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FRAMED SURFACES AND ONE-PARAMETER FAMILIES OF FRAMED CURVES IN EUCLIDEAN 3-SPACE

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Dedicated to Professor Goo Ishikawa on the occasion of his 60th birthday

Abstract. In this paper, we consider two objects as surfaces with singular points in Euclidean 3-space. One is the class of framed surfaces and the other is that of one-parameter families of framed curves. The basic invariants of a framed surface or the curvature of a one-parameter family of framed curves determine the surface and the moving frame up to congruence. We give relations between framed surfaces and one-parameter families of framed curves. In particular, a surface with corank one singularities can be considered as a one-parameter family of framed curves at least locally. Moreover, we give concrete examples of such surfaces with singular points described as one-parameter families of framed curves.

1. Introduction

Recently, differential geometry of curves and surfaces with singular points is extensively investigated (for instance, see [3, 4, 5, 6, 7, 8, 11, 13, 14, 17, 19, 20, 21, 24, 25, 26, 27, 29, 30, 31, 32, 33, 34]). All non-singular surfaces are locally diffeomorphic to each other. Therefore, a diffeomorphism on the target breaks down the differential geometry on surfaces in this sense.

In [34, 6], a normal form of cross caps is given by using a parameter change on the source and an isometry (a rotation) on the target. Moreover, normal forms of cuspidal edges, swallowtails and cuspidal cross caps are given in [20, 29, 24], respectively. By using the normal forms, they obtain $SO(3)$ invariants and give differential geometric properties of surfaces with singular points by using the invariants.

We treat surfaces with singular points, that is, singular surfaces more directly. As a way to study surfaces with singular points in Euclidean 3-space, we consider two approaches. One is to consider framed surfaces and the other is to use one-parameter families of framed curves. We give relations between these two objects.

A framed surface is a surface in Euclidean 3-space with a moving frame (cf. [10]). Framed surfaces may have singular points. By using the moving frames, the basic invariants and the curvatures of framed surfaces are introduced in [10].

On the other hand, a framed curve is a curve in Euclidean 3-space with a moving frame (cf. [12]). Framed curves may have singular points. Therefore, we may consider one-parameter families of framed curves as surfaces with singular points. In [27], the authors have considered one-parameter families of framed curves in order to define an envelope of a family of space curves. By using the moving frame, the curvature of a one-parameter family of framed curves is

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introduced in [27]. We review the theories for framed surfaces, framed curves and one-parameter families of framed curves in §2. The basic invariants of a framed surface or the curvature of a one-parameter family of framed curves determine the surface and the moving frame up to congruence. We give relations between framed surfaces and one-parameter families of framed curves in §3. We then prove that surfaces with corank one singularities can be considered as one-parameter families of framed curves at least locally (Theorem 4.1). As concrete examples of one-parameter families of framed curves, we give surfaces with first kind singularities (for example, cuspidal edges and cuspidal cross caps), second kind singularities (for example, swallowtails) and cross caps by using normal forms in §4. In general, non-degenerate singular points are also of corank one. Moreover, $\mathcal{A}$-simple singularities of a map from a 2-dimensional manifold to a 3-dimensional one are also of corank one, see [22]. Hence, it is possible to treat map germs of non-degenerate singular points and $\mathcal{A}$-simple singularities as one-parameter families of framed curves.

All maps and manifolds considered in this paper are differentiable of class C^∞.

2. Previous results

Let $\mathbb{R}^3$ be the 3-dimensional Euclidean space equipped with the inner product

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1b_1 + a_2b_2 + a_3b_3,$$

where $\boldsymbol{a} = (a_1, a_2, a_3)$ and $\boldsymbol{b} = (b_1, b_2, b_3) \in \mathbb{R}^3$. The norm of $\boldsymbol{a}$ is given by $|\boldsymbol{a}| = \sqrt{\boldsymbol{a} \cdot \boldsymbol{a}}$ and the vector product is given by

$$\boldsymbol{a} \times \boldsymbol{b} = \det \begin{pmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$

where $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ is the canonical basis of $\mathbb{R}^3$. Let S^2 be the unit sphere in $\mathbb{R}^3$, that is,

$$S^2 = \{\boldsymbol{a} \in \mathbb{R}^3 | |\boldsymbol{a}| = 1\}.$$

We denote the 3-dimensional smooth manifold $\{(\boldsymbol{a}, \boldsymbol{b}) \in S^2 \times S^2 | \boldsymbol{a} \cdot \boldsymbol{b} = 0\}$ by Δ.

Let U be a simply connected domain in $\mathbb{R}^2$ and I be an interval in $\mathbb{R}$. We quickly review the theories of framed surfaces, framed curves and one-parameter families of framed curves.

2.1. **Framed surfaces in Euclidean 3-space.** A framed surface in Euclidean 3-space is a smooth surface with a moving frame.

Definition 2.1. We say that $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$ is a *framed surface* if

$$\boldsymbol{x}_u(u, v) \cdot \boldsymbol{n}(u, v) = 0, \boldsymbol{x}_v(u, v) \cdot \boldsymbol{n}(u, v) = 0$$

for all $(u, v) \in U$, where $\boldsymbol{x}_u(u, v) = (\partial \boldsymbol{x}/\partial u)(u, v)$ and $\boldsymbol{x}_v(u, v) = (\partial \boldsymbol{x}/\partial v)(u, v)$. We say that $\boldsymbol{x} : U \to \mathbb{R}^3$ is a *framed base surface* if there exists $(\boldsymbol{n}, \boldsymbol{s}) : U \to \Delta$ such that $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ is a framed surface.

By definition, a framed base surface is a frontal. For definition and properties of frontals see [1, 2, 30]. On the other hand, a frontal is a framed base surface at least locally.

We denote $\boldsymbol{t}(u, v) = \boldsymbol{n}(u, v) \times \boldsymbol{s}(u, v)$. Then $\{\boldsymbol{n}(u, v), \boldsymbol{s}(u, v), \boldsymbol{t}(u, v)\}$ is a moving frame along $\boldsymbol{x}(u, v)$. Thus, we have the following systems of differential equations:

$$\begin{pmatrix} \boldsymbol{x}_u \\ \boldsymbol{x}_v \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} \boldsymbol{s} \\ \boldsymbol{t} \end{pmatrix}, \tag{1}$$

$$\begin{pmatrix} \boldsymbol{n}_u \\ \boldsymbol{s}_u \\ \boldsymbol{t}_u \end{pmatrix} = \begin{pmatrix} 0 & e_1 & f_1 \\ -e_1 & 0 & g_1 \\ -f_1 & -g_1 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{n} \\ \boldsymbol{s} \\ \boldsymbol{t} \end{pmatrix}, \quad \begin{pmatrix} \boldsymbol{n}_v \\ \boldsymbol{s}_v \\ \boldsymbol{t}_v \end{pmatrix} = \begin{pmatrix} 0 & e_2 & f_2 \\ -e_2 & 0 & g_2 \\ -f_2 & -g_2 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{n} \\ \boldsymbol{s} \\ \boldsymbol{t} \end{pmatrix}, \tag{2}$$

where $a_i, b_i, e_i, f_i, g_i : U \to \mathbb{R}, i = 1, 2$ are smooth functions, which we call *basic invariants* of the framed surface. We denote the matrices in the equalities (1) and (2) by $\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2$, respectively. We also call the matrices $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2)$ *basic invariants* of the framed surface $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$. Note that (u, v) is a singular point of $\boldsymbol{x}$ if and only if $\det \mathcal{G}(u, v) = 0$.

Considering the integrability conditions $\boldsymbol{x}_{uv} = \boldsymbol{x}_{vu}$ and $\mathcal{F}_{2,u} - \mathcal{F}_{1,v} = \mathcal{F}_1\mathcal{F}_2 - \mathcal{F}_2\mathcal{F}_1$, the basic invariants should satisfy the following conditions:

$$\begin{cases} a_{1v} - b_1 g_2 = a_{2u} - b_2 g_1, \\ b_{1v} - a_2 g_1 = b_{2u} - a_1 g_2, \\ a_1 e_2 + b_1 f_2 = a_2 e_1 + b_2 f_1, \end{cases} \qquad \begin{cases} e_{1v} - f_1 g_2 = e_{2u} - f_2 g_1, \\ f_{1v} - e_2 g_1 = f_{2u} - e_1 g_2, \\ g_{1v} - e_1 f_2 = g_{2u} - e_2 f_1. \end{cases} \tag{3}$$

Definition 2.2. Let $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}), (\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{n}}, \widetilde{\boldsymbol{s}}) : U \to \mathbb{R}^3 \times \Delta$ be framed surfaces. We say that $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ and $(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{n}}, \widetilde{\boldsymbol{s}})$ are *congruent as framed surfaces* if there exist a constant rotation $A \in SO(3)$ and a translation $\boldsymbol{a} \in \mathbb{R}^3$ such that

$$\widetilde{\boldsymbol{x}}(u, v) = A(\boldsymbol{x}(u, v)) + \boldsymbol{a}, \widetilde{\boldsymbol{n}}(u, v) = A(\boldsymbol{n}(u, v)), \widetilde{\boldsymbol{s}}(u, v) = A(\boldsymbol{s}(u, v)),$$

for all $(u, v) \in U$.

We have the existence and uniqueness theorems for framed surfaces in terms of basic invariants (cf. [10]).

Theorem 2.3 (Existence Theorem for framed surfaces)**.** *Let U be a simply connected domain in $\mathbb{R}^2$ and let $a_i, b_i, e_i, f_i, g_i : U \to \mathbb{R}, i = 1, 2$ be smooth functions with the integrability conditions (3). Then, there exists a framed surface $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$ whose associated basic invariants coincide with a_i, b_i, e_i, f_i, g_i, $i = 1, 2$.*

Theorem 2.4 (Uniqueness Theorem for framed surfaces)**.** *Let $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}), (\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{n}}, \widetilde{\boldsymbol{s}}) : U \to \mathbb{R}^3 \times \Delta$ be framed surfaces with basic invariants $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2), (\widetilde{\mathcal{G}}, \widetilde{\mathcal{F}}_1, \widetilde{\mathcal{F}}_2)$, respectively. Then $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ and $(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{n}}, \widetilde{\boldsymbol{s}})$ are congruent as framed surfaces if and only if the basic invariants $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2)$ and $(\widetilde{\mathcal{G}}, \widetilde{\mathcal{F}}_1, \widetilde{\mathcal{F}}_2)$ coincide.*

Let $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$ be a framed surface with basic invariants $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2)$. We consider rotations of the vectors $\boldsymbol{s}, \boldsymbol{t}$. We denote

$$\begin{pmatrix} \boldsymbol{s}^\theta(u, v) \\ \boldsymbol{t}^\theta(u, v) \end{pmatrix} = \begin{pmatrix} \cos\theta(u, v) & -\sin\theta(u, v) \\ \sin\theta(u, v) & \cos\theta(u, v) \end{pmatrix} \begin{pmatrix} \boldsymbol{s}(u, v) \\ \boldsymbol{t}(u, v) \end{pmatrix},$$

where $\theta : U \to \mathbb{R}$ is a smooth function. Then $\boldsymbol{n} \times \boldsymbol{s}^\theta = \boldsymbol{t}^\theta$ and $\{\boldsymbol{n}, \boldsymbol{s}^\theta, \boldsymbol{t}^\theta\}$ is also a moving frame along $\boldsymbol{x}$. It follows that $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}^\theta)$ is a framed surface. We call the frame $\{\boldsymbol{n}, \boldsymbol{s}^\theta, \boldsymbol{t}^\theta\}$ a *rotation frame* by θ of the framed surface $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$. We denote by $(\mathcal{G}^\theta, \mathcal{F}_1^\theta, \mathcal{F}_2^\theta)$ the basic invariants of $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}^\theta)$. By a direct calculation, we have the following.

