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THE EMBEDDED NASH PROBLEM OF BIRATIONAL MODELS OF RATIONAL TRIPLE SINGULARITIES

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1. Introduction

Given a variety X defined over an algebraically closed field of characteristic 0, we are often not able to exhibit an explicit resolution of its singularities; on the other hand there are infinitely many resolutions of singularities of X giving extra information which is not intrinsic to the singularity. The need for understanding the information which is common to all the resolutions of singularities of a given space X led Nash (in [22]) to study the arc space of X. See also [6, 24] for more details. This paper follows this line of thoughts. The difference here is that we are interested in the embedded resolutions of singularities of $X \subset \mathbb{A}^n$.

For this purpose, we replace the arc space X_∞ of X with the jet schemes of X: the arc space X_∞ of X is the space of germs of formal curves drawn on X. The jet schemes are a family of finite dimensional schemes indexed by integers which approximate the infinite dimensional arc space; for $m \in \mathbf{N}$, the m-th jet scheme X_m of X, can be thought of as the space of arcs in the ambient space $\mathbb{A}^n$ whose "contact" with X is greater or equal to $m+1$; this gives the intuition why these schemes should detect information about embedded resolutions of singularities. The main question considered in this paper is: can we construct an embedded resolution of singularities from the jet schemes of $X \subset \mathbb{A}^n$? More precisely, we ask the following much less optimistic question:

($\star$) *Can one construct an embedded resolution of singularities of $X \subset \mathbb{A}^n$ from the irreducible components of the space X_m^{Sing} of jets centered at the singular locus of $X \subset \mathbb{A}^n$?*

This question is studied in [18, 17, 15, 20]. In [20], the authors proved that the irreducible components of the jet schemes centered at the singular locus of a rational double point surface singularity (known also as "simple singularities" in the literature) give a minimal embedded resolution by a birational toric modification of the ambient space. Equivalently, a certain natural family of the irreducible components of the jet schemes of X centered at the singular point 0 X_m^0 is in bijection with the divisorial valuations whose center is a toric divisor on every toric embedded resolution; this bijection is actually a conceptual correspondence since one can associate with any irreducible component of X_m^0 a divisorial valuation centered at the origin of $\mathbb{A}^n$ (see [5]).

In general, such a statement is hopeless: indeed, even for an irreducible plane curve singularity (say, for the cusp $\{y^2-x^3=0\} \subset \mathbb{A}^2$), the irreducible components of the jet schemes centered at the origin give divisorial valuations which do not appear in the minimal embedded resolution of the curve singularity (in that case, the minimal embedded resolution makes sense and is unique).

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The answer to ($\star$) is no in general. Indeed, consider the three-dimensional variety defined by

$$X = \{x^2 + y^2 + z^2 + w^5 = 0\} \subset \mathbb{A}^4.$$

It has an isolated singularity at the origin 0. On the one hand, by a direct computation, we see that the jet schemes X_m^0 centered at 0 are irreducible for every $m \geq 1$. On the other hand, we have two exceptional (irreducible) divisors that appear on every embedded resolution of the singularity (at least those which correspond to the two essential divisors appearing in the abstract resolution of the origin 0) of X; these are the divisors associated with the monomial valuations on $k[x, y, z, w]$ defined by the vectors $(1, 1, 1, 1)$ and $(2, 2, 2, 1)$. The valuation associated with the vector $(2, 2, 2, 1)$ does not correspond to any of the schemes X_m^0 with $m \geq 1$. Note that this example is one of the counterexamples to the Nash problem given in [12]; note also that the Nash correspondence is bijective in dimension 2 [8, 9] but there are many counter-examples in higher dimension ([11, 7]). This suggests that a reasonable frame to study the question ($\star$) is the surface singularities.

In this paper we study the question ($\star$) for a family of hypersurface singularities whose normalizations are rational triple point singularities (RTP-singularities, for short). These hypersurfaces are classified in [1] and are called the non-isolated forms of RTP-singularities. We prove that, for this class of singularities, the answer to ($\star$) is positive. When X is of that type, we determine again a natural family of irreducible components of X_m^{Sing}, $m \geq 1$ whose associated divisorial valuations are monomial, hence defined by some vectors in $\mathbb{N}^3$. For all of the non-isolated forms of RTP-singularities except when X is of type $B_{k-1,2l-1}$, we show that these vectors give a regular subdivision Σ of the dual Newton fan of X and hence a nonsingular toric variety Z_Σ; since our singularities are Newton non-degenerate [27, 2, 1], this gives a birational toric morphism $Z_\Sigma \longrightarrow \mathbb{A}^3$ which is an embedded resolution of $X \subset \mathbb{A}^3$; the irreducible components of the exceptional divisor correspond to the natural set of irreducible components of X_m^{Sing}.

When X is of type $B_{k-1,2l-1}$, we again build a toric embedded resolution from the irreducible components of the jet schemes which does not factor through the toric map associated with the dual Newton fan (such resolutions of non-degenerate singularities also appear when one considers an embedded resolution in family [14]). This again shows mysteriously that the jet schemes tell us something about the "minimality" of the embedded resolution, as in the case of rational double point singularities.

The paper is organized as follows: Section 2 present a reminder on RTP-singularities. Section 3 is devoted to jet schemes and how one can associate a divisorial valuation with a component of the jet schemes; it also contains a summary of the approach to the embedded resolutions which will be constructed in the sequel. Each of the remaining sections is devoted to a class of RTP-singularities (given in the table of contents above): we compute each of the jet schemes and present the results in the jet graph (see Section 3). We then give the toric embedded resolution which comes from the jet schemes. We give the explicit computations with all details for the classes $E_{6,0}$ and $A_{k-1,l-1,m-1}$. For the other classes, except a subclass of the type B, we proceed similarly, so we present here only the results of the computations. The case $B_{k-1,2l-1}$ with $k \geq l$ is treated in detail as its behavior is completely different from the other cases. This is related to the fact that the abstract toric resolution of $B_{k-1,2l-1}$ which is obtained from a subdivision of the two dimensional cones of the dual Newton fan is not minimal [1].

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2. RTP-singularities

Let X denote a germ of a surface $(X,0) \subset (\mathbb{C}^N,0)$ having a singularity at 0. We say that the singularity of X is rational if $H^1(\tilde{X},\mathcal{O}_{\tilde{X}}) = 0$ where $\pi : \tilde{X} \longrightarrow X$ is a resolution of X. This definition does not depend on the resolution π. It is well known that the rational singularities of complex surfaces have nice combinatorial properties which can be computed via their resolutions. In [3], the rational singularities of multiplicity 3 are classified by their dual graphs associated with the irreducible components of the minimal resolutions. For short, we call RTP-singularities this class of rational singularities. They are among the surface singularities defined in $\mathbb{C}^4$ and, each of which is defined by three equations given in [26]. The classification problem of rational singularities of multiplicity $m \geq 3$ is well studied in [13] and [25].

In [1], the authors obtain the equations of a class of hypersurfaces in $\mathbb{C}^3$ having nonisolated singularities obtained by projecting the equations of RTP-singularities to a generic hyperplane in $\mathbb{C}^4$ and, they call them the non-isolated forms of RTP-singularities since the normalizations of these hypersurfaces in $\mathbb{C}^3$ are exactly the RTP-singularities. They also show that:

Theorem 2.1. *The RTP-singularities are non-degenerate with respect to their Newton polyhedron. In particular, they can be resolved by a toric birational map $Z \longrightarrow \mathbb{C}^4$.*

In [1], the dual graph of the minimal resolution for all RTP-singularities, except those of type $B_{k-1,2l-1}$ for $k \geq l$ (see Section 6) are constructed by refining the dual Newton fan of the corresponding non-isolated forms of RTP-singularity (see also [23, 27]). In the case of the nonisolated form of a rational singularity of type $B_{k-1,2l-1}$ with $k \geq l$, the resolution obtained by the subdivision of the corresponding dual Newton fan is not minimal: consider the vectors $R := (2l-2, 2, 2k+1)$, $Q := (2k-l+2, 1, 2k-l+2)$, $P := (l-1, 1, l-1)$, $V := (2k-l, 1, 2k-l+1)$ and $U := (l-1, 1, l)$ coming out in the subdivision of the dual Newton fan of that singularity:

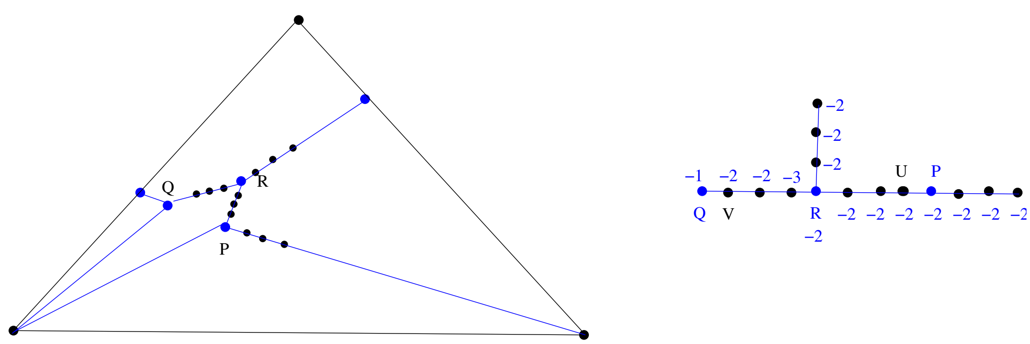


FIGURE 1. Dual Newton fan of a $B_{k-1,2l-1}$ singularity (with $k \geq l$), and its dual (abstract) resolution graph

Using [23], one can compute the self-intersections of the irreducible components of the exceptional divisors corresponding to these vectors; they are given by the number decorating the dual graph given on the right-hand side. We omit the genus decorations which are all 0 in this case. The exceptional component corresponding to the vector Q has self-intersection (-1); by Castelnuovo's criterion, (cf. for example [10], chapter V), that component can be contracted to a nonsingular point without creating singularities. If we continue to contract each (-1)-curve and neighboring components accordingly we obtain a (-3)-curve on the segment $[QR]$ and the dual graph of the minimal resolution of the RTP-singularity of type $B_{k-1,2l-1}$, $k \geq l$.

3. Jet schemes

Let k be an algebraically closed field of arbitrary characteristic and X be a k-algebraic variety. For $m \in \mathbb{N}$, the jet scheme X_m is the scheme representing the functor

$$F_m: \begin{array}{c} k\text{-}Schemes \to Sets \\ Spec(A) \mapsto Hom_k\left(Spec\left(A[t]/(t^{m+1})\right), X\right) \end{array}$$

where A is a k-algebra. The closed points of X_m are in bijection with the $k[t]/(t^{m+1})$ points of X. By definition, we have $X_0 = X$. Moreover, for $m, p \in \mathbb{N}$ with $m > p$, we have a canonical projection $\pi_{m,p} : X_m \longrightarrow X_p$ which is induced by the surjection $A[t]/(t^{m+1}) \longrightarrow A[t]/(t^{p+1})$. These morphisms are affine and verify $\pi_{m,p} \circ \pi_{q,m} = \pi_{q,p}$ for $p < m < q$; they define a projective system whose limit is a scheme that we denote X_∞ and which is called the arc space of X. Let us denote the canonical projection $\pi_{m,0} : X_m \longrightarrow X_0$ by π_m and, the canonical morphisms $X_\infty \longrightarrow X_m$ by Ψ_m.

We show here for a surface $X = \{f = 0\} \subset k^3$ (since the varieties that we are considering are defined this way) that the functor of the jet schemes is representable; this explains also how one determines jet schemes. We have

$$X = \operatorname{Spec}\frac{k[x,y,z]}{(f)}.$$

For a $k-$algebra A, an element γ in $F_m(\operatorname{Spec}(A))$ corresponds to a $k-$algebra homomorphism

$$\gamma^* : \frac{k[x,y,z]}{(f)} \longrightarrow \frac{A[t]}{(t^{m+1})}.$$

The data of such a γ is equivalent to the data of

$$\begin{aligned} \gamma^*(x) = x(t) = x_0 + x_1 t + \cdots + x_m t^m \in A[t]/(t^{m+1}), \\ \gamma^*(y) = y(t) = y_0 + y_1 t + \cdots + y_m t^m \in A[t]/(t^{m+1}), \\ \gamma^*(z) = z(t) = z_0 + z_1 t + \cdots + z_m t^m \in A[t]/(t^{m+1}); \end{aligned}$$

such that

$$f(x(t), y(t), z(t)) = F_0 + F_1 t + \cdots + F_m t^m + \cdots = 0 \bmod (t^{m+1}).$$

Here, for $i \geq 0, F_i$ is simply the coefficient of t^i in the expanding of $f(x(t), y(t), z(t))$.

Hence, the data of such a γ is equivalent to the data of $x_j, y_j, z_j \in A$ with $j = 0, \ldots, m$ such that $F_i(x_0, y_0, z_0, \ldots, x_i, y_i, z_i) = 0$ with $i = 0, \ldots, m$. This is equivalent to determining an A-point of the scheme

$$X_m := \operatorname{Spec}\frac{K[x_i, y_i, z_i; i = 0, \ldots, m]}{(F_0, \ldots, F_m)},$$

which then represents the functor F_m and, is by definition the m-th jet scheme of X.

From now on, we assume that X is a surface in $\mathbb{C}^3$ defined by $\{f(x,y,z) = 0\}$ and Y is a subvariety of X. Let $m \in \mathbb{N}$ We denote by $X_m^Y := \pi_m^{-1}(Y)$. We consider a special type of the irreducible components of $X_m^Y, m \in \mathbb{N}$ where Y is the singular locus of X or $Y \subset X$ is a curve contained in a coordinate hyperplane of $\mathbb{C}^3$. To such Y, we associate a divisorial valuation over $\mathbb{C}^3$ with an irreducible component $\mathcal{C}_m \subset X_m^Y$ in the following way.

Let $\psi_m^a : \mathbb{C}^3_\infty \longrightarrow \mathbb{C}^3_m$ be the truncation morphism associated with the ambient space $\mathbb{C}^3$, here the exponent "a" stands for *ambient map* . The morphism ψ_m^a is a trivial fibration, hence ${\psi_m^a}^{-1}(\mathcal{C}_m)$ is an irreducible cylinder in $\mathbb{C}^3_\infty$. Let η be the generic point of ${\psi_m^a}^{-1}(\mathcal{C}_m)$. By

Corollary 2.6 in [5], the map $\nu_{\mathcal{C}_m} : \mathbb{C}[x,y,z] \longrightarrow \mathbb{N}$ defined by $\nu_{\mathcal{C}_m}(h) = \mathrm{ord}_t h \circ \eta$ is a divisorial valuation on $\mathbb{C}^3$.

To each irreducible component $\mathcal{C}_m$ of X_m^Y, let us associate a vector, called the weight vector, in the following way:

$$v(\mathcal{C}_m) := (\nu_{\mathcal{C}_m}(x), \nu_{\mathcal{C}_m}(y), \nu_{\mathcal{C}_m}(z)) \in \mathbb{N}^3.$$

Now, we want to characterize the irreducible components of X_m^Y that will allow us to construct an embedded resolution of X: For $p \in \mathbb{N}$, we consider the following cylinder in the arc space:

$$Cont^p(f) = \{\gamma \in \mathbb{C}^3_\infty; \mathrm{ord}_t f \circ \gamma = p\}.$$

Definition 3.1. Let $X : \{f = 0\}$ be a surface in $\mathbb{C}^3$ and let Y be a subvariety of X.
(i) The elements of the set:

$$EC(X) := \{\text{Irreducible components } \mathcal{C}_m \text{ of } X_m^Y \text{ such that } {\psi_m^a}^{-1}(\mathcal{C}_m) \cap Cont^{m+1} f \neq \emptyset$$
$$\text{and } v(\mathcal{C}_m) \neq v(\mathcal{C}_{m-1}) \text{ for any component } \mathcal{C}_{m-1} \text{ verifying } \pi_{m,m-1}(\mathcal{C}_m) \subset \mathcal{C}_{m-1}, m \geqslant 1\}$$

are called the *essential components* for X.
(ii) the elements of the set of associated valuations:

$$EV(X) := \{\nu_{\mathcal{C}_m},\ \mathcal{C}_m \in EC(X)\},$$

are called *embedded-valuations* for X.

