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DIFFERENTIABLE EQUISINGULARITY OF HOLOMORPHIC FOLIATIONS

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Abstract. We prove that a C^∞ equivalence between germs of holomorphic foliations at $(\mathbb{C}^2,0)$ establishes a bijection between the sets of formal separatrices preserving equisingularity classes. As a consequence, if one of the foliations is of second type, so is the other and they are equisingular.

1. Introduction

A celebrated theorem of Zariski [13] asserts that two topological equivalent germs of curves at $(\mathbb{C}^2,0)$ are necessarily equisingular, that is, their desingularization by blow-ups are combinatorially isomorphic. In [1], the authors prove the following analogous result for holomorphic foliations at $(\mathbb{C}^2,0)$, valid for the generic class of *generalized curve* foliations:

Theorem A. *Let $\mathcal{F}$ and $\mathcal{F}'$ be topologically equivalent germs of holomorphic foliations at $(\mathbb{C}^2,0)$. Suppose that $\mathcal{F}$ is a generalized curve foliation. Then $\mathcal{F}'$ is also a generalized curve foliation. Besides, $\mathcal{F}$ and $\mathcal{F}'$ have isomorphic desingularizations.*

The proof of this theorem is based upon the following result, also proved in [1]:

Theorem B. *Let $\mathcal{F}$ be a generalized curve foliation at $(\mathbb{C}^2,0)$ and let $\mathrm{Sep}(\mathcal{F})$ be its set of separatrices. Then, a desingularization of $\mathrm{Sep}(\mathcal{F})$ is also a desingularization of $\mathcal{F}$.*

In fact, if $\mathcal{F}$ is topologically equivalent to $\mathcal{F}'$, we have that $\mathrm{Sep}(\mathcal{F})$ and $\mathrm{Sep}(\mathcal{F}')$ are also topological equivalent, since the separatrices of a generalized curve foliation are convergent. Therefore, Theorem A follows from Theorem B and Zariski's Theorem. In general, the validity of Theorem A outside the class of generalized curve foliations is a difficult open problem. Actually, such a result would imply the topological invariance of the algebraic multiplicity of a holomorphic foliation, which is also an open problem (see [9, 10, 11]). The desingularization of a germ of foliation $\mathcal{F}$ is closely related to the desingularization of its set of separatrices $\mathrm{Sep}(\mathcal{F})$ — including the purely formal ones —, although the conclusion of Theorem B is not always true if the hypothesis of generalized curve is removed. Another serious difficulty is the fact that the topological equivalence does not naturally map purely formal separatrices of $\mathcal{F}$ into purely formal separatrices of $\mathcal{F}'$, as in the case of convergent separatrices.

If the equivalence between $\mathcal{F}$ and $\mathcal{F}'$ is supposed to be C^∞, a correspondence among formal separatrices of both foliations can be established. Let Φ be such a C^∞ equivalence and consider its Taylor series $\hat{\Phi}$ as a real formal diffeomorphism of $(\mathbb{C}^2,0)$. Let S be a possibly formal separatrix of $\mathcal{F}$, which can be seen as a parametrized two-dimensional real formal surface at

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$(\mathbb{C}^2, 0)$. Then the formal composition $\hat{\Phi}(S)$ is a parametrized two-dimensional real formal surface at $(\mathbb{C}^2, 0)$. In this setting, we have:

Theorem I. *Let Φ be a C^∞ equivalence between two germs $\mathcal{F}$ and $\mathcal{F}'$ of singular holomorphic foliations at $(\mathbb{C}^2, 0)$. Let S be a separatrix of $\mathcal{F}$, considered as a parametrized two-dimensional real formal surface at $(\mathbb{C}^2, 0)$. Then the following properties hold:*

(1) The real formal surface $\hat{\Phi}(S)$ is a real formal reparametrization of some separatrix S' of $\mathcal{F}'$, denoted by $S' = \Phi_*(S)$.
(2) Let $\mathscr{S}$ be the reduced curve defined as the union of a finite collection $S_1, \ldots, S_m$ of separatrices of $\mathcal{F}$. Denote by $\mathscr{S}'$ the reduced curve defined as the union of $\Phi_*(S_1), \ldots, \Phi_*(S_m)$. Then $\mathscr{S}$ and $\mathscr{S}'$ are equisingular.

As a consequence of Theorem I, if $\mathcal{F}$ and $\mathcal{F}'$ are C^∞ equivalent foliations, then the sets of separatrices $\mathrm{Sep}(\mathcal{F})$ and $\mathrm{Sep}(\mathcal{F}')$ have isomorphic desingularizations. Taking into account that the property described in Theorem B is valid for the larger class of *second type* foliations (see [7]), we obtain the following equidesingularization result for C^∞ equivalent foliations:

Theorem II. *Let $\mathcal{F}$ and $\mathcal{F}'$ be two germs of holomorphic foliations at $(\mathbb{C}^2, 0)$ equivalent by a germ of C^∞ diffeomorphism. If $\mathcal{F}$ is a foliation of second type, then $\mathcal{F}'$ is of second type. Moreover, $\mathcal{F}$ and $\mathcal{F}'$ are equisingular.*

This paper is structured in the following way. In sections 2 and 3 we present basic definitions and some properties of second type foliations. Next, in sections 4 and 5, we introduce the notion of characteristic curves for germs of holomorphic foliations. These are one-dimensional real curves intrinsically associated to separatrices — both convergent and formal. Characteristic curves are invariant by C^∞ equivalences and enable us to establish a one to one correspondence among separatrices of two C^∞ equivalent foliations. This is done in section 6. Next, in section 7, we introduce the concept of formal real equivalence of formal complex curves and we show that this notion implies equisingularity (Theorem 7.2). In section 8, we present the proof of Theorem I. Finally, in section 9, we accomplish the proof of Theorem II.

2. Foliations, separatrices and desingularization

A germ of singular holomorphic foliation $\mathcal{F}$ at $(\mathbb{C}^2, 0)$ is the object defined by an equation of the form $\omega = 0$, where ω is a $1-$form $\omega = P(u,v)du + Q(u,v)dv$ — or, equivalently, by the orbits of the germ of holomorphic vector field $\mathbf{v} = -Q(u,v)\partial/\partial u + P(u,v)\partial/\partial v$ —, where $P, Q \in \mathbb{C}\{u,v\}$ are relatively prime, defining what we call a *reduced* equation. Two reduced $1-$forms ω and $\tilde{\omega}$ define the same foliation if and only if $\omega = u\,\tilde{\omega}$ for some unity $u \in \mathbb{C}\{u,v\}$. In general, we can assume that a $1-$form $\omega = P(u,v)du + Q(u,v)dv$ defines a foliation by taking as reduced equation $\omega/R = 0$, where $R = \gcd(P,Q)$.

A considerable amount of information on the local topology and dynamics of a foliation is given by their *separatrices*. A separatrix for a foliation $\mathcal{F}$ is an invariant irreducible formal curve. Algebraically, it is defined by an irreducible formal series $f \in \mathbb{C}[[u,v]]$, with $f(0,0) = 0$, satisfying

$$\omega \wedge df = fhdu \wedge dv$$

for some formal series $h \in \mathbb{C}[[u,v]]$. If f can be taken in $\mathbb{C}\{u,v\}$, the separatrix is said to be *analytic* or *convergent*. We denote by $\mathrm{Sep}(\mathcal{F})$ the set of separatrices of $\mathcal{F}$ at $0 \in \mathbb{C}^2$.

The singularity $0 \in \mathbb{C}^2$ for $\mathcal{F}$ is said to be *simple* if the linear part $\mathrm{D}\mathbf{v}(0)$ of a vector field $\mathbf{v}$ inducing $\mathcal{F}$ has eigenvalues $\lambda_1, \lambda_2 \in \mathbb{C}$ meeting one of the following conditions:
Case 1: $\lambda_1\lambda_2 \neq 0$ and $\lambda_1/\lambda_2 \notin \mathbb{Q}^+$. We say that $0 \in \mathbb{C}^2$ is *non-degenerate* or *complex hyperbolic*. The set of separatrices $\mathrm{Sep}(\mathcal{F})$ is formed by two transversal branches, both of them analytic.

Case 2: $\lambda_1 \neq 0$ and $\lambda_2 = 0$. This is called a *saddle-node* singularity, for which there are formal coordinates (u, v) such that $\mathcal{F}$ is induced by

$$\omega = v(1 + \lambda u^k)du + u^{k+1}dv, \tag{2.1}$$

where $\lambda \in \mathbb{C}$ and $k \in \mathbb{Z}_{>0}$. The curve $\{u = 0\}$, corresponding to the tangent direction defined by the non-zero eigenvalue, defines an analytic separatrix, called *strong*, whereas $\{v = 0\}$ is tangent to a possibly formal separatrix, called *weak* or *central*. The integer $k + 1 > 1$ is called *tangency index* of $\mathcal{F}$ with respect to the weak separatrix, or simply *weak index*, and will be denoted by $\mathrm{Ind}_0^w(\mathcal{F})$.

A global foliation $\mathcal{G}$ on a holomorphic surface M corresponds to the assignment, for each $p \in M$, of compatible local foliations $\mathcal{G}_p$. For instance, a holomorphic $1-$form ω on M defines a foliation $\mathcal{G}$ by taking $\mathcal{G}_p$ as the local foliation defined by the germification of ω at p. Let $\mathcal{F}$ be a local foliation at $(\mathbb{C}^2, 0)$ defined by the $1-$form ω and let $\pi : (M, E) \to (\mathbb{C}^2, 0)$ be a sequence of punctual blow-ups starting at $0 \in \mathbb{C}^2$. The pull-back $1-$form $\pi^*\omega$ defines a foliation $\tilde{\mathcal{F}} = \pi^*\mathcal{F}$ with isolated singularities on (M, E) called the *strict transform* of $\mathcal{F}$ by π. We have the definition:

Definition 2.1. Let $\mathcal{G}$ be a foliation on (M, E), where E is a normal crossings divisor. With respect to the pair $(\mathcal{G}, E)$, we say that $p \in E$ is

(1) a *regular* point, if there are local analytic coordinates (u, v) at p such that $E_p \subset \{uv = 0\}$, where E_p denotes the germ of E at p, and $\mathcal{G} : du = 0$;
(2) a *simple singularity*, if p is a simple singularity for $\mathcal{G}$ and $E_p \subset \mathrm{Sep}_p(\mathcal{G})$.

This allows us to present the notion of reduction of singularities of a foliation with respect to a normal crossings divisor:

Definition 2.2. Let $\mathcal{G}$ be a foliation on (M, E), where E is a normal crossings divisor. We say that $(\mathcal{G}, E)$ is *reduced* or *desingularized* if all points $p \in E$ are either regular or simple singularities for the pair $(\mathcal{G}, E)$. A *reduction of singularities* or *desingularization* for a germ of foliation $\mathcal{F}$ at $(\mathbb{C}^2, 0)$ is a morphism $\pi : (M, E) \to (\mathbb{C}^2, 0)$, formed by a composition of punctual blow-ups, such that $(\pi^*\mathcal{F}, E)$ is reduced.

For a local foliation $\mathcal{F}$ at $(\mathbb{C}^2, 0)$, there always exists a reduction of singularities (see [12] and [1]). Besides, there exists a *minimal* one, in the sense that it factorizes, by an additional sequence of blow-ups, any other reduction of singularities of $\mathcal{F}$. In the sequel, whenever we refer to a reduction of singularities, we mean a minimal one.