Proposition 2.5. *Under the above notations, the relations between the basic invariants $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2)$ and $(\mathcal{G}^\theta, \mathcal{F}_1^\theta, \mathcal{F}_2^\theta)$ are given by*

$$\mathcal{G}^\theta = \mathcal{G} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} a_1\cos\theta - b_1\sin\theta & a_1\sin\theta + b_1\cos\theta \\ a_2\cos\theta - b_2\sin\theta & a_2\sin\theta + b_2\cos\theta \end{pmatrix},$$

$$\mathcal{F}_1^\theta = \begin{pmatrix} 0 & e_1\cos\theta - f_1\sin\theta & e_1\sin\theta + f_1\cos\theta \\ -e_1\cos\theta + f_1\sin\theta & 0 & g_1 - \theta_u \\ -e_1\sin\theta - f_1\cos\theta & -g_1 + \theta_u & 0 \end{pmatrix},$$

$$\mathcal{F}_2^\theta = \begin{pmatrix} 0 & e_2\cos\theta - f_2\sin\theta & e_2\sin\theta + f_2\cos\theta \\ -e_2\cos\theta + f_2\sin\theta & 0 & g_2 - \theta_v \\ -e_2\sin\theta - f_2\cos\theta & -g_2 + \theta_v & 0 \end{pmatrix}.$$

2.2. **Framed curves in Euclidean 3-space.** A framed curve in Euclidean 3-space is a smooth curve with a moving frame.

Definition 2.6. We say that $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta$ is a *framed curve* if $\dot\gamma(t)\cdot\nu_1(t) = 0$ and $\dot\gamma(t)\cdot\nu_2(t) = 0$ for all $t \in I$. We say that $\gamma : I \to \mathbb{R}^3$ is a *framed base curve* if there exists $(\nu_1, \nu_2) : I \to \Delta$ such that (γ, ν_1, ν_2) is a framed curve.

We denote $\boldsymbol{\mu}(t) = \nu_1(t) \times \nu_2(t)$. Then $\{\nu_1(t), \nu_2(t), \boldsymbol{\mu}(t)\}$ is a moving frame along the framed base curve $\gamma(t)$ in $\mathbb{R}^3$ and we have the Frenet type formula,

$$\begin{pmatrix} \dot\nu_1(t) \\ \dot\nu_2(t) \\ \dot{\boldsymbol{\mu}}(t) \end{pmatrix} = \begin{pmatrix} 0 & \ell(t) & m(t) \\ -\ell(t) & 0 & n(t) \\ -m(t) & -n(t) & 0 \end{pmatrix} \begin{pmatrix} \nu_1(t) \\ \nu_2(t) \\ \boldsymbol{\mu}(t) \end{pmatrix}, \ \dot\gamma(t) = \alpha(t)\boldsymbol{\mu}(t)$$

where $\ell(t) = \dot\nu_1(t)\cdot\nu_2(t)$, $m(t) = \dot\nu_1(t)\cdot\boldsymbol{\mu}(t), n(t) = \dot\nu_2(t)\cdot\boldsymbol{\mu}(t)$ and $\alpha(t) = \dot\gamma(t)\cdot\boldsymbol{\mu}(t)$. We call the mapping (ℓ, m, n, α) *the curvature of the framed curve* (γ, ν_1, ν_2). Note that t_0 is a singular point of γ if and only if $\alpha(t_0) = 0$.

Definition 2.7. Let $(\gamma, \nu_1, \nu_2), (\widetilde\gamma, \widetilde\nu_1, \widetilde\nu_2) : I \to \mathbb{R}^3 \times \Delta$ be framed curves. We say that (γ, ν_1, ν_2) and $(\widetilde\gamma, \widetilde\nu_1, \widetilde\nu_2)$ are *congruent as framed curves* if there exist a constant rotation $A \in SO(3)$ and a translation $\boldsymbol{a} \in \mathbb{R}^3$ such that $\widetilde\gamma(t) = A(\gamma(t)) + \boldsymbol{a}$, $\widetilde{\nu_1}(t) = A(\nu_1(t))$ and $\widetilde{\nu_2}(t) = A(\nu_2(t))$ for all $t \in I$.

We have the existence and uniqueness theorems for framed curves in terms of the curvatures (cf. [12]).

Theorem 2.8 (Existence Theorem for framed curves)**.** *Let $(\ell, m, n, \alpha) : I \to \mathbb{R}^4$ be a smooth mapping. Then, there exists a framed curve $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta$ whose curvature is given by (ℓ, m, n, α).*

Theorem 2.9 (Uniqueness Theorem for framed curves)**.** *Let*

$$(\gamma, \nu_1, \nu_2), (\widetilde\gamma, \widetilde\nu_1, \widetilde\nu_2) : I \to \mathbb{R}^3 \times \Delta$$

be framed curves with curvatures $(\ell, m, n, \alpha), (\widetilde\ell, \widetilde m, \widetilde n, \widetilde\alpha)$, respectively. Then (γ, ν_1, ν_2) and $(\widetilde\gamma, \widetilde\nu_1, \widetilde\nu_2)$ are congruent as framed curves if and only if the curvatures (ℓ, m, n, α) and $(\widetilde\ell, \widetilde m, \widetilde n, \widetilde\alpha)$ coincide.

As a special case of a framed curve, let us consider a spherical Legendre curve, for details see [31]. We say that $(\gamma, \nu) : I \to \Delta$ is a spherical Legendre curve if $\dot\gamma(t)\cdot\nu(t) = 0$ for all $t \in I$. We call γ a frontal and ν a dual of γ.

We define $\boldsymbol{\mu}(t) = \gamma(t) \times \nu(t)$. Then $\boldsymbol{\mu}(t) \in S^2$, $\gamma(t)\cdot\boldsymbol{\mu}(t) = 0$ and $\nu(t)\cdot\boldsymbol{\mu}(t) = 0$ for all $t \in I$. It follows that $\{\gamma(t), \nu(t), \boldsymbol{\mu}(t)\}$ is a moving frame along the frontal $\gamma(t)$.

Let $(\gamma,\nu):I\to\Delta$ be a spherical Legendre curve. Then we have

$$\begin{pmatrix}\dot\gamma(t)\\ \dot\nu(t)\\ \dot{\boldsymbol{\mu}}(t)\end{pmatrix}=\begin{pmatrix}0&0&m(t)\\0&0&n(t)\\-m(t)&-n(t)&0\end{pmatrix}\begin{pmatrix}\gamma(t)\\ \nu(t)\\ \boldsymbol{\mu}(t)\end{pmatrix},$$

where $m(t)=\dot\gamma(t)\cdot\boldsymbol{\mu}(t)$ and $n(t)=\dot\nu(t)\cdot\boldsymbol{\mu}(t)$.

We say that the pair of functions (m,n) is the curvature of the spherical Legendre curve $(\gamma,\nu):I\to\Delta$.

2.3. **One-parameter families of framed curves in Euclidean 3-space.** We consider one-parameter families of framed curves in Euclidean 3-space. Let $(\gamma,\nu_1,\nu_2):U\to\mathbb{R}^3\times\Delta$ be a smooth mapping, where U is a simply connected domain in $\mathbb{R}^2$.

Definition 2.10. We say that $(\gamma,\nu_1,\nu_2):U\to\mathbb{R}^3\times\Delta$ is a *one-parameter family of framed curves with respect to* u (respectively, *with respect to* v) if $(\gamma(\cdot,v),\nu_1(\cdot,v),\nu_2(\cdot,v))$ is a framed curve for each v (respectively, $(\gamma(u,\cdot),\nu_1(u,\cdot),\nu_2(u,\cdot))$ is a framed curve for each u).

If $(\gamma,\nu_1,\nu_2):U\to\mathbb{R}^3\times\Delta$ is a one-parameter family of framed curves with respect to u, then we denote $\boldsymbol{\mu}(u,v)=\nu_1(u,v)\times\nu_2(u,v)$. It follows that $\{\nu_1(u,v),\nu_2(u,v),\boldsymbol{\mu}(u,v)\}$ is a moving frame along $\gamma(u,v)$. We have the Frenet type formula.

$$\begin{aligned}
\begin{pmatrix}\nu_{1u}(u,v)\\ \nu_{2u}(u,v)\\ \boldsymbol{\mu}_u(u,v)\end{pmatrix}&=\begin{pmatrix}0&\ell(u,v)&m(u,v)\\-\ell(u,v)&0&n(u,v)\\-m(u,v)&-n(u,v)&0\end{pmatrix}\begin{pmatrix}\nu_1(u,v)\\ \nu_2(u,v)\\ \boldsymbol{\mu}(u,v)\end{pmatrix},\\
\begin{pmatrix}\nu_{1v}(u,v)\\ \nu_{2v}(u,v)\\ \boldsymbol{\mu}_v(u,v)\end{pmatrix}&=\begin{pmatrix}0&L(u,v)&M(u,v)\\-L(u,v)&0&N(u,v)\\-M(u,v)&-N(u,v)&0\end{pmatrix}\begin{pmatrix}\nu_1(u,v)\\ \nu_2(u,v)\\ \boldsymbol{\mu}(u,v)\end{pmatrix},\\
\gamma_u(u,v)&=\alpha(u,v)\boldsymbol{\mu}(u,v),\\
\gamma_v(u,v)&=P(u,v)\nu_1(u,v)+Q(u,v)\nu_2(u,v)+R(u,v)\boldsymbol{\mu}(u,v),
\end{aligned}$$

where

$$\begin{aligned}
\ell(u,v)&=\nu_{1u}(u,v)\cdot\nu_2(u,v), & m(u,v)&=\nu_{1u}(u,v)\cdot\boldsymbol{\mu}(u,v),\\
n(u,v)&=\nu_{2u}(u,v)\cdot\boldsymbol{\mu}(u,v), & \alpha(u,v)&=\gamma_u(u,v)\cdot\boldsymbol{\mu}(u,v),\\
L(u,v)&=\nu_{1v}(u,v)\cdot\nu_2(u,v), & M(u,v)&=\nu_{1v}(u,v)\cdot\boldsymbol{\mu}(u,v),\\
N(u,v)&=\nu_{2v}(u,v)\cdot\boldsymbol{\mu}(u,v), & P(u,v)&=\gamma_v(u,v)\cdot\nu_1(u,v),\\
Q(u,v)&=\gamma_v(u,v)\cdot\nu_2(u,v), & R(u,v)&=\gamma_v(u,v)\cdot\boldsymbol{\mu}(u,v).
\end{aligned}$$

By $\gamma_{uv}(u,v)=\gamma_{vu}(u,v)$, $\nu_{1uv}(u,v)=\nu_{1vu}(u,v)$, $\nu_{2uv}(u,v)=\nu_{2vu}(u,v)$ and $\boldsymbol{\mu}_{uv}(u,v)=\boldsymbol{\mu}_{vu}(u,v)$, we have the integrability condition

$$\begin{aligned}
L_u(u,v)&=M(u,v)n(u,v)-N(u,v)m(u,v)+\ell_v(u,v),\\
M_u(u,v)&=N(u,v)\ell(u,v)-L(u,v)n(u,v)+m_v(u,v),\\
N_u(u,v)&=L(u,v)m(u,v)-M(u,v)\ell(u,v)+n_v(u,v),\\
P_u(u,v)&=Q(u,v)\ell(u,v)+R(u,v)m(u,v)-\alpha(u,v)M(u,v),\\
Q_u(u,v)&=-P(u,v)\ell(u,v)+R(u,v)n(u,v)-\alpha(u,v)N(u,v),\\
R_u(u,v)&=-P(u,v)m(u,v)-Q(u,v)n(u,v)+\alpha_v(u,v)
\end{aligned}\tag{4}$$

for all $(u,v)\in U$.

We call the mapping $(\ell, m, n, \alpha, L, M, N, P, Q, R)$ satisfying the integrability condition (4) the *curvature of the one-parameter family of framed curves with respect to* u of (γ, ν_1, ν_2).

