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ON THE INDEX OF PRINCIPAL FOLIATIONS OF SURFACES IN $\mathbb{R}^3$ WITH CORANK 1 SINGULARITIES

J. C. F. COSTA, L. F. MARTINS, AND J. J. NUÑO-BALLESTEROS

Abstract. It is well known that the index associated to the principal foliations at a cross-cap point is $\frac{1}{2}$. In this work we study the index for other corank 1 singularities from $(\mathbb{R}^2, 0)$ to $(\mathbb{R}^3, 0)$ which either are simple or are non-simple but included in strata of $\mathcal{A}_e$-codimension ≤ 3. We show that the index, under certain conditions, is always 0 or 1, bearing out that the Loewner conjecture could be true for all corank 1 singularities.

1. Introduction

The classical Loewner conjecture states that the index of the binary differential equation (BDE) which represents the equation of the principal directions of a smooth immersed surface in $\mathbb{R}^3$ at an isolated umbilic point is always less than or equal to 1. The Loewner conjecture is a stronger version of the famous Carathéodory conjecture, which claims that every smooth convex embedding of a 2-sphere in $\mathbb{R}^3$ must have at least two umbilics. In fact, since the sum of the indices of the umbilics of a compact immersed surface is equal to its Euler-Poincaré characteristic (according to the Poincaré-Hopf formula) it follows that the Loewner conjecture implies the Carathéodory conjecture, not only for a convex embedding of a 2-sphere, but for any immersion. The Loewner conjecture is true in the analytic case (cf. [19, 30]) but the smooth case is still open, as far as we know.

A natural question is whether or not the Loewner conjecture is still true when we consider a singular surface parametrised as the image of a smooth non-immersive map germ $f : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$. In fact, if the non-immersive point is isolated then we have a well defined BDE for the principal directions outside the origin and it makes sense to consider the index of the singular point of the BDE. By definition, the corank of f is the dimension of the kernel of its differential at the origin. When f has corank 2, then it is known that this conjecture is false, since it is not difficult to construct a surface with an isolated singular point of index two (see [11], Remark 4.7). However, we believe that if f has corank 1, then the index is always less than or equal to one and hence, the Loewner conjecture is also true in this case. The main purpose of this paper is to analyse many examples which support this conjecture.

The family of examples we consider here is taken from Mond's classification in [23], where he gives a classification under $\mathcal{A}$-equivalence (that is, changes of coordinates in the source and target) of all smooth germs $f : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$ which either are simple or are non-simple but included in strata of $\mathcal{A}_e$-codimension ≤ 3. All map germs in this list have corank 1, so we can use them to test our conjecture. Of course this list is far from being a complete classification,

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but they are the most natural examples to begin with the analysis. Our main result is that the index, under certain conditions, is always 0 or 1 in all these examples (Theorem 3.4).

In final of the paper we also consider generic deformations of the singular surface. Let

$$f_\lambda : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0}), \quad \lambda \in (-\varepsilon, \varepsilon),$$

be a generic deformation of a corank 1 map germ f, i.e. $f_0 = f$, f_λ is generic for $\lambda \neq 0$ and the map $(\lambda, t) \mapsto f_\lambda(t)$ is smooth. D. Mond showed in [24] how to count the number of cross-caps in f_λ. Using his result, we estimate the number of umbilic points that appear on the image of f_λ in a neighbourhood of its singular point (Proposition 5.3).

Some references for index of BDE's are [4, 6, 7, 8, 20].

2. Surfaces with corank 1 singularities

We shall consider surfaces in $\mathbb{R}^3$ defined as the image of a corank 1 smooth map $f : U \to \mathbb{R}^3$, where U is an open subset of $\mathbb{R}^2$. The differential geometry of singular surfaces has been an object of interest in the past decades and it can be considered with different approaches (cross-caps or Whitney umbrellas, cuspidal edges, swallowtails or more general types of fronts, etc.) For example, see [3, 10, 14, 16, 17, 22, 25, 26, 27, 28]. See also [21], where the authors studied in depth the geometry of surfaces in $\mathbb{R}^3$ with corank 1 singularities.

From the Singularity Theory point of view, if we are concerned in corank 1 map germs $(\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$ up to $\mathcal{A}$-equivalence then we have a classification list given by D. Mond in [23]. The Mond's classification is summarized in Table 1 for either simple map germs or non-simple but included in strata of $\mathcal{A}_e$-codimension ≤ 3. When k is even, S_k^+ is equivalent to S_k^-, and C_k^+ to C_k^-.

We recall that two map germs $f, g : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$ are said to be $\mathcal{A}$-*equivalent*, denoted by $f \sim g$, if there exist germs of diffeomorphisms $h : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^2, \mathbf{0})$ and $k : (\mathbb{R}^3, \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$ such that $g = k \circ f \circ h^{-1}$. For more details about definitions and notations from Singularity theory used in this work (such as, $\mathcal{A}_e$-codimension, simple germs, etc.), see [31].

Table 1: $\mathcal{A}$-classes of corank 1 map germs $(\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$ either simple or non-simple but included in strata of $\mathcal{A}_e$-codimension ≤ 3 (cf. [23]).

Germ	$\mathcal{A}_e$-codimension	Name
(x, y^2, xy)	0	Cross-cap (S_0)
$(x, y^2, y^3 \pm x^{k+1}y)$, $k \geq 1$	k	$S_k^\pm$
$(x, y^2, x^2y \pm y^{2k+1})$, $k \geq 2$	k	$B_k^\pm$
$(x, y^2, xy^3 \pm x^k y)$, $k \geq 3$	k	$C_k^\pm$
$(x, y^2, x^3y + y^5)$	4	F_4
$(x, xy + y^{3k-1}, y^3)$, $k \geq 2$	k	H_k
$(x, xy + y^3, xy^2 + ay^4)$, $a \neq 0, \frac{1}{2}, 1, \frac{3}{2}$	3	P_3

Surfaces in the same $\mathcal{A}$-orbit clearly have diffeomorphic image but not necessarily they have the same local differential geometry. So, we cannot take f as one of the normal forms in the above table. We need parametrisations for corank 1 surfaces in $\mathbb{R}^3$ obtained with changes of coordinates at source and target which preserve the geometry of the image.

The geometry of singular surfaces parametrised locally by a germ of a smooth function $\mathcal{A}$-equivalent to one of those in Table 1 is considered, for instance, in [10, 15, 26].

We summarize in the next result the partition of the set of all corank 1 map germs $f : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$ according to their 2-jets under the action of the group $\mathcal{A}^2$ (i.e., the group of 2-jets of diffeomorphisms in the source and target). We denote by $J^2(2,3)$ the space of

2-jets $j^2f(\mathbf{0})$ of map germs $f:(\mathbb{R}^2,\mathbf{0})\to(\mathbb{R}^3,\mathbf{0})$ and by $\Sigma^1J^2(2,3)$ the subset of 2-jets of corank 1.

Proposition 2.1. (Classification of 2-jets [23]) *There exist four orbits in $\Sigma^1J^2(2,3)$ under the action of $\mathcal{A}^2$, which are*

$$(x,y^2,xy),\ (x,y^2,0),\ (x,xy,0),\ (x,0,0).$$

The following result gives relevant parametrisations for corank 1 surfaces in $\mathbb{R}^3$ according to the classification given in Proposition 2.1. The cross-cap case, that is, when $j^2f(\mathbf{0})\sim(x,y^2,xy)$ is done in [10, 32], and the case $j^2f(\mathbf{0})\sim(x,0,0)$ is not of our interest here because f is a non-simple wich is included in a stratum of $\mathcal{A}_e$-codimension >3.

Proposition 2.2. ([15]) *Let $f:(\mathbb{R}^2,\mathbf{0})\to(\mathbb{R}^3,\mathbf{0})$ be a corank 1 map germ. Then, after using smooth changes of coordinates in the source and isometries in the target, we can reduce $j^kf(\mathbf{0})$ to the form*

$$(x,y)\ \mapsto\ \left(x\,,\ \frac{1}{2}\,y^2+\sum_{i=2}^{k}\frac{b_i}{i!}\,x^i\,,\ \frac{1}{2}\,a_{20}x^2+\sum_{i+j=3}^{k}\frac{a_{ij}}{i!j!}\,x^iy^j\right), \tag{1}$$

if $j^2f(\mathbf{0})$ is $\mathcal{A}$-equivalent to $(x,y^2,0)$, or

$$(x,y)\ \mapsto\ \left(x\,,\ xy+\sum_{i=3}^{k}\frac{b_i}{i!}\,y^i\,,\ \frac{1}{2}\,a_{20}x^2+\sum_{i+j=3}^{k}\frac{a_{ij}}{i!j!}\,x^iy^j\right), \tag{2}$$

if $j^2f(\mathbf{0})$ is $\mathcal{A}$-equivalent to $(x,xy,0)$, where b_i,a_{ij} are constants.

