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DUALITY ON GENERALIZED CUSPIDAL EDGES PRESERVING SINGULAR SET IMAGES AND FIRST FUNDAMENTAL FORMS

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Dedicated to Professor Toshizumi Fukui for his sixtieth birthday.

Abstract. In the second, fourth and fifth authors' previous work, a duality on generic real analytic cuspidal edges in the Euclidean 3-space $\boldsymbol{R}^3$ preserving their singular set images and first fundamental forms, was given. Here, we call this an "isometric duality". When the singular set image has no symmetries and does not lie in a plane, the dual cuspidal edge is not congruent to the original one. In this paper, we show that this duality extends to generalized cuspidal edges in $\boldsymbol{R}^3$, including cuspidal cross caps, and 5/2-cuspidal edges. Moreover, we give several new geometric insights on this duality.

Introduction

Consider a generic cuspidal edge germ f whose singular set image is a given space curve C. In the second, fourth and fifth authors' previous work [14], the existence of an isometric dual $\check{f}$ of f was shown, which is a cuspidal edge germ having the same first fundamental form as f. Roughly speaking, a cuspidal edge which has the same first fundamental form and the same singular set image as f but is not right equivalent to f, is called an "isomer" of f (see Definition 0.6 for details). The isometric dual $\check{f}$ is a typical example of isomers of f. Recently, the authors found that if we reverse the orientation of C, two other candidates of isomers of f denoted by f_* and $\check{f}_*$ are obtained by imitating the construction of $\check{f}$. These two map germs f_* and $\check{f}_*$ are cuspidal edge germs which are called the *inverse* and the *inverse dual* of f, respectively ($\check{f}_*$ is just the isometric dual of f_*). In this paper, we will show that all of isomers of f are right equivalent to one of

$$\check{f},\ f_*,\ \check{f}_*.$$

We will also determine the number of congruence classes in the set of isomers of f.

By the terminology "C^r-differentiable" we mean C^∞-differentiability if $r=\infty$ and real analyticity if $r=\omega$. We denote by $\boldsymbol{R}^3$ the Euclidean 3-space. Let U be a neighborhood of the origin $(0,0)$ in the uv-plane $\boldsymbol{R}^2$, and let $f: U \to \boldsymbol{R}^3$ be a C^r-map. Without loss of generality, we may assume $f(o)=\mathbf{0}$, where

$$o := (0,0), \qquad \mathbf{0} := (0,0,0). \tag{0.1}$$

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A point $p \in U$ is called a *singular point* if f is not an immersion at p. A singular point $p \in U$ is called a *cuspidal edge point* (resp. a *generalized cuspidal edge point*) if there exist local C^r-diffeomorphisms φ on $\boldsymbol{R}^2$ and Φ on $\boldsymbol{R}^3$ such that $\varphi(o) = p$, $\Phi(f(p)) = \mathbf{0}$ and

$$(f_{3/2} :=)(u, v^2, v^3) = \Phi \circ f \circ \varphi(u, v) \quad \big(\text{resp. } (u, v^2, v^3\alpha(u, v)) = \Phi \circ f \circ \varphi(u, v)\,\big),$$

where $\alpha(u, v)$ is a C^r-function. Similarly, a singular point $p \in U$ is called a $5/2$-*cuspidal edge point* (resp. a *fold singular point*) if there exist local C^r-diffeomorphisms φ on $\boldsymbol{R}^2$ and Φ on $\boldsymbol{R}^3$ such that $\varphi(o) = p$, $\Phi(f(p)) = \mathbf{0}$ and

$$(f_{5/2} :=)(u, v^2, v^5) = \Phi \circ f \circ \varphi(u, v) \quad \big(\text{resp. } (u, v^2, 0) = \Phi \circ f \circ \varphi(u, v)\,\big).$$

Also, a singular point $p \in U$ is called a *cuspidal cross cap point* if there exist local C^r-diffeomorphisms φ on $\boldsymbol{R}^2$ and Φ on $\boldsymbol{R}^3$ such that $\varphi(o) = p$, $\Phi(f(p)) = \mathbf{0}$ and

$$(f_{\mathrm{ccr}} :=)(u, v^2, uv^3) = \Phi \circ f \circ \varphi(u, v).$$

These singular points are all generalized cuspidal edge points.

Let $\mathcal{G}^r_{3/2}(\boldsymbol{R}^2_o, \boldsymbol{R}^3)$ (resp. $\mathcal{G}^r(\boldsymbol{R}^2_o, \boldsymbol{R}^3)$) be the set of germs of C^r-cuspidal edges (resp. generalized C^r-cuspidal edges) $f(u, v)$ satisfying $f(o) = \mathbf{0}$. We fix $l > 0$ and consider an embedding (i.e. a simple regular space curve)

$$\mathbf{c} : J \to \boldsymbol{R}^3 \qquad (J := [-l, l])$$

such that $\mathbf{c}(0) = \mathbf{0}$. We do not assume here that $u \mapsto \mathbf{c}(u)$ is the arc-length parametrization (if necessary, we assume this in latter sections). We denote by C the image of $\mathbf{c}$. Here, we ignore the orientation of C and think of it as the singular set image (i.e. the image of the singular set) of f. We let $\mathcal{G}^r_{3/2}(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C)$ (resp. $\mathcal{G}^r(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C)$) be the subset of $\mathcal{G}^r_{3/2}(\boldsymbol{R}^2_o, \boldsymbol{R}^3)$ (resp. $\mathcal{G}^r(\boldsymbol{R}^2_o, \boldsymbol{R}^3)$) such that the singular set image of f is contained in C (we call C the *edge* of f). Similarly, a subset of $\mathcal{G}^r(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C)$ denoted by

$$\mathcal{G}^r_{\mathrm{ccr}}(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C), \qquad (\text{resp. } \mathcal{G}^r_{5/2}(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C)\,)$$

consisting of germs of cuspidal cross caps (resp. 5/2-cuspidal edges) is also defined.

Throughout this paper, we assume the curvature function $\kappa(u)$ of $\mathbf{c}(u)$ satisfies

$$\kappa(u) > 0 \qquad (u \in J). \tag{0.2}$$

Let U be a neighborhood of $J \times \{0\}$ of $\boldsymbol{R}^2$ and $f : U \to \boldsymbol{R}^3$ a C^r-map consisting only of generalized cuspidal edge points along $J \times \{0\}$ such that

$$f(u, 0) = \mathbf{c}(u) \qquad (u \in J). \tag{0.3}$$

We denote by $\mathcal{G}^r(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$ the set of such f (f is called a *generalized cuspidal edge along* C). Like as the case of map germs at o, the sets

$$\mathcal{G}^r_{3/2}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C), \quad \mathcal{G}^r_{\mathrm{ccr}}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C), \quad \mathcal{G}^r_{5/2}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$$

are also canonically defined. For each point P on the edge C, the plane $\Pi(P)$ passing through P which is perpendicular to the curve C is called the *normal plane* of f at P. The section of the image of f by the normal plane $\Pi(P)$ of C at P is a planar curve with a singular point at P. We call this the *sectional cusp* of f at P. Moreover, we can find a tangent vector $\mathbf{v} \in T_P\boldsymbol{R}^3$ at P, which points in the tangential direction of the sectional cusp at P. We call $\mathbf{v}$ the *cuspidal direction* (cf. (3.6) and Figure 1). The angle θ_P of the cuspidal direction from the principal normal vector of C at P is called the *cuspidal angle*.

If we normalize the initial value $\theta_{\mathbf{c}(0)} \in (-\pi, \pi]$ at $\mathbf{c}(0)(= \mathbf{0})$, then the cuspidal angle

$$\theta(u) := \theta_{\mathbf{c}(u)} \qquad (u \in J)$$

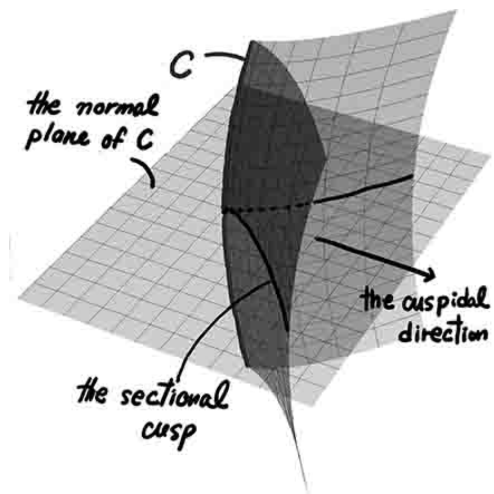

FIGURE 1. A cuspidal edge and its sectional cusp

at $\mathbf{c}(u)$ can be uniquely determined as a C^r-function on J. In [12, 16], the singular curvature $\kappa_s(u)$ and the limiting normal curvature $\kappa_\nu(u)$ along the edge $\mathbf{c}(u)$ are defined. In our present situation, they can be expressed as (cf. [3, Remark 1.9])

$$\kappa_s(u) := \kappa(u)\cos\theta(u), \qquad \kappa_\nu(u) := \kappa(u)\sin\theta(u) \qquad (u \in J). \tag{0.4}$$

By definition, $\kappa(u) = \sqrt{\kappa_s(u)^2 + \kappa_\nu(u)^2}$ holds on J. We say that $f \in \mathcal{G}^r(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C)$ is *generic* at o if

$$|\kappa_s(0)| < \kappa(0). \tag{0.5}$$

We denote by $\mathcal{G}^r_*(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C)$ the set of germs of generic generalized C^r-cuspidal edges in $\mathcal{G}^r(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C)$, and set

$$\begin{aligned} \mathcal{G}^r_{*,3/2}(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C) &:= \mathcal{G}^r_*(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C) \cap \mathcal{G}^r_{3/2}(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C), \\ \mathcal{G}^r_{*,\mathrm{ccr}}(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C) &:= \mathcal{G}^r_*(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C) \cap \mathcal{G}^r_{\mathrm{ccr}}(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C), \\ \mathcal{G}^r_{*,5/2}(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C) &:= \mathcal{G}^r_*(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C) \cap \mathcal{G}^r_{5/2}(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C). \end{aligned} \tag{0.6}$$

On the other hand, for $f \in \mathcal{G}^r(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$, we consider the condition

$$|\kappa_s(u)| < \kappa(u) \qquad (u \in J), \tag{0.7}$$

which implies that all singular points of f along the curve C are generic. We denote by

$$\mathcal{G}^r_*(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C) \tag{0.8}$$

the set of $f \in \mathcal{G}^r(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$ satisfying (0.7). Moreover, if

$$\max_{u \in J} |\kappa_s(u)| < \min_{u \in J} \kappa(u) \tag{0.9}$$

holds, then f is said to be *admissible*. We denote by

$$\mathcal{G}^r_{**}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C) \tag{0.10}$$

the set of admissible $f \in \mathcal{G}^r(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$. Then by imitating (0.6),

$$\mathcal{G}^r_{*,3/2}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C), \qquad \mathcal{G}^r_{**,3/2}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C) \tag{0.11}$$

are also defined. The following assertion is obvious:

Lemma 0.1. *Suppose that f belongs to $\mathcal{G}^r_{3/2}(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C)$ (resp. $\mathcal{G}^r_{*,3/2}(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C)$). Then there exists $\varepsilon(>0)$ such that f is an element of $\mathcal{G}^r_{3/2}(\boldsymbol{R}^2_{J(\varepsilon)}, \boldsymbol{R}^3, C)$ (resp. $\mathcal{G}^r_{**,3/2}(\boldsymbol{R}^2_{J(\varepsilon)}, \boldsymbol{R}^3, C)$), where $J(\varepsilon) := [-\varepsilon, \varepsilon]$.*

Let $\mathrm{O}(3)$ (resp. $\mathrm{SO}(3)$) be the orthogonal group (resp. the special orthogonal group) as the isometry group (resp. the orientation preserving isometry group) of $\boldsymbol{R}^3$ fixing the origin $\mathbf{0}$.

Definition 0.2. Suppose that f_i $(i = 1, 2)$ are generalized cuspidal edges belonging to $\mathcal{G}^r(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C)$ (resp. $\mathcal{G}^r(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$). Then the image of f_1 is said to have the *same image* as f_2 if there exists a neighborhood $U_i(\subset \boldsymbol{R}^2)$ of o (resp. $J \times \{0\}$) such that $f_1(U_1) = f_2(U_2)$. On the other hand, f_1 is said to be *congruent* to f_2 if there exists an orthogonal matrix $T \in \mathrm{O}(3)$ such that $T \circ f_1$ has the same image as f_2.

We then define the following two equivalence relations:

Definition 0.3. For a given f belonging to $\mathcal{G}^r(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C)$ (resp. $\mathcal{G}^r(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$), we denote by ds^2_f its first fundamental form. A generalized cuspidal edge g belonging to $\mathcal{G}^r(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C)$ (resp. $\mathcal{G}^r(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$) is said to be *right equivalent* to f if there exists a diffeomorphism φ defined on a neighborhood of o (resp. $J \times \{0\}$) in $\boldsymbol{R}^2$ such that $g = f \circ \varphi$.

Definition 0.4. For a given generalized cuspidal edge $f \in \mathcal{G}^r(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C)$ (resp. $\mathcal{G}^r(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$), we denote by ds^2_f its first fundamental form. A generalized cuspidal edge $g \in \mathcal{G}^r(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C)$ (resp. $\mathcal{G}^r(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$) is said to be *isometric* to f if there exists a diffeomorphism φ defined on a neighborhood of o (resp. $J \times \{0\}$) in $\boldsymbol{R}^2$ such that $\varphi^* ds^2_f = ds^2_g$.

In particular, we consider the case $f = g$. If $\varphi^* ds^2_f = ds^2_f$ and φ is not the identity map, then φ is called a *symmetry* of ds^2_f. Moreover, if φ reverses the orientation of the singular curve of f, then φ is said to be *effective*.

Remark 0.5. A cuspidal edge $g \in \mathcal{G}^r_{3/2}(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C)$ (resp. $\mathcal{G}^r_{3/2}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$) has the same image as a given germ $f \in \mathcal{G}^r_{3/2}(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C)$ (resp. $\mathcal{G}^r_{3/2}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$) if and only if g is right equivalent to f (cf. [10]).

If two generalized cuspidal edges $f, g \in \mathcal{G}^r(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C)$ (resp. $\mathcal{G}^r(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$) are right equivalent, then they are isometric each other. However, the converse may not be true. So we give the following:

Definition 0.6. For a given $f \in \mathcal{G}^r(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C)$ (resp. $\mathcal{G}^r(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$), a generalized cuspidal edge $g \in \mathcal{G}^r(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C)$ (resp. $\mathcal{G}^r(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$) is called an *isomer* of f (cf. [14]) if it satisfies the following conditions;

(1) g is isometric to f, and
(2) g is not right equivalent to f.

In this situation, we say that g is a *faithful isomer* of f if

- there exists a local diffeomorphism φ such that $\varphi^* ds^2_f = ds^2_g$, and
- the orientations of C induced by $u \mapsto f \circ \varphi(u, 0)$ and $u \mapsto g(u, 0)$ are compatible with respect to the one induced by $u \mapsto f(u, 0)$.

In [14, Corollary D], it was shown the existence of an involution

$$\mathcal{G}^\omega_{*,3/2}(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C) \ni f \mapsto \check{f} \in \mathcal{G}^\omega_{*,3/2}(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C). \tag{0.12}$$

To construct $\check{f}$, we need to apply the so-called Cauchy-Kowalevski theorem on partial differential equations of real analytic category (cf. Theorem 3.8). Here, $\check{f}$ is called the *isometric dual* of f, which satisfies the following properties:

(i) The first fundamental form of $\check{f}$ coincides with that of f.
(ii) The map $\check{f}$ is a faithful isomer of f.
(iii) If $\theta(P)$ is the cuspidal angle of f at $P(\in C)$, then $-\theta(P)$ is the cuspidal angle of $\check{f}$ at P.

In [14], a necessary and sufficient condition for a given positive semi-definite metric to be realized as the first fundamental form of a cuspidal edge along C is given. In this paper, we first prove the following using the method given in [14]:

Theorem I. *There exists an involution* (*called the first involution*)

$$\mathcal{I}_C : \mathcal{G}^{\omega}_{*}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C) \ni f \mapsto \check{f} \in \mathcal{G}^{\omega}_{*}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C) \tag{0.13}$$

defined on $\mathcal{G}^{\omega}_{*}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$ (*cf.* (0.8)) *satisfying the properties* (i), (ii) *and* (iii) *above. Moreover, regarding* f *and* $\check{f}$ *as map germs at* o (*cf. Lemma 0.1*), $\mathcal{I}_C$ *induces a map*

$$\mathcal{I}_o : \mathcal{G}^{\omega}_{*}(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C) \ni f \mapsto \check{f} \in \mathcal{G}^{\omega}_{*}(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C), \tag{0.14}$$

which gives a generalization of the map as in (0.12).