This means that the elements of $EV(X)$ appear in the embedded toric resolution of X. We will be interested in a subset of $EV(X)$, which gives us an embedded resolution. In the following sections, in order to determine such a subset when X is a non-isolated form of an RTP-singularity, we will study the m-th jet schemes of X, for $m \leq l$ with l large enough. We will encode the structure of these jet schemes by a levelled graph whose vertices correspond to the irreducible components of X_m^Y for an integer m; two vertices at the level m and $m-1$ are joined by an edge if the transition morphism $\pi_{m,m-1}$ sends the corresponding components one into the other [16]. An element of $EV(X)$ corresponding to a component $\mathcal{C}_m \in EC(X)$ is actually a monomial (or toric) valuation (see proposition 2.3 in [20]) and is defined by the vector $v(\mathcal{C}_m) = (a,b,c)$: this means that, for $h = \sum_{\{(i,j,k)\}} a_{(i,j,k)} x^i y^j z^k \in \mathbb{C}[x,y,z]$ we have:

$$\nu_{\mathcal{C}_m}(h) = \min_{\{(i,j,k) | a_{(i,j,k)} \neq 0\}} \{ai + bj + ck\}.$$

By subdividing the first quadrant of $\mathbb{R}^3$ using the vectors $v(\mathcal{C}_m)$ for some $\mathcal{C}_m \in EC(X)$, we obtain a fan Σ whose support is the first quadrant of $\mathbb{R}^3$ and whose one dimensional cones are generated by these $v(\mathcal{C}_m)$'s. Note that one can obtain different fans from a set of vectors in $\mathbb{R}^3$, depending on the way one relies the vertices and, some of them may not be regular, but here we are interested in finding a regular fan. Hence we have a proper birational map $\mu_\Sigma : Z_\Sigma \longrightarrow \mathbb{C}^3$ where Z_Σ is smooth and the irreducible components of the exceptional divisor of μ_Σ correspond to the vectors $v(\mathcal{C}_m)$ that we consider. More precisely, the divisorial valuations corresponding to the irreducible components of the exceptional divisor of μ_Σ are exactly the $\nu_{\mathcal{C}_m}$ associated with the components $\mathcal{C}_m$ that we consider.

We will find such a regular fan Σ for a non-isolated form X of an RTP-singularity which is not of the type $B_{k-1,2l-1}$ ($i.e$ we will construct Σ using the vectors of type $v(\mathcal{C}_m)$) that refines the dual Newton fan of $X \subset \mathbb{C}^3$. Thanks to Varchenko's theorem [27], this gives that μ_Σ is an embedded resolution of $X \subset \mathbb{C}^3$. On the other hand, for a $B_{k-1,2l-1}$-singularity, we cannot apply Varchenko's theorem because there is no Σ refining the dual Newton fan as described above; nevertheless we build a regular fan Σ satisfying the properties above and, we prove by studying the total transform of our singularity by μ_Σ that $\mu_\Sigma : Z_\Sigma \longrightarrow \mathbb{C}^3$ is an embedded resolution of the $B_{k-1,2l-1}$ singularity.

4. RTP-singularities of type $E_{6,0}$

The singularity of $X \subset \mathbb{C}^3$ defined by the equation:

$$f(x,y,z) = z^3 + y^3 z + x^2 y^2 = 0$$

is called $E_{6,0}$-type singularity. Its dual Newton fan is given in Figure 2.

In this section, we compute explicitly the m-th jet schemes (for $m \leq 18$) and we determine a subset of $EV(X)$ which gives a regular subdivision of the dual Newton fan as explained in the previous section. We represent the irreducible components as a graph in Figure 3, where we also weight the vertex associated with a component $\mathcal{C}_m$ by the vector $v(\mathcal{C}_m)$ also defined in the previous section. For a component $\mathcal{C}_m$ which projects by the maps $\pi_{m,m-1}$ given in Section 3 on a monomial component (*i.e.* a component whose associated valuation is monomial) $\mathcal{C}_{m-1}$, which is not itself monomial; we also weight the associated vertex by the unique non-monomial equation which, together with the hyperplane coordinates $\mathcal{C}_{m-1}$, defines $\mathcal{C}_m$. That helps for the computations of the irreducible components in the process. Here we do not pay much attention to the edges since they are not relevant for the problem at hand. The arrows in Figure 3 correspond to a component $\mathcal{C}_m$ such that the inverse image of a dense open set in it gives an irreducible component for every $n \geq m$. First let us fix some notations:

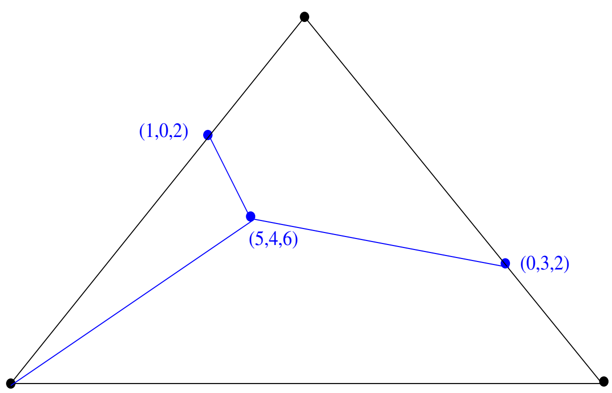


FIGURE 2. Dual Newton fan of $E_{6,0}$ singularity

Notation: Let

$$(*) \quad f\Big(\sum_{i=0}^{m} x_i t^i, \sum_{i=0}^{m} y_i t^i, \sum_{i=0}^{m} z_i t^i\Big) = \sum_{i=0}^{i=m} F_i t^i \quad \mod(t^{m+1}).$$

We know that (e.g. [20]) the m-th jet scheme X_m is defined by the ideal

$$I_m = (F_0, F_1, \ldots, F_m) \subset \mathbb{C}[x_i, y_i, z_i; i = 0, \ldots, m].$$

4.1. **Jet Schemes of $E_{6,0}$.** For $m \geq 1$, we will determine the irreducible components of the space of $m-$jets that projects on the singular locus of X, i.e. the irreducible components of $X_m^{Sing} := \pi_{m,0}^{-1}(V(y_0, z_0)) \subset X_m \subset \mathrm{Spec}(\mathbb{C}[x_i, y_i, z_i; i = 0, \ldots, m]) = \mathbb{C}_m^3$; here $V(I)$ denotes the variety defined by an ideal I and $\mathbb{C}_m^3$ is the m-th jet scheme of the affine three dimensional space $\mathbb{C}^3$; we insist here that when considering X_m^{Sing} for a given m, the symbol $V(I)$ designates the variety defined by an ideal I in $\mathbb{C}_m^3$. Recall that $\pi_{m_0} : X_m \longrightarrow X_0 = X$. We also insist on the fact that we consider only the reduced structure of these schemes.

For $m = 1$, we have $X_1^{Sing} = V(y_0, z_0) \subset \mathrm{Spec}(\mathbb{C}[x_i, y_i, z_i; i = 0, 1])$ because, if we put $y_0 = z_0 = 0$ in the equation $(*)$ we get $F_0 = F_1 = 0$ modulo the ideal (y_0, z_0). Hence, X_1^{Sing} consists of a unique irreducible component, denoted by $\mathcal{C}_{1,1}$. The weight vector of $\mathcal{C}_{1,1}$ is $(0, 1, 1)$.

For $m = 2$, we have $X_2^{Sing} = \pi_{2,1}^{-1}(\mathcal{C}_{1,1})$; this uses the fact $\pi_{2,0} = \pi_{1,0} \circ \pi_{2,1}$. A direct computation using the equation $(*)$ gives:

$$F_2 = x_0^2 y_1^2 \bmod (y_0, z_0).$$

Hence $X_2^{Sing} = V(y_0, z_0, x_0^2 y_1^2) \subset \mathrm{Spec}(\mathbb{C}[x_i, y_i, z_i; i = 0, 1, 2]) = \mathbb{C}_2^3$. We deduce that X_2^{Sing} has two irreducible components $\mathcal{C}_{2,1} := V(y_0, z_0, x_0)$ and $\mathcal{C}_{2,2} := V(y_0, z_0, y_1)$ both are sent via $\pi_{2,1}$ into $\mathcal{C}_{1,1}$; there weight vectors are respectively $(1, 1, 1)$ and $(0, 2, 1)$. These vectors are represented in Figure 3 at the levels $m = 1$ and $m = 2$.

For $m = 3$, using the fact $\pi_{3,0} = \pi_{2,0} \circ \pi_{3,2}$, it is sufficient to study $\pi_{3,2}^{-1}(\mathcal{C}_{2,j})$ with $j = 1, 2$ to understand X_3^{Sing}.

- To find $\pi_{3,2}^{-1}(\mathcal{C}_{2,1})$, we compute F_3 modulo the ideal (x_0, y_0, z_0) and we obtain:

  $$F_3 = z_1^3 \bmod (x_0, y_0, z_0);$$

  Hence we obtain that $\mathcal{C}_{3,1} := \pi_{3,2}^{-1}(\mathcal{C}_{2,1}) = V(x_0, y_0, z_0, z_1)$ is irreducible.
- Similarly, we obtain that $\mathcal{C}_{3,2} := \pi_{3,2}^{-1}(\mathcal{C}_{2,2}) = V(y_0, y_1, z_0, z_1)$ is irreducible.

So we have $X_3^{Sing} = \mathcal{C}_{3,1} \cup \mathcal{C}_{3,2}$ where $\mathcal{C}_{3,1}$ and $\mathcal{C}_{3,2}$ are both irreducible and clearly there is no inclusions between them: indeed, $\mathcal{C}_{3,1}$ is included in $V(x_0)$ but $\mathcal{C}_{3,2}$ is not and, $\mathcal{C}_{3,2}$ is included in $V(y_1)$ but $\mathcal{C}_{3,1}$ is not. We conclude that $\mathcal{C}_{3,1}$ and $\mathcal{C}_{3,2}$ are the irreducible components of X_3^{Sing}. Their associated weight vectors are respectively $(1, 1, 2)$ and $(0, 2, 2)$.

For $m = 4$, as in the previous case, it is sufficient to consider $\pi_{4,3}^{-1}(\mathcal{C}_{3,j})$, with $j = 1, 2$. As the computations go almost in the same way, we just announce what we obtain:

- To determine $\pi_{4,3}^{-1}(\mathcal{C}_{3,1})$ we compute F_4 modulo the ideal (x_0, y_0, z_0, z_1). We have

  $$F_4 = x_1^2 y_1^2 \bmod (x_0, y_0, z_0, z_1).$$

  Hence, $\pi_{4,3}^{-1}(\mathcal{C}_{3,1})$ has 2 irreducible components

  $$\mathcal{C}_{4,1} = V(x_0, y_0, y_1, z_0, z_1) \quad \text{and} \quad \mathcal{C}_{4,2} = V(x_0, y_0, x_1, z_0, z_1).$$
- Similarly we have $\pi_{4,3}^{-1}(\mathcal{C}_{3,2}) = \mathcal{C}_{4,1} \cup \mathcal{C}_{4,3}$ where $\mathcal{C}_{4,3} = V(y_0, y_1, y_2, z_1, z_0)$.

Then we get

$$X_4^{Sing} = \mathcal{C}_{4,1} \cup \mathcal{C}_{4,2} \cup \mathcal{C}_{4,3}$$

which is a decomposition into irreducible varieties. Using a similar argument as in the case of $m = 3$, we conclude that there are no mutual inclusions between these components; hence this is the decomposition into irreducible components. The corresponding weight vectors of $\mathcal{C}_{4,1}, \mathcal{C}_{4,2}$ and $\mathcal{C}_{4,3}$ are respectively $(1, 2, 2), (2, 1, 2)$ and $(0, 3, 2)$. Figure 3 encodes also this information.

For $m = 5$, we have

$$X_5^{Sing} = \pi_{5,4}^{-1}(\mathcal{C}_{4,1}) \cup \pi_{5,4}^{-1}(\mathcal{C}_{4,2}) \cup \pi_{5,4}^{-1}(\mathcal{C}_{4,3}) \subset \mathbb{C}_5^3.$$

- To determine $\pi_{5,4}^{-1}(\mathcal{C}_{4,1})$, we compute F_5 modulo $(x_0, y_0, y_1, z_0, z_1)$ that we find to be 0. We deduce, that $C_{5,1} := \pi_{5,4}^{-1}(\mathcal{C}_{4,1}) = V(x_0, y_0, y_1, z_0, z_1)$ is irreducible. A small attention here is needed: The varieties $\mathcal{C}_{4,1}$ and $\mathcal{C}_{5,1}$ are not the same; they are defined by the same equations but in different rings; they actually define the same valuation on $\mathbb{C}^3$ (see Proposition 2.3 in [20]).
- Computing F_5 modulo the ideal $(x_0, x_1, y_0, z_0, z_1)$, we find $F_5 = y_1^3 z_2 = 0$. So $\pi_{5,4}^{-1}(\mathcal{C}_{4,2})$ is the union of $V(x_0, x_1, y_0, y_1, z_0, z_1)$ and $\mathcal{C}_{5,2} := V(x_0, x_1, y_0, z_1, z_0, z_2)$.
- As for $\pi_{5,4}^{-1}(\mathcal{C}_{4,1})$, computing F_5 modulo $(y_0, y_1, y_2, z_0, z_1)$ we find zero. This gives that $\mathcal{C}_{5,3} := \pi_{5,4}^{-1}(\mathcal{C}_{4,3}) = V(y_0, y_1, z_0, z_1, z_2)$is irreducible.

Hence we obtain

$$X_5^{Sing} = \mathcal{C}_{5,1} \cup \mathcal{C}_{5,2} \cup V(x_0, x_1, y_0, y_1, z_0, z_1) \cup \mathcal{C}_{5,3}.$$

Since $V(x_0, x_1, y_0, y_1, z_0, z_1)$ is included in $\mathcal{C}_{5,1}$, the decomposition

$$X_5^{Sing} = \mathcal{C}_{5,1} \cup \mathcal{C}_{5,2} \cup \mathcal{C}_{5,3}$$

is the decomposition into the irreducible components. Moreover, the weight vectors of $\mathcal{C}_{5,j}$ for $j = 1, 2, 3$ are $(1, 2, 2), (2, 1, 3)$ and $(0, 3, 2)$ respectively.

For $m = 6$, we have

$$X_6^{Sing} = \pi_{6,5}^{-1}(\mathcal{C}_{5,1}) \cup \pi_{6,5}^{-1}(\mathcal{C}_{5,2}) \cup \pi_{6,5}^{-1}(\mathcal{C}_{5,3}) \subset \mathbb{C}_6^3.$$

- To determine $\pi_{6,5}^{-1}(\mathcal{C}_{5,1})$, we compute F_6 modulo the ideal $(x_0, y_0, y_1, z_0, z_1)$ and we find

  $$\mathcal{C}_{6,1} := \pi_{6,5}^{-1}(\mathcal{C}_{5,1}) = V(x_0, y_0, y_1, z_0, z_1, z_2^3 + x_1^2 y_2^2) \subset \mathbb{C}_6^3.$$

  Notice that $\mathcal{C}_{6,1}$ is isomorphic to the product of an affine space and the hypersurface defined by $\{z_2^3 + x_1^2 y_2^2 = 0\}$; this hypersurface is a Hirzebruch-Jung singularity which is well known to be an irreducible quasi-ordinary singularity [4]; in particular $\mathcal{C}_{6,1}$ is irreducible. Actually, we will see that $\mathcal{C}_{6,1}$ will give rise to an irreducible component of X_6^{Sing} whose weight vector is same as the weight vector associated with $\mathcal{C}_{5,1}$, so it is not an essential component (see definition above): the divisorial valuation associated with it is not monomial while a divisorial valuation associated with an essential component is monomial. Before we continue to study on X_6^{Sing}, let us consider $\pi_{m,6}^{-1}(\mathcal{C}_{6,1})$ for $m \geq 7$:

  For this, we will stratify $\mathcal{C}_{6,1}$ into its regular locus and its singular locus which are defined respectively by $x_1 = z_2 = 0$ and $y_2 = z_2 = 0$. The inverse images

  $$\pi_{7,6}^{-1}(\mathcal{C}_{6,1} \cap \{x_1 = z_2 = 0\}) \quad \text{and} \quad \pi_{7,6}^{-1}(\mathcal{C}_{6,1} \cap \{y_2 = z_2 = 0\})$$

  will give the irreducible components of X_7^{Sing} looking like the irreducible components that we have studied before which are the essential components, so give the new weight vectors. The inverse image of the regular part of $\mathcal{C}_{6,1}$ with respect to $\pi_{m,6}$, with $m \geq 7$ is equal to $\pi_{m,6}^{-1}(\mathcal{C}_{6,1} \cap \{z_2 \neq 0\})$; this latter is defined in $\mathbb{C}_m^3 \cap \{z_2 \neq 0\}$ by the ideal generated by $x_0, y_0, y_1, z_0, z_1, z_2^3 + x_1^2 y_2^2$ and

  $$F_j = c_j z_3 z_{j-3} + H_j(x_1, \ldots, x_{j-5}, y_2, \ldots, y_{j-4}, z_3, \ldots, z_{j-3}), c_j \in \mathbb{C}^*$$

  for $7 \leq j \leq m$. The functions F_j are linear as we can invert $c_j z_3 \neq 0$. Then the Zariski closure $\overline{\pi_{m,6}^{-1}(\mathcal{C}_{6,1} \cap \{z_2 \neq 0\})}$ is irreducible and, is actually an irreducible component of X_m^{Sing} for every $m \leq 7$. Note that the weight vector of $\overline{\pi_{m,6}^{-1}(\mathcal{C}_{6,1} \cap \{z_2 \neq 0\})}$ is $(1, 2, 2)$ which is same as the one for $\mathcal{C}_{6,1}$ and for $\mathcal{C}_{5,1}$; hence they don't give an essential component.

