Journal *of* Singularities
Volume 21 (2020), 282-300

Proc. of Geometric and Algebraic Singularity Theory, Będlewo, 2017
DOI: 10.5427/jsing.2020.21q

A DESCRIPTION OF A RESULT OF DELIGNE BY LOG HIGHER ALBANESE MAP

SAMPEI USUI

Dedicated to Goo Ishikawa on his sixtieth birthday

Abstract. In a joint work with Kazuya Kato and Chikara Nakayama, log higher Albanese manifolds were constructed as an application of log mixed Hodge theory with group action. In this framework, we describe a work of Deligne on some nilpotent quotients of the fundamental group of the projective line minus three points, where polylogarithms appear. As a result, we have q-expansions of higher Albanese maps at boundary points, i.e., log higher Albanese maps over the boundary.

0. Introduction

We review the results of [11]: - General theory of log mixed Hodge structures with polarizable graded quotients endowed with group actions. - Description of the functors represented by higher Albanese manifolds in terms of tensor functors. - Toroidal partial compactifications of higher Albanese manifolds to get log higher Albanese manifolds, and describe the functors represented by them.

We describe a result of Deligne in [3] about polylogarithms, which were appeared in higher Albanese maps, in terms of the log higher Albanese maps. The advantage of our formulation is that log higher Albanese maps have q expansions at the boundary points over which we observe directly $\zeta(n)$ $(n \geq 2)$ as values of polylogarithms.

For readers' convenience, we add as Appendix a summary of the related result of Deligne in [3].

Actually, for the present description in Section 3, it is enough to use the formulation of spaces of nilpotent orbits in [10] Part III. The formulation of them in [11] is reviewed in Sections 1 and 2 for further study in the case of higher Albanese manifolds with non-trivial Hodge structures.

1. Log mixed Hodge structures with group action

We review general formulations and results of log mixed Hodge structures with group action in [11] and [10] Parts III, IV, in a minimal size for the later use of this paper. The full version will be appeared in [10] Part V.

1.1. A *log structure* on a ringed space $(S, \mathcal{O}_S)$ consists of a sheaf of monoids M on S and a homomorphism $\alpha : M \to \mathcal{O}_S$ such that $\alpha^{-1}(\mathcal{O}_S^\times) \xrightarrow{\sim} \mathcal{O}_S^\times$.

1.2. Example. Let $S = \mathbf{C}$ and $\{0\}$ a divisor. The associated log structure is

$$M = \{f \in \mathcal{O}_S \mid f \text{ is invertible on } S \smallsetminus \{0\}\}.$$

2010 *Mathematics Subject Classification.* Primary 14C30; Secondary 14D07, 32G20.

Key words and phrases. Hodge theory, log Hodge structure, log higher Albanese map, polylogarithm, zeta value.

$S^{\log}$ is defined to be the set of all pairs (s,h) consisting of a point $s\in S$ and an argument function h which is a homomorphism $M_s\to\mathbf{S}^1$ of monoids whose restriction to $\mathcal{O}_{S,s}^{\times}$ is $u\mapsto u(s)/|u(s)|$.

In this case, the ringed space $(S^{\log},\mathcal{O}_S^{\log})$ is explained as follows. Let

$$\tilde{S}^{\log}:=\mathbf{C}\cup(\mathbf{R}\times i\infty)=\mathbf{R}\times i(\mathbf{R}\cup\infty)$$

endowed with coordinate function $z=x+iy$ $(-\infty<y\le\infty)$. Let $S^{\log}:=(\mathbf{C}\cup(\mathbf{R}\times i\infty))/\mathbf{Z}$, and consider maps $\tilde{S}^{\log}\to S^{\log}\to S: z=x+iy\mapsto(e^{-2\pi y},e^{2\pi ix})\mapsto q:=e^{2\pi iz}$. Note that $(e^{-2\pi y},e^{2\pi ix})$ is a polar coordinate extended over $-\infty<y\le\infty$, and $S^{\log}\to S$ is a real oriented blowing-up at $\{0\}$, which is proper. $h:M_0\to\mathbf{S}^1$ in $t:=(0,h)\in S^{\log}$ sends q to $e^{2\pi ix}$. Since z is considered as a branch of $(2\pi i)^{-1}\log(q)$, we have $\mathcal{O}_{S,t}^{\log}=\mathcal{O}_{S,0}[z]$ which is isomorphic to a polynomial algebra $\mathcal{O}_{S,0}[T]$ of one indeterminate T over $\mathcal{O}_{S,0}$ under $z\leftrightarrow T$ ([12] 2.2.5).

For more general and finer treatment, see [9], [12] 2.2.

1.3. Let G be a linear algebraic group over $\mathbf{Q}$. Let G_u be the unipotent radical of G and let $G_{\mathrm{red}}=G/G_u$. Let $\mathrm{Rep}(G)$ be the category of finite-dimensional linear representations of G over $\mathbf{Q}$.

1.4. Let $k_0:\mathbf{G}_m\to G_{\mathrm{red}}$ be a $\mathbf{Q}$-rational and central homomorphism. Assume that, for one (hence all) lifting $\mathbf{G}_{m,\mathbf{R}}\to G_{\mathbf{R}}$ of k_0, the adjoint action of $\mathbf{G}_{m,\mathbf{R}}$ on $\mathrm{Lie}\,(G_u)_{\mathbf{R}}=\mathbf{R}\otimes_{\mathbf{Q}}\mathrm{Lie}\,(G_u)$ is of weight ≤ -1.

Then, for any $V\in\mathrm{Rep}(G)$, the action of $\mathbf{G}_m$ on V via a lifting $\mathbf{G}_m\to G$ of k_0 defines an increasing filtration W on V over $\mathbf{Q}$, called *weight filtration*, which is independent of the lifting.

1.5. Assume that we are given a homomorphism $k_0:\mathbf{G}_m\to G_{\mathrm{red}}$ as in 1.4. A *G-mixed Hodge structure* (*G-MHS*, for short) *of type* k_0 is an exact $\otimes$-functor ([4] 2.7) from $\mathrm{Rep}(G)$ to the category of $\mathbf{Q}$-mixed Hodge structures keeping the underlying vector spaces with weight filtrations.

1.6. As in [2], let $S_{\mathbf{C}/\mathbf{R}}$ be the Weil restriction of scalars of $\mathbf{G}_m$ from $\mathbf{C}$ to $\mathbf{R}$. It represents the functor $A\mapsto(\mathbf{C}\otimes_{\mathbf{R}}A)^{\times}$ for commutative rings A over $\mathbf{R}$. In particular, $S_{\mathbf{C}/\mathbf{R}}(\mathbf{R})=\mathbf{C}^{\times}$, which is understood as $\mathbf{C}^{\times}$ regarded as an algebraic group over $\mathbf{R}$.

Let $w:\mathbf{G}_m\to S_{\mathbf{C}/\mathbf{R}}$ be the homomorphism induced from the natural map $A^{\times}\to(\mathbf{C}\otimes_{\mathbf{R}}A)^{\times}$.

1.7. The following (1) and (2) are equivalent:

(1) A finite-dimensional linear representation of $S_{\mathbf{C}/\mathbf{R}}$ over $\mathbf{R}$.

(2) A finite-dimensional $\mathbf{R}$-vector space V with a decomposition

$$V_{\mathbf{C}}:=\mathbf{C}\otimes_{\mathbf{R}}V=\bigoplus_{p,q\in\mathbf{Z}}V_{\mathbf{C}}^{p,q}$$

such that, for any p,q, $V_{\mathbf{C}}^{q,p}$ is complex conjugate of $V_{\mathbf{C}}^{p,q}$ (Hodge decomposition).

For a finite-dimensional linear representation V of $S_{\mathbf{C}/\mathbf{R}}$, the corresponding decomposition is defined by

$$V_{\mathbf{C}}^{p,q}:=\{v\in V_{\mathbf{C}}\mid [z]v=z^p\bar{z}^q v\text{ for }z\in\mathbf{C}^{\times}\}.$$

Here $[z]$ denotes $z\in\mathbf{C}^{\times}$ regarded as an element of $S_{\mathbf{C}/\mathbf{R}}(\mathbf{R})$.

1.8. Let H be a G-MHS of type k_0 (1.5). By 1.7 and Tannaka duality (cf. [4] 1.12 Théorème), the Hodge decompositions of gr^W of $H(V)$ for $V \in \mathrm{Rep}(G)$ give a homomorphism $S_{\mathbf{C}/\mathbf{R}} \to (G_{\mathrm{red}})_{\mathbf{R}}$ such that the composite $\mathbf{G}_m \overset{w}{\to} S_{\mathbf{C}/\mathbf{R}} \to (G_{\mathrm{red}})_{\mathbf{R}}$ coincides with k_0. We call this

$$S_{\mathbf{C}/\mathbf{R}} \to (G_{\mathrm{red}})_{\mathbf{R}}$$

the *associated homomorphism with* H.

1.9. Let $k_0 : \mathbf{G}_m \to G_{\mathrm{red}}$ be as in 1.4. Fix a homomorphism $h_0 : S_{\mathbf{C}/\mathbf{R}} \to (G_{\mathrm{red}})_{\mathbf{R}}$ such that $h_0 \circ w = k_0$.

G-mixed Hodge structure of type h_0 is a G-mixed Hodge structure of type k_0 (1.5) whose associated homomorphism $S_{\mathbf{C}/\mathbf{R}} \to (G_{\mathrm{red}})_{\mathbf{R}}$ (1.8) is $G_{\mathrm{red}}(\mathbf{R})$-conjugate to h_0.

The *period domain* $D = D(G, h_0)$ associated to (G, h_0) is defined to be the set of isomorphism classes of G-mixed Hodge structures of type h_0.

1.10. Usual period domains of Griffiths [5] and their generalization for mixed Hodge structures [13] are special cases of the present period domains.

Let $\Lambda = (H_0, W, (\langle\ ,\ \rangle_w)_w, (h^{p,q})_{p,q})$ be the Hodge data as usual as in [10] Part III. Let G be the subgroup of $\mathrm{Aut}(H_{0,\mathbf{Q}}, W)$ consisting of elements which induce *similitudes for* $\langle\ ,\ \rangle_w$ for each w. That is, $G := \{g \in \mathrm{Aut}(H_{0,\mathbf{Q}}, W) \mid$ for any w, there is a $t_w \in \mathbf{G}_m$ such that $\langle gx, gy\rangle_w = t_w\langle x, y\rangle_w$ for any $x, y \in \mathrm{gr}^W_w\}$. Let $G_1 := \mathrm{Aut}(H_{0,\mathbf{Q}}, W, (\langle\ ,\ \rangle_w)_w) \subset G$.

Let $D(\Lambda)$ be the period domain of [13]. Then $D(\Lambda)$ is identified with an open and closed part of D in this paper as follows.

Assume that $D(\Lambda)$ is not empty and fix an $\mathbf{r} \in D(\Lambda)$. Then the Hodge decomposition of $\mathrm{gr}^W\mathbf{r}$ induces $h_0 : S_{\mathbf{C}/\mathbf{R}} \to (G_{\mathrm{red}})_{\mathbf{R}}$. (We have $\langle [z]x, [z]y\rangle_w = |z|^{2w}\langle x, y\rangle_w$ for $z \in \mathbf{C}^\times$ (see 1.6 for $[z]$).) Consider the associated period domain D (1.9). Then D is a finite disjoint union of $G_1(\mathbf{R})G_u(\mathbf{C})$-orbits which are open and closed in D. Let $\mathcal{D}$ be the $G_1(\mathbf{R})G_u(\mathbf{C})$-orbit in D consisting of points whose associated homomorphisms $S_{\mathbf{C}/\mathbf{R}} \to (D_{\mathrm{red}})_{\mathbf{R}}$ are $(G_1/G_u)(\mathbf{R})$-conjugate to h_0. Then the map $H \mapsto H(H_{0,\mathbf{Q}})$ gives a $G_1(\mathbf{R})G_u(\mathbf{C})$-equivariant isomorphism $\mathcal{D} \simeq D(\Lambda)$.