Let $\pi : (M, E) \to (\mathbb{C}^2, 0)$ be a reduction of singularities for $\mathcal{F}$ and denote $\tilde{\mathcal{F}} = \pi^*\mathcal{F}$. The divisor $E = \pi^{-1}(0)$ is a finite union of components which are embedded projective lines, crossing normally at *corners*. The regular points of E are called *trace points*. A component $D \subset E$ can be:

(1) *non-dicritical*, if D is $\tilde{\mathcal{F}}$-invariant. In this case, D contains a finite number of simple singularities. Each trace singularity carries a separatrix transversal to E, whose projection by π is a branch in $\mathrm{Sep}(\mathcal{F})$.
(2) *dicritical*, if D is not $\tilde{\mathcal{F}}$-invariant. The definition of desingularization gives that D may intersect only non-dicritical components and that $\tilde{\mathcal{F}}$ is everywhere transverse do D. The π-image of a local leaf of $\tilde{\mathcal{F}}$ at each trace point of D belongs to $\mathrm{Sep}(\mathcal{F})$.

For each $B \in \mathrm{Sep}(\mathcal{F})$ we associate the trace point $\tau_E(B) \in E$ given by $\pi^*B \cap E$, where π^*B denotes the strict transform of B. We define $\mathrm{Sep}(D) = \{B \in \mathrm{Sep}(\mathcal{F}); \tau_E(B) \in D\}$ as the set of branches *attached* to the component $D \subset E$. We thus have a decomposition

$$\mathrm{Sep}(\mathcal{F}) = \mathrm{Iso}(\mathcal{F}) \cup \mathrm{Dic}(\mathcal{F}),$$

where

$$\mathrm{Iso}(\mathcal{F}) = \bigcup_{D \text{ non-dicritical}} \mathrm{Sep}(D) \qquad \text{and} \qquad \mathrm{Dic}(\mathcal{F}) = \bigcup_{D \text{ dicritical}} \mathrm{Sep}(D).$$

Separatrices in $\mathrm{Iso}(\mathcal{F})$, known as *isolated*, can be additionally classified in two types. A branch $B \in \mathrm{Iso}(\mathcal{F})$ is *strong* or of *Briot and Bouquet* type if either $\tau_E(B)$ is a non-degenerate singularity or if $\tau_E(B)$ is a saddle-node singularity with $\pi^* B$ as its strong separatrix. On the other hand, $B \in \mathrm{Iso}(\mathcal{F})$ is *weak* if $\tau_E(B)$ is a saddle-node singularity whose weak separatrix is $\pi^* B$. This classification engenders the decomposition $\mathrm{Iso}(\mathcal{F}) = \mathrm{Iso}^s(\mathcal{F}) \cup \mathrm{Iso}^w(\mathcal{F})$, where notation is self-evident. Note that $\mathrm{Iso}(\mathcal{F})$ is a finite set and all purely formal separatrices of $\mathcal{F}$ are contained in $\mathrm{Iso}^w(\mathcal{F})$.

On the other hand, if non-empty, $\mathrm{Dic}(\mathcal{F})$ is an infinite set of analytic separatrices, called *dicritical*. A foliation $\mathcal{F}$ may be classified either as *non-dicritical* — when $\mathrm{Sep}(\mathcal{F})$ is finite, which happens when $\mathrm{Dic}(\mathcal{F}) = \emptyset$ — or as *dicritical*, otherwise.

Let $\mathcal{F}$ be a foliation at $(\mathbb{C}^2, 0)$ with reduction of singularities $\pi : (M, E) \to (\mathbb{C}^2, 0)$. The *dual tree* associated to $\mathcal{F}$ is the acyclic, double weighted, directed graph $\mathbb{A}^*(\mathcal{F})$ defined in the following way:

(1) to each component $D \subset E$ we associate a vertex $v(D)$;
(2) to $v(D)$ we associate weights $n_1(D) \in \mathbb{Z}_{<0}$ and $n_2(D) \in \mathbb{N} \cup \{\infty\}$, where $n_1(D) = D \cdot D$ is the self-intersection number of D in M and $n_2(D) = \#\mathrm{Sep}(D)$;
(3) there is an arrow from $v(D_2)$ to $v(D_1)$ if and only if $D_2 \cap D_1 \neq \emptyset$ and D_2 results from a blow-up at a point in D_1.

The *valence* of a component $D \subset E$ is the number $\mathrm{Val}(D)$ of arrows of $\mathbb{A}^*(\mathcal{F})$ touching $v(D)$. In other words, it is the total number of components of E intersecting D other from D itself.

Definition 2.3. Two foliations $\mathcal{F}$ and $\mathcal{F}'$ are said to be *equisingular* or *equireducible* if $\mathbb{A}^*(\mathcal{F}) = \mathbb{A}^*(\mathcal{F}')$.

Let $\mathcal{F}$ be a foliation at $(\mathbb{C}^2, 0)$. A sequence of blow-ups $\pi : (M, E) \to (\mathbb{C}^2, 0)$ *desingularizes* $\mathrm{Sep}(\mathcal{F})$ if the transforms $\pi^* S$ of branches $S \in \mathrm{Sep}(\mathcal{F})$ are all disjoint and transverse to the regular part of E. We call this map, which is supposed to be minimal, an $\mathcal{S}$*-desingularization* or $\mathcal{S}$*-reduction* for $\mathcal{F}$. Following the same procedure as in the construction of $\mathbb{A}^*(\mathcal{F})$, we define the $\mathcal{S}$*-dual tree* of $\mathcal{F}$, denoted as $\mathbb{A}^*_{\mathcal{S}}(\mathcal{F})$, as the dual tree associated to the $\mathcal{S}$*-desingularization* of $\mathcal{F}$. In this case, $n_2(D)$ is the number of components of $\mathrm{Sep}(\mathcal{F})$ whose strict transforms by π pass through a component $D \subset E$. With this at hand, we have the following definitions:

Definition 2.4. A germ of foliation $\mathcal{F}$ is $\mathcal{S}$*-desingularizable* or $\mathcal{S}$*-reducible* if $\mathbb{A}^*_{\mathcal{S}}(\mathcal{F}) = \mathbb{A}^*(\mathcal{F})$, that is, an $\mathcal{S}$-desingularization actually is a desingularization for $\mathcal{F}$.

Definition 2.5. Two germs of foliations $\mathcal{F}$ and $\mathcal{F}'$ at $(\mathbb{C}^2, 0)$ are $\mathcal{S}$*-equisingular* or $\mathcal{S}$*-equireducible* if $\mathbb{A}^*_{\mathcal{S}}(\mathcal{F}) = \mathbb{A}^*_{\mathcal{S}}(\mathcal{F}')$, that is, if their sets of separatrices have equivalent desingularizations.

3. Second type foliations

We keep the notation $\pi : (M, E) \to (\mathbb{C}^2, 0)$ for the reduction of singularities of $\mathcal{F}$ and $\tilde{\mathcal{F}} = \pi^* \mathcal{F}$ for the strict transform foliation. We say that a saddle-node singularity for $\tilde{\mathcal{F}}$ is *tangent* if its weak separatrix is contained in E. Non-tangent saddle-nodes are also known as *well-oriented*. The following definition is due to J.-F. Mattei and E. Salem (see [7] and also [2], [4] [5]):

Definition 3.1. A foliation $\mathcal{F}$ at $(\mathbb{C}^2, 0)$ is of *second type* if there are no tangent saddle-nodes in its reduction of singularities.

The main property of second type foliations to be used in this article is the following result, which already appeared in [7, Th. 3.1.9] in the non-dicritical case:

Proposition 3.2. *Second type foliations are $\mathcal{S}$-desingularizable.*

Proof. We first remark that the same proof of Lemma 1 in [1] applies to the following more general statement: a second type foliation with exactly two smooth transversal formal separatrices is simple. The result then follows by the same arguments as in the proof of [1, Th. 2]. □

We establish the following definition:

Definition 3.3. Two germs of foliations $\mathcal{F}$ and $\mathcal{F}'$ at $(\mathbb{C}^2, 0)$ are topologically (respectively, C^∞) equivalent if there is a germ of homeomorphism (respectively, C^∞ diffeomorphism) $\Phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ which sends leaves of $\mathcal{F}$ on leaves of $\mathcal{F}'$.

The family of second type foliations contains the subclass of *generalized curve* foliations, characterized by the absence saddle-nodes in the desingularization. The property of being a generalized curve foliation is a topological invariant and topological equivalent generalized curves are equisingular. This is the main result in [1]. Indeed, the topology of a generalized curve foliation is closely related to its separatrix set, entirely formed by convergent curves. Our aim in Theorem II is to prove the equisingularity property for the family of second type foliations. If all separatrices of two second type foliations are convergent, then their topological equivalence implies equisingularity. Actually, there is a correspondence between homeomorphic separatrices for both foliations and the result follows from Zariski's equisingularity for curves in conjunction with the fact that a second type foliation is $\mathcal{S}$-desingularizable. However, in principle, a merely continuous equivalence map does not track purely formal separatrices. For this reason, in the statement of Theorem II, the regularity hypothesis on the equivalence map is strengthened and we ask for C^∞ equivalences.

The following object was defined in [4]. A more thorough study on its properties is found in [5]. Again, $\mathcal{F}$ is a germ of foliation at $(\mathbb{C}^2, 0)$ with reduction process $\pi : (M, E) \to (\mathbb{C}^2, 0)$.

Definition 3.4. A *balanced equation of separatrices* for $\mathcal{F}$ is a formal meromorphic function $\hat{F}$ whose associated divisor is

$$(\hat{F})_0 - (\hat{F})_\infty = \sum_{S \in \mathrm{Iso}(\mathcal{F})} (S) + \sum_{S \in \mathrm{Dic}(\mathcal{F})} a_S (S), \tag{3.1}$$

where the coefficients $a_S \in \mathbb{Z}$ are non-zero only for finitely many $S \in \mathrm{Dic}(\mathcal{F})$, and, for each dicritical component $D \subset E$, the following equality holds:

$$\sum_{S \in \mathrm{Sep(D)}} a_S = 2 - \mathrm{Val}(D). \tag{3.2}$$

Note that if $\mathcal{F}$ is non-dicritical, then a balanced equation is an equation for the set of separatrices.

We recall that the multiplicity $\rho(D)$ of a component $D \subset E$ is defined as the algebraic multiplicity of a curve γ at $(\mathbb{C}^2, 0)$ such that $\pi^*\gamma$ is transversal to D outside a corner of E. We have the following definition:

Definition 3.5. The *tangency excess* of $\mathcal{F}$ along E is the number

$$\tau_0(\mathcal{F}) = \sum_{q \in SN(\mathcal{F})} \rho(D_q)(\mathrm{Ind}_q^w(\tilde{\mathcal{F}}) - 1), \tag{3.3}$$

where $SN(\mathcal{F}) \subset E$ denotes the set of all tangent saddle-nodes, D_q is the component of E containing the weak separatrix of $\tilde{\mathcal{F}}$ at $q \in SN(\mathcal{F})$ and $\mathrm{Ind}_q^w(\tilde{\mathcal{F}}) > 1$ is the weak index.