They are encoded in Figure 3 by the dashed arrow which starts at the vertex weighted by the vector $(1,2,2)$ and the equation $z_2^3+x_1^2y_2^2=0$.

- To determine $\pi_{6,5}^{-1}(\mathcal{C}_{5,2})$, we compute F_6 modulo the ideal $(x_0,x_1,y_0,z_0,z_1,z_2)$ and we find that $F_6=y_1^2(z_2y_1+x_2^2)$. So $\pi_{6,5}^{-1}(\mathcal{C}_{5,2})$ is the union of $\mathcal{C}_{6,2}:=V(x_0,x_1,y_0,y_1,z_0,z_1,z_2)$ and $\mathcal{C}_{6,3}:=V(x_0,x_1,y_0,z_1,z_0,z_2,z_3y_1+x_2^2)$ which are both irreducible. We note that $\pi_{m,6}^{-1}(\mathcal{C}_{6,3})$ is irreducible for every $m\geq 7$ and gives rise to an irreducible component of X_m^{Sing} for every $m\geq 7$. The irreducibility of the inverse image results from the fact that $\mathcal{C}_{6,3}$ is the product of an affine space and an A_1-singularity and the jet schemes of such singularity are irreducible [21, 19] (what applies here for A_1 is also true for any rational singularity). The components of $\pi_{m,6}^{-1}(\mathcal{C}_{6,3})$ are not the essential components, they are associated with non-monomial valuations and they have the same weight vector, namely $(2,1,3)$. They are encoded in Figure 3 (to the most right of the graph) by the dashed arrow which starts at the vertex weighted by the vector $(2,1,3)$ and the equation $x_2^2+z_3y_1=0$.
- To determine $\pi_{6,5}^{-1}(\mathcal{C}_{5,3})$, we compute F_6 modulo the ideal (y_0,y_1,y_2,z_0,z_1) and we find that $F_6=z_2^3+x_0^2y_3^3$. Hence
$$\mathcal{C}_{6,4}:=\pi_{6,5}^{-1}(\mathcal{C}_{5,3})=V(y_0,y_1,y_2,z_0,z_1,z_2^3+x_0^2y_3^2)$$
is irreducible. By the same argument as in the case of $\pi_{m,6}^{-1}(\mathcal{C}_{6,1})$, the inverse images $\pi_{7,6}^{-1}(\mathcal{C}_{6,4}\cap\{x_0=z_2=0\})$ and $\pi_{7,6}^{-1}(\mathcal{C}_{6,4}\cap\{y_3=z_2=0\})$ will give rise to the irreducible components of X_7^{Sing}; the Zariski closure $\overline{\pi_{m,6}^{-1}(\mathcal{C}_{6,4}\cap\{z_2\neq 0\})}$ is irreducible and is actually an irreducible component of X_m^{Sing} for every $m\geq 7$. This is encoded in Figure 3 by the dashed arrow starting at the vertex weighted by the vector $(0,3,2)$ and the equation $z_2^3+x_0^2y_3^2$.

To summarize, we obtain $X_6^{Sing}=\mathcal{C}_{6,1}\cup\mathcal{C}_{6,2}\cup\mathcal{C}_{6,3}\cup\mathcal{C}_{6,4}$ where each $\mathcal{C}_{6,j}$ for $j=1\dots,4$ is irreducible. Obviously, $\mathcal{C}_{6,2}\subset\mathcal{C}_{6,1}$ and, using the same argument as in the case of $m=3$, we verify that there is no inclusion among the remaining $\mathcal{C}_{6,j}$'s. Hence we get the irreducible decomposition
$$X_6^{Sing}=\mathcal{C}_{6,1}\cup\mathcal{C}_{6,3}\cup\mathcal{C}_{6,4}$$
with the respective weight vectors $(1,2,2),(2,1,3)$ and $(0,3,2)$.

For $m=7$, by the above discussions, we have a stratification
$$X_7^{Sing}=\pi_{7,6}^{-1}(\mathcal{C}_{6,1}\cap\{x_1=z_2=0\})\cup\pi_{7,6}^{-1}(\mathcal{C}_{6,1}\cap\{y_2=z_2=0\})\cup\overline{\pi_{7,6}^{-1}(\mathcal{C}_{6,1}\cap\{z_2\neq 0\})}\cup$$
$$\pi_{7,6}^{-1}(\mathcal{C}_{6,3})\cup\pi_{7,6}^{-1}(\mathcal{C}_{6,4}\cap\{y_3=z_2=0\})\cup\overline{\pi_{m,6}^{-1}(\mathcal{C}_{6,4}\cap\{z_2\neq 0\})}$$
which is the decomposition into irreducible components; indeed, on the one hand using the same argument as for $m=3$, there is no inclusions between $\pi_{7,6}^{-1}(\mathcal{C}_{6,3})$ and the other components; on the other hand, the other components are clearly not equal, this means that there are no inclusions between them because they are irreducible and they have the same dimension (actually codimension 7 in $\mathbb{C}_7^3$).

Note that the codimension is easy to compute since the equations are either hyperplane coordinates in $\mathbb{C}_7^3$ or we consider the closure of a constructible set which is defined by hyperplane coordinates and by linear equations. The weight vectors are respectively $(2,2,3)$, $(1,3,3)$, $(1,2,2)$, $(2,1,3)$, $(0,4,3)$, and $(0,3,3)$. Moreover we have
$$\pi_{7,6}^{-1}(\mathcal{C}_{6,4}\cap\{x_0=z_2=0\})=\pi_{7,6}^{-1}(\mathcal{C}_{6,1}\cap\{y_2=z_2=0\}).$$

We should also note that although $\mathcal{C}_{6,2}$ is not an irreducible component, its inverse image $\pi_{7,6}^{-1}(\mathcal{C}_{6,2})$ which is equal to $\pi_{7,6}^{-1}(\mathcal{C}_{6,1} \cap \{y_2 = z_2 = 0\})$ gives an irreducible component.

We have gone through the arguments which allow to determine all the irreducible components of X_m^{Sing} for $m \leq 18$. This is encoded in Figure 3. Note that 18 is the quasi-degree of the weighted homogeneous polynomial defining our singularity.
One last important thing is that the axis $Y = \{x = z = 0\}$ is drawn on our singularity. We determine the essential components of $X_m^Y, m \geq 0$, we find $V(x_0, z_0) \subset X_0 \subset \mathbb{C}_0^3$ and $V(x_0, z_0, z_1) \subset X_1 \subset \mathbb{C}_1^3$ whose weight vectors are respectively $(1,0,1), (1,0,2)$.

To conclude, the essential components are the irreducible components of X_m^Z (where Z is the singular locus of X or $Z = Y$ is the y-axis) whose defining equations are hyperplane coordinates and, their associated valuations are monomial and determined with their weight vectors. Hence we get the graph in Figure 3 for the jet schemes.

Proposition 4.1. *For an $E_{6,0}$-singularity, the monomial valuations associated with the vectors* $(0,1,1), (0,2,1), (1,1,1), (0,3,2), (1,1,2), (1,2,2), (2,1,2), (2,1,3), (2,2,3), (3,2,3), (3,2,4), (3,3,4), (4,3,5), (5,4,6)$ *belong to $EV(X)$.*

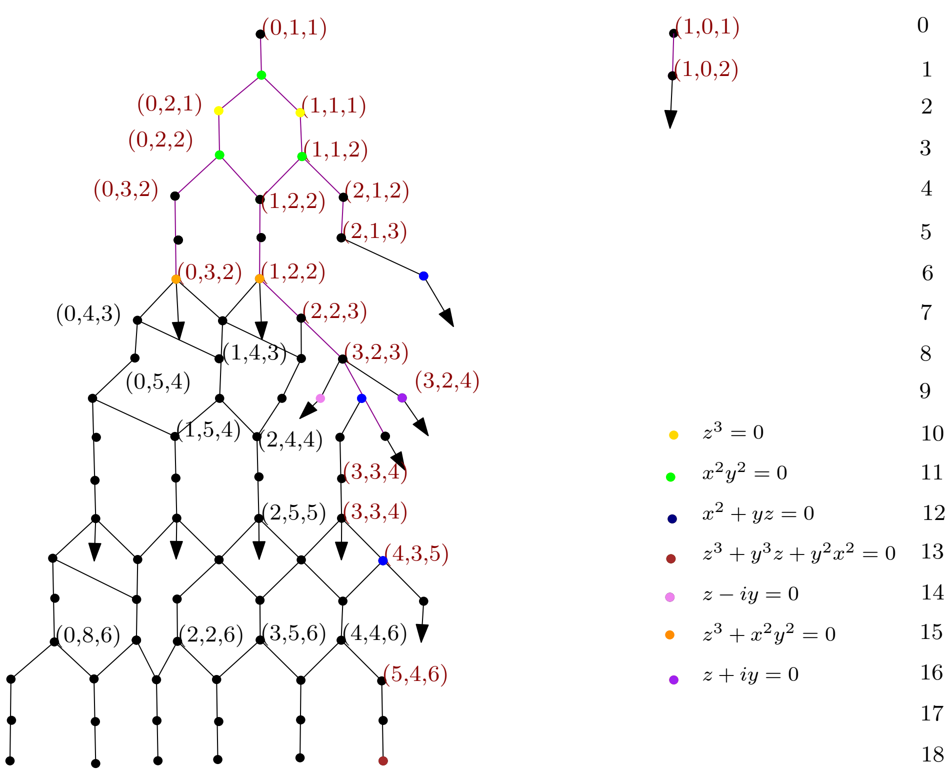


FIGURE 3. Jets schemes of $E_{6,0}$

4.2. **Toric Embedded Resolution of $E_{6,0}$.** Now we are ready to announce the main result for the surface X of type $E_{6,0}$-singularity.

Theorem 4.2. *There exists a toric birational map $\mu_\Sigma : Z_\Sigma \longrightarrow \mathbb{C}^3$ which is an embedded resolution of $X \subset \mathbb{C}^3$ such that the components of the exceptional divisor of μ_Σ correspond to the irreducible components of the m-th jet schemes of X (centered at the singular locus and the intersection of X with the coordinate hyperplane). Moreover this yields a construction (not canonical) of μ_Σ.*

Proof. By [27, 23, 1] (see also [20] for a summary), an embedded resolution of $X \subset \mathbb{C}^3$ can be obtained by constructing a regular subdivision of the dual Newton fan of $X \subset \mathbb{C}^3$. The dual Newton fan Σ for $E_{6,0}$ is presented in Figure 2 .

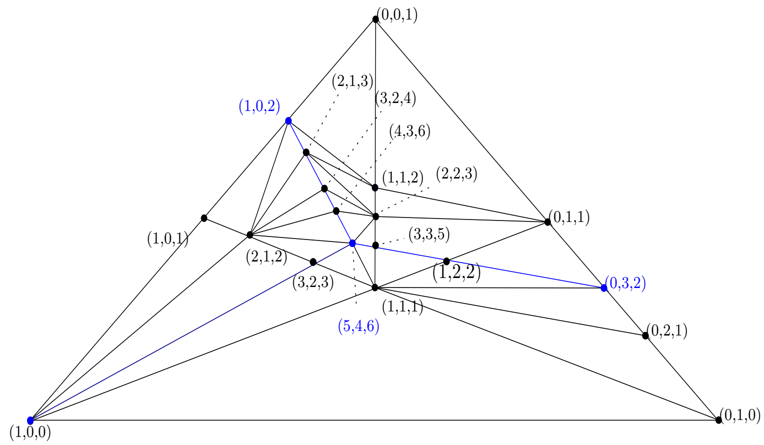


FIGURE 4. An embedded resolution of $E_{6,0}$

In Figure 4, we give a regular subdivision Σ where the rays (cones of dimension 1) are the lines supported by the vectors given in proposition 4.1. To see that this is a regular subdivision, it is sufficient to show that each cone is regular (means that the determinant of the matrix whose columns are any three vectors generating a cone of Σ equals 1). Moreover the 1-dimensional cones (rays) are in bijective correspondence with the components of the exceptional divisors. □

5. RTP-SINGULARITIES OF TYPE $A_{k-1,l-1,m-1}$

The singularity of $X \subset \mathbb{C}^3$ defined by the equations:

- $k \geq \ell \geq m$,
$$z^3 + xz^2 - (x + y^k + y^\ell + y^m)y^k z + y^{2k+\ell} = 0,$$
- $k = \ell < m$,
$$z^3 + (x - y^k)z^2 - (x + y^k + y^m)y^k z + y^{2k+m} = 0.$$

is called $A_{k-1,l-1,m-1}$-type singularity where $k, \ell, m \geq 1$.

5.1. **Jet Schemes and toric Embedded resolution of $A_{k-1,l-1,m-1}$ when $k = l \leq m$.** The singular locus is $\{y = z = 0\}$. So we compute the jets schemes over $\{y = z = 0\}$. The graph representing the irreducible components of the jet schemes of $A_{k-1,l-1,m-1}$ is in Figure 5

Theorem 5.1. *With the preceding notation, the monomial valuations associated with the vectors*

- $(0, 1, 1), (0, 1, 2), \ldots (0, 1, k + m)$

- $(s, 1, s), \ldots, (s, 1, m+k-s)$ $1 \leq s \leq k-1$
- $(k, 1, k), \ldots, (k, 1, m)$

belong to $EV(X)$. Moreover, these vectors give a toric birational map $\mu_\Sigma : Z_\Sigma \longrightarrow \mathbb{C}^3$ which is an embedded resolution of $X \subset \mathbb{C}^3$ (in the neighborhood of the origin) such that the components of the exceptional divisor of μ_Σ correspond to the monomial valuations defined by them; hence they correspond to the irreducible components of the m-th jet schemes of X (centered at the singular locus and the intersection of X with the coordinate hyperplanes).

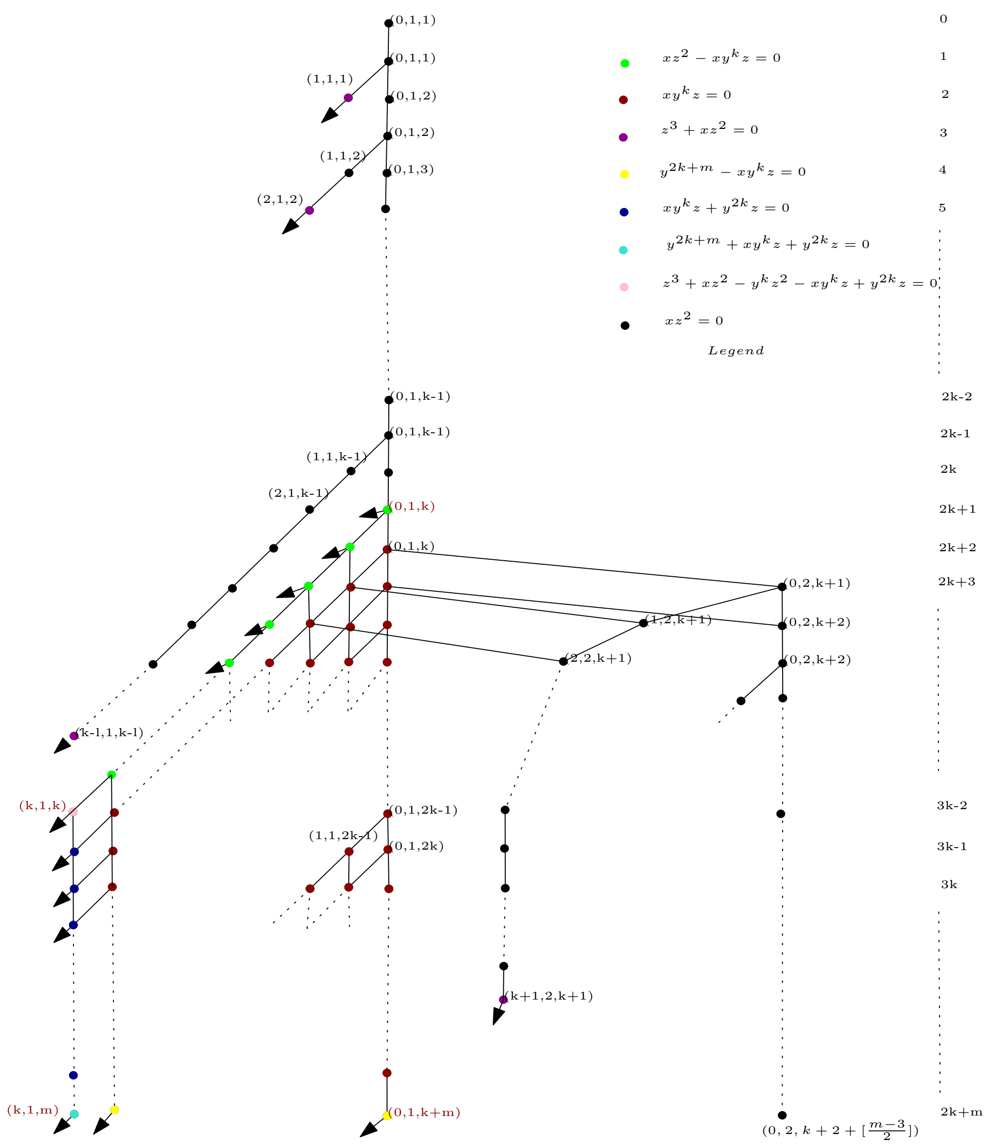


FIGURE 5. Jets schemes of $A_{k-1,k-1,m-1}$

Proof. The first part of the theorem results from the jet graph. Before showing that the given vectors give a simplicial regular decomposition of the dual Newton fan of $A_{k-1,k-1,m-1}$, let us study their positions in the fan:

- for all $0 \leq s \leq k$, we have $(s,1,k) \in [(0,1,k),(k,1,k)]$
- for all $0 \leq s \leq k$, we have $(s,1,k+m-s) \in [(0,1,k+m),(k,1,m)]$:

$$\begin{vmatrix} k & 0 & s \\ 1 & 1 & 1 \\ m & k+m & k+m-s \end{vmatrix} = \begin{vmatrix} k & 0 & s \\ 0 & 1 & 1 \\ -k & k+m & -s \end{vmatrix} = 0$$

- the vectors $(\alpha,1,l+\alpha+1)$ for all $0 \leq \alpha \leq k$ are aligned, for each $0 \leq l \leq k$.