Definition 2.11. Let $(\gamma, \nu_1, \nu_2), (\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2) : U \to \mathbb{R}^3 \times \Delta$ be one-parameter families of framed curves with respect to u. We say that (γ, ν_1, ν_2) and $(\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2)$ are *congruent as one-parameter families of framed curves* if there exist a constant rotation $A \in SO(3)$ and a translation $\boldsymbol{a} \in \mathbb{R}^3$ such that $\widetilde{\gamma}(u, v) = A(\gamma(u, v)) + \boldsymbol{a}$, $\widetilde{\nu}_1(u, v) = A(\nu_1(u, v))$ and $\widetilde{\nu}_2(u, v) = A(\nu_2(u, v))$ for all $(u, v) \in U$.

We also have the existence and uniqueness theorems for one-parameter families of framed curves in terms of curvatures (cf. [27]).

Theorem 2.12 (Existence Theorem for one-parameter families of framed curves)**.**
Let $(\ell, m, n, \alpha, L, M, N, P, Q, R) : I \to \mathbb{R}^{10}$ *be a smooth mapping satisfying the integrability condition* (4)*. Then, there exists a one-parameter family of framed curves with respect to* u*,* $(\gamma, \nu_1, \nu_2) : U \to \mathbb{R}^3 \times \Delta$ *whose curvature is given by* $(\ell, m, n, \alpha, L, M, N, P, Q, R)$*.*

Theorem 2.13 (Uniqueness Theorem for one-parameter families of framed curves)**.** *Let*

$$(\gamma, \nu_1, \nu_2), (\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2) : U \to \mathbb{R}^3 \times \Delta$$

be one-parameter families of framed curves with respect to u *with curvatures*

$$(\ell, m, n, \alpha, L, M, N, P, Q, R), (\widetilde{\ell}, \widetilde{m}, \widetilde{n}, \widetilde{\alpha}, \widetilde{L}, \widetilde{M}, \widetilde{N}, \widetilde{P}, \widetilde{Q}, \widetilde{R}),$$

respectively. Then (γ, ν_1, ν_2) *and* $(\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2)$ *are congruent as one-parameter families of framed curves if and only if the curvatures* $(\ell, m, n, \alpha, L, M, N, P, Q, R)$ *and* $(\widetilde{\ell}, \widetilde{m}, \widetilde{n}, \widetilde{\alpha}, \widetilde{L}, \widetilde{M}, \widetilde{N}, \widetilde{P}, \widetilde{Q}, \widetilde{R})$ *coincide.*

Let $(\gamma, \nu_1, \nu_2) : U \to \mathbb{R}^3 \times \Delta$ be a one-parameter family of framed curves with respect to u with curvature $(\ell, m, n, \alpha, L, M, N, P, Q, R)$. For the normal plane of $\gamma(u, v)$, spanned by $\nu_1(t, \lambda)$ and $\nu_2(t, \lambda)$, there are other frames by rotations. We define $(\nu_1^\theta(u, v), \nu_2^\theta(u, v)) \in \Delta$ by

$$\begin{pmatrix} \nu_1^\theta(u, v) \\ \nu_2^\theta(u, v) \end{pmatrix} = \begin{pmatrix} \cos\theta(u, v) & -\sin\theta(u, v) \\ \sin\theta(u, v) & \cos\theta(u, v) \end{pmatrix} \begin{pmatrix} \nu_1(u, v) \\ \nu_2(u, v) \end{pmatrix},$$

where $\theta : U \to \mathbb{R}$ is a smooth function. Then $(\gamma, \nu_1^\theta, \nu_2^\theta) : U \to \mathbb{R}^3 \times \Delta$ is also a one-parameter family of framed curves with respect to u and

$$\boldsymbol{\mu}^\theta(u, v) = \nu_1^\theta(u, v) \times \nu_2^\theta(u, v) = \nu_1(u, v) \times \nu_2(u, v) = \boldsymbol{\mu}(u, v).$$

Proposition 2.14. *Under the above notation, the curvature*

$$(\ell^\theta, m^\theta, n^\theta, \alpha^\theta, L^\theta, M^\theta, N^\theta, P^\theta, Q^\theta, R^\theta)$$

of $(\gamma, \nu_1^\theta, \nu_2^\theta)$ *is given by*

$$\begin{aligned} &(\ell - \theta_u, m\cos\theta - n\sin\theta, m\sin\theta + n\cos\theta, \alpha, L - \theta_v, M\cos\theta - N\sin\theta, \\ &M\sin\theta + N\cos\theta, P\cos\theta - Q\sin\theta, P\sin\theta + Q\cos\theta, R). \end{aligned}$$

We call the moving frame $\{\nu_1^\theta(u, v), \nu_2^\theta(u, v), \boldsymbol{\mu}(u, v)\}$ the rotated frame along $\gamma(u, v)$ by $\theta(u, v)$.

We also have similar results for the case of one-parameter families of framed curves with respect to v.

3. Relations between framed surfaces and one-parameter families of framed curves

3.1. **Framed surfaces as one-parameter families of framed curves.**

Let

$$(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$$

be a framed surface with basic invariants $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2)$. We denote $\boldsymbol{t} = \boldsymbol{n} \times \boldsymbol{s}$. We give conditions for the framed surface to be a one-parameter family of framed curves.

Lemma 3.1. *Under the above notations, we have the following.*

(1) *$(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ is a one-parameter family of framed curves with respect to u if and only if $a_1(u, v) = 0$ for all $(u, v) \in U$.*

(2) *$(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{t})$ is a one-parameter family of framed curves with respect to u if and only if $b_1(u, v) = 0$ for all $(u, v) \in U$.*

(3) *$(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ is a one-parameter family of framed curves with respect to v if and only if $a_2(u, v) = 0$ for all $(u, v) \in U$.*

(4) *$(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{t})$ is a one-parameter family of framed curves with respect to v if and only if $b_2(u, v) = 0$ for all $(u, v) \in U$.*

Proof. (1) If $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ is a one-parameter family of framed curves with respect to u, then $\boldsymbol{x}_u(u, v) \cdot \boldsymbol{n}(u, v) = 0$ and $\boldsymbol{x}_u(u, v) \cdot \boldsymbol{s}(u, v) = 0$ for all $(u, v) \in U$. Since $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ is a framed surface, the condition $\boldsymbol{x}_u(u, v) \cdot \boldsymbol{n}(u, v) = 0$ holds. Hence, the condition $\boldsymbol{x}_u(u, v) \cdot \boldsymbol{s}(u, v) = 0$ for all $(u, v) \in U$ is equivalent to $a_1(u, v) = 0$ for all $(u, v) \in U$.

The other cases can be proved similarly. □

Proposition 3.2. *Under the above notations, we have the following.*

(1) *Suppose that there exist smooth functions $k_1, k_2 : U \to \mathbb{R}$ such that*

$$(k_1(u, v), k_2(u, v)) \neq (0, 0)$$

and

$$k_1(u, v)a_1(u, v) + k_2(u, v)b_1(u, v) = 0$$

for all $(u, v) \in U$. Then there exist smooth functions $\theta, \varphi : U \to \mathbb{R}$ such that $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}^\theta)$ and $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{t}^\varphi)$ are one-parameter families of framed curves with respect to u.

(2) *Suppose that there exist smooth functions $k_1, k_2 : U \to \mathbb{R}$ such that*

$$(k_1(u, v), k_2(u, v)) \neq (0, 0)$$

and $k_1(u, v)a_2(u, v) + k_2(u, v)b_2(u, v) = 0$ for all $(u, v) \in U$. Then there exist smooth functions $\theta, \varphi : U \to \mathbb{R}$ such that $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}^\theta)$ and $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{t}^\varphi) : U \to \mathbb{R}^3 \times \Delta$ are one-parameter families of framed curves with respect to v.

Proof. (1) We take a smooth function $\theta : U \to \mathbb{R}$ which satisfies the condition

$$(\cos\theta(u, v), \sin\theta(u, v)) = \left(\frac{k_1(u, v)}{\sqrt{k_1^2(u, v) + k_2^2(u, v)}}, \frac{-k_2(u, v)}{\sqrt{k_1^2(u, v) + k_2^2(u, v)}} \right).$$

Then by Proposition 2.5,

$$\begin{aligned} a_1^\theta(u, v) &= a_1(u, v)\cos\theta(u, v) - b_1(u, v)\sin\theta(u, v) \\ &= \frac{1}{\sqrt{k_1^2(u, v) + k_2^2(u, v)}} \left(k_1(u, v)a_1(u, v) + k_2(u, v)b_1(u, v)\right) \\ &= 0. \end{aligned}$$

By Lemma 3.1 (1), $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}^\theta)$ is a one-parameter family of framed curves with respect to u. Moreover, we take a smooth function $\varphi : U \to \mathbb{R}$ which satisfies the condition

$$(\cos\varphi(u,v), \sin\varphi(u,v)) = \left(\frac{k_2(u,v)}{\sqrt{k_1^2(u,v)+k_2^2(u,v)}}, \frac{k_1(u,v)}{\sqrt{k_1^2(u,v)+k_2^2(u,v)}} \right).$$

Then by Proposition 2.5,

$$\begin{aligned} b_1^\varphi(u,v) &= a_1(u,v)\sin\varphi(u,v) + b_1(u,v)\cos\varphi(u,v) \\ &= \frac{1}{\sqrt{k_1^2(u,v)+k_2^2(u,v)}} \left(k_1(u,v)a_1(u,v) + k_2(u,v)b_1(u,v)\right) \\ &= 0. \end{aligned}$$

By Lemma 3.1 (2), $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{t}^\varphi)$ is a one-parameter family of framed curves with respect to u.

(2) We can prove the assertion by a similar calculation. □

We give a relation between basic invariants of a framed surface and the curvature of the one-parameter family of framed curves under a condition.

Proposition 3.3. *Let $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$ be a framed surface with basic invariants $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2)$. Suppose that $a_1(u,v) = 0$ for all $(u,v) \in U$. Then the curvature of the one-parameter family of framed curves with respect to u of $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$ is given by*

$$\begin{aligned} &(\ell(u,v), m(u,v), n(u,v), \alpha(u,v), L(u,v), M(u,v), N(u,v), P(u,v), Q(u,v), R(u,v)) \\ &= (e_1(u,v), f_1(u,v), g_1(u,v), b_1(u,v), e_2(u,v), f_2(u,v), g_2(u,v), 0, a_2(u,v), b_2(u,v)). \end{aligned}$$

Proof. By definitions of basic invariants and the curvature, we have

$$\begin{aligned} \ell(u,v) &= \boldsymbol{n}_u(u,v)\cdot\boldsymbol{s}(u,v) = e_1(u,v), & m(u,v) &= \boldsymbol{n}_u(u,v)\cdot\boldsymbol{t}(u,v) = f_1(u,v), \\ n(u,v) &= \boldsymbol{s}_u(u,v)\cdot\boldsymbol{t}(u,v) = g_1(u,v), & \alpha(u,v) &= \boldsymbol{x}_u(u,v)\cdot\boldsymbol{t}(u,v) = b_1(u,v), \\ L(u,v) &= \boldsymbol{n}_v(u,v)\cdot\boldsymbol{s}(u,v) = e_2(u,v), & M(u,v) &= \boldsymbol{n}_v(u,v)\cdot\boldsymbol{t}(u,v) = f_2(u,v), \\ N(u,v) &= \boldsymbol{s}_v(u,v)\cdot\boldsymbol{t}(u,v) = g_2(u,v), & P(u,v) &= \boldsymbol{x}_v(u,v)\cdot\boldsymbol{n}(u,v) = 0, \\ Q(u,v) &= \boldsymbol{x}_v(u,v)\cdot\boldsymbol{s}(u,v) = a_2(u,v), & R(u,v) &= \boldsymbol{x}_v(u,v)\cdot\boldsymbol{n}(u,v) = b_2(u,v). \end{aligned}$$

□

We give examples of framed surfaces which are not a one-parameter family of framed curves with respect to u nor v as follows.