1.11. Fix a homomorphism $h_0 : S_{\mathbf{C}/\mathbf{R}} \to (G_{\mathrm{red}})_{\mathbf{R}}$ as in 1.9.

Let $\mathcal{C}$ be the category of triples (V, W, F) consisting of a finite-dimensional $\mathbf{Q}$-vector space V, an increasing filtration W on V (called the weight filtration), and a decreasing filtration F on $V_{\mathbf{C}}$ (called the Hodge filtration).

Let Y be the set of all isomorphism classes of exact $\otimes$-functors from $\mathrm{Rep}(G)$ to $\mathcal{C}$ preserving the underlying vector spaces with weight filtrations.

Then $G(\mathbf{C})$ acts on Y by changing the Hodge filtration F. We have $D \subset Y$ and D is a $G(\mathbf{R})G_u(\mathbf{C})$-orbit in Y (cf. [11] Proposition 3.2.5). We define $\check{D} := G(\mathbf{C})D$ in Y. Thus

$$D \subset \check{D} = G(\mathbf{C})D \subset Y.$$

$\check{D}$ is a $G(\mathbf{C})$-homogeneous space and D is an open subspace. Hence $\check{D}$ and D are complex analytic manifolds.

1.12. Let $h_0 : S_{\mathbf{C}/\mathbf{R}} \to (G_{\mathrm{red}})_{\mathbf{R}}$ be as in 1.9. Let C be the image of $i \in \mathbf{C}^\times = S_{\mathbf{C}/\mathbf{R}}(\mathbf{R})$ by h_0 in $(G_{\mathrm{red}})(\mathbf{R})$ (Weil operator). We say that h_0 is *$\mathbf{R}$-polarizable* if $\{a \in (G_{\mathrm{red}})'(\mathbf{R}) \mid Ca = aC\}$ is a maximal compact subgroup of $(G_{\mathrm{red}})'(\mathbf{R})$. Here $(G_{\mathrm{red}})'$ denotes the commutator subgroup of G_{red}.

1.13. Let $h_0 : S_{\mathbf{C}/\mathbf{R}} \to (G_{\mathrm{red}})_{\mathbf{R}}$ be $\mathbf{R}$-polarizable (1.12).

Let Γ be a subgroup of $G(\mathbf{Q})$ for which there is a faithful $V \in \mathrm{Rep}(G)$ and a $\mathbf{Z}$-lattice L in V such that L is stable under the action of Γ.

Then the following holds ([11] Proposition 3.3.4):
(1) The action of Γ on D is proper, and the quotient space $\Gamma \setminus D$ is Hausdorff.
(2) If Γ is torsion-free and if $\gamma p = p$ with $\gamma \in \Gamma$ for some $p \in D$, then $\gamma = 1$.
(3) If Γ is torsion-free, then the projection $D \to \Gamma \setminus D$ is a local homeomorphism.

1.14. Let (G, h_0) be as above.

A *nilpotent cone* is a cone σ over $\mathbf{R}_{\geq 0}$ in $\mathrm{Lie}\,(G)_{\mathbf{R}}$ generated by a finite number of mutually commuting elements such that, for any $V \in \mathrm{Rep}(G)$, the image of σ under the induced map $\mathrm{Lie}\,(G)_{\mathbf{R}} \to \mathrm{End}_{\mathbf{R}}(V)$ consists of nilpotent operators.

For $F \in \check{D}$ and a nilpotent cone σ, $(\sigma, \exp(\sigma_{\mathbf{C}})F)$ is a *nilpotent orbit* if it satisfies the following conditions: Take a generators $N_1, \dots, N_n \in \mathrm{Lie}\,(G)_{\mathbf{R}}$ of the cone σ.

(1) (admissibility) There is a faithful $V \in \mathrm{Rep}(G)$ such that the relative monodromy weight filtrations $M(N_j, W)$ on V exist for all $1 \leq j \leq n$.

(2) (Griffiths transversality) $N_j F^p \subset F^{p-1}$ for any $1 \leq j \leq n$, $p \in \mathbf{Z}$.

(3) (limit mixed Hodge property) $\exp(\sum_{j=1}^n iy_j N_j)F \in D$ if $y_j \in \mathbf{R}_{>0}$ are sufficiently large.

This is well-defined, i.e., independent of choices of generators $N_1, \dots, N_n$.

Note that, for admissibility, the above condition (1) is enough under the assumption of $\mathbf{R}$-polarizability (cf. [7], [10] III Proposition 1.3.4, Remark in 2.2.2, [8] Proposition 2.1.10).

1.15. A *weak fan* Σ *in* $\mathrm{Lie}\,(G)$ is a nonempty set of sharp rational nilpotent cones satisfying the conditions that it is closed under taking faces and that any $\sigma, \sigma' \in \Sigma$ coincide if they have a common interior point and if there is an $F \in \check{D}$ such that both $(\sigma, \exp(\sigma_{\mathbf{C}})F)$ and $(\sigma', \exp(\sigma'_{\mathbf{C}})F)$ are nilpotent orbits.

For a weak fan Σ in $\mathrm{Lie}\,(G)$, let D_Σ be the set of all nilpotent orbits $(\sigma, \exp(\sigma_{\mathbf{C}})F)$ with $\sigma \in \Sigma$ and $F \in \check{D}$.

1.16. Let Γ be a subgroup of $G(\mathbf{Q})$ as in 1.13.

A weak fan Σ in 1.15 is said to be *strongly compatible with* Γ if Σ is stable under the adjoint action of Γ and each cone $\sigma \in \Sigma$ is generated over $\mathbf{R}_{\geq 0}$ by log of $\exp(\sigma) \cap \Gamma$.

1.17. $\mathcal{B}$ denotes the category of locally ringed spaces with a covering by open sets each of which has the strong topology in an analytic space. $\mathcal{B}(\log)$ denotes the category of objects of $\mathcal{B}$ endowed with an fs log structure. For precise definitions of these, see [12] 3.2.4, [10] Part III 1.1.

1.18. Let $S \in \mathcal{B}(\log)$. A **Q**-*log mixed Hodge structure* (**Q**-*LMH*, for short) *with* **R**-*polarizable graded quotients on* S is $(H_{\mathbf{Q}}, W, H_{\mathcal{O}}, F)$ consisting of locally constant sheaf $H_{\mathbf{Q}}$ with an increasing filtration W of $H_{\mathbf{Q}}$ on $(S^{\log}, \mathcal{O}_S^{\log})$, locally free sheaf $H_{\mathcal{O}}$ with a decreasing filtration F of $H_{\mathcal{O}}$ on $(S, \mathcal{O}_S)$ such that gr_F^p is locally free for all p, and a specified isomorphism $\mathcal{O}_S^{\log} \otimes_{\mathbf{Q}} H_{\mathbf{Q}} \simeq \mathcal{O}_S^{\log} \otimes_{\mathcal{O}_S} H_{\mathcal{O}}$, whose pullbacks to each fs log point $s \in S$ satisfy the following conditions (1)–(3).

(1) (admissibility) Let $N_1, \dots, N_n$ be a generator of the local monodromy cone

$$C(s) := \mathrm{Hom}\,(M_s/\mathcal{O}_s^\times, \mathbf{R}_{\geq 0}) \subset \pi_1(s^{\log}).$$

The relative monodromy weight filtrations $M(N_j, W)$ exists for all $1 \leq j \leq n$.

(2) (Griffiths transversality) $\nabla F^p \subset \omega_s^{1,\log} \otimes_{\mathcal{O}_s} F^{p-1}$ for all $p \in \mathbf{Z}$.

(3) ($\mathbf{R}$-polarizability on graded quotients) For each $w \in \mathbf{Z}$, there is a non-degenerate $(-1)^w$-symmetric bilinear form $\langle\ ,\ \rangle_w : H(\mathrm{gr}_w^W)_{\mathbf{R}} \times H(\mathrm{gr}_w^W)_{\mathbf{R}} \to \mathbf{R}$ over $\mathbf{R}$ such that the quadruple $(H(\mathrm{gr}_w^W), \langle\ ,\ \rangle_w, H(\mathrm{gr}_w^W)_{\mathcal{O}}, F(\mathrm{gr}_w^W))$ is an $\mathbf{R}$-polarized log Hodge structure of weight w on s. The last part means as follows. Let $q_1, \dots, q_r \in M_s \smallsetminus \mathcal{O}_s^\times$ whose classes generate the monoid $M_s/\mathcal{O}_s^\times$. For $t \in s^{\log}$ and $a \in \mathrm{sp}(t)$ with $\exp(a(\log(q_j)))$ sufficiently small for all

$1 \le j \le r$, $(H(\mathrm{gr}^W_w), \langle\ ,\ \rangle_w, H(\mathrm{gr}^W_w)_{\mathcal{O}}, F(\mathrm{gr}^W_w)(a))$ is an $\mathbf{R}$-polarized Hodge structure. Here we use $H(\mathrm{gr}^W_w)_{\mathbf{R}} := \mathbf{R} \otimes_{\mathbf{Q}} H(\mathrm{gr}^W_w)$, $H(\mathrm{gr}^W_w)_{\mathcal{O}} := \mathcal{O}_s \otimes_{\mathbf{Q}} H(\mathrm{gr}^W_w)$, $F(\mathrm{gr}^W_w) := F(H(\mathrm{gr}^W_w)_{\mathcal{O}})$.

Note that, in [12] Definition 2.4.7, [10] Part III 1.3.2, rational polarizations on graded quotients were used. But, in the present paper, we use $\mathbf{R}$-polarizability on graded quotients. Even under this latter condition, the proof of [10] Part III Proposition 1.3.4 works.

1.19. Definition. Given (G, h_0) and Γ as in 1.13. Let $S \in \mathcal{B}(\log)$.

A *G-log mixed Hodge structure with a Γ-level structure on S* is (H, μ) consisting of an exact $\otimes$-functor $H : \mathrm{Rep}(G) \to \mathrm{LMH}(S); (V, W) \mapsto (V, W, F)$ and a global section μ of the quotient sheaf $\Gamma\backslash\mathcal{I}$.

Here $\mathcal{I}$ is the following sheaf on $S^{\log}$. For an open set U of $S^{\log}$, $\mathcal{I}(U)$ is the set of all isomorphisms $H_{\mathbf{Q}}|_U \xrightarrow{\sim} \mathrm{id}$ of $\otimes$-functors from $\mathrm{Rep}(G)$ to the category of local systems of $\mathbf{Q}$-modules V on U preserving the weight filtration W.

1.20. Let (G, h_0) be as in 1.13 and let Γ and Σ be as in 1.16.

A G-LMH H on S with a Γ-level structure μ is said to be *of type* (h_0, Σ) if the following (i) and (ii) are satisfied for any $s \in S$ and any $t \in s^{\log}$. Take a $\otimes$-isomorphism $\tilde{\mu}_t : H_{\mathbf{Q},t} \xrightarrow{\sim} \mathrm{id}$ which belongs to μ_t.

(i) There is a cone $\sigma \in \Sigma$ such that the logarithm of the action of the cone $\mathrm{Hom}\,((M_S/\mathcal{O}_S^\times)_s, \mathbf{N}) \subset \pi_1(s^{\log})$ on $H_{\mathbf{Q},t}$ is contained, via $\tilde{\mu}_t$, in $\sigma \subset \mathrm{Lie}\,(G)_{\mathbf{R}}$.

(ii) Let $\sigma \in \Sigma$ be the smallest cone satisfying (i). Let $a : \mathcal{O}^{\log}_{S,t} \to \mathbf{C}$ be a ring homomorphism which induces the evaluation $\mathcal{O}_{S,s} \to \mathbf{C}$ at s and consider the Hodge filtration F of the functor $V \mapsto \tilde{\mu}_t a(H(V))$ in Y. Then this functor belongs to $\check{D}$ and (σ, F) generates a nilpotent orbit.