Note that $\tau_0(\mathcal{F}) \geq 0$ and, by definition, $\tau_0(\mathcal{F}) = 0$ if and only if $SN(\mathcal{F}) = \emptyset$, that is, if and only if $\mathcal{F}$ is of second type.

The *algebraic multiplicity* of a foliation $\mathcal{F}$ having $\omega = Pdx + Qdy = 0$ as a reduced equation is the integer $\nu_0(\mathcal{F}) = \min(\nu_0(P), \nu_0(Q))$. The tangency excess measures the extent that a balanced equation of separatrices computes the algebraic multiplicity of a foliation. This is expressed in the following fact, whose proof is found in [4]:

Proposition 3.6. *Let $\mathcal{F}$ be a foliation on $(\mathbb{C}^2, 0)$ with $\hat{F}$ as a balanced equation of separatrices. Denote by $\nu_0(\mathcal{F})$ and $\nu_0(\hat{F})$ their algebraic multiplicities. Then*

$$\nu_0(\mathcal{F}) = \nu_0(\hat{F}) - 1 + \tau_0(\mathcal{F}).$$

We have, as a consequence:

Corollary 3.7. *With the above notation,*

$$\nu_0(\mathcal{F}) = \nu_0(\hat{F}) - 1$$

if and only if $\mathcal{F}$ is a second type foliation.

4. Pseudo-analytic curves

We begin with a definition:

Definition 4.1. Consider $\gamma : [0, \epsilon) \to \mathbb{R}^k$ ($k \in \mathbb{N}$) with $\gamma(0) = 0$. We say that the series

$$\hat{\gamma} = \sum_{j=1}^{\infty} a_j t^j \; (a_j \in \mathbb{R}^k)$$

is the Taylor series of γ at $0 \in \mathbb{R}$ if, for each $n \in \mathbb{N}$, there is a function $\gamma_n : [0, \epsilon) \to \mathbb{R}^k$ with $||\gamma_n(t)|| = o(t^n)$ and such that

$$\gamma(t) = \sum_{j=1}^{n} a_j t^j + \gamma_n(t) = p_n(t) + \gamma_n(t). \tag{4.1}$$

We say that $\hat{\gamma}$ is *non-degenerate* if $a_j \neq 0$ for some $j \in \mathbb{N}$.

Observe that we are considering a Taylor series for γ at $t = 0$, even though we do not ask it to be of class C^∞ in $[0, \epsilon)$. However, for γ as above,

$$D\gamma(0) = \lim_{t \to 0} \gamma(t)/t = \lim_{t \to 0} (a_1 + \gamma_1(t)/t) = a_1.$$

This gives, in particular, that γ is continuous at $t = 0$. It is also easy to see that, when γ is non-degenerate, $\gamma(t) \neq 0$ for all $t \neq 0$ sufficiently small. Functions with Taylor series as in Definition 4.1 are stable under composition by diffeomorphisms. The proof of this fact, that we present below for the sake of completeness, is literally the same as that for the analogous result in the C^∞ category.

Proposition 4.2. *Suppose that $\gamma : [0, \epsilon) \to \mathbb{R}^k$ has a non-degenerate Taylor series $\hat{\gamma}$ at $0 \in \mathbb{R}$. Let U and U' be neighborhoods of $0 \in \mathbb{R}^k$ such that U contains the image of γ. Let $\Phi : U \to U'$ be a C^∞ diffeomorphism with $\Phi(0) = 0$. Then the curve $\Phi \circ \gamma$ has a non-degenerate Taylor series at $0 \in \mathbb{R}^k$ which is given by the formal composition $\hat{\Phi} \circ \hat{\gamma}$, where $\hat{\Phi}$ is the Taylor series of Φ at $0 \in \mathbb{R}^k$.*

Proof. For a fixed $n \in \mathbb{N}$, we write the Taylor formula of order n for Φ at $0 \in \mathbb{R}^k$:

$$\Phi(h) = \Phi'(0) \cdot h + \frac{1}{2}\Phi''(0) \cdot h^{(2)} + \cdots + \frac{1}{n!}\Phi^{(n)}(0) \cdot h^{(n)} + r(h) = P_n(h) + r(h),$$

where $\Phi^{(j)} \in \mathcal{L}_j(\mathbb{R}^k, \mathbb{R}^k)$ denotes the j-th derivative of Φ, $h^{(j)} = (h, \ldots, h) \in (\mathbb{R}^k)^j$, and $r : U \subset \mathbb{R}^k \to \mathbb{R}^k$ is a map such that $\lim_{h\to 0} r(h)/||h||^n = 0$. Inserting (4.1), we find an expression of the form

$$\Phi \circ \gamma(t) = P_n \circ p_n(t) + L_n(t) \cdot \gamma_n(t) + r \circ \gamma(t),$$

where $L_n : [0, \epsilon) \to \mathcal{L}_1(\mathbb{R}^k, \mathbb{R}^k)$ is a map such that $\lim_{t\to 0} L_n(t) = L_n(0) = \Phi'(0)$. Denote by $p_n^*(t)$ the truncation of $P_n \circ p_n(t)$ in degree n and by $q_n^*(t)$ the remaining terms, all of them of degree at least $n + 1$. We then have a decomposition $\Phi \circ \gamma(t) = p_n^*(t) + \gamma_n^*(t)$, where $\gamma_n^*(t) = q_n^*(t) + L_n(t) \cdot \gamma_n(t) + r \circ \gamma(t)$. In order to prove the proposition, we have to check that $\lim_{t\to 0} \gamma_n^*(t)/t^n = 0$, which reduces to showing that $\lim_{t\to 0} r \circ \gamma(t)/t^n = 0$. But, since γ is non-degenerate, we can write

$$\lim_{t\to 0} \frac{r \circ \gamma(t)}{t^n} = \lim_{t\to 0} \frac{r(\gamma(t))}{||\gamma(t)||^n} \left\| \frac{\gamma(t)}{t} \right\|^n = 0,$$

which holds since $\lim_{t\to 0} \gamma(t)/t = a_1$. □

This proposition allows us to establish the following definition:

Definition 4.3. Let M be a C^∞ manifold of dimension $k \in \mathbb{N}$ and consider the C^∞ curve $\gamma : [0, \epsilon) \to M$ with $\gamma(0) = p \in M$. We say that γ is *pseudo-analytic* at $p \in M$ if, for some C^∞ chart ψ with $\psi(p) = 0 \in \mathbb{R}^k$, the curve $\psi \circ \gamma$ has a non-degenerate Taylor series at $0 \in \mathbb{R}^k$.

As a direct consequence of Proposition 4.2 we have:

Proposition 4.4. *Let M and M' be C^∞ manifolds of dimension $k \in \mathbb{N}$ and let $\Phi : M \to M'$ be a C^∞ diffeomorphism with $\Phi(p) = p'$. Suppose that $\gamma : [0, \epsilon) \to M$ is pseudo-analytic at $p = \gamma(0)$. Then $\Phi \circ \gamma$ is pseudo-analytic at $p' \in M'$.*

The advantage of defining pseudo-analytic curves in the more general setting of curves with Taylor series is that we gain their stability under real blow-ups and, as a consequence, the *iterated tangents* property as defined in [3]. This is formalized in the following:

Proposition 4.5. *Suppose that $\gamma : [0, \epsilon) \to M$ is injective and pseudo-analytic at $p = \gamma(0)$. Let $\pi : \tilde{M} \to M$ be the punctual real blow-up at $p \in M$. Then there exists $\tilde{p} \in \pi^{-1}(p)$ such the curve $\tilde{\gamma} = \pi^{-1} \circ \gamma : (0, \epsilon) \to \tilde{M}$ can be continuously extended by defining $\tilde{\gamma}(0) = \tilde{p}$. Moreover, the extended curve $\tilde{\gamma} : [0, \epsilon) \to \tilde{M}$ is injective and pseudo-analytic at $\tilde{p}$. Clearly, this proposition holds if π is any finite composition of real blow-ups at $p \in M$.*

Proof. Take C^∞ coordinates $(x_1, \ldots, x_k)$ at $p \in M$. Then $\gamma = (\gamma_1, \ldots, \gamma_k)$ has a Taylor series

$$\left(\sum_{j=\nu}^{\infty} a_1^j t^j, \ldots, \sum_{j=\nu}^{\infty} a_k^j t^j \right)$$

with $(a_1^\nu, \ldots, a_k^\nu) \neq 0$. Of course we can assume that $a_1^\nu \neq 0$. If $a_2^\nu \neq 0$, define the diffeomorphism

$$\psi\colon (x_1, \ldots, x_k) \mapsto (x_1, x_2 - \frac{a_2^\nu}{a_1^\nu} x_1, x_3, \ldots, x_k)$$

and consider

$$\psi \circ \gamma(t) = (\tilde{\gamma_1}(t), \tilde{\gamma_2}(t), \ldots, \tilde{\gamma_k}(t)).$$

Then it is easy to see that $\mathrm{ord}(\hat{\tilde{\gamma}}_2) > \nu$, where $\hat{\tilde{\gamma}}_2$ is the Taylor series of $\tilde{\gamma}_2$. Therefore, by changing coordinates if necessary we can assume that $a_1^\nu \neq 0$ and $a_j^\nu = 0$ for $j = 2, \ldots, k$. Thus, for $t > 0$ small we have

$$\pi^{-1} \circ \gamma(t) = \left(\gamma_1(t), \frac{\gamma_2(t)}{\gamma_1(t)}, \ldots, \frac{\gamma_k(t)}{\gamma_1(t)} \right),$$

which clearly tends to $(0, \ldots, 0)$ as $t \to 0$. Since γ_1 has a non-degenerate Taylor series, it suffices to show that $\frac{\gamma_j(t)}{\gamma_1(t)}$ has a Taylor series for $j = 2, \ldots, k$. We will show that the formal quotient $\sum\limits_{j=0}^{\infty} q_j t^j$ of $\sum\limits_{j=\nu}^{\infty} a_2^j t^j$ by $\sum\limits_{j=\nu}^{\infty} a_1^j t^j$ is the Taylor series of $\frac{\gamma_2}{\gamma_1}$ at $t = 0$; the other cases are equal. Fix $n \in \mathbb{N}$. It is sufficient to show that

$$R := \frac{\gamma_2(t)}{\gamma_1(t)} - \sum_{j=0}^{n} q_j t^j = o(t^n).$$

We can express

$$\gamma_1(t) = \sum_{j=\nu}^{\nu+n} a_1^j t^j + f_1(t),$$

$$\gamma_2(t) = \sum_{j=\nu}^{\nu+n} a_2^j t^j + f_2(t),$$

where $f_1(t), f_2(t) = o(t^{\nu+n})$. Then

$$\frac{R}{t^n} = \frac{\gamma_2(t) - \gamma_1(t) \sum\limits_{j=0}^{n} q_j t^j}{t^n \gamma_1(t)} \tag{4.2}$$

$$= \frac{\left(\sum_{j=\nu}^{\nu+n} a_2^j t^j + f_2(t) \right) - \left(\sum_{j=\nu}^{\nu+n} a_1^j t^j + f_1(t) \right) \sum\limits_{j=0}^{n} q_j t^j}{t^n \gamma_1(t)} \tag{4.3}$$

$$= \frac{o(t^{\nu+n})}{t^n \gamma_1(t)} = \frac{o(t^{\nu+n})}{t^{\nu+n}} \frac{1}{\gamma_1(t)/t^\nu} \to 0 \text{ as } t \to 0. \tag{4.4}$$

