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SINGULAR WELSCHINGER INVARIANTS

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ABSTRACT. We suggest an invariant way to enumerate nodal and nodal-cuspidal real deformations of real plane curve singularities. The key idea is to assign Welschinger signs to the counted deformations. Our invariants can be viewed as a local version of Welschinger invariants enumerating real plane rational curves.

INTRODUCTION

Gromov-Witten invariants of the plane can be identified with the degrees of the Severi varieties, which parameterize irreducible plane curves of given degree and genus. As a local version, one can consider a versal deformation of an isolated plane curve singularity $(C,z)\subset\mathbb{C}^2$ with base $B(C,z)\simeq(\mathbb{C}^s,0)$, and the following strata in $B(C,z)$:

$$EG^i_{C,z},\quad 1\le i\le\delta(C,z),\tag{1}$$

parameterizing deformations with the total δ-invariant greater or equal to i;

$$EC^k_{C,z},\quad 0\le k\le\kappa(C,z)-2\delta(C,z),\tag{2}$$

parameterizing deformations with the total δ-invariant equal to $\delta(C,z)$ and the total κ-invariant equal to $2\delta(C,z)+k$ (a necessary information on δ- and κ-invariants can be found in [7] or [9, Section 3.4]). Note also that $EC^0_{C,z}=EG^{\delta(C,z)}_{C,z}$.

The strata (1) are called *Severi loci*; among them, $\mathcal{D}_{C,z}:=EG^1_{C,z}$ is the discriminant hypersurface in $B(C,z)$, and $EG_{C,z}:=EG^{\delta(C,z)}_{C,z}$ is the so-called *equigeneric locus*. We call the strata (2) *generalized equiclassical loci*, and among them $EC_{C,z}:=EC^{\kappa(C,z)-2\delta(C,z)}_{C,z}$ is the so-called *equiclassical locus*. The incidence relations are as follows:

$$EG^i_{C,z}\supsetneq EG^{i+1}_{C,z}\supsetneq EC^k_{C,z}\supsetneq EC^{k+1}_{C,z}$$

for all $1\le i<\delta(C,z)$ and $1\le k<\kappa(C,z)-2\delta(C,z)$. All these loci are pure-dimensional germs of complex spaces (cf. [14, 15]).

A natural problem is to compute the multiplicities of $EG^i_{C,z},EC^k_{C,z}$ for all i,k [1]. This problem was solved for the equigeneric stratum $EG_{C,z}$ in [8]. In the particular case of an irreducible germ with one Puiseux pair, i.e., the germ topologically equivalent to $x^p+y^q=0$, $2\le p<q$, $\gcd(p,q)=1$, one has (see [3, Proposition 4.3] and [8, Section G])

$$\operatorname{mult}EG_{C,z}=\frac{1}{p+q}\binom{p+q}{p}.$$

The multiplicities of all Severi loci $EG^i_{C,z}$ were expressed in [15] in terms of the Euler characteristics of the punctual Hilbert schemes on curve germs representing a given singularity. The

[1]We understand the multiplicity of a point of an algebraic variety embedded into an affine space as the intersection number at this point with a generic smooth germ of the complementary dimension (cf. [13, Chapter 5, Definition 5.9]).

multiplicities of the equiclassical loci $EC^k_{C,z}$ are not known except for the case of the smoothness mentioned in [6, Theorems 2 and 27].

The multiplicity admits an enumerative interpretation: it can be regarded as the number of intersection points of a locus $V \subset B(C,z)$ with a generic affine subspace $L \subset B(C,z)$ of the complementary dimension (equal to $\operatorname{codim} V$) chosen to be transversal to the tangent cone $\widehat{T}_0 V$.

The **goal** of this note is to define *real multiplicities* of the Severi loci (1) and of the generalized equiclassical loci (2). Let the singularity (C,z) be real[2]. Then the Severi loci and the generalized equiclassical loci are defined over the reals. Thus, given such a locus V, we count real intersection points of V with a generic real affine subspace $L \subset B(C,z)$ of the complementary dimension. Our **main result** is that, in certain cases, the count of real intersection points of V and L equipped with Welschinger-type signs (cf. [17, 18]) is invariant, i.e., does not depend on the choice of L. We were motivated by [11, Lemma 15], which, in fact, states the existence of a Welschinger type invariant for the equigeneric stratum $EG_{C,z}$. In this note, we go further and prove the existence of similar Welschinger type invariants for $EG^{\delta(C,z)-1}_{C,z}$ (see Proposition 3.2 in Section 3) and for $EG^1_{C,z} = \mathcal{D}_{C,z} \subset B(C,z)$ (see Proposition 3.3 in Section 3) as well as for all the loci $EC^k_{C,z}$ (see Proposition 4.1 in Section 4).

We remark that a similar enumeration of real plane rational curves with at least one cusp is not invariant, i.e., depends on the choice of point constraints (cf. [19]).

As an example, we perform computations for singularities of type A_n and real isolated singularities (see Section 5). The problem of computation of the Welschinger type invariants for arbitrary real singularities (even for quasihomogeneous singularities) remains widely open. A possible relation to enumerative invariants of (global) plane algebraic curves could be a key to this problem.

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1. Singular Welschinger numbers

We shortly recall definitions and basic properties of objects of our interest. Details can be found in [7] and [9, Chaper II].

Let (C,z) be the germ of a plane complex analytic curve C at its isolated singular point $z=(0,0)\in\mathbb{C}^2$, which is given by an analytic equation $f(x,y)=0$, $f\in\mathbb{C}\{x,y\}$. We shortly call it *singularity*. The Milnor ball $D(C,z)\subset\mathbb{C}^2$ is a closed ball centered at z such that $C\cap D(C,z)$ is closed and smooth outside z with the boundary $\partial(C\cap D(C,z))\subset\partial D(C,z)$, and the intersection of C with any 3-sphere in $D(C,z)$ centered at z is transversal. Pick an integer $N>0$ and consider the small neighborhood $B(C,z)$ of 0 in the space (which is a $\mathbb{C}$-algebra) $R(C,z) := \mathbb{C}\{x,y\}/(\langle f\rangle + \mathfrak{m}_z^N)$, where $\mathfrak{m}_z \subset \mathbb{C}\{x,y\}$ is the maximal ideal. We can suppose that, for any $\varphi\in B(C,z)$, the curve $\mathcal{C}_\varphi := \{f+\varphi=0\}\cap D(C,z)$ has only isolated singularities in $D(C,z)$, is smooth along $\partial D(C,z)$, and intersects the sphere $\partial D(C,z)$ transversally. It is well-known that the deformation $\pi:\mathcal{C}\to B(C,z)$ of (C,z), where $\pi^{-1}(\varphi)=\mathcal{C}_\varphi$, is versal for $N>\mu(C,z)+1$ (cf. [2, Page 165] or [7, Section 3]). The space $B(C,z)$ contains the equigeneric stratum $EG_{C,z}\subset B(C,z)$, formed by $\varphi\in B(C,z)$ such that $\mathcal{C}_\varphi$ has the total δ-invariant equal to $\delta(C,z)$ (the maximal possible value), the equiclassical locus $EC_{C,z}\subset EG_{C,z}\subset B(C,z)$, formed

[2]Under the *real* object we always understand a complex object invariant with respect to the complex conjugation, while the real point set of a real variety (or a germ of variety) is marked by the prefix $\mathbb{R}$.

by $\varphi \in EG_{C,z}$ such that $\mathcal{C}_\varphi$ has the total κ-invariant equal to $\kappa(C,z)$ (also the maximal possible value), and the discriminant

$$\mathcal{D}_{C,z} = \{\varphi \in B(C,z) \ : \ \mathcal{C}_\varphi \text{ is singular}\}.$$

The following statement summarizes some known facts on the above loci (see [7, Theorems 1.1, 1.3, 4.15, 4.17, 5.5, Corollary 5.13] and [6, Theorems 2 and 27]).

Lemma 1.1.