The existence of the map $\mathcal{I}_o$ follows also from [5, Theorem B], since $\check{f}$ is strongly congruent to f in the sense of [5, Definition 3]. However, the existence of the map $\mathcal{I}_C$ itself does not follow from [5], since $\check{f}$ given in Theorem I is not a map germ at o, but a map germ along the curve C. Some variants of this result for germs of swallowtails and cuspidal cross caps were given in [5, Theorem B] using a method different from [14]. (For swallowtails, the duality corresponding to the above properties (i), (ii) and (iii) are not obtained, see item (4) below.) The authors find Theorem I to be suggestive of the following geometric problems:

(1) How many right equivalence classes of isomers of f exist other than $\check{f}$?
(2) When are isomers non-congruent to each other?
(3) The existence of the isometric dual can be proved by applying the Cauchy-Kowalevski theorem. So we need to assume that the given generalized cuspidal edges are real analytic. It is then natural to ask if one can find a new method for constructing the isometric dual in the C^∞-differentiable category.
(4) Can one extend isometric duality to a much wider class, say, for swallowtails?

In this paper, we show the following:

- For a given generalized cuspidal edge $f \in \mathcal{G}^{\omega}_{**}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$, there exists a unique generalized cuspidal edge $f_* \in \mathcal{G}^{\omega}_{**}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$ (called the *inverse* of f) having the same first fundamental form as f along the space curve $\mathbf{c}(-u)$ whose cuspidal angle has the same sign as that of f. Moreover, any isomers of f are right equivalent to one of $\{f, \check{f}, f_*, \check{f}_*\}$ (see Theorem II), where $\check{f}_* := \mathcal{I}_C(f_*)$ is called the *inverse dual* of f.
- The four maps f, $\check{f}$, f_*, $\check{f}_*$ are non-congruent in general. Moreover, the right equivalence classes and congruence classes of these four surfaces are determined in terms of the properties of C and ds^2_f (cf. Theorems III and IV).
- Suppose that the image of a C^∞-differentiable cuspidal edge f is invariant under a non-trivial symmetry $T \in \mathrm{SO}(3)$ (cf. Definition 1.2) of $\boldsymbol{R}^3$. Then explicit construction of $\check{f}$ without use of the Cauchy-Kowalevski theorem is given (see Example 5.3).

About the last question (4), the authors do not know whether the isomers of a given swallowtail will exist in general, since the method given in this paper does not apply directly. So it left here as an open problem. (A possible isometric deformations of swallowtails are discussed in authors' previous work [5].)

The paper is organized as follows: In Section 1, we explain our main results. In Section 2, we review the definition and properties of Kossowski metrics. In Section 3, we prove Theorem I as a modification of the proof of [14]. In Section 4, we recall a representation formula for generalized cuspidal edges given in Fukui [3], and prove Theorem II. In Section 5, we investigate the properties of generic cuspidal edges with symmetries. Moreover, we prove Theorems III and IV. Several examples are given in Section 6. Finally, in the appendix, a representation formula for generalized cusps in the Euclidean plane is given.

1. Results

Let ds^2 be a C^r-differentiable positive semi-definite metric on a C^r-differentiable 2-manifold M^2. A point $o \in M^2$ is called a *regular point* of ds^2 if it is positive definite at o, and is called a *singular point* (or a *semi-definite point*) if ds^2 is not positive definite at o. Kossowski [8] defined a certain kind of positive semi-definite metrics called "Kossowski metrics" (cf. Section 2). We let ds^2 be such a metric. Then for each singular point $o \in M^2$, there exists a regular curve $\gamma : (-\varepsilon, \varepsilon) \to M^2$ such that $\gamma(0) = o$ and γ parametrizes the singular set of ds^2 near o. Such a curve is called the *singular curve* of ds^2 near o. In this situation, if $ds^2(\gamma'(0), \gamma'(0))$ does not vanish, then we say that "ds^2 is of type I at o". The first fundamental forms (i.e. the induced metrics) of germs of generalized cuspidal edges are Kossowski metrics of type I (cf. Proposition 3.1).

Setting $M^2 := (\boldsymbol{R}^2; u, v)$, we denote by $\mathcal{K}^r_{\mathrm{I}}(\boldsymbol{R}^2_o)$ the set of germs of C^r-Kossowski metrics of type I at $o := (0, 0)$. We fix such a $ds^2 \in \mathcal{K}^r_{\mathrm{I}}(\boldsymbol{R}^2_o)$. Then the metric is expressed as

$$ds^2 = E du^2 + 2F du dv + G dv^2,$$

and there exists a C^r-function λ such that $EG - F^2 = \lambda^2$. Let K be the Gaussian curvature of ds^2 defined at points where ds^2 is positive definite. Then

$$\hat{K} := \lambda K \tag{1.1}$$

can be considered as a C^r-differentiable function defined on a neighborhood $U(\subset \boldsymbol{R}^2)$ of o (cf. [12, 5]). If $\hat{K}$ vanishes (resp. does not vanish) at a singular point $q \in U$ of ds^2, then ds^2 is said to be *parabolic* (resp. *non-parabolic*) at q (see Definition 2.6). We denote by $\mathcal{K}^r_*(\boldsymbol{R}^2_o)$ (resp. $\mathcal{K}^r_p(\boldsymbol{R}^2_o)$) the set of germs of non-parabolic (resp. parabolic) C^r-Kossowski metrics of type I at o. The subset of $\mathcal{K}^r_p(\boldsymbol{R}^2_o)$ defined by

$$\mathcal{K}^r_{p,*}(\boldsymbol{R}^2_o) := \{ds^2 \in \mathcal{K}^r_p(\boldsymbol{R}^2_o)\,;\, \hat{K}'(o) \neq 0\}$$
$$\Big(= \{ds^2 \in \mathcal{K}^r_{\mathrm{I}}(\boldsymbol{R}^2_o)\,;\, \hat{K}(o) = 0,\ \hat{K}'(o) \neq 0\}\Big)$$

plays an important role in this paper, where $\hat{K}' = \partial \hat{K}/\partial u$. Metrics belonging to $\mathcal{K}^r_{p,*}(\boldsymbol{R}^2_o)$ are called *p-generic*. On the other hand, if $\hat{K}$ vanishes identically along the singular curve of $ds^2 \in \mathcal{K}^r_{\mathrm{I}}(\boldsymbol{R}^2_o)$, we call ds^2 an *asymptotic* Kossowski metric of type I. We let $\mathcal{K}^r_a(\boldsymbol{R}^2_o)$ be the set of germs of such metrics. This terminology comes from the following two facts:

- for a regular surface, a direction where the normal curvature vanishes is called an asymptotic direction, and
- the induced metric of a cuspidal edge whose limiting normal curvature κ_ν vanishes identically along its singular set belongs to $\mathcal{K}^r_a(\boldsymbol{R}^2_o)$. (Such a cuspidal edge is called an *asymptotic cuspidal edge*, see Proposition 4.12.)

By definition, we have

$$\mathcal{K}^r_*(\boldsymbol{R}^2_o) \cap \mathcal{K}^r_p(\boldsymbol{R}^2_o) = \emptyset, \qquad \mathcal{K}^r_*(\boldsymbol{R}^2_o) \cup \mathcal{K}^r_p(\boldsymbol{R}^2_o) = \mathcal{K}^r_{\mathrm{I}}(\boldsymbol{R}^2_o),$$
$$\mathcal{K}^r_a(\boldsymbol{R}^2_o) \subset \mathcal{K}^r_p(\boldsymbol{R}^2_o) \subset \mathcal{K}^r_{\mathrm{I}}(\boldsymbol{R}^2_o).$$

For $ds^2 \in \mathcal{K}^r_a(\boldsymbol{R}^2_o)$, the Gaussian curvature K can be extended on a neighborhood of o as a C^r-differentiable function. Let $\eta \in T_o\boldsymbol{R}^2$ be the null vector at the singular point o of the asymptotic Kossowski metric ds^2. If

$$dK(\eta)(o) \neq 0, \tag{1.2}$$

then ds^2 is said to be *a-generic*, and we denote by $\mathcal{K}^r_{a,*}(\boldsymbol{R}^2_o)(\subset \mathcal{K}^r_a(\boldsymbol{R}^2_o))$ the set of germs of a-generic asymptotic C^r-Kossowski metrics. Considering the first fundamental form ds^2_f of f, we can define a map

$$\mathcal{J}_o : \mathcal{G}^r_*(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C) \ni f \mapsto ds^2_f \in \mathcal{K}^r_{\mathrm{I}}(\boldsymbol{R}^2_o). \tag{1.3}$$

Theorem II. *There exists an involution* (*called the second involution*)

$$\mathcal{I}^*_C : \mathcal{G}^\omega_{**}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C) \ni f \mapsto f_* \in \mathcal{G}^\omega_{**}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$$

defined on $\mathcal{G}^\omega_{**}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$ (*cf.* (0.11)) *satisfying the following properties:*

(1) *f_* has the same first fundamental form as f, and is a non-faithful isomer of f,*
(2) *$\mathcal{I}^*_C \circ \mathcal{I}_C = \mathcal{I}_C \circ \mathcal{I}^*_C$, where $\mathcal{I}_C$ is the first involution as in Theorem* I.
(3) *Regarding f and f_* as map germs at o (cf. Lemma 0.1), $\mathcal{I}^*_C$ canonically induces a map*

$$\mathcal{I}^*_o : \mathcal{G}^\omega_*(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C) \ni f \mapsto f_* \in \mathcal{G}^\omega_*(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C) \tag{1.4}$$

such that $\mathcal{J}_o \circ \mathcal{I}^*_o = \mathcal{J}_o$ and $\mathcal{I}^*_o \circ \mathcal{I}_o = \mathcal{I}_o \circ \mathcal{I}^*_o$.
(4) *Suppose that g belongs to $\mathcal{G}^\omega_*(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C)$ (resp. $\mathcal{G}^\omega_{**}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$). If the first fundamental form of g is isometric to that of f, then g is right equivalent to one of $f, \check{f}, f_*$ and $\check{f}_*$.*

Recently, Fukui [3] gave a representation formula for generalized cuspidal edges along their edges in $\boldsymbol{R}^3$. (In [3], a similar formula for swallowtails is also given, although it is not applied in this paper.) We denote by $C^r(\boldsymbol{R}_o)$ (resp. $C^r(\boldsymbol{R}^2_o)$) the set of C^r-function germs at the origin of $\boldsymbol{R}$ (resp. $\boldsymbol{R}^2$). We fix a generalized cuspidal edge $f \in \mathcal{G}^r(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C)$ arbitrarily. The sectional cusp of f at $\mathbf{c}(u)$ induces a function $\mu(u,t) \in C^r(\boldsymbol{R}^2_o)$ which is called the "extended half-cuspidal curvature function" giving the normalized curvature function of the sectional cusp at $\mathbf{c}(u)$ (see the appendix). The value

$$\kappa_c(u) := \frac{\mu(u,0)}{2} \tag{1.5}$$

coincides with the cuspidal curvature at the singular point of the sectional cusp, and so it is called the *cuspidal curvature function* of f (cf. [12]). In Section 4, we give a Björling-type representation formula for cuspidal edges (cf. Proposition 4.3), which is a modification of the formula given in Fukui [3]. (In fact, Fukui [3] expressed the sectional cusp as a pair of functions, but did not use the function μ.) Fukui [3] explained several geometric invariants of cuspidal edges in terms of κ_s, κ_ν and θ. In Section 4, using several properties of modified Fukui's formula together with the proof of Theorem I, we reprove the following assertion which determine the images of the maps $\mathcal{I}_o$ and $\mathcal{J}_o$ (the assertions for the map $\mathcal{I}^*_o$ are not given in [14, 5, 6]):

Fact 1.1. *The maps $\mathcal{I}_o$, $\mathcal{I}^*_o$ and $\mathcal{J}_o$ (cf. (0.14), (1.3) and (1.4)) satisfy the followings:*

(1) *These two maps $\mathcal{I}_o$ and $\mathcal{I}^*_o$ are involutions on $\mathcal{G}^\omega_{*,3/2}(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C)$, and $\mathcal{J}_o$ maps $\mathcal{G}^\omega_{*,3/2}(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C)$ onto $\mathcal{K}^\omega_*(\boldsymbol{R}^2_o)$ (cf.* [14, Theorem 12]).

(2) *The two maps* $\mathcal{I}_o$ *and* $\mathcal{I}_o^*$ *are involutions on* $\mathcal{G}^{\omega}_{*,\mathrm{ccr}}(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C)$, *and* $\mathcal{J}_o$ *maps* $\mathcal{G}^{\omega}_{*,\mathrm{ccr}}(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C)$ *onto* $\mathcal{K}^{\omega}_{p,*}(\boldsymbol{R}^2_o)$ (*cf.* [5, Theorem A]).
(3) *The two maps* $\mathcal{I}_o$ *and* $\mathcal{I}_o^*$ *are involutions on* $\mathcal{G}^{\omega}_{*,5/2}(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C)$, *and* $\mathcal{J}_o$ *maps* $\mathcal{G}^{\omega}_{*,5/2}(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C)$ *onto* $\mathcal{K}^{\omega}_{a,*}(\boldsymbol{R}^2_o)$ (*cf.* [6, Theorem 5.6]).

We may assume that the origin $\mathbf{0}$ is the midpoint of C, and give here the following terminologies:

Definition 1.2. The curve C admits a *symmetry* at $\mathbf{0}$ if there exists $T \in \mathrm{O}(3)$ such that $T(C) = C$ and T is not the identity. Moreover, T is said to be *trivial* if $T(P) = P$ for all $P \in C$. A symmetry of C which is not trivial is called a *non-trivial symmetry*. (Obviously, each non-trivial symmetry reverses the orientation of C.) A non-trivial symmetry is called *positive* (resp. *negative*) if $T \in \mathrm{SO}(3)$ (resp. $T \in \mathrm{O}(3) \setminus \mathrm{SO}(3)$).

If C lies in a plane, then there exists a reflection $S \in \mathrm{O}(3)$ with respect to the plane. Then S is a trivial symmetry of C. We prove the following assertion.

Theorem III. *Let* $f \in \mathcal{G}^{\omega}_{**,3/2}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$, *that is,* f *is admissible. Then the number of the right equivalence classes of* f, $\check{f}$, f_* *and* $\check{f}_*$ *is four if and only if* ds^2_f *has no symmetries* (*cf. Definition 0.4*).

Moreover, we can prove the following:

Theorem IV. *Let* $f \in \mathcal{G}^{\omega}_{**,3/2}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$. *Then the number* N_f *of the congruence classes of the images of* $f, \check{f}$, f_* *and* $\check{f}_*$ *satisfies the following properties:*

(1) *If* C *has no non-trivial symmetries, and also* ds^2_f *has no symmetries, then* $N_f = 4$,
(2) *if not the case in* (1), *it holds that* $N_f \le 2$,
(3) $N_f = 1$ *if and only if*
  (a) C *lies in a plane and has a non-trivial symmetry,*
  (b) C *lies in a plane and* ds^2_f *has a symmetry, or*
  (c) C *has a positive symmetry and* ds^2_f *also has a symmetry.*

2. Kossowski metrics

In this section, we quickly review several fundamental properties of Kossowski metrics.

Definition 2.1. Let p be a singular point of a given positive semi-definite metric ds^2 on M^2. Then a non-zero tangent vector $\boldsymbol{v} \in T_pM^2$ is called a *null vector* if

$$ds^2(\boldsymbol{v}, \boldsymbol{v}) = 0. \tag{2.1}$$

Moreover, a local coordinate neighborhood $(U; u, v)$ is called *adjusted* at $p \in U$ if $\partial_v := \partial/\partial v$ gives a null vector of ds^2 at p.

It can be easily checked that (2.1) implies that $ds^2(\boldsymbol{v}, \boldsymbol{w}) = 0$ for all $\boldsymbol{w} \in T_pM^2$. If $(U; u, v)$ is a local coordinate neighborhood adjusted at $p \in U$, then $F(p) = G(p) = 0$ holds, where

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2. \tag{2.2}$$

Definition 2.2. A singular point $p \in M^2$ of a C^r-differentiable positive semi-definite metric ds^2 on M^2 is called *K-admissible* if there exists a local coordinate neighborhood $(U; u, v)$ adjusted at p satisfying

$$E_v(p) = 2F_u(p), \qquad G_u(p) = G_v(p) = 0, \tag{2.3}$$

where E, F, G are the C^r-functions on U given in (2.2).