□

5. Pseudo-analytic curves in complex surfaces

Let V be a regular complex surface and consider a curve $\gamma : [0, \epsilon) \to V$ pseudo-analytic at $p = \gamma(0) \in V$. In local holomorphic coordinates at p the Taylor series of γ is given as

$$\hat{\gamma} = (\sum_{j=1}^{\infty} a_j t^j + i \sum_{j=1}^{\infty} b_j t^j, \sum_{j=1}^{\infty} c_j t^j + i \sum_{j=1}^{\infty} d_j t^j),$$

where $a_j, b_j, c_j, d_j \in \mathbb{R}$. Then, if we set $\alpha_j = a_j + i b_j$ and $\beta_j = c_j + i d_j$, we can write

$$\hat{\gamma} = (\sum_{j=1}^{\infty} \alpha_j t^j, \sum_{j=1}^{\infty} \beta_j t^j).$$

Thus, converting the real variable $t \in \mathbb{R}$ into a complex variable $z \in \mathbb{C}$, we can view the Taylor series of γ at $p \in V$ as the formal complex parametrized curve

$$(\sum_{j=1}^{\infty} \alpha_j z^j, \sum_{j=1}^{\infty} \beta_j z^j).$$

Definition 5.1. Let γ be a pseudo-analytic curve at $p \in V$ and let $\mathcal{C}$ be a formal parametrized complex curve at $p \in V$. We say that γ is *asymptotic to $\mathcal{C}$ at $p \in V$* if $\hat{\gamma}$ is a formal reparametrization of $\mathcal{C}$, that is, if there exists a formal invertible complex series $\hat{\psi} = \sum_{j=1}^{\infty} \sigma_j w^j$ such that $\hat{\gamma} = \mathcal{C} \circ \hat{\psi}$.

In particular, by taking $\hat{\psi}$ as the identity in the definition, we trivially have that a pseudo analytic curve γ is always asymptotic to the formal parametrized curve defined by its Taylor series $\hat{\gamma}$. The next result shows that a pseudo analytic curve in a complex surface has the *complex iterated tangents* property:

Proposition 5.2. *Let γ be an injective pseudo-analytic curve at $p \in V$, which is asymptotic to a formal complex curve $\mathcal{C}$ at $p \in V$. Let $\pi\colon \tilde{V} \to V$ be the punctual complex blow-up at $p \in V$. Then there exists $\tilde{p} \in \pi^{-1}(p)$ such that the curve $\tilde{\gamma} = \pi^{-1} \circ \gamma : (0, \epsilon) \to \tilde{V}$ can be continuously extended by defining $\tilde{\gamma}(0) = \tilde{p}$. Moreover:*

(1) the extended curve $\tilde{\gamma} : [0, \epsilon) \to \tilde{V}$ is injective and pseudo-analytic at $\tilde{p}$;
(2) $\tilde{\gamma}$ is asymptotic at $\tilde{p}$ to the strict transform of $\mathcal{C}$ by π.

Clearly, this proposition holds if π is any finite composition of complex blow-ups at $p \in V$.

Proof. Take holomorphic coordinates (z_1, z_2) at $p \in V$. Then $\gamma = (\gamma_1, \gamma_2)$ has a Taylor series

$$\left(\sum_{j=\nu}^{\infty} a_1^j t^j, \sum_{j=\nu}^{\infty} a_2^j t^j \right)$$

with $a_1^j, a_2^j \in \mathbb{C}$ and $(a_1^\nu, a_2^\nu) \neq 0$. Of course we can assume that $a_1^\nu \neq 0$. Moreover, as in the proof of Proposition 4.5, by changing coordinates if necessary we can assume that $a_2^\nu = 0$. Thus, for $t > 0$ small we have

$$\pi^{-1} \circ \gamma(t) = \left(\gamma_1(t), \frac{\gamma_2(t)}{\gamma_1(t)} \right),$$

which clearly tends to $(0, 0)$ as $t \to 0$. Since γ_1 has a non-degenerate Taylor series, in order to prove item (1) it suffices to show that $\frac{\gamma_2(t)}{\gamma_1(t)}$ has a Taylor series at $t = 0$. In fact, proceeding as in Proposition 4.5, we show that the Taylor series of $\frac{\gamma_2(t)}{\gamma_1(t)}$ is given by the formal quotient of $\sum_{j=\nu}^{\infty} a_2^j t^j$ by $\sum_{j=\nu}^{\infty} a_1^j t^j$ and this also implies item (2). $\square$

The iterated application of Proposition 5.2 allows us to associate to γ a sequence of infinitely near points $\{p_n\}_{n \geq 0}$, with $p_0 = p$, which coincides with the sequence of infinitely near points of the formal parametrized complex curve $\mathcal{C}$ to which it is asymptotic.

Definition 5.3. Let $\mathcal{F}$ be a one-dimensional holomorphic foliation on a regular complex surface V, with a singularity at $p \in V$. Consider a C^∞ curve $\gamma : [0, \epsilon) \to V$ with $\gamma(0) = p$. We say that γ is a *characteristic curve* of $\mathcal{F}$ at p if the following properties hold:

(1) γ is injective and pseudo-analytic at $p \in V$;
(2) $\gamma((0, \epsilon))$ is contained in a leaf of $\mathcal{F}$.

We emphasize that, in the above definition, the pseudo-analytic curve is required to be C^∞. The following is a simple exercise: if $\gamma : [0, \epsilon) \to \mathbb{R}^k$ is a C^∞ pseudo-analytic curve having Taylor series $\hat{\gamma}(t)$, then its curve of derivatives $D\gamma : [0, \epsilon) \to \mathbb{R}^k$ is C^∞ and has the formal derivative $D\hat{\gamma}(t)$ as Taylor series. This is used next in order to prove that a characteristic curve of a foliation is canonically asymptotic to a separatrix:

Proposition 5.4. *Let $\mathcal{F}$ be a one-dimensional holomorphic foliation on a complex regular surface V, with a singularity at $p \in V$. If $\gamma\colon [0,\epsilon) \to V$ is a characteristic curve of $\mathcal{F}$ at p, then $\hat{\gamma}$ is a parametrized formal separatrix of $\mathcal{F}$.*

Proof. Let ω be a holomorphic $1-$form defining $\mathcal{F}$ near $p \in V$. We work in holomorphic coordinates at $p \in V$ and consider the curve of derivatives $D\gamma : [0,\epsilon) \to V$ whose Taylor series is $D\hat{\gamma}$. Since $\gamma(t)$ is contained in a leaf of $\mathcal{F}$ for $t \in (0,\epsilon)$, we have that

$$\omega(\gamma(t)) \cdot D\gamma(t) = 0 \text{ for all } t \in [0,\epsilon).$$

Then the Taylor series of $\omega(\gamma(t)) \cdot D\gamma(t)$ at $t = 0$ is null, that is,

$$\hat{\omega}(\hat{\gamma}) \cdot D\hat{\gamma} = 0.$$

□

We finish this section by providing, in three examples, an analysis of characteristic curves according to the types of separatrices to which they are asymptotic.

Example 5.5. *Characteristic curves asymptotic to a dicritical separatrix.* Let γ be a characteristic curve asymptotic to a dicritical separatrix S of a foliation $\mathcal{F}$. After the desingularization of $\mathcal{F}$, the strict transforms of S and γ tend to a trace point in a dicritical component. Thus, the only possibility is that γ is a curve contained in S.

Example 5.6. *Characteristic curves asymptotic to a strong separatrix.* Let γ be a characteristic curve asymptotic to a strong separatrix S of a foliation $\mathcal{F}$. Let $\pi : (M,E) \to (\mathbb{C}^2,0)$ be the desingularization of $\mathcal{F}$ and denote by $\tilde{\mathcal{F}} = \pi^*\mathcal{F}$ the strict transform of $\mathcal{F}$. Then the strict transform $\tilde{S} = \pi^* S$ is a separatrix of some reduced singularity $p \in E$ of $\tilde{\mathcal{F}}$. Since the separatrix S is strong, by doing one more blow-up at p if it were a saddle-node, we can assume that the singularity at p is non-degenerate. Moreover, by performing some additional blow-ups if necessary, we can assume that the ratio of eigenvalues of the singularity at p has a negative real part. Thus, we can take holomorphic local coordinates (u,v) at p such that:

(1) the foliation $\tilde{\mathcal{F}}$ at p is generated by a 1-form

$$\omega = udv - vQ(u,v)du,$$

where $Q(u,v) = \lambda + \ldots$, $\mathrm{Re}(\lambda) < 0$;

(2) E is given by $\{u = 0\}$;

(3) $\tilde{S}$ is given by $\{v = 0\}$.

Let $\tilde{\gamma}$ be the strict transform of γ by π. By Proposition 5.2, $\tilde{\gamma}$ is a characteristic curve of $\tilde{\mathcal{F}}$ asymptotic to $\tilde{S}$. We will prove that $\tilde{\gamma}$ is contained in $\tilde{S}$. If we express $\tilde{\gamma}(t) = (u(t),v(t))$, $t \in [0,\epsilon)$, since $\tilde{\gamma}$ is tangent to the foliation $\tilde{\mathcal{F}}$, we have that

$$u(t)v'(t) - v(t)Q\big(u(t),v(t)\big)u'(t) = 0.$$

Then, if we define $r(t) = |v(t)|^2$, a straightforward computation gives us that

$$r' = 2|v|^2\mathrm{Re}\big(\frac{u'}{u}Q\big).$$

Since $\tilde{\gamma}$ has a non-zero Taylor series and $\tilde{\gamma}$ is asymptotic to $\{v = 0\}$, we see that $u(t)$ has a non-zero Taylor series $\hat{u}(t) = \sum_{j \geq n} a_j t^j$, $a_j \in \mathbb{C}$, $a_n \neq 0$, $n \in \mathbb{N}$. From this we easily obtain that

$$\frac{u'(t)}{u(t)} = \frac{1}{t}\big(n + o(t)\big)$$

and therefore

$$r' = 2|v|^2 \frac{1}{t} \mathrm{Re}\Big(\big(n + o(t)\big)Q\Big)$$

for all $t \in (0, \epsilon)$. Suppose that $\tilde{\gamma}$ is not contained in $\tilde{S}$. Then we have $|v(t)| > 0$ for all $t \in (0, \epsilon)$. Thus, since

$$\mathrm{Re}\Big(\big(n + o(t)\big)Q\Big) \to n\mathrm{Re}(\lambda) < 0,$$

for $t > 0$ small enough we have that $r'(t) < 0$. But this is a contradiction, since that $r(0) = 0$ and $r(t) > 0$ for $t \in (0, \epsilon)$. Therefore, we conclude that a strong separatrix contains all its asymptotic characteristic curves.