Let $f:(\mathbb{R}^2,\mathbf{0})\to(\mathbb{R}^3,\mathbf{0})$ be a map germ of corank 1 and let $j^kf(\mathbf{0})$ be given by (1) in Proposition 2.2. Then, the conditions for f to be $\mathcal{A}$-equivalent to S_k,B_k,C_k or F_4 are as follows (see [15, 26]):

$$\begin{array}{ll}
S_1: & a_{03}\neq0,a_{21}\neq0,\\
S_{k\geq2}: & a_{03}\neq0,a_{21}=\cdots=a_{k1}=0,a_{(k+1)1}\neq0,\\
B_2: & a_{03}=0,a_{21}\neq0,3a_{05}a_{21}-5a_{13}^2\neq0,\\
B_{k\geq3}: & a_{03}=0,a_{21}\neq0,3a_{05}a_{21}-5a_{13}^2=0,\\
 & \xi_3=\cdots=\xi_{k-1}=0,\xi_k\neq0,\\
C_{k\geq3}: & a_{03}=0,a_{21}=\cdots=a_{(k-1)1}=0,a_{k1}\neq0,a_{13}\neq0,\\
F_4: & a_{03}=0,a_{21}=0,a_{31}\neq0,a_{13}=0,a_{05}\neq0,
\end{array} \tag{3}$$

where ξ_m depends on the $(2m+1)$-jet of the third component of (1) in Proposition 2.2 (see [15]).

If f is such that the $j^kf(\mathbf{0})$ is given by (2) in Proposition 2.2, then the conditions for f to be $\mathcal{A}$-equivalent to H_k or P_3 can be deduced in a similar way (see for instance [26]). In particular, we distinguish between the H_k and P_3 singularities by looking at the coefficient a_{03}. We have:

$$\begin{array}{ll}
H_{k\geq2}: & a_{03}\neq0,\\
P_3: & a_{03}=0.
\end{array} \tag{4}$$

In order to characterize completely the H_k and P_3 singularities some more conditions are necessary (see [26]). Since these other conditions are not used here in our calculations, we will omit them except for the condition $a_{04}-3a_{12}b_3\neq0$ for P_3-singularity which we show now. In fact, let f be $\mathcal{A}$-equivalent to P_3. We compute the double point curve of $f(x,y)=(x,p(x,y),q(x,y))$, which is defined by equations:

$$\frac{p(x,y)-p(x,u)}{y-u}=\frac{q(x,y)-q(x,u)}{y-u}=0.$$

This gives us the two following equations:

$$
\begin{aligned}
&24x + 4b_3(u^2 + uy + y^2) + b_4(u^3 + u^2y + uy^2 + y^3) + \text{ h.o.t.} = 0,\\
&a_{04}(u^3 + u^2y + uy^2 + y^3) + 4xa_{13}(u^2 + uy + y^2)\\
&\quad + 6x(a_{22}x + 2a_{12})(u + y) + 12a_{21}x^2 + 4a_{31}x^3 + \text{ h.o.t.} = 0,
\end{aligned}
$$

where h.o.t. means “higher-order terms”.

Now, using the first equation to eliminate the variable x, one obtains a curve in the plane (y, u) which is isomorphic to the double point curve:

$$\mathcal{W} = 1/24(u + y)(a_{04}(u^2 + y^2) - 2a_{12}b_3(u^2 + uy + y^2) + \text{ h.o.t.} = 0.$$

We know from [24] that if f is $\mathcal{A}$-equivalent to P_3, then the Milnor number of $\mathcal{W}$ at the origin must be equal to 4. This implies that $\mathcal{W}$ is 3-determined and thus, its 3-jet has to be nondegenerate. In other words, the discriminant of $j^3\mathcal{W}(\mathbf{0})$ must be different of 0, that is,

$$(a_{04} - 3a_{12}b_3)(a_{04} - a_{12}b_3) \neq 0$$

holds. In particular, $a_{04} - 3a_{12}b_3 \neq 0$.

3. Index of lines of curvature

Let $f : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a smooth map given by $f(x, y) = (f_1(x, y), f_2(x, y), f_3(x, y))$. The first and the second fundamental forms for f are given, respectively, by

$$I = E\,dx^2 + 2F\,dxdy + G\,dy^2 \quad \text{and} \quad II = L\,dx^2 + 2M\,dxdy + N\,dy^2$$

where

$$E = \langle f_x, f_x\rangle \ , \quad F = \langle f_x, f_y\rangle \ , \quad G = \langle f_y, f_y\rangle \ ,$$

$$L = \frac{\det(f_x, f_y, f_{xx})}{\sqrt{EG - F^2}} \ , \quad M = \frac{\det(f_x, f_y, f_{xy})}{\sqrt{EG - F^2}} \ , \quad N = \frac{\det(f_x, f_y, f_{yy})}{\sqrt{EG - F^2}} \ ,$$

and the subscripts denote partial derivatives. It follows that L, M, N are only defined if the denominator does not vanish; that is, at the regular points of f because $EG-F^2 = \|f_x \times f_y\| \neq 0$ only in these points. For situations which include the case where f may have singularities, we can define

$$L' = \det(f_x, f_y, f_{xx}) \ , \quad M' = \det(f_x, f_y, f_{xy}) \ , \quad N' = \det(f_x, f_y, f_{yy}) \ , \tag{5}$$

and work with this functions instead of L, M, N.

We recall that *umbilics points* are regular points of f in which the second fundamental form is proportional to the first. Then, the rank of the matrix

$$\begin{pmatrix} E & F & G \\ L' & M' & N' \end{pmatrix} \tag{6}$$

is not maximal either at an umbilic or at a singular point of f.

Suppose that (x, y) is a regular point of f which is not umbilic. Then the principal directions of f at (x, y) are defined as the directions determined by the eigenvectors of the second fundamental form at (x, y). The equation of the principal directions of f is given by the binary differential equation (BDE)

$$(FN' - GM')\,dy^2 + (EN' - GL')\,dxdy + (EM' - FL')\,dx^2 = 0 \ . \tag{7}$$

Thus, the principal directions define a pair of orthogonal line fields on the surface, which are singular either at an umbilic or at a singular point of f.

The equation (7) can be seen as a particular case of a positive quadratic differential form (PQD) on $M = f(U)$ in the sense of [13], that is, as a quadratic differential form ω such that for every point p in M the subset $\omega(p)^{-1}(0)$ of the tangent plane T_pM of M at p is either: (i) the union of two transversal lines (in this case p is called a regular point of ω), or (ii) all T_pM (in this case p is called a singular point of ω). In local coordinates (x, y), a PQD form is given by

$$\omega = A(x,y)dy^2 + B(x,y)dxdy + C(x,y)dx^2, \tag{8}$$

where A, B, C are smooth functions, called the *coefficients* of the PQD, such that $B^2 - 4AC \geq 0$. Because (8) is a PQD, $B^2 - 4AC = 0$ if and only if $A = B = C = 0$ ([13]). The points where $A = B = C = 0$ are the singular points of ω and the set

$$\Delta = \{(x, y) \in U; B^2 - 4AC(x, y) = 0\}$$

which is called the discriminant of the PQD coincides with its singular set. (For a general quadratic differential equation which is not necessarily a PQD, the discriminant Δ is different from the set of singular points of the equation; see for example the survey paper [29].)

Therefore, if ω is the PQD (7) associated to f then $(x, y) \in \triangle$ if and only if (x, y) is an umbilic or singular point of f (and hence a singular point of ω), which can be easily seen from the matrix (6). Then, all important features of the equation (8) occur on the discriminant. Taking an isolated singular point p of ω, we can consider the index at p associated with any of the lines of principal curvature determined by ω, which is denoted in the literature by $\mathrm{ind}(\omega, p)$ but we shall denote here by $\mathrm{ind}_{\mathcal{P}}(f, p)$ in order to specify f and with $\mathcal{P}$ indicating principal, as a reference for the equation (7). This means the number of turns of the line field when we run through a small circle centered at p. For instance, we can easily to compute the index of the three types of generic umbilics classified by Darboux (see, for instance, [2, 9, 12, 14]): the lemon (or D_1), the monster (or D_2) and the star (or D_3), which are $1/2$, $1/2$ and $-1/2$, respectively. Moreover, from the description for the principal lines at a cross-cap point p of f, whose configuration can be found in [12], for example, we deduce that the index $\mathrm{ind}_{\mathcal{P}}(f, p)$ is $1/2$ (see Figure 1).