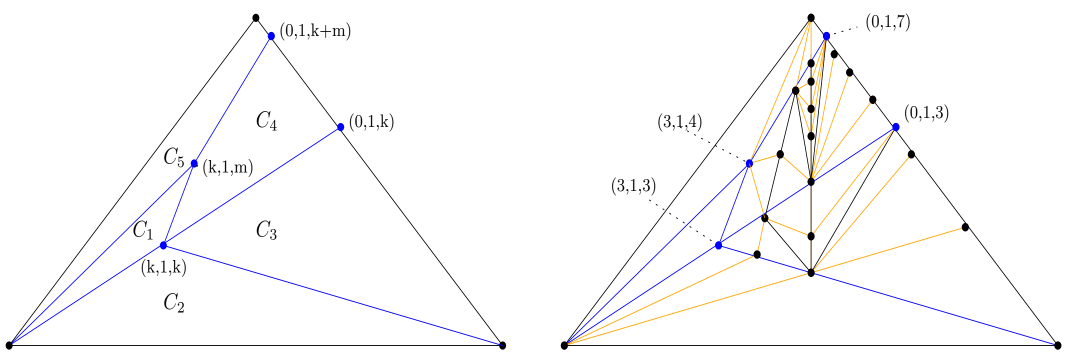


FIGURE 6. Dual Newton fan of $A_{k-1,k-1,m-1}$ and an embedded resolution for $A_{2,2,3}$

Now let us decompose each subcone C_i into regular cones:

Decomposition of C_1: The cone C_1 contains the vectors $(k,1,\beta)$ for $k \leq \beta \leq m-1$. They are on the skeleton of the fan. For $k \leq \beta \leq m-1$, we have: $\begin{vmatrix} k & k & 1 \\ 1 & 1 & 0 \\ \beta & \beta+1 & 0 \end{vmatrix} = 1.$

Decomposition of C_2: The cone C_2 contains the vectors $(1,1,1),\ldots,(k,1,k)$ which are on the skeleton. For $0 \leq \alpha \leq k-1$ we have: $\begin{vmatrix} 1 & \alpha & \alpha+1 \\ 0 & 1 & 1 \\ 0 & \alpha & \alpha+1 \end{vmatrix} = 1.$

Decomposition of C_3: To decompose the cone C_3, we first add successively an edge between the vectors $(k-1,1,k),(k-2,1,k-2),(k-3,1,k),\ldots$ with the last vector being $(0,1,k)$ if k is odd and with $(0,1,0)$ if k is even. Then we obtain that the vectors $(\alpha,1,\alpha),\ldots,(\alpha,1,k)$ are in the same triangles (see Figure 7). Now let us add those vectors and the vectors on the associated edges successively.

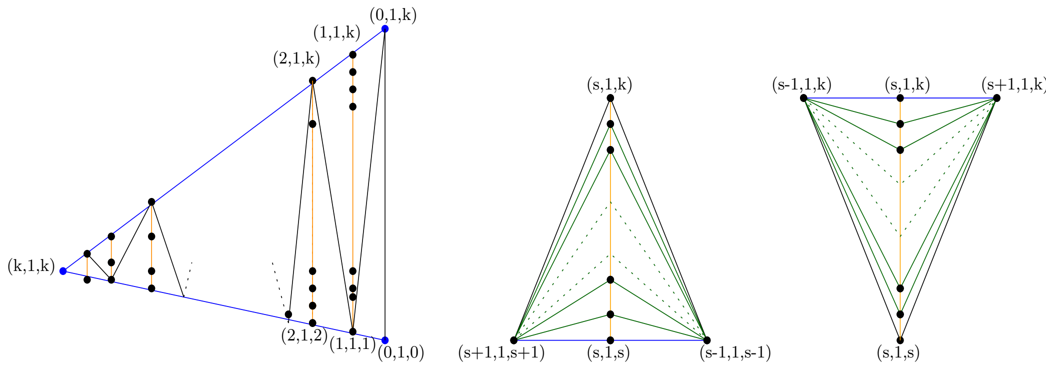


FIGURE 7. Decomposition of the cone C_3 and of its two types of subcones

Each new subcone will be regular as we only have one of the following two cases:

- **Case 1**: for $\alpha \le \beta \le k-1$ we have

$$\begin{vmatrix} \alpha-1 & \alpha & \alpha \\ 1 & 1 & 1 \\ k & \beta & \beta+1 \end{vmatrix} = 1 \quad \text{and} \quad \begin{vmatrix} \alpha+1 & \alpha & \alpha \\ 1 & 1 & 1 \\ k & \beta & \beta+1 \end{vmatrix} = 1$$

- **Case 2**:

$$\begin{vmatrix} \alpha+1 & \alpha & \alpha \\ 1 & 1 & 1 \\ \alpha+1 & \beta & \beta+1 \end{vmatrix} = 1 \quad \text{and} \quad \begin{vmatrix} \alpha-1 & \alpha & \alpha \\ 1 & 1 & 1 \\ \alpha-1 & \beta & \beta+1 \end{vmatrix} = -1$$

Decomposition of C_4: The cone C_4 is decomposed by adding successively the edges between the vectors $(k,1,m),(k-1,1,k),(k-2,1,m+2),\dots$ with the last vector being $(1,1,k)$ if k is odd and with $(1,1,k+m-1)$ if k is even. Then let us add successively the vectors and the associated edges $(s,1,\alpha)$ for $k \le \alpha \le k+m-s$.

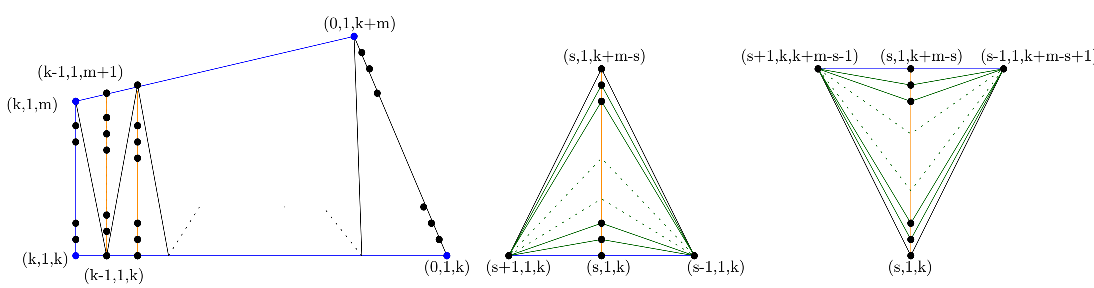


FIGURE 8. Decomposition of the cone C_4 and of its two types of subcones

Each new subcone will be regular as we have: for $0 \leq s \leq k-1$ and for $k \leq \beta \leq k+m-s$,

$$\begin{vmatrix} s-1 & s & s \\ 1 & 1 & 1 \\ k & \beta & \beta+1 \end{vmatrix} = 1 \quad and \quad \begin{vmatrix} s+1 & s & s \\ 1 & 1 & 1 \\ k & \beta & \beta+1 \end{vmatrix} = 1$$

or

$$\begin{vmatrix} s+1 & s & s \\ 1 & 1 & 1 \\ k+m-s-1 & \beta & \beta+1 \end{vmatrix} = 1 \quad and \quad \begin{vmatrix} s-1 & s & s \\ 1 & 1 & 1 \\ k+m-s+1 & \beta & \beta+1 \end{vmatrix} = 1$$

Decomposition of C_5: The cone C_5 contains the vectors $(s, 1, k+m-s)$ for $0 \leq s \leq k$ which are on the skeleton. For $0 \leq \alpha \leq k$, we have:

$$\begin{vmatrix} 0 & \alpha & \alpha+1 \\ 0 & 1 & 1 \\ 1 & k+m-\alpha & k+m-1-\alpha \end{vmatrix} = 1, \begin{vmatrix} k & 1 & 0 \\ 1 & 0 & 0 \\ m & 0 & 1 \end{vmatrix} = 1.$$

□

5.2. **Jet Schemes and toric Embedded resolution of** $A_{k-1,l-1,m-1}$ **when** $k \geq l \geq m$**.** The graph representing the irreducible components of the jet schemes of $A_{k-1,l-1,m-1}$ projecting on the singular locus $\{y = z = 0\}$ is given by Figure 9 below.

Theorem 5.2. *Let $X \subset \mathbb{C}^3$ be a surface of type $A_{k-1,l-1,m-1}$ with $k \geq l \geq m$. The monomial valuations associated with the vectors:*

- $(0, 1, 1), (0, 1, 2), \ldots (0, 1, k+l)$
- $(s, 1, s), \ldots, (s, 1, l+k-s)$ $1 \leq s \leq m-1$
- $(m, 1, m), \ldots, (m, 1, k+l-m)$
- $(m+r, 1, m+r), \ldots, (m+r, 1, k-r)$ *with* $1 \leq r \leq E(\frac{k-m}{2})$

belong to $EV(X)$. Moreover, these vectors give a toric birational map $\mu_\Sigma : Z_\Sigma \longrightarrow \mathbb{C}^3$ which is an embedded resolution of $X \subset \mathbb{C}^3$ (in the neighborhood of the origin) such that the components of the exceptional divisor of μ_Σ correspond to the monomial valuations defined by the vectors; hence they correspond to irreducible components of the m-th jet schemes of X (centered at the singular locus and the intersection of X with the coordinate hyperplanes).

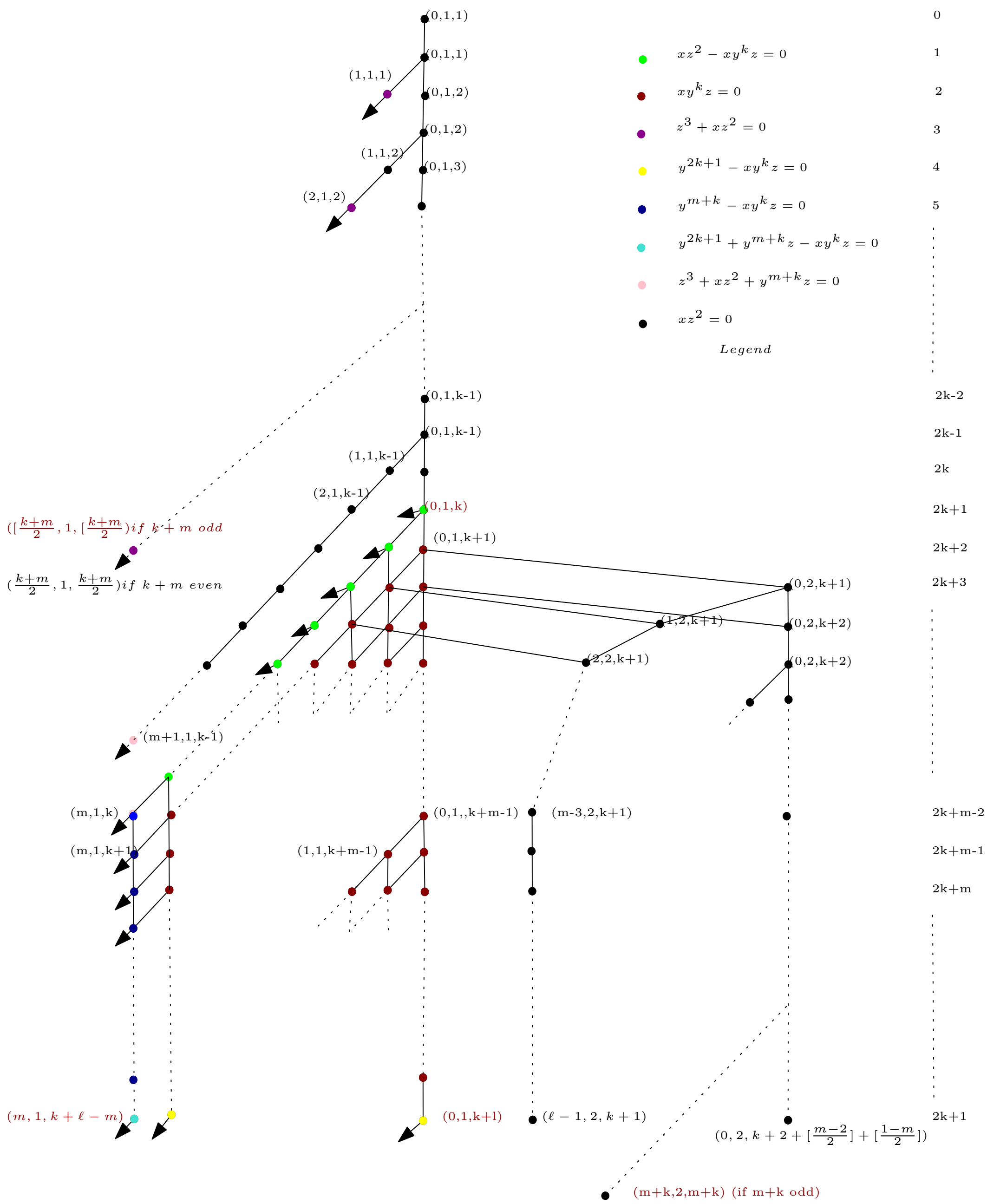


FIGURE 9. Jets schemes of $A_{k-1,l-1,m-1}$

Proof. As above, we first study the positions of the vectors given in theorem 5.2:

- $(m+r,1,k-r) \in [(m+k,2,m+k),(m,1,k)]$:

$$\begin{vmatrix} m+k & m+k & m \\ 1 & 2 & 1 \\ k+r & m+k & k \end{vmatrix} = \begin{vmatrix} r & k & m \\ 0 & 1 & 1 \\ -r & m & k \end{vmatrix} = \begin{vmatrix} r & k & m \\ 0 & 1 & 1 \\ 0 & m+k & k+m \end{vmatrix} = 0$$

- $(\alpha,1,k+l-\alpha) \in [(m,1,k+l-m),(0,1,k+l)]$ for $0 \leq \alpha \leq m$:

$$\begin{vmatrix} m & 0 & \alpha \\ 1 & 1 & 1 \\ k+l-m & k+l & k+l-\alpha \end{vmatrix} = \begin{vmatrix} m & 0 & \alpha \\ 0 & 1 & 0 \\ -m & k+l & -\alpha \end{vmatrix} = 0$$

If $\frac{m+k}{2} \in \mathbb{Z}$, then the dual fan can be decomposed in the same way as for the case $A_{k-1,k-1,m-1}$. Otherwise, we have to show the subcones containing the vector $(m+k,2,m+k)$ are regular. In this case $E(\frac{k-m}{2}) = \frac{k-m-1}{2}$ and $(m+E(\frac{k-m}{2}),1,m+E(\frac{k-m}{2})) = (\frac{k+m-1}{2},1,\frac{k+m-1}{2})$. We have :

$$\begin{vmatrix} \frac{k+m-1}{2} & \frac{k+m-1}{2} & m+k \\ 1 & 1 & 2 \\ \frac{k+m+1}{2} & \frac{k+m-1}{2} & m+k \end{vmatrix} = \begin{vmatrix} 0 & \frac{k+m-1}{2} & m+k \\ 0 & 1 & 2 \\ 1 & \frac{k+m-1}{2} & m+k \end{vmatrix} = 1 \ \textit{and} \ \begin{vmatrix} 1 & \frac{k+m-1}{2} & m+k \\ 0 & 1 & 2 \\ 0 & \frac{k+m-1}{2} & m+k \end{vmatrix} = 1.$$

