Journal *of* Singularities
Volume 21 (2020), 205-211

Proc. of Geometric and Algebraic Singularity Theory, Będlewo, 2017
DOI: 10.5427/jsing.2020.21l

KATO'S CHAOS CREATED BY QUADRATIC MAPPINGS ASSOCIATED WITH SPHERICAL ORTHOTOMIC CURVES

TAKASHI NISHIMURA

Abstract. In this paper, we first show that for a given generic spherical curve $\gamma : I \to S^n$ and a generic point $P \in S^n$, the spherical orthotomic curve relative to γ and P naturally yield a simple quadratic mapping $\Phi_P : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$. Since S^n is compact and $\Phi_P|_{S^n} : S^n \to S^n$ is the spherical counterpart of the trivial expanding mapping $x \mapsto 2x$, it is natural to expect a chaotic behavior for the iteration of $\Phi_P|_{S^n}$. Accordingly, we show that $\Phi_P|_{S^n}$ (and incidentally $\Phi_P|_{D^{n+1}}$ as well) actually creates Kato's chaos. Therefore, by investigating spherical orthotomic curves, an example of singular quadratic mapping creating Kato's chaos is naturally obtained.

1. Introduction

Throughout this paper, let n be a non-negative integer. In addition, let S^n, D^{n+1} be the unit sphere and the unit closed disk of $\mathbb{R}^{n+1}$ respectively.

Let I be an interval. In [1], for a given plane unit-speed curve $\gamma : I \to \mathbb{R}^2$ and a given point $P \in \mathbb{R}^2$, the pedal curve $ped_{\gamma,P} : I \to \mathbb{R}^2$ and the orthotomic curve $ort_{\gamma,P} : I \to \mathbb{R}^2$ are defined as follows:

$$\begin{aligned} ped_{\gamma,P}(s) &= P + ((\gamma(s) - P) \cdot N(s))\, N(s), \\ ort_{\gamma,P}(s) &= P + 2\,((\gamma(s) - P) \cdot N(s))\, N(s). \end{aligned}$$

Here, $N(s)$ is the unit normal vector to γ at $\gamma(s)$. For instance, let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be a parabola defined by $\gamma(t) = \left(t, t^2 - \frac{1}{4}\right)$ and let P be the origin $(0, 0)$. Let $\ell : \mathbb{R} \to \mathbb{R}$ be the arc-length of γ measured from $\gamma(0)$. Then, $ped_{\gamma\circ\ell^{-1},P}$ is just the affine tangent line to the parabola $\gamma \circ \ell^{-1}$ at $\gamma \circ \ell^{-1}(0)$ and $ort_{\gamma\circ\ell^{-1},P}$ is merely the directrix of the parabola with the focal point P. From this elementary example, in general, the orthotomic curve for a given unit-speed curve γ may be considered as a generalization of the directrix of a parabola in some sense. Moreover, as explained in pp. 175–177 in [1], orthotomic curves have a seismic application. This is a very interesting and very important practical application of orthotomic curves. Since pedal curves seem to be well-studied rather than orthotomic curves, we are interested in how to obtain the orthotomic curve from the pedal curve for a given unit-speed curve γ and a point P. By definition, it follows

$$\frac{ort_{\gamma,P}(s) + P}{2} = ped_{\gamma,P}(s)$$

and thus $ort_{\gamma,P}(s) = 2ped_{\gamma,P}(s) - P$. Therefore, by using the simple mapping $F_P : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$F_P(x) = 2x - P,$$

we have the following:

$$ort_{\gamma,P}(s) = F_P \circ ped_{\gamma,P}(s).$$

2010 *Mathematics Subject Classification.* 37D45, 54H20, 26A18, 39B12 .

Key words and phrases. Kato's chaos, Sensitivity, Accessibility, Quadratic mapping, Spherical orthotomic curve.

Since F_P is nothing but the radial expansion with factor 2 with respect to the point P, the study of orthotomic curves may be completely reduced to the study of pedal curves in the plane curve case.