Example 3.4. Let $\boldsymbol{x} : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$\boldsymbol{x}(u,v) = \begin{cases} \left(e^{-\frac{1}{u^2}-\frac{1}{v^2}} \cos\frac{1}{u^2}\cos\frac{1}{v^2}, e^{-\frac{1}{u^2}-\frac{1}{v^2}} \sin\frac{1}{u^2}\sin\frac{1}{v^2}, 0 \right) & (u, v \neq 0), \\ (0,0,0) & (u = 0 \text{ or } v = 0). \end{cases}$$

Then $\boldsymbol{x}$ is a smooth mapping. Moreover, if we take $\boldsymbol{n}(u,v) = (0,0,1)$ and $\boldsymbol{s}(u,v) = (1,0,0)$, then $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : \mathbb{R}^2 \to \mathbb{R}^3 \times \Delta$ is a framed surface.

Next, we show that $\boldsymbol{x}$ is not a one-parameter family of framed curves with respect to u nor v. If $u, v \neq 0$, then we have

$$\begin{aligned} \boldsymbol{x}_u(u,v) &= \frac{2e^{-\frac{1}{u^2}-\frac{1}{v^2}}}{u^3} \left(\left(\cos\frac{1}{u^2} + \sin\frac{1}{u^2}\right)\cos\frac{1}{v^2}, \left(\sin\frac{1}{u^2} - \cos\frac{1}{u^2}\right)\sin\frac{1}{v^2}, 0 \right), \\ \boldsymbol{x}_v(u,v) &= \frac{2e^{-\frac{1}{u^2}-\frac{1}{v^2}}}{v^3} \left(\left(\cos\frac{1}{v^2} + \sin\frac{1}{v^2}\right)\cos\frac{1}{u^2}, \left(\sin\frac{1}{v^2} - \cos\frac{1}{v^2}\right)\sin\frac{1}{u^2}, 0 \right). \end{aligned}$$

For $v \in \mathbb{R}$ with $\cos(1/v^2)\sin(1/v^2) \neq 0$, $\lim_{u\to 0+0} \boldsymbol{x}_u(u,v)/|\boldsymbol{x}_u(u,v)|$ does not exist. Hence there does not exist $(\nu_1^u, \nu_2^u) : \mathbb{R}^2 \to \Delta$ such that $(\boldsymbol{x}, \nu_1^u, \nu_2^u)$ is a one-parameter family of framed curves with respect to u (cf. [9]). Also, for $u \in \mathbb{R}$ with $\cos(1/u^2)\sin(1/u^2) \neq 0$, $\lim_{v\to 0+0} \boldsymbol{x}_v(u,v)/|\boldsymbol{x}_v(u,v)|$ does not exist. Hence, there does not exist $(\nu_1^v, \nu_2^v) : \mathbb{R}^2 \to \Delta$ such that $(\boldsymbol{x}, \nu_1^v, \nu_2^v)$ is a one-parameter family of framed curves with respect to v. In particular, $\boldsymbol{x}$ is not a one-parameter family of framed base curves with respect to u nor v around $(0,0)$.

A singular point of a mapping $\boldsymbol{x} : U \to \mathbb{R}^3$ is a $D_4^\pm$ singularity if $\boldsymbol{x}$ at the point is $\mathcal{A}$-equivalent (equivalent by diffeomorphisms of the source and of the target) to the map germ $(u,v) \mapsto (uv, u^2 \pm 3v^2, u^2v \pm v^3)$ at $(0,0)$ (cf. [2, 28]).

Example 3.5 ($D_4^\pm$ singularity)**.** Let $\boldsymbol{x}^\pm : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$\boldsymbol{x}^\pm(u,v) = (uv, u^2 \pm 3v^2, u^2v \pm v^3).$$

Define $\boldsymbol{n} : \mathbb{R}^2 \to S^2$ by $\boldsymbol{n}(u,v) = (2u, v, -2)/\sqrt{4u^2+v^2+4}$. Since $\boldsymbol{x}_u^\pm(u,v) = (v, 2u, 2uv)$ and $\boldsymbol{x}_v^\pm(u,v) = (u, \pm 6v, u^2 \pm 3v^2)$, $\boldsymbol{x}_u^\pm(u,v) \cdot \boldsymbol{n}(u,v) = \boldsymbol{x}_v^\pm(u,v) \cdot \boldsymbol{n}(u,v) = 0$ for all $(u,v) \in \mathbb{R}^2$. It follows that $(\boldsymbol{x}^\pm, \boldsymbol{n}) : \mathbb{R}^2 \to \mathbb{R}^3 \times S^2$ is a Legendre immersion. However, $\boldsymbol{x}^\pm$ are not one-parameter families of framed base curves with respect to u nor v around $(0,0)$.

We give an example of a framed surface which is also a one-parameter family of framed curves with respect to u and v, respectively.

Example 3.6. Let m_1, n_1, k_1, m_2, n_2 and k_2 be positive integers with

$$m_1 = n_1 + k_1 \quad \text{and} \quad m_2 = n_2 + k_2.$$

Let $\boldsymbol{x} : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$\boldsymbol{x}(u,v) = \left(\frac{1}{n_1}u^{n_1}, \frac{1}{m_1}u^{m_1} + \frac{1}{n_2}v^{n_2}, \frac{1}{m_2}v^{m_2}\right).$$

Define $(\boldsymbol{n}, \boldsymbol{s}) : \mathbb{R}^2 \to \Delta$ by

$$\boldsymbol{n}(u,v) = \frac{(u^{k_1}v^{k_2}, -v^{k_2}, 1)}{\sqrt{u^{2k_1}v^{2k_2} + v^{2k_2} + 1}}, \ \boldsymbol{s}(u,v) = \frac{(1, u^{k_1}, 0)}{\sqrt{u^{2k_1}+1}}.$$

Since

$$\boldsymbol{x}_u(u,v) = (u^{n_1-1}, u^{m_1-1}, 0) = u^{n_1-1}(1, u^{k_1}, 0), \ \boldsymbol{x}_v(u,v) = (0, v^{n_2-1}, v^{m_2-1}) = v^{n_2-1}(0, 1, v^{k_2}),$$

we have $\boldsymbol{x}_u(u,v) \cdot \boldsymbol{n}(u,v) = \boldsymbol{x}_v(u,v) \cdot \boldsymbol{n}(u,v) = 0$ for all $(u,v) \in \mathbb{R}^2$. It follows that $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ is a framed surface. If $n_1, n_2 > 1$, then $(0,0)$ is a corank 2 singular point of $\boldsymbol{x}$. Moreover, define $(\nu_1^u, \nu_2^u) : \mathbb{R}^2 \to \Delta$ and $(\nu_1^v, \nu_2^v) : \mathbb{R}^2 \to \Delta$ by

$$\nu_1^u(u,v) = \frac{(-u^{k_1}, 1, 0)}{\sqrt{u^{2k_1}+1}}, \ \nu_2^u(u,v) = (0,0,1), \ \nu_1^v(u,v) = \frac{(0, -v^{k_2}, 1)}{\sqrt{v^{2k_2}+1}}, \ \nu_2^v(u,v) = (1,0,0).$$

Then $(\boldsymbol{x}, \nu_1^u, \nu_2^u)$ and $(\boldsymbol{x}, \nu_1^v, \nu_2^v)$ are one-parameter families of framed curves with respect to u and v, respectively.

3.2. **One-parameter families of framed curves as framed surfaces.** First, we consider a one-parameter family of framed curves with respect to u. We give conditions for the surface to be a framed base surface. In this section, we use the following notations. Let

$$(\boldsymbol{x}, \nu_1^u, \nu_2^u) : U \to \mathbb{R}^3 \times \Delta$$

be a one-parameter family of framed curves with respect to u with curvature

$$(\ell^u, m^u, n^u, \alpha^u, L^u, M^u, N^u, P^u, Q^u, R^u).$$

Lemma 3.7. *Under the above notations, we have the following.*

(1) $(\boldsymbol{x}, \nu_1^u, \nu_2^u) : U \to \mathbb{R}^3 \times \Delta$ *is a framed surface if and only if* $P^u(u,v) = 0$ *for all* $(u,v) \in U$.

(2) $(\boldsymbol{x}, \nu_2^u, \nu_1^u) : U \to \mathbb{R}^3 \times \Delta$ *is a framed surface if and only if* $Q^u(u,v) = 0$ *for all* $(u,v) \in U$.

Proof. (1) Since $(\boldsymbol{x}, \nu_1^u, \nu_2^u)$ is a one-parameter family of framed curves with respect to u, we have $\boldsymbol{x}_u(u,v) \cdot \nu_1^u(u,v) = 0$ for all $(u,v) \in U$. Since $\boldsymbol{x}_v(u,v) \cdot \nu_1^u(u,v) = P^u(u,v)$, $(\boldsymbol{x}, \nu_1^u, \nu_2^u)$ is a framed surface if and only if $P^u(u,v) = 0$ for all $(u,v) \in U$.

(2) We can prove the assertion by a similar calculation. □

Proposition 3.8. *Under the above notations, suppose that there exist smooth functions* $k_1, k_2 : U \to \mathbb{R}$ *such that* $(k_1(u,v), k_2(u,v)) \neq (0,0)$ *and* $k_1(u,v)P^u(u,v) + k_2(u,v)Q^u(u,v) = 0$ *for all* $(u,v) \in U$. *Then there exist smooth functions* $\theta, \varphi : U \to \mathbb{R}$ *such that* $(\boldsymbol{x}, \nu_1^{u,\theta}, \nu_2^{u,\theta})$ *and* $(\boldsymbol{x}, \nu_2^{u,\varphi}, \nu_1^{u,\varphi}) : U \to \mathbb{R}^3 \times \Delta$ *are framed surfaces.*

Proof. We take a smooth function $\theta : U \to \mathbb{R}$ which satisfies the condition

$$(\cos\theta(u,v), \sin\theta(u,v)) = \left(\frac{k_1(u,v)}{\sqrt{k_1^2(u,v) + k_2^2(u,v)}}, \frac{-k_2(u,v)}{\sqrt{k_1^2(u,v) + k_2^2(u,v)}} \right).$$

Then by Proposition 2.14,

$$\begin{aligned} P^{u,\theta}(u,v) &= P^u(u,v)\cos\theta(u,v) - Q^u(u,v)\sin\theta(u,v) \\ &= \frac{1}{\sqrt{k_1^2(u,v) + k_2^2(u,v)}}(k_1(u,v)P^u(u,v) + k_2(u,v)Q^u(u,v)) \\ &= 0. \end{aligned}$$

By Lemma 3.7 (1), $(\boldsymbol{x}, \nu_1^{u,\theta}, \nu_2^{u,\theta})$ is a framed surface. Moreover, we take a smooth function $\varphi : U \to \mathbb{R}$ which satisfies the condition

$$(\cos\varphi(u,v), \sin\varphi(u,v)) = \left(\frac{k_2(u,v)}{\sqrt{k_1^2(u,v) + k_2^2(u,v)}}, \frac{k_1(u,v)}{\sqrt{k_1^2(u,v) + k_2^2(u,v)}} \right).$$

Then by Proposition 2.14,

$$\begin{aligned} Q^{u,\theta}(u,v) &= P^u(u,v)\sin\theta(u,v) + Q^u(u,v)\cos\theta(u,v) \\ &= \frac{1}{\sqrt{k_1^2(u,v) + k_2^2(u,v)}}(k_1(u,v)P^u(u,v) + k_2(u,v)Q^u(u,v)) \\ &= 0. \end{aligned}$$

By Lemma 3.7 (2), $(\boldsymbol{x}, \nu_2^{u,\varphi}, \nu_1^{u,\varphi})$ is a framed surface. □

Next, we consider one-parameter families of framed curves with respect to u and v. We give conditions for the surface to be a framed base surface.