If (H, μ) is of type (h_0, Σ), we have a map $S \to \Gamma \setminus D_\Sigma$, called the *period map* associated to (H, μ), which sends $s \in S$ to the class of the nilpotent orbit $(\sigma, Z) \in D_\Sigma$ which is obtained in (ii).

1.21. Let (G, h_0) be as in 1.13 and let Γ and Σ be as in 1.16.

Introduce on $\Gamma \setminus D_\Sigma$ the *strong topology*, that is the strongest topology for which the period map $S \to \Gamma \setminus D_\Sigma$ is continuous for all (S, H, μ), and introduce a sheaf of holomorphic functions $\mathcal{O}$ and a log structure M.

Theorem 1.22. *Let (G, h_0, Γ, Σ) be as in 1.21. Assume that h_0 is $\mathbf{R}$-polarizable (1.12). Then*

(1) *$\Gamma \setminus D_\Sigma$ is Hausdorff.*

From hereafter, assume that Γ is neat.

(2) *$\Gamma \setminus D_\Sigma$ is a log manifold (*[10] Part III 1.1.5*). In particular, $\Gamma \setminus D_\Sigma$ belongs to $\mathcal{B}(\log)$.*

(3) *$\Gamma \setminus D_\Sigma$ represents the contravariant functor from $\mathcal{B}(\log)$ to* (Set):

$S \mapsto \{$isomorphism class of G-LMH over S with a Γ-level structure of type (h_0, Σ) $\}$.

(4) *Let S be a connected, log smooth, fs log analytic space, and let U be the open subspace of S consisting of all points of S at which the log structure of S is trivial. Assume that $S \smallsetminus U$ is a smooth divisor.*

Let (H, μ) be a G-MHS over U of type h_0 (1.9) endowed with a Γ-level structure (1.19). Let $\varphi : U \to \Gamma \setminus D$ be the associated period map. Assume that (H, μ) extends to a G-LMH over S with a Γ-level structure of type (h_0, Σ).

Then, φ extends to a morphism $\overline{\varphi}$ in $\mathcal{B}(\log)$ in the following commutative diagram:

$$\begin{array}{ccc} S & \xrightarrow{\overline{\varphi}} & \Gamma \setminus D_\Sigma \\ \bigcup & & \bigcup \\ U & \xrightarrow{\varphi} & \Gamma \setminus D. \end{array}$$

2. Log higher Albanese manifolds

We review here formulations and results of higher Albanese manifolds in [6] and of log higher Albanese manifolds in [11].

2.1. Let X be a connected smooth algebraic variety over $\mathbf{C}$. Fix $b \in X$. Let Γ be a quotient group of $\pi_1(X,b)$ which is torsion-free and nilpotent.

Let $\mathcal{G} = \mathcal{G}_\Gamma$ be the unipotent algebraic group over $\mathbf{Q}$ whose Lie algebra is defined as follows: Let I be the augmentation ideal $\mathrm{Ker}(\mathbf{Q}[\Gamma] \to \mathbf{Q})$ of $\mathbf{Q}[\Gamma]$. Then $\mathrm{Lie}\,(\mathcal{G})$ is the $\mathbf{Q}$-subspace of $\mathbf{Q}[\Gamma]^\wedge := \varprojlim_n \mathbf{Q}[\Gamma]/I^n$ generated by all $\log(\gamma)$ $(\gamma \in \Gamma)$. The Lie product of $\mathrm{Lie}\,(\mathcal{G})$ is defined by $[x,y] = xy - yx$. We have $\Gamma \subset \mathcal{G}(\mathbf{Q})$.

2.2. Let $\pi_1 = \pi_1(X,b)$. Let J be the augmentation ideal $\mathrm{Ker}(\mathbf{Q}[\pi_1] \to \mathbf{Q})$ of $\mathbf{Q}[\pi_1]$. For a positive integer n, let Γ_n be the image of $\pi_1 \to \mathbf{Q}[\pi_1]/J^n$. Then $\mathrm{Lie}\,(\mathcal{G}_{\Gamma_n})$ has a mixed Hodge structure induced from de Rham theory on the path spaces over X by Chen's iterated integral.

For a given Γ as in 2.1, there exists $n \geq 1$ such that Γ is a quotient of Γ_n. Hereafter we assume that $\mathrm{Lie}\,(\mathcal{G}_\Gamma)$ has a quotient mixed Hodge structure of the one on $\mathrm{Lie}\,(\mathcal{G}_{\Gamma_n})$. Note that this mixed Hodge structure on $\mathrm{Lie}\,(\mathcal{G}_\Gamma)$ is independent of the choice of n.

We note that there is an insufficient statement on mixed Hodge structure on $\mathrm{Lie}\,(\mathcal{G}_\Gamma)$ in [11] 6.1.2. The authors of [11] agreed to correct this part, so as to assume the existence of this mixed Hodge structure on $\mathrm{Lie}\,(\mathcal{G}_\Gamma)$ as above in the present paper.

Let $\mathcal{G} = \mathcal{G}_\Gamma$. Let $F^0\mathrm{Lie}\,(\mathcal{G})_{\mathbf{C}}$ be the 0-th Hodge filter on $\mathrm{Lie}\,(\mathcal{G})_{\mathbf{C}}$ and let $F^0\mathcal{G}(\mathbf{C})$ be the corresponding subgroup of $\mathcal{G}(\mathbf{C})$. The *higher Albanese manifold* associated to (X,Γ) is defined in [6] as

$$A_{X,\Gamma} := \Gamma \setminus \mathcal{G}(\mathbf{C})/F^0\mathcal{G}(\mathbf{C}).$$

2.3. Take a $\mathbf{Q}$-MHS V_0 with polarizable gr^W having the $\mathbf{Q}$-MHS on $\mathrm{Lie}\,(\mathcal{G})_{\mathbf{Q}}$ with $\mathcal{G} = \mathcal{G}_\Gamma$ in 2.2 as a direct summand.

Let $Q \subset \mathrm{Aut}(V_{0,\mathbf{Q}})$ be the *Mumford-Tate group* associated to V_0, i.e., the Tannaka group of the Tannaka category $\langle V_0 \rangle$ generated by V_0: $\langle V_0 \rangle \xrightarrow{\sim} \mathrm{Rep}(Q)$. Explicitly, it is the smallest $\mathbf{Q}$-subgroup Q of $\mathrm{Aut}(V_{0,\mathbf{Q}})$ such that $Q_{\mathbf{R}}$ contains the image of the homomorphism $h : S_{\mathbf{C}/\mathbf{R}} \to \mathrm{Aut}(V_{0,\mathbf{R}})$ and such that $\mathrm{Lie}\,(Q)_{\mathbf{R}}$ contains δ. Here h and δ are determined by the canonical splitting of the $\mathbf{Q}$-MHS V_0 ([1], [10] Part II 1.2).

The action of Q on $\mathrm{Lie}\,(\mathcal{G})$ induces an action of Q on $\mathcal{G}$. By this, define a semi-direct product G of Q and $\mathcal{G}$:

$$1 \to \mathcal{G} \to G \to Q \to 1.$$

We have $\mathcal{G} \subset G_u$. We have $h_0 : S_{\mathbf{C}/\mathbf{R}} \to (Q_{\mathrm{red}})_{\mathbf{R}} = (G_{\mathrm{red}})_{\mathbf{R}}$ by using the Hodge decomposition of $\mathrm{gr}^W V_0$.

2.4. Let D_G (resp. D_Q) be the period domain D for G (resp. Q) and h_0 in 2.3. We have a canonical map $\Gamma \setminus D_G \to D_Q$ induced by $G \to Q$.

Let $b_Q \in D_Q$ be the isomorphism class of the evident functor $\mathrm{Rep}(Q) \to \mathbf{Q}$-MHS by definition in 2.3, and let $b_G \in D_G$ be the isomorphism class of $\mathrm{Rep}(G) \to \mathrm{Rep}(Q) \overset{b_Q}{\to} \mathbf{Q}$-MHS via the section $Q \hookrightarrow G$.

Let $\mathcal{D}$ be the fiber of the map $D_G \to D_Q$ over b_Q.

Theorem 2.5. *The map $G_u(\mathbf{C}) \to D_G$; $g \mapsto g \cdot b_G$ induces an isomorphism*

$$A_{X,\Gamma} = \Gamma \setminus \mathcal{G}(\mathbf{C})/F^0\mathcal{G}(\mathbf{C}) \simeq \Gamma \setminus \mathcal{D}$$

of analytic manifolds.

2.6. Let $\mathcal{C}_{X,\Gamma}$ be the category of variations of $\mathbf{Q}$-MHS $\mathcal{H}$ on X satisfying the following three conditions:

(1) For any $w \in \mathbf{Z}$, $\mathrm{gr}^W_w \mathcal{H}$ is a constant polarizable Hodge structure.

(2) $\mathcal{H}$ is *good at infinity* in the sense of [6] (1.5), i.e., there exists a smooth compactification $\overline{X}$ of X with normal crossing boundary divisor $\overline{X} \smallsetminus X$ such that the Hodge filtration bundles extend to sub-bundles of the canonical extension of $\mathcal{O}$-module of $\mathcal{H}$ which induce the corresponding thing for each $\mathrm{gr}^W_w \mathcal{H}$, and that, for the nilpotent logarithm N_j of a local monodromy transformation about a component of $\overline{X} \smallsetminus X$, the relative monodromy weight filtration $M(N_j, W)$ exists.

(3) The monodromy action of $\pi_1(X, b)$ factors through Γ.

Hain and Zucker showed

Theorem 2.7. ([6] (1.6) Theorem). *The category $\mathcal{C}_{X,\Gamma}$ is equivalent to the category of $\mathbf{Q}$-MHS V with polarizable $\mathrm{gr}^W V$ endowed with an action of* $\mathrm{Lie}\,(\mathcal{G})$ *such that* $\mathrm{Lie}\,(\mathcal{G}) \otimes V \to V$ *is a homomorphism of MHS.*

2.8. Define a contravariant functor $\mathcal{F}_\Gamma : \mathcal{B}(\log) \to \mathrm{Sets}$ as follows: For $S \in \mathcal{B}(\log)$, $\mathcal{F}_\Gamma(S)$ is the set of isomorphism classes of pairs (H, μ) of an exact $\otimes$-functor $H : \mathcal{C}_{X,\Gamma} \to \mathrm{MHS}(S)$ and a Γ-level structure μ satisfying the following condition (i). Here a Γ-*level structure* means a global section of the sheaf $\Gamma \setminus \mathcal{I}$, where $\mathcal{I}$ is the sheaf of functorial $\otimes$-isomorphisms $H(\mathcal{H})_{\mathbf{Q}} \overset{\sim}{\to} \mathcal{H}(b)_{\mathbf{Q}}$ of $\mathbf{Q}$-local systems preserving weight filtrations.

(i) For any $\mathbf{Q}$-MHS h, we have a functorial $\otimes$-isomorphism $H(h_X) \cong h_S$ such that the induced isomorphism of local systems $H(h_X)_{\mathbf{Q}} \cong h_{\mathbf{Q}} = h_X(b)_{\mathbf{Q}}$ belongs to μ. Here h_X (resp. h_S) denotes the constant variation (resp. family) of $\mathbf{Q}$-MHS over X (resp. S) associated to h.

Theorem 2.9. *Let the notation be as in 2.8. The functor $\mathcal{F}_\Gamma$ is represented by $A_{X,\Gamma} \simeq \Gamma \setminus \mathcal{D}$.*

This follows from Theorem 2.5 and Theorem 2.7.

Let $\varphi : X \to A_{X,\Gamma}$ be the higher Albanese map.