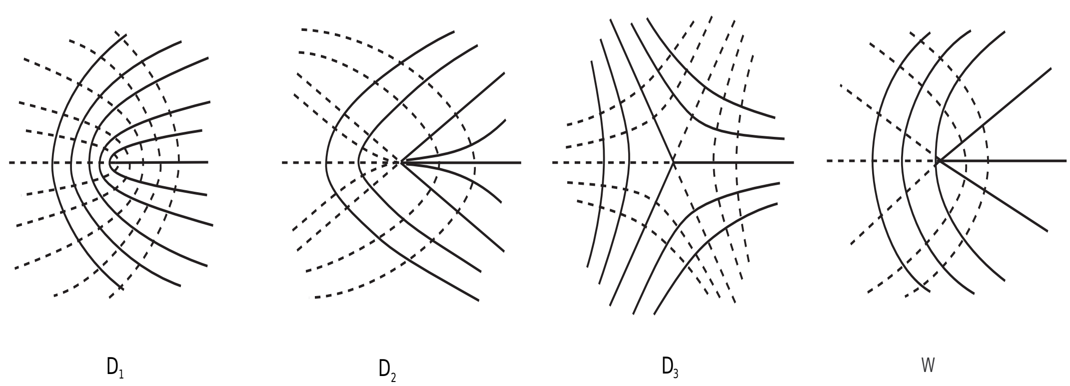

FIGURE 1. From left to right: configuration of integral curves of the principal directions at generic umbilics D_1, D_2 and D_3, and of the principal lines at a cross-cap point of f, W.

In order to consider the index $\mathrm{ind}_{\mathcal{P}}(f, p)$ it is necessary to have p as an isolated singular point of ω (for example, we should eliminate the possibility of the existence of a sequence of umbilic points on the smooth part of the surface that converges to p). We shall consider this question.

For this, we use the following lemma which shows that the index of an isolated singular point of a PQD is related to the mapping degree, given in terms of the coefficients of ω.

Lemma 3.1. ([18], Part 2, VIII, 2.3) *Let p be an isolated singular point of the positive quadratic differential form $\omega = A(x,y)\,dy^2 + B(x,y)\,dxdy + C(x,y)\,dx^2$. Then,*

$$\operatorname{ind}(\omega, p) = -\frac{1}{2}\deg((A,B),p) = -\frac{1}{2}\deg((B,C),p)$$

where $\deg((A,B),p)$ and $\deg((B,C),p)$ denote the mapping degrees of the maps (A,B) and (B,C), respectively, at p.

Let $h : (\mathbb{R}^n, \mathbf{0}) \to (\mathbb{R}^n, \mathbf{0})$ be a continuous map such that $\mathbf{0}$ is isolated in $h^{-1}(\mathbf{0})$. The degree $\deg(h,\mathbf{0})$ of h at $\mathbf{0}$ is defined as follows: choose a ε-ball B^n_ε centered at $\mathbf{0}$ in $\mathbb{R}^n$ so small that $h^{-1}(\mathbf{0}) \cap B^n_\varepsilon = \{\mathbf{0}\}$ and let S^{n-1}_ε be the $(n-1)$-sphere centered at the origin of radius ε. Choose an orientation of each copy of $\mathbb{R}^n$. Then the degree of h at $\mathbf{0}$ is the degree of the mapping $\frac{h}{\|h\|} : S^{n-1}_\varepsilon \to S^{n-1}$ ($S^{n-1} \subset \mathbb{R}^n$ is the unit standard sphere), where the spheres are oriented as $(n-1)$-spheres in $\mathbb{R}^n$. If h is differentiable, this degree can be computed as the sum of the signs of the Jacobian determinant of h (i.e., of its derivative) at all the h-preimages near $\mathbf{0}$ of a regular value of h near $\mathbf{0}$.

We also recall that $h : (\mathbb{R}^n, \mathbf{0}) \to (\mathbb{R}^n, \mathbf{0})$ is a *quasi-homogeneous map germ with weight* $\boldsymbol{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$ and *quasi-degree* $\boldsymbol{d} = (d_1, \ldots, d_n) \in \mathbb{N}^n$ if

$$h_i(\lambda^{a_1}x_1, \lambda^{a_2}x_2, \ldots, \lambda^{a_n}x_n) = \lambda^{d_i} h_i(x_1, x_2, \ldots, x_n)$$

for each $i = 1, 2, \ldots, n$ and all $\lambda > 0$. We say that a smooth function has *quasi-order* m if all monomials in its Taylor expression have quasi-degree greater than or equal to m. We say that h is a *semi-quasi-homogeneous map with weight* $\boldsymbol{a}$ and *quasi-degree* $\boldsymbol{d}$ if $h = g + G$ with g a quasi-homogeneous map germ with weight $\boldsymbol{a}$ and quasi-degree $\boldsymbol{d}$ such that $\mathbf{0}$ is isolated in $g^{-1}(\mathbf{0})$, and each component G_i of G has quasi-order greater than d_i, $i = 1, 2, \ldots, n$.

The following theorem shows that for semi-quasi-homogeneous map germs, the degree at a zero coincides with the degree at this zero of its quasi-semi-homogeneous part.

Theorem 3.2. ([5]) *With the above notations, let $h = g + G$ be a semi-quasi-homogeneous map germ. Then $\mathbf{0}$ is isolated in $h^{-1}(0)$ and*

$$\deg(h, \mathbf{0}) = \deg(g, \mathbf{0}).$$

Before giving the results about the index of the lines of curvature for a corank 1 surface, we present an illustrative example explaining all our calculations.

Example 3.3. *Let S_1^+-standard be the map germ given by $(x, y^2, y^3 + x^2y)$ as in Table 1. The coefficients of its first and second fundamental forms are, respectively:*

$$E = 1 + 4x^2y^2, \quad F = 2xy(x^2 + 3y^2), \quad G = 4y^2 + (x^2 + 3y^2)^2$$

and

$$L' = 4y^2, \quad M' = 4xy, \quad N' = -2x^2 + 6y^2.$$

Let $Ady^2 + Bdxdy + Cdx^2 = 0$ the BDE of the principal directions of S_1^+-standard. Then, from (7) we have that

$$A = -8x^5y - 16xy^3 - 24x^3y^3,\ B = -2x^2 + 6y^2 - 12x^4y^2 - 16y^4 - 36y^6,\ C = 4xy + 8x^3y^3 - 24xy^5.$$

Consider the map germ $h = (B, C) : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^2, \mathbf{0})$ given by $h = g + G$ taking

$$g(x,y) = (-2x^2 + 6y^2, 4xy) \text{ and } G(x,y) = (-12x^4y^2 - 16y^4 - 36y^6, 8x^3y^3 - 24xy^5).$$

In this case, g is a homogeneous map germ, $\mathbf{0}$ is isolated in $g^{-1}(\mathbf{0})$ and each component G_i of G has quasi-order greater than 2. Then, by Theorem 3.2, the degree of h in $\mathbf{0}$ coincides with the degree of g in $\mathbf{0}$. It is easy to calculate the degree of g in $\mathbf{0}$, which is -2. Hence, by Lemma 3.1, the index of the BDE associated to S_1^+-standard is 1.

The case S_1^--standard is analogous. Repeating this same sketch of calculations, we can conclude that the index of the BDE associated to S_1^--standard is 0. Figure 2 shows S_1^- and S_1^+-standards surfaces with their lines of curvatures.

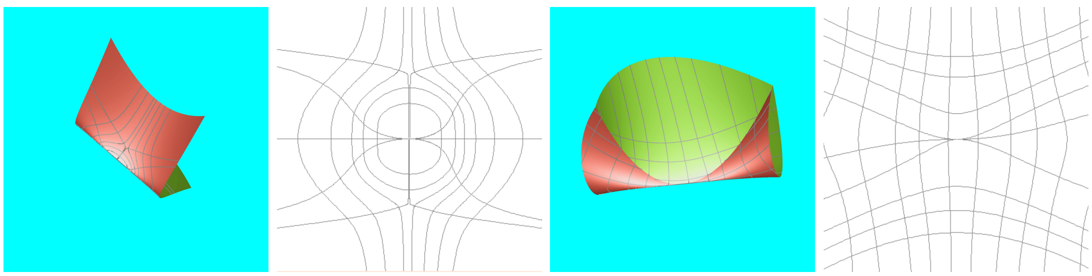

FIGURE 2. Standards S_1^+ and S_1^- surfaces and their lines of curvature.

In Proposition 2.1 are listed four orbits in $\Sigma^1 J^2(2,3)$ under the action of group $\mathcal{A}^2$, with the first one corresponding the known case of the cross-cap (cf. [10, 32]) and the fourth orbit listed corresponding to a non-simple germ wich is included in a stratum of $\mathcal{A}_e$-codimension > 3. Then it is just remaining to consider two cases in the 2-jet classification: $(x, y^2, 0)$ and $(x, xy, 0)$. The next theorem complete the study of the index of an isolated singular point of a BDE which represents the equation of the principal directions of a corank 1 simple map germ $f : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$ or non simple but in strata of $\mathcal{A}_e$-codimension ≤ 3.