Let $(\boldsymbol{x}, \nu_1^u, \nu_2^u) : U \to \mathbb{R}^3 \times \Delta$ be a one-parameter family of framed curves with respect to u and $(\boldsymbol{x}, \nu_1^v, \nu_2^v) : U \to \mathbb{R}^3 \times \Delta$ be a one-parameter family of framed curves with respect to v, respectively. We denote $\boldsymbol{\mu}^u = \nu_1^u \times \nu_2^u$ and $\boldsymbol{\mu}^v = \nu_1^v \times \nu_2^v$.

Proposition 3.9. *Under the above notations, we have the following.*

(1) *Suppose that* $\boldsymbol{\mu}^u(u,v)$ *and* $\boldsymbol{\mu}^v(u,v)$ *are linearly independent for all* $(u,v) \in U$, *that is, if* $k_1(u,v)\boldsymbol{\mu}^u(u,v) + k_2(u,v)\boldsymbol{\mu}^v(u,v) = 0$ *for all* $(u,v) \in U$, *where* $k_1, k_2 : U \to \mathbb{R}$ *are smooth functions, then* $(k_1(u,v), k_2(u,v)) = (0,0)$ *for all* $(u,v) \in U$. *Then there exists a smooth mapping* $(\boldsymbol{n}, \boldsymbol{s}) : U \to \Delta$ *such that* $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ *is a framed surface.*

(2) *Suppose that* $\boldsymbol{\mu}^u(u,v)$ *and* $\boldsymbol{\mu}^v(u,v)$ *are linearly dependent for all* $(u,v) \in U$*, that is, there exist smooth functions* $k_1, k_2 : U \to \mathbb{R}$ *such that* $(k_1(u,v), k_2(u,v)) \neq (0,0)$ *and*

$$k_1(u,v)\boldsymbol{\mu}^u(u,v) + k_2(u,v)\boldsymbol{\mu}^v(u,v) = 0$$

for all $(u,v) \in U$*. Then there exists a smooth mapping* $(\boldsymbol{n}, \boldsymbol{s}) : U \to \Delta$ *such that* $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ *is a framed surface.*

Proof. (1) Since $\boldsymbol{\mu}^u(u,v)$ and $\boldsymbol{\mu}^v(u,v)$ are linearly independent, we can define the smooth mapping $(\boldsymbol{n}, \boldsymbol{s}) : U \to \Delta$ by

$$\boldsymbol{n}(u,v) = \frac{\boldsymbol{\mu}^u(u,v) \times \boldsymbol{\mu}^v(u,v)}{|\boldsymbol{\mu}^u(u,v) \times \boldsymbol{\mu}^v(u,v)|}, \ \boldsymbol{s}(u,v) = \boldsymbol{\mu}^u(u,v).$$

It follows that

$$\begin{aligned}
\boldsymbol{x}_u(u,v) \cdot \boldsymbol{n}(u,v) &= \alpha^u(u,v)\boldsymbol{\mu}^u(u,v) \cdot (\boldsymbol{\mu}^u(u,v) \times \boldsymbol{\mu}^v(u,v)/|\boldsymbol{\mu}^u(u,v) \times \boldsymbol{\mu}^v(u,v)|) = 0,\\
\boldsymbol{x}_v(u,v) \cdot \boldsymbol{n}(u,v) &= \alpha^v(u,v)\boldsymbol{\mu}^v(u,v) \cdot (\boldsymbol{\mu}^u(u,v) \times \boldsymbol{\mu}^v(u,v)/|\boldsymbol{\mu}^u(u,v) \times \boldsymbol{\mu}^v(u,v)|) = 0.
\end{aligned}$$

Moreover, $\boldsymbol{n}(u,v) \cdot \boldsymbol{s}(u,v) = (\boldsymbol{\mu}^u(u,v) \times \boldsymbol{\mu}^v(u,v)/|\boldsymbol{\mu}^u(u,v) \times \boldsymbol{\mu}^v(u,v)|) \cdot \boldsymbol{\mu}^u(u,v) = 0$. Therefore, $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$ is a framed surface.

(2) By the assumption and $\boldsymbol{\mu}^u(u,v), \boldsymbol{\mu}^v(u,v) \in S^2$, if $k_1(p) = 0$ (respectively, $k_2(p) = 0$), then $k_2(p) = 0$ (respectively, $k_1(p) = 0$). It follows that $k_1(u,v) \neq 0$ and $k_2(u,v) \neq 0$ for all $(u,v) \in U$. Then we have $\boldsymbol{\mu}^v(u,v) = \pm\boldsymbol{\mu}^u(u,v)$. We define the smooth mapping $(\boldsymbol{n}, \boldsymbol{s}) : U \to \Delta$ by $\boldsymbol{n}(u,v) = \nu_1^u(u,v)$, $\boldsymbol{s}(u,v) = \boldsymbol{\mu}^u(u,v)$. Then $\boldsymbol{x}_u(u,v) \cdot \boldsymbol{n}(u,v) = 0$ and

$$\boldsymbol{x}_v(u,v) \cdot \boldsymbol{n}(u,v) = \alpha^v(u,v)\boldsymbol{\mu}^v(u,v) \cdot \nu_1^u(u,v) = \pm\alpha^v(u,v)\boldsymbol{\mu}^u(u,v) \cdot \nu_1^u(u,v) = 0.$$

Moreover, $\boldsymbol{n}(u,v) \cdot \boldsymbol{s}(u,v) = \nu_1^u(u,v) \cdot \boldsymbol{\mu}^u(u,v) = 0$. Therefore, $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$ is a framed surface. $\Box$

We give an example of a one-parameter family of framed curves with respect to u and v which is not a framed base surface.

Example 3.10 (A cross cap)**.** Let $\boldsymbol{x} : \mathbb{R}^2 \to \mathbb{R}^3$ be given by $\boldsymbol{x}(u,v) = (u+v, (u+v)v, v^2)$. Note that $\boldsymbol{x}$ is diffeomorphic to the cross cap $\widetilde{\boldsymbol{x}}(u,v) = (u, uv, v^2)$ by using the parameter change $\phi(u,v) = (u+v, v)$. Since $\boldsymbol{x}_u(u,v) = (1, v, 0)$, if we consider the smooth mapping $(\nu_1^u, \nu_2^u) : \mathbb{R}^2 \to \Delta$ defined by

$$\nu_1^u(u,v) = \frac{(-v, 1, 0)}{\sqrt{1+v^2}}, \ \nu_2^u(u,v) = (0,0,1),$$

then $\boldsymbol{x}_u(u,v) \cdot \nu_1^u(u,v) = 0$, $\boldsymbol{x}_u(u,v) \cdot \nu_2^u(u,v) = 0$ and $\nu_1^u(u,v) \cdot \nu_2^u(u,v) = 0$ for all $(u,v) \in \mathbb{R}^2$. Hence, $(\boldsymbol{x}, \nu_1^u, \nu_2^u)$ is a one-parameter family of framed curves with respect to u. Moreover, since $\boldsymbol{x}_v(u,v) = (1, u+2v, 2v)$, if we consider the smooth mapping $(\nu_1^v, \nu_2^v) : \mathbb{R}^2 \to \Delta$ defined by

$$\nu_1^v(u,v) = \frac{(-(u+2v), 1, 0)}{\sqrt{1+(u+2v)^2}}, \ \nu_2^v(u,v) = \frac{(2v, 2v(u+2v), -1-(u+2v)^2)}{\sqrt{(1+(u+2v)^2)(1+(u+2v)^2+4v^2)}},$$

then $\boldsymbol{x}_v(u,v) \cdot \nu_1^v(u,v) = 0$, $\boldsymbol{x}_v(u,v) \cdot \nu_2^v(u,v) = 0$ and $\nu_1^v(u,v) \cdot \nu_2^v(u,v) = 0$ for all $(u,v) \in \mathbb{R}^2$. Hence, $(\boldsymbol{x}, \nu_1^v, \nu_2^v)$ is a one-parameter family of framed curves with respect to v. However, the cross cap is not a frontal at $(0,0)$ (cf. [6]). Hence $\boldsymbol{x}$ is not a framed base surface. Since

$$\boldsymbol{\mu}^u(u,v) = \frac{(1, v, 0)}{\sqrt{1+v^2}}, \ \boldsymbol{\mu}^v(u,v) = -\frac{(1, u+2v, 2v)}{\sqrt{1+(u+2v)^2+4v^2}},$$

the conditions in Proposition 3.9 are not satisfied around $(0,0)$.

4. Surfaces with corank one singular points

We consider surfaces with corank one singular points from the view point of one-parameter families of framed curves.

If $(0,0)$ is a corank one singular point of $\boldsymbol{x}$, then

$$\boldsymbol{x}(u,v)=(u,f(u,v),g(u,v)) \quad \text{or} \quad \boldsymbol{x}(u,v)=(v,f(u,v),g(u,v))$$

around $(0,0)$ by using a parameter change (a one-parameter parameter change).

Theorem 4.1. *Let $\boldsymbol{x}:U\to\mathbb{R}^3$ be a smooth mapping and $p\in U$ be a corank one singular point. Suppose that $\boldsymbol{x}$ is given by the form $\boldsymbol{x}(u,v)=(u,f(u,v),g(u,v))$.*

(1) There exists a smooth mapping $(\nu_1^u,\nu_2^u):U\to\Delta$ such that $(\boldsymbol{x},\nu_1^u,\nu_2^u)$ is a one-parameter family of framed curves with respect to u.

(2) If there exist smooth functions $k_1,k_2:U\to\mathbb{R}$ such that $(k_1(u,v),k_2(u,v))\neq(0,0)$ and $k_1(u,v)f_v(u,v)+k_2(u,v)g_v(u,v)=0$ for all $(u,v)\in U$, then there exists a smooth mapping $(\nu_1^v,\nu_2^v):U\to\Delta$ such that $(\boldsymbol{x},\nu_1^v,\nu_2^v)$ is a one-parameter family of framed curves with respect to v. Conversely, if there exists a smooth mapping $(\nu_1^v,\nu_2^v):U\to\Delta$ such that $(\boldsymbol{x},\nu_1^v,\nu_2^v)$ is a one-parameter family of framed curves with respect to v, then there exist smooth function germs $k_1,k_2:(U,p)\to\mathbb{R}$ such that $(k_1(u,v),k_2(u,v))\neq(0,0)$ and $k_1(u,v)f_v(u,v)+k_2(u,v)g_v(u,v)=0$ around p.

Proof. (1) Since $\boldsymbol{x}_u(u,v)=(1,f_u(u,v),g_u(u,v))$, we consider smooth mappings

$$\nu_1^u(u,v)=\frac{(-f_u(u,v),1,0)}{\sqrt{1+f_u^2(u,v)}},\ \nu_2^u(u,v)=\frac{(-g_u(u,v),-f_u(u,v)g_u(u,v),1+f_u^2(u,v))}{\sqrt{(1+f_u^2(u,v)+g_u^2(u,v))(1+f_u^2(u,v))}}.$$

By a direct calculation, we have

$$\boldsymbol{x}_u(u,v)\cdot\nu_1^u(u,v)=0,\quad \boldsymbol{x}_u(u,v)\cdot\nu_2^u(u,v)=0,\ \text{and}\quad \nu_1^u(u,v)\cdot\nu_2^u(u,v)=0$$

for all $(u,v)\in U$. Hence, $(\boldsymbol{x},\nu_1^u,\nu_2^u):U\to\mathbb{R}^3\times\Delta$ is a one-parameter family of framed curves with respect to u.

(2) Since $\boldsymbol{x}_v(u,v)=(0,f_v(u,v),g_v(u,v))$, we consider smooth mappings

$$\nu_1^v(u,v)=\frac{(0,k_1(u,v),k_2(u,v))}{\sqrt{k_1^2(u,v)+k_2^2(u,v)}},\ \nu_2^v(u,v)=(1,0,0).$$

By a direct calculation, we have

$$\boldsymbol{x}_v(u,v)\cdot\nu_1^v(u,v)=0,\quad \boldsymbol{x}_v(u,v)\cdot\nu_2^v(u,v)=0,\ \text{and}\quad \nu_1^v(u,v)\cdot\nu_2^v(u,v)=0$$

for all $(u,v)\in U$. Hence, $(\boldsymbol{x},\nu_1^v,\nu_2^v):U\to\mathbb{R}^3\times\Delta$ is a one-parameter family of framed curves with respect to v.