2.10. Let Σ be a weak fan in $\mathrm{Lie}\,(G)$ such that $\sigma \subset \mathrm{Lie}\,(\mathcal{G})_{\mathbf{R}}$ for any $\sigma \in \Sigma$. Assume that Σ and Γ are strongly compatible. Let $\Gamma \setminus D_{G,\Sigma} \to D_Q$ be a canonical morphism induced by $G \to Q$. Define

$$A_{X,\Gamma,\Sigma} := (\text{the fiber of } \Gamma \setminus D_{G,\Sigma} \to D_Q \text{ over } b_Q) \in \mathcal{B}(\log)$$

Define a contravariant functor $\mathcal{F}_{\Gamma,\Sigma} : \mathcal{B}(\log) \to \mathrm{Sets}$ as follows: For $S \in \mathcal{B}(\log)$, $\mathcal{F}_{\Gamma,\Sigma}(S)$ is the set of isomorphism classes of pairs (H, μ) consisting of an exact $\otimes$-functor $H : \mathcal{C}_{X,\Gamma} \to \mathrm{LMH}(S)$ and a Γ-level structure μ satisfying the condition (i) in 2.8 and also the following condition (ii).

(ii) The following (ii-1) and (ii-2) are satisfied for any $s \in S$ and any $t \in s^{\log}$. Let

$$\tilde{\mu}_t : H(\mathcal{H})_{\mathbf{Q},t} \cong \mathcal{H}(b)_{\mathbf{Q}}$$

be a functorial $\otimes$-isomorphism which belongs to μ_t.

(ii-1) There is a $\sigma \in \Sigma$ such that the logarithm of the action of the local monodromy cone $\mathrm{Hom}\,((M_S/\mathcal{O}_S^\times)_s, \mathbf{N}) \subset \pi_1(s^{\log})$ on $H_{\mathbf{Q},t}$ is contained, via $\tilde{\mu}_t$, in $\sigma \subset \mathrm{Lie}\,(\mathcal{G})_{\mathbf{R}}$.

(ii-2) Let $\sigma \in \Sigma$ be the smallest cone which satisfies (ii-1) and let $a : \mathcal{O}^{\log}_{S,t} \to \mathbf{C}$ be a ring homomorphism which induces the evaluation $\mathcal{O}_{S,s} \to \mathbf{C}$ at s. Then, for each $\mathcal{H} \in \mathcal{C}_{X,\Gamma}$, $(\sigma, \tilde{\mu}_t(a(H(\mathcal{H}))))$ generates a nilpotent orbit in the sense of [10] Part III, 2.2.2.

Theorem 2.11. *Let the notation be as in 2.9 and 2.10.*

(1) *The functor $\mathcal{F}_{\Gamma,\Sigma}$ is represented by $A_{X,\Gamma,\Sigma}$.*

(2) *Let $\overline{X}$ be a smooth algebraic variety over $\mathbf{C}$ which contains X as a dense open subset such that the complement $\overline{X} \smallsetminus X$ is a smooth divisor. Endow $\overline{X}$ with the log structure associated to this divisor. Assume that Σ is the fan consisting of all rational nilpotent cones in $\mathrm{Lie}\,(\mathcal{G})_{\mathbf{R}}$ of rank ≤ 1 (denoted by Ξ in [11] 6.2.5). Then, the* higher Albanese map *$\varphi : X \to A_{X,\Gamma}$ extends uniquely to a morphism $\overline{\varphi} : \overline{X} \to A_{X,\Gamma,\Sigma}$ of log manifolds.*

Since an object of $\mathcal{C}_{X,\Gamma}$ is good at infinity (2.6), it extends to an LMH over $\overline{X}$. Hence (2) follows from (1) and the general theorem 1.22 (4).

3. Description of a result of Deligne by log higher Albanese map

For a group $\Gamma^{(n)}$ in 3.3 below, Deligne [3] showed that polylogarithms appear in the higher Albanese map $X \to A_{X,\Gamma^{(n)}}$ (cf. Section A below). Here we describe them in our framework in [11] (Section 2 in the present paper).

3.1. Let $X := \mathbf{P}^1(\mathbf{C}) \smallsetminus \{0, 1, \infty\} \subset \overline{X} := \mathbf{P}^1(\mathbf{C})$ with affine coordinate x. Let $b := (0, 1)$ the "tangential base point" over $0 \in \overline{X}$ with tangent $v_0 \in T_0(\overline{X}) = \mathrm{Hom}_{\mathbf{C}}(m_0/m_0^2, \mathbf{C})$ defined by $v_0(x) = 1$ in [3] Section 15. This is understood in log geometry in the following way. Let $y = (0, h) \in \overline{X}^{\log}$ be the point lying over $0 \in X$, where $h : M^{\mathrm{gp}}_{\overline{X},0} = \mathcal{O}^{\times}_{\overline{X},0} \times x^{\mathbf{Z}} \to \mathbf{S}^1$ is the argument function which is a group homomorphism sending $f \in \mathcal{O}^{\times}_{\overline{X},0}$ to $f(0)/|f(0)|$ and x to $v_0(x)/|v_0(x)| = 1$ ([11] 6.3.7). Let $u_0 \in \mathcal{O}^{\log}_{\overline{X},y}$ be the branch of $\log(x)$ having real value on $\mathbf{R}_{>0}$. (This u_0 is the branch denoted by $f \in \mathcal{O}^{\log}_{\overline{X},y}$ in [11] 6.3.7 (ii), and u_0 can be also regarded as the function $2\pi iz$ on $\tilde{S}^{\log}$ in 1.1.1.) Then the corresponding base point in the boundary in our sense is $b = (y, a)$, where $a : \mathcal{O}^{\log}_{\overline{X},y} = \mathbf{C}\{x\}[u_0] \to \mathbf{C}$ is the specialization which is a ring homomorphism sending x to 0 and u_0 to $a(u_0) = \log(v_0(x)) = \log(1) = 0$ ([11] 6.3.7 (ii)).

See [11] 6.3.6, 6.3.7 for more general description of the above correspondence of boundary points.

3.2. The inclusion $X \subset \mathbf{G}_m(\mathbf{C}) = \mathbf{C}^{\times}$ induces $\pi_1(X, b) \to \pi_1(\mathbf{G}_m(\mathbf{C}), b) = \mathbf{Z}(1)$. Let K be its kernel, and let $\Gamma := \pi_1(X, b)/[K, K]$ and $\Gamma_1 := K/[K, K]$. Then

$$1 \to \Gamma_1 \to \Gamma \to \mathbf{Z}(1) \to 1.$$

3.3. Let $Z^n\Gamma$ be the descending central series of Γ defined by $Z^{n+1}\Gamma := [Z^n\Gamma, \Gamma]$ starting with $Z^1\Gamma = \Gamma$.

Let $\Gamma^{(n)} := \Gamma/Z^{n+1}(\Gamma)$ and $\Gamma_1^{(n)} := \mathrm{Image}\,(\Gamma_1 \to \Gamma^{(n)})$. Let $\gamma_0, \gamma_1 \in \Gamma^{(n)}$ be the classes of small loops anticlockwise around 0 and clockwise around 1, respectively. Then, we have

$$\Gamma^{(n)} = \langle \gamma_0, \gamma_1 \rangle,\ (\mathrm{ad}\,\gamma_0)^{k-1}\gamma_1\ (1 \leq k \leq n) \text{ are commutative, } \Gamma_1^{(n)} = \textstyle\sum_{k=1}^{n} \mathbf{Z}(\mathrm{ad}\,\gamma_0)^{k-1}\gamma_1.$$

3.4. Let $\Lambda = (V, W, (\langle\ ,\ \rangle_w)_{w \in \mathbf{Z}}, (h^{p,q})_{p,q \in \mathbf{Z}})$ be as follows. V is a free $\mathbf{Z}$-module with basis $e_1, e_2, e_3, \ldots, e_{n+1}$. W is a weight filtration on $V_{\mathbf{Q}}$ defined by

$$\begin{aligned} W_{-2n-1} = 0 \subset W_{-2n} = W_{-2n+1} = \mathbf{Q}e_1 \subset W_{-2n+2} = W_{-2n+3} = W_{-2n+1} + \mathbf{Q}e_2 \\ \subset \cdots \subset W_0 = W_{-1} + \mathbf{Q}e_{n+1} = V_{\mathbf{Q}}. \end{aligned}$$

$\langle\ ,\ \rangle_w : \mathrm{gr}^W_w(V_{\mathbf{Q}}) \times \mathrm{gr}^W_w(V_{\mathbf{Q}}) \to \mathbf{Q}$ $(w \in \mathbf{Z})$ are the $\mathbf{Q}$-bilinear forms characterized by

$$\langle e_{n+1+k}, e_{n+1+k}\rangle_{2k} = 1$$

for $k = 0, -1, \ldots, -n$. $h^{k,k} = 1$ for $k = 0, -1, \ldots, -n$, and $h^{p,q} = 0$ for the other (p, q).

Let $D(\Lambda)$ be the period domain in [10] Part III with universal Hodge filtration F:

$$F^1 = 0 \subset F^0 = \mathbf{C}(e_{n+1} + \sum_{n\geq j\geq 1} a_{j,n+1}e_j) \subset F^{-1} = F^0 + \mathbf{C}(e_n + \sum_{n-1\geq j\geq 1} a_{j,n}e_j)$$

$$\subset \cdots \subset F^{-n} = F^{-n+1} + \mathbf{C}e_1 = V_{\mathbf{C}}.$$

3.5. Let $\mathcal{G}$ be the unipotent group $\mathcal{G}$ in 2.1 for $\Gamma^{(n)}$. Define an action of $\mathrm{Lie}\,(\mathcal{G})$ on $V_{\mathbf{Q}}$ by $N_0 = \log(\gamma_0)$, $N_1 = \log(\gamma_1)$:

$$N_0 e_j = e_{j-1}\ (j = 2, \ldots, n), \quad N_0 e_j = 0\ (j = 1, n+1),$$

$$N_1 e_{n+1} = -e_n, \quad N_1 e_j = 0\ (j = 1, 2, \ldots, n).$$

Then

$$(-N_0 + N_1)^j = (-N_0)^j + (-\mathrm{Ad}N_0)^{j-1}N_1 \quad (1 \leq j \leq n+1).$$

3.6. Let X be as in 3.1 and $\Gamma^{(n)}$ be as in 3.3. We consider the higher Albanese manifold $A_{X,\Gamma^{(n)}}$ of X by using the base point b in 3.1.

The $\mathbf{Q}$-MHS on $\mathrm{Lie}\,(\mathcal{G})$ is as follows: N_0 and N_1 are of Hodge type $(-1, -1)$ and compatible with bracket and hence $F^0\mathcal{G}(\mathbf{C}) = \{1\}$. Thus the higher Albanese manifold is

$$A_{X,\Gamma^{(n)}} = \Gamma^{(n)} \backslash \mathcal{G}(\mathbf{C}).$$

Lemma 3.7. *Let F and N_j $(j = 0, 1)$ be as in 3.4 and in 3.5.*

(i) *We have the following.*

(1) (N_0, F) *satisfies the Griffiths transversality if and only if*

$$a_{k,n+1} = 0 \quad (2 \leq k \leq n); \quad a_{1,k} = a_{l-k+1,l} \quad (2 \leq k < l \leq n).$$

(2) (N_1, F) *satisfies the Griffiths transversality if and only if*

$$a_{k,n} = 0 \quad (1 \leq k \leq n-1).$$

(3) $(-N_0 + N_1, F)$ *satisfies the Griffiths transversality if and only if*

$$a_{1,k} = a_{l-k+1,l} \quad (2 \leq k < l \leq n+1).$$

(ii) *The following three conditions are equivalent.*

(1) *The Lie action* $\mathrm{Lie}\,(\mathcal{G}) \otimes V \to V$ *in 3.5 is a homomorphism of MHS with respect to the MHS on* $\mathrm{Lie}\,(\mathcal{G})$ *in 3.6 and the MHS* (V, W, F) *in 3.4.*

(2) *For $j = 0$ and 1, (N_j, F) satisfies the Griffiths transversality.*

(3) $a_{j,k} = 0$ *unless* $(j, k) = (1, n+1)$.

The assertions are easily verified by direct computation.