□

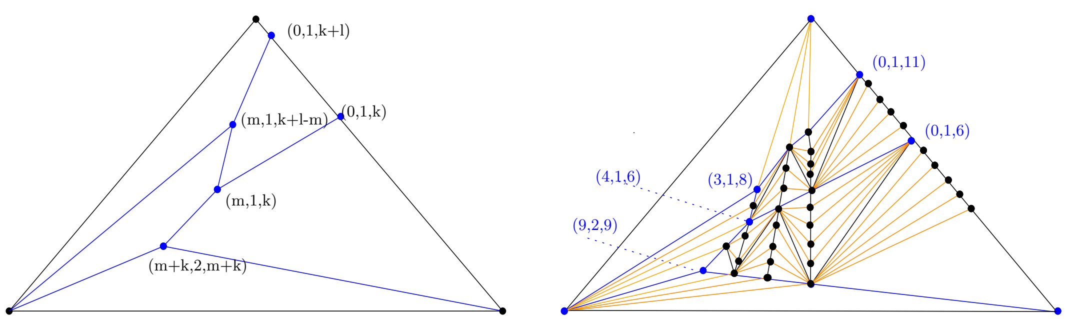


FIGURE 10. Dual Newton fan of $A_{k-1,l-1,m-1}$ with $k \geq l \geq m$, and a resolution of $A_{5,4,2}$

6. Jet Schemes and Toric Embedded Resolution of $B_{k-1,m}$

The singularity of $X \subset \mathbb{C}^3$ defined by the equations:

- $m = 2\ell$,
$$z^3 + xz^2 - (y^{k+1} + y^{\ell})y^k z - xy^{2k+1} = 0,$$
- $m = 2\ell - 1$,
$$z^3 + (x - y^{\ell-1})z^2 - y^{2k+1}z - xy^{2k+1} = 0.$$

is called $B_{k-1,m}$-type singularity with $k \geq 2$ and $m \geq 3$.
In the case where $m = 2l$, the jet schemes and the toric embedded resolution behaves as in the case of $A_{m,k,l}$; so, let's just present the jet graph presenting the irreducible components of the jets schemes projecting over the singular locus $\{y = z = 0\}$ and the axis $\{x = z = 0\}$ included in X:

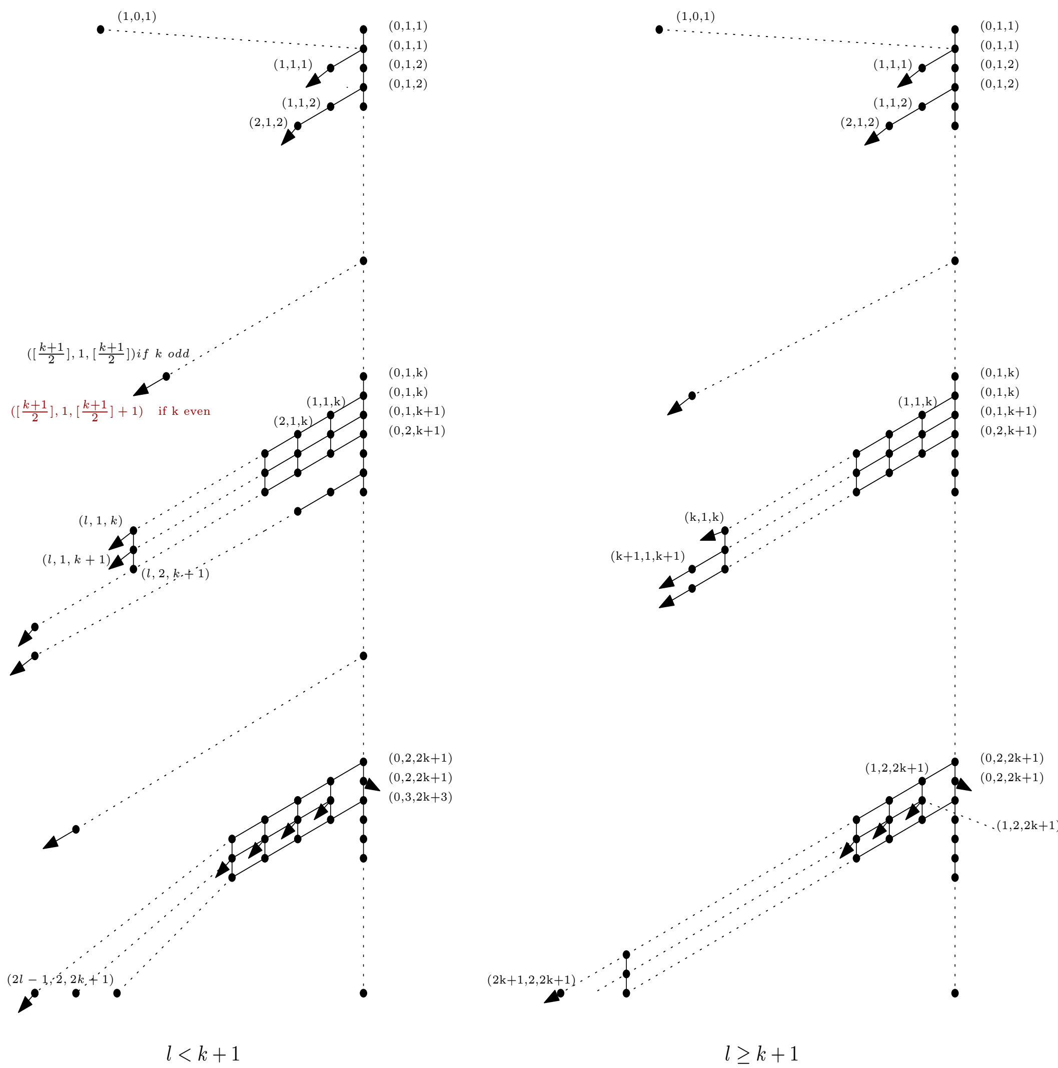


FIGURE 11. Jets schemes of $B_{k-1,m}$ when $m = 2l$

Theorem 6.1. *Let $X \subset \mathbb{C}^3$ be a surface of type $B_{k-1,2l}$. The monomial valuations associated with the vectors:*

- $(0,1,1),(0,1,2),\dots,(0,1,k+1)$
- $(1,1,1),\dots,(1,1,k+1)$
- $\dots$
- $(l,1,l),\dots,(l,1,k+1)$
- $(l+1,1,l+1),\dots,(l+1,1,k-1)$
- $(l+2,1,l+2),\dots,(l+1,1,k-2)$
- $\dots$

- $(E((l+k)/2), 1, E((l+k)/2))$, *and* $(E((l+k)/2), 1, E((l+k)/2)+1)$ *if* $k+l$ *is odd.*
- $(0, 2, 2k+1) \dots (2l-l, 2, 2k+1)$

belong to $EV(X)$*. Moreover, these vectors give a toric birational map* $\mu_\Sigma : Z_\Sigma \longrightarrow \mathbb{C}^3$ *which is an embedded resolution of* $X \subset \mathbb{C}^3$ *(in the neighborhood of the origin) such that the components of the exceptional divisor of* μ_Σ *correspond to the monomial valuations defined by them; hence they correspond to the irreducible components of the* m*-th jet schemes of* X *(centered at the singular locus and the intersection of* X *with the coordinate hyperplanes).*

The vectors given in the theorem allows us to decompose the corresponding dual Newton fan into regular subcones and find an embedded resolution of the singularity.

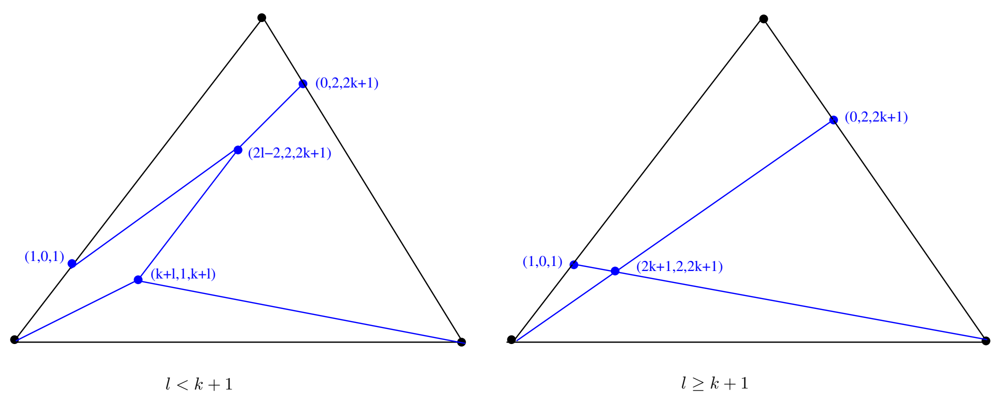


FIGURE 12. Dual Newton fans of $B_{k-1,m}$ when $m = 2l$

Two embedded resolutions for two special cases look as the following:

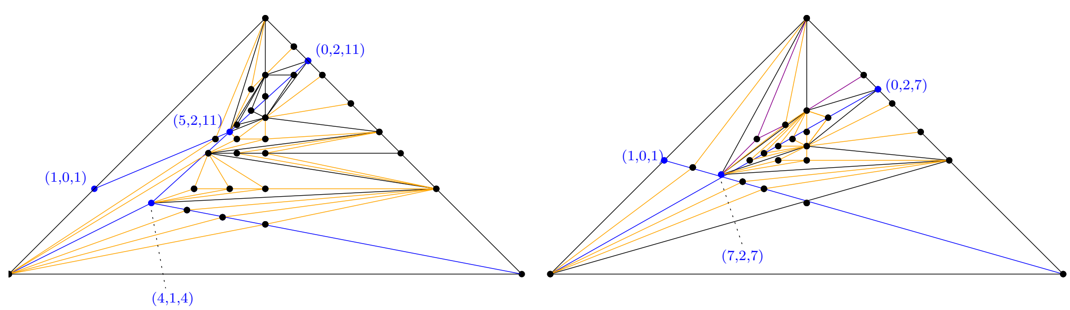


FIGURE 13. Embedded resolution of $B_{4,6}$ and of $B_{2,10}$

In the case of $B_{k-1,2l-1}$, there is an amazing subclass (see Section 2 below) for which the jet schemes give a resolution which is not a subdivision of the dual Newton fan of the singularity. So this case needs to be treated in details. There are two sub-cases to be considered which are the cases $k+1 \leq l$ and $k \geq l$.

Let us first treat the case $k+1 \leq l$: we start by computing the irreducible components of the jet schemes projecting on the singular locus $\{y = z = 0\}$ and the axis $\{x = z = 0\}$ included in X. And, by computing the associated vectors we obtain Figure 14:

Theorem 6.2. *Let X be of type $B_{k-1,2l-1}$ with $k+1 \leq l$. The monomial valuations associated with the vectors:*

- $(0,1,1),(0,1,2),\dots,(0,1,k+1)$
- $(1,1,1),\dots,(1,1,k+1)$
- $\dots$
- $(k,1,k),(k,1,k+1)$
- $(k+1,1,k+1)$
- $(0,2,2k+1)\dots(2k+1,2,2k+1)$

belong to $EV(X)$. Moreover there exists a toric birational map $\mu_\Sigma : Z_\Sigma \longrightarrow \mathbb{C}^3$ which is an embedded resolution of $X \subset \mathbb{C}^3$ such that the irreducible components of the exceptional divisor of μ_Σ correspond to the irreducible components of the m-th jet schemes of X (centered at the singular locus and the intersection of X with the coordinate hyperplane). This yields a construction of μ_Σ (not canonical).

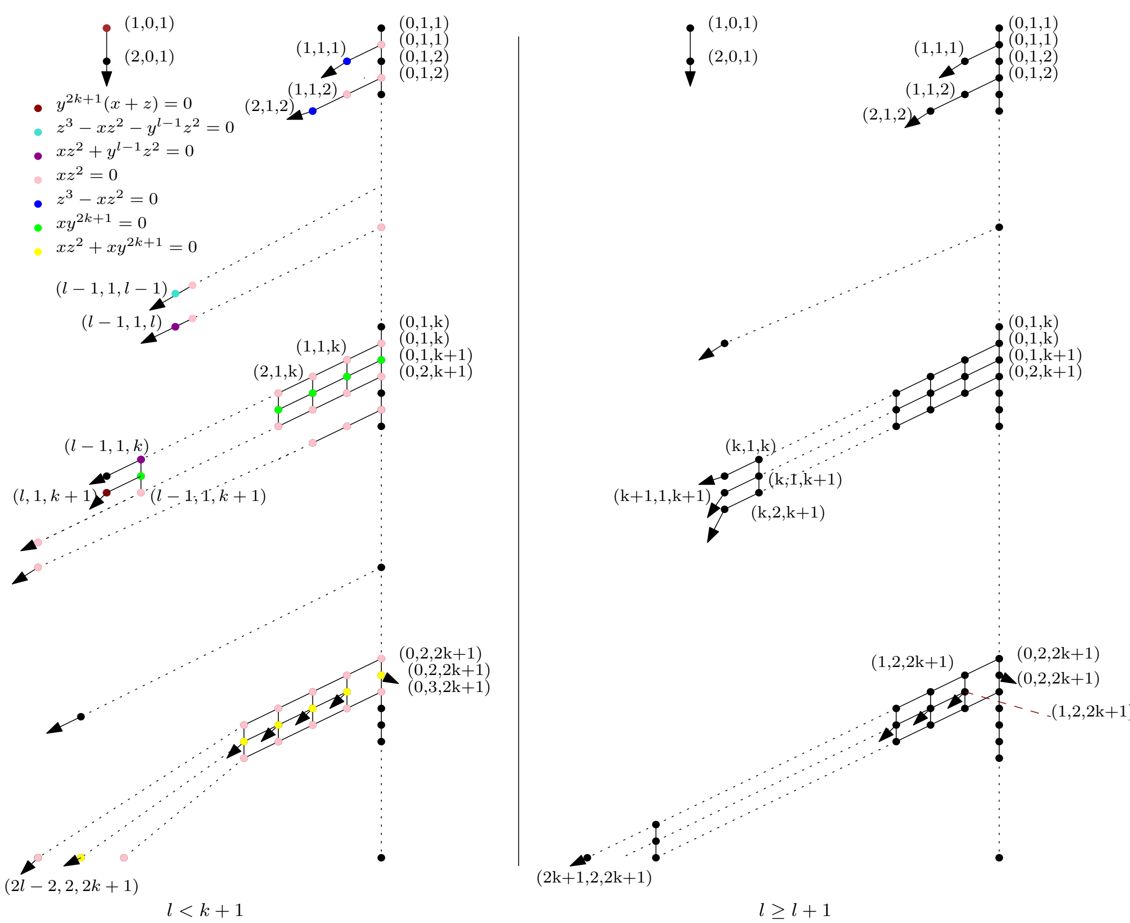


FIGURE 14. Jets schemes of $B_{k-1,2l-1}$

The computations are similar to the case $B_{k-1,2l}$; the associated vectors with the jet schemes give a subdivision of the dual Newton fan, thus an embedded resolution, of the singularity.

Theorem 6.3. *Let X be of type $B_{k-1,2l-1}$ for $l \leq k$. The monomial valuations associated with the vectors*

- $(0,1,1),(0,1,2),\dots,(0,1,k+1)$
- $(1,1,1),\dots,(1,1,k+1)$
- $\dots$
- $(l-1,1,l-1),\dots,(l-1,1,k+1)$
- $(l,1,k+1)$
- $(0,2,2k+1)\dots(2l-2,2,2k+1)$

belong to $EV(X)$. Moreover there exists a toric birational map $\mu_\Sigma : Z_\Sigma \longrightarrow \mathbb{C}^3$ which is an embedded resolution of $X \subset \mathbb{C}^3$ such that the irreducible components of the exceptional divisor of μ_Σ correspond to the irreducible components of the m-th jet schemes of X (centered at the singular locus and the intersection of X with the coordinate hyperplane). This yields a construction of μ_Σ (not canonical).

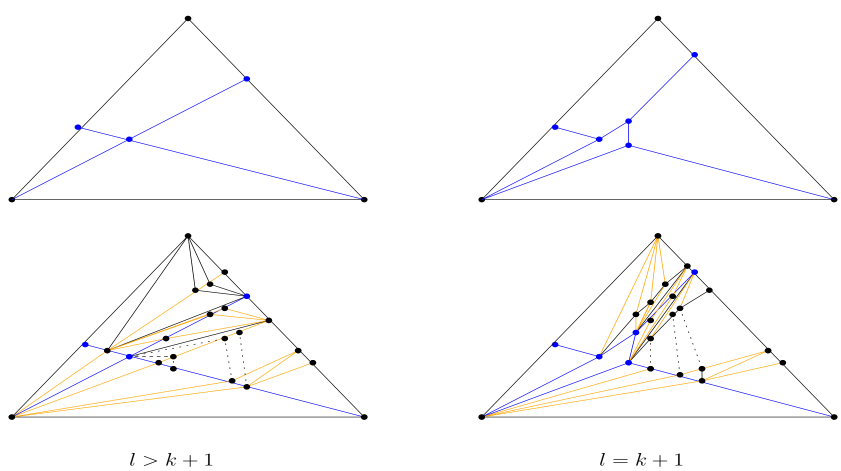


FIGURE 15. Dual Newton fan of $B_{k-1,2l-1}$ for $l > k+1$ (*resp.* $l = k+1$) and an embedded resolution

In the case where X is of type $B_{k-1,2l-1}$ with $l \leq k$, the corresponding dual Newton fan of the singularity is given with the right-hand figure of Figure 16.