Conversely, suppose that $(\boldsymbol{x},\nu_1^v,\nu_2^v):U\to\mathbb{R}^3\times\Delta$ is a one-parameter family of framed curves with respect to v. We denote

$$\nu_1^v(u,v)=(\nu_{11}^v(u,v),\nu_{12}^v(u,v),\nu_{13}^v(u,v))$$

and $\nu_2^v(u,v)=(\nu_{21}^v(u,v),\nu_{22}^v(u,v),\nu_{23}^v(u,v))$. It follows that

$$\begin{aligned}\boldsymbol{x}_v(u,v)\cdot\nu_1^v(u,v) &= \nu_{12}^v(u,v)f_v(u,v)+\nu_{13}^v(u,v)g_v(u,v)=0,\\ \boldsymbol{x}_v(u,v)\cdot\nu_2^v(u,v) &= \nu_{22}^v(u,v)f_v(u,v)+\nu_{23}^v(u,v)g_v(u,v)=0.\end{aligned}$$

If $(\nu_{12}^v(p),\nu_{13}^v(p))\neq(0,0)$, then $(\nu_{12}^v(u,v),\nu_{13}^v(u,v))\neq(0,0)$ around p. If we consider $(k_1,k_2)=(\nu_{12}^v,\nu_{13}^v)$, then the condition is satisfied. On the other hand, if $(\nu_{12}^v(p),\nu_{13}^v(p))=(0,0)$, then $\nu_1^v(p)=(\pm1,0,0)$. Since $\nu_1^v(p)\cdot\nu_2^v(p)=0$, we have $(\nu_{22}^v(p),\nu_{23}^v(p))\neq(0,0)$. It follows that $(\nu_{22}^v(u,v),\nu_{23}^v(u,v))\neq(0,0)$ around p. If we consider $(k_1,k_2)=(\nu_{22}^v,\nu_{23}^v)$, then the condition is satisfied. □

Remark 4.2. Suppose that $\boldsymbol{x} : U \to \mathbb{R}^3$ is given by $\boldsymbol{x}(u, v) = (u, f(u, v), g(u, v))$ and there exists a smooth mapping $(\nu_1^v, \nu_2^v) : U \to \Delta$ such that $(\boldsymbol{x}, \nu_1^v, \nu_2^v)$ is a one-parameter family of framed curves with respect to v. Then $(f, g) : U \to \mathbb{R}^2$ is a one-parameter family of frontal curves with respect to v around $p \in U$. For definition and properties of one-parameter families of frontal curves (Legendre curves) see [16, 32]. Conversely, if $(f, g) : U \to \mathbb{R}^2$ is a one-parameter family of frontal curves with respect to v, then there exists a smooth mapping $(\nu_1^v, \nu_2^v) : U \to \Delta$ such that $(\boldsymbol{x}, \nu_1^v, \nu_2^v)$ is a one-parameter family of framed curves with respect to v by Theorem 4.1. Also see [18].

Proposition 4.3. (1) *Let $(\boldsymbol{x}, \nu_1^u, \nu_2^u) : U \to \mathbb{R}^3 \times \Delta$ be given by $\boldsymbol{x}(u, v) = (u, f(u, v), g(u, v))$,*

$$\nu_1^u(u,v) = \frac{(-f_u(u,v), 1, 0)}{\sqrt{1 + f_u^2(u,v)}}, \ \nu_2^u(u,v) = \frac{(-g_u(u,v), -f_u(u,v)g_u(u,v), 1 + f_u^2(u,v))}{\sqrt{(1 + f_u^2(u,v) + g_u^2(u,v))(1 + f_u^2(u,v))}}.$$

Then the curvature of the one-parameter family of framed curves with respect to u, $(\boldsymbol{x}, \nu_1^u, \nu_2^u)$ is given by

$$\begin{aligned}
\ell^u(u,v) &= \nu_{1u}^u(u,v) \cdot \nu_2^u(u,v) = \frac{f_{uu}(u,v)g_u(u,v)}{(1 + f_u^2(u,v))\sqrt{1 + f_u^2(u,v) + g_u^2(u,v)}},\\
m^u(u,v) &= \nu_{1u}^u(u,v) \cdot \boldsymbol{\mu}^u(u,v) = \frac{-f_{uu}(u,v)}{\sqrt{(1 + f_u^2(u,v))(1 + f_u^2(u,v) + g_u^2(u,v))}},\\
n^u(u,v) &= \nu_{2u}^u(u,v) \cdot \boldsymbol{\mu}^u(u,v) = \frac{-g_{uu}(u,v) + f_{uu}(u,v)f_u(u,v)g_u(u,v) - f_u^2(u,v)g_{uu}(u,v)}{(1 + f_u^2(u,v) + g_u^2(u,v))\sqrt{1 + f_u^2(u,v)}},\\
\alpha^u(u,v) &= \boldsymbol{x}_u(u,v) \cdot \boldsymbol{\mu}^u(u,v) = \sqrt{1 + f_u^2(u,v) + g_u^2(u,v)},\\
L^u(u,v) &= \nu_{1v}^u(u,v) \cdot \nu_2^u(u,v) = \frac{f_{uv}(u,v)g_u(u,v)}{(1 + f_u^2(u,v))\sqrt{1 + f_u^2(u,v) + g_u^2(u,v)}},\\
M^u(u,v) &= \nu_{1v}^u(u,v) \cdot \boldsymbol{\mu}^u(u,v) = \frac{-f_{uv}(u,v)}{\sqrt{(1 + f_u^2(u,v))(1 + f_u^2(u,v) + g_u^2(u,v))}},\\
N^u(u,v) &= \nu_{2v}^u(u,v) \cdot \boldsymbol{\mu}^u(u,v) = \frac{-g_{uv}(u,v) + f_{uv}(u,v)f_u(u,v)g_u(u,v) - f_u^2(u,v)g_{uv}(u,v)}{(1 + f_u^2(u,v) + g_u^2(u,v))\sqrt{1 + f_u^2(u,v)}},\\
P^u(u,v) &= \boldsymbol{x}_v(u,v) \cdot \nu_1^u(u,v) = \frac{f_v(u,v)}{\sqrt{1 + f_u^2(u,v)}},\\
Q^u(u,v) &= \boldsymbol{x}_v(u,v) \cdot \nu_2^u(u,v) = \frac{-f_u(u,v)g_u(u,v)f_v(u,v) + g_v(u,v) + f_u^2(u,v)g_v^2(u,v)}{\sqrt{(1 + f_u^2(u,v))(1 + f_u^2(u,v) + g_u^2(u,v))}},\\
R^u(u,v) &= \boldsymbol{x}_v(u,v) \cdot \boldsymbol{\mu}^u(u,v) = \frac{f_u(u,v)f_v(u,v) + g_u(u,v)g_v(u,v)}{\sqrt{1 + f_u^2(u,v) + g_u^2(u,v)}}.
\end{aligned}$$

(2) *Suppose that there exist smooth functions $k_1, k_2 : U \to \mathbb{R}$ such that $(k_1(u, v), k_2(u, v)) \neq (0, 0)$ and $k_1(u, v)f_v(u, v) + k_2(u, v)g_v(u, v) = 0$ for all $(u, v) \in U$. Let $(\boldsymbol{x}, \nu_1^v, \nu_2^v) : U \to \mathbb{R}^3 \times \Delta$ be given by $\boldsymbol{x}(u, v) = (u, f(u, v), g(u, v))$,*

$$\nu_1^v(u,v) = \frac{(0, k_1(u,v), k_2(u,v))}{\sqrt{k_1^2(u,v) + k_2^2(u,v)}}, \ \nu_2^v(u,v) = (1, 0, 0)$$

Then the curvature of the one-parameter family of framed curves with respect to v, $(\boldsymbol{x}, \nu_1^v, \nu_2^v)$ is given by

$$\begin{aligned}
\ell^v(u,v) &= \nu_{1v}^v(u,v) \cdot \nu_2^v(u,v) = 0,\\
m^v(u,v) &= \nu_{1v}^v(u,v) \cdot \boldsymbol{\mu}^v(u,v) = \frac{k_{1v}(u,v)k_2(u,v) - k_{2v}(u,v)k_1(u,v)}{(k_1^2(u,v) + k_2^2(u,v))},\\
n^v(u,v) &= \nu_{2v}^v(u,v) \cdot \boldsymbol{\mu}^v(u,v) = 0,\\
\alpha^v(u,v) &= \boldsymbol{x}_v(u,v) \cdot \boldsymbol{\mu}^v(u,v) = \frac{k_2(u,v)f_v(u,v) - k_1(u,v)g_v(u,v)}{\sqrt{k_1^2(u,v) + k_2^2(u,v)}},\\
L^v(u,v) &= \nu_{1u}^v(u,v) \cdot \nu_2^v(u,v) = 0,\\
M^v(u,v) &= \nu_{1u}^v(u,v) \cdot \boldsymbol{\mu}^v(u,v) = \frac{k_{1u}(u,v)k_2(u,v) - k_{2u}(u,v)k_1(u,v)}{(k_1^2(u,v) + k_2^2(u,v))},\\
N^v(u,v) &= \nu_{2u}^v(u,v) \cdot \boldsymbol{\mu}^v(u,v) = 0,\\
P^v(u,v) &= \boldsymbol{x}_u(u,v) \cdot \nu_1^v(u,v) = \frac{f_u(u,v)k_1(u,v) + g_u(u,v)k_2(u,v)}{\sqrt{k_1^2(u,v) + k_2^2(u,v)}},\\
Q^v(u,v) &= \boldsymbol{x}_u(u,v) \cdot \nu_2^v(u,v) = 1,\\
R^v(u,v) &= \boldsymbol{x}_u(u,v) \cdot \boldsymbol{\mu}^v(u,v) = \frac{f_u(u,v)k_2(u,v) - g_u(u,v)k_1(u,v)}{\sqrt{k_1^2(u,v) + k_2^2(u,v)}}.
\end{aligned}$$

Proof. (1) By definition, we have

$$\boldsymbol{\mu}^u(u,v) = \nu_1^u(u,v) \times \nu_2^u(u,v) = \frac{(1, f_u(u,v), g_u(u,v))}{\sqrt{1 + f_u^2(u,v) + g_u^2(u,v)}}.$$

By a direct calculation, we have the curvature.

(2) By definition, we have

$$\boldsymbol{\mu}^v(u,v) = \nu_1^v(u,v) \times \nu_2^v(u,v) = \frac{(0, k_2(u,v), -k_1(u,v))}{\sqrt{k_1^2(u,v) + k_2^2(u,v)}}.$$

By a direct calculation, we have the curvature. □

For the surface $\boldsymbol{x}(u,v) = (u, f(u,v), g(u,v))$, if we consider a parameter change

$$\phi(u,v) = (u+v, v),$$

then we have $\boldsymbol{x} \circ \phi(u,v) = (u+v, \tilde{f}(u,v), \tilde{g}(u,v))$. Then we have the following corollary.

Corollary 4.4. *Let $\boldsymbol{x} : U \to \mathbb{R}^3$ be a smooth mapping given by the form*

$$\boldsymbol{x}(u,v) = (u+v, f(u,v), g(u,v)).$$

Then there exist smooth mappings $(\nu_1^u, \nu_2^u) : U \to \Delta$ and $(\nu_1^v, \nu_2^v) : U \to \Delta$ such that $(\boldsymbol{x}, \nu_1^u, \nu_2^u)$ and $(\boldsymbol{x}, \nu_1^v, \nu_2^v)$ are one-parameter families of framed curves with respect to u and v, respectively.

By a similar calculation of Theorem 4.1 (2), we also have the following result (cf. [23, Proposition 3.4]).