Similarly, in the case of S^n, by obtaining the orthotomic curve from the pedal curve for a given spherical unit-speed curve γ and a point P, we can get an expanding mapping $S^n \to S^n$ with similar properties as the above F_P. However, in this case, the space S^n is compact. Thus, this expanding mapping $S^n \to S^n$ is expected to have some kneading effect. This expectation leads us to study the iteration of this mapping. In order to get the expanding mapping $S^n \to S^n$, for a generic unit-speed curve $\gamma : I \to S^n$ and a generic point $P \in S^n$, the pedal curve $ped_{\gamma,P} : I \to S^n$ and the orthotomic curve $ort_{\gamma,P} : I \to S^n$ need to be defined reasonably. In [5, 6], a reasonable definition of spherical unit speed curve is given; and then for a spherical unit speed curve $\gamma : I \to S^n$ and a generic point $P \in S^n$, the spherical pedal curve $ped_{\gamma,P} : I \to S^n$ is defined reasonably. Notice that the well-definedness of $ped_{\gamma,P} : I \to S^n$ implies $P \cdot ped_{\gamma,P}(s) \neq 0$ for any $s \in I$ (see [5, 6]). Thus, by using the following relation which is reasonable in S^n,

$$\frac{ort_{\gamma,P}(s) + P}{2} = (P \cdot ped_{\gamma,P}(s))\, ped_{\gamma,P}(s),$$

the spherical orthotomic curve $ort_{\gamma,P} : I \to S^n$ is naturally defined as follows:

$$ort_{\gamma,P}(s) = 2\,(P \cdot ped_{\gamma,P}(s))\, ped_{\gamma,P}(s) - P.$$

Therefore, by using the mapping $\Phi_P : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ defined by

$$\Phi_P(x) = 2(P \cdot x)x - P,$$

the orthotomic curve is obtained from the pedal curve as follows:

$$ort_{\gamma,P}(s) = \Phi_P \circ ped_{\gamma,P}(s).$$

As in the following lemma, both $\Phi_P|_{S^n}$ and $\Phi_P|_{D^{n+1}}$ $(n \geq 0)$ are endomorphisms. Thus, $\Phi_P|_{S^n}$ $(n \geq 1)$ may be regarded as the spherical counterpart of the expansion F_P. By combining these facts and the compactness of S^n (resp., D^{n+1}), it is expected that not only $\Phi_P|_{S^n}$ but also $\Phi_P|_{D^{n+1}}$ may have a chaotic behavior of some kind.

Lemma 1. *For any $P \in S^n$, the following three hold:*

(1) $\Phi_P(S^n) \subset S^n$ *for any* $n \geq 0$.
(2) $\Phi_P(S^n) \supset S^n$ *for any* $n \geq 1$.
(3) $\Phi_P(D^{n+1}) = D^{n+1}$ *for any* $n \geq 0$.

For the proof of Lemma 1, see Section 2. The following two examples, too, show that for both $\Phi_P|_{S^n}$ and $\Phi_P|_{D^{n+1}}$, the chaotic behavior of their iteration deserves to be investigated.

Example 1. Suppose that $n = 1$ and $P = (1, 0)$. Then, $\Phi_P(x) = (2x_1^2 - 1, 2x_1x_2)$, where $x = (x_1, x_2)$. If x belongs to S^1, x may be written as $x = (\cos\theta, \sin\theta)$. Then,

$$\Phi_P|_{S^1}(\cos\theta, \sin\theta) = \left(2\cos^2\theta - 1, 2\cos\theta\sin\theta\right) = (\cos 2\theta, \sin 2\theta).$$

Thus, the restricted mapping $\Phi_P|_{S^n}$ in this case is exactly the same mapping given in Chapter 1, Example 3.4 of Devaney's well-known book [2].