(1) The stratum $EG_{C,z}$ satisfies the following properties:

(i) *$EG_{C,z}$ is irreducible of codimension $\delta(C,z)$;*

(ii) *$EG_{C,z}$ is smooth iff all irreducible components of (C,z) (which we call local branches of (C,z)) are smooth;*

(iii) *the normalization of $EG_{C,z}$ is smooth and projects one-to-one onto $EG_{C,z}$;*

(iv) *the tangent cone $\widehat{T}_0 EG_{C,z}$ is the linear space $J^{cond}_{C,z}/\mathfrak{m}_z^N$ of codimension $\delta(C,z)$ in $R(C,z)$, where $J^{cond}_{C,z} \subset \mathbb{C}\{x,y\}/\langle f\rangle$ is the conductor ideal;*

(v) *$EG_{C,z}$ contains an open dense subset $EG^*_{C,z}$ that parameterizes the curves $\mathcal{C}_\varphi$ having $\delta(C,z)$ nodes as their only singularities.*

(2) The stratum $EC_{C,z}$ satisfies the following properties:

(i) *$EC_{C,z}$ is irreducible of codimension $\kappa(C,z)-\delta(C,z)$;*

(ii) *$EC_{C,z}$ is smooth iff each local branch of (C,z) either is smooth, or has topological type $x^m + y^{m+1} = 0$ with $m \ge 2$;*

(iii) *the normalization of $EC_{C,z}$ is smooth and projects one-to-one onto $EC_{C,z}$;*

(iv) *the tangent cone $\widehat{T}_0 EC_{C,z}$ is the linear space $J^{ec}_{C,z}/\mathfrak{m}_z^N$ of codimension $\kappa(C,z)-\delta(C,z)$ in $R(C,z)$, where $J^{ec}_{C,z} \subset \mathbb{C}\{x,y\}/\langle f\rangle$ is the equiclassical ideal;*

(v) *the locus $EC_{C,z}$ contains an open dense subset $EC^*_{C,z}$ that parameterizes the curves $\mathcal{C}_\varphi$ having $3\delta(C,z)-\kappa(C,z)$ nodes and $\kappa(C,z)-2\delta(C,z)$ ordinary cusps as their only singularities.*

(3) The discriminant $\mathcal{D}_{C,z}$ is an irreducible hypersurface with the tangent cone

$$\widehat{T}_0\mathcal{D}_{C,z} = \mathfrak{m}_z/(\langle f\rangle + \mathfrak{m}_z^N).$$

An open dense subset $\mathcal{D}^*_{C,z} \subset \mathcal{D}_{C,z}$ parameterizes the curves $\mathcal{C}_\varphi$ having one node and no other singularities.

In the same way one can establish some of these properties for the Severi loci (1) and the generalized equiclassical loci (2).

Lemma 1.2.

(1) Each Severi locus $EG^i_{C,z}$ is a (possibly reducible) germ of a complex space of pure codimension i. An open dense subset $EG^{i,*}(C,z) \subset EG^i_{C,z}$ parameterizes curves with i nodes as their only singularities.

(2) Each generalized equiclassical locus $EC^k_{C,z}$ is a (possibly reducible) germ of a complex space of pure codimension $\delta(C,z)+k$. An open dense subset $EC^{k,*}_{C,z} \subset EC^k_{C,z}$ parameterizes curves with $\delta(C,z)-k$ nodes and k ordinary cusps as their only singularities.

It is well-known that $\operatorname{mult}\mathcal{D}_{C,z} = \mu(C,z)$ (the Milnor number), $\operatorname{mult} EG_{C,z}$ has been computed in [8] as the Euler characteristic of an appropriate compactified Jacobian.

Now we switch to the real setting. We call the complex space V real if it is invariant under the (natural) action of the complex conjugation and denote by $\mathbb{R}V$ its real point set. Suppose that (C,z) is real.

Definition 1.3.

(1) Let $\mathcal{V} \subset B(C,z)$ be either a Severi locus $EG^i_{C,z}$, $1 \le i \le \delta(C,z)$, or a generalized equiclassical locus $EC^k_{C,z}$, $1 \le k \le \kappa(C,z) - 2\delta(C,z)$, and let $V \subset \mathcal{V}$ be the union of some real irreducible components of $\mathcal{V}$. We say that V is appropriate for real enumeration (briefly, RE-appropriate) if the tangent cone $\widehat{T}_0 V$ is a linear subspace of $R(C,z)$ of dimension $\dim V$.

(2) For an RE-appropriate locus $V \subset B(C,z)$, we choose a real linear subspace $L_0 \subset R(C,z)$ of dimension $\dim L_0 = \mathrm{codim}_{B(C,z),\mathbb{C}} V$, which meets $\widehat{T}_0 V$ only at the origin, and a neighborhood $U \subset B(C,z)$ of the origin such that $L_0 \cap \mathcal{V} \cap U = \{0\}$. Pick also a real affine space $L \subset R(C,z)$ of dimension $\dim L = \dim_{\mathbb{R}} L_0$ sufficiently close to L_0 and such that the intersection $L \cap V \cap U$ consists of $\mathrm{mult}\, V$ distinct element which all belong to $V^* := V \cap \mathcal{V}^*$.

Now we define the following quantity

$$W(C,z,V,L) = \sum_{\varphi \in L \cap \mathbb{R}V \cap U} w(\varphi), \quad \text{where } w(\varphi) = (-1)^{s(\varphi) + ic(\varphi)},$$

with $s(\varphi)$ being the number of real elliptic[3] nodes of $\mathcal{C}_\varphi$ and $ic(\varphi)$ the number of pairs of complex conjugate cusps of $\mathcal{C}_\varphi$. In case of $V = EG_{C,z}$ or $EC_{C,z}$, we write $W^{eg}(C,z,L)$ or $W^{ec}(C,z,L)$, respectively.

In what follows we examine the dependence on L and prove some invariance statements.

2. Singular Welschinger invariant $W^{eg}(C,z)$

The following statement is a consequence of [11, Lemma 15]. We provide a proof, since in a similar manner we treat other instances of the invariance. By Lemma 1.1(1), the locus $EG_{C,z}$ is RE-appropriate, thus, we can speak of the numbers $W^{eg}(C,z,L)$.

Proposition 2.1. *Given a real singularity (C,z), the quantity $W^{eg}(C,z,L)$ does not depend on the choice of L.*

Proof. We follow the argument from the original proof of Welschinger's invariance theorem, see [17, Section 3] and [18, Section 2.3].

Let $L'_0, L''_0 \subset R(C,z)$ be two real linear subspaces of dimension $\delta(C,z)$ transversally intersecting $\widehat{T}_0 EG_{C,z}$ at the origin, and let $L', L'' \subset R(C,z)$ be real affine subspaces of dimension $\delta(C,z)$, which are sufficiently close to L'_0, L''_0, respectively, in the sense of Definition 1.3. We can connect the pairs (L'_0, L') and (L''_0, L'') by a generic smooth path $(L_0(t), L(t))_{t\in[0,1]}$ consisting of real linear subspaces $L_0(t)$ of $R(C,z)$ of dimension $\delta(C,z)$, which are transversal to $T_0 EG_{C,z}$, and real affine subspaces $L(t)$ of dimension $\delta(C,z)$ sufficiently close to $L_0(t)$ in the sense of Definition 1.3, $0 \le t \le 1$. It follows from Lemma 1.1(1) that, for all $t \in [0,1]$, the space $L(t)$ intersects $EG_{C,z}$ transversally at each element of $L(t) \cap EG_{C,z}$. Furthermore, all but finitely many spaces $L(t)$ intersect $EG_{C,z}$ along $EG^*_{C,z}$, transversally at each intersection point. The remaining finite subset $F \subset (0,1)$ is such that, for any $\hat{t} \in F$, the intersection $L(\hat{t}) \cap EG_{C,z}$ consists of elements of $EG^*_{C,z}$ and one real element φ belonging to a codimension one substratum of $EG_{C,z}$. The classification of these codimension one substrata is known (see, for instance [7, Theorem 1.4]): an element φ of such a substratum is as follows:

(n1) either $\mathcal{C}_\varphi$ has an ordinary cusp A_2 and $\delta(C,z) - 1$ nodes,
(n2) or $\mathcal{C}_\varphi$ has a tacnode A_3 and $\delta(C,z) - 2$ nodes,
(n3) or $\mathcal{C}_\varphi$ has a triple point D_4 and $\delta(C,z) - 3$ nodes.