Proposition 4.5. *Let $\boldsymbol{x} : U \to \mathbb{R}^3$ be a smooth mapping and $p \in U$ be a corank one singular point. Suppose that $\boldsymbol{x}$ is given by the form $\boldsymbol{x}(u,v) = (u, f(u,v), g(u,v))$. Then there exist smooth functions $k_1, k_2 : U \to \mathbb{R}$ such that $(k_1(u,v), k_2(u,v)) \neq (0,0)$ and*

$$k_1(u,v)f_v(u,v) + k_2(u,v)g_v(u,v) = 0$$

for all $(u,v) \in U$ if and only if there exists a smooth mapping $\boldsymbol{n} : U \to S^2$ such that $(\boldsymbol{x}, \boldsymbol{n})$ is a Legendre surface.

Proof. Suppose that $k_1(u,v)f_v(u,v)+k_2(u,v)g_v(u,v)=0$ for all $(u,v)\in U$. Since
$$\boldsymbol{x}_u(u,v)=(1,f_u(u,v),g_u(u,v))$$
and $\boldsymbol{x}_v(u,v)=(0,f_v(u,v),g_v(u,v))$, we define $\boldsymbol{n}:U\to S^2$ by
$$\boldsymbol{n}(u,v)=\frac{(-k_1(u,v)f_u(u,v)-k_2(u,v)g_u(u,v),k_1(u,v),k_2(u,v))}{\sqrt{(k_1(u,v)f_u(u,v)+k_2(u,v)g_u(u,v))^2+k_1^2(u,v)+k_2^2(u,v)}}.$$
Then $\boldsymbol{x}_u(u,v)\cdot\boldsymbol{n}(u,v)=0$ and $\boldsymbol{x}_v(u,v)\cdot\boldsymbol{n}(u,v)=0$ for all $(u,v)\in U$. Hence, $(\boldsymbol{x},\boldsymbol{n})$ is a Legendre surface.

Conversely, suppose that $(\boldsymbol{x},\boldsymbol{n}):U\to\mathbb{R}^3\times S^2$ is a Legendre surface. We denote
$$\boldsymbol{n}(u,v)=(n_1(u,v),n_2(u,v),n_3(u,v)).$$
By definition, we have
$$\begin{aligned}\boldsymbol{x}_u(u,v)\cdot\boldsymbol{n}(u,v)&=n_1(u,v)+f_u(u,v)n_2(u,v)+g_u(u,v)n_3(u,v)=0,\\ \boldsymbol{x}_v(u,v)\cdot\boldsymbol{n}(u,v)&=f_v(u,v)n_2(u,v)+g_v(u,v)n_3(u,v)=0.\end{aligned}$$
If $n_2(u,v)=n_3(u,v)=0$, then $n_1(u,v)=0$. It contradicts the fact that $\boldsymbol{n}(u,v)\in S^2$. Hence $(n_2(u,v),n_3(u,v))\neq(0,0)$ for all $(u,v)\in U$ and $f_v(u,v)n_2(u,v)+g_v(u,v)n_3(u,v)=0$. □

By Theorem 4.1 (2) and Proposition 4.5, we have the following corollary.

Corollary 4.6. *Let $\boldsymbol{x}:(U,p)\to\mathbb{R}^3$ be a smooth mapping germ and p be a corank one singular point. Suppose that $\boldsymbol{x}$ is given by the form $\boldsymbol{x}(u,v)=(u,f(u,v),g(u,v))$. The following are equivalent:*

(1) *There exists a smooth mapping germ $(\nu_1^v,\nu_2^v):(U,p)\to\Delta$ such that $(\boldsymbol{x},\nu_1^v,\nu_2^v)$ is a one-parameter family of framed curves with respect to v.*

(2) *There exists a smooth mapping germ $\boldsymbol{n}:(U,p)\to S^2$ such that $(\boldsymbol{x},\boldsymbol{n})$ is a Legendre surface.*

(3) *There exists a smooth mapping germ $(\boldsymbol{n},\boldsymbol{s}):(U,p)\to\Delta$ such that $(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s})$ is a framed surface.*

We consider concrete examples of one-parameter families of framed curves. We give cuspidal edges, swallowtails and cuspidal cross caps which are generic singularities of frontals. Since these are frontals, they are also framed surfaces at least locally. Moreover, we consider cross caps and ruled surfaces as one-parameter families of framed curves.

We say that a singular point of a mapping $\boldsymbol{x}:U\to\mathbb{R}^3$ is a *cuspidal edge* (respectively, *swallowtail, cuspidal cross cap* or *cross cap*) if $\boldsymbol{x}$ at the point is $\mathcal{A}$-equivalent to the map germ $(u,v)\mapsto(u,v^2,v^3)$ (respectively, $(u,4v^3+2uv,3v^4+uv^2),(u,v^2,uv^3)$ or (u,uv,v^2)) at $(0,0)$.

Let $\boldsymbol{x}:U\to\mathbb{R}^3$ be the frontal of a Legendre surface $(\boldsymbol{x},\boldsymbol{n})$, where U is a domain in $\mathbb{R}^2$. We define the discriminant function $\lambda:U\to\mathbb{R}$ by $\lambda(u,v)=\det(\boldsymbol{x}_u,\boldsymbol{x}_v,\boldsymbol{n})(u,v)$ where (u,v) is a coordinate system on U. When a singular point p of $\boldsymbol{x}$ is non-degenerate, that is, $d\lambda(p)\neq0$, there exists a smooth parametrization $\delta(t):(-\varepsilon,\varepsilon)\to U$, $\delta(0)=p$ of the singular set $S(\boldsymbol{x})$. We call the curve $\delta(t)$ the singular curve of $\boldsymbol{x}$. Moreover, there exists a smooth vector field $\eta(t)$ along δ satisfying that $\eta(t)$ generates $\ker d\boldsymbol{x}_{\delta(t)}$.

Remark 4.7. If a singular point p is non-degenerate of $(\boldsymbol{x},\boldsymbol{n})$, then p is also of corank one. Hence $\boldsymbol{x}$ is a one-parameter family of framed base curves around p.

A non-degenerate singular point p is called of *first kind* (respectively, of *second kind*) if $\eta\lambda(p)\neq0$ (respectively, $\eta\lambda(p)=0$ and $\eta\eta\lambda(p)\neq0$), see [29, 21].

Now we define a function $\phi_x(t)$ on $(-\epsilon,\epsilon)$ by $\phi_x(t)=\det((\boldsymbol{x}\circ\delta)',\boldsymbol{n}\circ\delta,d\boldsymbol{n}(\eta))(t)$. Using these notations, we have the following result (see [15] for example).

Theorem 4.8 ([4], [17])**.** *Let $(\boldsymbol{x}, \boldsymbol{n}) : U \to \mathbb{R}^3$ be a Legendre surface and $p \in U$ be a non-degenerate singular point of $\boldsymbol{x}$. Then the following assertions hold.*

(1) *If $\eta\lambda(p) \neq 0$, then $\boldsymbol{x}$ to be a front near p if and only if $\phi_x(0) \neq 0$ holds.*

(2) *The map germ $\boldsymbol{x}$ at p is $\mathcal{A}$-equivalent to the cuspidal edge if and only if $\boldsymbol{x}$ to be front near p and $\eta\lambda(p) \neq 0$ hold.*

(3) *The map germ $\boldsymbol{x}$ at p is $\mathcal{A}$-equivalent to the swallowtail if and only if $\boldsymbol{x}$ to be front near p and $\eta\lambda(p) = 0$ and $\eta\eta\lambda(p) \neq 0$ hold.*

(4) *The map germ $\boldsymbol{x}$ at p is $\mathcal{A}$-equivalent to the cuspidal cross cap if and only if $\eta\lambda(p) \neq 0$, $\phi_x(0) = 0$ and $\phi'_x(0) \neq 0$ hold.*

Here, $\eta\lambda : U \to \mathbb{R}$ means the directional derivative of λ by the vector field $\tilde{\eta}$, where $\tilde{\eta}$ is an extended vector field of η to U.

4.1. **First kind singularities.** We consider first kind singularities. A normal form of the first kind singularities is given in [24].

Proposition 4.9 (R. Oset Sinha, K. Saji [24])**.** *Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a frontal with a normal unit vector field ν. Let 0 be a singular point of the first kind. Then there exist a coordinate system (u, v) on $(\mathbb{R}^2, 0)$ and an isometry germ $\Phi : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ satisfying that*

$$\Phi \circ f(u,v) = \left(u, a(u) + \frac{v^2}{2}, b_0(u) + b_1(u)v^2 + b_2(u)v^3 + b_3(u,v)v^4\right),$$

where a, b_0, b_1, b_2, b_3 be smooth functions satisfying that $a(0) = a'(0) = b_0(0) = b_0'(0) = b_1(0) = 0$.

By using Proposition 4.9, we have the following.

Proposition 4.10. *Let $\boldsymbol{x} : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be given by $\boldsymbol{x}(u,v) = \Phi \circ f(u,v)$ in Proposition 4.9. Then there exist smooth mappings $(\nu_1^u, \nu_2^u) : (\mathbb{R}^2, 0) \to \Delta$ and $(\nu_1^v, \nu_2^v) : (\mathbb{R}^2, 0) \to \Delta$ such that $(\boldsymbol{x}, \nu_1^u, \nu_2^u)$ and $(\boldsymbol{x}, \nu_1^v, \nu_2^v) : (\mathbb{R}^2, 0) \to \mathbb{R}^3 \times \Delta$ are one-parameter families of framed curve germs with respect to u and v, respectively.*

Proof. By Theorem 4.1 (1), there exists a smooth mapping $(\nu_1^u, \nu_2^u) : (\mathbb{R}^2, 0) \to \Delta$ such that $(\boldsymbol{x}, \nu_1^u, \nu_2^u)$ is a one-parameter family of framed curve germs with respect to u.

We denote

$$\begin{aligned} f(u,v) &= a(u) + \frac{v^2}{2}, \\ g(u,v) &= b_0(u) + b_1(u)v^2 + b_2(u)v^3 + b_3(u,v)v^4. \end{aligned}$$

Then $f_v(u,v) = v$ and $g_v(u,v) = 2b_1(u)v + 3b_2(u)v^2 + b_{3v}(u,v)v^4 + 4b_3(u,v)v^3$. Hence, if we consider $k_1(u,v) = 2b_1(u) + 3b_2(u)v + b_{3v}(u,v)v^3 + 4b_3(u,v)v^2$ and $k_2(u,v) = -1$, then $(k_1(u,v), k_2(u,v)) \neq (0,0)$ and $k_1(u,v)f_v(u,v) + k_2(u,v)g_v(u,v) = 0$. By Theorem 4.1 (2), there exists a smooth mapping $(\nu_1^v, \nu_2^v) : (\mathbb{R}^2, 0) \to \Delta$ such that $(\boldsymbol{x}, \nu_1^v, \nu_2^v)$ is a one-parameter family of framed curve germs with respect to v. □

We treat cuspidal edges and cuspidal cross caps as concrete examples of the first kind singularities in the following. A normal form of the cuspidal cross cap is given in [24]. They consider folding mappings. Here we give the following normal form similarly to cuspidal edges in [20].