Example 2. Suppose that $n = 0$. Then, P is 1 or -1, and $\Phi_P(x) = 2x^2 - 1$ or $-2x^2 + 1$. Define the affine transformation $h_P : \mathbb{R} \to \mathbb{R}$ as follows:

$$h_P(x) = \begin{cases} -2x + 1 & (\text{if } P = 1), \\ 2x - 1 & (\text{if } P = -1). \end{cases}$$

Then, in each case, it is easily seen that $h_P^{-1} \circ \Phi_P \circ h_P(x) = 4x(1 - x)$. Therefore, in each case, $\Phi_P|_{D^1}$ has the same dynamics as Chapter 1, Example 8.9 of [2].

From Examples 1 and 2, it seems meaningful to study the chaotic behavior of iteration for $\Phi_P|_{S^n} : S^n \to S^n$ ($n \geq 1$) or $\Phi_P|_{D^{n+1}} : D^{n+1} \to D^{n+1}$ ($n \geq 0$), which is the main purpose of this paper.

Definition 1. Let (X, d) be a metric space with metric d and let $f : X \to X$ be a continuous mapping.

(1) The mapping f is said to be *sensitive* if there is a positive number $\lambda > 0$ such that for any $x \in X$ and any neighborhood U of x in X, there exists a point $y \in U$ and a non-negative integer $k \geq 0$ such that $d(f^k(x), f^k(y)) > \lambda$, where f^k stands for $\underbrace{f \circ \cdots \circ f}_{k\text{-tuples}}$.

(2) The mapping f is said to be *transitive* if for any non-empty open subsets $U, V \subset X$, there exists a positive integer $k > 0$ such that $f^k(U) \cap V \neq \emptyset$.

(3) The mapping f is said to be *accessible* if for any $\lambda > 0$ and any non-empty open subsets $U, V \subset X$, there exist two points $u \in U$, $v \in V$ and a positive integer $k > 0$ such that $d(f^k(u), f^k(v)) \leq \lambda$.

(4) The mapping f is said to be *topologically mixing* if for any non-empty open subsets $U, V \subset X$, there exists a positive integer $k > 0$ such that $f^m(U) \cap V \neq \emptyset$ for any $m \geq k$.

(5) The mapping f is said to be *chaotic in the sense of Devaney* ([2]) if f is sensitive, transitive and the set consisting of periodic points of f is dense in X.

(6) The mapping f is said to be *chaotic in the sense of Kato* ([3]) if f is sensitive and accessible.

By definition, it is clear that if a mapping $f : X \to X$ is topologically mixing, then it is transitive. Moreover, by [3], it is known that if a mapping $f : X \to X$ is topologically mixing, then it is chaotic in the sense of Kato. Although Kato's chaos has been well-investigated (for instance, see [3, 4, 7]), elementary examples which are singular and not transitive seem to have been desired. Theorem 1 gives such examples.

Theorem 1. (1) *Let P be a point of S^1.*

(1-1) *The endomorphism $\Phi_P|_{S^1} : S^1 \to S^1$ is chaotic in the sense of Devaney. Moreover, it is chaotic in the sense of Kato.*

(1-2) *The endomorphism $\Phi_P|_{D^2} : D^2 \to D^2$ is chaotic in the sense of Kato although it is not chaotic in the sense of Devaney.*

(2) *Let P be a point of S^0. Then, $\Phi_P|_{D^1} : D^1 \to D^1$ is chaotic in the sense of Devaney. Moreover, it is chaotic in the sense of Kato.*

(3) *Let m be an integer such that $m \geq 2$. Moreover, let P be a point of S^m. Then, both $\Phi_P|_{D^{m+1}} : D^{m+1} \to D^{m+1}$ and $\Phi_P|_{S^m} : S^m \to S^m$ are chaotic in the sense of Kato.*