[3] A real node is called elliptic if it is equivariantly isomorphic to $x^2 + y^2 = 0$.

In cases (n2) and (n3), the stratum $EG_{C,z}$ is smooth at φ (cf. Lemma 1.1(1ii)), and the deformation of $\mathcal{C}_\varphi$ under the variation of $L(t)$ induces independent equivariant deformations of all (smooth) local branches of $\mathcal{C}_\varphi$ at the non-nodal singular point. Then the exponent $s(\psi)$ (see Definition 1.3) for any real nodal curve $\mathcal{C}_\psi$, $\psi \in EG_{C,z}$ close to $\mathcal{C}_\varphi$ always equals modulo 2 the number of elliptic nodes of $\mathcal{C}_\varphi$ plus the intersection number of complex conjugate local branches of $\mathcal{C}_\varphi$ at the non-nodal singular point. Thus, the crossing of these strata does not affect $W^{eg}(C, z, L(t))$.

In case (n1), the germ of $B(C, z)$ at φ can be represented as

$$B(A_2) \times B(A_1)^{\delta(C,z)-1} \times (\mathbb{C}^{n-\delta(C,z)-1}, 0)$$

(cf. [9, Proposition I.1.14 and Theorem I.1.15] and [11, Lemma 13]), where $n = \dim B(C, z)$, $B(A_2) \simeq (\mathbb{C}^2, 0)$ is a miniversal deformation base of an ordinary cusp, which we without loss of generality can identify with the base of the deformation $\{y^2 - x^3 - \alpha x - \beta \ : \ \alpha, \beta \in (\mathbb{C}^2, 0)\}$, and $B(A_1) \simeq (\mathbb{C}, 0)$ stands for the versal deformation of an ordinary node. Here

$$(EG_{C,z}, \varphi) = EG(A_2) \times EG(A_1)^{\delta(C,z)-1} \times (\mathbb{C}^{n-\delta(C,z)-1}, 0),$$

where

$$EG(A_2) = \left\{ \frac{\alpha^3}{27} - \frac{\beta^2}{4} \right\} \subset B(A_2), \quad EG(A_1) = \{0\} \subset B(A_1),$$

$$\widehat{T}_\varphi EG_{C,z} = \widehat{T}_0 EG(A_2) \times \{0\}^{\delta(C,z)-1} \times \mathbb{C}^{n-\delta(C,z)-1}.$$

Then the transversality of the intersection of $L(\hat{t})$ and $\widehat{T}_\varphi EG_{C,z}$ yields that the family $\{L(t)\}_{|t-\hat{t}|<\eta}$ projects to the family of smooth curves $\{L^1(t)\}_{|t-\hat{t}|<\eta}$ transversal to

$$\widehat{T}_0 EG(A_2) = \{\beta = 0\}.$$

It is easy to see that either $L^1(t)$ does not intersect $EG(A_2)$ in real points, or it intersects $EG(A_2)$ in two real points (α_1, β_1), (α_2, β_2) with $\beta_1 < 0 < \beta_2$, where the former point corresponds to a real curve with a hyperbolic node in a neighborhood of the cusp, while the latter one - to a real curve with an elliptic node. Hence, the Welschinger signs of these intersections of $L^1(t)$ with $EG(A_2)$ cancel out, which confirms the constancy of $W^{eg}(C, z, L(t))$, $|t - \hat{t}| < \eta$, in the considered wall-crossing. $\square$

We mention also two more useful properties of the invariant $W^{eg}(C, z)$.

Lemma 2.2.

(1) The number $W^{eg}(C, z)$ is an invariant of a real equisingular deformation class. That is, if $(C_t, z)_{t\in[0,1]}$ is an equisingular[4] family of real singularities, then $W^{eg}(C_0, z) = W^{eg}(C_1, z)$.

(2) Let $(C, z) = \bigcup_i (C_i, z)$ be the decomposition of a real singularity (C, z) into irreducible over $\mathbb{R}$ components. Then $W^{eg}(C, z) = \prod_i W^{eg}(C_i, z)$.

Proof.

(1) Suppose that $(C_t, z)_{t\in[0,1]}$ is given by a family $f_t(x, y)$ of real analytic in $D(C, z)$ functions analytically depending on the parameter t. Fix $N > \mu(C, z) + 1$. Then we have a smooth family $\widetilde{B} \to [0, 1]$ with fibers $\widetilde{B}_t = B(C_t, z) = \mathbb{C}\{x, y\}/(\langle f_t\rangle + \mathfrak{m}^N)$ being versal deformation bases of (C_t, z) for each $t \in [0, 1]$, as well as the inscribed equidimensional family $\widetilde{EG} \to [0, 1]$ with fibers

[4] "Equisingular" means "preserving the (complex) topological type".

$\widetilde{EG}_t = EG_{C_t,z} \subset B(C_t, z)$ for each $t \in [0,1]$. Respectively, we have a smooth, inscribed into $\widetilde{B}$ family $\mathcal{T} \to [0,1]$ with the fibers $\mathcal{T}_t = \widehat{T}_0 EG_{C_t,z} \subset B(C_t, z)$, and the following holds

$$T_{(0,t)}\widetilde{EG} = T_{(0,t)}\mathcal{T} = \widehat{T}_0 EG_{C_t,z} \oplus T_t[0,1], \quad \text{for all } t \in [0,1].$$

Hence, there exists a smooth, inscribed into $\widetilde{B}$ family $\widetilde{L}_0 \to [0,1]$ with fibers $L_0(t) \subset B(C_t, z)$ being real linear subspaces of dimension $\delta(C_t, z) = \delta(C, z)$ transversally intersecting $\widehat{T}_0 EG_{C_t,z}$ in $B(C_t, z)$ for each $t \in [0,1]$. Now, given $\tau \in [0,1]$ and a real affine space $L(\tau) \subset B(C_\tau, z)$ of dimension $\delta(C, z)$ sufficiently close to $L_0(\tau)$ and intersecting $EG_{C_\tau,z}$ transversally in mult $EG_{C_\tau,z}$ elements that belong to $EG^*_{C_\tau,z}$, we can include $L(\tau)$ into a family $\mathcal{L}(\tau) \to (\tau - \zeta, \tau + \zeta)$, where $0 < \zeta \ll 1$, with fibers $\mathcal{L}(\tau)_t$ affine subspaces of $B(C_t, z)$ of dimension $\delta(C, z)$ sufficiently close to $L_0(t)$ and intersecting $EG_{C_t,z}$ transversally along $EG^*_{C_t,z}$ [5]. We can choose finitely many subfamilies like that defined on segments $[\tau_i, \tau_{i+1}]$, $0 \le i < m$, $\tau_0 = 0$, $\tau_m = 1$. The constancy of $W^{eg}(C_t, z) = W^{eg}(C_t, z, L(t))$ along each segment is evident and the equality for the common endpoint τ_i of the two segments $[\tau_{i-1}, \tau_i]$, $[\tau_i, \tau_{i+1}]$ follows from Proposition 2.1.