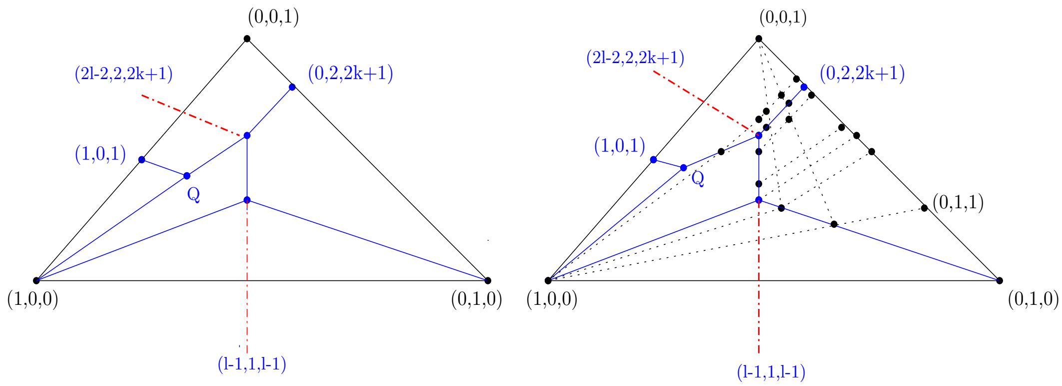


FIGURE 16. Dual Newton fan of $B_{k-1,2l-1}$ with $l \leq k$ and it is with the vectors of Theorem 6.3

Remark 6.4. The set of vectors above does not contain the vector $Q = (2k - l + 2, 1, 2k - l + 2)$, thus the decomposition obtained by these vectors will not be a regular decomposition of the dual Newton fan of the singularity.

Proof. Consider the polygons

$$J = [(1,0,0),(0,1,0),(0,0,1),(1,0,1),(l,1,k+1),(2l-2,2,2k+1),(l-1,1,l-1)]$$

and

$$K = [(1,0,0),(1,0,1),(l,1,k+1),(2l-2,2,2k+1),(l-1,1,l-1)]$$

in the dual Newton fan of the singularity. In J, the vectors obtained from the jet schemes give a regular subdivision of this polygon (following the computations of $B_{k-1,2l}$). As J is a sub-polygon of the fan, the strict transform of X is regular on these charts. In K, we find a subdivision by adding an edge from $(1,0,0)$ to $(l,1,k+1)$, another edge from $(1,0,0)$ to $(l-1,1,s)$ for $l-1 \leq s \leq k$ and another edge from $(l,1,k+1)$ to $(l-1,1,k)$. In this way, we obtain a regular subdivision of K.

Since K is not compatible with the dual Newton fan, we cannot use Varchenko's theorem to deduce the smoothness of the strict transform of X in the charts corresponding to the subdivision of K by the toric map. So, we should prove this fact:

- For this, let us first consider the cone $[(1,0,0),(l-1,1,s),(l-1,1,s+1)]$ for $l-1 \leq s < k$; the monoidal transformation corresponding to it is:

$$\begin{cases} x = x_1 y_1^{l-1} z_1^{l-1} \\ y = y_1 z_1 \\ z = y_1^s z_1^{s+1} \end{cases}$$

Then the total transform of $B_{k-1,2l-1}$ is defined by:

$$\{y_1^{2s+l-1} z_1^{2s+l+1}(y_1^{s-l+1} z_1^{s-l+2} - x_1 - y_1^{2k-s-l+2} z_1^{2k-s-l+1} - x_1 y_1^{2k-2s+1} z_1^{2k+2s-1}) = 0\}$$

The strict transform is smooth and transversal to the exceptional divisors defined by $y_1 = 0$ and $z_1 = 0$.

- Now let us consider the cone $[(1,0,0),(l-1,1,k),(l,1,k+1)]$; the monoidal transformation corresponding to it is:

$$\begin{cases} x = x_1 y_1^{l-1} z_1^{l} \\ y = y_1 z_1 \\ z = y_1^{k} z_1^{k+1} \end{cases}$$

Then the total transform of $B_{k-1,2l-1}$ is :

$$\{y_1^{2k+l-1} z_1^{2k+l+1}(y_1^{k-l+1} z_1^{k-l} - x_1 z_1 - 1 - y_1^{k-l+2} z_1^{k-l+1} - x_1 y_1) = 0\}.$$

The strict transform is smooth and transversal to the exceptional divisors defined by $y_1 = 0$ and $z_1 = 0$.

- Finally let us consider the cone $[(1,0,0),(1,0,1),(l,1,k+1)]$; the monoidal transformation corresponding to it is:

$$\begin{cases} x = x_1 y_1 z_1^{l} \\ y = z_1 \\ z = y_1 z_1^{k+1} \end{cases}$$

Then the total transform of $B_{k-1,2l-1}$ is :

$$\{y_1 z_1^{2k+l+1}(y_1^2 z_1^{k+2-l} - x_1 y_1^2 z_1 - y_1 - z_1^{k-l+1} - x_1) = 0\}.$$

The strict transform is smooth and transversal to the exceptional divisors defined by $y_1 = 0$ and $z_1 = 0$.

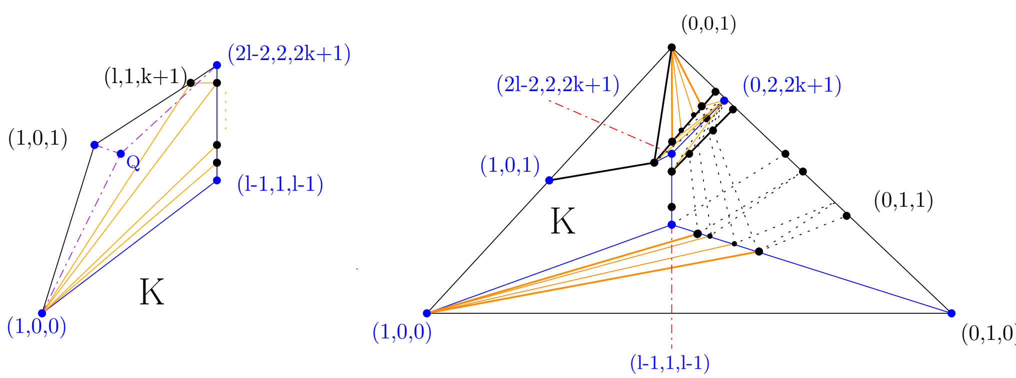


FIGURE 17. The polygon K and an embedded resolution of $B_{k-1,2l-1}$ with $l \leq k$

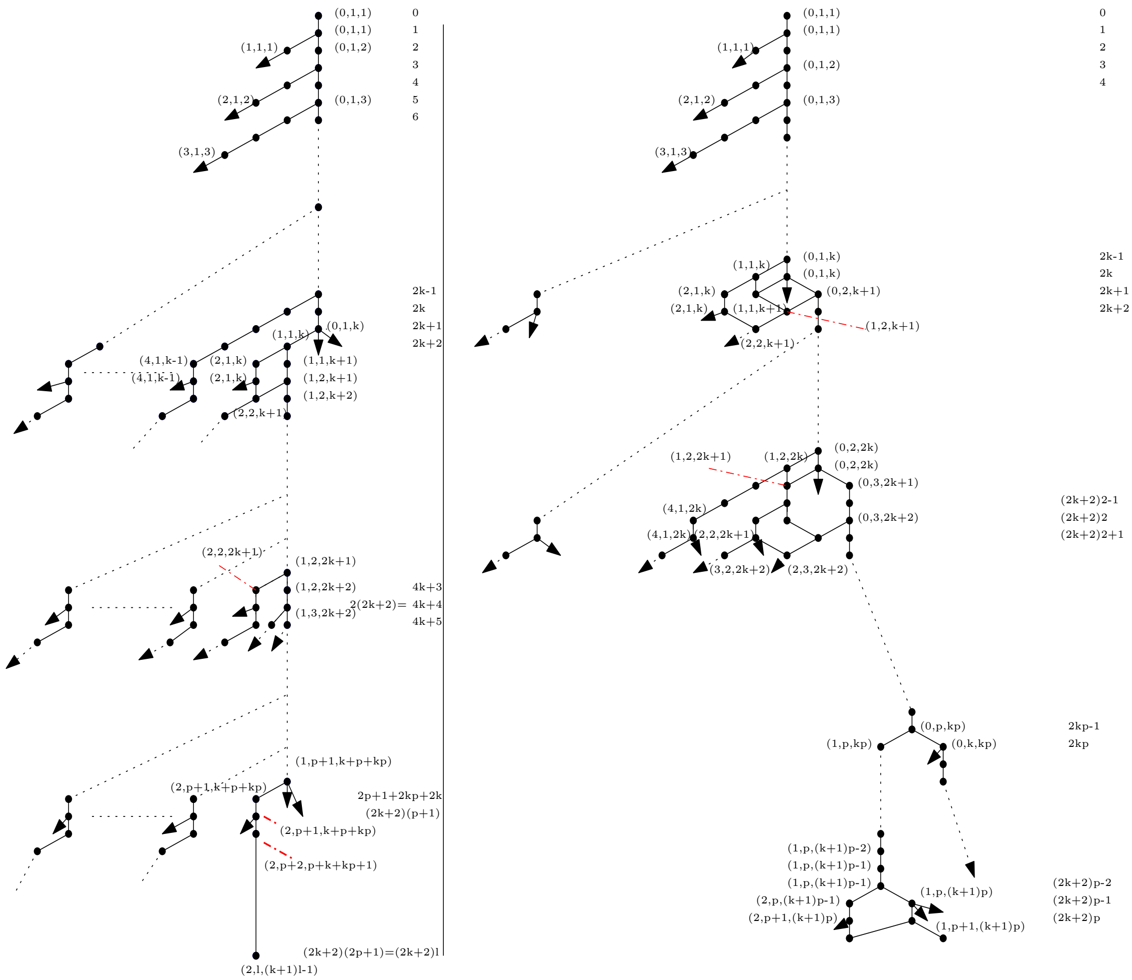
□

7. Jet Schemes and Toric Embedded Resolution of $C_{k-1,l+1}$

The singularity of $X \subset \mathbb{C}^3$ defined by the equation:

$$z^3 + xz^2 - \ell x^{\ell-1} y^{2k} z - (x^{\ell} + y^2) y^{2k} = 0$$

is called $C_{k-1,l+1}$-type singularity where $k \geq 1$ and $\ell \geq 2$. For $k = 3q-1$, we obtain the jet graph given in Figure 18 which represents the irreducible components of the jet schemes of $C_{k-1,l+1}$ projecting on the singular locus $\{y = z = 0\}$.



$C_{k-1,2p+1}$ $C_{k-1,2p+2}$

FIGURE 18. Jet schemes of $C_{k-1,2p+2}$ with $k = 3q-1$

Theorem 7.1. *Let X be a surface singularity of type $C_{k-1,l+1}$. The monomial valuations associated with the vectors:*

- *for* $k = 3q-1$ *and* $l = 2p$
  - $(0,1,1),(0,1,2)...(0,1,k)$
  - $(1,1,1),...,(1,1,k),(1,1,k+1)$
  - $(2,1,2),...,(2,1,k)$
  - $(3,1,3),...,(3,1,k-1)$
  - $(4,1,4),...,(4,1,k-1)$
  - ...

   - $(2q-1, 1, 2q-1), (2q-1, 1, 2q)$
   - $(2q, 1, 2q)$
   - $(1, 1, k), (1, 2, 2(k+1)-1), \ldots, (1, p, (k+1)p-1)$
   - $(2, 1, k), (2, 2, 2(k+1)-1), (2, 3, 3(k+1)-1), \ldots, (2, l, (k+1)l-1)$
   - $(1, 1, k+1), (1, 2, 2(k+1)), \ldots, (1, p, (k+1)p)$
- *for* $k = 3q-1$ *and* $l = 2p+1$
   - $(0, 1, 1), (0, 1, 2)...(0, 1, k)$
   - $(1, 1, 1), ..., (1, 1, k), (1, 1, k+1)$
   - $(2, 1, 2), ..., (2, 1, k)$
   - $(3, 1, 3), ..., (3, 1, k-1)$
   - $(4, 1, 4), ..., (4, 1, k-1)$
   - $\ldots$
   - $(2q-1, 1, 2q-1), (2q-1, 1, 2q)$
   - $(2q, 1, 2q)$
   - $(1, 1, k), (1, 2, 2(k+1)-1), \ldots, (1, p, (k+1)p-1), (1, p, (k+1)(p+1)-1)$
   - $(2, 1, k), (2, 2, 2(k+1)-1), (2, 3, 3(k+1)-1), \ldots, (2, l, (k+1)l-1)$
   - $(1, 1, k+1), (1, 2, 2(k+1)), \ldots, (1, p, (k+1)p)$

belong to $EV(X)$*. Moreover there exists a toric birational map* $\mu_\Sigma : Z_\Sigma \longrightarrow \mathbb{C}^3$ *which is an embedded resolution of* $X \subset \mathbb{C}^3$ *such that the irreducible components of the exceptional divisor of* μ_Σ *correspond to the irreducible components of the* m*-th jet schemes of* X *(centered at the singular locus and the intersection of* X *with the coordinate hyperplane). This yields a construction (not canonical) of* μ_Σ.

The embedded resolutions are represented on the figure below.

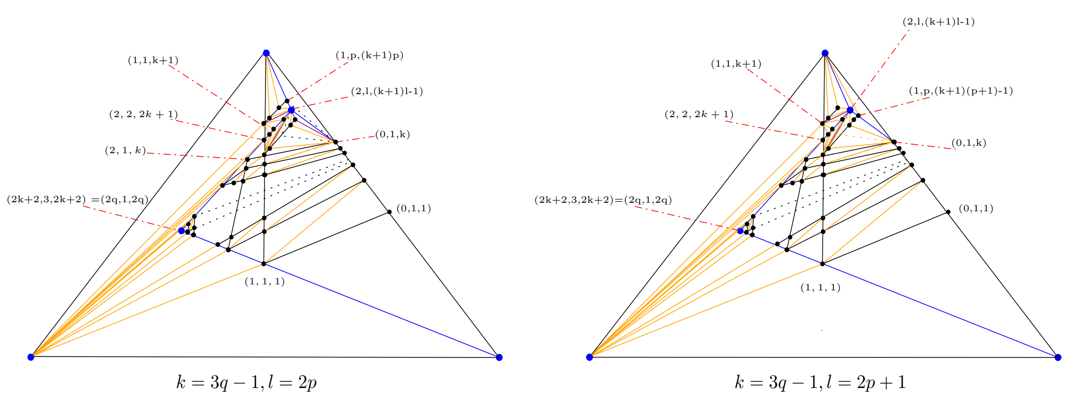


FIGURE 19. An embedded resolution of $C_{k-1,l+1}$ when $k = 3q-1$ and $l = 2p$ or $l = 2p+1$

8. Jet Schemes and Toric Embedded Resolution of D_{k-1}

The singularity of $X \subset \mathbb{C}^3$ defined by the equation:

$$z^3 + (x + y^{2k})z^2 + (2xy^k - y^2)y^k z + x^2 y^{2k} = 0$$

is called D_{k-1}-type singularity with $k \geq 1$. The jet graph is given in Figure 20 where the irreducible components of the jet schemes of D_{k-1} projecting on the singular locus $\{y = z = 0\}$ and the axis $\{x = z = 0\}$ included in X:

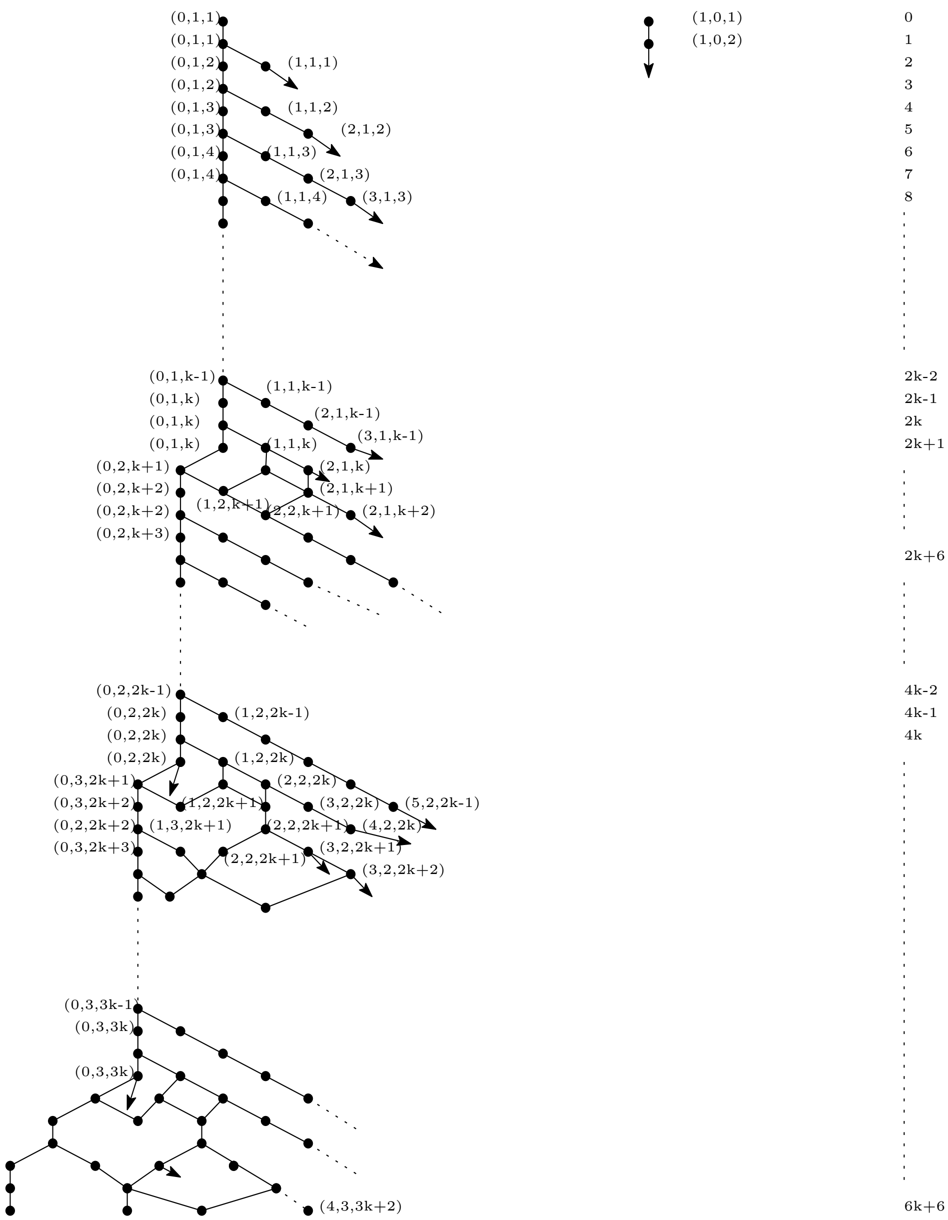


Figure 20. Jet schemes of D_{k-1}

Theorem 8.1. *Let X be a surface singularity of type D_{k-1}. The monomial valuations associated with the following vectors belong to $EV(X)$. Moreover there exists a toric birational map $\mu_\Sigma : Z_\Sigma \longrightarrow \mathbb{C}^3$ which is an embedded resolution of $X \subset \mathbb{C}^3$ such that the irreducible components of the exceptional divisor of μ_Σ correspond to the irreducible components of the m-th jet schemes of X (centered at the singular locus and the intersection of X with the coordinate hyperplane). This yields a construction (not canonical) of μ_Σ.*

- $(1,0,1),(1,0,2)$
- $(0,1,1),(0,1,2)...(0,1,k)$
- $(1,1,1),...,(1,1,k),(2,2,2k+1)),(1,1,k+1)$
- $(2,1,2),...,(2,1,k+2)$
- $(3,1,3),...,(3,1,k-1)$
- $\ldots$
- $(m,1,m),(m,1,m+1),(m,1,m+2)$
- $(m+1,1,m+1)$
- $(3,2,2k+1),(3,2,2k+2)$

When k is odd, we should add two more vectors: $(m+1,1,m+2),(k+2,2,k+2)$, where $m = E(\frac{k}{2})$.

These vectors placed in the dual Newton fan give the regular subdivision:

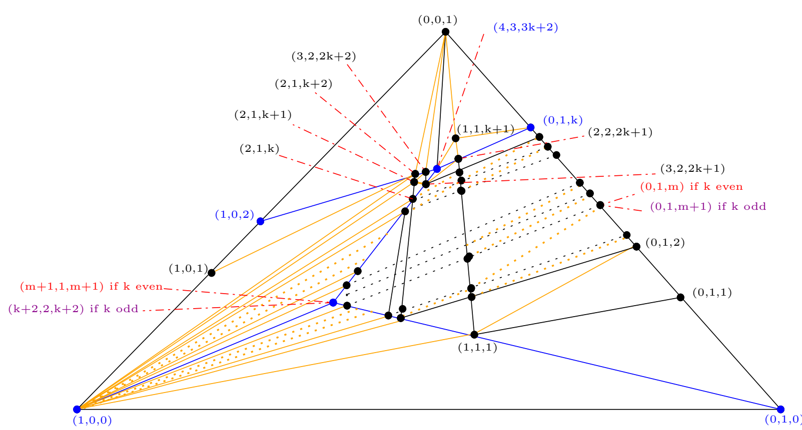


FIGURE 21. Embedded resolutions of D_{k-1} for $k = 2m$ and $k = 2m + 1$

9. Jet Schemes and Toric Embedded Resolution of $E_{7,0}$

The singularity of $X \subset \mathbb{C}^3$ defined by the equation $z^3 + x^2yz + y^4 = 0$ is called an $E_{7,0}$-type singularity. The singular locus is $\{y = z = 0\}$.

Theorem 9.1. *Let X be a surface singularity of type $E_{7,0}$. The monomial valuations associated with the vectors: $\{(0,1,1),(0,2,1),(0,1,2),(0,1,3)$, $(1,1,1)$, $(1,1,2)$, $(1,2,2)$, $(1,2,3)$, $(1,2,4)$, $(2,2,3)$, $(2,3,4)$, $(2,3,5)$, $(3,3,4)$, $(3,4,5)$, $(3,4,6)$, $(4,5,7)$, $(5,6,8)\}$ belong to $EV(X)$. There exists a toric birational map $\mu_\Sigma : Z_\Sigma \longrightarrow \mathbb{C}^3$ which is an embedded resolution of $X \subset \mathbb{C}^3$ such that the irreducible components of the exceptional divisor of μ_Σ correspond to the irreducible components of the m-th jet schemes of X (centered at the singular locus and the intersection of X with the coordinate hyperplane). Moreover this yields a construction (not canonical) of μ_Σ.*

Following almost the same process as in the case of $E_{6,0}$, we continue until $m = 22$ to obtain the following jet graph:

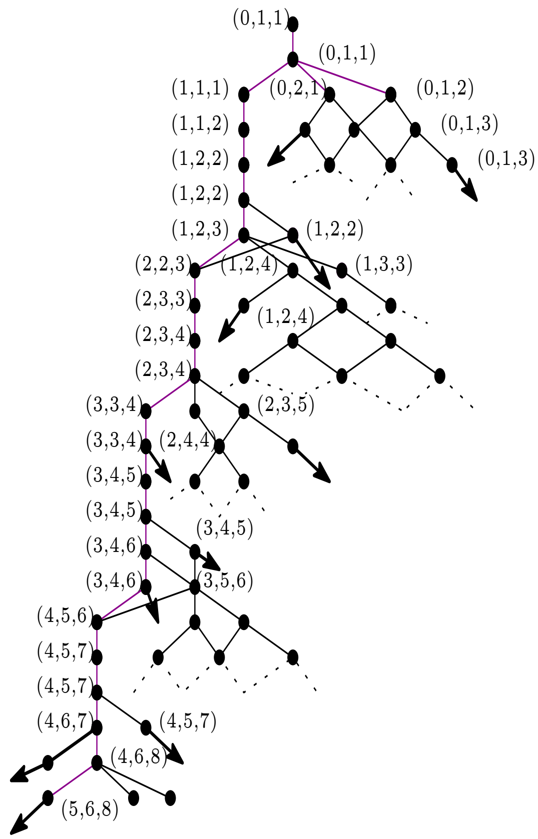


FIGURE 22. Jet schemes of $E_{7,0}$

The vectors corresponding to the irreducible jet schemes give the following subdivision, which is an embedded resolution of X:

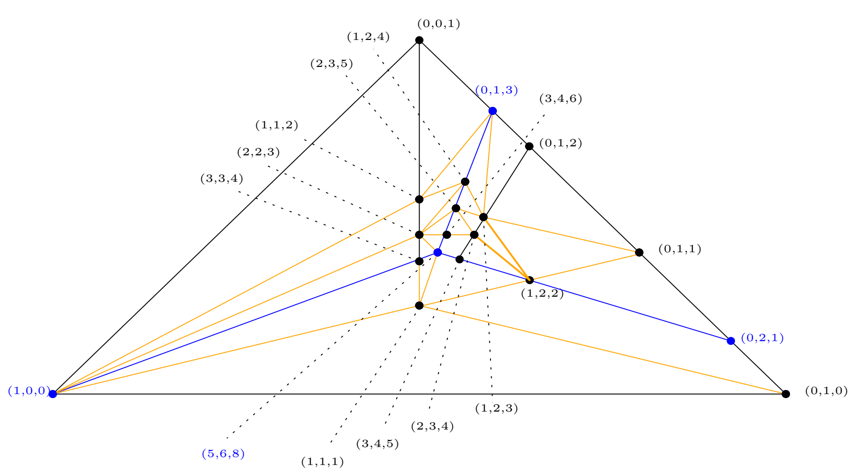


FIGURE 23. An embedded resolution of $E_{7,0}$

10. Jet Schemes and Toric Embedded Resolution of $E_{0,7}$

The singularity of $X \subset \mathbb{C}^3$ defined by the equation:

$$z^3 + y^5 + x^2y^2 = 0$$

is called $E_{0,7}$-type singularity. The singular locus is $\{y = z = 0\}$. The jet graph representing the irreducible jet schemes is obtained as:

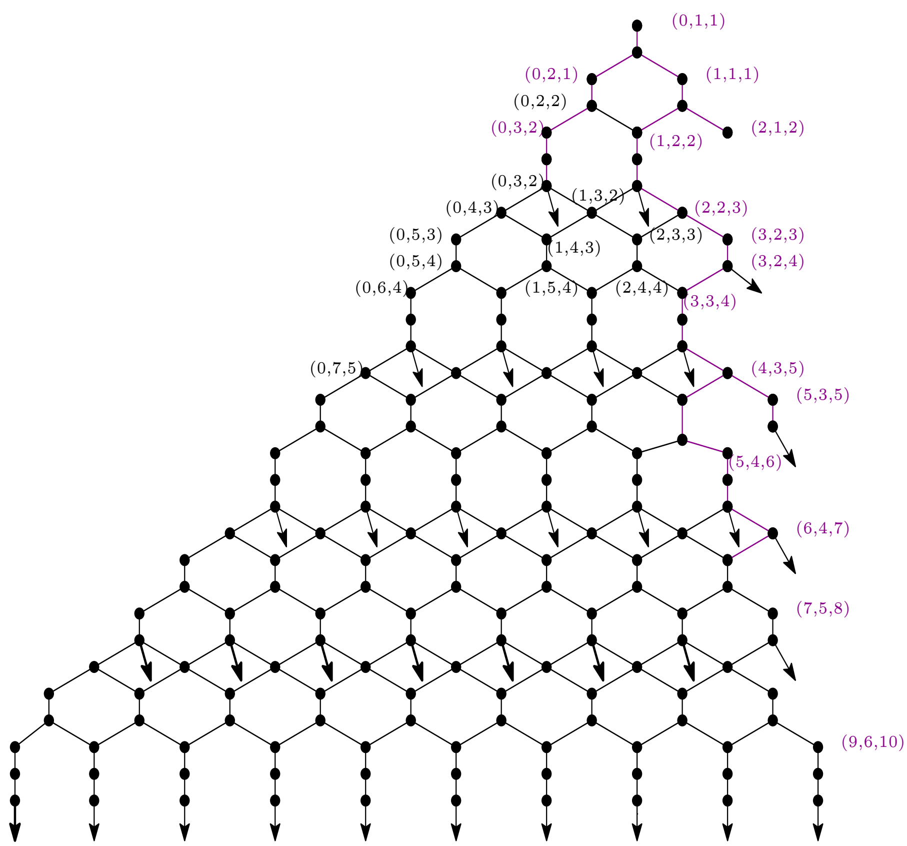


Figure 24. Jet schemes of $E_{0,7}$

Theorem 10.1. *Let X be a surface of type $E_{0,7}$. The monomial valuations associated with the vectors* $\{(0,1,1),(0,2,1),(1,1,1),\ (0,3,2),\ (1,1,2),\ (1,2,2),\ (2,1,2),\ (2,2,3),\ (3,2,3),\ (3,2,4),\ (3,3,4),\ (4,3,5),\ (5,3,5))\ (5,4,6),(6,4,7),(7,5,8),(9,6,10)\}$ *belong to $EV(X)$. There exists a toric birational map $\mu_\Sigma : Z_\Sigma \longrightarrow \mathbb{C}^3$ which is an embedded resolution of $X \subset \mathbb{C}^3$ such that the irreducible components of the exceptional divisor of μ_Σ correspond to the irreducible components of the m-th jet schemes of X (centered at the singular locus and the intersection of X with the coordinate hyperplane). Moreover this yields a construction (not canonical) of μ_Σ.*

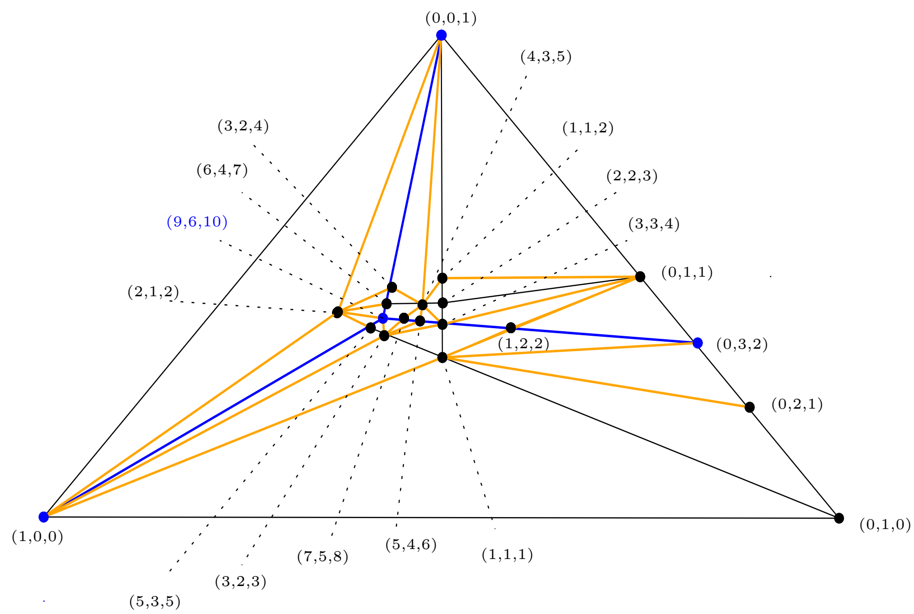


FIGURE 25. An embedded resolution of $E_{0,7}$

11. Jet Schemes and Toric Embedded Resolution of F_{k-1}

The singularity of $X \subset \mathbb{C}^3$ defined by the equation:

$$z^3 + (x + y^{2k})z^2 + 2xy^{2k}z + (x^2 + y^3)y^{2k} = 0$$

is called F_{k-1}-type singularity. The singular locus is $\{y = z = 0\}$.

Theorem 11.1. *Let X be a surface singularity of type F_{k-1}. The monomial valuations associated with the vectors:*

- $(0, 1, 1), \dots, (0, 1, k)$
- $(1, 1, 1), \dots, (1, 1, k + 1)$
- $(2, 1, 2), \dots, (2, 1, k + 1)$
- $(3, 1, 3), \dots, (3, 1, k)$
- $\dots$
- $(a, 1, b)$
- $(2, 2, 2k + 1), (3, 2, 2k + 2), (4, 2, 2k + 1), (6, 2, 2k) \dots (c, 2, d)$
- $(4, 3, 3k + 2), (5, 3, 3k + 2), (7, 3, 3k + 1), (9, 3, 3k) \dots (2k + 3, 3, 2k + 3)$
- $(3k + 2, 3, 3k + 2)$ *if* $k = 3m + 1$

with

- $(a, 1, b) = (\frac{2k+3}{3}, 1, \frac{2k+3}{3})$ *and* $(c, 2, d) = (\frac{4k+6}{3}, 2, \frac{4k+6}{3})$ *if* $k = 3m$ *for* $m \in \mathbb{N}$*;*
- $(a, 1, b) = (\frac{2k+1}{3}, 1, \frac{2k+4}{3})$ *and* $(c, 2, d) = (\frac{4k+2}{3}, 2, \frac{4k+8}{3})$ *if* $k = 3m + 1$ *for* $m \in \mathbb{N}$
- $(a, 1, b) = (\frac{2k-1}{3}, 1, \frac{2k+5}{3})$ *and* $(c, 2, d) = (\frac{4k+4}{3}, 2, \frac{4k+7}{3})$ *if* $k = 3m + 2$ *for* $m \in \mathbb{N}*$

belong to $EV(X)$. Moreover there exists a birational map $\mu_{\Sigma} : Z_{\Sigma} \longrightarrow \mathbb{C}^3$ which is an embedded resolution of $X \subset \mathbb{C}^3$ such that the irreducible components of the exceptional divisor of μ_{Σ} correspond to the irreducible components of the m-th jet schemes of X (centered at the singular locus and the intersection of X with the coordinate hyperplane). Moreover this yields a construction (not canonical) of μ_{Σ}.