(4) *Let m be an integer such that $m \geq 2$. Moreover, let P be a point of S^m. Then, neither $\Phi_P|_{D^{m+1}} : D^{m+1} \to D^{m+1}$ nor $\Phi_P|_{S^m} : S^m \to S^m$ is transitive. In particular, neither $\Phi_P|_{D^{m+1}} : D^{m+1} \to D^{m+1}$ nor $\Phi_P|_{S^m} : S^m \to S^m$ is chaotic in the sense of Devaney.*

This paper is organized as follows. In Section 2, the proof of Lemma 1 is given. Theorem 1 is proved in Section 3. Section 4 is an appendix where geometric properties of Φ_P are given though some of properties of Φ_P given in Section 4 already appear implicitly in Sections 2 and 3.

2. Proof of Lemma 1

2.1. **Proof of the assertion (1) of Lemma 1.** Let x be a point of S^n. Then, $x \cdot x = 1$ and we have the following:

$$\begin{aligned}\Phi_P(x)\cdot\Phi_P(x) &= (2(x\cdot P)x - P)\cdot(2(x\cdot P)x-P)\\ &= 4(x\cdot P)^2(x\cdot x) - 4(x\cdot P)^2 + (P\cdot P)\\ &= 4(x\cdot P)^2 - 4(x\cdot P)^2 + 1 = 1.\end{aligned}$$

This completes the proof of the assertion (1). □

2.2. **Proof of the assertion (2) of Lemma 1.** Let y be a point of S^n. Suppose that $y \neq -P$. Set

$$x = \frac{\frac{y+P}{2}}{||\frac{y+P}{2}||}.$$

Then, it follows

$$\begin{aligned}2(x\cdot P)x - P &= 2\left(\frac{\frac{y+P}{2}}{||\frac{y+P}{2}||}\cdot P\right)\frac{\frac{y+P}{2}}{||\frac{y+P}{2}||} - P\\ &= \frac{2}{||y+P||^2}\left((y\cdot P)+1\right)(y+P) - P\\ &= \frac{1}{(1+(y\cdot P))}\left((y\cdot P)+1\right)(y+P) - P\\ &= (y+P) - P = y.\end{aligned}$$

Next, suppose that $y = -P$. Let x be a point of S^n such that $x \cdot P = 0$. Then,

$$2(x\cdot P)x - P = -P = y.$$

Therefore, we have the assertion (2). □

2.3. **Proof of the assertion (3) of Lemma 1.** Let x be a point of $\mathbb{R}^{n+1}$ such that $x \cdot x < 1$. Then, we have

$$\Phi_P(x)\cdot\Phi_P(x) < 4(x\cdot P)^2 - 4(x\cdot P)^2 + 1 = 1.$$

Conversely, let y be a point satisfying $y \cdot y < 1$. Notice that in this case $(y\cdot P)+1 \geq -||y||+1 > 0$ and $1 + ||y||^2 + 2(y\cdot P) \geq 1 + ||y||^2 - 2||y|| = (1-||y||)^2 > 0$. Set

$$a = \sqrt{\frac{1+||y||^2+2(y\cdot P)}{2(y\cdot P)+2}} \text{ and } x = a\frac{\frac{y+P}{2}}{||\frac{y+P}{2}||}.$$

Then,

$$\begin{aligned}2(x\cdot P)x - P &= 2\left(a\frac{\frac{y+P}{2}}{||\frac{y+P}{2}||}\cdot P\right)a\frac{\frac{y+P}{2}}{||\frac{y+P}{2}||} - P\\ &= \frac{2a^2}{||y+P||^2}\left((y\cdot P)+1\right)(y+P) - P\\ &= \frac{2a^2}{(1+||y||^2+2(y\cdot P))}\left((y\cdot P)+1\right)(y+P) - P\\ &= (y+P) - P = y.\end{aligned}$$