Example 5.7. *Characteristic curves asymptotic to a weak separatrix.* Consider a saddle-node foliation $\mathcal{F}$ at $0 \in \mathbb{C}^2$ whose strong separatrix is contained in $\{(u, v) \colon u = 0\}$. Then there is a formal series $\hat{s}(u) = \sum_{j=1}^{\infty} c_j u^j$ such that the weak separatrix S of $\mathcal{F}$ is given by $v = \hat{s}(u)$. It is known that there exists a constant $\vartheta > 0$ depending only on the analytic type of the saddle-node such that, given $\eta \in \mathbb{C}^*$, we can find a holomorphic function f defined on a sector of the form

$$V = \{re^{i\theta}\eta \in \mathbb{C} : 0 < r < \epsilon, \, -\vartheta < \theta < \vartheta\} \quad (\epsilon > 0)$$

such that:

(1) the graph $\{(u, f(u)) : u \in V\}$ is contained in a leaf of the foliation $\mathcal{F}$;

(2) the function f has the series $\sum\limits_{j=1}^{\infty} c_j u^j$ as asymptotic expansion at $0 \in \mathbb{C}$.

Define

$$\gamma(t) = (\eta t, f(\eta t))$$

for $t \geq 0$ small and observe that the Taylor series of γ at $t = 0$ is given by $(\eta t, \hat{s}(\eta t))$. Therefore γ is a characteristic curve of $\mathcal{F}$ asymptotic to the separatrix S. We remark that the function f above can be non-unique, for example if η corresponds to a "node" direction of the saddle-node. Thus, even if the weak separatrix is convergent, it does not contain all its asymptotic characteristic curves.

6. Correspondence of separatrices by a C^∞ equivalence

The main result of this section is the following:

Theorem 6.1. *Let Φ be a C^∞ equivalence between two germs $\mathcal{F}$ and $\mathcal{F}'$ of singular holomorphic foliations at $(\mathbb{C}^2, 0)$. Then, given a formal separatrix S of $\mathcal{F}$, there exists a unique formal separatrix S' of $\mathcal{F}'$, denoted by $S' = \Phi_*(S)$, such that for any characteristic curve γ of $\mathcal{F}$ asymptotic to S we have that $\gamma' = \Phi(\gamma)$ is a characteristic curve of $\mathcal{F}'$ asymptotic to S'.*

If $S \in \mathrm{Sep}(\mathcal{F})$ and γ is a characteristic curve of $\mathcal{F}$ asymptotic to S, it is straightforward from the definitions that $\gamma' = \Phi(\gamma)$ is a characteristic curve of $\mathcal{F}'$. Thus, by Proposition 5.4, the formal series $\hat{\gamma}'$ is a separatrix of $\mathcal{F}'$. However, it is not immediately clear that the element defined by $\hat{\gamma}'$ in $\mathrm{Sep}(\mathcal{F}')$ is independent of γ. It is worth remarking that Theorem 6.1 depends heavily on the fact that the real diffeomorphism Φ is an equivalence between holomorphic foliations. For instance, let Φ be any real linear diffeomorphism of $\mathbb{C}^2$ such that $\Phi(x + iy, 0) = (x, y) \in \mathbb{C}^2$, $x, y \in \mathbb{R}$. Then the curves $\gamma_a(t) = (t + ati, 0)$, with $a \in \mathbb{R}$, are all asymptotic to the first complex axis. However, their images by Φ are the curves $\gamma_a'(t) = (t, at)$, which are asymptotic to complex lines in a non-constant family. As we will see, the statement of Theorem 6.1 is non-trivial only when S is a weak separatrix. In this case, the main point in its proof is to show that the map $\gamma \mapsto \hat{\gamma}' = \widehat{\Phi(\gamma)}$, defined in the family of characteristic curves of $\mathcal{F}$ asymptotic to S and taking values in the finite set $\mathrm{Iso}(\mathcal{F}')$ of isolated separatrices of $\mathcal{F}'$, is continuous and, hence, constant. Before passing to the proof of the theorem, we state two preliminary results:

Lemma 6.2. *Fix $\nu \in \mathbb{N}$. Then, for each $j \in \mathbb{N}$ there exists a complex polynomial P_j in $2j+1$ variables such that, if*

$$\hat{S} = (\sum_{j=\nu}^{\infty} a_j z^j, \sum_{j=\nu}^{\infty} b_j z^j)$$

is a parametrized formal complex curve with $a_\nu = c^\nu$, $c \neq 0$, and we set

$$\sigma_j = P_j(c, \frac{1}{c}, a_{\nu+1}, \ldots, a_{\nu+j-1}, b_\nu, \ldots, b_{\nu+j-1}),$$

then S can be reparametrized as

$$(x^\nu, \sum_{j=\nu}^{\infty} \sigma_j x^j),$$

that is, there is an invertible series $\hat{\psi}(x)$ such that $\hat{S}(\hat{\psi}(x)) = (x^\nu, \sum_{j=\nu}^{\infty} \sigma_j x^j)$.

Proof. This lemma is only a Puiseux's Parametrization, putting in evidence the dependence of the final coefficients in terms of the initial ones. □

Proposition 6.3. *Let Φ be a C^∞ equivalence between two holomorphic foliations with isolated singularity at $0 \in \mathbb{C}^2$ and let $J\colon \mathbb{C}^2 \to \mathbb{C}^2$ be the complex conjugation. Then either $D\Phi(0)$ or $D\Phi(0) \circ J$ is a $\mathbb{C}$-linear isomorphism of $\mathbb{C}^2$.*

Proof. See [10, Lemma 4.3].

□

Proof of Theorem 6.1. Let γ be a characteristic curve of $\mathcal{F}$ asymptotic to $S \in \mathrm{Sep}(\mathcal{F})$. As we commented above, $\Phi(\gamma)$ is a characteristic curve of $\mathcal{F}'$ asymptotic to the formal separatrix given by its Taylor series. If S is convergent, standard arguments prove that $\Phi(S)$ is also a convergent separatrix. Hence, if we take $S' = \Phi(S)$, the proof of the theorem will be easy in the following cases:

(1) S is a dicritical separatrix;
(2) S is a strong separatrix.

In both cases, if γ is a characteristic curve asymptotic to S, by examples 5.5 and 5.6 we have that $\gamma \subset S$. Since $\Phi(\gamma)$ is contained in S', then $\Phi(\gamma)$ is a characteristic curve asymptotic to S'.

We begin the proof of the remaining case. Let $\pi : (M, E) \to (\mathbb{C}^2, 0)$ be the reduction of singularities of $\mathcal{F}$. Then, the strict transform $\tilde{S} = \pi^* S$ is the weak separatrix of a saddle-node singularity at some trace point $p \in E$. Clearly the strong separatrix at p is contained in E. Let (u, v) be local holomorphic coordinates at $p \in E$ such that:

- $p \simeq (0, 0)$;
- E is given by $\{u = 0\}$.

Then there exists a formal series $\hat{s}(u) = \sum_{j=1}^{\infty} c_j u^j$ such that $\tilde{S}$ is given by $v = \hat{s}(u)$. Let γ be a characteristic curve asymptotic to S and let $\tilde{\gamma}(t) = (u(t), v(t))$ be the strict transform of γ by π. Then $\tilde{\gamma}$ has a non-degenerate Taylor series given by

$$(\hat{u}, \hat{v}) = \left(\sum_{j=1}^{\infty} u_j t^j, \sum_{j=1}^{\infty} v_j t^j \right), \text{ where } u_j, v_j \in \mathbb{C}.$$

Since $\tilde{\gamma}$ is asymptotic to $\tilde{S}$ at p — which means that $\tilde{\gamma}$ is a formal reparametrization of $\tilde{S}$ — we deduce that $u_1 \neq 0$ and $\hat{v} = \hat{s} \circ \hat{u}$. Let $\hat{\pi} = \hat{\pi}_\nu + \hat{\pi}_{\nu+1} + \ldots$ be the Taylor series of π at p in the coordinates (u, v), where the $\hat{\pi}_\nu$ is the initial part of π. It is easy to see that the initial part of $\hat{\tilde{\gamma}} = (\hat{u}, \hat{s} \circ \hat{u})$ is given by $\hat{\tilde{\gamma}}_1 = (u_1 t, c_1 u_1 t)$. Then the initial part of $\hat{\gamma} = \hat{\pi} \circ \hat{\tilde{\gamma}}$ is $\hat{\pi}_\nu(u_1, c_1 u_1)t^\nu$. Since Φ is a diffeomorphism, its initial part $\hat{\Phi}_1$ is an isomorphism, so

$$\hat{\Phi}_1 \circ \hat{\pi}_\nu(u_1, c_1 u_1) \neq 0.$$

Then the initial part of $\hat{\gamma'} = \hat{\Phi} \circ \hat{\gamma}$ is

$$\hat{\gamma'}_\nu = \hat{\Phi}_1 \circ \hat{\pi}_\nu(u_1, c_1 u_1)t^\nu.$$

Thus, we can write

$$\hat{\gamma'} = (\sum_{j=\nu}^{\infty} a_j t^j, \sum_{j=\nu}^{\infty} b_j t^j), \tag{6.1}$$

where the coefficients a_j and b_j are polynomials in the coefficients of $\operatorname{Re}(\hat{u})$ and $\operatorname{Im}(\hat{u})$ and $(a_\nu, b_\nu) \neq 0$. If $J : \mathbb{C}^2 \to \mathbb{C}^2$ is the complex conjugation, by Proposition 6.3 we have that either Φ_1 or $\Phi_1 \circ J$ is a $\mathbb{C}$-linear isomorphism. Then there is $(a, b) \in \mathbb{C}^2 \backslash \{0\}$ such that

- $(a_\nu, b_\nu) = (au_1^\nu, bu_1^\nu)$, or
- $(a_\nu, b_\nu) = (a\bar{u_1}^\nu, b\bar{u_1}^\nu)$.

Both cases are similar, so we only deal with the first one. Of course we can suppose that $a \neq 0$, so $a_\nu \neq 0$ for all $u_1 \in \mathbb{C}^*$. Since γ' is a characteristic curve of $\mathcal{F}'$, by Proposition 5.4, the formal curve

$$\hat{\gamma'} = (\sum_{j=\nu}^{\infty} a_j z^j, \sum_{j=\nu}^{\infty} b_j z^j)$$

is a parametrization of a formal separatrix $S'_{\hat{u}}$ of $\mathcal{F}'$. Moreover, $S'_{\hat{u}}$ is an isolated separatrix, otherwise γ' should be contained in a dicritical separatrix of $\mathcal{F}'$. We can apply Lemma 6.2 in order to obtain a parametrization

$$S'_{\hat{u}} = (x^\nu, \sum_{j=\nu}^{\infty} \sigma_j x^j),$$

where the coefficients σ_j are polynomials in $1/u_1$ and in the coefficients of $\operatorname{Re}(\hat{u})$ and $\operatorname{Im}(\hat{u})$. Consider the map

$$\phi \colon \hat{u} \mapsto (\sigma_\nu, \sigma_{\nu+1}, \ldots),$$

whose domain consists of all formal series $\hat{u}$ obtained as above. Clearly we can identify the set of formal complex series in one variable with $\mathbb{C}^\mathbb{N}$. Then the function ϕ is defined in some subset $\hat{U}$ of $\mathbb{C}^* \times \mathbb{C}^\mathbb{N}$ and takes values in the set

$$\Sigma = \{(\sigma_\nu, \sigma_{\nu+1}, \ldots) \in \mathbb{C}^\mathbb{N} \colon (x^\nu, \sum_{j=\nu}^{\infty} \sigma_j x^j) \in \operatorname{Iso}(\mathcal{F}')\},$$

where $\operatorname{Iso}(\mathcal{F}')$ is the set of isolated separatrices of $\mathcal{F}'$. Observe that the set Σ is finite and ϕ is continuous if we consider the product topology in $\mathbb{C}^\mathbb{N}$. Then it is sufficient to prove that $\hat{U}$ is connected, because in this case the map ϕ is constant and we can define $S' = S'_{\hat{u}}$ for any $\hat{u} \in \hat{U}$. We will prove that $\hat{U}$ is path connected. From Example 5.7, for any $u_1 \in \mathbb{C}^*$ there exists a characteristic curve of $\mathcal{F}$ whose strict transform by π has a Taylor series at $p \in M$ given by $(u_1 t, \hat{s}(u_1 t))$. This shows that $C^* := \mathbb{C}^* \times \{0\}^\mathbb{N}$ is contained in $\hat{U}$. Since this set is path connected, it suffices to show that any $\hat{u} \in \hat{U}$ can be connected to some point in C^* by a continuous path. Fix $\hat{u} = (u_1, \ldots) \in \hat{U}$. Then there exists a characteristic curve γ of $\mathcal{F}$ such that its strict transform $\tilde{\gamma} = (u(t), v(t))$ by π has $(\hat{u}, \hat{s} \circ \hat{u})$ as its Taylor series at $p \in M$. We can assume that the image of $u(t)$ is contained in a sector of the form

$$V = \{re^{i\theta} \in \mathbb{C} : 0 < r < \epsilon,\ a < \theta < b\}\ (\epsilon, a, b > 0)$$

such that there exists a function $f \in \mathcal{O}(V)$ with the following properties:

- the graph $\{(u, f(u)) : u \in V\}$ is contained in a leaf of the foliation;
- the function f has the series $\hat{s} = \sum_{j=1}^{\infty} c_j u^j$ as asymptotic expansion at $0 \in \mathbb{C}$.

Consider $\tilde{\gamma_0}(t) = (u(t), f(u(t)))$ and $\tilde{\gamma_1}(t) = (u_1 t, f(u_1 t))$ and observe the following:

- $\tilde{\gamma_0}$ and $\tilde{\gamma_1}$ are the strict transforms by π of characteristic curves of $\mathcal{F}$ asymptotic to S;
- $\tilde{\gamma_0}$ has $(\hat{u}, \hat{s} \circ \hat{u})$ as its Taylor series;
- If $\hat{u}_1 := (u_1, 0, \dots) \in C^*$, then $\tilde{\gamma_1}$ has $(\hat{u}_1, \hat{s} \circ \hat{u}_1)$ as its Taylor series.

Define the family of curves

$$\Gamma_s(t) = \Big((1-s)u(t) + su_1 t, f\big((1-s)u(t) + su_1 t\big)\Big),\ \ s \in [0, 1].$$

It is easy to see that

- $\Gamma_0 = \tilde{\gamma_0}$ and $\Gamma_1 = \tilde{\gamma_1}$;
- each Γ_s is the strict transform of a characteristic curve asymptotic to S;
- $\hat{\Gamma_s} = (\hat{u}_s, \hat{s} \circ \hat{u}_s)$, where $\hat{u}_s = u_1 + \sum_{j=2}^{\infty} (1-s) u_j t^j$.

Then $\hat{u}_s$ defines a continuous path connecting $\hat{u}$ to $\hat{u}_1 \in C^*$. □

7. Formal real equivalence and equisingularity for curves

Let Φ be a C^∞ germ of diffeomorphism of $(\mathbb{C}^2, 0)$. In the previous section, we dealt with the correspondence between formal real parametrized curves at $(\mathbb{C}^2, 0)$ given by $\hat{\gamma} \mapsto \hat{\gamma}' = \Phi \circ \hat{\gamma}$, which determines a correspondence between formal complex parametrized curves simply by the formal replacement of the real parameter by a complex one. In full generality, this procedure does not preserve equisingularity classes. For instance, the germ of diffeomorphism defined by

$$\Phi(x_1 + iy_1, x_2 + iy_2) = (x_1 + iy_2, y_1 + ix_2)$$

induces, in the above way, a correspondence between the non-equisingular complex parametrized curves $\hat{\gamma}(z) = ((1+i)z^2, (1+i)z^3)$ and $\hat{\gamma}'(z) = (z^2 + iz^3, z^2 + iz^3)$ (which is $z \to (z, z)$, after reparametrization). However, when Φ is a C^∞ equivalence between two germs of holomorphic foliations $\mathcal{F}$ and $\mathcal{F}'$, the correspondence Φ_* between separatrices provided by Theorem 6.1 preserves equisingularity classes. In order to prove this, we first introduce the concept of formal real equivalence for formal complex curves and prove, in Theorem 7.2 below, that this notion implies the equisingularity property. Eventually, in Theorem 8.1 of the next section, we achieve our goal by proving that separatrices which corresponds by Φ_* are actually formally real equivalent.

We first set a definition: a *formal parametrized real surface* at $(\mathbb{C}^2, 0)$ is a non-zero series in two variables of the form

$$\sum_{j,k \in \mathbb{N}} a_{jk} x^j y^k, \tag{7.1}$$

where $a_{jk} \in \mathbb{C}^2$ for all $j, k \in \mathbb{N}$. When x, y are real coordinates and the series is convergent, this object can be interpreted as the map $(x, y) \in \mathbb{R}^2 \mapsto \sum_{j,k \in \mathbb{N}} a_{jk} x^j y^k \in \mathbb{C}^2$, which defines a parametrized real analytic surface. Naturally, a formal parametrized complex curve $\sum_{j \in \mathbb{N}} \alpha_j z^j$ $(\alpha_j \in \mathbb{C}^2)$ is also a formal parametrized real surface if we do the substitution $z = x + iy$.

A *formal real reparametrization* of the surface (7.1) is any series obtained by a substitution $(x,y)=\Psi(\bar{x},\bar{y})$, where Ψ is a formal diffeomorphism of $(\mathbb{R}^2,0)$.

Definition 7.1. Let $\hat{\Phi}$ be a formal diffeomorphism of $(\mathbb{R}^4,0)$. Let $\sigma(z)=\sum_{j\in\mathbb{N}}\sigma_j z^j$ and $\sigma'(z)=\sum_{j\in\mathbb{N}}\sigma'_j z^j$ be two formal parametrized irreducible complex curves at $(\mathbb{C}^2,0)$. We say that $\hat{\Phi}$ is a *formal real equivalence* between σ and σ' if the formal parametrized real surface $\hat{\Phi}\circ\sigma$ is a formal real reparametrization of σ'. In this situation we also say that σ and σ' are *formally real equivalent* by $\hat{\Phi}$. In general, we say that $\hat{\Phi}$ is a formal real equivalence between two reduced formal complex curves $\mathscr{C}$ and $\mathscr{C}'$ at $(\mathbb{C}^2,0)$ if there is a bijection between the irreducible components of $\mathscr{C}$ with the irreducible components of $\mathscr{C}'$ such that each pair of corresponding irreducible components are formally real equivalent by $\hat{\Phi}$.

Theorem 7.2. *Let $\hat{\Phi}$ be a formal real equivalence between two germs of reduced formal complex curves $\mathscr{C}$ and $\mathscr{C}'$ at $(\mathbb{C}^2,0)$. Then $\mathscr{C}$ and $\mathscr{C}'$ are equisingular.*

Proof. Let $\xi^1,\ldots,\xi^m$ and $\xi'^1,\ldots,\xi'^m$ be the irreducible components of $\mathscr{C}$ and $\mathscr{C}'$ respectively and assume that $\hat{\Phi}$ maps ξ^k to ξ'^k for $k=1,\ldots,m$. Let

$$\sigma_k(z)=\sum_{j\geq 1}a_j(k)z^j,\ a_j(k)\in\mathbb{C}^2$$

be a formal parametrization of ξ^k. Then $\hat{\Phi}\circ\sigma_k$ is a real formal parametrization of the irreducible component ${\xi'}^k$ of $\mathscr{C}'$, that is, there exists a real formal diffeomorphism $\psi_k\colon(\mathbb{C},0)\to(\mathbb{C},0)$ such that $\hat{\Phi}\circ\sigma_k\circ\psi_k(z)$ is a complex formal parametrization of ${\xi'}^k$. Given $n\in\mathbb{N}$, let ξ_n^k and ${\xi'}_n^k$ be the complex curves defined by the n-jets of $\sigma_k(z)$ and $\hat{\Phi}\circ\sigma_k\circ\psi_k(z)$ respectively. Let $\mathscr{C}_n$ and $\mathscr{C}'_n$ be the reduced curves whose irreducible components are $\{\xi_n^k\colon k=1,\ldots,m\}$ and $\{{\xi'}_n^k\colon k=1,\ldots,m\}$, respectively. We know that for n large enough:

(1) $\mathscr{C}$ and $\mathscr{C}_n$ are equisingular;
(2) $\mathscr{C}'$ and $\mathscr{C}'_n$ are equisingular.

Then it is sufficient to prove that the analytic curves $\mathscr{C}_n$ and $\mathscr{C}'_n$ are topologically equivalent for n large enough. For the sake of simplicity we denote ξ^1, ${\xi'}^1$, ξ_n^1, ${\xi'}_n^1$, σ_1 and ψ_1 by ξ, ξ', ξ_n, ξ'_n, σ and ψ, respectively. Then $\hat{\Phi}\circ\sigma\circ\psi(z)$ is a complex formal parametrization of ξ' and ξ_n is defined by the n-jet σ_n of σ. Since the curves ξ_n and ξ'_n are analytic, we will use the same notation for the sets defined by these curves. If Φ_n is the n-jet of Φ, the singular real surface $\mathscr{S}$ given by $\Phi_n(\xi_n)$ is asymptotic to $\hat{\Phi}\circ\sigma\circ\psi(z)$ up to order n. In fact, if we consider the n-jet ψ_n of ψ, the real parametrization $\Phi_n\circ\sigma_n\circ\psi_n(z)$ of $\Phi_n(\xi_n)$ has a Taylor series coinciding with the Taylor series of $\hat{\Phi}\circ\sigma\circ\psi(z)$ up to order n. After a finite sequence of complex blow-ups $\pi\colon(M,E)\to(\mathbb{C}^2,0)$, the strict transform $\tilde{\xi'}$ of ξ' is a regular formal curve transverse to the exceptional divisor E at a point p. Let (u,v) be holomorphic coordinates on a neighborhood of p such that:

- $p\simeq(0,0)$;
- the exceptional divisor E is given by $\{u=0\}$;
- the curve $\tilde{\xi'}$ is given by a formal equation $v=\sum\limits_{j\geq 1}c_j u^j$.

The following properties hold for n large enough:

- the strict transform $\tilde{\xi}'_n$ of ξ'_n by π intersects E at the point p and is given by an analytic equation of the form $v=\zeta(u)=c_1u+o(u)$ near p;
- the strict transform $\tilde{\mathscr{S}}$ of $\mathscr{S}$ by π intersects E at the point p and is given by a C^∞ equation of the form $v=f(u)=c_1u+o(u)$ near p.

Given $\epsilon > 0$, there is a set $D = \{|u| \leq a, |v| \leq b\}$, with $0 < a, b < \epsilon$, such that

- $\tilde{\xi'_n} \cap D = \{v = \zeta(u); |u| \leq a\}$;
- $\tilde{\mathscr{S}} \cap D = \{v = f(u); |u| \leq a\}$;
- $|\zeta(u)|, |f(u)| < b$ for all $|u| \leq a$.