Theorem 4.11. (1) **[L. Martins, K. Saji** [20]**]** *Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a cuspidal edge germ. Then there exist a diffeomorphism germ $\phi : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ and isometry germ $\Phi : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ satisfying that*

$$\Phi \circ f \circ \phi(u,v) = \left(u, \frac{a_{20}}{2}u^2 + \frac{a_{30}}{6}u^3 + \frac{1}{2}v^2, \frac{b_{20}}{2}u^2 + \frac{b_{12}}{2}uv^2 + \frac{b_{03}}{6}v^3\right) + h(u,v)$$

$(b_{03} \neq 0, b_{20} \geq 0)$, *where*

$$h(u,v) = (0, u^4h_1(u), u^4h_2(u) + u^2v^2h_3(u) + uv^3h_4(u) + v^5h_5(u,v)),$$

with $h_1(u), h_2(u), h_3(u), h_4(u), h_5(u,v)$ *are smooth functions.*

(2) *Let* $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ *be a cuspidal cross cap germ. Then there exist a diffeomorphism germ* $\phi : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ *and isometry germ* $\Phi : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ *satisfying that*

$$\begin{aligned}\Phi \circ f \circ \phi(u,v) \quad = \quad & \left(u, \frac{a_{20}}{2}u^2 + \frac{a_{30}}{6}u^3 + \frac{a_{40}}{24}u^4 + \frac{1}{2}v^2,\right. \\ & \left.\frac{b_{20}}{2}u^2 + \frac{b_{30}}{6}u^3 + \frac{b_{40}}{24}u^4 + \frac{b_{12}}{2}uv^2 + \frac{b_{13}}{6}uv^3 + \frac{b_{04}}{24}v^4\right) + h(u,v),\end{aligned}$$

$(b_{13} \neq 0, b_{20} \geq 0)$, *where*

$$h(u,v) = (0, u^5h_1(u), u^5h_2(u) + u^3v^2h_3(u) + u^2v^3h_4(u,v) + v^5h_5(v)),$$

with $h_1(u), h_2(u), h_3(u), h_4(u,v), h_5(v)$ *are smooth functions.*

Proof. (2) Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a cuspidal cross cap germ and ν be a unit normal of f. By using the same method in [20], we may assume that a null vector field η is given by the form ∂_v on $S(f)$ and the singular curve $\delta(t)$ is given by the form $(t, 0)$. Moreover, we may assume that

$$f(u,v) = (u, a_1(u) + v^2/2, b_1(u) + v^2b_2(u) + v^3b_3(u,v)), \tag{5}$$

where a_1, b_1, b_2 and b_3 are smooth functions, $a_1(0) = a_1'(0) = b_1(0) = b_1'(0) = b_2(0) = 0$ and a_1' means the derivation of a_1 with respect to u for example. By a direct calculation, we obtain $\nu(u,v) = \mathcal{N}(u,v)\widetilde{\nu}(u,v)$ where

$$\begin{aligned}\widetilde{\nu}(u,v) \quad = \quad & (a_1'(u)(2b_2(u) + 3vb_3(u,v) + v^2b_{3,v}(u,v)) - (b_1'(u) + v^2b_2'(u) + v^3b_{3,u}(u,v)), \\ & -(2b_2(u) + 3vb_3(u,v) + v^2b_{3,v}(u,v)), 1)\end{aligned}$$

and $\mathcal{N}(u,v) = 1/|\widetilde{\nu}(u,v)|$. Then $\phi_f(t) = \det(f(t,0), \nu(t,0), \nu_v(t,0)) = 3\mathcal{N}(t,0)b_3(t,0)$. Since f is not a front and Theorem 4.8 (1), we have $\phi_f(0) = 3b_3(0,0) = 0$, that is, $b_3(0,0) = 0$. Moreover, under this condition, $\phi_f'(t) = 3\mathcal{N}(t,0)b_{3,u}(t,0)$. Since f is a cuspidal cross cap germ and Theorem 4.8 (4), we have $\phi_f'(0) = 3\mathcal{N}(0,0)b_{3,u}(0,0) \neq 0$, that is, $b_{3,u}(0,0) \neq 0$. Hence, we have $b_3(u,v) = ua_4(u,v) + b_4(v)$, where a_4 and b_4 are smooth functions, $a_4(0,0) \neq 0$ and $b_4(0) = 0$. Substituting this equation to (5), we have

$$f(u,v) = (u, a_1(u) + v^2/2, b_1(u) + v^2b_2(u) + uv^3a_4(u,v) + v^3b_4(v)),$$

where $a_1(0) = a_1'(0) = b_1(0) = b_1'(0) = b_2(0) = b_4(0) = 0$ and $a_4(0,0) \neq 0$. By rotations $(u,v) \mapsto (-u,-v)$ and $(x,y,z) \mapsto (-x,y,z)$, we may assume $b_1''(0) \geq 0$. Summarizing up the above argument, we have the normal form of cuspidal cross cap. $\square$

By Corollary 4.6, or by using Theorem 4.11, we have the following.

Proposition 4.12. *Let* $\boldsymbol{x} : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ *be given by* $\boldsymbol{x}(u,v) = \Phi \circ f \circ \phi(u,v)$ *in Theorem 4.11 (1) or (2). Then there exist smooth mappings* $(\nu_1^u, \nu_2^u) : (\mathbb{R}^2, 0) \to \Delta$ *and* $(\nu_1^v, \nu_2^v) : (\mathbb{R}^2, 0) \to \Delta$ *such that* $(\boldsymbol{x}, \nu_1^u, \nu_2^u)$ *and* $(\boldsymbol{x}, \nu_1^v, \nu_2^v) : (\mathbb{R}^2, 0) \to \mathbb{R}^3 \times \Delta$ *are one-parameter families of framed curve germs with respect to* u *and* v*, respectively.*

4.2. **Second kind singularities.**

Proposition 4.13 (K. Saji [29]). *For any functions g and h satisfying $g_{vvv}(0,0) > 0$, $g(0,0) = h(0,0) = 0$, $g_u(0,0) - g_{vv}(0,0) = 0$, $h_u(0,0) - h_{vv}(0,0) = 0$ and $h_{vvv}(0,0) = 0$,*

$$f(u,v) = \Bigg(u, \left(\frac{v^2}{2} - u\right) g_{vv}(u,v) - v g_v(u,v) + g(u,v),$$
$$\left(\frac{v^2}{2} - u\right) h_{vv}(u,v) - v h_v(u,v) + h(u,v)\Bigg)$$

is a frontal satisfying that 0 is a singular point of the second kind, and $f_u(0,0) = (1,0,0)$, a null vector field $\eta = \partial_v$, the singular set $S(f) = \{v^2/2 - u = 0\}$. Moreover, if $h_{vvvv}(0,0) \neq 0$, then 0 is a swallowtail. Conversely, for any singular point of second kind p of a frontal $f : U \to \mathbb{R}^3$, there exists a coordinate system (u,v) on U, and an orientation preserving isometry Φ on $\mathbb{R}^3$ such that $\Phi \circ f(u,v)$ can be written in the above form.

By using Proposition 4.13, we have the following.

Proposition 4.14. *Let $\boldsymbol{x} : U \to \mathbb{R}^3$ be given by $\boldsymbol{x}(u,v) = \Phi \circ f(u,v)$ in Proposition 4.13. Then there exist smooth mappings $(\nu_1^u, \nu_2^u) : U \to \Delta$ and $(\nu_1^v, \nu_2^v) : U \to \Delta$ such that $(\boldsymbol{x}, \nu_1^u, \nu_2^u)$ and $(\boldsymbol{x}, \nu_1^v, \nu_2^v) : U \to \mathbb{R}^3 \times \Delta$ are one-parameter families of framed curve germs with respect to u and v around p, respectively.*

Proof. By Theorem 4.1 (1), there exists a smooth mapping $(\nu_1^u, \nu_2^u) : U \to \Delta$ such that $(\boldsymbol{x}, \nu_1^u, \nu_2^u)$ is a one-parameter family of framed curve germs with respect to u.

By a direct calculation, we have

$$\boldsymbol{x}_v(u,v) = \left(0, \left(\frac{v^2}{2} - u\right) g_{vvv}(u,v), \left(\frac{v^2}{2} - u\right) h_{vvv}(u,v)\right).$$

Since $g_{vvv}(0,0) > 0$, we have $g_{vvv}(u,v) \neq 0$ around $p \in U$. Hence, if we consider $(k_1(u,v), k_2(u,v))$ $= (-h_{vvv}(u,v), g_{vvv}(u,v))$, then $(k_1(u,v), k_2(u,v)) \neq (0,0)$ and

$$k_1(u,v) f_v(u,v) + k_2(u,v) g_v(u,v) = 0.$$

By Theorem 4.1 (2), there exists a smooth mapping $(\nu_1^v, \nu_2^v) : U \to \Delta$ such that $(\boldsymbol{x}, \nu_1^v, \nu_2^v)$ is a one-parameter family of framed curve germs with respect to v around p. □

4.3. **Cross caps.** The cross cap map germ is not a frontal. However, the generic singularities from 2-dimensional manifolds to 3-dimensional one are cross caps. In [6, 34, 11], they investigate cross caps from the view point of differential geometry.

Proposition 4.15 (J. M. West [34], T. Fukui, M. Hasegawa [6]). *Let $g : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a smooth map with a cross cap at $(0,0)$. Then there are a rotation $T : \mathbb{R}^3 \to \mathbb{R}^3$ and a diffeomorphism $\phi : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ so that*

$$T \circ g \circ \phi(u,v) = \left(u, uv + B(v) + O(u,v)^{k+1}, \sum_{j=2}^{k} A_j(u,v) + O(u,v)^{k+1}\right) \ (k \geq 3),$$

where

$$B(v) = \sum_{i=3}^{k} \frac{b_i}{i!} v^i \quad \textit{and} \quad A_j(u,v) = \sum_{i=0}^{j} \frac{a_{i,j-i}}{i!(j-i)!} u^i v^{j-i} \ \textit{with } a_{02} \neq 0.$$

By Theorem 4.1 (1), we have the following.

Proposition 4.16. *Let $\boldsymbol{x} : U \to \mathbb{R}^3$ be given by $\boldsymbol{x}(u,v) = T \circ g \circ \phi(u,v)$ in Proposition 4.15. Then there exists a smooth mapping $(\nu_1^u, \nu_2^u) : U \to \Delta$ such that $(\boldsymbol{x}, \nu_1^u, \nu_2^u) : U \to \mathbb{R}^3 \times \Delta$ is a one-parameter family of framed curve germs with respect to u.*

Moreover, the $\mathcal{A}$-simple singularities of a map from a 2-dimensional manifold to a 3-dimensional one are also of corank one, see [22]. These are also one-parameter families of framed base curves.

4.4. **Ruled surfaces.** We consider ruled surfaces as follows. Let $\gamma : I \to \mathbb{R}^3$ be a smooth curve and $(\delta, \nu) : I \to \Delta$ a spherical Legendre curve with the curvature (m, n), see §2.2 (cf. [31]). We define a *ruled surface* $\boldsymbol{x} : \mathbb{R} \times I \to \mathbb{R}^3$ by $\boldsymbol{x}(u,v) = \gamma(v) + u\delta(v)$. We denote $\boldsymbol{\mu}(v) = \delta(v) \times \nu(v)$.

Since ruled surfaces are constructed by a one-parameter family of straight lines, these are one-parameter families of framed curves.

Proposition 4.17. *Under the above notations, there exists a smooth mapping*

$$(\nu_1^u, \nu_2^u) : \mathbb{R} \times I \to \Delta$$

such that $(\boldsymbol{x}, \nu_1^u, \nu_2^u)$ is a one-parameter family of framed curves with respect to u with the curvature

$$\begin{aligned}(\ell^u(u,v), m^u(u,v), n^u(u,v), \alpha^u(u,v), L^u(u,v), M^u(u,v), N^u(u,v), P^u(u,v), Q^u(u,v), R^u(u,v)) \\ = (0, 0, 0, 1, n(v), 0, -m(v), \dot\gamma(v) \cdot \nu(v), \dot\gamma(v) \cdot \boldsymbol{\mu}(v) + um(v), \dot\gamma(v) \cdot \delta(v)).\end{aligned}$$

Proof. Since $\boldsymbol{x}_u(u,v) = \delta(v)$, if we take $\nu_1^u(u,v) = \nu(v)$, $\nu_2^u(u,v) = \boldsymbol{\mu}(v)$, then

$$(\boldsymbol{x}, \nu_1^u, \nu_2^u) : \mathbb{R} \times I \to \mathbb{R}^3 \times \Delta$$

is a one-parameter family of framed curves with respect to u. By a direct calculation, we have the curvature. □

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