Therefore, the assertion (3) holds. □

3. Proof of Theorem 1

3.1. **Proof of the assertion (1) of Theorem 1.** We first show the assertion (1-1). Let x be a point of S^1. Set

$$P = (\cos\alpha, \sin\alpha) \text{ and } x = (\cos\theta, \sin\theta).$$

Then, it is easily seen that

$$\begin{aligned} & \Phi_P(\cos\theta, \sin\theta) \\ = \; & 2\left((\cos\alpha, \sin\alpha) \cdot (\cos\theta, \sin\theta)\right)(\cos\theta, \sin\theta) - (\cos\alpha, \sin\alpha) \\ = \; & (\cos(2\theta - \alpha), \sin(2\theta - \alpha)). \end{aligned}$$

It follows $\Phi_P^k(\cos(\theta+\alpha), \sin(\theta+\alpha)) = \left(\cos(2^k\theta+\alpha), \sin(2^k\theta+\alpha)\right)$ and therefore, by the same argument as in Example 8.6 of [2], $\Phi_P|_{S^1}$ is chaotic in the sense of Devaney. In order to show that $\Phi_P|_{S^1}$ is chaotic in the sense of Kato, it is sufficient to show that $\Phi_P|_{S^1}$ is accessible, which is easily seen by the above formula.

Next, we show the assertion (1-2). Since $\mathbb{R}^2$ may be regarded as $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$, the given point $P \in S^1$ is naturally considered as a point of S^2. Then, $\Phi_P|_{S^2}$ and $\Phi_P|_{D^2}$ are semi-conjugate. Thus, the assertion (1-2) easily follows from the assertions (3) and (4) for $\Phi_P|_{S^2}$. □

3.2. **Proof of the assertion (2) of Theorem 1.** By Subsection 3.1 and Example 8.9 of [2], $\Phi_P|_{D^1}$ is chaotic in the sense of Devaney. Moreover, it is easily seen that the property of accessibility is preserved by semi-conjugacy. Thus, $\Phi_P|_{D^1}$ is chaotic in the sense of Kato as well. □

3.3. **Proof of the assertion (3) of Theorem 1.** Let Q be a point of $S^m - \{P, -P\}$. Set

$$P_Q^{\perp} = \frac{Q - (P \cdot Q)P}{||Q - (P \cdot Q)P||}.$$

Then, it follows $P_Q^{\perp} \in S^m$ and $P \cdot P_Q^{\perp} = 0$. Let x be a point of the circle $S^m \cap (\mathbb{R}P + \mathbb{R}P_Q^{\perp})$. Then, x may be written as $x = \cos\theta \; P + \sin\theta \; P_Q^{\perp}$. Then, it is easily seen that

$$\Phi_P(\cos\theta \; P + \sin\theta \; P_Q^{\perp}) = \cos 2\theta \; P + \sin 2\theta \; P_Q^{\perp}.$$

Hence, for any non-empty open neighborhood U of Q in S^m there exists a positive integer i such that the circle $S^m \cap (\mathbb{R}P + \mathbb{R}P_Q^{\perp})$ is contained in $\Phi_P^i(U)$. Therefore, $\Phi_P|_{S^m}$ is sensitive.

Next, take another point R. By the same argument as above, it is seen that for any non-empty open neighborhood V of R in S^m there exists a positive integer j such that the circle $S^m \cap (\mathbb{R}P + \mathbb{R}P_Q^{\perp})$ is contained in $\Phi_P^j(V)$. Set $k = \max(i, j)$. Then, it follows

$$P \in \Phi_P^k(U) \cap \Phi_P^k(V).$$

Hence, $\Phi_P|_{S^m}$ is accessible.