The jet graph representing the irreducible components of the jet schemes projecting on the singular locus is given by:

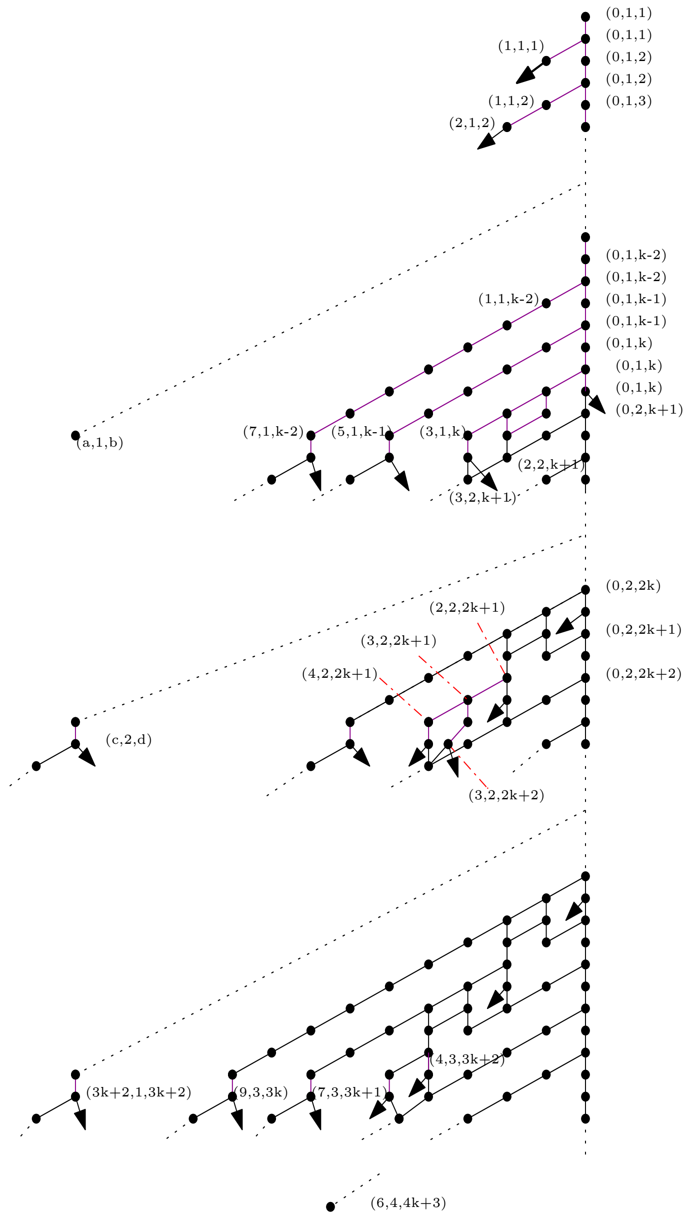


FIGURE 26. Jet schemes of F_{k-1}

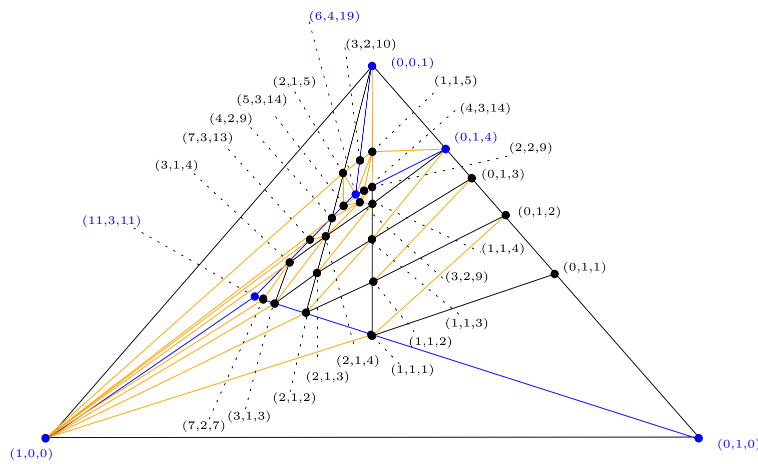


Figure 27. An embedded resolution of F_3

12. Jet Schemes and Toric Embedded Resolution of H_n

The singularity of $X \subset \mathbb{C}^3$ defined by the equation:

- $z^3 + x^2y(x + y^{k-1}) = 0$ where $n = 3k - 1$
- $z^3 + xy^kz + x^3y = 0$ where $n = 3k$
- $z^3 + xy^{k+1}z + x^3y^2 = 0$ where $n = 3k + 1$

is called H_n-type singularity.

Theorem 12.1. *Let X be a surface of type H_n. The monomial valuations associated with the vectors:*

1. $n = 3k - 1$
   - $(2, 0, 1), (3, 0, 2)$
   - $(0, 1, 1), (1, 1, 2), \dots (k - 1, 1, k)$
   - $(0, 2, 1), (1, 2, 2), \dots (2k - 2, 2, 2k - 1)$
   - $(0, 3, 1), (1, 3, 2), \dots (3k - 3, 3, 3k - 2)$
   - $(1, 0, 1), (1, 1, 1), (2, 1, 2), \dots (k, 1, k)$
2. $n = 3k$
   - $(2, 0, 1))$
   - $(0, 1, 1), (1, 1, 2), \dots (k, 1, k + 1)$
   - $(0, 2, 1), (1, 2, 2), \dots (2k - 1, 2, 2k)$
   - $(0, 3, 1), (1, 3, 2), \dots (3k - 2, 3, 3k - 1)$
   - $(1, 0, 1), (1, 1, 1), (2, 1, 2), \dots (k, 1, k)$
3. $n = 3k - 1$
   - $(0, 1, 1), (1, 1, 2), \dots (k, 1, k + 1)$
   - $(0, 2, 1), (1, 2, 2), \dots (2k, 2, 2k + 1)$

- $(0,3,2),(1,3,3),\dots(3k-1,3,3k+1)$
- $(1,0,1),(1,0,2),(2,0,1)$
- $(1,1,1),(2,1,2),\dots(k,1,k)$

belong to $EV(X)$. Moreover there exists a toric birational map $\mu_\Sigma : Z_\Sigma \longrightarrow \mathbb{C}^3$ which is an embedded resolution of $X \subset \mathbb{C}^3$ such that the irreducible components of the exceptional divisor of μ_Σ correspond to the irreducible components of the m-th jet schemes of X (centered at the singular locus and the intersection of X with the coordinate hyperplane). This yields a construction (not canonical) of μ_Σ.

The tree representing the irreducible components of the jets schemes projecting on the singular locus $\{x = z = 0\}$ and the axis $\{y = z = 0\}$ included in X is the following:

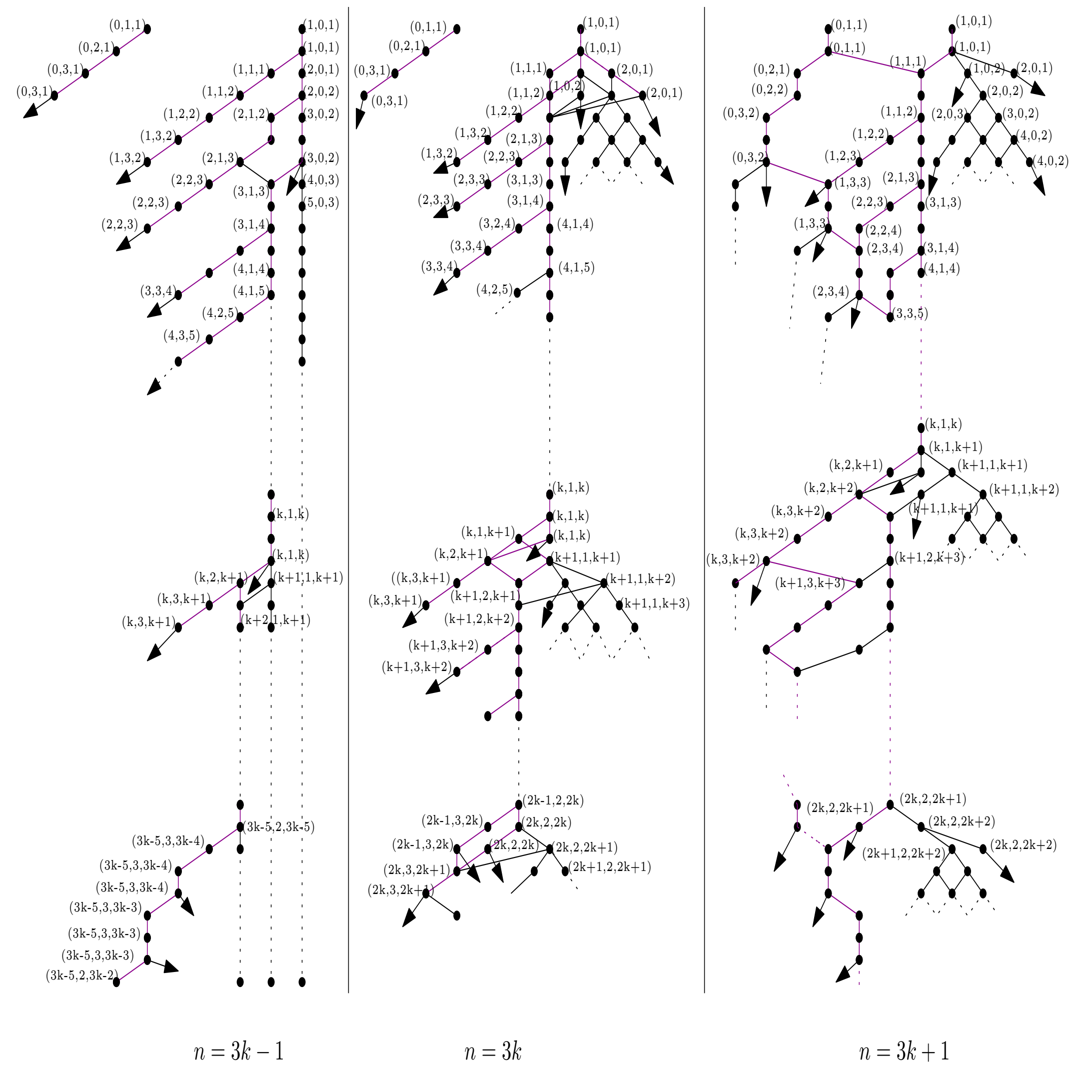


FIGURE 28. Jets schemes of H_n

An embedded resolution for each case is represented on the figure below:

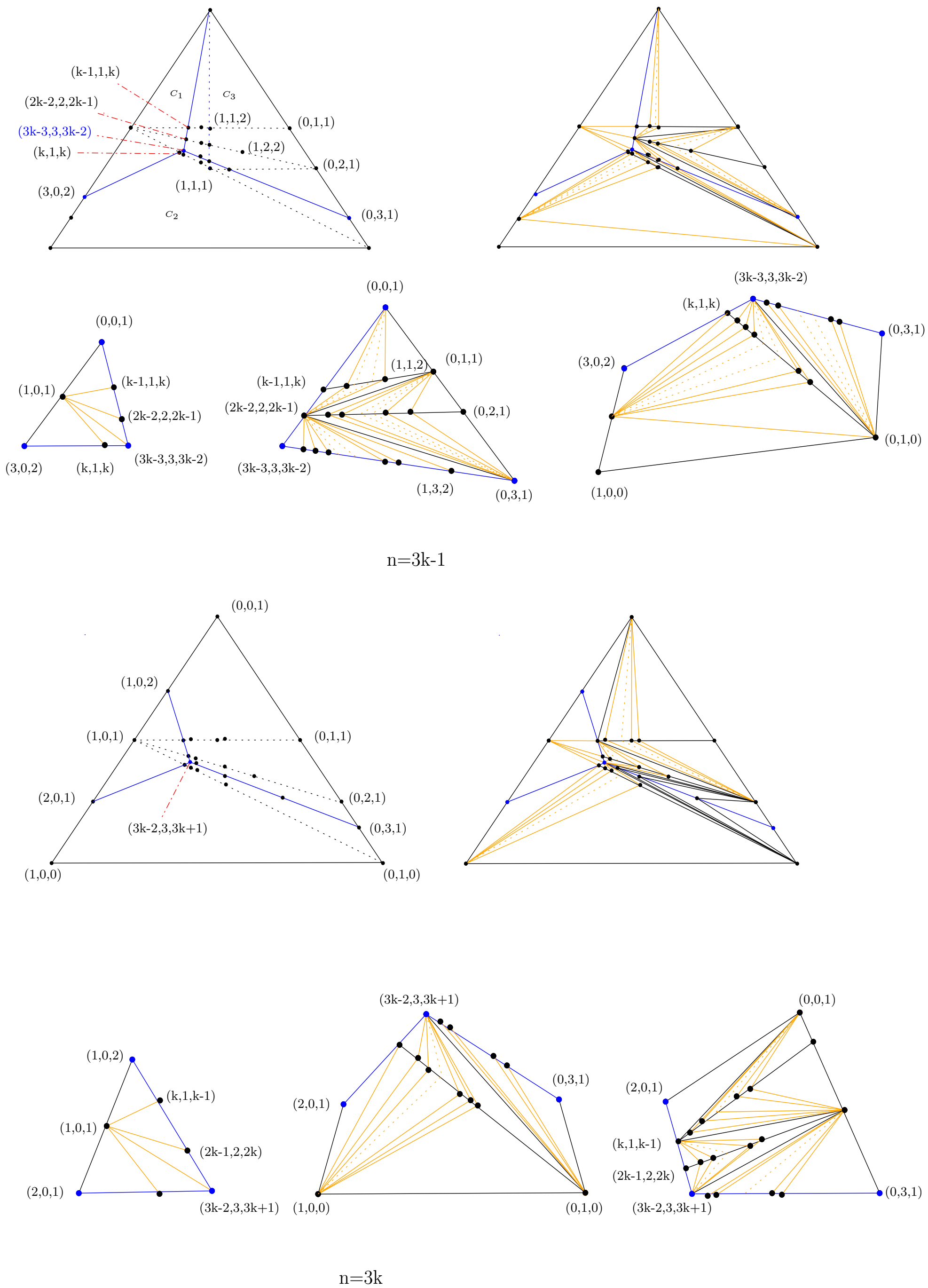
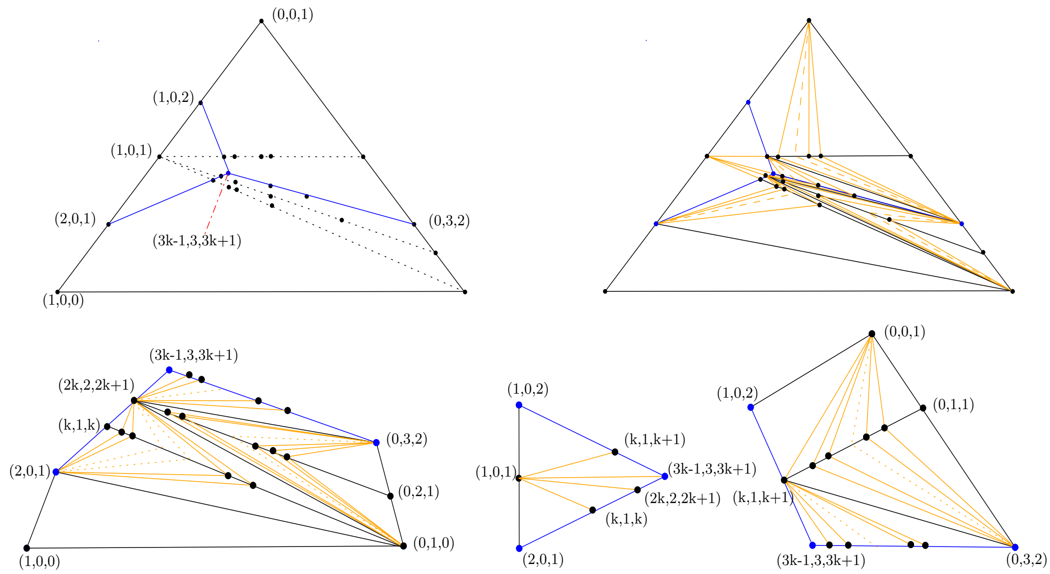


FIGURE 29. Embedded resolutions of H_n



$n = 3k + 1$

FIGURE 30. Embedded resolutions of H_{3k+1}

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