of $N_j \times (0, 1)$ where each N_j is a connected compact branched one-manifold of M. Thus, if $P : M \to L$ is the quotient mapping that identifies each component of a level set of f to a point and $x_i = P(X_i)$ then it follows that $L - \{x_i\}$ is a finite set of open intervals. Hence, since L is compact, it is a graph.

Since f decreases along orbits of X_t then the Lyapunov graph L associated to X_t and f has no oriented cycle. □

On the other hand, to construct a flow that satisfies a given dynamics, a great combinatorial tool is an abstract Lyapunov graph which can aggregate topological and dynamical information.

Definition 5.2. *An abstract Lyapunov graph is a finite connected oriented graph L which possesses no oriented cycles and with each vertex labelled with the numerical Conley indices. Each edge a that is incoming (resp. outgoing) i.e., positively incident to v (resp. negatively incident* *to v) will be labelled with a nonnegative integer b_a^+ (resp. b_a^-) where $a \in \{1,\dots,e^+\}$ (resp. $a \in \{1,\dots,e^-\}$), which we refer to as the weight on an edge.*

The question becomes: once necessary conditions on Lyapunov graphs are found, are they sufficient to realize an abstract graph as a GS flow on a manifold?

Theorem 5.3. *A Lyapunov graph L of a Gutierrez-Sotomayor flow X_t with simple singularities on M, satisfies the following conditions:*

(1) *If a vertex v is labelled with a repelling (attracting) singularity then:*
   (a) *If $p \in M(\mathcal{R})$ then $e_v^- = 1$ and $b_1^- = 1$ ($e_v^+ = 1$ and $b_1^+ = 1$).*
   (b) *If $p \in M(\mathcal{C})$ then $e_v^- = 2$ and $b_1^- = b_2^- = 1$ ($e_v^+ = 2$ and $b_1^+ = b_2^+ = 1$).*
   (c) *If $p \in M(\mathcal{W})$ then $e_v^- = 1$ and $b_1^- = 2$ ($e_v^+ = 1$ and $b_1^+ = 2$).*
   (d) *If $p \in M(\mathcal{D})$ then $e_v^- = 1$ and $b_1^- = 3$ ($e_v^+ = 1$ and $b_1^+ = 3$).*
   (e) *If $p \in M(\mathcal{T})$ then $e_v^- = 1$ and $b_1^- = 7$ ($e_v^+ = 1$ and $b_1^+ = 7$).*

(2) *If a vertex v is labelled with a saddle singularity p then:*
   (a) *If $p \in M(\mathcal{R})$ then $1 \le e_v^- \le 2$ and $1 \le e_v^+ \le 2$.*
   (b) *If $p \in M(\mathcal{C})$ then $1 \le e_v^- \le 2$ and $1 \le e_v^+ \le 2$.*
   (c) *If $p \in M(\mathcal{W})$ then*
      (i) *If p is of type* **si** *then $e_v^- = 1$ and $1 \le e_v^+ \le 2$.*
      (ii) *If p is of type* **se** *then $1 \le e_v^- \le 2$ and $e_v^+ = 1$.*
   (d) *If $p \in M(\mathcal{D})$ then*
      (i) *If p is of type* **as** *then $1 \le e_v^- \le 2$ and $e_v^+ = 1$.*
      (ii) *If p is of type* **rs** *then $e_v^- = 1$ and $1 \le e_v^+ \le 2$.*
      (iii) *If p is of type* **si** *then $1 \le e_v^- \le 2$ and $1 \le e_v^+ \le 4$.*
      (iv) *If p is of type* **se** *then $1 \le e_v^- \le 4$ and $1 \le e_v^+ \le 2$.*
   (e) *If $p \in M(\mathcal{T})$ then*
      (i) *If p is of type* **ssa** *then $1 \le e_v^- \le 2$ and $e_v^+ = 1$.*
      (ii) *If p is of type* **ssr** *then $e_v^- = 1$ and $1 \le e_v^+ \le 2$.*

All weights on the entering and exiting edges of v must satisfy the table.