Moreover, under the identification of S^m and $S^m \times \{0\} (\subset S^{m+1})$, the given point $P \in S^m$ is considered as a point of S^{m+1}. Then, $\Phi_P|_{S^{m+1}}$ and $\Phi_P|_{D^{m+1}}$ are semi-conjugate. Thus, $\Phi_P|_{D^{m+1}}$ is also sensitive and accessible. Therefore, both $\Phi_P|_{S^m}$ and $\Phi_P|_{D^{m+1}}$ are chaotic in the sense of Kato. □

3.4. **Proof of the assertion (4) of Theorem 1.** Let Q, R be points of S^m so that P, Q, R are linearly independent. Then, R does not belong to the circle $S^m \cap (\mathbb{R}P + \mathbb{R}P_Q^{\perp})$ where $P_Q^{\perp}$ is the point constructed in Subsection 3.3. Thus, by the argument given in Subsection 3.3, there exist sufficiently small neighborhoods U (resp., V) of Q (resp., R) in S^m such that $\Phi_P^{\ell}(U) \cap V = \emptyset$ for any $\ell \geq 0$. Hence, $\Phi_P|_{S^m}$ is never transitive.

Again, under the identification of S^m and $S^m \times \{0\}(\subset S^{m+1})$, the given point $P \in S^m$ is considered as a point of S^{m+1}. Then, $\Phi_P|_{S^{m+1}}$ and $\Phi_P|_{D^{m+1}}$ are semi-conjugate. Thus, even $\Phi_P|_{D^{m+1}}$ is not transitive. □

4. Some properties of Φ_P

In this section, following the referee's suggestions, the geometric structure of Φ_P is studied.

Proposition 1. *Let P, $h : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be a point of $\mathbb{R}^{n+1}$ and an orthogonal linear mapping respectively. Set $\widetilde{P} = h(P)$. Then, the following equality holds:*

$$\Phi_{\widetilde{P}} \circ h = h \circ \Phi_P.$$

Proof. Let A be the orthogonal matrix corresponding to h. For any $x \in \mathbb{R}^{n+1}$, we have the following:

$$\begin{aligned} \Phi_{\widetilde{P}} \circ h(x) &= \Phi_{\widetilde{P}}(xA) \\ &= 2\left(\widetilde{P} \cdot xA\right) xA - \widetilde{P} \\ &= 2\left(PA \cdot xA\right) xA - PA \\ &= \left(2\left(P \cdot x\right) x - P\right) A \\ &= h \circ \Phi_P(x). \end{aligned}$$

□

Corollary 1. *Let P be a point of S^n and let $h : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be an orthogonal linear mapping such that $h(P) = (1, 0, \ldots, 0)$. Then, $h \circ \Phi_P \circ h^{-1}$ is the following mapping where $x = (x_1, x_2, \ldots, x_{n+1})$:*

$$h \circ \Phi_P \circ h^{-1}(x_1, x_2, \ldots, x_{n+1}) = (2x_1^2 - 1, 2x_1x_2, \ldots, 2x_1x_{n+1}).$$

Notice that if we understand that $x_2 \in \mathbb{R}^n$, then the form of Φ_P in Example 1 is exactly the same as the form of $h \circ \Phi_P \circ h^{-1}$ in Corollary 1. Moreover, the following holds.

Proposition 2. *Let P be a point of $\mathbb{R}^{n+1} - \{\mathbf{0}\}$. Then, the mapping Φ_P preserves any 2-dimensional linear subspace that contains P. Moreover, the restrictions of Φ_P to such linear subspaces are conjugated to each other.*

Proof. The proof of the first assertion of Proposition 2 is implicitly given in Subsection 3.3 although in Subsection 3.3 P is a point of S^n. Thus, it is omitted to give it here.

We show the second assertion of Proposition 2 by using the same symbols as in Subsection 3.3. Let $\widetilde{Q}$ be a point of $S^n - \left(\mathbb{R}P + \mathbb{R}P_Q^{\perp}\right)$ and let $h : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be an orthogonal linear mapping such that $h(P) = P$ and $h(Q) = \widetilde{Q}$. Then, it is trivially seen that h maps the 2-dimensional linear space $\left(\mathbb{R}P + \mathbb{R}P_Q^{\perp}\right)$ to $\left(\mathbb{R}P + \mathbb{R}P_{\widetilde{Q}}^{\perp}\right)$. Moreover, by Proposition 1, the following equality holds:

$$\Phi_{\widetilde{P}} \circ h = h \circ \Phi_P.$$

Therefore, the second assertion of Proposition 2 holds. □

Proposition 2 reduces the study of dynamical system of Φ_P to the 2-dimensional case, which is given in Example 1.