Theorem 3.4. *Let $f : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$ be a corank 1 simple map germ or non-simple strata of $\mathcal{A}_e$-codimension ≤ 3. Consider $j^k f(\mathbf{0})$ as in Proposition 2.2. If $a_{12}^2 - a_{21}a_{03} \neq 0$ then $\mathbf{0} \in \mathbb{R}^2$ is an isolated singular point of the BDE associated to f given in (7) and*

$$\mathrm{ind}_{\mathcal{P}}(f, \mathbf{0}) = \begin{cases} 0 & if \quad a_{21}a_{03} < 0 \\ 0 & if \quad a_{12}^2 > a_{21}a_{03} \\ 1 & if \quad a_{12}^2 < a_{21}a_{03} \end{cases}$$

if $j^2 f(\mathbf{0})$ has type $(x, y^2, 0)$ and

$$\mathrm{ind}_{\mathcal{P}}(f, \mathbf{0}) = \begin{cases} 0 & if \quad a_{21}a_{03} \leq 0 \\ 1 & if \quad a_{21}a_{03} > 0 \end{cases}$$

if $j^2 f(\mathbf{0})$ has type $(x, xy, 0)$.

Proof. Under hypothesis, we just need to consider map germs which are $\mathcal{A}$-equivalent to one of those given in Table 1 and such that the 2-jet has the type $(x, y^2, 0)$ or $(x, xy, 0)$. We divide the proof in three parts. In all of them, we start with the following procedure:

Given $f : (\mathbb{R}^3, \mathbf{0}) \to (\mathbb{R}^2, \mathbf{0})$, we first calculate the coefficients E, F, G, L', M', N' associated to f; second we get the BDE expression of the principal directions of f given by (7), denoted here by $Ady^2 + Bdxdy + Cdx^2 = 0$.

These calculations can be done quickly using for instance the Mathematica software. Thus, they will be omitted here.

Part 1. $j^2 f(\mathbf{0}) \sim (x, y^2, 0)$**.**

• Consider f $\mathcal{A}$-equivalent to S_1 given in Table 1 (this means f is $\mathcal{A}$-equivalent to S_1^+ or S_1^-). The conditions on the coefficients of a S_1-singularity are $a_{03} \neq 0$ and $a_{21} \neq 0$. We use the same procedure given in Example 3.3. After calculating the coefficients of the first and second fundamental forms associated to f and getting the BDE expression of the principal directions of f, let us take the semi-quasi-homogeneous map $h = (B, C) : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^2, \mathbf{0})$. So, in this case we can consider $h = g + G$, where

$$g(x, y) = (-\frac{a_{21}}{2}x^2 + \frac{a_{03}}{2}y^2, a_{21}xy + a_{12}y^2)$$

is quasi-homogeneous (in fact homogeneous) and G has higher order terms. We call resultant of g to the resultant of the two components of g (with respect to one of the variables). The resultant of g is given by the expression $a_{12}^2 - a_{21}a_{03}$ (which is not zero by hypothesis) then we can conclude that $\mathbf{0}$ is isolated in $g^{-1}(\mathbf{0})$. Therefore, by Theorem 3.2, $\mathbf{0}$ is isolated in $h^{-1}(\mathbf{0})$ and the degree of h in $\mathbf{0}$ coincides with the degree of g in $\mathbf{0}$. Now we apply Lemma 3.1 to calculate the index $\mathrm{ind}_{\mathcal{P}}(f, \mathbf{0})$. To do this, let us calculate the degree of g at $\mathbf{0}$. Since $a_{03}, a_{21} \neq 0$, it may occur:

$$\text{(i)} \quad a_{21}a_{03} < 0 \quad \text{or} \quad \text{(ii)} \quad a_{21}a_{03} > 0.$$

Taking the following change of coordinates in the source of g

$$\begin{cases} X = a_{21}x + a_{12}y \\ Y = y \end{cases}$$

it holds that

$$g \sim (-\frac{1}{2a_{21}}(X^2 - 2a_{12}XY + (a_{12}^2 - a_{21}a_{03})Y^2), XY).$$

Taking now the change of coordinates in the target $k_1(u, v) = (-2a_{21}u, v)$, we have

$$g \sim (X^2 - 2a_{12}XY + (a_{12}^2 - a_{21}a_{03})Y^2), XY).$$

After one more change of coordinates in the target given by $k_2(u, v) = (u + 2a_{12}v, v)$, it holds that

$$g \sim (X^2 + (a_{12}^2 - a_{21}a_{03})Y^2, XY) = \tilde{g}(X, Y).$$

Due the previous change of coordinates, it follows that

$$\deg\,(g, \mathbf{0}) = -\mathrm{sgn}(2X^2 - 2(a_{12}^2 - a_{21}a_{03})Y^2)\deg\,(\tilde{g}, \mathbf{0}),$$

where sgn denotes the sign of a function.

If $a_{21}a_{03} < 0$ then $a_{12}^2 - a_{21}a_{03} > 0$. So, $\tilde{g}$ is not surjective and thus $\deg\,(\tilde{g}, \mathbf{0}) = 0$. Hence $\deg\,(g, \mathbf{0}) = 0$ and thus $\mathrm{ind}_{\mathcal{P}}(f, \mathbf{0}) = 0$.

If $a_{21}a_{03} > 0$, we have two possibilities: $a_{12}^2 > a_{21}a_{03}$ or $a_{12}^2 < a_{21}a_{03}$. If $a_{12}^2 > a_{21}a_{03}$ then $a_{12}^2 - a_{21}a_{03} > 0$ and as already done, $\mathrm{ind}_{\mathcal{P}}(f, \mathbf{0}) = 0$. If $a_{12}^2 < a_{21}a_{03}$ then $a_{12}^2 - a_{21}a_{03} < 0$. In this case, the Jacobian determinant of $\tilde{g}$ is equal to

$$2X^2 - 2(a_{12}^2 - a_{21}a_{03})Y^2 > 0$$

for any (X, Y).

Taking any regular value of $\tilde{g}$, there always exist two $\tilde{g}$-preimages for which the signs of the Jacobian determinants are 1. Hence $\deg\,(\tilde{g}, \mathbf{0}) = 2$, which implies that $\deg\,(g, \mathbf{0}) = -2$ and thus $\mathrm{ind}_{\mathcal{P}}(f, \mathbf{0}) = 1$.

• Consider f $\mathcal{A}$-equivalent to S_k given in Table 1, for any $k \geq 2$. By conditions on the coefficients of a S_k-singularity given in (3) and by hypothesis $a_{12}^2 - a_{21}a_{03} \neq 0$, one has that $a_{12} \neq 0$.

We reproduce the same steps as in the previous case. From the coefficients B and C of (7) for f, we can take, for all $k \geq 2$, the semi-quasi-homogeneous map $h = (B, C) : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^2, \mathbf{0})$ given by $h = g + G$, where

$$g(x, y) = \left(-\frac{a_{(k+1)1}}{(k+1)!} x^{k+1} + \frac{a_{03}}{2} y^2, a_{12} y^2 \right) \tag{9}$$

is quasi-homogeneous and G has higher-order terms. In the expression of the resultant of g appears just a_{12}, which is not zero in this case. Therefore, for all $k \geq 2$, the map germ g in (9) is clearly not surjective and hence its degree is 0. By Theorem 3.2, $\deg(h, \mathbf{0}) = \deg(g, \mathbf{0}) = 0$. As consequence, by Lemma 3.1, the $\mathrm{ind}_{\mathcal{P}}(f, \mathbf{0}) = 0$.

• Consider f $\mathcal{A}$-equivalent to B_k given in Table 1, for any $k \geq 2$. A B_k-singularity is characterized by conditions which appear in (3). Since $a_{03} = 0$, the general hypothesis reduces to $a_{12} \neq 0$. We proceed in the same way as in the previous cases, following the same steps.

In this case, for all $k \geq 2$, we can take the semi-quasi-homogeneous map

$$h = (B, C) : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^2, \mathbf{0})$$

given by $h = g + G$, where

$$g(x, y) = \left(-\frac{1}{2} a_{21} x^2, a_{12} y^2 + a_{21} xy \right) \tag{10}$$

is homogeneous and G has higher-order terms. The resultant of g is given by $a_{12}^2 a_{21}$ which is not zero. Therefore, for all $k \geq 2$, the map germ g in (10) clearly is not surjective and hence its degree is 0. Then, again we have that $\mathrm{ind}_{\mathcal{P}}(f, \mathbf{0}) = 0$.

• Consider f $\mathcal{A}$-equivalent to C_k given in Table 1, for any $k \geq 3$. A C_k-singularity is characterized by conditions

$$a_{03} = 0, a_{21} = a_{31} = \cdots = a_{(k-1)1} = 0, a_{k1} \neq 0 \text{ and } a_{13} \neq 0.$$

Then, the general hypothesis again reduces to $a_{12} \neq 0$. Proceeding in the same way as in the previous cases, for all $k \geq 3$, we can take the semi-quasi-homogeneous map

$$h = (B, C) : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^2, \mathbf{0})$$

such that $h = g + G$, where

$$g(x, y) = \left(-\frac{1}{k!} a_{k1} x^k, a_{12} y^2 \right), \tag{11}$$

is quasi-homogeneous and G has higher-order terms.

In the expression of the resultant of g just appears a_{12}, which is not zero. Therefore, for all $k \geq 3$, the map germ g in (11) clearly is not surjective and hence its degree is 0 from which one concludes that $\mathrm{ind}_{\mathcal{P}}(f, \mathbf{0}) = 0$.