$M(\mathcal{G})$	type	e_v^-	e_v^+	weights
$p \in M(\mathcal{R})$	**a**	0	1	$b_1^+ = 1$
	s	1	1	$b_1^- = b_1^+$
	s	1	2	$b_1^- = b_1^+ + b_2^+ - 1$
	s	2	1	$b_1^+ = b_1^- + b_2^- - 1$
	r	1	0	$b_1^- = 1$
$p \in M(\mathcal{C})$	**a**	0	2	$b_1^+ = b_2^+ = 1$
	s	1	1	$b_1^- = b_1^+$
	s	2	2	$b_1^- + b_2^- = b_1^+ + b_2^+$
	r	2	0	$b_1^- = b_2^- = 1$

$p \in M(\mathcal{W})$	**a**	0	1	$b_1^+ = 2$
	$\mathbf{s_u}$	1	1	$b_1^- = b_1^+ + 1$
	$\mathbf{s_u}$	1	2	$b_1^- = b_1^+ + b_2^+$
	$\mathbf{s_s}$	1	1	$b_1^+ = b_1^- + 1$
	$\mathbf{s_s}$	2	1	$b_1^+ = b_1^- + b_2^-$
	r	1	0	$b_1^- = 2$
$p \in M(\mathcal{D})$	**a**	0	1	$b_1^+ = 3$
	sa	1	1	$b_1^+ = b_1^- + 2$
	sa	2	1	$b_1^+ = b_1^- + b_2^- + 1$
	sr	1	1	$b_1^- = b_1^+ + 2$
	sr	1	2	$b_1^- = b_1^+ + b_2^+ + 1$
	$\mathbf{ss_u}$	1	1	$b_1^- = b_1^+ + 2$
	$\mathbf{ss_u}$	1	2	$b_1^- = b_1^+ + b_2^+ + 1$
	$\mathbf{ss_u}$	1	3	$b_1^- = b_1^+ + b_2^+ + b_3^+$
	$\mathbf{ss_u}$	1	4	$b_1^- = b_1^+ + b_2^+ + b_3^+ + b_4^+ - 1$
	$\mathbf{ss_u}$	2	1	$b_1^+ = b_1^- + b_2^- - 3$
	$\mathbf{ss_u}$	2	2	$b_1^- + b_2^- = b_1^+ + b_2^+ + 2$
	$\mathbf{ss_u}$	2	3	$b_1^- + b_2^- = b_1^+ + b_2^+ + b_3^+ + 1$
	$\mathbf{ss_u}$	2	4	$b_1^- + b_2^- = b_1^+ + b_2^+ + b_3^+ + b_4^+$
	$\mathbf{ss_s}$	1	1	$b_1^+ = b_1^- + 2$
	$\mathbf{ss_s}$	1	2	$b_1^- = b_1^+ + b_2^+ - 3$
	$\mathbf{ss_s}$	2	1	$b_1^+ = b_1^- + b_2^- + 1$
	$\mathbf{ss_s}$	2	2	$b_1^+ + b_2^+ = b_1^- + b_2^- + 2$
	$\mathbf{ss_s}$	3	1	$b_1^+ = b_1^- + b_2^- + b_3^-$
	$\mathbf{ss_s}$	3	2	$b_1^+ + b_2^+ = b_1^- + b_2^- + b_3^- + 1$
	$\mathbf{ss_s}$	4	1	$b_1^+ = b_1^- + b_2^- + b_3^- + b_4^- - 1$
	$\mathbf{ss_s}$	4	2	$b_1^+ + b_2^+ = b_1^- + b_2^- + b_3^- + b_4^-$
	r	1	0	$b_1^- = 3$
$p \in M(\mathcal{T})$	**a**	0	1	$b_1^+ = 7$
	ssa	1	1	$b_1^+ = b_1^- + 2$
	ssa	2	1	$b_1^+ = b_1^- + b_2^- + 1$
	ssr	1	1	$b_1^- = b_1^+ + 2$
	ssr	1	2	$b_1^- = b_1^+ + b_2^+ + 1$
	r	1	0	$b_1^- = 7$

Proof. First, we prove the inequalities on the degree of the vertices v in L.

Let L be a Lyapunov graph associated to a Gutierrez-Sotomayor flow X_t and f a Lyapunov function on a two-manifold with simple singularities M. If p is a singularity such that $f(p) = c$, denote by N_1 the component of $f^{-1}([c-\epsilon, c+\epsilon])$, with $\epsilon > 0$ sufficiently small so that it contains only one singular point p. Let $N_0 = N_1 \cap f^{-1}(c-\epsilon)$. Then (N_1, N_0) is an index pair for p.