3.8. For any fixed $a \in \mathbf{C}$, denote by $F(a)$ the Hodge filtration in 3.7 (ii) (3) with $a_{1,n+1} = a$. By the action in 3.5, we define

$$\mathcal{D} := \exp\left(\mathbf{C}N_0 + \sum_{k=1}^{n} \mathbf{C}(\mathrm{Ad}N_0)^{k-1}N_1\right) F(a) \subset D(\Lambda).$$

Then, this $\mathcal{D}$ coincides with $\mathcal{D}$ in 2.4. Hence $\mathcal{G}(\mathbf{C}) \simeq \mathcal{D}$ and $A_{X,\Gamma^{(n)}} \simeq \Gamma^{(n)} \backslash \mathcal{D}$ as complex analytic manifolds.

3.9. Let $\varphi : X \to A_{X,\Gamma^{(n)}} \simeq \Gamma^{(n)} \setminus \mathcal{D}$ be the composite of higher Albanese map and the isomorphism in 3.8. Let $F(x)$ be the pullback by φ of the universal Hodge filtration on $\Gamma^{(n)} \setminus \mathcal{D}$.

Since $F(x)$ is rigid by Theorem 2.7, we consider a connection equation:

$$dF(x) = \omega F(x), \quad \omega := (2\pi i)^{-1}\frac{dx}{x}N_0 + (2\pi i)^{-1}\frac{dx}{1-x}N_1.$$

That is,

$$da_{k-1,k}(x) = (2\pi i)^{-1}\frac{dx}{x} \quad (2 \le k \le n),$$

$$da_{n,n+1}(x) = -(2\pi i)^{-1}\frac{dx}{1-x},$$

$$da_{j,k}(x) = (2\pi i)^{-1}a_{j+1,k}(x)\frac{dx}{x} \quad (3 \le k \le n+1,\ 1 \le j \le k-2).$$

3.10. This system is solved by iterated integrals. The solutions are

$$a_{j,k}(x) = \frac{1}{(k-j)!}((2\pi i)^{-1}\log(x))^{k-j} \quad (2 \le k \le n,\ 1 \le j \le k-1),$$

$$a_{j,n+1}(x) = -(2\pi i)^{-n-1+j}l_{n+1-j}(x) \quad (1 \le j \le n).$$

Here the $l_j(x)$ are polylogarithms, in particular $l_1(x) = -\log(1-x)$.

3.11. Table of solutions:

$$\begin{pmatrix} 1 & a_{1,2} & \dots & a_{1,n} & a_{1,n+1} \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & 0 & \ddots & a_{n-1,n} & a_{n-1,n+1} \\ \vdots & \vdots & \ddots & 1 & a_{n,n+1} \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & (2\pi i)^{-1}\log(x) & \dots & \frac{((2\pi i)^{-1}\log(x))^{n-1}}{(n-1)!} & -(2\pi i)^{-n}l_n(x) \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & 0 & \ddots & (2\pi i)^{-1}\log(x) & -(2\pi i)^{-2}l_2(x) \\ \vdots & \vdots & \ddots & 1 & -(2\pi i)^{-1}l_1(x) \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

Note that, for $1 \le j \le n$,

$$\exp((2\pi i)^{-1}\log(x)N_0)e_j = e_j + (2\pi i)^{-1}\log(x)e_{j-1} + \dots + \frac{1}{(j-1)!}((2\pi i)^{-1}\log(x))^{j-1}e_1,$$

for $j = n+1$,

$$\exp\left(-\sum_{n \ge k \ge 1}(2\pi i)^{-k}l_k(x)(\mathrm{Ad}N_0)^{k-1}N_1\right)e_{n+1} = e_{n+1} - \left(\sum_{n \ge k \ge 1}(2\pi i)^{-k}l_k(x)e_{n+1-k}\right).$$

3.12. For $\alpha, \beta, \lambda_2, \dots, \lambda_n \in \mathbf{C}$, let $F = F(\alpha, \beta, \lambda_2, \dots, \lambda_n)$ be a Hodge filtration:

$$F^1 = 0 \subset F^0 = \mathbf{C}(e_{n+1} + \beta e_n + \lambda_2 e_{n-1} + \dots + \lambda_n e_1)$$

$$\subset F^{-1} = F^0 + \mathbf{C}\left(e_n + \alpha e_{n-1} + \frac{\alpha^2}{2!}e_{n-2} + \dots + \frac{\alpha^{n-1}}{(n-1)!}e_1\right) \subset \cdots$$

$$\subset F^{-n+1} = F^{-n+2} + \mathbf{C}(e_2 + \alpha e_1) \subset F^{-n} = F^{-n+1} + \mathbf{C}e_1 = V_{\mathbf{C}}.$$

3.13. Let $\varphi : X \to A_{X,\Gamma^{(n)}} \simeq \Gamma^{(n)} \setminus \mathcal{D}$ be the higher Albanese map in 3.9. We have a commutative diagram

$$\begin{array}{ccccc} & & \tilde\varphi(X) & \subset & \mathcal{D} \\ & \tilde\varphi \nearrow & \downarrow & & \downarrow \\ X & \xrightarrow[\varphi]{\sim} & \varphi(X) & \subset & \Gamma^{(n)} \setminus \mathcal{D} \end{array}$$

where $\tilde\varphi : X \to \mathcal{D}$ is a multi-valued map corresponding to the Hodge filtration

$$x \mapsto F((2\pi i)^{-1} \log(x), -(2\pi i)^{-1} l_1(x), \dots, -(2\pi i)^{-n} l_n(x))$$

in the notation in 3.12. $\tilde\varphi(X) \to X$ and $\tilde\varphi(X) \to \varphi(X)$ are $\Gamma^{(n)}$-torsors and $\varphi : X \xrightarrow{\sim} \varphi(X)$ is an isomorphism.

3.14. Let Σ be the set of all cones of the form $\mathbf{R}_{\geq 0} N$ with $N \in \mathrm{Lie}\,(\mathcal{G})$. We consider the extended period domain $D(\Lambda)_\Sigma$ in [10] Part III. This is only a set. By using the strong topology ([12] Section 3.1), the quotient $\Gamma^{(n)} \setminus D(\Lambda)_\Sigma$ has a structure of a log manifold. Define $\Gamma^{(n)} \setminus \mathcal{D}_\Sigma$ to be the closure of $\Gamma^{(n)} \setminus \mathcal{D}$ in $\Gamma^{(n)} \setminus D(\Lambda)_\Sigma$. This inherits a structure of log manifold. We have $A_{X,\Gamma^{(n)},\Sigma} \simeq \Gamma^{(n)} \setminus \mathcal{D}_\Sigma$ in the category $\mathcal{B}(\log)$.

Let $N \in \mathrm{Lie}\,(\mathcal{G})$ and $\sigma := \mathbf{R}_{\geq 0} N$. Let Γ_σ be the group generated by the monoid $\Gamma^{(n)} \cap \exp(\sigma)$. If we use as Σ the fan consisting of the cone σ and 0, also denoted by σ by abuse of notation, we have $A_{X,\Gamma_\sigma,\sigma} \simeq \Gamma_\sigma \setminus \mathcal{D}_\sigma$ in the category $\mathcal{B}(\log)$.

3.15. Let N_0 be as in 3.5 and set $\sigma_0 = \mathbf{R}_{\geq 0} N_0$. Let $F = F(\alpha, \beta, \lambda_2, \dots, \lambda_n)$ be as in 3.12. By Lemma 3.7 (i) (1), (N_0, F) satisfies the Griffiths transversality if and only if

$$\beta = \lambda_2 = \cdots = \lambda_{n-1} = 0.$$

If this is the case, (N_0, F) generates a σ_0-nilpotent orbit, since admissibility and $\mathbf{R}$-polarizability on gr^W trivially hold. We describe the local structure of $\Gamma_{\sigma_0} \setminus \mathcal{D}_{\sigma_0}$ near the image p_0 of this nilpotent orbit.

Let $Y := \{(q, \beta, \lambda_2, \dots, \lambda_n) \in \mathbf{C}^{n+1} \mid \beta = \lambda_2 = \cdots = \lambda_{n-1} = 0 \text{ if } q = 0\}$ be the log manifold with the strong topology, with the structure sheaf of rings which is the inverse image of the sheaf of holomorphic functions on $\mathbf{C}^{n+1}$, and with the log structure generated by q. Then there is an open neighborhood U of $(0, 0, \dots, 0, \lambda_n)$ in $\mathbf{C}^{n+1}$ and an open immersion

$$Y \cap U \hookrightarrow \Gamma_{\sigma_0} \setminus \mathcal{D}_{\sigma_0}$$

of log manifolds which sends

$$(q, \beta, \lambda_2, \dots, \lambda_n) \in Y \cap U$$

with $q \neq 0$ to the class of $F(\alpha, \beta, \lambda_2, \dots, \lambda_n)$, where $\alpha \in \mathbf{C}$ is such that $q = e^{2\pi i \alpha}$, and which sends $(0, 0, \dots, 0, \lambda_n)$ to p_0.

3.16. Near $x = 0$, a nilpotent orbit in naive sense is

$$\exp((2\pi i)^{-1} \log(x) N_0) F(0, 0, \dots, 0, \lambda_n^0) = F((2\pi i)^{-1} \log(x), 0, \dots, 0, \lambda_n^0), \tag{1}$$

where $\lambda_n^0 = -(2\pi i)^{-n} l_n(0)$. The corresponding "higher Albanese map" (i.e., local version about 0 of $\tilde\varphi$ in 3.13) is

$$F((2\pi i)^{-1} \log(x), -(2\pi i)^{-1} l_1(x), \dots, -(2\pi i)^{-n} l_n(x)) \tag{2}$$

under the condition $l_j(0) = 0$ $(1 \leq j \leq n-1)$. These two are asymptotic when x goes to the boundary point $b = (y, a)$ with $y = (0, h) \in \overline{X}^{\log}$ and a being the specialization at y in 3.1.

3.17. As above, let u_0 be the branch of $\log(x)$ in 3.1 and T an indeterminate over $\mathcal{O}_{\overline{X},0}$. Then, by 1.1.1, we have an isomorphism $\mathcal{O}^{\log}_{\overline{X},y} = \mathcal{O}_{\overline{X},0}[u_0] \simeq \mathcal{O}_{\overline{X},0}[T]$ of $\mathcal{O}_{\overline{X},0}$-algebras under $(2\pi i)^{-1}u_0 \leftrightarrow T$. Consider an $\mathcal{O}_{\overline{X},0}$-algebra homomorphism $\mathcal{O}_{\overline{X},0}[T] \to \mathcal{O}_{\overline{X},0}$, $T \mapsto x$.

Under the initial condition in 3.16 given by the base point b in 3.1, we have

$$l_j(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^j} \quad (1 \le j \le n-1), \quad l_n(x) = c + \sum_{k=1}^{\infty} \frac{x^k}{k^n}$$

on a simply connected neighborhood $\overline{X}_0$ of $0 \in \overline{X}$, where $c := -(2\pi i)^n \lambda_n^0$.

Let $\alpha = (2\pi i)^{-1}\log(x)$. Then, as

$$x \to 0,\ \exp(-\alpha N_0)(F \text{ in } 3.16\ (2))$$

converges to $F(0,0,\dots,0,\lambda_n^0)$ in $\mathcal{D}$ (3.8), and hence the class of (F in 3.16 (2)) converges to the class p_0 (3.15) of the nilpotent orbit $(\sigma_0, \exp(\sigma_{0,\mathbf{C}})F(0,0,\dots,0,\lambda_n^0))$ in $\Gamma_{\sigma_0} \setminus \mathcal{D}_{\sigma_0}$. We thus have an extension of the higher Albanese map over $\overline{X}_0$ (Theorem 2.11 (2)):

$$\overline{\varphi}_0 : \overline{X}_0 \to \Gamma_{\sigma_0} \setminus \mathcal{D}_{\sigma_0}.$$

This is a morphism in the category $\mathcal{B}(\log)$. The log structure on the source (resp. the target) is given by x (resp. q). The pullback of the universal log mixed Hodge structure on the target coincides with the log mixed Hodge structure on the source.