(2) The second statement of the lemma follows from the fact that an equigeneric deformation of (C, z) induces independent equigeneric deformations of the components (C_i, z) and vice versa (see [16, Theorem 1, page 73], [5, Corollary 3.3.1], and also [9, Theorem II.2.56]), and from the fact that the deformed components (C_i, z) and (C_j, z), $i \ne j$, can intersect only in hyperbolic real nodes and in complex conjugate nodes. $\Box$

3. Singular Welschinger invariants associated with $EG^{\delta(C,z)-1}_{C,z}$ and $\mathcal{D}_{C,z}$

The key ingredient of the proof of Proposition 2.1 is that the tangent cone to the equigeneric stratum $EG_{C,z}$ is a linear space of dimension equal to $\dim EG_{C,z}$. We intend to establish a similar statement for $EG^{\delta(C,z)-1}_{C,z}$.

Recall the following fact used in the sequel: By [14, Theorem 1.1] the closure of each irreducible component of $EG^{\delta(C,z)-1}_{C,z}$ contains $EG_{C,z}$, and a generic element of such a component can be obtained by smoothing out a node of an element of $EG^*_{C,z}$.

Lemma 3.1. *Under the following assumptions a real substratum $V \subset EG^{\delta(C,z)-1}_{C,z}$ is RE-appropriate:*

- (i) *(C, z) contains a real singular local branch (C', z), and $V \subset EG^{\delta(C,z)-1}_{C,z}$ is the union of those irreducible components of $EG^{\delta(C,z)-1}_{C,z}$, which parameterize nodal curves obtained from the curves $\mathcal{C}_\varphi$, $\varphi \in EG^*_{C,z}$, by smoothing out a real node on the component $\mathcal{C}'_\varphi$ of $\mathcal{C}_\varphi$ corresponding to the local branch (C', z);*
- (ii) *(C, z) contains a pair of complex conjugate local branches (C', z), (C'', z), and $V \subset EG^{\delta(C,z)-1}_{C,z}$ is the union of those irreducible components of $EG^{\delta(C,z)-1}_{C,z}$, which parameterize nodal curves obtained from the curves $\mathcal{C}_\varphi$, $\varphi \in EG^*_{C,z}$, by smoothing out a real intersection point of the components $\mathcal{C}'_\varphi, \mathcal{C}''_\varphi$ of $\mathcal{C}_\varphi$ corresponding to the local branches (C', z), (C'', z).*

Proof. (i) Notice, first, that $V \simeq EG^{\delta(C',z)-1}_{C',z} \times EG_{C'',z}$, where (C'', z) is the union of the local branches of (C, z) different from (C', z) (cf. the proof of Lemma 2.2(2)). Hence, we can just assume that (C, z) is irreducible.

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[5]Here, the number of intersections equals mult $EG_{C_\tau,z}$, which, in fact, proves that mult $EG_{C,z}$ is an invariant of the topological type of the singularity (C, z).

If for $\varphi \in EG^{\delta(C,z)-1}_{C,z}$, the curve $\mathcal{C}_\varphi$ has precisely $\delta(C,z)-1$ nodes as its only singularities, then the tangent space $T_\varphi EG^{\delta(C,z)-1}_{C,z}$ can be identified with the space of elements $\psi \in R(C,z)$ vanishing at the nodes of $\mathcal{C}_\varphi$. It has codimension $\delta(C,z)-1$, and the following bound for the intersection holds:

$$(\mathcal{C}_\psi \cdot \mathcal{C}_\varphi) \geq 2 \cdot \#(\text{number of nodes of } \mathcal{C}_\varphi) = 2\delta(C,z)-2,$$

for all $\psi \in T_\varphi EG^{\delta(C,z)-1}_{C,z}$. Hence, any limit of a sequence of the tangent spaces $T_\varphi EG^{\delta(C,z)-1}_{C,z}$ as $\varphi \to 0$ is contained in the linear space

$$\left\{\psi \in R(C,z) \ : \ \operatorname{ord} \psi\big|_{(C,z)} \geq 2\delta(C,z)-2\right\}$$

of codimension at most $\delta(C,z)-1$. By [4, Proposition 5.8.6] we have

$$\left\{\psi \in R(C,z) \ : \ \operatorname{ord} \psi\big|_{(C,z)} \geq 2\delta(C,z)-1\right\}$$
$$= \left\{\psi \in R(C,z) \ : \ \operatorname{ord} \psi\big|_{(C,z)} \geq 2\delta(C,z)\right\} = J^{cond}_{C,z}/\mathfrak{m}^N_z.$$

Hence

$$\operatorname{codim}\left\{\psi \in R(C,z) \ : \ \operatorname{ord} \psi\big|_{(C,z)} \geq 2\delta(C,z)-2\right\}$$
$$\geq \operatorname{codim}\left\{\psi \in R(C,z) \ : \ \operatorname{ord} \psi\big|_{(C,z)} \geq 2\delta(C,z)-1\right\} - 1$$
$$= \operatorname{codim} J^{cond}_{C,z}/\mathfrak{m}^N_z - 1 = \delta(C,z)-1,$$

and we are done.

(ii) As in the previous case, we can assume that $(C,z) = (C',z) \cup (C'',z)$. The above argument yields that the limits of sequences of the tangent spaces $T_\varphi \mathbb{R}V$ as $\varphi \in \mathbb{R}V^*$ tends to 0, are contained in the linear subspace

$$\left\{\psi \in \mathbb{R}R(C,z) \ : \ \operatorname{ord} \psi\big|_{(C,z)} \geq 2\delta(C,z)-2\right\}$$

of real codimension at most $\delta(C,z)-1$. Since the elements $\psi \in \mathbb{R}R(C,z)$ are real and the branches C', C'' are complex conjugate, we conclude that the above space coincides with the space

$$\left\{\psi \in \mathbb{R}R(C,z) \ : \ \operatorname{ord} \psi\big|_{(C',z)} = \operatorname{ord} \psi\big|_{(C'',z)} \geq 2\delta(C',z) + (C' \cdot C'')_z - 1\right\}.$$

So, it remains to show that its codimension equals exactly $\delta(C,z)-1$. To this end, we shall prove that the complex codimension of the space

$$\left\{\psi \in R(C,z) \ : \ \operatorname{ord} \psi\big|_{(C',z)} = \operatorname{ord} \psi\big|_{(C'',z)} \geq 2\delta(C',z) + (C' \cdot C'')_z - 1\right\}$$

is at least $\delta(C,z)-1$. Namely, we just impose an extra linear condition and show that the resulting space

$$\Lambda = \Big\{\psi \in R(C,z) \ : \ \operatorname{ord} \psi\big|_{(C',z)} \geq 2\delta(C',z) + (C' \cdot C'')_z$$
$$\operatorname{ord} \psi\big|_{(C'',z)} \geq 2\delta(C'',z) + (C' \cdot C'')_z - 1\Big\}$$

has codimension $\geq \delta(C,z)$. Write $f = f'f''$, where $f' = 0$ and $f'' = 0$ are equations of (C',z), (C'',z), respectively. By the Noether's theorem in the form of [10, Theorem II.2.1.26], any $\psi \in \Lambda$ can be represented as $\psi = af' + bf''$, where $a, b \in R(C,z)$ and

$$\operatorname{ord} a\big|_{(C'',z)} \geq 2\delta(C'',z)-1, \quad \operatorname{ord} b\big|_{(C',z)} \geq 2\delta(C',z).$$

By [4, Proposition 5.8.6], the former inequality yields

$$\operatorname{ord} a\big|_{(C'',z)} \geq 2\delta(C'',z),$$

which finally implies that $\Lambda\subset J^{cond}_{C,z}/\mathfrak{m}_z^N$, and hence $\operatorname{codim}\Lambda\ge\delta(C,z)$. □

Proposition 3.2. *Let $V\subset EG^{\delta(C,z)-1}_{C,z}$ satisfy the hypotheses of one of the cases in Lemma 3.1. Then $W(C,z,V,L)$ does not depend on the choice of the real affine space L as in Definition 1.3(2).*

Proof. We closely follow the argument in the proof of Proposition 2.1. If the general element of a codimension one substratum of V has the total δ-invariant equal to $\delta(C,z)-1$, then by [7, Theorem 1.4] this substratum satisfies conditions of one of the cases (n1)-(n3) as in the proof of Proposition 2.1. If the total δ-invariant of the general element of a codimension one substratum of V equals $\delta(C,z)$, then it belongs to $EG_{C,z}$, and for the dimension reason we have

(n4) the substratum is $EG_{C,z}$ (i.e., for its generic element φ the curve $\mathcal{C}_\varphi$ has $\delta(C,z)$ nodes).