If ds_f^2 is the induced metric of a C^r-map $f : U \to \boldsymbol{R}^3$ and $f_v(p) = \mathbf{0}$, then (2.3) is satisfied automatically (cf. Proposition 3.1). The property (2.3) does not depend on the choice of a local coordinate system adjusted at p, as shown in [8] and [4, Proposition 2.7]. In fact, a coordinate-free treatment for the K-admissibility of singular points is given in [8] and [4, Definition 2.3].

Definition 2.3. A positive semi-definite C^r-differentiable metric ds^2 is called a *Kossowski metric* if each singular point $p \in M^2$ of ds^2 is K-admissible and there exists a C^r-function $\lambda(u, v)$ defined on a local coordinate neighborhood $(U; u, v)$ of p such that

$$EG - F^2 = \lambda^2 \qquad (\text{on } U), \tag{2.4}$$

$$(\lambda_u(p), \lambda_v(p)) \neq (0, 0), \tag{2.5}$$

where E, F, G are C^r-functions on U given in (2.2).

The above function λ is determined up to $\pm$-ambiguity (see [5, Proposition 3]). We call such a λ the *signed area density function* of ds^2 with respect to the local coordinate neighborhood $(U; u, v)$. The following fact is known (cf. [8, 16]).

Fact 2.4. *Let ds^2 be a C^r-differentiable Kossowski metric defined on a domain U of the uv-plane. Then the 2-form $d\hat{A} := \lambda du \wedge dv$ on U is defined independently of the choice of adjusted local coordinates (u, v).*

We call $d\hat{A}$ the *signed area form* of ds^2. Let K be the Gaussian curvature defined on the complement of the singular set of ds^2.

Fact 2.5 ([8] and [4, Theorem 2.15])**.** *The 2-form $\Omega := K d\hat{A}$ can be extended as a C^r-differential form on U.*

Definition 2.6. We call Ω the *Euler form* of ds^2. If Ω vanishes (resp. does not vanish) at a singular point $p \in U$ of ds^2, then p is called a *parabolic point* (resp. *non-parabolic point*).

The following fact is also known (cf. [8, 4, 5]).

Fact 2.7. *Let p be a singular point of a Kossowski metric ds^2. Then the null space* (*i.e. the subspace generated by null vectors at p*) *of ds^2 is 1-dimensional.*

By applying the implicit function theorem for λ (cf. (2.5)), there exists a regular curve $\gamma(t)$ ($|t| < \varepsilon$) in the uv-plane (called the *singular curve*) parametrizing the singular set of ds^2 such that $\gamma(0) = p$. Then there exists a C^r-differentiable non-zero vector field $\eta(t)$ along $\gamma(t)$ which points in the null direction of the metric ds^2. We call $\eta(t)$ a *null vector field* along the singular curve $\gamma(t)$.

Definition 2.8 ([4])**.** A singular point $p \in M^2$ of a Kossowski metric ds^2 is said to be of *type I* or an A_2 *point* if the derivative $\gamma'(0)$ of the singular curve at p (called the *singular direction* at $\gamma(t)$) is linearly independent of the null vector $\eta(0)$. Moreover, ds^2 is called of *type I* if all of the singular points of ds^2 are of type I.

3. Generalized cuspidal edges

Fix a bounded closed interval $J(\subset \boldsymbol{R})$ and consider a C^r-embedding $\mathbf{c} : J \to \boldsymbol{R}^3$ with arc-length parameter. We assume that the curvature function $\kappa(u)$ of $\mathbf{c}(u)$ is positive everywhere. We fix a C^r-map $\tilde{f} : \tilde{U} \to \boldsymbol{R}^3$ defined on a domain $\tilde{U}$ in the xy-plane $\boldsymbol{R}^2$ containing $J_1 \times \{0\}$ such that each point of $J_1 \times \{0\}$ is a generalized cuspidal edge point and

$$\tilde{f}(J_1 \times \{0\}) = C \qquad (C := \mathbf{c}(J)),$$

where J_1 is a bounded closed interval in $\boldsymbol{R}$. Such an $\tilde{f}$ is called a *generalized cuspidal edge along* C. For such an $\tilde{f}$, there exists a diffeomorphism

$$\varphi : U \ni (u,v) \mapsto (x(u,v), y(u,v)) \in \varphi(U)(\subset \tilde{U})$$

such that

$$f(u,v) := \tilde{f}(x(u,v), y(u,v)) \tag{3.1}$$

satisfies

$$f(u,v) = \mathbf{c}(u) + \frac{v^2}{2}\hat{\xi}(u,v), \tag{3.2}$$

where $\hat{\xi}(u,0)$ gives a vector field along $\mathbf{c}$ which is linearly independent of $\mathbf{c}'(u)$.

Proposition 3.1. *The induced metrics of C^r-differentiable generalized cuspidal edges are C^r-differentiable Kossowski metrics whose singular points are of type I.*

Proof. Let f be a generalized cuspidal edge as in (3.2), and let $ds_f^2 = Edu^2 + 2Fdudv + Gdv^2$ be the first fundamental form of f. Then

$$E = f_u \cdot f_u, \quad F = f_u \cdot f_v, \quad G := f_v \cdot f_v$$

hold, where "$\cdot$" is the inner product of $\boldsymbol{R}^3$. Since $f_v(u,0) = \mathbf{0}$, one can easily check (2.3). By (3.2), we have

$$EG - F^2 = |f_u \times f_v|^2 = v^2 \left| \left(\mathbf{c}' + \frac{v^2}{2}\hat{\xi}_u\right) \times \left(\hat{\xi} + \frac{v}{2}\hat{\xi}_v\right) \right|^2,$$

where $\times$ denotes the cross product in $\boldsymbol{R}^3$. Since two vectors $\mathbf{c}'(u)$, $\hat{\xi}(u,0)$ are linearly independent, the function λ on U given by

$$\lambda := v\lambda_0, \qquad \lambda_0 := \left| \left(\mathbf{c}' + \frac{v^2}{2}\hat{\xi}_u\right) \times \left(\hat{\xi} + \frac{v}{2}\hat{\xi}_v\right) \right| \tag{3.3}$$

is C^r-differentiable and $\lambda_0(u,0) \neq 0$. Moreover, λ^2 coincides with $EG - F^2$. Since $\lambda_v \neq 0$, ds_f^2 is a Kossowski metric. Since $f_v(u,0) = \mathbf{0}$, $\partial_v := \partial/\partial v$ gives the null-direction, which is linearly independent of the singular direction ∂_u. So all singular points of ds_f^2 are of type I. $\square$

Let ds_f^2 be the induced metric of C^r-differentiable generalized cuspidal edge $f \in \mathcal{G}^r(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$. We set $\hat{K}(:= \lambda K)$ (cf. (1.1)), where K is the Gaussian curvature of ds_f^2 defined at points where ds_f^2 is positive definite. As mentioned in the introduction, $\hat{K}$ can be extended as a C^r-function on U. Moreover, $\check{K} := vK$ also can be considered as a C^r-function on U (cf. [12, 5]).

Corollary 3.2. *The following assertions hold:*

(1) $\hat{K}(u,0) \neq 0$ *if and only if* $\check{K}(u,0) \neq 0$, *and*
(2) $\hat{K}_u(u,0) \neq 0$ *if and only if* $\check{K}_u(u,0) \neq 0$, *under the assumption* $\hat{K}(u,0) = 0$.

Proof. By (3.3), we have the expression $\lambda = v\lambda_0$, where $\lambda_0(u,0) \neq 0$. So if we set $\check{K} = vK$, then $\hat{K} = \lambda_0\check{K}$, and $\hat{K}(u,0) = \lambda_0(u,0)\check{K}(u,0)$ hold, and so the first assertion is obvious. Differentiating $\hat{K} = \lambda_0\check{K}$, we have

$$\hat{K}_u = (\lambda_0)_u\check{K} + \lambda_0\check{K}_u.$$

Since $\hat{K}(u,0) = 0$ implies $\check{K}(u,0) = 0$, we have $\hat{K}_u(u,0) = \lambda_0(u,0)\check{K}_u(u,0)$, proving the second assertion. $\square$

Remark 3.3. For a generalized cuspidal edge f,

$$\nu(u,v) := \frac{(2\mathbf{c}'(u) + v^2\hat{\xi}_u(u,v)) \times (2\hat{\xi}(u,v) + v\hat{\xi}_v(u,v))}{|(2\mathbf{c}'(u) + v^2\hat{\xi}_u(u,v)) \times (2\hat{\xi}(u,v) + v\hat{\xi}_v(u,v))|}$$

gives a C^r-differentiable unit normal vector field on U. So f is a frontal map.

Definition 3.4. A parametrization (u,v) of $f \in \mathcal{G}^r(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$ is called an *adapted coordinate system* (cf. [12, Definition 3.7]) if

(1) $f_v(u,0) = \mathbf{0}$ and $|f_u(u,0)| = |f_{vv}(u,0)| = 1$ along the u-axis,
(2) $f_{vv}(u,0)$ is perpendicular to $f_u(u,0)$.

To show the existence of an adapted coordinate system, we prepare the following under the assumption that the curve $\mathbf{c}(u)$ is real analytic:

Lemma 3.5 ([5, Proposition 6])**.** *Let ds^2 be a C^ω-differentiable Kossowski metric defined on an open subset $U(\subset \boldsymbol{R}^2)$. Suppose that $\gamma : J \to U$ is a real analytic singular curve with respect to ds^2 such that*

$$ds^2(\gamma'(t), \gamma'(t)) > 0 \qquad (t \in J). \tag{3.4}$$

Then, for each $t_0 \in J$, there exists a C^ω-differentiable local coordinate system $(V; u, v)$ containing $(t_0, 0)$ such that $V \subset U$ and the coefficients E, F, G of the first fundamental form

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2$$

satisfy the following three conditions:

(1) *$\gamma(u) = (u,0)$, $E(u,0) = 1$ and $E_v(u,0) = 0$ hold along the u-axis,*
(2) *$F(u,v) = 0$ on V, and*
(3) *there exists a C^ω-function G_0 defined on V such that $G(u,v) = v^2 G_0(u,v)/2$ and $G_0(u,0) = 2$.*

Proof. Applying [5, Proposition 6] at the point $(t_0, 0)$ on a singular curve of ds^2, we obtain the desired local coordinate system. □

Corollary 3.6. *For each generalized cuspidal edge $f \in \mathcal{G}^\omega(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$ along C and for each singular point p of f, there exists a local coordinate neighborhood $(V; u, v)$ of p such that the restriction $f|_V$ of f is parametrized by an adapted coordinate system.*

Proof. We let ds^2_f be the first fundamental form of $f(x,y)$. By Lemma 3.5, we obtain a parameter change $(x,y) \mapsto (u(x,y), v(x,y))$ on a neighborhood of p such that the new parameter (u,v) of $f(u,v)$ defined by (3.1) satisfies (1)-(3) of Lemma 3.5 for the first fundamental form ds^2_f of f. Then we can show that this new coordinate system (u,v) is the desired one: Since the u-axis is the singular set of ds^2_f, we have $f_v(u,0) = \mathbf{0}$. On the other hand, $f_u(u,0) \cdot f_u(u,0) = E(u,0) = 1$ and

$$f_{vv}(u,0) \cdot f_u(u,0) = \left.\frac{\partial F(u,v)}{\partial v}\right|_{v=0} = 0. \tag{3.5}$$

Finally, we have

$$f_{vv}(u,0) \cdot f_{vv}(u,0) = \frac{1}{2}\left.\frac{\partial^2 G(u,v)}{\partial v^2}\right|_{v=0} = \frac{G_0(u,0)}{2} = 1,$$

proving the assertion. □

From now on, we assume that $f(u,v)$ is parametrized by the local coordinate system as in Definition 3.4. Then u is the arc-length parameter of the edge $\mathbf{c}(u) := f(u,0)$. In this section, we assume that the curvature function $\kappa(u)$ of $\mathbf{c}(u)$ is positive for each u. Then the torsion function $\tau(u)$ is well-defined. We can take the unit tangent vector $\mathbf{e}(u) := \mathbf{c}'(u)$ ($' = d/du$), and the unit principal normal vector $\mathbf{n}(u)$ satisfying $\mathbf{c}''(u) = \kappa(u)\mathbf{n}(u)$. We set

$$\mathbf{b}(u) := \mathbf{e}(u) \times \mathbf{n}(u),$$

which is the binormal vector of $\mathbf{c}(u)$. Since $f_{vv}(u,0)$ is perpendicular to $\mathbf{e}(u)$, we can write

$$f_{vv}(u,0) = \cos\theta(u)\mathbf{n}(u) - \sin\theta(u)\mathbf{b}(u), \tag{3.6}$$

which is called the *cuspidal direction*. As defined in the introduction,

- the plane $\Pi(\mathbf{c}(u))$ passing through $\mathbf{c}(u)$ spanned by $\mathbf{n}(u)$ and $\mathbf{b}(u)$ is the normal plane of the space curve $\mathbf{c}(u)$,
- the section of the image of f by $\Pi(\mathbf{c}(u))$ is a plane curve, which is called the *sectional cusp* at $\mathbf{c}(u)$, and
- the vector $f_{vv}(u,0)$ points in the tangential direction of the sectional cusp at $\mathbf{c}(u)$. So we call $\theta(u)$ the *cuspidal angle function*.
- By using $\theta(u)$, the *singular curvature* κ_s and the *limiting normal curvature* κ_ν along the edge of f (cf. [16]) are given in (0.4).

The following fact is important:

Lemma 3.7 ([16])**.** *The singular curvature is intrinsic. In particular, it is defined along the singular curve with respect to a given Kossowski metric* (cf. [4, (2.17)])*. More precisely,*

$$\kappa_s(u) = \frac{-E_{vv}(u,0)}{2} \tag{3.7}$$

holds, where (u,v) is the coordinate system as in Lemma 3.5.

Proof. As shown in [16, Proposition 1.8], κ_s is expressed as

$$\kappa_s = \frac{-F_vE_u + 2EF_{uv} - EE_{vv}}{2E^{3/2}\lambda_v}, \tag{3.8}$$

where (u,v) is a local coordinate system such that the u-axis is the singular set and ∂_v points in the null direction. If (u,v) is the local coordinate system as in Lemma 3.5, then $F = 0$, $\lambda = v\sqrt{EG_0}$ and $E(u,0) = 1$ hold. So we can obtain (3.7). □

We now prove the following theorem under the assumption that the curve $\mathbf{c}$ is real analytic:

Theorem 3.8. *We let U be an open subset of the uv-plane $\boldsymbol{R}^2$ containing $J\times\{0\}$ and ds^2 a real analytic Kossowski metric satisfying* (3.4)*. Suppose that the curvature function κ of the curve $\mathbf{c}$ is positive everywhere and the absolute value of the singular curvature $\kappa_s(u)$ of ds^2 along the singular curve*

$$J \ni u \mapsto (u,0) \in U$$

is less than $\kappa(u)$ for each $u \in J$. Then there exist two real analytic generalized cuspidal edges g_+, g_- defined on an open subset $V(\subset U)$ containing $J\times\{0\}$ satisfying the following properties:

1. *The maps $u \mapsto g_+(u,0)$ and $u \mapsto g_-(u,0)$ parametrize C, which induce the same orientation as $\mathbf{c} : J \to \boldsymbol{R}^3$.*
2. *ds^2 is the common first fundamental form of g_+ and g_-.*
3. *g_- is a faithful isomer of g_+.*
4. *If $\kappa_\nu^\pm : J \to \boldsymbol{R}$ are the limiting normal curvature functions of $g_\pm$, then $\kappa_\nu^- = -\kappa_\nu^+$ holds on J.*

(5) *If ds^2 is non-parabolic at $(u,0)$, then g_+ and g_- have cuspidal edges at $(u,0)$.*

Moreover, suppose that $h : U \to \boldsymbol{R}^3$ is a generalized cuspidal edge whose first fundamental form is ds^2. If $u \mapsto h(u,0)$ parametrizes C giving the same orientation as $\mathbf{c} : J \to \boldsymbol{R}^3$, then h coincides with g_+ or g_-.

We prove this theorem from here on out, as a modification of the proof given in [14].