3.18. By using log mixed Hodge theory, 3.16 is described as follows.

Taking the images of the nilpotent orbit in naive sense 3.16 (1) and the "higher Albanese map" 3.16 (2), we have their real analytic extensions with boundary

$$\overline{\nu}_0^{\log},\ \overline{\varphi}_0^{\log} : \overline{X}_0^{\log} \to (\Gamma_{\sigma_0} \setminus \mathcal{D}_{\sigma_0})^{\log}.$$

Here, $\overline{X}_0^{\log}$ is like Example 1.1.1, and $(\Gamma_{\sigma_0} \setminus \mathcal{D}_{\sigma_0})^{\log}$ coincides with the moduli of nilpotent i-orbits $\Gamma_{\sigma_0} \setminus \mathcal{D}^{\sharp}_{\sigma_0}$ in the present situation ([10] III Theorem 2.5.6).

Let $\tilde{\overline{X}}_0^{\log}$ be the universal covering of $\overline{X}_0^{\log}$. The above maps are still lifted to

$$\tilde{\overline{\nu}}_0^{\log},\ \tilde{\overline{\varphi}}_0^{\log} : \tilde{\overline{X}}_0^{\log} \to \mathcal{D}^{\sharp}_{\sigma_0}.$$

The boundary point b in 3.16 can be understood as the point

$$b = (z = 0 + i\infty) = (u_0 = -\infty + i0) \in \tilde{\overline{X}}_0^{\log}.$$

We have $(\exp(-(2\pi i)^{-1}\log(x)N_0)(3.16\ (2)))(b) = F(0,0,\dots,0,\lambda_n^0)$, and

$$\tilde{\overline{\nu}}_0^{\log}(b) = \tilde{\overline{\varphi}}_0^{\log}(b) = (\text{nilpotent } i\text{-orbit generated by } (N_0, F(0,0,\dots,0,\lambda_n^0))) \in \mathcal{D}^{\sharp}_{\sigma_0}.$$

3.19. Let now $\sigma_1 = \mathbf{R}_{\ge 0}N_1$ for N_1 in 3.5. Let $F = F(\alpha, \beta, \lambda_2, \dots, \lambda_n)$ be as in 3.12. By Lemma 3.7 (i) (2), (N_1, F) satisfies the Griffiths transversality if and only if $\alpha = 0$. If this is the case, (N_1, F) generates a σ_1-nilpotent orbit, since admissibility and $\mathbf{R}$-polarizability on gr^W trivially hold. We have a similar description of the local structure of $\Gamma_{\sigma_1} \setminus \mathcal{D}_{\sigma_1}$ near the image p_1 of this nilpotent orbit.

Let Y be the log manifold $\{(\alpha, q, \lambda_2, \dots, \lambda_n) \in \mathbf{C}^{n+1} \mid \alpha = 0 \text{ if } q = 0\}$ with the strong topology, the structure sheaf and the log structure defined by q. Then there is an open neighborhood U of $(0, 0, \lambda_2, \dots, \lambda_n)$ in $\mathbf{C}^{n+1}$ and an open immersion

$$Y \cap U \hookrightarrow \Gamma_{\sigma_1} \setminus \mathcal{D}_{\sigma_1}$$

of log manifolds which sends

$$(\alpha, q, \lambda_2, \dots, \lambda_n) \in Y \cap U$$

with $q \neq 0$ to the class of $F(\alpha, \beta, \lambda_2, \dots, \lambda_n)$, where $\beta \in \mathbf{C}$ is such that $q = e^{2\pi i \beta}$, and which sends $(0, 0, \lambda_2, \dots, \lambda_n)$ to p_1.

3.20. We assume the initial condition in 3.16. Near $x = 1$, a nilpotent orbit in naive sense is

$$\text{(1)} \qquad \exp((2\pi i)^{-1}\log(1-x)N_1) \cdot F(0, 0, -(2\pi i)^{-2}\zeta(2), \dots, -(2\pi i)^{-n}(c + \zeta(n)))$$
$$= F(0, -(2\pi i)^{-1}l_1(x), -(2\pi i)^{-2}\zeta(2), \dots, -(2\pi i)^{-n}(c + \zeta(n))).$$

The corresponding "higher Albanese map" (i.e., local version about 1 of $\tilde{\varphi}$ in 3.13) is

$$\text{(2)} \qquad F((2\pi i)^{-1}\log(x), -(2\pi i)^{-1}l_1(x), \dots, -(2\pi i)^{-n}l_n(x)).$$

These two are asymptotic when x goes to the tangential boundary point $\tilde{p}_1 := (1, -1)$ with tangent $v_1 \in T_1(\overline{X}) = \mathrm{Hom}_{\mathbf{C}}(m_1/m_1^2, \mathbf{C})$ defined by $v_1(1-x) = -1$. This is the boundary point in our sense described as follows. Let u_1 be the branch of $\log(1-x)$ having real value on $\mathbf{R}_{<1}$. Then the corresponding point in the boundary in our sense is $\tilde{p}_1 = (y, a)$ with $y = (1, h) \in \overline{X}^{\log}$ such that the argument function $h : M^{\mathrm{gp}}_{\overline{X},1} = \mathcal{O}^{\times}_{\overline{X},1} \times (1-x)^{\mathbf{Z}} \to \mathbf{S}^1$ is a group homomorphism sending $f \in \mathcal{O}^{\times}_{\overline{X},1}$ to $f(1)/|f(1)|$ and $1 - x$ to $v_1(1-x)/|v_1(1-x)| = -1$, and the specialization $a : \mathcal{O}^{\log}_{\overline{X},y} = \mathbf{C}\{1-x\}[u_1] \to \mathbf{C}$ is a ring homomorphism sending $1 - x$ to 0 and u_1 to

$$a(u_1) = -a(-u_1) = \log(v_1(-(1-x))) = \log(1) = 0$$

(cf. [11] 6.3.7 (ii)).

3.21. As above, let u_1 be the branch of $\log(1-x)$ and T an indeterminate over $\mathcal{O}_{\overline{X},1}$. Then, by 1.1.1, we have an isomorphism

$$\mathcal{O}^{\log}_{\overline{X},y} = \mathcal{O}_{\overline{X},1}[u_1] \simeq \mathcal{O}_{\overline{X},1}[T]$$

of $\mathcal{O}_{\overline{X},1}$-algebras under $(2\pi i)^{-1}u_1 \leftrightarrow T$. Consider an $\mathcal{O}_{\overline{X},1}$-algebra homomorphism

$$\mathcal{O}_{\overline{X},1}[T] \to \mathcal{O}_{\overline{X},1}, \ T \mapsto 1 - x.$$

Let $\beta = (2\pi i)^{-1}\log(1-x)$. Then, as $x \to 1$ in $\overline{X}$ along the real axis starting from b over 0 to 1, $\exp(-\beta N_1)(F$ in 3.20 (2)) converges to $F(0, 0, -(2\pi i)^{-2}\zeta(2), \dots, -(2\pi i)^{-n}(c + \zeta(n)))$ in $\mathcal{D}$ (3.8), and hence the class of (F in 3.20 (2)) converges to the class p_1 (3.19) of the nilpotent orbit

$$(\sigma_1, \exp(\sigma_{1,\mathbf{C}})F(0, 0, -(2\pi i)^{-2}\zeta(2), \dots, -(2\pi i)^{-n}(c + \zeta(n))))$$

in $\Gamma_{\sigma_1} \backslash \mathcal{D}_{\sigma_1}$. We thus have an extension of the higher Albanese map over a simply connected neighborhood $\overline{X}_1$ of 1 in $\overline{X}$ (Theorem 2.11 (2)):

$$\overline{\varphi}_1 : \overline{X}_1 \to \Gamma_{\sigma_1} \backslash \mathcal{D}_{\sigma_1}.$$

This is a morphism in the category $\mathcal{B}(\log)$. The log structure on the source (resp. the target) is given by $1-x$ (resp. q). The pullback of the universal log mixed Hodge structure on the target coincides with the log mixed Hodge structure on the source.

3.22. By using log mixed Hodge theory, 3.20 is described as follows.

Taking the images of the nilpotent orbit in naive sense 3.20 (1) and the "higher Albanese map" 3.20 (2), we have their real analytic extensions with boundary

$$\overline{\nu}_1^{\log}, \ \overline{\varphi}_1^{\log} : \overline{X}_1^{\log} \to (\Gamma_{\sigma_1} \backslash \mathcal{D}_{\sigma_1})^{\log}.$$

Here, $\overline{X}_1^{\log}$ is similar to Example 1.1.1 over $x = 1$, and $(\Gamma_{\sigma_1} \backslash \mathcal{D}_{\sigma_1})^{\log}$ coincides with the moduli of nilpotent i-orbits $\Gamma_{\sigma_1} \backslash \mathcal{D}^{\sharp}_{\sigma_1}$ in the present situation ([10] III Theorem 2.5.6).

Let $\tilde{\overline{X}}_1^{\log}$ be the universal covering of $\overline{X}_1^{\log}$. The above maps are still lifted to

$$\tilde{\overline{\nu}}_1^{\log},\ \tilde{\overline{\varphi}}_1^{\log} : \tilde{\overline{X}}_1^{\log} \to \mathcal{D}^{\sharp}_{\sigma_1}.$$

The boundary point $\tilde{p}_1$ in 3.20 can be understood as the point

$$\tilde{p}_1 = (z_1 = 0 + i\infty) = (u_1 = -\infty + i0) \in \tilde{\overline{X}}_1^{\log},$$

(where $2\pi i z_1 := u_1$). We have

$$(\exp(-(2\pi i)^{-1}\log(1-x)N_1)(3.20\,(2)))(\tilde{p}_1) = F(0,0,-(2\pi i)^{-2}\zeta(2),\dots,-(2\pi i)^{-n}(c+\zeta(n))),$$

and $\tilde{\overline{\nu}}_1^{\log}(\tilde{p}_1) = \tilde{\overline{\varphi}}_1^{\log}(\tilde{p}_1) \in \mathcal{D}^{\sharp}_{\sigma_1}$ is the nilpotent i-orbit generated by

$$(N_1, F(0,0,-(2\pi i)^{-2}\zeta(2),\dots,-(2\pi i)^{-n}(c+\zeta(n)))).$$

3.23. In order to describe the local structure near $x = \infty$, we take a local coordinate $\xi := x^{-1}$.

By abuse of notation, let $F(\xi)$ be the pullback of the universal Hodge filtration by the composite $\varphi : X \to A_{X,\Gamma^{(n)}} \simeq \Gamma^{(n)} \setminus \mathcal{D}$ of higher Albanese map and the isomorphism in 3.8.