The analysis of the cases (n1)-(n3) literally coincides with that in the proof of Proposition 2.1. In case (n4), the germ of $\mathbb{R}V$ at φ consists of k pairwise transversal smooth real germs of codimension $\delta(C,z)-1$ in $\mathbb{R}R(C,z)$. Under the hypotheses of Lemma 3.1(i), k is the number of real nodes of the component $\mathcal{C}'_\varphi$, and each component of $\mathbb{R}V$ parameterizes deformations of $\mathcal{C}_\varphi$ smoothing out a specific real node of $\mathcal{C}'_\varphi$. Under the hypotheses of Lemma 3.1(ii), k is the number of real intersection points of the components $\mathcal{C}'_\varphi$ and $\mathcal{C}''_\varphi$, and each component of V parameterizes a deformation of $\mathcal{C}_\varphi$ smoothing out one of these k real itersection points. For any (smooth) component M of the germ of V at φ, the intersection of $L(t)\cap M$, $0<|t-\hat t|<\eta$, yields a curve $\mathcal{C}_\psi$ whose Welschinger sign depends only on the real nodes of $\mathcal{C}_\varphi$ different from the specific smoothed out node, and hence does not depend on t. □

By Lemma 1.1(3), the tangent cone $\widehat{T}_0\mathcal{D}_{C,z}$ is a hyperplane. As in the preceding case, this yields

Proposition 3.3. *Given a real singularity (C,z), the number*

$$W^{discr}(C,z,L):=W(C,z,\mathcal{D}_{C,z},L)$$

does not depend on the choice of a real line L.

The proof literally follows the argument in the proof of Propositions 2.1 and 3.2.

4. Singular Welschinger invariants associated with $EC^k_{C,z}$

We start with the equiclassical stratum $EC_{C,z}$, which is the most interesting.

Proposition 4.1.

(1) Given a real singularity (C,z), the number $W^{ec}(C,z,L)$ does not depend on the choice of L.

(2) The number $W^{ec}(C,z)$ is an invariant of a real equisingular deformation class. That is, if $(C_t,z)_{t\in[0,1]}$ is an equisingular family of real singularities, then $W^{ec}(C_0,z)=W^{ec}(C_1,z)$.

(3) Let $(C,z)=\bigcup_i(C_i,z)$ be the decomposition of a real singularity (C,z) into irreducible over $\mathbb{R}$ components. Then $W^{ec}(C,z)=\prod_i W^{ec}(C_i,z)$.

Proof. Again the proof follows the argument in the proof of Proposition 2.1. So, we accept the initial setting and the notations in the proof of Proposition 2.1. Then we study the wall-crossings that correspond to codimension one substrata in $EC_{C,z}$. If $\varphi\in EC_{C,z}$ is a general element of a codimension one substratum, then

(n1') either $\mathcal{C}_\varphi$ has $3\delta(C,z)-\kappa(C,z)-1$ nodes and $\kappa(C,z)-2\delta(C,z)+1$ cusps,

(n2') or $\mathcal{C}_\varphi$ has $3\delta(C,z)-\kappa(C,z)-2$ nodes, $\kappa(C,z)-2\delta(C,z)$ cusps, and one tacnode A_3,

(n3') or $\mathcal{C}_\varphi$ has $3\delta(C,z)-\kappa(C,z)-3$ nodes, $\kappa(C,z)-2\delta(C,z)$ cusps, and one triple point D_4,
(c1') or $\mathcal{C}_\varphi$ has $3\delta(C,z)-\kappa(C,z)-1$ nodes, $\kappa(C,z)-2\delta(C,z)-1$ cusps, and one singularity A_4,
(c2') or $\mathcal{C}_\varphi$ has $3\delta(C,z)-\kappa(C,z)-2$ nodes, $\kappa(C,z)-2\delta(C,z)-1$ cusps, and one singularity D_5,
(c3') or $\mathcal{C}_\varphi$ has $3\delta(C,z)-\kappa(C,z)-1$ nodes, $\kappa(C,z)-2\delta(C,z)-2$ cusps, and one singularity E_6.

Indeed, if the degeneration of a general element of $EC_{C,z}$ involves only nodes, then [7, Theorem 1.4] yields that the only possible limit elements must be of types (n1'), (n2'), (n3'). An analysis of these wall-crossings is completely similar to that for the wall-crossing (n1), (n2), (n3), respectively, studied in the proof of Proposition 2.1. Thereby we conclude the constancy of $W^{ec}(C,z,L(t))$, $|t-t^*|<\eta$ in the wall-crossings (n1'), (n2'), (n3').

Next we explain why (c1'), (c2'), (c3') are the only possible codimension one substrata of $EC_{C,z}$ that involve cusps of the degenerating elements of $\mathbb{R}EC^*_{C,z}$. To this end, we show that, any other collection of singularities of $\mathcal{C}_\varphi$ can be deformed into $3\delta(C,z)-\kappa(C,z)$ nodes and $\kappa(C,z)-2\delta(C,z)$ cusps in two successive non-equisingular deformations.

Before we proceed further, we explain the existence of each of the deformations described below. First, this existence can be deduced from the openness of versality as in [9, Theorem II.1.15]. Another explanation is that the versal deformation of the given singularity, by construction, is induced by the space of plane curves of degree $N-1$, and hence the deformations of all possible multisingularites coming from the given singularity appear to be versal as N is large enough, for instance, $4(N-1)-4>\mu(C,z)$ (cf. [10, Section 4.4.1, Proposition 4.4.2]).

By our assumption, at least one of the non-nodal-cuspidal singularities of $\mathcal{C}_\varphi$ must contain a singular local branch. Thus,

- if $\mathcal{C}_\varphi$ has at least two non-nodal-cuspidal singularity, we, first, deform one such singularity into nodes and cusps (along its equiclassical deformation), then all other singularities;
- if the non-nodal-cuspidal singularity of $\mathcal{C}_\varphi$ has at least three local branches (one of which denoted P is singular), we, first, shift away a branch, different from P, then equiclassically deform the obtained curve into a nodal-cuspidal one;
- if the non-nodal-cuspidal singularity of $\mathcal{C}_\varphi$ has two singular branches P_1, P_2, we, first, shift P_2 so that P_2 remains centered at a smooth point of P_1, then equiclassically deform the obtained curve into a nodal-cuspidal one;
- if the non-nodal-cuspidal singularity of $\mathcal{C}_\varphi$ has two branches, P_1 smooth and P_2 singular, which is different from an ordinary cusp, then we, first, equiclassically deform the local branch P_2 into nodes and (necessarily appearing) cusps, while keeping one cusp centered on P_1, then deform the obtained triple singularity into nodes and one cusp;
- if the non-nodal-cuspidal singularity of $\mathcal{C}_\varphi$ has two branches, P_1 smooth and P_2 singular of type A_2, which is tangent to P_1, then we, first, rotate P_1 so that it becomes transversal to P_2, then deform the obtained singularity D_5 into two nodes and one cusp;
- if the non-nodal-cuspidal singularity of $\mathcal{C}_\varphi$ is unibranch either of multiplicity $m\ge 3$ and not of the topological type $y^m+x^{m+1}=0$, or of multiplicity 2 and not of type A_4, then we, first, equigenerically deform this singularity into some nodes and a singularity of topological type $y^m+x^{m+1}=0$, if $m\ge 3$, or a singularity A_4, if $m=2$ (this can be done by the blow-up construction as in the proof of [1, Theorem 1], see also [12, Section 2.1]), then equiclassically deform the obtained curve into a nodal-cuspidal one;
- if the non-nodal-cuspidal singularity of $\mathcal{C}_\varphi$ is of the topological type $y^m+x^{m+1}=0$, $m\ge 4$, then the codimension of its equisingular stratum in a versal deformation base

equals $\frac{m^2+3m}{2}-3$, while the codimension of the equiclassical stratum equals

$$\kappa(\{y^m+x^{m+1}=0\})-\delta(\{y^m+x^{m+1}=0\})=\frac{m^2+m}{2}-1$$
$$=\left(\frac{m^2+3m}{2}-3\right)-(m-2)\le\left(\frac{m^2+3m}{2}-3\right)-2.$$

Now we analyze the wall-crossings of type (c1'), (c2'), and (c3') as described above.