We have the following:

Claim. There is a homeomorphism $\tilde{h}\colon D \to D$ satisfying $\tilde{h}(\tilde{\mathscr{S}} \cap D) = \tilde{\xi'_n} \cap D$ such that $\tilde{h}(u,v) = (u,v)$ if $|v| = b$.

Proof. Without loss of generality, we can suppose that $\zeta(u) = 0$ for all $|u| \leq a$. For each u, with $|u| \leq a$, consider the complex disc $\mathbb{D}_u = \{(u,v) \in \mathbb{C}^2; |v| \leq b\}$. It suffices to produce a continuous family of homeomorphisms $h_u : \mathbb{D}_u \to \mathbb{D}_u$ such that $h_u(f(u)) = 0$ and $h_u(v) = v$ if $|v| = b$. We then have our claim by setting $\tilde{h}(u,v) = (u, h_u(v))$. In order to produce this family, fix $b' < b$ such that $|f(u)| \leq b'$ for all u and take a bump function $\rho : [0,\infty) \to \mathbb{R}$ such that $\rho \equiv 1$ in $[0,b')$, $\rho > 0$ in $[0,b)$ and $\rho = 0$ in $[b,\infty)$. Define a (real) vector field on $\mathbb{D}_u$ by $\vec{X}_u(v) = -\rho(|v|)f(u)$. We can take h_u as the flow of this vector field at time $t = 1$. □

The above homeomorphism is extended as a homeomorphism between two neighborhoods of E by setting $\tilde{h} = \mathrm{id}$ outside D, where id stands for the identity map. Then the map $h = \pi \circ \tilde{h} \circ \pi^{-1}$ defines a homeomorphism between two neighborhoods U_1 and U_2 of $0 \in \mathbb{C}^2$ such that:

- $h(\mathscr{S} \cap U_1) = \xi'_n \cap U_2$;
- $h(\pi(D)) = \pi(D)$ and $h = \mathrm{id}$ outside $\pi(D)$.

Thus, the map $\mathfrak{h} = h \circ \Phi_n$ is a topological equivalence between ξ_n and ξ'_n. Moreover, since $\mathfrak{h}$ coincides with Φ_n outside $\pi(D)$, a similar construction as above can be successively made in an infinitesimal neighborhood of each irreducible component of $\mathscr{C}$ in order to obtain, for n large enough, a topological equivalence $\mathfrak{h}$ between $\mathscr{C}$ and $\mathscr{C}'$. □

We close this section by establishing a kind of "factorization" theorem for a real parametrization of an irreducible complex curve. In more precise terms, suppose that ξ is an irreducible curve at $(\mathbb{C}^2, 0)$ defined by the formal equation $F(u,v) = 0$. Let

$$\Gamma = (f(x,y), g(x,y))$$

be a formal parametrized real surface at $(\mathbb{C}^2, 0)$ whose "image" is contained in ξ, that is, such that $F(f,g) = 0$. Then, Lemma 7.3 asserts that Γ is a formal real reparametrization of a Puiseux parametrization of ξ. This result and its Corollary 7.4 will be important in the proof of Theorem I.

Lemma 7.3. *Let F be an irreducible element in $A = \mathbb{C}[[x,y]]$ and let $\sigma \in \mathbb{C}[[z]]$, $n \in \mathbb{N}$ be such that $(z^n, \sigma(z))$ is a Puiseux parametrization for the formal curve $F = 0$. Let $f, g \in A$ be such that $F(f,g) = 0$. Then there exists a series $\psi \in A$ such that*

$$(f,g) = \big(\psi^n, \sigma(\psi)\big).$$

Proof. We will first show that it is sufficient to prove that f has an n^{th} root in A. Suppose that there exists $\phi \in A$ such that $\phi^n = f$. By Puiseux's Theorem we have that

$$F(t^n, y) = U \prod_{\xi^n = 1} \big(y - \sigma(\xi t)\big), \tag{7.2}$$

where U is a unit in $\mathbb{C}[[t,y]]$. Since $F(\phi^n, g) = F(f,g) = 0$, we conclude from equation (7.2) that $g = \sigma(\xi\phi)$ for some ξ such that $\xi^n = 1$. Therefore it suffices to take $\psi = \xi\phi$.

Let us prove that f has an n^{th} root in A. We exclude the trivial case $n = 1$ and suppose by contradiction that f has no n^{th} root in A. Denote by Q the field of fractions of A. At first we will show that, without loss of generality, we can assume that the polynomial $z^n - f$ is irreducible in $Q[z]$. Let $d \in \mathbb{N}$ be the greatest divisor of n such that f has a d^{th} root in A. Then there exists $\tilde{f} \in A$ such that $f = \tilde{f}^d$. Since $F(\tilde{f}^d, g) = 0$, for some irreducible factor $\tilde{F}$ of $F(x^d, y)$ we have $\tilde{F}(\tilde{f}, g) = 0$. If we set $\tilde{n} = \frac{n}{d}$, since $f = \tilde{f}^d$ has no n^{th} root in A, we have that $\tilde{f}$ has no $\tilde{n}^{\text{th}}$ root in A. Therefore, we have that $\tilde{F}$, $\tilde{f}$ and g satisfy the hypothesis of the lemma and $\tilde{f}$ has no $\tilde{n}^{\text{th}}$ root in A. Moreover, from the maximality of d we see that, for any divisor $k \neq 1$ of $\tilde{n}$, the series $\tilde{f}$ has no k^{th} root in A. Thus, without loss of generality we can assume that f has no k^{th} root in A for any divisor $k \neq 1$ of n. This implies that, for any divisor $k \neq 1$ of n, the element f has no k^{th} root in the field Q of fractions of A. From this we conclude that the polynomial $z^n - f$ is irreducible in $Q[z]$ (see, for instance, [6, Ch. VI.9]).

Given any $h \in A$, define $h^*(t,x) = h(x, tx)$. If we write $h = \sum_{j=0}^{\infty} h_j$, where h_j is a homogeneous polynomial of degree j in $\mathbb{C}[x,y]$, we obtain that

$$h^*(t,x) = \sum_{j=0}^{\infty} h_j(1,t)x^j.$$

Notice that $h_j(1,t)$ is a polynomial of degree at most j, so that the map $h \mapsto h^*$ defines an isomorphism from A into a ring A^* contained in the ring $K[[x]]$ of formal power series with coefficients in the field $K = \mathbb{C}(t)$ of complex rational functions in the variable t. In particular, if $f = \sum\limits_{j \geq \nu} f_j$ with $f_\nu \neq 0$, we obtain that

$$f^* = \sum_{j \geq \nu}^{\infty} f_j(1,t)x^j, \quad f_\nu(1,t) \neq 0.$$

If $\bar{K}$ is the algebraic closure of K, we know that the series f^* has an n^{th} root ψ in $\bar{K}[[x]]$. Then ψ is a root of the polynomial $z^n - f^* \in A^*[z]$. Since the polynomial $z^n - f$ is irreducible in $Q[z]$, it follows that n is the minimum degree of a non-zero polynomial in $A^*[z]$ having ψ as a root. Since $\mathbb{C} \subset \bar{K}$, we will consider F as an element in $\bar{K}[[x,y]]$. Then, since $F(f^*, g^*) = 0$ and $f^* = \psi^n$, it follows from Puiseux's Theorem in $\bar{K}[[x,y]]$ that $g^* = \sigma(\xi\psi)$ for some ξ, $\xi^n = 1$. Without loss of generality we can assume that $\xi = 1$. If we do the substitution $\psi^n = f^*$ in the equation $g^* = \sigma(\psi)$, for some series $\sigma_1, \ldots, \sigma_{n-1} \in \mathbb{C}[[z]]$ we obtain an equation of the form

$$-g^* + \sigma_1(f^*)\psi + \ldots + \sigma_{n-1}(f^*)\psi^{n-1} = 0.$$

Thus, since $\sigma_j(f^*) = \big(\sigma_j(f)\big)^* \in A^*$, we have that ψ is a root of the polynomial

$$P = -g^* + \sigma_1(f^*)z + \ldots + \sigma_{n-1}(f^*)z^{n-1} \in A^*[z].$$

Then, since n is the minimum degree of a polynomial in $A^*[z]$ vanishing on ψ, we conclude that $P = 0$. Then $g = 0$ and consequently we have the equation $F(f, 0) = 0$. Therefore, if we express

$$F(x,y) = \sum_{j \geq 0} s_j(x)y^j$$

with $s_j(x) \in \mathbb{C}[[x]]$, we obtain that $s_0(f) = 0$. This implies that $s_0 = 0$, because $f \neq 0$. Then, since F is irreducible, we have that $F = Uy$ for some unit $U \in A$. But this implies that $n = 1$, which is a contradiction. □

Let F be an irreducible element in $\mathbb{C}[[x,y]]$. We say that a formal parametrized complex curve

$$\Gamma(z) = \sum_{j \in \mathbb{N}} a_j z^j,\ a_j \in \mathbb{C}^2$$

is a complex parametrization of the curve $F = 0$ if $\Gamma \neq 0$ and we have $F(\Gamma(z)) = 0$. We say that the complex parametrization Γ is reducible if there exist another formal parametrized complex curve $\tilde{\Gamma}$ and an element $\varphi \in \mathbb{C}[[z]]$ with $\mathrm{ord}(\varphi) > 1$ such that $\Gamma(z) = \tilde{\Gamma}(\varphi(z))$. Otherwise we say that Γ is an irreducible complex parametrization of $F = 0$. As a consequence of Lemma 7.3, we have:

Corollary 7.4. *Let F be an irreducible element in $\mathbb{C}[[x, y]]$ and let $\sigma \in \mathbb{C}[[z]]$, $n \in \mathbb{N}$ be such that $(z^n, \sigma(z))$ is a Puiseux parametrization for the formal curve $F = 0$. Let Γ be any irreducible complex parametrization of $F = 0$. Then there exists a formal complex diffeomorphism $\varphi \in \mathbb{C}[[z]]$ such that*

$$\Gamma = \big(\varphi^n, \sigma(\varphi)\big).$$

Proof. Let $\Gamma = (f, g)$, where $f, g \in \mathbb{C}[[z]]$. Since $F(f, g) = 0$ and (f, g) can be considered as a formal real surface, by Lemma 7.3 there exists $\psi \in \mathbb{C}[[x, y]]$ such that

$$(f, g) = \big(\psi^n, \sigma(\psi)\big).$$

Since $(z^n, \sigma(z))$ is a Puiseux parametrization for the curve $F = 0$, this curve is different from the y-axis and therefore $f \neq 0$. Then $\psi \neq 0$ and, since

$$\psi^n = f \in \mathbb{C}[[z]],$$

we deduce that ψ is in fact a non-null complex series: there exists $\varphi \in \mathbb{C}[[z]]$, $\varphi \neq 0$ such that

$$\psi(x, y) = \varphi(x + iy).$$

Then we have that

$$\Gamma = \big(\varphi^n, \sigma(\varphi)\big)$$

and, since Γ is an irreducible complex parametrization, we conclude that $\mathrm{ord}(\varphi) = 1$ and therefore φ is a formal complex diffeomorphism.