Remark 3.9. For each $t_0 \in J$, we can take a connected local coordinate neighborhood $(V(t_0); u, v)$ of $(t_0, 0)$ satisfying (1), (2) and (3) of Lemma 3.5. Since J is compact, we can find finite points $t_1, ..., t_k \in J$ such that $\{V(t_j)\}_{j=1}^k$ covers the singular curve $J \times \{0\}$. It is sufficient to prove Theorem 3.8 by replacing U by each $V(t_j)$ $(j = 1, ..., k)$. (In fact, the assertion of Theorem 3.8 contains the uniqueness of $g_\pm$ on each $V(t_j)$, and so $g_\pm$ obtained in $V(t_j)$ can be uniquely extended to $V(t_j) \cup V(t_{j+1})$ for each $j = 1, ..., k-1$.)

The statements of Theorem 3.8 are properties of the maps $g_\pm$ which do not depend on the choice of a local coordinate system containing $J \times \{0\}$. As explained in Remark 3.9, we may assume the existence of a local coordinate system $(U; u, v)$ satisfying (1), (2) and (3) of Lemma 3.5, without loss of generality. Then U contains a bounded closed interval I on the u-axis such that $I \times \{0\}$ gives the singular set of ds^2. We now show the existence of a real analytic generalized cuspidal edge $g(u,v)$ such that $g(u,0) = \mathbf{c}(u)$, $g_v(u,0) = \mathbf{0}$ and

$$g_u \cdot g_u = E, \quad g_u \cdot g_v = 0, \quad g_v \cdot g_v = G,$$

which is defined on a neighborhood of $I \times \{0\}$ in U using the Cauchy-Kowalevski theorem. (We remark that $\mathbf{c}(u)$ is parametrized as an arc-length parameter.) As in Lemma 3.5, we can write $G = v^2 G_0/2$. The following lemma holds:

Lemma 3.10. *If there exists a real analytic generalized cuspidal edge $g\,(= g_\pm)$ as in Theorem 3.8, then it is a solution of the following system of partial differential equations*

$$\begin{cases} g_v &= v\zeta, \\ \xi_v & (= g_{uv}) = v\zeta_u, \\ \zeta_v &= \dfrac{1}{4}\left((\zeta, g_u, \xi_u)^T\right)^{-1}\left((G_0)_v, -v(G_0)_u, 2r - v(G_0)_{uu} + 4v\zeta_u\cdot\zeta_u\right)^T \end{cases} \tag{3.9}$$

of unknown $\boldsymbol{R}^3$-valued functions g, ξ, ζ with the initial data

$$g(u,0) = \mathbf{c}(u), \quad \xi(u,0) = \mathbf{c}'(u)(= g_u(u,0)), \quad \zeta(u,0) = \mathbf{x}(u), \tag{3.10}$$

on I, where A^T denotes the transpose of a 3×3-matrix A and

$$\mathbf{x}(u) := \cos\theta(u)\mathbf{n}(u) \mp \sin\theta(u)\mathbf{b}(u), \quad \cos\theta(u) := \frac{\kappa_s(u)}{\kappa(u)}. \tag{3.11}$$

Remark 3.11. Since $g_v = v\zeta$ and $\xi_v = v\zeta_u$, we have $\xi_v = v\zeta_u = g_{uv}$. Thus, the initial condition $\xi(u,0) = g_u(u,0)$ yields $\xi(u,v) = g_u(u,v)$.

Proof of Lemma 3.10. Since ds^2 is real analytic, E and G are real analytic functions. Since $g_v(u,0) = \mathbf{0}$, we can write

$$g_v(u,v) = v\zeta(u,v),$$

where $\zeta(u,v)$ is a real analytic function defined on a neighborhood of $I \times \{0\}$ in $\boldsymbol{R}^2$. Then

$$\zeta_v \cdot \zeta = \frac{(\zeta\cdot\zeta)_v}{2} = \frac{(G_0)_v}{4}. \tag{3.12}$$

On the other hand, since

$$vg_u \cdot \zeta = g_u \cdot g_v = 0, \tag{3.13}$$

we have $g_u \cdot \zeta = 0$. Differentiating this, we have

$$0 = v(\zeta \cdot g_u)_v = v\zeta_v \cdot g_u + v\zeta \cdot g_{uv} = v\zeta_v \cdot g_u + g_v \cdot g_{uv} = v\zeta_v \cdot g_u + \frac{G_u}{2}.$$

Since $G = v^2 G_0/2$, we have

$$\zeta_v \cdot g_u = -\frac{v}{4}(G_0)_u. \tag{3.14}$$

We now obtain information on $\zeta_v \cdot g_{uu}$. It holds that

$$v\zeta \cdot g_{uu} = g_v \cdot g_{uu} = (g_v \cdot g_u)_u - g_{uv} \cdot g_u = -g_{uv} \cdot g_u = -\frac{E_v}{2},$$

that is, we obtain

$$\zeta \cdot g_{uu} = -\frac{E_v}{2v}. \tag{3.15}$$

On the other hand, we have that

$$\begin{aligned}
\zeta \cdot g_{uu} + v\zeta_v \cdot g_{uu} = g_{vv} \cdot g_{uu} &= (g_{vv} \cdot g_u)_u - g_{vvu} \cdot g_u \\
&= \{(g_v \cdot g_u)_v - (g_v \cdot g_{uv})\}_u - (g_{uv} \cdot g_u)_v + g_{uv} \cdot g_{uv} \\
&= (-G_u/2)_u - (E_v/2)_v + g_{uv} \cdot g_{uv}.
\end{aligned}$$

This, together with (3.15), gives the following identity

$$\zeta_v \cdot g_{uu} = \frac{E_v - vE_{vv}}{2v^2} - v\frac{(G_0)_{uu}}{4} + v\zeta_u \cdot \zeta_u. \tag{3.16}$$

Since $E_v(u, 0) = 0$, the function E_v/v is a real analytic function, and the function

$$r(u, v) := \frac{E_v - vE_{vv}}{v^2} = \left(\frac{-E_v}{v}\right)_v \tag{3.17}$$

is also real analytic. By (3.13), (3.14) and (3.16), we have the third equality of (3.9) under the assumption that the 3×3 matrix

$$M(u, v) := (\zeta, g_u, \xi_u)$$

is regular, where $\xi := g_u$. The map g must have the initial data (3.10), where

$$\mathbf{x}(u) = \zeta(u, 0) = \lim_{v \to 0} \frac{g_v(u, v)}{v} = g_{vv}(u, 0).$$

By (3.6), $\mathbf{x}(u)$ can be written in the form

$$(\mathbf{x}_+(u) :=)\mathbf{x}(u) = \cos\theta(u)\mathbf{n}(u) - \sin\theta(u)\mathbf{b}(u), \tag{3.18}$$

where $\theta(u)$ is the function defined by (3.11) and $\kappa(u)$ (resp. $\kappa_s(u)$) is the curvature function of $\mathbf{c}(u)$ (resp. the singular curvature function defined by (3.7)). In fact, since the singular curvature κ_s of ds^2 is less than κ on I, there exists a real analytic angular function $\theta : I \to \boldsymbol{R}$ satisfying (3.11) and

$$0 < |\theta(u)| < \frac{\pi}{2} \qquad (u \in I).$$

Moreover, such a θ is determined up to a $\pm$-ambiguity. In particular,

$$(\mathbf{x}_-(u) :=)\mathbf{x}(u) = \cos\theta(u)\mathbf{n}(u) + \sin\theta(u)\mathbf{b}(u) \tag{3.19}$$

is the other possibility. $\square$

We now return to the proof of Theorem 3.8. We have

$$\begin{aligned}(M(u,0)=)\ &(\zeta(u,0),\, g_u(u,0),\, g_{uu}(u,0)) \\ &= (\cos\theta(u)\mathbf{n}(u) - \sin\theta(u)\mathbf{b}(u),\, \mathbf{e}(u),\, \kappa(u)\mathbf{n}(u)).\end{aligned}$$

Since the singular curvature of ds^2 satisfies $|\kappa_s| < \kappa$ on I, the function $\sin\theta$ does not vanish on I. Thus the matrix $M(u,0)$ is regular for each $u \in I$. We can then apply the Cauchy-Kowalevski theorem (cf. [9]) for the system of partial differential equations (3.9) with initial data (3.10) and obtain a unique real analytic solution (g,ξ,ζ) of (3.9) defined on a neighborhood of $I\times\{0\}$ in $\boldsymbol{R}^2$. Thus, we obtained the existence of real analytic generalized cuspidal edges $g_\pm(u,v)$ corresponding to the initial data $\mathbf{x}_\pm(u)$. By the above construction of these $g_\pm$, the functions $\pm\theta$ coincide with the cuspidal angles of $g_\pm$, respectively. To accomplish the proof of Theorem 3.8, we need to verify that the first fundamental forms of $g_\pm$ coincide with ds^2. To show this, we consider the case $g = g_+$ with initial condition $\mathbf{x}(u) := \mathbf{x}_+(u)$, without loss of generality. The third equation of (3.9) yields $\zeta_v\cdot\zeta = (G_0)_v/4$, and hence we have $(\zeta\cdot\zeta - G_0/2)_v = 0$. Since

$$\zeta(u,0)\cdot\zeta(u,0) - \frac{G_0(u,0)}{2} = \mathbf{x}(u)\cdot\mathbf{x}(u) - 1 = 0,$$

the Cauchy-Kowalevski theorem yields that

$$\zeta\cdot\zeta = \frac{G_0}{2}. \tag{3.20}$$

Hence, by the first equation of (3.9), we have

$$g_v\cdot g_v = \frac{v^2G_0}{2} = G. \tag{3.21}$$

On the other hand, using (3.9), we have

$$(\xi - g_u)_v = \xi_v - g_{uv} = v\zeta_u - (g_v)_u = v\zeta_u - (v\zeta)_u = 0.$$

The initial condition $\xi(u,0) = g_u(u,0)$ yields that $g_u = \xi$. Then $g_{uv} = \xi_v = v\zeta_u$ and

$$g_{uv}\cdot\zeta = v\zeta_u\cdot\zeta = v\frac{(\zeta\cdot\zeta)_u}{2} = \frac{v(G_0)_u}{4}$$

hold. Using this, we have

$$(g_u\cdot\zeta)_v = g_{uv}\cdot\zeta + g_u\cdot\zeta_v = \frac{v(G_0)_u}{4} - \frac{v(G_0)_u}{4} = 0.$$

Since $g_u(u,0)\cdot\zeta(u,0) = 0$, we can conclude that $g_u\cdot\zeta = 0$, that is,

$$g_u\cdot g_v = 0 \tag{3.22}$$

is obtained. We now prepare the following:

Lemma 3.12. *Suppose that (which is one of the conditions in (3.9))*

$$\zeta_v\cdot\xi_u (= \zeta_v\cdot g_{uu}) = \frac{2r - v(G_0)_{uu} + 4v\zeta_u\cdot\zeta_u}{4}.$$

Then the initial condition (3.18) implies the following identity

$$\frac{E_v}{2} + v\zeta\cdot\xi_u = 0. \tag{3.23}$$

Proof. Using (3.20), we have that

$$\begin{aligned}(\zeta\cdot\xi_u)_v &= \zeta_v\cdot\xi_u+\zeta\cdot\xi_{uv}=\zeta_v\cdot\xi_u+\zeta\cdot g_{uuv}=\zeta_v\cdot\xi_u+\zeta\cdot(v\zeta_{uu})\\ &=\frac{1}{4}\left(2r-v(G_0)_{uu}+4v\zeta_u\cdot\zeta_u\right)+\zeta\cdot(v\zeta_{uu})\\ &=\frac{r}{2}-\frac{v}{2}(G_0)_{uu}+v(\zeta_u\cdot\zeta_u+\zeta\cdot\zeta_{uu})\\ &=\frac{r}{2}-\frac{v}{4}(\zeta\cdot\zeta)_{uu}+\frac{v}{2}(\zeta\cdot\zeta)_{uu}=\frac{r}{2}.\end{aligned}$$

By (3.17),

$$\left(\zeta\cdot\xi_u+\frac{E_v}{2v}\right)_v=0$$

holds. On the other hand, we have

$$\begin{aligned}\zeta(u,0)\cdot\xi_u(u,0)&=\mathbf{x}(u)\cdot g_{uu}(u,0)=(\cos\theta(u)\mathbf{n}(u)-\sin\theta(u)\mathbf{b}(u))\cdot\mathbf{c}''(u)\\ &=\big(\cos\theta(u)\mathbf{n}(u)-\sin\theta(u)\mathbf{b}(u)\big)\cdot(\kappa(u)\mathbf{n}(u))=\kappa(u)\cos\theta(u)\\ &=\kappa(u)\frac{\kappa_s(u)}{\kappa(u)}=\kappa_s(u)=\frac{-E_{vv}(u,0)}{2}=\lim_{v\to0}\frac{-E_v(u,v)}{2v}.\end{aligned}$$

So we obtain (3.23). □

We again return to the proof of Theorem 3.8. By (3.23), we have

$$\frac{1}{2}(g_u\cdot g_u)_v=g_{uv}\cdot g_u=(g_v\cdot g_u)_u-g_v\cdot g_{uu}=-g_v\cdot g_{uu}=\frac{E_v}{2}.$$

This, with the initial condition $g_u(u,0)\cdot g_u(u,0)=\mathbf{c}'(u)\cdot\mathbf{c}'(u)=1$ implies

$$g_u\cdot g_u=E. \tag{3.24}$$

By (3.24), (3.22) and (3.21), we can conclude that ds^2 coincides with the first fundamental form of $g=g_+$, which implies the existence and uniqueness of $g=g_+$. Replacing θ by $-\theta$, we also obtain the existence and uniqueness of $g=g_-$. Since the cuspidal angles of $g_\pm$ are distinct, the image of g_- does not coincide with g_+. Since the orientation of $u\mapsto g_-(u,0)$ is compatible with that of the curve $u\mapsto g_+(u,0)$, the map g_- is a faithful isomer of g_+.

Here, we suppose ds^2 is non-parabolic at $(u,0)$, then g_+ and g_- are wave fronts by [5, Proposition 4 (o)]. Since ds^2 is of type I, the criterion of cuspidal edges given in [5, Proposition 4 (i)] yields that g_+ and g_- are both cuspidal edges.

Finally, the last assertion of Theorem 3.8 follows from the uniqueness of the system of partial equations (3.9) as a consequence of the Cauchy-Kowalevski theorem, proving Theorem 3.8.