In case (c1'), the miniversal unfolding of an A_4 singularity $y^2=x^5$ is given by the family $y^2=x^5+a_3x^3+a_2x^2+a_1x+a_0$ with the base $B=\{(a_0,...,a_3)\in(\mathbb{C}^4,0)\}$. The equiclassical locus $EC\subset B$ is the projection of the curve $\{y^2=(x-2\lambda)^3(x+3\lambda)^2\}_{\lambda\in(\mathbb{C},0)}$ to B, and it is given by

$$\alpha_3=-5\lambda^2,\quad \alpha_2=-45\lambda^3,\quad \alpha_1=0,\quad \alpha_0=108\lambda^5,$$

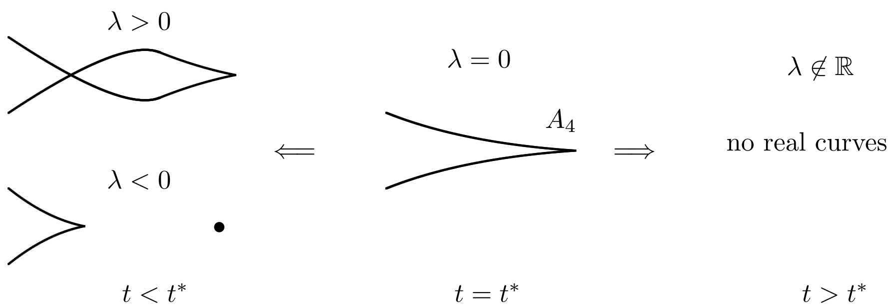
and hence this curve has an ordinary cusp at the origin with the tangent cone

$$\widehat{T}_0EC=\{\alpha_2=\alpha_1=\alpha_0=0\}.$$

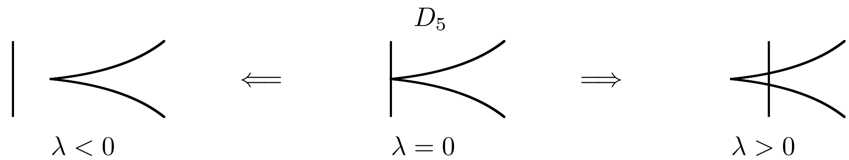
Without loss of generality we can suppose that the projection of the family of affine spaces $\{L(t)\}_{|t-t^*|<\eta}$ to B is given by $L'(t)=\{\alpha_3=t-t^*\}$, $|t-t^8|<\eta$. For $t<t^*$, the intersection $EC\cap L'(t)$ consists of two real elements, one corresponding to a curve with a cusp and a hyperbolic node and the other corresponding to a curve with a cusp and an elliptic node, while, for $t>t^*$, the intersection consists of two complex conjugate elements (see Figure 1(1)), and hence the constancy of $W^{ec}(C,z,L(t))$, $|t-t^*|<\eta$, follows.

In case (c2'), the equiclassical locus in a miniversal deformation base of a singularity D_5 given, say, by $x(y^2-x^3)=0$ is smooth and can be modeled by the family $\{(x-\lambda)(y^2-x^3)=0\}_{|\lambda|<\eta}$. So, in the considered wall-crossing a real curve with a cusp and two hyperbolic nodes turns into a curve with a cusp and two complex conjugate nodes (see Figure 1(b)), and hence the constancy of $W^{ec}(C,z,L(t))$, $|t-t^*|<\eta$, follows.

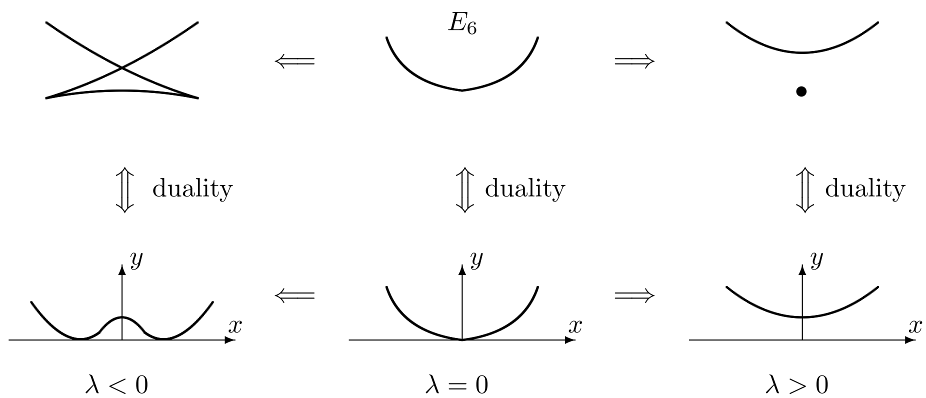
In case (c3'), again the equiclassical locus in a miniversal deformation base of a singularity E_6 is smooth (see [6, Theorem 27]) and one-dimensional. Hence, there are two kinds of real equiclassical deformations of E_6, and we claim that one half-branch of $\mathbb{R}EC(E_6)$ parameterizes curves with two real cusps and one hyperbolic node, while the other half-branch parameterizes curves with two complex conjugate cusps and one elliptic node. This immediately would yield the constancy of $W^{ec}(C,z,L(t))$, $|t-t^*|<\eta$. To prove the claim, we suppose that the E_6 singularity is given by $y^3+x^4=0$ and consider the versal deformation generated by the monomials in the Newton triangle $T_{3,4}=\text{conv}\{(0,0),(0,3),(4,0)\}$. The base of this deformation is 11-dimensional and it can be regarded as the product of a miniversal deformation base and $(\mathbb{C}^5,0)$ (see [9, Proposition II.1.14]). The equiclassical locus EC is then a smooth 6-dimensional germ, which contains a smooth 5-dimensional equisingular locus ES. The complement $\mathbb{R}EC\setminus\mathbb{R}ES$ consists of two components. To complete the proof of the claim, we exhibit a one-dimensional subfamily of $\mathbb{R}EC$ that crosses $\mathbb{R}ES$ and parameterizes curves of two types asserted in the claim. Namely, consider the family of real curves $\{y=(x^2+\lambda)^2\}_{-\eta<\lambda<\eta}$. In the projective plane, these are quartic curves with E_6 singularity at infinity, and the x-axis is their common bitangent; furthermore, for $\lambda<0$, the curve has two real inflexion points, while for $\lambda>0$, the real point set is convex. By Plücker formulas, the family of dual curves is the family of real plane quartics with a smooth point where the tangent intersects the curve with multiplicity 4, i.e., the family of quartics with the Newton triangle $T_{3,4}$. Moreover, for $\lambda<0$ we obtain quartics with two real cusps and a hyperbolic node, while for $\lambda>0$, quartics with two imaginary cusps and an elliptic node (see Figure 1(c)). □



(a) Type (c1'): A_4-wall



(b) Type (c2'): D_5-wall



(c) Type (c3'): E_6-wall

FIGURE 1. Walls of types (c1'), (c2'), (c3')

The other loci $EC^k_{C,z}$, $1 \le k < \kappa(C,z) - 2\delta(C,z)$, may be reducible. Assume that

$$(C,z) = (C_1,z) \cup ... \cup (C_s,z)$$

is the splitting into irreducible (over $\mathbb{C}$) components. Given a partition $\overline{k} = (k_1, ..., k_s)$ such that

$$k_1 + ... + k_s = k, \quad 0 \le k_i \le \kappa(C_i,z) - 2\delta(C_i,z), \ i = 1, ..., s, \tag{3}$$

we define the substratum $EC^{\overline{k}}_{C,z} \subset EC^k_{C,z}$, which is the union of those irreducible components of $EC^k_{C,z}$ whose generic elements φ are such that $\mathcal{C}_\varphi = \mathcal{C}_{1,\varphi} \cup ... \cup \mathcal{C}_{s,\varphi}$ with $\mathcal{C}_{i,\varphi} \in EC^{k_i}_{C_i,z}$, $i = 1, ..., s$.