Since $d\log(x) = -d\log(\xi)$ and $-d\log(x-1) = d\log(\xi) - d\log(1-\xi)$, a connection equation in 3.9 now is

$$dF(\xi) = \omega F(\xi), \quad \omega := (2\pi i)^{-1}\frac{d\xi}{\xi}(-N_0 + N_1) + (2\pi i)^{-1}\frac{d\xi}{1-\xi}N_1.$$

That is,

$$da_{k-1,k}(\xi) = -(2\pi i)^{-1}\frac{d\xi}{\xi} \quad (2 \le k \le n),$$

$$da_{n,n+1}(\xi) = -(2\pi i)^{-1}\frac{d\xi}{\xi} - (2\pi i)^{-1}\frac{d\xi}{1-\xi},$$

$$da_{j,k}(\xi) = -(2\pi i)^{-1}a_{j+1,k}(\xi)\frac{d\xi}{\xi} \quad (3 \le k \le n+1,\ 1 \le j \le k-2).$$

3.24. This system is solved by iterated integrals as before, and the solutions are

$$a_{j,k}(\xi) = \frac{1}{(k-j)!}(-(2\pi i)^{-1}\log(\xi))^{k-j} \quad (2 \le k \le n,\ 1 \le j \le k-1),$$

$$a_{j,n+1}(\xi) = \frac{1}{(n+1-j)!}(-(2\pi i)^{-1}\log(\xi))^{n+1-j} + (-(2\pi i)^{-1})^{n+1-j}l_{n+1-j}(\xi) \quad (1 \le j \le n).$$

3.25. Table of solutions:

$$\begin{pmatrix} 1 & a_{1,2} & \dots & a_{1,n} & a_{1,n+1} \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & 0 & \ddots & a_{n-1,n} & a_{n-1,n+1} \\ \vdots & \vdots & \ddots & 1 & a_{n,n+1} \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -(2\pi i)^{-1}\log(\xi) & \dots & \frac{(-(2\pi i)^{-1}\log(\xi))^{n-1}}{(n-1)!} & \frac{(-(2\pi i)^{-1}\log(\xi))^{n}}{n!} + (-(2\pi i)^{-1})^n l_n(\xi) \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & 0 & \ddots & -(2\pi i)^{-1}\log(\xi) & \frac{(-(2\pi i)^{-1}\log(\xi))^{2}}{2!} + (-(2\pi i)^{-1})^2 l_2(\xi) \\ \vdots & \vdots & \ddots & 1 & -(2\pi i)^{-1}\log(\xi) - (2\pi i)^{-1} l_1(\xi) \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

3.26. Let now $\sigma_\infty = \mathbf{R}_{\geq 0}N_\infty$ with $N_\infty := -N_0 + N_1$ for N_0, N_1 in 3.5. Let

$$F = F(-\alpha', \beta', \lambda'_2, \dots, \lambda'_n)$$

be as in 3.12. By Lemma 3.7 (i) (3), (N_∞, F) satisfies the Griffiths transversality if and only if $\beta' = -\alpha', \lambda'_2 = \frac{(-\alpha')^2}{2!}, \dots, \lambda'_{n-1} = \frac{(-\alpha')^{n-1}}{(n-1)!}$. If this is the case, (N_∞, F) generates a σ_∞-nilpotent orbit, since admissibility and $\mathbf{R}$-polarizability on gr^W trivially hold. We describe the local structure of $\Gamma_{\sigma_\infty} \setminus \mathcal{D}_{\sigma_\infty}$ near the image p_∞ of this nilpotent orbit.

Let

$$Y := \left\{(q', \beta', \lambda'_2, \dots, \lambda'_n) \in \mathbf{C}^{n+1} \,|\, \beta' = -\alpha', \lambda'_2 = \frac{(-\alpha')^2}{2!}, \dots, \lambda'_{n-1} = \frac{(-\alpha')^{n-1}}{(n-1)!} \text{ if } q' = 0\right\}$$

be the log manifold with the strong topology, with the structure sheaf of rings which is the inverse image of the sheaf of holomorphic functions on $\mathbf{C}^{n+1}$, and with the log structure generated by q'. Then there is an open neighborhood U of $(0, 0, \dots, 0, \lambda'_n)$ in $\mathbf{C}^{n+1}$ and an open immersion

$$Y \cap U \hookrightarrow \Gamma_{\sigma_\infty} \setminus \mathcal{D}_{\sigma_\infty}$$

of log manifolds which sends $(q', \beta', \lambda'_2, \dots, \lambda'_n) \in Y \cap U$ with $q' \neq 0$ to the class of

$$F(-\alpha', \beta', \lambda'_2, \dots, \lambda'_n),$$

where $\alpha' \in \mathbf{C}$ is such that $q' = e^{2\pi i \alpha'}$, and which sends $(0, 0, \dots, 0, \lambda'_n)$ to p_∞.

3.27. Near $x = \infty$, i.e., $\xi = 0$, a nilpotent orbit in naive sense is

(1) $$\exp((2\pi i)^{-1}\log(\xi)N_\infty)F(0, 0, \dots, 0, \lambda'^0_n)$$

$$= F\Big(-(2\pi i)^{-1}\log(\xi), -(2\pi i)^{-1}\log(\xi), \frac{(-(2\pi i)^{-1}\log(\xi))^2}{2!}, \dots, \frac{(-(2\pi i)^{-1}\log(\xi))^n}{n!} + \lambda'^0_n\Big),$$

where $\lambda'^0_n = (-(2\pi i)^{-1})^n l_n(0)$. The corresponding "higher Albanese map" (i.e., local version about ∞ of $\tilde{\varphi}$ in 3.13) is

(2) $$F\Big(-(2\pi i)^{-1}\log(\xi), -(2\pi i)^{-1}\log(\xi) - (2\pi i)^{-1}l_1(\xi), \dots, \frac{(-(2\pi i)^{-1}\log(\xi))^n}{n!} + (-(2\pi i)^{-1})^n l_n(\xi)\Big)$$

under the condition $l_j(0) = 0$ $(1 \leq j \leq n-1)$. These two are asymptotic when ξ goes to the boundary point b' described as follows.

Changing ∞ and ξ into 0 and x, respectively, $b' = (\infty, 1)$ corresponds to the tangential boundary point $(0, 1)$ of Deligne, i.e., b' is the tangential base point over $\infty \in \overline{X}$ with tangent $v' \in T_\infty(\overline{X}) = \mathrm{Hom}_{\mathbf{C}}(m_\infty/m_\infty^2, \mathbf{C})$ defined by $v'(\xi) = 1$.

This corresponds to our boundary point $b' = (y', a')$ with $y' = (\infty, h') \in \overline{X}^{\log}$ described as follows. Let u' be the branch of $\log(\xi)$ having real value on $\mathbf{R}_{>0}$. The argument function

$$h' : M^{\mathrm{gp}}_{\overline{X},\infty} = \mathcal{O}^{\times}_{\overline{X},\infty} \times \xi^{\mathbf{Z}} \to \mathbf{S}^1$$

is a group homomorphism sending $f \in \mathcal{O}^{\times}_{\overline{X},\infty}$ to $f(\xi = 0)/|f(\xi = 0)|$ and ξ to $v'(\xi)/|v'(\xi)| = 1$, and the specialization $a' : \mathcal{O}^{\log}_{\overline{X},y'} = \mathbf{C}\{\xi\}[u'] \to \mathbf{C}$ is a ring homomorphism sending ξ to 0 and u' to $a'(u') = \log(v'(\xi)) = \log(1) = 0$.

3.28. As above, let u' be the branch of $\log(\xi)$ and T an indeterminate over $\mathcal{O}_{\overline{X},\infty}$. Then, by 1.1.1, we have an isomorphism $\mathcal{O}^{\log}_{\overline{X},y'} = \mathcal{O}_{\overline{X},\infty}[u'] \simeq \mathcal{O}_{\overline{X},\infty}[T]$ of $\mathcal{O}_{\overline{X},\infty}$-algebras under $(2\pi i)^{-1}u' \leftrightarrow T$. Consider an $\mathcal{O}_{\overline{X},\infty}$-algebra homomorphism $\mathcal{O}_{\overline{X},\infty}[T] \to \mathcal{O}_{\overline{X},\infty}$, $T \mapsto \xi$.

Let $\alpha' = (2\pi i)^{-1}\log(\xi)$. Then, as $\xi \to 0$, $\exp(-\alpha' N_\infty)(F$ in 3.27 (2)) converges to

$$F(0, 0, \ldots, 0, \lambda'^{0}_{n})$$

in $\mathcal{D}$ (3.8), and hence the class of (F in 3.27 (2)) converges to the class p_∞ (3.26) of the nilpotent orbit $(\sigma_\infty, \exp(\sigma_{\infty,\mathbf{C}})F(0, 0, \ldots, 0, \lambda'^{0}_{n}))$ in $\Gamma_{\sigma_\infty} \setminus \mathcal{D}_{\sigma_\infty}$. We thus have an extension of the higher Albanese map over $\overline{X}_\infty$ (Theorem 2.11 (2)):

$$\overline{\varphi}_\infty : \overline{X}_\infty \to \Gamma_{\sigma_\infty} \setminus \mathcal{D}_{\sigma_\infty}.$$

This is a morphism in the category $\mathcal{B}(\log)$. The log structure on the source (resp. the target) is given by ξ (resp. q). The pullback of the universal log mixed Hodge structure on the target coincides with the log mixed Hodge structure on the source.

3.29. By using log mixed Hodge theory, 3.27 is described as follows.

Taking the images of the nilpotent orbit in naive sense 3.27 (1) and the "higher Albanese map" 3.27 (2), we have their real analytic extensions with boundary

$$\overline{\nu}^{\log}_\infty,\ \overline{\varphi}^{\log}_\infty : \overline{X}^{\log}_\infty \to (\Gamma_{\sigma_\infty} \setminus \mathcal{D}_{\sigma_\infty})^{\log}.$$

Here, $\overline{X}^{\log}_\infty$ is like Example 1.1.1, and $(\Gamma_{\sigma_\infty} \setminus \mathcal{D}_{\sigma_\infty})^{\log}$ coincides with the moduli of nilpotent i-orbits $\Gamma_{\sigma_\infty} \setminus \mathcal{D}^\sharp_{\sigma_\infty}$ in the present situation ([10] III Theorem 2.5.6).

Let $\tilde{\overline{X}}^{\log}_\infty$ be the universal covering of $\overline{X}^{\log}_\infty$. The above maps are still lifted to

$$\tilde{\overline{\nu}}^{\log}_\infty,\ \tilde{\overline{\varphi}}^{\log}_\infty : \tilde{\overline{X}}^{\log}_\infty \to \mathcal{D}^\sharp_{\sigma_\infty}.$$

The boundary point b' in 3.27 can be understood as the point

$$b' = (z' = 0 + i\infty) = (u' = -\infty + i0) \in \tilde{\overline{X}}^{\log}_\infty$$

(where $2\pi i z' := u'$). We have $(\exp(-(2\pi i)^{-1}\log(\xi)N_\infty)(3.27\,(2)))(b') = F(0, 0, \ldots, 0, \lambda'^{0}_{n})$, and

$$\tilde{\overline{\nu}}^{\log}_\infty(b') = \tilde{\overline{\varphi}}^{\log}_\infty(b') = (\text{nilpotent } i\text{-orbit generated by } (N_\infty, F(0, 0, \ldots, 0, \lambda'^{0}_{n}))) \in \mathcal{D}^\sharp_{\sigma_\infty}.$$

3.30. For any $\sigma \in \Sigma$, $\Gamma_\sigma \setminus \mathcal{D}_\sigma \to \Gamma^{(n)} \setminus \mathcal{D}_\Sigma$ is a local homeomorphism. This is analogously proved as [12] Theorem A (iv).

Summing-up, we have a global extension over $\overline{X}$ of the higher Albanese map which is an isomorphism over its image:

$$\overline{\varphi} : \overline{X} \xrightarrow{\sim} \overline{\varphi}(\overline{X}) \subset A_{X,\Gamma^{(n)},\Sigma} \simeq \Gamma^{(n)} \setminus \mathcal{D}_\Sigma.$$

3.31. To study analytic continuations and extensions of polylogarithms in the spaces of nilpotent i-orbits $D^\sharp_\Sigma$, in the spaces of SL(2)-orbits $D_{\mathrm{SL}(2)}$, and in spaces of Borel-Serre orbits D_{BS} is an interesting problem. See [10] for these extended period domains and their relations which are described as a fundamental diagram.

A. Summary of a result of Deligne in [3]

We add here a summary of a result of Deligne in [3] for readers' convenience.

A.1. Just as 3.1–3.2, consider the situation $X := \mathbf{P}^1(\mathbf{C}) \smallsetminus \{0,1,\infty\} \subset \overline{X} := \mathbf{P}^1(\mathbf{C})$. Let $b := (0,1)$ the "tangential base point" over $0 \in \overline{X}$ with tangent 1.