Since p is a singularity then $\partial N_1 \neq \emptyset$, thus, $H_2(N_1) = 0$. Also, N_1 is connected, thus $\widetilde{H}_0(N_1) = 0$. Let v be the vertex of L labelled with p then dim $H_0(N_0) = e_v^-$ and if $N_0 \neq \emptyset$ then dim $\widetilde{H}_0(N_0) = e_v^- - 1$.

Hence, for N_0 we have the following long exact sequence:

$$0 \longrightarrow CH_2(p) \xrightarrow{\partial_2} H_1(N_0) \xrightarrow{i_1} H_1(N_1) \xrightarrow{p_1} CH_1(p) \xrightarrow{\partial_1} \widetilde{H}_0(N_0) \longrightarrow 0.$$

Secondly, we prove the conditions on the weights of the edges incident to v.

The Theorem 4.2 relates the first Betti number of the boundary components that are entering sets and exiting sets for the flow, ∂N_1^+ and ∂N_1^-, the isolating block (N_1, N_0) of a singularity, $p \in M$, of X_t with the number of boundary components of N_1 and the numerical Conley indices of $p \in M$. Since the fixed point $p \in M$ corresponds to a vertex v on the Lyapunov graph, ∂N_1^+ (∂N_1^-) corresponds to edges positively (negatively) incident to v then Theorem 4.2 relates the degree (of the entering and exiting edges) of v, to the weights on the edges (entering and exiting) incidents to v and the numerical Conley index with which v was labelled.

$$(h_2 - h_1 + h_0) - (h_2 - h_1 + h_0)^* = e_v^+ - \mathcal{B}^+ - e_v^- + \mathcal{B}^- \tag{8}$$

where $\mathcal{B}^+ = \sum_{k=1}^{e_v^+} b_k^+$ and $\mathcal{B}^- = \sum_{k=1}^{e_v^-} b_k^-$.

Considering all the possibilities for e^+, e^- in the inequalities involving the degree of v and using the above equations, we obtain the weights on the table and the result follows.

□

Example 5.4. *Gluing the isolating blocks to obtain a Gutierrez-Sotomayor flow.*

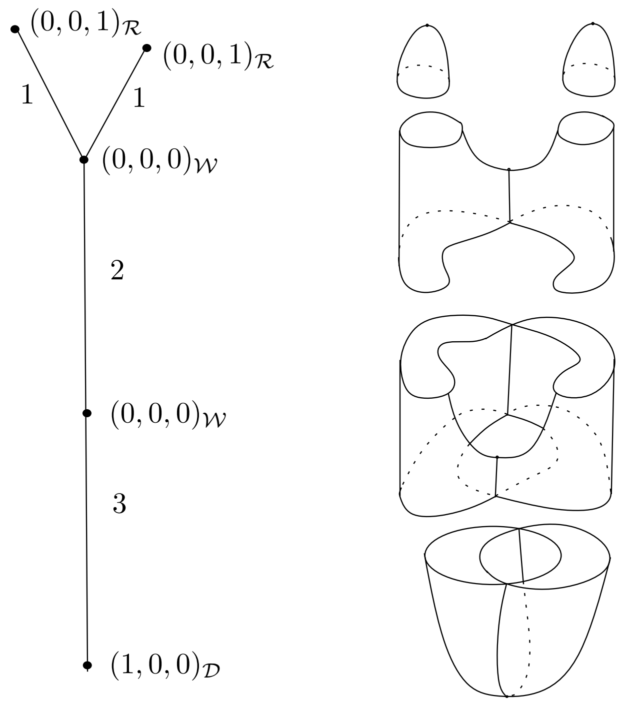

FIGURE 39. An abstract Lyapunov graph and its realization as a GS flow.

We conclude this paper with a couple of remarks. Example 5.4 suggests that one may be able to find sufficient conditions on abstract Lyapunov graphs in order to check their realizability. See Figure 39. This has not yet been done and remains an open question.

Also, one would like to include in a study similar to this one, the inclusion of periodic orbits and singular cycles.

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H. Montúfar, Faculty of Mathematics, Federal University of Uberlândia, Uberlândia - Brazil
Email address: montufar@ufu.br

K. A. de Rezende, Institute of Mathematics and Scientific Computing, University of Campinas, Campinas - Brazil
Email address: ketty@unicamp.br