• Consider f $\mathcal{A}$-equivalent to F_4 given in Table 1. The F_4-singularity is characterized by conditions

$$a_{03} = a_{21} = a_{13} = 0, a_{31} \neq 0 \text{ and } a_{05} \neq 0.$$

Then, the general hypothesis again reduces for $a_{12} \neq 0$. In this case, we can take the semi-quasi-homogeneous map $h = (B, C) : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^2, \mathbf{0})$ given by $h = g + G$, where

$$g(x, y) = \left(-\frac{1}{6} a_{31} x^3, a_{12} y^2 \right) \tag{12}$$

is quasi-homogeneous and G has higher-order terms.

The resultant of g is given by a_{12} which is not zero. Therefore, the map germ g in (12) is also non surjective and hence its degree is 0. Thus $\mathrm{ind}_{\mathcal{P}}(f, \mathbf{0}) = 0$.

Part 2. $j^2f(\mathbf{0}) \sim (x, xy, 0)$ **and** f **is a simple map germ.**

In this case f is $\mathcal{A}$-equivalent to H_k given in Table 1, with $k \geq 2$. We have already seen in Section 2 that a necessary condition to H_k-singularity occurs is $a_{03} \neq 0$.

In this case, we can take the semi-quasi-homogeneous map $h = (B, C) : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^2, \mathbf{0})$, $h = g + G$, where

$$g(x, y) = \left(a_{12}x^2 + a_{03}xy, \frac{1}{2}a_{21}x^2 - \frac{1}{2}a_{03}y^2\right) \tag{13}$$

is homogeneous and G has higher-order terms.

The resultant of g is given by the expression $-a_{03}^2(a_{12}^2 - a_{21}a_{03})$, which is not zero. Then $\mathbf{0}$ is isolated in $g^{-1}(\mathbf{0})$.

Consider the following change of coordinates in the source of g:

$$\begin{cases} X = x \\ Y = a_{12}x + a_{03}y\,. \end{cases}$$

Then

$$g \sim \left(XY, \frac{1}{2a_{03}}\left(-(a_{12}^2 - a_{21}a_{03})X^2 + 2a_{12}XY - Y^2\right)\right).$$

Taking another change of coordinates $k_1(u, v) = (u, 2a_{03}v)$, now in the target, it holds that

$$g \sim (XY, -(a_{12}^2 - a_{21}a_{03})X^2 + 2a_{12}XY - Y^2).$$

After one more change of coordinates in the target given by $k_2(u, v) = (u, v - 2a_{12}u)$, we have

$$g \sim (XY, -(a_{12}^2 - a_{21}a_{03})X^2 + 2a_{12}XY - Y^2) = \tilde{g}(X, Y).$$

Due the previous changes of coordinates applied in g, it follows that $\deg(g, \mathbf{0}) = \deg(\tilde{g}, \mathbf{0})$, which does not depend on the sign of a_{03}.

If $a_{12}^2 - a_{21}a_{03} > 0$ then $\tilde{g}$ is not surjective. In fact, take for instance $(0, \epsilon) \in \mathbb{R}^2, \epsilon > 0$ small enough. Then there is not (X, Y) such that $\tilde{g}(X, Y) = (0, \epsilon)$. Suppose by absurd that

$$XY = 0 \;\text{ and }\; -(a_{12}^2 - a_{21}a_{03})X^2 + 2a_{12}XY - Y^2 = \epsilon.$$

From the first expression, $X = 0$ or $Y = 0$. If $X = 0$, then the second equation reduces to $-Y^2 = \epsilon > 0$. Otherwise, if $Y = 0$, then we obtain $-(a_{12}^2 - a_{21}a_{03})X^2 = \epsilon > 0$ while $(a_{12}^2 - a_{21}a_{03}) > 0$.

Thus, $\tilde{g}$ is not surjective and $\deg(\tilde{g}, \mathbf{0}) = 0 = \deg(g, \mathbf{0})$. Hence, $\operatorname{ind}_{\mathcal{P}}(f, \mathbf{0}) = 0$.

If $a_{12}^2 - a_{21}a_{03} < 0$, the Jacobian determinant of $\tilde{g}$ is equal to

$$-2Y^2 + 2(a_{12}^2 - a_{21}a_{03})X^2 < 0.$$

For any regular value of $\tilde{g}$, there always exist two $\tilde{g}$-preimages for which the sign of the Jacobian determinants of $\tilde{g}$ are -1. Hence $\deg(\tilde{g}, \mathbf{0}) = -2$, which implies that $\deg(g, \mathbf{0}) = -2$ and thus $\operatorname{ind}_{\mathcal{P}}(f, \mathbf{0}) = 1$.

Part 3. $j^2f(\mathbf{0}) \sim (x, xy, 0)$ **and** f **is a non-simple strata of** $\mathcal{A}_e$**-codimension** ≤ 3**.**

In this case f is $\mathcal{A}$-equivalent to P_3 given in Table 1. We have already seen in Section 2 that necessary conditions to P_3-singularity occurs are $a_{03} = 0$ and $a_{04} - 3a_{12}b_3 \neq 0$.

In this case, we can take the semi-quasi-homogeneous map $h = (A, B) : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^2, \mathbf{0})$ given by $h = g + G$, where

$$g(x, y) = \left(\frac{1}{2}a_{21}x^2 + \left(-\frac{1}{6}a_{04} + \frac{1}{2}a_{12}b_3\right)y^3, a_{12}x^2\right),$$

is quasi-homogeneous with weight $(3, 2)$ and quasi-degree $(6, 6)$ and G has only higher-order terms. Moreover, since $a_{12} \neq 0$ and $a_{04} - 3a_{12}b_3 \neq 0$, the resultant of g given by $a_{12}^2(a_{04} - 3a_{12}b_3)^2$

is not zero. Therefore $g^{-1}(\mathbf{0}) = \mathbf{0}$. In particular, h is semi-quasi-homogeneous and $\deg(h) = \deg(g) = 0$ because g is not surjective. So, $\mathrm{ind}_{\mathcal{P}}(f, \mathbf{0}) = 0$. $\square$

From Theorem 3.4 it holds that:

Corollary 3.5. *Let f be a map germ in the $\mathcal{A}$-class of one of the map germs given in Table 1, with $j^k f(\mathbf{0})$ as in Proposition 2.2. Suppose that $a_{12}^2 - a_{21}a_{03} \neq 0$.*
(i) *If $f \sim S_1^{\pm}$ or H_k then $\mathrm{ind}_{\mathcal{P}}(f, \mathbf{0}) = 0$ or 1.*
(ii) *If $f \sim S_{k\geq 2}^{\pm}, B_k^{\pm}, C_k^{\pm}, F_4$ or P_3 then $\mathrm{ind}_{\mathcal{P}}(f, \mathbf{0}) = 0$.*

Remark 3.6. *It follows from Theorem 3.4 that for any corank 1 map germ f satisfying its hypothesis, the singularity of the BDE of the principal directions of f is an isolated point, i.e. there is not sequence of umbilic points on the smooth part of the surface that converges to the singular point of the surface.*

4. Geometric interpretation of the condition $a_{12}^2 - a_{21}a_{03} \neq 0$

Let $f : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$ be a corank 1 map germ whose 2-jet has $\mathcal{A}^2$-type either $(x, y^2, 0)$ or $(x, xy, 0)$. We want to analyze the circles which have a special contact with f at the origin. To do this, we need to look at the singularity type of the contact map germ $C_{\mathbf{v},\mathbf{u}} : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^2, \mathbf{0})$ given by

$$C_{\mathbf{v},\mathbf{u}}(x, y) = (\langle f(x, y), \mathbf{v}\rangle, \|f(x, y) - \mathbf{u}\|^2 - \|\mathbf{u}\|^2),$$

where $\mathbf{v}, \mathbf{u} \in \mathbb{R}^3$, $\|\mathbf{v}\| = 1$ is the unit normal vector of the circle and $\mathbf{u}$ is its centre. Note that the first component is nothing but the height function which measures the contact of f with the normal plane to $\mathbf{v}$ and the second component the squared distance function which measures the contact of f with the sphere of centre $\mathbf{u}$.

In order to consider the desired contact we use the umbilic curvature, the binormal and asymptotic directions defined in [21], which are related to contact properties of the surface given by f with planes and spheres. The umbilic curvature κ_u is an important second-order invariant of the f: when it is non-zero, then $1/\kappa_u$ is the radius of the unique sphere with umbilical contact (that is, contact of type $\Sigma^{2,2}$ in Thom-Boardman terminology) with the surface at the singular point. See [21] for details.

We recall that a map germ $g : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^2, \mathbf{0})$ has type $\Sigma^{2,1}$ if and only if its 2-jet is equivalent to $(x^2, 0)$.