Lemma 4.2. *In the above notation, the tangent cone* $\widehat{T}_0 EC^{\overline{k}}_{C,z}$ *is a linear subspace of* $R(C,z)$ *of codimension* $k+\delta(C,z)=\operatorname{codim} EC^{\overline{k}}_{C,z}$ *(i.e., the locus* $EC^{\overline{k}}_{C,z}$ *is RE-appropriate).*

Proof. It is sufficient to treat the case of an irreducible singularity (C,z). Let φ be a generic element of a component of $EC^k_{C,z}$. The tangent space $T_\varphi EC^k_{C,z}$ at φ can be identified with the space

$$\{\psi\in R(C,z)\ :\ \psi(\mathrm{Sing}(\mathcal{C}_\varphi))=0,\qquad \operatorname{ord}\psi\big|_P\ge 3 \text{ for each cuspidal local branch } P\},$$

and hence the limit of each sequence of tangent spaces $T_\varphi EC^k_{C,z}$ as $\varphi\to 0$ is contained in the linear space

$$\{\psi\in R(C,z)\ :\ \operatorname{ord}\psi\big|_{C,z}\ge 2\delta(C,z)+k\}.$$

It remains to notice that

$$\operatorname{codim}\{\psi\in R(C,z)\ :\ \operatorname{ord}\psi\big|_{C,z}\ge 2\delta(C,z)+k\}=\delta(C,z)+k.$$

The latter follows, for instance, from [4, Propositions 5.8.6 and 5.8.7]. □

As a corollary we obtain

Proposition 4.3. *Given a real singularity* (C,z) *splitting into irreducible (over* $\mathbb{C}$*) irreducible components* (C_i,z), $i=1,...,s$, *and a sequence* $\overline{k}=(k_1,...,k_s)$ *satisfying (3) and an extra condition* $k_i=k_j$ *as long as* (C_i,z) *and* (C_j,z) *are complex conjugate, the locus* $EC^{\overline{k}}_{C,z}$ *is real, and the number* $W(C,z,EC^{\overline{k}}_{C,z},L)$ *does not depend on the choice of* L.

The proof literally coincides with the proof of Proposition 4.1.

5. Examples

5.1. **Singularities of type A_n.** A complex singularity of type A_n is analytically isomorphic to the canonical one $\{y^2-x^{n+1}=0\}\subset(\mathbb{C}^2,0)$, and its miniversal deformation can be chosen to be

$$\left\{y^2-x^{n+1}-\sum_{i=0}^{n-1}a_ix^i=0\right\}_{a_0,\dots,a_{n-1}\in(\mathbb{C},0)}$$

with the base $B(A_n)=\{(a_0,...,a_{n-1})\in(\mathbb{C}^n,0)\}$.

Lemma 5.1. *(1) For any* $n\ge 1$, *and* $1\le i\le\delta(A_n)=\left[\frac{n+1}{2}\right]$,

$$\widehat{T}_0 EG^i_{A_n}=\{a_0=...=a_{i-1}=0\}\subset B(A_n) \qquad (4)$$

the linear subspace of codimension $i=\operatorname{codim} EG^i_{A_n}$.

(2) If n *is odd, then* $EC_{A_n}=EG_{A_n}$. *If* $n=2k$, *then*

$$\widehat{T}_0 EC_{A_n}=\{a_0=...=a_k=0\}\subset B(A_n).$$

Proof. Let (C,z) be a canonical singularity of type A_n. The tangent space to $EG^i_{C,z}$ at a generic element φ consists of $\psi\in B(C,z)$ such that $\mathcal{C}_\psi$ passes through all i nodes of $\mathcal{C}_\varphi$, and hence, $(\mathcal{C}_\psi\cdot\mathcal{C}_\varphi)_{D(C,z)}\ge 2i$. It follows that the limit of any sequence of these tangent spaces as $\varphi\to 0$ is contained in the linear space $\{\psi\in B(C,z)\ :\ (\mathcal{C}_\psi\cdot C)_z\ge 2i\}$, which one can easily identify with the space in the right-hand side of (4). So, the first claim of the lemma follows for the dimension reason. The same argument settles the second claim. □

Proposition 5.2. *For any $n \ge 1$ and $k \ge 1$, we have*

$$\mathrm{mult}\, EG^i_{A_n} = \binom{n+1-i}{i}, \quad \text{for all} \quad i = 1, ..., \delta(A_n) = \left[\frac{n+1}{2}\right],$$

$$\text{and} \quad \mathrm{mult}\, EC(A_{2k}) = k.$$

Remark 5.3. *The multiplicities* $\mathrm{mult}\, EG^i_{A_n}$ *were computed in* [15, Section 5, page 540]. *Here we provide another, more explicit computation, which will be used below for computing singular Welschinger invariants.*

Proof. (1) If $n+1 = 2i$, then $EG^i_{A_n} = EG(A_n) = EC(A_n)$ is smooth; hence, the multiplicity equals 1. Thus, suppose that $n+1 > 2i$. By Lemma 5.1(1), the question on $\mathrm{mult}\, EG^i_{A_n}$ reduces to the following one: How many polynomials $P(x)$ of degree $\le i-1$ satisfy the condition

$$x^{n+1} + x^i + P(x) = Q(x)^2 R(x), \tag{5}$$

where Q, R are monic polynomials of degree i, $n+1-2i$, respectively?

Combining relation (5) with its derivative, we obtain

$$(n+1-i)x^i + ((n+1)P - xP') = ((n+1)QR - 2xQ'R - xQR')\, Q,$$

which immediately yields

$$(n+1)QR - 2xQ'R - xQR' = n+1-i. \tag{6}$$

Substituting

$$Q(x) = x^i + \sum_{j=1}^{i} \alpha_j x^{i-j}, \quad R(x) = x^{n+1-2i} + \sum_{j=1}^{n+1-2i} \beta_j x^{n_1-2i-j}$$

into (6), we obtain that the terms of the top degree $n+1-i$ cancel out, while the coefficients of x^m, $m = 0, ..., n-i$, yield the system of equations

$$\begin{cases} 2\alpha_1 + \beta_1 = 0, \\ 2j\alpha_j + j\beta_j + \sum_{0<m<j} c_{jm}\alpha_{j-m}\beta_m = 0, \quad j = 2, ..., n-i, \\ (n+1)\alpha_i\beta_{n+1-2i} = n+1-i, \end{cases} \tag{7}$$

where we assume $\alpha_j = 0$ as $j > i$ and $\beta_j = 0$ as $j > n+1-2i$.

Suppose that $\deg Q = i \ge \deg R = n+1-2i$. From the $(n+1-2i)$ first equations in (7) we express β_j as a polynomial in $\alpha_1, ..., \alpha_i$ of homogeneity degree j, while α_m has weight m, for all $j = 1, ..., n+1-2i$. Substituting these expressions into the other equations, we obtain a system of i equations in $\alpha_1, ..., \alpha_i$ of homogeneity degrees $n+2-2i, ..., n+1-i$, respectively. Thus, (cf. the computation in [8, Section G, Example 1]) the number of solutions (counted with multiplicities) appears to be

$$\frac{(n+2-2i) \cdot ... \cdot (n+1-i)}{i!} = \binom{n+1-i}{i}$$

as required. In the same way we treat the case when $\deg Q = i \le \deg R = n+1-2i$.