By the above proof of Theorem 3.8, we obtain the following:

Corollary 3.13. *The cuspidal angle of g_- is $-\theta$, where θ is the cuspidal angle of g_+. In particular, g_- is a faithful isomer of g_+ since $\sin\theta\neq0$.*

We next prove the following:

Lemma 3.14. *Let U be an open subset of the uv-plane $\boldsymbol{R}^2$ containing $J\times\{0\}$, and let ds^2 be a real analytic Kossowski metric of type I defined on U satisfying (1)–(3) of Lemma 3.5. Suppose that the singular set of ds^2 consists only of non-parabolic points. If there exist open subsets $V_i(\subset U)$ $(i=1,2)$ containing $J\times\{0\}$ and a diffeomorphism $\varphi:V_1\to V_2$ such that $\varphi^*ds^2=ds^2$ and $\varphi(u,0)=(u,0)$ hold for $u\in J$, then $V_1=V_2$ and φ is the identity map.*

Proof. Let $\mathbf{c}(u)$ $(u \in J)$ be a space curve satisfying the assumption of Theorem 3.8, and let g_+ be one of cuspidal edges realizing ds^2 as in Theorem 3.8. Since $g_+ \circ \varphi$ and g_+ have the common first fundamental form ds^2, the last assertion of Theorem 3.8 yields that $g_+ \circ \varphi$ coincides with either g_+ or g_-. Since $g_+ \circ \varphi$ and g_+ have the same image, they have a common cuspidal angle at each point of C. So there exists a symmetry T of C such that $T \circ g_+ \circ \varphi = g_+$. Suppose T is not the identity map. Since $\varphi(u,0) = (u,0)$, φ maps the domain $D_+ := \{v > 0\}$ to $D_- := \{v < 0\}$. However, it is impossible, because $\varphi^* ds^2 = ds^2$ and the Gaussian curvature on D_+ takes the opposite sign of that on D_- (cf. [5, (1.14)]). Thus, T is the identity map and $g_+ \circ \varphi = g_+$ holds. Since the singular set of g_+ consists of cuspidal edge points, g_+ is injective, and φ must be the identity map. □

Proposition 3.15. *Let ds^2 be a real analytic Kossowski metric belonging to $\mathcal{K}^\omega_*(\boldsymbol{R}^2_o)$. Suppose that φ is a local C^ω-diffeomorphism satisfying $\varphi^* ds^2 = ds^2$ and $\varphi(o) = o$ which is not the identity map. Then φ is an involution which reverses the orientation of the singular curve. Moreover, such a φ is uniquely determined.*

Proof. We can take a local coordinate system satisfying (1)–(3) of Lemma 3.5. Since $\varphi(o) = o$, the fact that $u \mapsto (u,0)$ is the arc-length parametrization with respect to ds^2 yields that either $\varphi(u,0) = (u,0)$ or $\varphi(u,0) = (-u,0)$ holds. If $\varphi(u,0) = (u,0)$, then by Lemma 3.14, φ is the identity map, a contradiction. So we have $\varphi(u,0) = (-u,0)$. This means that φ reverses the orientation of the singular curve. In this situation, we have $\varphi \circ \varphi(u,0) = (u,0)$. Applying Lemma 3.14 again, $\varphi \circ \varphi$ is the identity map, that is, φ is an involution. We next suppose that ψ is another local C^ω-diffeomorphism satisfying $\psi^* ds^2 = ds^2$ and $\psi(o) = o$. Then $\varphi \circ \psi(u) = (u,0)$ holds, and Lemma 3.14 yields that $\varphi \circ \psi$ is the identity map. So ψ must coincide with φ. □

Corollary 3.16. *Let ds^2_f be a real analytic Kossowski metric as the first fundamental form of $f \in \mathcal{G}^\omega_{*,3/2}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$. Suppose that φ is a C^ω-symmetry of ds^2_f, then it is effective and is an involution reversing the orientation of the singular curve.*

Proof. Without loss of generality, we may assume that the parameters (u,v) of $f(u,v)$ satisfy (1)-(3) of Lemma 3.5 for ds^2_f. Let P be the midpoint of C with respect to the arc-length parameter. Then there exists $c \in J$ such that $f(c,0) = P$. Thinking $o := (c,0)$, we may regard f belongs to $\mathcal{G}^\omega_{*,3/2}(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C)$. Since $f \in \mathcal{G}^\omega_{*,3/2}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$, by restricting f to a neighborhood of o, the metric ds^2_f can be considered as an element of $\mathcal{K}^\omega_*(\boldsymbol{R}^2_o)$ (cf. [5, (2) of Theorem A]). So the symmetry φ of ds^2_f satisfies the desired property by Proposition 3.15. Since φ is real analytic, the property is extended on a tubular neighborhood of the singular curve. □

Moreover, the following important property for symmetries of Kossowski metrics is obtained:

Theorem 3.17. *Let p be a singular point of a real analytic Kossowski metric ds^2 which is an accumulation point of non-parabolic singular points of type I. Suppose that φ is a local C^ω-diffeomorphism fixing p satisfying $\varphi^* ds^2 = ds^2$. Then φ is an involution and reverses the orientation of the singular curve if it is not the identity map.*

Proof. Let $\gamma(t)$ be a real analytic parametrization of the singular curve of the real analytic Kossowski metric ds^2 such that $\gamma(0) = p$. We let $\{p_n\}_{n=1}^\infty$ be a sequence of non-parabolic points converging to p. Since γ is real analytic, the existence of such a sequence implies that, for sufficiently small $\varepsilon(>0)$, $\gamma((-\varepsilon,0) \cup (0,\varepsilon))$ consists of non-parabolic points of type I. Then

$$s(t) := \int_0^t \sqrt{ds^2(\gamma'(u), \gamma'(u))}\, du \qquad (t \in (-\varepsilon, \varepsilon))$$

is a monotone increasing function of t, giving a continuous parametrization of γ. Using this parameter s, either $\varphi\circ\gamma(s)=\gamma(s)$ or $\varphi\circ\gamma(s)=\gamma(-s)$ holds. If the former case happens, then applying Proposition 3.15 at a non-parabolic point $\gamma(s)$ ($s\neq 0$), φ must be the identity map on a neighborhood of $\gamma(s)$. Since φ is real analytic, it must be the identity map on a neighborhood of p.

We next consider the case that $\varphi\circ\gamma(s)=\gamma(-s)$. Then $\varphi\circ\varphi\circ\gamma(s)=\gamma(s)$, and the above argument implies that φ is an involution, proving the assertion. $\square$

Proof of Theorem I. Let ds^2_f be the first fundamental form of f. Then ds^2_f is a Kossowski metric of type I, by Proposition 3.1. Since f belongs to $\mathcal{G}^\omega_*(\boldsymbol{R}^2_J,\boldsymbol{R}^3,C)$ (cf. (0.5)), the singular curvature κ_s of ds^2_f is less than κ on J. By Theorem 3.8, there exist two generalized cuspidal edges $g_+,g_-\in\mathcal{G}^\omega_*(\boldsymbol{R}^2_J,\boldsymbol{R}^3,C)$ whose first fundamental forms coincide with ds^2_f. Since ds^2_f is the first fundamental form of f, the last assertion of Theorem 3.8 yields that either $f=g_+$ or $f=g_-$ holds. Without loss of generality, we may set $f=g_+$, then $\check f:=g_-$ is the desired isometric dual of f. The remaining assertions for $f\in\mathcal{G}^\omega_*(\boldsymbol{R}^2_o,\boldsymbol{R}^3,C)$ follow from Lemma 0.1. $\square$

Definition 3.18. For each $f\in\mathcal{G}^\omega_*(\boldsymbol{R}^2_o,\boldsymbol{R}^3,C)$ (resp. $f\in\mathcal{G}^\omega_*(\boldsymbol{R}^2_J,\boldsymbol{R}^3,C)$), we call the above $\check f\in\mathcal{G}^\omega_*(\boldsymbol{R}^2_o,\boldsymbol{R}^3,C)$ (resp. $\check f\in\mathcal{G}^\omega_*(\boldsymbol{R}^2_J,\boldsymbol{R}^3,C)$) the *isometric dual* of f.

4. A representation formula for generalized cuspidal edges

We set $J=[-l,l]$ ($l>0$). Let $\mathbf{c}:J\to\boldsymbol{R}^3$ be an embedding with arc-length parameter whose curvature function $\kappa(u)$ is positive everywhere. We denote by $\mathbf{e}(u):=\mathbf{c}'(u)$, and by C the image of $\mathbf{c}$. We let $\mathbf{n}(u)$ and $\mathbf{b}(u)$ be the unit principal normal vector field and unit binormal vector field of $\mathbf{c}(u)$, respectively. We fix a sufficiently small $\delta(>0)$ and consider a map given by

$$f(u,v):=\mathbf{c}(u)+(A(u,v),B(u,v))\begin{pmatrix}\cos\theta(u) & -\sin\theta(u)\\ \sin\theta(u) & \cos\theta(u)\end{pmatrix}\begin{pmatrix}\mathbf{n}(u)\\ \mathbf{b}(u)\end{pmatrix}, \tag{4.1}$$

where $u\in J$ and $|v|<\delta$. Here $A(u,v)$, $B(u,v)$ and $\theta(u)$ are C^r-functions, and satisfy

$$A(u,0)=A_v(u,0)=0,\quad A_{vv}(u,0)\neq 0,\quad B(u,0)=B_v(u,0)=B_{vv}(u,0)=0.$$

Then it can be easily checked that any generalized cuspidal edges along C are right equivalent to one of such a map. Moreover, if $B_{vvv}(u,0)\neq 0$, then f is a cuspidal edge along C. The function $\theta(u)$ is called the *cuspidal angle* at $\mathbf{c}(u)$. Let $\kappa(u)$ be the curvature of $\mathbf{c}(u)$. Then the C^r-functions defined by

$$\kappa_s(u)=\kappa(u)\cos\theta(u),\qquad \kappa_\nu(t)=\kappa(u)\sin\theta(u) \tag{4.2}$$

give the *singular curvature* and the *limiting normal curvature* respectively. The map germ f can be determined by

$$(\theta(u),\ A(u,v),\ B(u,v)).$$

We call these functions *Fukui's data*.

Definition 4.1. In the expression (4.1), if

- u is an arc-length parameter of $\mathbf{c}$,
- for each $u\in J$, the map $(-\delta,\delta)\ni t\mapsto(A(u,t),B(u,t))\in\boldsymbol{R}^2$ is a generalized cusp at $t=0$ (called a *sectional cusp at* u), and t gives a normalized half-arc-length parameter (see the appendix),

then the expression (4.1) of f by setting $v=t$ as the normalized half-arc-length parameter is called the *normal form* of a generalized cuspidal edge.

We now fix such a normal form f. We set

$$\begin{pmatrix}\mathbf{v}_2(u)\\ \mathbf{v}_3(u)\end{pmatrix}=\begin{pmatrix}\cos\theta(u) & -\sin\theta(u)\\ \sin\theta(u) & \cos\theta(u)\end{pmatrix}\begin{pmatrix}\mathbf{n}(u)\\ \mathbf{b}(u)\end{pmatrix}, \tag{4.3}$$

then we have

$$f(u,t)=\mathbf{c}(u)+A(u,t)\mathbf{v}_2(u)+B(u,t)\mathbf{v}_3(u). \tag{4.4}$$

Definition 4.2. Let (a,b) $(a<b)$ be an interval on $\boldsymbol{R}$, and $\delta\in(0,\infty]$ a positive number. A C^r-differentiable ($r=\infty$ or $r=\omega$) quadruple $(\kappa,\tau,\theta,\hat{\mu})$ is called a *fundamental data* (or a *modified Fukui-data*) if

- $\kappa:(a,b)\to\boldsymbol{R}$ is a C^r-function such that $\kappa>0$,
- $\tau,\theta:(a,b)\to\boldsymbol{R}$ and $\hat{\mu}:(a,b)\times(-\delta,\delta)\to\boldsymbol{R}$ are C^r-functions.

Summarizing the above discussions, one can easily show the following representation formula for generalized cuspidal edges, which is a mixture of Fukui's representation formula as in [3, (1.1)] for generalized cuspidal edges and a representation formula for cusps in the appendix (cf. Lemma A.1):

Proposition 4.3. *Let $(\kappa,\tau,\theta,\hat{\mu})$ be a given fundamental data and $\mathbf{c}(u)$ $(u\in J)$ the space curve with arc-length parameter whose curvature function and torsion function are $\kappa(u)$ and $\tau(u)$. Then,*

$$f(u,t):=\mathbf{c}(u)+(A(u,t),B(u,t))\begin{pmatrix}\cos\theta(u) & -\sin\theta(u)\\ \sin\theta(u) & \cos\theta(u)\end{pmatrix}\begin{pmatrix}\mathbf{n}(u)\\ \mathbf{b}(u)\end{pmatrix} \tag{4.5}$$

gives a generalized cuspidal edge written in a normal form along $C:=\mathbf{c}(J)$, where (A,B) is given by

$$(A(u,t),B(u,t))=\int_0^t v(\cos\lambda(u,v),\sin\lambda(u,v))dv,\ \ \lambda(u,t):=\int_0^t\hat{\mu}(u,v)dv. \tag{4.6}$$

Moreover,

(1) *θ gives the cuspidal angle of f along $\mathbf{c}$,*
(2) *$t\mapsto\hat{\mu}(u,t)$ is the function given in (A.2) for the sectional cusp of f at u.*

Furthermore, any generalized cuspidal edge along C is right equivalent to such an f constructed in this manner (see also Remark 0.5).

Remark 4.4. Let $\mathbf{c}_0(u)$ be a space curve parametrized by the arc-length parameter u defined on an interval $J:=[-l,l]$ $(l>0)$, whose curvature function and torsion function are $\kappa(u)$ and $\tau(u)$, respectively. We assume that $\mathbf{c}_0(0)=\mathbf{0}$. Suppose that $C:=\mathbf{c}_0(J)$ admits a non-trivial symmetry T. Since $\mathbf{0}$ is the midpoint of C and is fixed by T, we may assume that $T\in\mathrm{O}(3)$ and set $\sigma:=\det(T)\in\{1,-1\}$. Then $\mathbf{c}_1(u):=T\mathbf{c}_0(-u)$ is a space curve whose curvature function and torsion function are $\kappa(u)$ and $\sigma\tau(u)$ respectively. We denote by $\mathbf{e}_i(u)(:=\mathbf{c}_i'(u))$, $\mathbf{n}_i(u)$ and $\mathbf{b}_i(u)$ $(i=0,1)$ the unit tangent vector, unit principal normal vector and unit binormal vector of $\mathbf{c}_i(u)$, respectively. Differentiating $T\circ\mathbf{c}_0(u)=\mathbf{c}_1(u)$, we have

$$\begin{aligned}&T\mathbf{e}_0(-u)=T\circ\mathbf{c}_0'(-u)=-\mathbf{c}_1'(u)=-\mathbf{e}_1(u),\\ &\kappa_0(-u)T\mathbf{n}_0(-u)=T\circ\mathbf{c}_0''(-u)=\mathbf{c}_1''(u)=\kappa_1(u)\mathbf{n}_1(u).\end{aligned}$$

In particular, $T\mathbf{e}_0(-u)=-\mathbf{e}_1(u)$, $T\mathbf{n}_0(-u)=\mathbf{n}_1(u)$ and $\kappa_0(-u)=\kappa_1(u)$ hold, where κ_i $(i=1,2)$ is the curvature function of $\mathbf{c}_i$. Since $\sigma:=\det(T)\in\{1,-1\}$, we have

$$\mathbf{b}_0=\mathbf{e}_0\times\mathbf{n}_0=(-T\mathbf{e}_1)\times(T\mathbf{n}_1)=-T\,(\mathbf{e}_1\times\mathbf{n}_1)=-\sigma T\mathbf{b}_1.$$

Using this, one can also obtain the relation $-\sigma\tau_0(-u) = \tau_1(u)$, where τ_i $(i = 1, 2)$ is the torsion function of $\mathbf{c}_i$. We set

$$f_i := \mathbf{c}_i + (A_i, B_i)\begin{pmatrix}\cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i\end{pmatrix}\begin{pmatrix}\mathbf{n}_i \\ \mathbf{b}_i\end{pmatrix} \qquad (i = 0, 1),$$

and suppose

$$A_0(-u, t) = A_1(u, t), \quad B_0(-u, t) = -\sigma B_1(u, t), \quad \theta_0(-u) = -\sigma\theta_1(u).$$

Then

$$\begin{aligned}
&T \circ f_0(-u, t) \\
&= T\mathbf{c}_0(-u) + (A_0(-u, t), B_0(-u, t))\begin{pmatrix}\cos\theta_0(-u) & -\sin\theta_0(-u) \\ \sin\theta_0(-u) & \cos\theta_0(-u)\end{pmatrix}\begin{pmatrix}T\mathbf{n}_0(-u) \\ T\mathbf{b}_0(-u)\end{pmatrix} \\
&= \mathbf{c}_1(u) + (A_1(u, t), -\sigma B_1(u, t))\begin{pmatrix}\cos(-\sigma\theta_1(u)) & -\sin(-\sigma\theta_1(u)) \\ \sin(-\sigma\theta_1(u)) & \cos(-\sigma\theta_1(u))\end{pmatrix}\begin{pmatrix}\mathbf{n}_1(u) \\ -\sigma\mathbf{b}_1(u)\end{pmatrix} \\
&= \mathbf{c}_1(u) + (A_1(u, t), B_1(u, t))\begin{pmatrix}\cos\theta_1(u) & -\sin\theta_1(u) \\ \sin\theta_1(u) & \cos\theta_1(u)\end{pmatrix}\begin{pmatrix}\mathbf{n}_1(u) \\ \mathbf{b}_1(u)\end{pmatrix} = f_1(u, t).
\end{aligned}$$

Thus, we obtain the relation $f_1(u, t) = T \circ f_0(-u, t)$. In particular, f_1 has the same first fundamental form as f_0. Moreover,

(a) if $T \in \mathrm{SO}(3)$, then the cuspidal angle of f_1 takes opposite sign of that of f_0. By the uniqueness of the isometric dual of f_0 (cf. Theorem 3.8), $\check{f}_0(u, t) = f_1(u, t) = T \circ f_0(-u, t)$ holds, that is, f_1 is the faithful isomer (i.e. the isometric dual) of f_0.

(b) if $T \in \mathrm{O}(3) \setminus \mathrm{SO}(3)$, then the cuspidal angle of f_1 coincides with that of f_0. Then $f_0(u, t) = f_1(u, t) = T \circ f_0(-u, t)$ holds (cf. Theorem 3.8), that is, the image of f_0 is invariant by T.