Lemma 4.1. *Let $f : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$ be a corank 1 map germ with $j^k f(\mathbf{0})$ as in Proposition 2.2 and with non-zero umbilic curvature κ_u at the origin.*

(i) *If $j^2 f(\mathbf{0}) \sim (x, y^2, 0)$, there are exactly two circles with contact of type $\Sigma^{2,1}$ with f at the origin, given by $\mathbf{u} = (0, 0, 1/a_{20})$ and $\mathbf{v} = (0, 0, 1)$ or $\mathbf{v} = (0, -a_{20}, b_2)/\sqrt{a_{20}^2 + b_2^2}$.*
(ii) *If $j^2 f(\mathbf{0}) \sim (x, xy, 0)$, there is exactly one circle with contact of type $\Sigma^{2,1}$ with f at the origin, given by $\mathbf{u} = (0, 0, 1/a_{20})$ and $\mathbf{v} = (0, 0, 1)$.*

Proof. Notice that the circle determined by $\mathbf{u}, \mathbf{v}$ has contact of type $\Sigma^{2,1}$ if and only if the sphere of centre $\mathbf{u}$ has umbilical contact and the plane normal to $\mathbf{v}$ is binormal (i.e., it has a degenerate contact $\Sigma^{2,1}$). Then, our results follow from the analysis of contacts with spheres and planes in [21], where the umbilic curvature at the origin is $\kappa_u(\mathbf{0}) = |a_{20}|$. $\square$

We observe that if $j^2 f(\mathbf{0}) \sim (x, y^2, xy)$ then there is not circle with contact of type $\Sigma^{2,1}$ with f at the origin (because there is not sphere with contact of type $\Sigma^{2,2}$ with f, see [21] for details). The circles with contact of type $\Sigma^{2,1}$ with f given in the above lemma will be called $\Sigma^{2,1}$-*circles* for simplicity.

Definition 4.2. Let $g : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^2, \mathbf{0})$ be a map germ of type $\Sigma^{2,1}$. We say that g is $\Sigma^{2,1}$-*generic* if it is $\mathcal{A}$-equivalent to a finitely determined map germ of the form

$$(x^2, c_0x^3 + 3c_1x^2y + 3c_2xy^2 + c_3y^3),$$

for some $c_0, c_1, c_2, c_3 \in \mathbb{R}$.

Remark 4.3. *It follows from the definition that if $j^3g(\mathbf{0}) = (x^2, c_0x^3+3c_1x^2y+3c_2xy^2+c_3y^3)$, then a necessary condition for g being $\Sigma^{2,1}$-generic is that $c_2^2 - c_1c_3 \neq 0$. In fact, a necessary condition for finite determinacy for map germs $(\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^2, \mathbf{0})$ is that its Jacobian determinant has to be non-degenerate. A simple computation shows that the Jacobian determinant of $j^3g(\mathbf{0})$ is $6x(c_1x^2 + 2c_2xy + c_3y^2)$, so we must have $c_3 \neq 0$ and $c_2^2 - c_1c_3 \neq 0$.*

Corollary 4.4. *Let $f : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$ be a corank 1 map germ with $j^kf(\mathbf{0})$ as in Proposition 2.2 and with non-zero umbilic curvature κ_u at the origin. Assume that the $\Sigma^{2,1}$-circles of f have $\Sigma^{2,1}$-generic contact. Then, $a_{12}^2 - a_{21}a_{03} \neq 0$.*

Proof. It is easy to show that for $\mathbf{u} = (0, 0, 1/a_{20})$ and $\mathbf{v} = (0, 0, 1)$, we have:

$$j^3C_{\mathbf{v},\mathbf{u}}(\mathbf{0}) = \left(\frac{1}{2}a_{20}x^2, -\frac{1}{3a_{20}}(a_{30}x^3 + 3a_{21}x^2y + 3a_{12}xy^2 + a_{03}y^3)\right).$$

When $j^2f(\mathbf{0}) \sim (x, y^2, 0)$ and we consider $\mathbf{u} = (0, 0, 1/a_{20})$ and $\mathbf{v} = (0, -a_{20}, b_2)/\sqrt{a_{20}^2 + b_2^2}$, we get

$$j^3C_{\mathbf{v},\mathbf{u}}(\mathbf{0}) = \left(-\frac{1}{2\sqrt{a_{20}^2 + b_2^2}}a_{20}y^2, -\frac{1}{3a_{20}}(a_{30}x^3 + 3a_{21}x^2y + 3a_{12}xy^2 + a_{03}y^3)\right).$$

So the result follows from Remark 4.3.

□

5. Umbilics and cross-caps of generic deformations

Let $f : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a smooth map. It was shown in [12] that f is principally structurally stable at an umbilic point if and only if it is one of the Darbouxian umbilics D_i, $i = 1, 2, 3$ (see also [2]). Furthermore, the unique stable singularity for f is a cross-cap point.

The map f is said to be *generic* if the ulfoldings

$$D : \mathbb{R}^3 \times U \to \mathbb{R}^3 \times \mathbb{R}, \quad (\mathbf{u}, (x, y)) \mapsto (\mathbf{u}, d_{\mathbf{u}}(x, y)), \quad d_{\mathbf{u}}(x, y) = \frac{1}{2}\|f(x, y) - \mathbf{u}\|^2$$

and

$$H : S^2 \times U \to S^2 \times \mathbb{R}, \quad (\mathbf{v}, (x, y)) \mapsto (\mathbf{v}, h_{\mathbf{v}}(x, y)), \quad h_{\mathbf{v}}(x, y) = \langle f(x, y), \mathbf{v}\rangle$$

are generic in the Thom-Boardman sense (see [11] for details). So, if the map f is not generic, we can take a generic deformation $f_\lambda : U_0 \subset U \to \mathbb{R}^3$, $\lambda \in (-\varepsilon, \varepsilon)$, of f, i.e. $f_0 = f$, f_λ is generic for $\lambda \neq 0$ and the map $(\lambda, t) \mapsto f_\lambda(t)$ is smooth, and the index $\mathrm{ind}_{\mathcal{P}}(f, p)$ is equal to $(D_1 + D_2 - D_3 + W)/2$, where D_1, D_2, D_3 also denote the number of umbilics of each type and W the number of cross-caps points that appear in f_λ near p, for $\lambda \neq 0$ small enough.

When $f : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$ is a corank 1 map germ and f_λ, $\lambda \in (-\varepsilon, \varepsilon)$, is a generic deformation of f, D. Mond showed in [24] how to count the number of cross-caps in f_λ. More precisely, it is showed the following possibilities for W in f_λ according the $\mathcal{A}$-types of f given in

Table 1:

$$
\begin{array}{ll}
S_k^{\pm}, k \geq 1 : & W = \begin{cases} 2n\,;\, n = 0,1,\ldots,\frac{k+1}{2} & \text{if } k \text{ is odd} \\ 2n+1\,;\, n = 0,1,\ldots,\frac{k}{2} & \text{if } k \text{ is even} \end{cases} \\
B_k^{\pm}, H_k, k \geq 2 : & W = 0, 2 \\
C_k^{\pm}, k \geq 3 : & W = \begin{cases} 2n+1\,;\, n = 0,1,\ldots,\frac{k-1}{2} & \text{if } k \text{ is odd} \\ 2n\,;\, n = 0,1,\ldots,\frac{k}{2} & \text{if } k \text{ is even} \end{cases} \\
F_4, P_3 : & W = 1, 3.
\end{array}
$$

As an immediate consequence of this, we obtain some information about the number of umbilic points in f_λ. In fact, this number is equal to $D_1 + D_2 + D_3 = 2(\mathrm{ind}_{\mathcal{P}}(f, \mathbf{0}) + D_3) - W$. So, if W is even (respec. odd), the number of umbilic points that appear in f_λ is even (resp. odd). Consequently, we have:

Lemma 5.1. *Let $f : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$ be a corank 1 map germ simple or non-simple but including in strata of $\mathcal{A}_e$-codimension ≤ 3. If $f_\lambda : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$ is a generic deformation of f then the number of umbilic points that appear in f_λ near $\mathbf{0}$, for λ small enough, is:*
(i) *even if $f \sim S_k^{\pm}$ (with k odd), $B_k^{\pm}$, $C_k^{\pm}$ (with k even) or H_k;*
(ii) *odd if $f \sim S_k^{\pm}$ (with k even), $C_k^{\pm}$ (with k odd), F_4 or P_3.*

We shall give more precise information about the number of umbilic points in f_λ. Before stating the result and proving it, we need recall some facts about multiplicity for special types of singular points of a map.

Given a smooth map germ $f : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$, we say that $\mathbf{0}$ is a 2-*rounding* of f if $\mathbf{0}$ is either a 2-*flattening* (that is, there is a unit vector $\mathbf{v} \in \mathbb{R}^3$ such that $\mathbf{0}$ is a singularity of type $\Sigma^{2,2}$ of $h_{\mathbf{v}}$) or a *non-flat* 2-*rounding* (that is, it is not a 2-flattening and there is $\mathbf{u} \in \mathbb{R}^3$ such that $\mathbf{0}$ is a singularity of type $\Sigma^{2,2}$ of $d_{\mathbf{u}}$). It is known that a regular (resp. singular) point of f is a 2-rounding if and only if it is an umbilic point (resp. it is not a cross-cap point). See [11] for details. So, since a generic deformation of f only has umbilics of type D_i, $i = 1, 2, 3$, and cross-caps, and since cross-caps are not 2-roundings, then in order to estimate the number of umbilic points in f_λ it is enough to estimate the number of its 2-roundings, which is denoted by $n_{\mathcal{R}}(f_\lambda, \mathbf{0})$.