□

8. C^∞ EQUIVALENCES OF FOLIATIONS AND EQUISINGULARITY OF THE SET OF SEPARATRICES

This section is devoted to prove Theorem I.

Theorem 8.1. *Let Φ be a C^∞ equivalence between two germs $\mathcal{F}$ and $\mathcal{F}'$ of singular holomorphic foliations at $(\mathbb{C}^2, 0)$. Let S be a formal separatrix of $\mathcal{F}$ and let $S' = \Phi_*(S)$ be the corresponding separatrix of $\mathcal{F}'$ according to Theorem 6.1. Then $\hat{\Phi}$ is a formal real equivalence between S and S'.*

Proof. Take coordinates (z, w) in $(\mathbb{C}^2, 0)$ an suppose that S' is defined by a formal equation $F = 0$, where $F \in \mathbb{C}[[z, w]]$ is irreducible. As a first step, considering S as a formal real surface, we will prove that $F \circ \hat{\Phi} \circ S = 0$. Since this is obvious if S is convergent, we assume that S is a weak separatrix. Let $\pi : (M, E) \to (\mathbb{C}^2, 0)$ be the reduction of singularities of $\mathcal{F}$. Then, the strict transform $\tilde{S} = \pi^* S$ is the weak separatrix of a saddle-node singularity at some $p \in E$. Let (u, v) be local holomorphic coordinates at $p \in E$ such that:

(1) $p \simeq (0, 0)$;
(2) E is given by $\{u = 0\}$.

There exists a formal series $\hat{s}(u) = \sum_{j=1}^{\infty} c_j u^j$ such that $\tilde{S}$ is given by $v = \hat{s}(u)$, hence the separatrix $S(u) = \pi\big(u, \hat{s}(u)\big)$ is parametrized as a real surface by

$$S(x, y) = \pi(x + iy, \hat{s}(x + iy)).$$

In order to prove that $F\circ\hat{\Phi}\circ S=0$ it is sufficient to show that, if $\alpha,\beta\in\mathbb{R}^*$ are arbitrarily chosen, then the series

$$f(t):=F\circ\hat{\Phi}\circ S(\alpha t,\beta t)$$

is null. If we set $\eta=\alpha+i\beta$, the series f can be expressed as

$$f=F\circ\hat{\Phi}\circ S(\eta t).$$

As we have seen in Example 5.7, we know that $S(\eta t)$ is the Taylor series of a characteristic curve γ of $\mathcal{F}$ asymptotic to S. Then, since $\gamma':=\Phi(\gamma)$ is a characteristic curve of $\mathcal{F}'$ asymptotic to S', we deduce that $\hat{\gamma'}=\hat{\Phi}\circ S(\eta t)$ is a complex formal parametrization of S' and therefore

$$f=F\circ\hat{\Phi}\circ S(\eta t)=F\circ\hat{\gamma'}=0.$$

Without loss of generality we can assume that both curves S and S' are tangent to the z axis, which implies the following properties:

(1) S has a Puiseux parametrization $(T^n,\sigma(T))$, where n is the multiplicity of the curve S and $\sigma\in\mathbb{C}[[T]]$, $\operatorname{ord}(\sigma)>n$;
(2) S' has a Puiseux parametrization $(T^{n'},\sigma'(T))$, where n' is the multiplicity of the curve S' and $\sigma'\in\mathbb{C}[[T]]$, $\operatorname{ord}(\sigma')>n'$.

By Proposition 6.3 and without loss of generality — the other case is similar — we can assume that $\hat{\Phi}(z,w)$ has a complex linear part

$$\hat{\Phi}_1(z,w)=(az+bw,cz+dw),\ ad-bc\neq 0. \tag{8.1}$$

Suppose that

$$S(u)=\Big(\sum_{j\geq\bar{n}}\alpha_j u^j,\sum_{j\geq\bar{n}}\beta_j u^j\Big),\ (\alpha_{\bar{n}},\beta_{\bar{n}})\neq(0,0). \tag{8.2}$$

Since it is an irreducible parametrization, by Corollary 7.4 there exists a formal diffeomorphism $\varphi\in\mathbb{C}[[u]]$ such that $S=\big(\varphi^n,\sigma(\varphi)\big)$, hence $\bar{n}=n$ and $\beta_n=0$. Therefore, from (8.1) and (8.2) above we have that the initial part of $\hat{\Phi}\circ S$ is complex and is given by

$$\big(\hat{\Phi}\circ S\big)_1=\big(a\alpha_n u^n,c\alpha_n u^n\big).$$

Since S' is tangent to the z axis, we have that F has an initial part of the form $F_N=\mu y^N$, $\mu\neq 0$, $N\in\mathbb{N}$. Then, since $F\circ\hat{\Phi}\circ S=0$ implies $F_N\circ\big(\hat{\Phi}\circ S\big)_1=0$, we deduce that $c=0$ and consequently $a\neq 0$. By Lemma 7.3, since $F\big(\hat{\Phi}\circ S\big)=0$, there exists $\psi\in\mathbb{C}[[x,y]]$ such that

$$\hat{\Phi}\circ S(x+iy)=\big(\psi^{n'},\sigma'(\psi)\big).$$

Then, since

$$\big(\hat{\Phi}\circ S\big)_1=\big(a\alpha_n u^n,0\big),$$

the initial part ψ_ν, $\nu\in\mathbb{N}$, of ψ satisfies the equality $a\alpha_n u^n=\psi_\nu^{n'}$. Then $n=n'\nu$ and therefore $n'\leq n$. A similar argument using the inverse diffeomorphism Φ^{-1} allows us to conclude that $n=n'$, $\nu=1$ and, consequently, ψ has a linear part of the form $\sqrt[n]{a\alpha_n}(x+iy)$. Thus ψ is a formal real diffeomorphism and therefore $\hat{\Phi}\circ S$ is a formal real reparametrization of S'. □

Proof of Theorem I. It is a direct consequence of Theorems 8.1 and 7.2. □

9. The proof of Theorem II

Let $\mathcal{F}$ and $\mathcal{F}'$ be germs of foliations, equivalent by a germ of C^∞ diffeomorphism

$$\Phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0).$$

Let $S \in \operatorname{Sep}(\mathcal{F})$ be a branch of separatrix and $S' = \Phi_* S \in \operatorname{Sep}(\mathcal{F}')$ be the corresponding separatrix given by Theorem I. This result also asserts that, if $\mathscr{S} = \cup_{i=1}^k S_i$ is a reduced curve formed by the union of a finite number of branches in $\operatorname{Sep}(\mathcal{F})$, setting $\mathscr{S}' = \Phi_* \mathscr{S} = \cup_{i=1}^k \Phi_* S_i$, then $\mathscr{S}$ and $\mathscr{S}'$ are equisingular. As a consequence, we have that $S \in \operatorname{Iso}(\mathcal{F})$ if and only if $S' \in \operatorname{Iso}(\mathcal{F}')$ and $S \in \operatorname{Dic}(\mathcal{F})$ if and only if $S' \in \operatorname{Dic}(\mathcal{F}')$. We have clearly the following more general fact:

Proposition 9.1. *$\mathcal{F}$ and $\mathcal{F}'$ are $\mathcal{S}$-equisingular.*

Suppose now that $\hat{F}$ is a balanced equation of separatrices for $\mathcal{F}$, whose divisor is as in (3.1). We define $\hat{F}' = \Phi_* \hat{F}$ as any formal meromorphic function corresponding to the following divisor

$$(\hat{F}')_0 - (\hat{F}')_\infty \ = \sum_{S \in \operatorname{Iso}(\mathcal{F})} (S') + \sum_{S \in \operatorname{Dic}(\mathcal{F})} a_S (S').$$

Lemma 9.2. *Let $\mathcal{F}$ and $\mathcal{F}'$ be germs of foliations, equivalent by a germ of C^∞ diffeomorphism $\Phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$. Let $S \in Dic(\mathcal{F})$ and $S' = \Phi_* S \in Dic(\mathcal{F}')$ be corresponding separatrices, attached to dicritical components D and D' of the desingularizations of $\mathcal{F}$ and $\mathcal{F}'$. Then $Val(D) = Val(D')$.*

Proof. This is a consequence of Proposition 9.1 and of the following fact: if $\pi : (M, E) \to (\mathbb{C}^2, 0)$ is the reduction of singularities for $\mathcal{F}$ and $\mathcal{D} \subset E$ is the union of all dicritical components, then each connected component of $E \setminus \mathcal{D}$ carries a separatrix of $\mathcal{F}$ (see [8, Prop. 4]). □

This lemma allows us to prove the following:

Proposition 9.3. *Let $\mathcal{F}$ and $\mathcal{F}'$ be germs of foliations, equivalent by a germ of C^∞ diffeomorphism $\Phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$. If $\hat{F}$ is a balanced equation of separatrices for $\mathcal{F}$, then $\hat{F}' = \Phi_* \hat{F}$ is a balanced equation of separatrices for $\mathcal{F}'$. Besides, $\nu_0(\hat{F}) = \nu_0(\hat{F}')$.*

Proof. The isolated separatrices of $\mathcal{F}$ and $\mathcal{F}'$ are in correspondence by Φ, so that they appear in the zero divisor of both balanced equations with coefficient 1. Similarly, there is a correspondence between dicritical separatrices of $\mathcal{F}$ and $\mathcal{F}'$, which, by Lemma 9.2, are attached to dicritical components having the same valences. Therefore, $\hat{F}'$ is a balanced equation for $\mathcal{F}'$. Finally, the equality of the algebraic multiplicities follows from the equisingularity property given by Theorem I. □

The final ingredient for the proof of Theorem II is the following result of [10]:

Theorem 9.4. *Let $\mathcal{F}$ and $\mathcal{F}'$ be germs at $(\mathbb{C}^n, 0)$ of C^1 equivalent one dimensional foliations. Then $\nu_0(\mathcal{F}) = \nu_0(\mathcal{F}')$.*

This enables to prove the following:

Proposition 9.5. *The tangency excess $\tau_0(\mathcal{F})$ is a C^∞ invariant.*

Proof. Let Φ be a C^∞ equivalence between $\mathcal{F}$ and $\mathcal{F}'$. We have $\nu_0(\mathcal{F}) = \nu_0(\mathcal{F}')$ by the previous theorem. Moreover, Proposition 9.3 gives that if $\hat{F}$ is a balanced equation of separatrices for $\mathcal{F}$, then $\hat{F}' = \Phi_* \hat{F}$ is a balanced equation of separatrices for $\mathcal{F}'$ and $\nu_0(\hat{F}) = \nu_0(\hat{F}')$. The result then follows from Proposition 3.6. □

We are now ready to complete the proof of Theorem II:

Proof of Theorem II. Let $\mathcal{F}$ and $\mathcal{F}'$ be C^∞ equivalent foliations. Being $\mathcal{F}$ of second type, it holds $\tau_0(\mathcal{F}) = 0$. Consequently, by Proposition 9.5, $\tau_0(\mathcal{F}') = 0$ and $\mathcal{F}'$ is also of second type. Hence, both $\mathcal{F}$ and $\mathcal{F}'$ are $\mathcal{S}$-desingularizable by Proposition 3.2. The proof is accomplished by using the fact that C^∞ equivalent foliations are $\mathcal{S}$-equisingular (Proposition 9.1). $\square$

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