Remark 4.5. Let $f(u, t)$ be a generalized cuspidal edge associated to the data

$$(\kappa(u), \tau(u), \theta(u), \hat{\mu}(u, t)).$$

Then $f_\#(u, t) := f(-u, t)$ is also a generalized cuspidal edge along the same space curve as f but with the reversed orientation. If we set $\mathbf{c}_\#(u) := \mathbf{c}(-u)$, then $\mathbf{c}_\#(u) = f_\#(u, 0)$ holds. By a similar calculation like as in Remark 4.4, one can easily verify that $(\kappa(-u), -\tau(-u), -\theta(-u), \hat{\mu}(-u, t))$ gives the fundamental data of $f_\#(u, t)$.

We next prove Theorem II in the introduction.

Proof of Theorem II. We fix $f \in \mathcal{G}^{\omega}_{**}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$ arbitrarily. We denote by ds^2_f the first fundamental form of f. Since f is admissible, the singular curvature $\kappa_s(u)$ satisfies (0.9), and so (0.7) holds. By Theorem 3.8, there exist two distinct generalized cuspidal edges $g_\pm$ whose first fundamental forms coincide with ds^2_f such that $g_+ = f$, and $u \mapsto g_-(u, 0)$ has the same orientation as that of $u \mapsto f(u, 0)$. Since f is admissible, the singular curvature κ_s is determined only by ds^2_f. Thus $g_\pm$ belong to $\mathcal{G}^{\omega}_{**}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$. By the proof of Theorem I, we know that $\check{f} := g_-$ gives the isometric dual of f.

On the other hand, we replace u with $-u$ (that is, the orientation of C is reversed). Since f is admissible, it holds that

$$0 < |\kappa_s(u)| \le \min_{u \in J} \kappa(u) < \kappa(-u) \qquad (u \in J).$$

So, applying Theorem 3.8 again, there exist two distinct generalized cuspidal edges

$$h_\pm \in \mathcal{G}^{\omega}_{**}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$$

such that $u \mapsto h_\pm(u,0)$ have the same orientation as that of $u \mapsto f(-u,0)$. Then ds^2_f gives the common first fundamental form of the generalized cuspidal edges $h_\pm$. By (3.11), we may assume that the cuspidal angle $\theta_*(u)$ (resp. $-\theta_*(u)$) ($\theta_*(u)\theta(u) > 0$) of h_+ (resp. h_-) satisfies

$$\cos\theta_*(u) = \frac{\kappa_s(u)}{\kappa(-u)}.$$

Since the orientation of the singular curves of $h_\pm$ is opposite of that of f, the two maps $h_\pm$ are non-faithful isomers of f. We set

$$f_* := h_+ \text{ (the inverse), and } \check{f}_* := h_- \text{ (the inverse dual).}$$

By the above Remark 4.5, the cuspidal angle of $f_\#(u,v) := f(-u,v)$ is $-\theta(-u)$, the cuspidal angle $\theta_*(u)$ takes opposite sign of that of $f_\#(u,v)$. So the image of f does not coincide with that of f_*. Hence f_* is an isomer of f.

By our construction of f_*, (1), (2) and (3) are obvious. So we prove (4). We suppose that the first fundamental form of a generalized cuspidal edge $k \in \mathcal{G}^\omega_{**}(\boldsymbol{R}^2_I, \boldsymbol{R}^3, C)$ is isometric to ds^2_f. (The case that $k \in \mathcal{G}^\omega_*(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C)$ is obtained by Lemma 0.1.) Since the first fundamental form is determined independently of a choice of local coordinate system, we have $\mathcal{J}_C(f\circ\varphi) = \mathcal{J}_C(f)\circ\varphi$, where φ is a diffeomorphism on a certain tubular neighborhood of $J \times \{0\}$. So we may assume that $ds^2_k = ds^2_f$ without loss of generality. Then k must coincide with one of $\{g_+, g_-, h_+, h_-\}$, because of the uniqueness of the solution of (3.9) with initial condition (3.10). □

Definition 4.6. We call the above f_* and $\check{f}_*$ the *inverse* and the *inverse dual* of $f \in \mathcal{G}^\omega_{**}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$, respectively.

We next give criteria of a given germ of generalized cuspidal edge to be a cuspidal edge, cuspidal cross cap or 5/2-cuspidal edge in terms of the extended half-cuspidal curvature function $\hat{\mu}$.

Proposition 4.7. *Let $f \in \mathcal{G}^r(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$ be the generalized cuspidal edge associated to a fundamental data $(\kappa, \tau, \theta, \hat{\mu})$. Then*

1. *f gives a cuspidal edge along the u-axis if $\hat{\mu}(u,0) \neq 0$,*
2. *f gives a cuspidal cross cap at o if $\hat{\mu}(0,0) = 0$ and $\hat{\mu}_u(0,0) \neq 0$,*
3. *f gives a 5/2-cuspidal edge along the u-axis if $\hat{\mu}(u,0) = 0$ and $\hat{\mu}_{vv}(u,0) \neq 0$.*

The first and the second assertions have been proved in [3, Proposition 1.6].

Proof. We may assume that f is written in a normal form. The first assertion follows from (1) of Proposition A.2. The second assertion follows from the criterion for cuspidal cross caps given in [2], but can be proved much easier using (2) of [3, Proposition 4.4]. The third assertion is a consequence of (2) of Proposition A.2. □

To compute the first and the second fundamental forms of f in terms of fundamental data, the following Frenet-type formula for singular curves is convenient.

Lemma 4.8 (Izumiya-Saji-Takeuchi [7] and Fukui [3])**.** *The following formula holds (cf. (4.3)):*

$$\begin{pmatrix} \mathbf{e}' \\ \mathbf{v}_2' \\ \mathbf{v}_3' \end{pmatrix} = \begin{pmatrix} 0 & \kappa\cos\theta & \kappa\sin\theta \\ -\kappa\cos\theta & 0 & \tau-\theta' \\ -\kappa\sin\theta & -(\tau-\theta') & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e} \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{pmatrix}. \tag{4.7}$$

This formula can be rewritten as (cf. (4.3))

$$\begin{pmatrix} \mathbf{e}' \\ \mathbf{v}_2' \\ \mathbf{v}_3' \end{pmatrix} = \begin{pmatrix} 0 & \kappa_s & \kappa_\nu \\ -\kappa_s & 0 & \kappa_t \\ -\kappa_\nu & -\kappa_t & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e} \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{pmatrix},$$

which is the one given in Izumiya-Saji-Takeuchi [7, Proposition 3.1], where κ_t is the *cusp-directional torsion* defined in [11] and has the expression (cf. [3, Page 7])

$$\kappa_t = \tau - \theta'. \tag{4.8}$$

Using Lemma 4.8, one can easily obtain the following by a straightforward computation:

Proposition 4.9 (Fukui [3])**.** *The first fundamental form $ds_f^2 = Edu^2 + 2Fdudt + Gdt^2$ of f as in (4.5) is given by*

$$\begin{aligned} E &= (1 - (A\cos\theta + B\sin\theta)\kappa)^2 + (A_u + (\theta' - \tau)B)^2 + (B_u - (\theta' - \tau)A)^2, \\ F &= A_t(A_u + (\theta' - \tau)B) + B_t(B_u - (\theta' - \tau)A), \quad G = t^2, \end{aligned} \tag{4.9}$$

where κ, τ, θ are functions of u and A, B are functions of (u, t).

Proof. Differentiating $f = \mathbf{c} + A\mathbf{v}_2 + B\mathbf{v}_3$, we have

$$\begin{aligned} f_u &= (1 - (A\cos\theta + B\sin\theta)\kappa)\mathbf{e} + (A_u + (\theta' - \tau)B)\mathbf{v}_2 + (B_u - (\theta' - \tau)A)\mathbf{v}_3, \\ f_t &= A_t\mathbf{v}_2 + B_t\mathbf{v}_3. \end{aligned}$$

Since $E = f_u \cdot f_u$, $F = f_u \cdot f_t$ and $G = f_t \cdot f_t$, we obtain the assertion. □

We can write

$$\hat{\mu}(u, t) = \mu_0(u) + \mu_1(u)t + \mu_2(u)t^2 + \mu_3(u, t)t^3,$$

and then Lemma A.1 yields that

$$A = \frac{t^2}{2} - \frac{\mu_0(u)^2}{8}t^4 - \frac{\mu_0(u)\mu_1(u)}{10}t^5 + t^6 a_6(t, u), \tag{4.10}$$

$$B = \frac{\mu_0(u)}{3}t^3 + \frac{\mu_1(u)}{8}t^4 + \frac{2\left(-\mu_0(u)^3 + 2\mu_2(u)\right)}{30}t^5 + t^6 b_6(t, u), \tag{4.11}$$

where $a_6(t, u)$ and $b_6(t, u)$ denote C^r-functions.

Corollary 4.10. *The Gaussian curvature K of ds_f^2 satisfies*

$$K(u, t) = \frac{K_0(u)}{t} + K_1(u) + K_2(u)t + K_3(u, t)t^2,$$

where

$$\begin{aligned} K_0 &:= \mu_0\kappa_\nu, \quad K_1 := -\kappa_s\mu_0^2 - \kappa_t^2 + \kappa_\nu\mu_1, \\ K_2 &:= -\frac{\kappa_\nu\mu_0^3}{2} + \frac{\kappa_s\kappa_\nu\mu_0}{2} - \frac{3\kappa_s\mu_0\mu_1}{2} + \kappa_\nu\mu_2 - 2\mu_0'\kappa_t + \frac{\mu_0}{2}\kappa_t', \end{aligned}$$

and $K_3(u, t)$ is a C^r-function. Here κ_s, κ_ν and κ_t are defined in (0.4) and (4.8). Moreover, $\mu_0 = \kappa_c/2$ (cf. (1.5)) and $\kappa_t' = d\kappa_t(u)/du$.

Fukui [3, Theorem 1.8] has already determined the first two terms K_0 and K_1. So the essential part of the above corollary is the statement for K_2.

Proof. One can obtain this formula by computing the sectional curvature of ds_f^2, or alternatively, one can get it by computing the second fundamental form of f as Fukui did in [3]. In each approach, (4.10) and (4.11) play crucial roles. □

As a consequence of this corollary, the first term

$$K_0 := \mu_0\kappa_\nu = \frac{\kappa_c\kappa_\nu}{2}$$

defined in [12] is an intrinsic invariant, which is called the *product curvature*. The second term K_1 is an intrinsic invariant. We consider the term K_2. Since $K_0 = \kappa_c\kappa_\nu/2$, and since μ_0 is equal to the cuspidal curvature κ_c, the fact that κ_s and $\kappa_c\kappa_\nu$ are intrinsic yields that

$$\tilde{K}_2 := -\frac{\kappa_\nu\mu_0^3}{2} - \frac{3\kappa_s\mu_0\mu_1}{2} + \kappa_\nu\mu_2 - 2\mu_0'\kappa_t + \frac{\mu_0}{2}\kappa_t'$$

is also an intrinsic invariant. Using this, we can prove the following assertion:

Proposition 4.11. *Let $f \in \mathcal{G}^r(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$ be the generalized cuspidal edge associated to a fundamental data $(\kappa, \tau, \theta, \hat{\mu})$ satisfying $\sin\theta \neq 0$. Then*

(1) *f gives a cuspidal edge along the u-axis if $K_0(u) \neq 0$,*
(2) *f gives a cuspidal cross cap at $u = 0$ if $K_0(0) = 0$ and $dK_0(0)/du = 0$, and*
(3) *f gives a $5/2$-cuspidal edge along the u-axis if $K_0(u) = 0$ and $K_2(u) \neq 0$.*

In particular, these conditions depend only on the first fundamental form of f.

Proof. Since $\sin\theta(u) \neq 0$, we have $\kappa_\nu(u) \neq 0$. Since $K_0 = \mu_0\kappa_\nu$, $K_0(u) = 0$ if and only if $\mu_0(u) = 0$. Since $\mu_0(u) = \hat{\mu}(u,0)(= \kappa_c(u))$, the first and second assertions follow from (1) and (2) of Proposition 4.7, respectively. On the other hand, if $\mu_0(= \kappa_c)$ is identically zero, then $K_2 = \kappa_\nu\mu_2$. So $K_2(u) \neq 0$ if and only if $\mu_2(u) \neq 0$. Thus, the third assertion immediately follows from (3) of Proposition 4.7. □

We now prove Fact 1.1 in the introduction.

Proof of Fact 1.1. Since $\sin\theta \neq 0$ if and only if $\kappa_\nu \neq 0$, the assertions (1) and (2) follow from Theorem 3.8. We next prove (3). We remark that

$$\begin{aligned}
\mathcal{K}^\omega_*(\boldsymbol{R}^2_o) &= \{ds^2_f \in \mathcal{K}^\omega_{\mathrm{I}}(\boldsymbol{R}^2_o)\,;\, K_0(0) \neq 0\},\\
\mathcal{K}^\omega_{p,*}(\boldsymbol{R}^2_o) &= \{ds^2_f \in \mathcal{K}^\omega_{\mathrm{I}}(\boldsymbol{R}^2_o)\,;\, K_0(0) = 0,\ dK_0(0)/du \neq 0\},\\
\mathcal{K}^\omega_{a,*}(\boldsymbol{R}^2_o) &= \{ds^2_f \in \mathcal{K}^\omega_{\mathrm{I}}(\boldsymbol{R}^2_o)\,;\, K_0(u) = 0,\ K_2(0) \neq 0\}
\end{aligned}$$

hold in terms of our coordinates (u,t). We have shown the following (cf. Propositions 4.7 and 4.11).

- $K_0(0) \neq 0$ if and only if $\mu_0(0)(= \kappa_c(0)) \neq 0$.
- $K_0(0) = 0$ and $dK_0(0)/du \neq 0$ if and only if $\mu_0(0)(= \kappa_c(0)) = 0$ and $d\mu_0(0)/du \neq 0$.
- $K_0(u) = 0$ and $K_2(0) \neq 0$ if and only if $\mu_0(u) = 0$ and $\mu_2(0) \neq 0$.

By Corollary 3.2, the following assertions hold:

- $\hat{K}(o) \neq 0$ if and only if $K_0(0) \neq 0$.
- $\hat{K}(o) = 0$ and $\partial\hat{K}(o)/\partial u \neq 0$ if and only if $K_0(0) = 0$ and $dK_0(0)/du \neq 0$.

So the first fundamental form ds^2_f of f belongs to $\mathcal{K}^\omega_*(\boldsymbol{R}^2_o)$ (resp. $\mathcal{K}^\omega_{p,*}(\boldsymbol{R}^2_o)$) if and only if $\mu_0(0)(= \kappa_c(0)) \neq 0$ (resp. $\mu_0(0)(= \kappa_c(0)) = 0$ and $d\mu_0(0)/du \neq 0$). On the other hand, ds^2_f belongs to $\mathcal{K}^\omega_{a,*}(\boldsymbol{R}^2_o)$ if and only if $\mu_0(u) = 0$ and $\mu_1(0) \neq 0$. In fact, $\eta := \partial/\partial t$ gives the null direction of f along the u-axis (as the singular curve of ds^2_f), and we have (cf. (1.2)) $dK(\eta) = K_t(u,0) = K_2(u)$. □

Finally, we consider the cuspidal edges with vanishing limiting normal curvature: A cuspidal edge is called *asymptotic* if its first fundamental form is asymptotic (see Section 1), which is equivalent to the condition that the cuspidal angle $\theta(u)$ of f is constantly equal to 0 or π along its edge.

If f is an asymptotic cuspidal edge, the singular curvature κ_s, limiting normal curvature κ_ν and cusp-directional torsion κ_t satisfy

$$\kappa_s = \varepsilon\kappa, \quad \kappa_\nu = 0, \quad \kappa_t = \tau, \tag{4.12}$$

where $\varepsilon := \cos\theta \,(\in \{1, -1\})$. So we get the following:

Proposition 4.12. *Let $f \in \mathcal{G}^r_{3/2}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$ be a cuspidal edge associated to a fundamental data $(\kappa, \tau, \theta, \hat{\mu})$. If $\sin\theta$ vanishes identically, then*

1. *the limiting normal curvature κ_ν vanishes identically,*
2. *the first fundamental form of f is an asymptotic Kossowski metric, and*
3. *the Gaussian curvature K of f can be extended across its singular set as a C^r-function.*

Moreover, the sign of K coincides with the sign of $(K_1 =) - \varepsilon\kappa\mu_0^2 - \tau^2$ whenever $K_1 \neq 0$, where $\varepsilon := \cos\theta$.

As an application, we first consider the case K vanishes identically.