The number $n_{\mathcal{R}}(f_\lambda, \mathbf{0})$ is related with the multiplicity of $\mathbf{0}$ as a rounding of f, $\mu_{\mathcal{R}}(f, \mathbf{0})$, as follows:

$$n_{\mathcal{R}}(f_\lambda, \mathbf{0}) \leq \mu_{\mathcal{R}}(f, \mathbf{0}) \quad \text{and} \quad n_{\mathcal{R}}(f_\lambda, \mathbf{0}) \equiv \mu_{\mathcal{R}}(f, \mathbf{0}) \,(\mathrm{mod}\, 2),$$

for λ small enough, if $\mu_{\mathcal{R}}(f, \mathbf{0})$ is finite (see Theorem 2.9 of [11]), where

$$\mu_{\mathcal{R}}(f, \mathbf{0}) = \dim_{\mathbb{R}} \frac{C^\infty(\mathbb{R}^2, \mathbf{0})}{\mathcal{R}(f, \mathbf{0})} \ ,$$

with $C^\infty(\mathbb{R}^2, \mathbf{0})$ being the ring of germs at $\mathbf{0}$ of smooth real-valued functions on $\mathbb{R}^2$ and $\mathcal{R}(f, \mathbf{0})$ the ideal generated by the germs at $\mathbf{0}$ of the 4-minors of the matrix given by

$$\begin{pmatrix} {f_1}_x & {f_2}_x & {f_3}_x & 0 \\ {f_1}_y & {f_2}_y & {f_3}_y & 0 \\ {f_1}_{xx} & {f_2}_{xx} & {f_3}_{xx} & E \\ {f_1}_{xy} & {f_2}_{xy} & {f_3}_{xy} & F \\ {f_1}_{yy} & {f_2}_{yy} & {f_3}_{yy} & G \end{pmatrix},$$

where $f = (f_1, f_2, f_3)$. See [11] for details.

We also recall that if $h : (\mathbb{R}^n, \mathbf{0}) \to (\mathbb{R}^n, \mathbf{0})$ is a smooth map germ with $\mathbf{0}$ isolated in $h^{-1}(\mathbf{0})$, then the multiplicity $\mu(h, \mathbf{0})$ of h at $\mathbf{0}$ is defined by

$$\mu(h, \mathbf{0}) = \dim_{\mathbb{R}} \frac{C^\infty(\mathbb{R}^n, \mathbf{0})}{\langle h \rangle},$$

where $\langle h\rangle$ is the ideal generated by the components of h. It is known that $\mu(h,\mathbf{0})$ is the number of complex h-preimages near $\mathbf{0}$ of a regular value of h near $\mathbf{0}$. If $h=(h_1,\dots,h_n)$, with each h_i being a homogeneous polynomial such that $\mathbf{0}$ is isolated in $h^{-1}(\mathbf{0})$, it is well known that $\mu(h,\mathbf{0})=d_1\cdots d_n$, where d_i is the degree of each h_i. On the other hand, writing $h=g+G$, where $g=(g_1,\dots,g_n)$ with g_i being the first non-zero jet of h_i, then $\mu(h,\mathbf{0})=\mu(g,\mathbf{0})$, if $\mathbf{0}$ is isolated in $g^{-1}(\mathbf{0})$. When $\mathbf{0}$ is not isolated in $g^{-1}(\mathbf{0})$ in the above construction, we can take a suitable selection of weights associated with any variable in order to make possible a different decomposition $h=g'+G'$ satisfying $\mu(h,\mathbf{0})=\mu(g',\mathbf{0})$. In fact, it is valid the same statement of Theorem 3.2, with multiplicity instead of index (see Remark 3.1 of [5]). Furthermore, one shall use the following result:

Proposition 5.2. ([1, 5]) *Using the above notations, let $h=g+G$ be a semi-quasi-homogeneous map germ with weight $\mathbf{a}=(a_1,\dots,a_n)$ and quasi-degree $\mathbf{d}=(d_1,\dots,d_n)$. Suppose that $\mu(h,\mathbf{0})<\infty$. Then*

$$\mu(h,\mathbf{0})=\mu(g,\mathbf{0})=\frac{d_1\cdots d_n}{a_1\cdots a_n}.$$

Let us denote by $\mathit{\Sigma} D_i$ the number of umbilic points of f_λ, that is, $\mathit{\Sigma} D_i=D_1+D_2+D_3$. So, one gets the following result:

Proposition 5.3. *Under the same assumptions in Theorem 3.4, if the umbilic curvature of f is non-zero at the origin and f_λ is a generic deformation of f, then the number of umbilic points of f_λ, for λ small enough, if finite, satifies:*

(i) $f\sim S_k^{\pm}, k\geq 1$: $\quad \mathit{\Sigma} D_i\leq k+1$ *with* $\mathit{\Sigma} D_i\equiv k+1\,(\mathrm{mod}\,2)$.
(ii) $f\sim C_k^{\pm}, k\geq 3$: $\quad \mathit{\Sigma} D_i\leq k$ *with* $\mathit{\Sigma} D_i\equiv k\,(\mathrm{mod}\,2)$.
(iii) $f\sim B_k^{\pm}$ *or* $H_k, k\geq 2$: $\quad \mathit{\Sigma} D_i = 0$ *or* 2.
(iv) $f\sim F_4$ *or* P_3: $\mathit{\Sigma} D_i=1$ *or* 3.

Furthermore, $D_3\geq W$ when $\mathrm{ind}_{\mathcal{P}}(f,\mathbf{0})=0$, and $D_3\geq \frac{W}{2}$ when $\mathrm{ind}_{\mathcal{P}}(f,\mathbf{0})=1$.

Proof. We shall count the number $n_{\mathcal{R}}(f_\lambda,\mathbf{0})$ of 2-roundings of f_λ. Let us take $f=(x,f_2,f_3)$ as in Proposition 2.2.

Since f is not a cross-cap and $\kappa_u(\mathbf{0})=|a_{20}|\neq 0$, it follows from Corollary 2.17 of [21] that $\mathbf{0}$ is a non-flat 2-rounding of f. From [11] we conclude that $\mathcal{R}(f,\mathbf{0})=\langle P_y,P_{xy}\rangle$ if $j^2f(\mathbf{0})\sim(x,y^2,0)$ and $\mathcal{R}(f,\mathbf{0})=\langle P_y,P_{yy}\rangle$ if $j^2f(\mathbf{0})\sim(x,xy,0)$, where

$$P_y=\begin{vmatrix} f_x & 0\\ f_{xx} & E\\ f_{xy} & F\\ f_{yy} & G\end{vmatrix},\quad P_{xy}=\begin{vmatrix} f_x & 0\\ f_y & 0\\ f_{xx} & E\\ f_{yy} & G\end{vmatrix}\quad\text{and}\quad P_{yy}=\begin{vmatrix} f_x & 0\\ f_y & 0\\ f_{xx} & E\\ f_{xy} & F\end{vmatrix}.$$

Let $h:(\mathbb{R}^2,\mathbf{0})\to(\mathbb{R}^2,\mathbf{0})$ given by $h=(P_y,P_{xy})$ or (P_y,P_{yy}). Then

$$\mu_{\mathcal{R}}(f,\mathbf{0})\;=\;\dim_{\mathbb{R}}\frac{C^\infty(\mathbb{R}^2,\mathbf{0})}{\langle h\rangle}\;=\;\mu(h,\mathbf{0}).$$

• Let us suppose that $j^2f(\mathbf{0})\sim(x,y^2,0)$.
If $f\sim S_1^{\pm}$ or $B_k^{\pm}$ then $a_{21}\neq 0$. After some calculations we take $h=g+G$, where

$$g(x,y)=(-a_{21}x-a_{12}y,\frac{1}{2}a_{21}x^2-\frac{1}{2}a_{03}y^2)$$

and G has higher-order terms. Since the resultant of g is given by the expression $\frac{1}{2}a_{21}(a_{12}^2-a_{21}a_{03})$ and $a_{12}^2-a_{21}a_{03}\neq 0$ by hypothesis, we have that $\mathbf{0}$ is isolated in $g^{-1}(\mathbf{0})$ and it holds that

$$n_{\mathcal{R}}(f_\lambda,\mathbf{0}) \leq \mu_{\mathcal{R}}(f,\mathbf{0}) \,=\, \mu(g,\mathbf{0}) \,=\, 2\,.$$

By Lemma 5.1, ΣD_i is even for $S_1^{\pm}$ and $B_k^{\pm}$ and, therefore, $\Sigma D_i = 0, 2$.