The final assertion is for the mapping Φ_P where $P = (1, 0, \ldots, 0)$.

Proposition 3. *Let $P = (1, 0, \ldots, 0) \in S^n$ and let $\Phi_P : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be the mapping defined by*

$$\Phi_P(x_1, x_2, \ldots, x_{n+1}) = (2x_1^2 - 1, 2x_1x_2, \ldots, 2x_1x_{n+1}).$$

Let $(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}$ be a point such that

$$\varphi(x_1, x_2, \ldots, x_{n+1}) = x_1^2 + \mu\left(x_2^2 + \cdots + x_{n+1}^2\right) = 1,$$

where μ is a positive real number. Then, $\varphi \circ \Phi_P(x_1, x_2, \ldots, x_{n+1}) = 1$. In other words, Φ_P preserves the level set $\varphi^{-1}(1)$.

Proof. Assume that $\varphi(x_1, x_2, \ldots, x_{n+1}) = 1$. Then,

$$\begin{aligned}
\varphi \circ \Phi_P(x_1, x_2, \ldots, x_{n+1}) &= \left(2x_1^2 - 1\right)^2 + \mu\left((2x_1x_2)^2 + \cdots + (2x_1x_{n+1})^2\right) \\
&= 4x_1^4 - 4x_1^2 + 1 + 4\mu\left(x_1^2x_2^2 + \cdots x_1^2x_{n+1}^2\right) \\
&= 4x_1^4 - 4x_1^2\left(1 - \mu\left(x_2^2 + \cdots + x_{n+1}^2\right)\right) + 1 \\
&= 4x_1^4 - 4x_1^4 + 1 \\
&= 1.
\end{aligned}$$

□

Notice that Φ_P does not necessarily preserve other level sets $\varphi^{-1}(c)$ $(c \neq 1)$. The case $\mu = 1$ of Proposition 3 suggests (1) of Lemma 1.

Acknowledgement

The author would like to express his sincere gratitude to the referee for careful reading of this paper and making invaluable suggestions. This work was partially supported by JSPS KAKENHI Grant Number JP17K05245.

References

[1] J. W. Bruce and P. J. Giblin, *Curves and Singularities (second edition)*, Cambridge University Press, 1992.
[2] R. Devaney, *An Introduction to Chaotic Dynamical Systems*, Addison-Wesley, Reading, MA, 2nd ed., 1989.
[3] H. Kato, *Everywhere chaotic homeomorphisms on manifolds and k-dimensional Menger manifolds*, Topology Appl., **72** (1996), 1–17. DOI: 10.1016/0166-8641(96)00008-9
[4] R. Li, H. Wang and Y. Zhao, *Kato's chaos in duopoly games*, Chaos, Solitons and Fractals, **84** (2016), 69–72. DOI: 10.1016/j.chaos.2016.01.006
[5] T. Nishimura, *Normal forms for singularities of pedal curves produced by non-singular dual curve germs in S^n*, Geom. Dedicata, **133** (2008), 59–66. DOI: 10.1007/s10711-008-9233-5
[6] T. Nishimura, *Singularities of pedal curves produced by singular dual curve germs in S^n*, Demonstratio Math., **43** (2010), 447–459. DOI: 10.1515/dema-2013-0240
[7] L. Wang, J. Liang and Z. Chu, *Weakly mixing property and chaos*, Arch. Math., **109** (2017), 83–89. DOI: 10.1007/s00013-017-1044-1

Takashi Nishimura: Research Institute of Environment and Information Sciences, Yokohama National University, Yokohama 240-8501, Japan
Email address: nishimura-takashi-yx@ynu.ac.jp