Corollary 4.13. *Let $f \in \mathcal{G}^r_{3/2}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$ be the cuspidal edge whose Gaussian curvature K vanishes identically. Then C is a regular space curve whose torsion function does not vanish, and f is the tangential developable of C. In particular, f has no isomers.*

Proof. Since K vanishes identically, the identity $-\varepsilon\kappa\mu_0^2 = \tau^2$ holds along C. Since f is a cuspidal edge, μ_0 has no zeros, and the left hand side does not vanish. Thus, the torsion function τ of C also has no zeros. Since f is a wave front, its principal directions along C are well-defined (cf. [13, Proposition 1.6]). Moreover, each singular point of f is disjoint from umbilical set (cf. [13, Proposition 1,10]), and the zero principal curvature direction is uniquely determined at each point of C. Moreover, it can be easily seen that this direction must be the tangential direction of C. Since K vanishes identically, f must be a ruled surface (cf. [13, Proposition 2.2]), so it must be the tangential developable of C. □

Remark 4.14. The standard cuspidal edge $f_0(t) = (u^2, u^3, v)$ does not satisfy the assumption of Corollary 4.13, since the singular set image is a line.

We next consider the case $K > 0$. If $\theta = \pi$ and μ_0 is sufficiently large, then the Gaussian curvature K near the singular set can be positive. So we can construct cuspidal edges with $K > 0$. The following assertion is an immediate consequence of Proposition 4.12.

Corollary 4.15. *Let $f \in \mathcal{G}^r_{3/2}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$ be the cuspidal edge whose Gaussian curvature K is bounded near singular set and positive, then it is asymptotic satisfying $\theta = \pi$ and $\kappa_s < 0$.*

The negativity of κ_s has been pointed out in [16]. Although Theorem 3.8 does not cover the case $\kappa_\nu = 0$, Brander [1] showed the existence of cuspidal edges in the case of $K = 1$ along a given space curve C of $\kappa_\nu > 0$ using the loop group theory.

5. Relationships among isomers

In this section, we show several properties of isomers, and prove the last two statements in the introduction. We fix a space curve $\mathbf{c}(u)$ satisfying $\mathbf{c}(0) = \mathbf{0}$ which is parametrized by arc-length defined on a closed interval $J := [-l, l]$ $(l > 0)$ whose curvature function $\kappa(u)$ is positive everywhere. We prove the following:

Proposition 5.1. *Let $f \in \mathcal{G}^\omega_{*,3/2}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$. Then $\check{f}$ is congruent (cf. Definition 0.2) to f if and only if*

1. *C lies in a plane, or*

(2) *C has a positive non-trivial symmetry and the first fundamental form ds^2_f has an effective symmetry* (*cf. Definition 0.4*).

Proof. We suppose that $\check{f}$ is congruent to f. By Remark 4.5, it is sufficient to consider the case that C does not lie in any plane. By Remark 0.5, there exist an isometry T on $\boldsymbol{R}^3$ and a diffeomorphism φ defined on a neighborhood of the singular curve of f such that

$$T \circ f \circ \varphi = \check{f}. \tag{5.1}$$

We consider the case that T fixes each point of C. Then C must lie in a plane, a contradiction. So T is a non-trivial symmetry of C, that is, it reverses the orientation of C. We suppose that T is a negative symmetry. Then (b) of Remark 4.4 implies that the image of f coincides with that of $T \circ f$. Since the image of $\check{f}$ is different from that of f, this case never happens. So T must be a positive symmetry, and then φ gives an effective symmetry of ds^2_f.

Conversely, if C has a positive non-trivial symmetry and the first fundamental form ds^2_f has an effective symmetry φ, then $T \circ f \circ \varphi$ is a faithful isomer of f as seen in (a) of Remark 4.4. Since such an isomer is uniquely determined (cf. Theorem 3.8), we have (5.1). □

Remark 5.2. Suppose that C is planar and S is the reflection with respect to the plane containing C. For each $f \in \mathcal{G}^r_{*,3/2}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$, $S \circ f$ gives a faithful isomer of f. Moreover, if f is real analytic (i.e. $r = \omega$), then we have $\check{f} = S \circ f$ (cf. Definition 3.18).

Example 5.3. Let $f \in \mathcal{G}^\infty_*(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$ be an admissible generalized cuspidal edge whose fundamental data is $(\kappa, \tau, \theta, \hat{\mu})$ $(\tau \neq 0)$. Suppose that κ, τ and θ are constant, and the extended half-cuspidal curvature function $\hat{\mu}$ does not depend on u. In this case, without assuming the real analyticity of f, we can show the existence of an isometry $T \in \mathrm{SO}(3)$ and an effective symmetry φ of ds^2_f such that $T \circ f \circ \varphi$ gives a faithful isomer of f as follows: In fact, in this case C has the constant curvature κ and the constant torsion τ. Since $\tau \neq 0$, C is a helix in $\boldsymbol{R}^3$ and there exists a 180°-rotation $T \in \mathrm{SO}(3)$ with respect to the principal normal line at $\mathbf{0} \in C$ such that $T(C) = C$. By the first part of Proposition 5.10, it is sufficient to show that the first fundamental form

$$ds^2_f = E(t)du^2 + 2F(t)dudt + G(t)dt^2$$

of f admits an effective symmetry φ as an involution. In fact, if such a φ exists, then $(\check{f} :=) T \circ f \circ \varphi$ gives the isometric dual of f. In this situation, two functions A, B can be expressed as (cf. (4.9) and (4.6)) $A(t) := t^2\alpha(t)$ and $B(t) := t^3\beta(t)$, where $\alpha(t)$ and $\beta(t)$ are C^r-functions. By Proposition 4.9,

- $E(t)$ is positive for each t,
- there exists a C^∞-function $F_0(t)$ such that $F(t) = t^4F_0(t)$, and $G(t) = t^2$.

Setting

$$\omega_1 = \sqrt{E(t)}\left(du + \frac{F(t)}{E(t)}dt\right), \qquad \omega_2 = t\sqrt{\frac{E(t) - t^6F_0(t)^2}{E(t)}}dt,$$

we have $ds^2_f = (\omega_1)^2 + (\omega_2)^2$. Moreover, if we set

$$x(u,t) := u + \int_0^t \frac{F(v)}{E(v)}dv, \qquad y(t) := \int_0^t \sqrt{\frac{E(v) - v^6F_0(v)^2}{E(v)}}dv. \tag{5.2}$$

Then we can take (x, y) as a new local coordinate system centered at $(0,0)$, and t can be considered as a function of y. So we can write $t = t(y)$, and

$$ds^2_f = E(y)dx^2 + t(y)^2dy^2.$$

So the local diffeomorphism $\varphi : (x, y) \mapsto (-x, y)$ gives an effective symmetry of ds^2_f.

Regarding the fact that the fundamental data of f is $(\kappa, \tau, \theta, \mu)$, we show in later that $\check{f}$ is right equivalent to the cuspidal edge whose fundamental data of $(\kappa, \tau, -\theta, \mu)$, see Proposition 6.1.

Proof of Theorem III. Suppose that ds^2_f admits a symmetry φ. Then this symmetry is effective (cf. Corollary 3.16). So, $f \circ \varphi$ and $\check{f} \circ \varphi$ must be right equivalent to $\check{f}_*$ and f_*, respectively. In particular, the number of right equivalence classes of $f, \check{f}, f_*, \check{f}_*$ is two.

Conversely, we suppose that two of $\{f, \check{f}, f_*, \check{f}_*\}$ are right equivalent. Replacing f by $\check{f}$, f_*, $\check{f}_*$, we may assume that one of the right equivalent pair is f and the other is $g \in \{\check{f},\ f_*,\ \check{f}_*\}$. Without loss of generality, we may assume that f is written in a normal form. Since $\check{f}$ cannot be right equivalent to f, the map g must be right equivalent to f_* or $\check{f}_*$, that is, there exists a local diffeomorphism φ such that $g = f \circ \varphi$, which implies $\varphi^* ds^2_f = ds^2_f$. If φ is an identity map, then $g = f$ holds. However, it contradicts the fact that $u \mapsto f(u, 0)$ and $u \mapsto f_*(u, 0) = \check{f}_*(u, 0)$ give mutually distinct orientations to C. So, by Corollary 3.16, φ must be an effective symmetry of ds^2_f. □

Corollary 5.4. *Let $f \in \mathcal{G}^{\omega}_{**,3/2}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$. Suppose that*

(1) *C is planar and does not admit any non-trivial symmetry at $\mathbf{0}$, and*
(2) *ds^2_f admits no effective symmetries (cf. Definition 0.4).*

Then

- *$\check{f} := S \circ f$ holds, where $S \in \mathrm{O}(3)$ is the reflection with respect to the plane containing C,*
- *the isometric dual, inverse and the inverse dual are given by $S \circ f$, f_* and $S \circ f_*$, respectively. Moreover, f_* is not congruent to f.*

In particular, the four maps consist of two congruence classes.

Proof. As seen in Remark 5.2, $\check{f} := S \circ f$ holds. We next prove the second assertion. Since C lies in a plane, $\mathcal{I}_C(f) = S \circ f$ holds. By applying Theorem II, the right equivalence classes of $\mathcal{J}_C^{-1}(\mathcal{J}_C(f))$ are represented by $\{f,\ S \circ f,\ f_*,\ S \circ f_*\}$. It is sufficient to show that f_* is not congruent to f. If not, then, by Remark 0.5, there exist $T \in \mathrm{O}(3)$ and a diffeomorphism φ defined on a neighborhood of the singular curve of f such that $T \circ f_* \circ \varphi = f$. In particular, $\varphi^* ds^2_f = ds^2_f$ holds. By (1), T is not non-trivial. So, φ must be an effective symmetry, contradicting (2). □

We next consider the case that ds^2_f has an effective symmetry.

Proposition 5.5. *Let $f \in \mathcal{G}^{\omega}_{**,3/2}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$. Suppose that*

(1) *C is non-planar and does not admit any non-trivial symmetry at $\mathbf{0}$,*
(2) *ds^2_f admits an effective symmetry φ.*

Then $\check{f}(:= \mathcal{I}_C(f))$ is not congruent to f, and $\check{f}$, $\check{f} \circ \varphi$ and $f \circ \varphi$ give the isometric dual, inverse and inverse dual, respectively.

Proof. By Proposition 5.1, $\check{f}$ is not congruent to f. Since $\check{f} \circ \varphi$ (resp. $f \circ \varphi$) has the same first fundamental form as f, the fact that φ is effective yields that it coincides with either f_* or $\check{f}_*$. Since the cuspidal angle of $\check{f} \circ \varphi$ (resp. $f \circ \varphi$) takes the opposite sign (resp. the same sign) of that of f (cf. Remark 4.5), we have $f_* = \check{f} \circ \varphi$ (resp. $\check{f}_* = f \circ \varphi$). □

Corollary 5.6. *Let $f \in \mathcal{G}^{\omega}_{**,3/2}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$. Suppose that*

(1) *C is planar and does not admit any non-trivial symmetry at the origin $\mathbf{0}$,*
(2) *ds^2_f admits an effective symmetry φ.*

Then

- $\check{f} = S \circ f$ *holds, where* $S \in \mathrm{O}(3)$ *is the reflection with respect to the plane containing* C.
- *Moreover,* $S \circ f$, $S \circ f \circ \varphi$, $f \circ \varphi$ *give the isometric dual, inverse and inverse dual, respectively.*

As a consequence, all of isomers are congruent to f.

Proof. As we have seen in Remark 5.2, $\check{f} = S \circ f$ holds. Since $S \circ f \circ \varphi$ (resp. $f \circ \varphi$) has the same first fundamental form as f, the fact that φ is effective yields it coincides with f_* or $\check{f}_*$. Since the sign of cuspidal angle of $S \circ f \circ \varphi$ (resp. $f \circ \varphi$) along the curve $\mathbf{c}_\#(u) := \mathbf{c}(-u)$ takes the opposite sign (resp. the same sign) of that of f, we have $f_* = S \circ f \circ \varphi$ (resp. $\check{f}_* = f \circ \varphi$). Finally, it is obvious that the four maps are congruent. So the proposition is proved. $\square$

We then consider the case that C has a non-trivial symmetry.

Proposition 5.7. *Let* $f \in \mathcal{G}^{\omega}_{**,3/2}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$. *Suppose that*

(1) C *is non-planar and admits a non-trivial symmetry* $T \in \mathrm{O}(3)$ *at* $\mathbf{0}$,
(2) ds^2_f *does not admit any effective symmetries.*

Then

- $\check{f} := \mathcal{I}_C(f)$ *is not congruent to* f, *and*
- $T \circ \check{f}$, $T \circ f$ *are the inverse and inverse dual, respectively.*

In particular, f, $\check{f}$, $T \circ \check{f}$ *and* $T \circ f$ *consist of two congruence classes.*

Proof. By Proposition 5.1, $\check{f}$ is not congruent to f. So the assertion can be shown easily. $\square$

We get the following corollary.

Corollary 5.8. *Let* $f \in \mathcal{G}^{\omega}_{**,3/2}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$. *Suppose that* C *lies in a plane and admits a non-trivial symmetry* T *at the origin* $\mathbf{0}$. *Then* $\check{f} = S \circ f$ *holds, and* $T \circ f$, $S \circ T \circ f$ *give the inverse and the inverse dual of* f, *where* S *is a reflection with respect to the plane. As a consequence,* $f, \check{f}, f_*, \check{f}_*$ *belong to a single congruence class.*

Proof. Obviously, $\check{f} = S \circ f$ holds (cf. Remark 5.2). On the other hand, $T \circ f$ gives a non-faithful isomer, and its isometric dual $S \circ T \circ f$ also gives another non-faithful isomer. $\square$

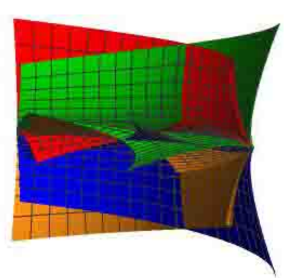

FIGURE 2. The four cuspidal edges given in Example 5.9

Example 5.9. We set

$$f(u,v) := \Big(\varphi(u,v)\cos u - 1,\ \varphi(u,v)\sin u,\ v^3 u + 2v^3 - v^2\Big),$$

where $\varphi(u,v) := -v^3u - 2v^3 - v^2 + 1$. Then, it has cuspidal edge singularities along

$$\mathbf{c}(u)\,(:= f(u,0)) = (\cos u - 1, \sin u, 0).$$

By setting,

$$S := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad T := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$S \circ f$ is the faithful isomer, and $T \circ f$, $TS \circ f$ are non-faithful isomers. We remark that f is associated to Fukui's data (θ, A, B) given by

$$\theta = \frac{\pi}{4}, \quad A(u,v) := \sqrt{2}v^2, \quad B(u,v) := \sqrt{2}v^3(u+2).$$

Finally, we consider the case that C and ds^2_f admit a symmetry and an effective symmetry, respectively.

Proposition 5.10. *Let $f \in \mathcal{G}^{\omega}_{**,3/2}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$. Suppose that*

(1) *C is non-planar and admits a non-trivial symmetry $T \in \mathrm{O}(3)$ at $\mathbf{0}$,*
(2) *ds^2_f admits an effective symmetry φ.*

Then any isomer of f is right equivalent to one of $\check{f}$, $\check{f} \circ \varphi$, $f \circ \varphi$. Moreover,

- *if T is positive (i.e. $T \in \mathrm{SO}(3)$), then $\check{f} = T \circ f \circ \varphi$, and*
- *if T is negative (i.e. $T \notin \mathrm{SO}(3)$), then $\check{f}$ is not congruent to f.*

Proof. We set $g := T \circ f \circ \varphi$. If T is positive, then g is a faithful isomer of f as shown in Remark 4.4. On the other hand, if T is negative, then $\check{f}$ is not congruent to f by Proposition 5.1 and so it not congruent to f. □

Proof of Theorem IV*.* We suppose that C has no non-trivial symmetries, and also ds^2_f has no symmetries. If two of $\{f, \check{f}, f_*, \check{f}_*\}$ are mutually congruent, replacing f by one of its isomers, we may assume that f is congruent to g, where g is one of $\{\check{f}, f_*, \check{f}_*\}$. By Proposition 5.1, we may assume that $g = f_*$ or $g = \check{f}_*$. Suppose that g is congruent to f. Then (cf. Remark 0.5) there exist a non-trivial symmetry $T \in \mathrm{O}(3)$ of C and a local diffeomorphism φ such that

$$T \circ g \circ \varphi = f.$$

Since C has no non-trivial symmetries, and ds^2_f has also no symmetries, φ is the identity map and T is not a non-trivial symmetry. However, this contradicts the fact that $u \mapsto f(u,0)$ and $u \mapsto f_*(u,0) = \check{f}_*(u,0)$ give mutually distinct orientations to C. So we obtained (1).