If $f \sim S_{k\geq 2}$, $C_k^{\pm}$ or F_4 then we reproduce the same steps as in previous case, taking an apropriated g such that $h = g + G$ satisfies the Corollary 5.2, getting after calculations the desired results.

• Let us suppose now that $j^2 f(\mathbf{0}) \sim (x, xy, 0)$. We take $f \sim H_k$ or P_3, depending on a_{03} is non-zero or zero, respectively. Since $h = (P_y, P_{yy})$, we take $h = g + G$, where

$$g(x,y) = \left(a_{12}x + a_{03}y, -\frac{1}{2}a_{21}x^2 + \frac{1}{2}a_{03}y^2\right)$$

when f is of H_k type, or $g(x,y) = (a_{12}x + (\frac{1}{2}a_{04} - a_{12}b_3)y^2, -\frac{1}{2}a_{21}x^2)$, when f is of P_3 type with $a_{21} \neq 0$, or $g(x,y) = (a_{12}x + (\frac{1}{2}a_{04} - a_{12}b_3)y^2, (\frac{1}{6}a_{04} - \frac{1}{2}a_{12}b_3)y^3)$, when f is of P_3 type with $a_{21} = 0$, with G having higher-order terms. Since $a_{12} \neq 0$ from hypothesis, and $a_{04} - 3a_{12}b_3 \neq 0$ when f is of P_3 type, which appear in the expression of the resultant of g, then we conclude that h is semi-quasi-homogeneous and so, it follows that $\mu(h,\mathbf{0}) = \mu(g,\mathbf{0}) = 2$ if $f \sim H_k$, and $\mu(h,\mathbf{0}) = \mu(g,\mathbf{0}) = 4$ if $f \sim P_3$ type. So, the result on ΣD_i follows from Lemma 5.1.

For the second part of the proposition, it is enough to use the relation

$$D_1 + D_2 - D_3 = 2\,\mathrm{ind}_{\mathcal{P}}(f,\mathbf{0}) + W.$$

□

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References

[1] V. Arnold, A. Varchenko and S. Gusein-Zade, *Singularités des applications différentiables*, Mir, Moscow, 1982.

[2] J. W. Bruce and D. L. Fidal, *On binary differential equations and umbilics*, Proc. Royal Soc. Edinburgh Sect A, 111 (1989), 147-168.

[3] J. W. Bruce and J. M. West, *Functions on a crosscap*, Math. Proc. Cambridge Philos. Soc., 123 (1998), 19-39.

[4] J. W. Bruce and F. Tari, *On the multiplicity of implicit differential equations*, J. Diff. Eq. 148 (1998), 122-147. DOI: 10.1006/jdeq.1998.3454

[5] A. Cima, A. Gasull, J. Torregrosa, *On the relation between index and multiplicity*, J. London. Math. Soc. (2) 57 (1998), 757-768. DOI: 10.1112/s0024610798006188

[6] L S. Challapa, *Index of quadratic differentail forms*, Real and complex singularities, Contemp. Math. 459, Amer. Math. Soc., Providence, RI (2008), 179-193. DOI: 10.1090/conm/459/08970

[7] L. S. Challapa, *Invariants of binary differential equations*, J. Dyn. Control Syst. 15 (2009), no. 2, 157-176. DOI: 10.1007/s10883-009-9066-z

[8] L. S. Challapa and M. A. S. Ruas, *Index of an implicit differential equation*, Publ. Mat. 54 (2010), no. 1, 173-186.

[9] G. Darboux, *Leçons sur la théorie générales des surfaces*, vol. 4, Gauthier-Villars, Paris, 1896.

[10] T. Fukui and M. Hasegawa, *Fronts of Whitney umbrella - a differential geometric approach via blowing up*, J. Singularities 4 (2012), 35-67. DOI: 10.5427/jsing.2012.4c

[11] T. Fukui and J.J. Nuño-Ballesteros, *Isolated roundings and flattenings of submanifolds in Euclidean spaces*, Tohoku Math. J. 57 (2005), 469-503. DOI: 10.2748/tmj/1140727069

[12] R. Garcia, J. Sotomayor and C. Gutierrez, *Lines of principal curvature around umbilics and Whitney umbrellas*, Tohoku Math. J. (2) 52 (2000), 163-172. DOI: 10.2748/tmj/1178224605

[13] V. Guiñez, *Positive quadratic differential forms and foliations with singularities on surfaces*, Trans. Amer. Math. Soc. 309 (1988), 477-502. DOI: 10.1090/s0002-9947-1988-0961601-4

[14] C. Gutierrez and J. Sotomayor, *Lines of principal curvature for mappings with Whitney umbrella singularities*, Tohoku Math. J. (2) 38 (1986), 551-559. DOI: 10.2748/tmj/1178228407

[15] M. Hasegawa, *Functions on singular surfaces*, Math. Anal. Lab. Records, Kyoto University Research Information Repository (2013), 1868: 1–8.
[16] M. Hasegawa, A. Honda, K. Naokawa, M. Umehara and K. Yamada, *Intrinsic invariants of cross caps*, Selecta Math. (N.S.) 20 (2014), no. 3, 769-785. DOI: 10.1007/s00029-013-0134-6
[17] M. Hasegawa, A. Honda, K. Naokawa, K. Saji, M. Umehara and K. Yamada, *Intrinsic properties of surfaces with singularities*, Internat. J. Math. 26 (2015), no. 4, 1540008, 34 pp. DOI: 10.1142/s0129167x1540008x
[18] H. Hopf, *Differential geometry in the large*, Lecture Notes in Math. 1000, Springer, New York, 1983.
[19] V. V. Ivanov, *An analytic conjecture of Carathéodory*, Siberian Math. J. 43 (2002), 251-322.
[20] S. Izumiya, M. C. R. Fuster, M. A. S. Ruas and F. Tari, *Differential geometry from a singularity theory viewpoint*, World Scientific Pub. Co. Pte. Ltd., Hackesanck, NJ, 2016. DOI: /doi.org/10.1142/9108
[21] L. F. Martins and J. J. Nuño-Ballesteros, *Contact properties of surfaces in* $\mathbb{R}^3$ *with corank 1 singularities*, Tohoku Math. J. (2) 67 (2015), no. 1, 105-124. DOI: 10.2748/tmj/1429549581
[22] L. F. Martins and K. Saji, *Geometric invariants of cuspidal edges*, Canadian J. Math. 68 (2016), no. 2, 445-462. DOI: /10.4153/cjm-2015-011-5
[23] D. Mond, *On the classification of germs of maps from* $\mathbb{R}^2$ *to* $\mathbb{R}^3$, Proc. London Math. Soc. (3) 50 (1985), 333-369. DOI: 10.1112/plms/s3-50.2.333
[24] D. Mond, *Some remarks on the geometry and classification of germs of maps from surfaces to 3-space*, Topology (26) 3 (1987), 361-383. DOI: 10.1016/0040-9383(87)90007-3
[25] J. J. Nuño-Ballesteros and F. Tari, *Surfaces in* $\mathbb{R}^4$ *and their projections to 3-spaces*, Proc. Roy. Soc. Edinburgh Sect. A, 137 (2007), 1313-1328.
[26] R. Oset Sinha and F. Tari, *Projections of surfaces in* $\mathbb{R}^4$ *to* $\mathbb{R}^3$ *and the geometry of their singular images*, Rev. Mat. Iberoam. 31 (2015), no. 1, 33-50. DOI: 10.4171/rmi/825
[27] R. Oset Sinha and F. Tari, *On the flat geometry of cuspidal edges*, Osaka J. Math. 55 (2018), no. 3, 393-421.
[28] K. Saji, M. Umehara and K. Yamada, *The geometry of fronts*, Ann. of Math. (2) 169 (2009), 491-529. DOI: 10.4007/annals.2009.169.491
[29] F. Tari, *Pairs of foliations on surfaces*, Real and complex singularities, 305-337, London Math. Soc. Lecture Note Ser., 380, Cambridge Univ. Press, Cambridge, 2010.
[30] C. J. Titus, *A proof of a conjecture of Loewner and of the conjecture of Caratheodory on umbilic points*, Acta Math. 131 (1973), 43-77. DOI: 10.1007/bf02392036
[31] C. T. C. Wall, *Finite determinacy of smooth map-germs*, Bull. London Math. Soc. 13 (1981), no. 6, 481–539.
[32] J. West, *The differential geometry of the cross-cap*, Ph.D. Thesis, Liverpool University, 1995.

J. C. F. Costa, Departamento de Matemática, UNESP - Universidade Estadual Paulista, R. Cristóvão Colombo, 2265, CEP 15054-000, São José do Rio Preto-SP, Brazil
Email address: joao.costa@unesp.br

L. F. Martins, Departamento de Matemática, UNESP - Universidade Estadual Paulista, R. Cristóvão Colombo, 2265, CEP 15054-000, São José do Rio Preto-SP, Brazil
Email address: luciana.martins@unesp.br

J. J. Nuño-Ballesteros, Departament de Geometria i Topologia, Universitat de València, Campus de Burjassot 46100, Spain
Email address: Juan.Nuno@uv.es