Consider the quotient group Γ of $\pi_1(X,b)$ as in [3] 16.14 (cf. 3.2): The inclusion

$$X \subset \mathbf{G}_m(\mathbf{C}) = \mathbf{C}^\times$$

induces $\pi_1(X,b) \to \pi_1(\mathbf{G}_m(\mathbf{C}),b) = \mathbf{Z}(1)_B$ (suffix B means Betti, cf. [3]). Let

$$K := \mathrm{Ker}\,(\pi_1(X,b) \to \mathbf{Z}(1)_B).$$

Let $\Gamma := \pi_1(X,b)/[K,K]$ and $\Gamma_1 := K/[K,K]$. Then, we have an exact sequence

$$1 \to \Gamma_1 \to \Gamma \to \mathbf{Z}(1)_B \to 1.$$

A.2. ([3] 16.15). Let $\mu_0, \mu_1 : \mathbf{Z}(1)_B \to \Gamma$ be the monodromies around $0,1$, respectively. Take a generator u of $\mathbf{Z}(1)_B$ (e.g. $u = 2\pi i$), put $a_j = \mu_j(u)$ $(j = 0,1)$. Then, $\Gamma = \langle a_0, a_1\rangle$ with relation: conjugates of a_1 are commutative.

Γ_1 is a representation of $\mathbf{Z}(1)_B$ with basis (conjugates of a_1) under the action

$$\gamma \mapsto \mu_0(t)\gamma\mu_0(t)^{-1} \ (\gamma \in \Gamma_1, t \in \mathbf{Z}(1)_B),$$

i.e., $\Gamma_1 = \mathbf{Z}[\mathbf{Z}(1)_B] \cdot a_1$, where $\sum_k c_k(a_0^k a_1 a_0^{-k}) = \sum_k c_k \cdot (2\pi i \cdot k) \cdot a_1$.

These are described as

$$\Gamma_1 = \mathbf{Z}[\mathbf{Z}(1)_B] \cdot a_1 \simeq \mathbf{Z}[u,u^{-1}] \cdot \frac{du}{u}, \quad \Gamma = \mathbf{Z}(1)_B \ltimes \Gamma_1,$$

$$\sum_k c_k(a_0^k a_1 a_0^{-k}) = \sum_k c_k \cdot (2\pi i \cdot k) \cdot a_1 \simeq \sum_k c_k u^k \frac{du}{u}$$

([3] 16.16). Action of $\mathbf{Z}(1)_B$ on Γ_1 is given by multiplication in $\mathbf{Z}[\mathbf{Z}(1)_B] = \mathbf{Z}[u,u^{-1}]$.

A.3. The descending central series of Γ induces a filtration on Γ_1:

$$Z^N(\Gamma) \cap \Gamma_1 = ((u-1)^{N-1}) \cdot \frac{du}{u} \quad (N \geq 1)$$

Let $\Gamma^{(N)} := \Gamma/Z^{N+1}(\Gamma)$ and $\Gamma_1^{(N)} := \mathrm{Image}(\Gamma_1 \to \Gamma^{(N)})$. Then

$$\Gamma_1^{(N)} = \mathbf{Z}[u,u^{-1}]/(u-1)^N \cdot \frac{du}{u}.$$

Put $u = e^v$ and hence $v = \log u$. Then

$$\mathbf{Q} \otimes \Gamma_1^{(N)} = \mathbf{Q}[u,u^{-1}]/(u-1)^N \cdot \frac{du}{u} = \mathbf{Q}[v]/(v^N) \cdot dv$$

and we have

$$\Gamma_1^{(N)} = \left\{ \sum_{k=0}^{N-1} c_k \exp(kv) dv \,\middle|\, c_k \in \mathbf{Z} \right\}.$$

A.4. As groups, identify

$$\varphi : \mathbf{Q}[v]/(v^N)\cdot dv \xrightarrow{\sim} \prod_1^N \mathbf{Q}(n)_B : \quad v^{n-1}dv = \frac{1}{n}dv^n \mapsto u^{\otimes n}.$$

Then

$$\sum_{k=0}^{N-1} c_k \exp(kv)dv \overset{\varphi}{\mapsto} \sum_{k=0}^{N-1} c_k \left(\sum_{n=0}^{N-1}\frac{1}{n!}k^n u^{\otimes n}\right)\otimes u = \sum_{n=1}^{N}\left(\sum_{k=0}^{N-1} c_k \frac{k^{n-1}}{(n-1)!}\right)u^{\otimes n}.$$

Hence

Proposition A.4.1. ([3] 16.17). $(n-1)!\cdot \mathrm{pr}_n \circ \varphi(\Gamma_1^{(N)}) = \mathbf{Z}(n)_B$.

A.5. ([3] 16.12). Define a Lie algebra action of $\mathbf{Q}(1)$ on $\prod_1^N \mathbf{Q}(n)$ by

$$a * (b_1, b_2, \dots, b_N) = (0, ab_1, \dots, ab_{N-1}),$$

and $\mathbf{Q}(1) \ltimes \prod_1^N \mathbf{Q}(n)$ the associated semi-direct product of Lie algebra.

Let $\mu_0, \mu_1 : \mathbf{Q}(1) \to \mathbf{Q}(1) \ltimes \prod_1^N \mathbf{Q}(n)$ be morphisms of Lie algebras such that μ_0 is the identity onto the first factor $\mathbf{Q}(1)$ and μ_1 is the identity onto the factor $\mathbf{Q}(1)$ in the product $\prod_1^N \mathbf{Q}(n)$.

By abuse of notation, let $\mu_0, \mu_1 : \mathbf{Q}(1) \to \mathbf{Q}\otimes \mathrm{Lie}\,\Gamma^{(N)}$. Then there exists a unique Lie algebra isomorphism respecting each μ_0, μ_1:

$$\mathbf{Q}(1) \ltimes \prod_1^N \mathbf{Q}(n) \xrightarrow{\sim} \mathbf{Q}\otimes \mathrm{Lie}\,\Gamma^{(N)} = \mathbf{Q}(1) \ltimes (\mathbf{Q}\otimes \mathrm{Lie}\,\Gamma_1^{(N)})$$

which is given by μ_0 and $\nu_n := (\mathrm{ad}\,\mu_0)^{n-1}(\mu_1)$ $(1 \le n \le N)$.

A.6. Let $\mathrm{Lie}\,U_{\mathrm{DR}}^{(N)}$ be the de Rham realization of iterated Tate motive in [3] 16.13. Let

$$e_\alpha := \mu_\alpha(1) \in \mathrm{Lie}\,U_{\mathrm{DR}}^{(N)} \quad \big(1 = \exp(2\pi i) \in \mathbf{Q}(1)_{\mathrm{DR}},\ \alpha = 0, 1\big).$$

Take coordinates $(u, (v_n)_{1\le n\le N})$ of $U_{\mathrm{DR}}^{(N)}$ as follows:

$$(u, (v_n)_n) \mapsto \exp(ue_0)\exp\left(\sum_{n=1}^N v_n (\mathrm{Ad}e_0)^{n-1}(e_1)\right).$$

Lemma A.6.1. ([3] 19.3.1). *Let $z \in \mathbf{C}^\times \smallsetminus \mathbf{R}_{\ge 1}$. The end point of the image in $U_{DR}^{(N)}(\mathbf{C})$ of the line segment from $(0, z)$ to z has coordinates $u = 0$, $v_n = -l_n(z)$.*

Proof. Let $z_1, z_2 \in \mathbf{C}^\times \smallsetminus \mathbf{R}_{\ge 1}$. Take a path from z_1 to z_2, and take an iterated integral $I_{z_1}^{z_2}$ of

$$dI(t) = \left(\frac{dt}{t}e_0 + \frac{dt}{t-1}e_1\right)\cdot I(t)$$

for $I(t) = 1 + ue_0 + \sum_n v_n(\mathrm{Ad}e_0)^{n-1}(e_1)$. Note

$$e_0 * e_0 = e_0,\ e_0 * (\mathrm{Ad}e_0)^{n-1}(e_1) = (\mathrm{Ad}e_0)^n(e_1) \ \ (1 \le n \le N),$$

$$e_1 * e_0 = 0,\ e_1 * e_1 = e_1,\ e_1 * (\mathrm{Ad}e_0)^{n-1}(e_1) = 0 \ \ (2 \le n \le N).$$

The corresponding differential equation is

$$du = \frac{dt}{t}, \quad dv_1 = \frac{dt}{t-1}, \quad dv_n = v_{n-1}\frac{dt}{t}.$$

Take $I(z_1) = \text{identity} \in U_{\mathrm{DR}}^{(N)}(\mathbf{C})$ as an initial condition and consider z_2 as a variable.

If z_1 is a tangential base point $(0, \tau)$ ([3] Section 15), replace the initial condition by

$$I(t) \exp\left(-\log\left(\frac{t}{\tau}\right)\right) \to \text{identity} \quad \text{as } t \to 0.$$

For the line segment from $(0, z)$ to z, we have

$$u = \log\left(\frac{t}{z}\right), \quad v_n = -l_n(t). \quad \square$$

Acknowlegements. The author thanks Kazuya Kato and Chikara Nakayama for joint research. This note grew up in the preparation for and during the workshop "Hodge theory and algebraic geometry" held at the end of August, 2017, Tokyo Denki University. The author thanks Tetsushi Ito, Satoshi Minabe, Taro Fujisawa, and Atsushi Ikeda for the good occasion and helpful discussions. The author thanks the referee for careful reading and valuable comments.

S. Usui was partially supported by JSPS Grants-in-Aid for Scientific Research (C) 17K05200.

References

[1] E. Cattani, A. Kaplan and W. Schmid, *Degeneration of Hodge structures*, Ann. of Math. **123** (1986), 457–535.

[2] P. Deligne, *Travaux de Shimura*, Séminaire Bourbaki (1970/71), Exp. 389, Lecture Notes in Math., **244**, Springer (1971), 123–165. DOI: 10.1007/bfb0058700

[3] P. Deligne, *Groupe fondamental de la droite projective moins trois points*, in "Galois group over **Q**", MSRI Publ. 16 (1989), 79–297. DOI: 10.1007/978-1-4613-9649-9_3

[4] P. Deligne, *Catégories tannakiennes*, in Grothendieck Festschrift, Springer (1990), 111–195.

[5] P. A. Griffiths, *Periods of integrals on algebraic manifolds. I. Construction and properties of modular varieties*, Amer. J. Math. **90** (1968), 568–626.

[6] R. M. Hain and S. Zucker, *Unipotent variation of mixed Hodge structure*, Invent. Math. **88** (1987), 83–124. DOI: 10.1007/bf01405093

[7] M. Kashiwara, *A study of variation of mixed Hodge structure*, Publ. RIMS, Kyoto Univ. **22** (1986), 991–1024.

[8] K. Kato, *On* SL(2)*-orbit theorems*, Kyoto J. Math. **54** (2014), 841–861. DOI: 10.1215/21562261-2801840

[9] K. Kato and C. Nakayama, *Log Betti cohomology, log étale cohomology, and log de Rham cohomology of log schemes over* **C**, Kodai Math. J. **22** (1999), 161–186. DOI: 10.2996/kmj/1138044041

[10] K. Kato, C. Nakayama and S. Usui, *Classifying spaces of degenerating mixed Hodge structures, I.*, Adv. Stud. Pure Math. **54** (2009), 187–222, *II.*, Kyoto J. Math. **51** (2011), 149–261, *III.*, J. Algebraic Geometry **22** (2013), 671–772, *IV.*, Kyoto J. Math. **58** (2018), 289-426, *V.*, in preparation.

[11] K. Kato, C. Nakayama and S. Usui, *Extended period domains, Algebraic groups, and higher Albanese manifolds*, in "Hodge theory and L^2-analysis", ALM 39 (2017), 451–473.

[12] K. Kato and S. Usui, *Classifying spaces of degenerating polarized Hodge structures*, Ann. Math. Studies **169**, Princeton University Press (2009). DOI: 10.1515/9781400837113

[13] S. Usui, *Variation of mixed Hodge structure arising from family of logarithmic deformations. II. Classifying space*, Duke Math. J. **51-4** (1984), 851–875. DOI: 10.1215/s0012-7094-85-05227-5

Sampei Usui, Graduate School of Science, Osaka University, Toyonaka, Osaka, 560-0043, Japan
Email address: usui@math.sci.osaka-u.ac.jp