The assertion (2) follows from Corollaries 5.4, 5.6, 5.8 and Propositions 5.5, 5.7, and 5.10, by using the fact that any symmetries of ds^2_f are effective (cf. Corollary 3.16).

Finally, suppose that $N_f = 1$. We first consider the case that C lies in a plane. If C has no non-trivial symmetries and ds^2_f has also no symmetries, then $N_f = 2$ holds by Corollary 5.4. So either C or ds^2_f has a symmetry. If C has a symmetry, then $N_f = 1$ by Corollary 5.8 (this corresponds to the case (a)). On the other hand, if C has no non-trivial symmetries and ds^2_f also has a symmetry φ, then φ is effective (cf. Corollary 3.16). So, Corollary 5.6 yields that $N_f = 1$. (This corresponds to the case (b). In fact, we denote by T_0 the reflection with respect to the plane containing C. We let T_1 be a non-trivial symmetry of C. If T_1 is positive, then (b) holds obviously. On the other hand, if T_1 is negative, then $T_0 \circ T_1$ is a positive symmetry and (b) holds.)

So we may assume that C does not lie in any planes. The assumption $N_f = 1$ implies $\check{f}$ must congruent to f. By Proposition 5.1, this holds only when (c) happens, since C does not lie in any planes. □

6. Examples

One method to give a numerical approximation of a isometric dual g of a real analytic cuspidal edge f is to determine the Taylor expansion of $g(u,v)$ at $v=0$ along the u-axis as a singular set so that $g=\mathcal{I}_C(f)$. In [14, Page 85], we give a numerical approximation of the isometric dual of

$$f_0(u,v)=\left(u,-\frac{v^2}{2}+\frac{u^3}{6},\frac{u^2}{2}+\frac{u^3}{6}+\frac{v^3}{6}\right).$$

We denote by C the image of singular curve $u\mapsto f_0(u,0)$. In the figure of the isometric dual $g_0=\mathcal{I}_C(f_0)$ given in [14, Figure 2], the surface g_0 seems like it is lying on the almost opposite side of f_0. This is the reason why the cuspidal angle $\theta(u)$ of $f_0(u,v)$ is $\pi/2$ at $u=0$. The red lines of Figure 3 (left) indicates the section of f_0, g_0 at $u=-1/4$. The orange (resp. blue) surface corresponds to f_0 (resp. g_0). We can recognize that the cuspidal angle takes value less than $\pi/2$, that is, the normal direction of g_0 is linearly independent of that of f_0 at $(u,v)=(-1/4,0)$. On the other hand, Figure 3 (right) indicates the images of the numerical approximations of the two non-faithful isomers f_1, g_1 of f_0.

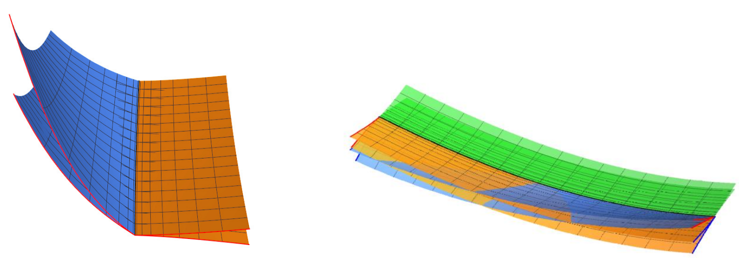

Figure 3. The images of f_0, g_0 (left), and the images of f_0, f_1, g_1 (right), where f_0 is indicated as the orange surfaces.

By Proposition 4.9, one can easily observe that the first fundamental form of $f_{-\theta}$ does not coincide with that of f_θ. This means that the image of $f_{-\theta}$ cannot coincide with that of f_θ nor $\check{f}_\theta$. However, one might expect the possibility that $f_{-\theta}$ is an isomer of f_θ. Here, we consider the case that the space curve C has a non-trivial symmetry T. In this case, we know that f, $\check{f}$, $T\circ f$, $T\circ\check{f}$ are only the possibilities of isomers. Thus, if $f_{-\theta}$ is an isomer of f_θ, then it must be congruent to either f or $\check{f}$. We give here the following two propositions which are related to one of these possibilities (by the following Proposition 6.1, Example 5.3 is just the case that $f_{-\theta}$ is right equivalent to $\check{f}$.)

Proposition 6.1. *Let C be a space curve which admits a non-trivial symmetry $T\in\mathrm{SO}(3)$ at $\mathbf{0}$, and let $f:=f_\theta\in\mathcal{G}^\infty(\boldsymbol{R}^2_J,\boldsymbol{R}^3,C)$ be a generalized cuspidal edge as in the formula (4.1) such that*

- *$T\circ f(-u,0)=f(u,0)$, and*
- *the cuspidal angle θ satisfies $\theta(u)=\sigma\theta(-u)$ where $\sigma\in\{+,-\}$.*

Suppose that $A(u,v)$ and $B(u,v)$ satisfy one of the following two conditions:

(1) *$A(-u,-v)=A(u,v)$ and $B(-u,-v)=-B(u,v)$ or*
(2) *$A(-u,v)=A(u,v)$ and $B(-u,v)=-B(u,v)$.*

Then $f_\theta = T\circ f_{-\sigma\theta}\circ\varphi$ holds, where $\varphi(u,v)=(-u,-v)$ (resp. $\varphi(u,v)=(-u,v)$) in the case of (1) (resp. (2)). In particular, $f_{-\theta}$ is a right equivalent to $\check{f}$ if $\sigma=+$, and the image of f is invariant under T if $\sigma=-$.

Proof. We consider the case $\sigma=+$, that is, $\theta(u)=\theta(-u)$. Since $T\circ\mathbf{c}(-u)=\mathbf{c}(u)$ and $T\in \mathrm{SO}(3)$ (cf. Remark 4.4),

$$-T\mathbf{e}(-u)=\mathbf{e}(u),\quad T\mathbf{n}(-u)=\mathbf{n}(u),\quad \mathbf{b}(u)=-T\mathbf{b}(-u).$$

In the case of (1) (resp. (2)), we set $\varphi(u,v):=(-u,-v)$ (resp. $\varphi(u,v):=(-u,v)$). Then $A\circ\varphi(u,v)=A(u,v)$ and $B\circ\varphi(u,v)=-B(u,v)$ hold, and so

$$\begin{aligned}T\circ f_\theta\circ\varphi &= \mathbf{c}+(A,-B)\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\mathbf{n}\\ -\mathbf{b}\end{pmatrix}\\ &= \mathbf{c}+(A,B)\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\mathbf{n}\\ \mathbf{b}\end{pmatrix}=f_{-\theta},\end{aligned}$$

proving the relation $f_\theta=T\circ f_{-\sigma\theta}\circ\varphi$. The case $\theta(u)=-\theta(-u)$ is proved in the same way.

We then consider the case that $\sigma=1$. In this case, $f_\theta=T\circ f_{-\theta}\circ\varphi$ holds. Since T is an isometry of $\boldsymbol{R}^3$, we have $\varphi^* ds^2_f = ds^2_g$, where $f:=f_{-\theta}$ and $g=f_{-\theta}$. So g is isometric to f. Since the cuspidal angle of g takes the opposite sign of that of f, the image of g does not coincide with f. So g is a faithful isomer of f. Then the uniqueness of the faithful isomer of f (cf. Theorem 3.8) yields that g is right equivalent to $\check{f}$. □

Similarly, the following assertion holds.

Proposition 6.2. *Let C be a space curve which admits a non-trivial symmetry $T\in \mathrm{O}(3)\setminus\mathrm{SO}(3)$ at $\mathbf{0}$, and let $f:=f_\theta\in\mathcal{G}^\infty(\boldsymbol{R}^2_J,\boldsymbol{R}^3,C)$ be a generalized cuspidal edge as in the formula (4.1) such that*

- *$T\circ f(-u,0)=f(u,0)$, and*
- *the cuspidal angle θ satisfies $\theta(u)=\sigma\theta(-u)$, where $\sigma\in\{+,-\}$.*

Suppose that $A(u,v)$ and $B(u,v)$ satisfy one of the following two conditions:

1. *$A(-u,-v)=A(u,v)$ and $B(-u,-v)=B(u,v)$,*
2. *$A(-u,v)=A(u,v)$ and $B(-u,v)=B(u,v)$.*

Then $f_\theta=T\circ f_{\sigma\theta}\circ\varphi$ holds, where $\varphi(u,v)=(-u,-v)$ (resp. $\varphi(u,v)=(-u,v)$) in the case of (1) (resp. (2)). In particular, $f_{-\theta}$ is right equivalent to $\check{f}$ if $\sigma=-$, and the image of f is invariant under T if $\sigma=+$.

Proof. Like as in the case of the proof of Proposition 6.1, $-T\mathbf{e}(-u)=\mathbf{e}(u)$ and $T\mathbf{n}(-u)=\mathbf{n}(u)$ hold. Since $\det(T)=-1$, we have $T\mathbf{b}(-u)=\mathbf{b}(u)$. In the case of (1) (resp. (2)), we set $\varphi(u,v):=(-u,-v)$ (resp. $\varphi(u,v):=(-u,v)$), then the relation $f_\theta=T\circ f_{\sigma\theta}\circ\varphi$ is obtained like as in the case of the proof of Proposition 6.1. One can also obtain the last assertion imitating the corresponding argument in the proof of Proposition 6.1. □

Example 6.3. Let a,b be real numbers so that $a>0$ and $b\neq 0$. Then

$$\mathbf{c}(u):=\left(a\cos\left(\frac{u}{c}\right)-a,\ a\sin\left(\frac{u}{c}\right),\ \frac{bu}{c}\right)\qquad(u\in\boldsymbol{R})$$

gives a helix of constant curvature $\kappa:=a/c^2$ and constant torsion $\tau:=b/c^2$, where $c:=\sqrt{a^2+b^2}$. At the point $\mathbf{0}:=\mathbf{c}(0)$ on the helix, $\mathbf{c}$ satisfies $T(\mathbf{c}(\boldsymbol{R}))=\mathbf{c}(\boldsymbol{R})$, where $T\in\mathrm{SO}(3)$ is the $180°$-rotation with respect to the line passing through the origin $\mathbf{0}$ which is parallel to the principal normal vector $\mathbf{n}(0)$. We set $a=b=1$, $\theta=\pi/4$. By setting

$$(A_1,B_1):=(v^2,v^3),\quad (A_2,B_2):=(v^2,v^5),\quad (A_3,B_3):=(v^2,uv^3).$$

The surfaces $g_{i,\pm} := f_{\pm\pi/4}$ $(i = 1, 2, 3)$ associated to the Fukui data $(\mathbf{c}, \pm\pi/4, A_i, B_i)$ correspond to cuspidal edges, 5/2-cuspidal edges, and cuspidal cross caps, respectively. The first two cases satisfy (1) of Proposition 6.1 and the third case satisfies (2) of Proposition 6.1. So $g_{i,-}$ $(i = 1, 2, 3)$ is a faithful isomer of $g_{i,+}$.

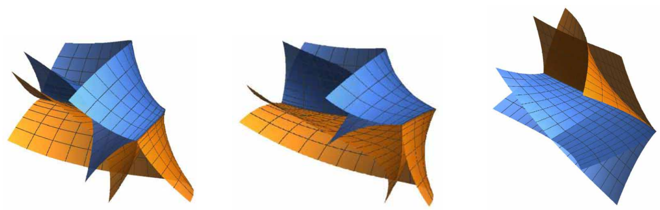

FIGURE 4. The images of cuspidal edges $g_{1,\pm}$ (left), 5/2-cuspidal edges $g_{2,\pm}$ (center) and cuspidal cross caps $g_{3,\pm}$ (right) given in Example 6.3.
(The orange surfaces correspond to $g_{i,+}$ and the blue surfaces correspond to $g_{i,-}$ for $i = 1, 2, 3$.)

Finally, we consider the case of fold singularities:

Example 6.4. We let $\mathbf{c}(u)$ be a C^∞-regular space curve with positive curvature κ and torsion τ. If we set

$$g_\pm(u, v) := \mathbf{c}(u) + \frac{v^2}{2}(\cos\theta\mathbf{n}(u) \mp \sin\theta\mathbf{b}(u)),$$

then it can be easily checked that g_- is a faithful isomer of g_+, where θ is a constant. These two surfaces can be extended to the following regular ruled surfaces:

$$\tilde{g}_\pm = \mathbf{c}(u) + \frac{v}{2}(\cos\theta\mathbf{n}(u) \mp \sin\theta\mathbf{b}(u)).$$

APPENDIX A. A REPRESENTATION FORMULA FOR GENERALIZED CUSPS

A plane curve $\sigma : J \to \boldsymbol{R}^2$ is said to have a singular point at $t = t_0$ if $\dot\sigma(t_0) = \mathbf{0}$ (the dot means d/dt). The singular point $t = t_0$ is called a *generalized cusp* if $\ddot\sigma(t_0) \neq \mathbf{0}$. In this situation, it is well-known that

(i) $t = t_0$ is a cusp if and only if $\ddot\sigma(t_0), \dddot\sigma(t_0)$ are linearly independent,
(ii) (cf. [15]) $t = t_0$ is a 5/2-cusp if and only if $\ddot\sigma(t_0), \dddot\sigma(t_0)$ are linearly dependent and

$$3\det(\ddot\sigma(t_0), \sigma^{(5)}(t_0))\ddot\sigma(t_0) - 10\det(\ddot\sigma(t_0), \sigma^{(4)}(t_0))\dddot\sigma(t_0) \neq \mathbf{0}.$$

From now on, we set $t_0 = 0$. The arc-length parameter $s(t)$ of σ given by

$$s(t) := \int_0^t |\dot\sigma(u)|du$$

is not smooth at $t = 0$, but if we set $w := \operatorname{sgn}(t)\sqrt{|s(t)|}$, then this gives a parametrization of σ near $t = 0$, which is called the *half-arc-length parameter* of σ near $t = 0$ in [17]. However, for our purpose, as Fukui [3] did, the parameter

$$v := \sqrt{2}w = \operatorname{sgn}(t)\left(2\int_0^t |\dot\sigma(u)|du\right)^{1/2} \tag{A.1}$$

called the *normalized half-arc-length parameter* is convenient, since it is compatible with the property $|f_{vv}| = 1$ for adapted coordinate systems (cf. Definition 3.4) of generalized cuspidal edges. This normalized half-arc-length parameter can be characterized by the property that $v^2/2$ gives the arc-length parameter of σ. Then by [17, Theorem 1.1], we can write

$$\sigma(v) = \int_0^v u(\cos\theta(u), \sin\theta(u))du, \qquad \theta(v) = \int_0^v \hat{\mu}(u)du. \tag{A.2}$$

We need the following lemma, which can be proved by a straightforward computation.

Lemma A.1. *Let v be the normalized half-arc-length parameter of the generalized cusp $\sigma(w)$ at $w = 0$. Then there exists an orientation preserving isometry T of $\boldsymbol{R}^2$ such that*

$$T \circ \sigma(v) = \Big(\frac{v^2}{2} - \frac{\mu_0^2 v^4}{8} - \frac{\mu_0\mu_1 v^5}{10}, \frac{\mu_0 v^3}{3} + \frac{\mu_1 v^4}{8} + \frac{(-\mu_0^3 + 2\mu_2)v^5}{30}\Big) + o(v^5), \tag{A.3}$$

where

$$\hat{\mu}(v) = \sum_{j=0}^{2} \mu_j v^j + o(v^3),$$

and $o(v^5)$ (resp. $o(v^3)$) is a term higher than v^5 (resp. v^3).

Using this with (i) and (ii), one can easily obtain the following assertion:

Proposition A.2. *Let v be the normalized half-arc-length parameter of the generalized cusp $\sigma(w)$ at $w = 0$. Then*

(1) *$w = 0$ is a cusp of σ if and only if $\mu_0 \neq 0$, and*
(2) *$w = 0$ is a $5/2$-cusp of σ if and only if $\mu_0 = 0$ and $\mu_2 \neq 0$.*

It is remarkable that the coefficient μ_1 does not affect the criterion for $5/2$-cusps. In this case, $\mu_0 = 0$ holds, and μ_1 and μ_2 are proportional to the "secondary cuspidal curvature" and the "bias" of $\sigma(t)$ at $t = 0$, respectively. Geometric meanings for these two invariants for $5/2$-cusps can be found in [6, Proposition 2.2].

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