(2) For $n = 2k$, by Lemma 5.1(2), the question on $\mathrm{mult}\, EC(A_{2k})$ reduces to the following one: How many polynomials $P(x)$ of degree k satisfy the condition

$$x^{2k+1} + x^{k+1} + P(x) = Q(x)^2 (x+\beta)^3, \tag{8}$$

where $Q(x)$ is a monic polynomial of degree $k-1$?

The preceding argument subsequently gives an equation

$$(2k+1)(x+\beta)Q - 3xQ - 2Q'(x+\beta) = k$$

with $Q(x)=x^{k-1}+\sum_{j=1}^{k-1}\alpha_j x^{k-1-j}$, which develops into the system

$$\begin{cases} 2\alpha_1+3\beta=0, \\ (2j+2)\alpha_j+(2j+3)\alpha_{j-1}\beta=0, \quad j=2,...,k-1, \\ (2k+1)\alpha_{k-1}\beta=k+1, \end{cases} \tag{9}$$

admitting a simplification of the form

$$\alpha_j=\nu_j\beta^j,\ j=1,...,k-1, \quad (2k+1)\nu_{k-1}\beta^k=k+1$$

with some $\nu_1,...,\nu_{k-1}\in\mathbb{Q}$. So, we finally obtain k solutions as required. $\square$

Now we pass to the real setting. The complex singularity of type A_n has a unique real form $y^2=x^{2k+1}$ if $n=2k$, and has two real forms $y^2=x^{2k}$ and $y^2=-x^{2k}$ (denoted by A^h_{2k-1} and A^e_{2k-1}, respectively) if $n=2k-1$.

Lemma 5.4. *(1) For all $k\ge 1$ and $i=1,...,k$, there exist singular Welschinger invariants*

$$W\left(A^h_{2k-1}, EG^i_{A^h_{2k-1}}\right),\ W\left(A^e_{2k-1}, EG^i_{A^e_{2k-1}}\right),\ \text{and } W\left(A_{2k}, EG^i_{A_{2k}}\right). \tag{10}$$

(2) Furthermore,

$$W^{eg}(A^e_{2k-1})=(-1)^k, \quad W^{eg}(A^h_{2k-1})=1,$$

$$W^{eg}(A_{2k})=\begin{cases}0, & k\equiv 1 \mod 2,\\ 1, & k\equiv 0 \mod 2,\end{cases}$$

$$W^{ec}(A_{2k})=\begin{cases}0, & k\equiv 0 \mod 2,\\ 1, & k\equiv 1 \mod 2.\end{cases}$$

Proof. The existence of the invariants (10) follows from Lemma 5.1 and the argument used in the proof of Propositions 2.1 and 3.2.

Since $\operatorname{mult} EG(A_{2k-1})=1$, we have $W^{eg}=\pm 1$ for A^h_{2k-1} and A^e_{2k-1}. More precisely, an equigeneric nodal deformation of A^h_{2k-1} has the form $y^2-Q(x)^2=0$, $\deg Q=k$, and hence it has only hyperbolic real nodes, i.e., $W^{eg}(A^h_{2k-1})=1$, while an equigeneric nodal deformation of A^e_{2k-1} has the form $y^2+Q(x)^2=0$, $\deg Q=k$, and hence it has only elliptic real nodes, whose number is of the same parity as k, i.e., $W^{eg}(A^e_{2k-1})=(-1)^k$.

Consider singularities A_{2k}. For $EG(A_{2k})=EG^k_{A_{2k}}$, system (7) takes the form

$$\begin{cases} 2\alpha_1+\beta_1=0, \\ 2j\alpha_j+(2j-1)\alpha_{j-1}\beta_1=0, \quad j=2,...,k, \\ (2k+1)\alpha_k\beta_1=k+1, \end{cases}$$

which yields

$$\alpha_j=\lambda_j\beta_1^j,\ (-1)^j\lambda_j>0,\ j=1,...,k, \quad \lambda_k\beta^{k+1}=\frac{k+1}{2k+1}.$$

So, if k is odd, we have no real solutions, and hence $W^{eg}(A_{2k})=0$. If k is even, then we have a unique real solution such that $\beta_1>0$ and $(-1)^j\alpha_j>0$. That is, $Q(x)$ has only positive real roots (if any), and hence the curve $y^2-(x+\beta_1)Q(x)^2=0$ has only hyperbolic real nodes, i.e., $W^{eg}(A_{2k})=1$.

In the same manner we analyze system (9) and obtain the values of $W^{ec}(A_{2k})$ as stated in the lemma. $\square$

5.2. **Real isolated singularities.** We call a real singularity (C,z) *real isolated*, if it has no real local branches, i.e., it is the union of pairs of complex conjugate local branches. Picking a branch in each pair, we form a complex singularity (C',z) and then represent $(C,z)=(C',z)\cup(\overline{C}',z)$.

Lemma 5.5. *For a real isolated singularity (C,z), we have*

$$W^{eg}(C,z)=(-1)^{(C'\cdot\overline{C}')_z}\cdot\operatorname{mult}EG_{C,z},$$

$$W^{ec}(C,z)=(-1)^{\kappa(C',z)+(C'\cdot\overline{C}')_z}\cdot\operatorname{mult}EC_{C,z}.$$

Proof. Consider the locus $\Sigma(C',\overline{C}')\subset B(C,z)$ parameterizing those deformations of (C,z), in which the components of (C',z) do not glue up with components of $(\overline{C}',z)$. This is the germ of a smooth real subvariety of $B(C,z)$ naturally diffeomorphic to $B(C',z)\times B(\overline{C}',z)$. By a conjugation-invariant coordinate change in $B(C,z)$ we can turn $\Sigma(C',\overline{C}')$ into a linear subspace of $B(C,z)$ and make the diffeomorphism $B(C',z)\times B(\overline{C}',z)\xrightarrow{\sim}\Sigma(C',\overline{C}')$ linear. We then can identify $B(C,z)$ with $\Sigma(C',\overline{C}')\oplus\Lambda$, where $\Lambda\subset B(C,z)$ is a real linear subspace of dimension $\operatorname{codim}_{B(C,z)}\Sigma(C',\overline{C}')$ transversally intersecting $\Sigma(C',\overline{C}')$.

Observe that $EG_{C,z}\subset\Sigma(C',\overline{C}')$ and that the isomorphism $B(C',z)\times B(\overline{C}',z)\xrightarrow{\sim}\Sigma(C',\overline{C}')$ takes $EG_{C',z}\times EG_{\overline{C}',z}$ onto $EG_{C,z}$. Consider now a linear subspace $L'_0\subset B(C',z)$ of dimension $\delta(C',z)$ transversal to $\widehat{T}_0EG_{C',z}$ in $B(C',z)$ and a sufficiently close to L'_0 generic affine subspace $L'\subset B(C',z)$ of the same dimension $\delta(C',z)$. It follows that the affine subspace $(L'\times\overline{L}')\oplus\Lambda$ transversally intersects $EG_{C,z}$ in $\operatorname{mult}EG_{C,z}=(\operatorname{mult}EG_{C',z})^2$ points, among which $\operatorname{mult}EG_{C',z}$ are real. It remains to notice that each curve corresponding to a real intersection point has only elliptic and complex conjugate nodes, and the number of the elliptic nodes has the parity of $(C'\cdot\overline{C}')_z$.

Precisely the same argument with the replacement of EG by EC proves the second formula in the lemma. The sign in the right-hand side is obtained from the fact that each curve corresponding to a real intersection point of $(L'\times\overline{L}')\oplus\Lambda$ with $EC_{C,z}=EC_{C',z}\times EG_{\overline{C}',z}$ has $(C'\cdot\overline{C}')_z$ mod 2 elliptic nodes and $\kappa(C',z)-2\delta(C',z)$ pairs of complex conjugate cusps. □

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