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Special volume in honor of the life and mathematics of Egbert Brieskorn

Editors:
Gert-Martin Greuel
Helmut A. Hamm
Lê Dũng Tráng

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Preface

The proposal to prepare a special volume in the Journal of Singularities with contributions to the memory of Egbert Brieskorn was made by Andrew Ranicki in autumn 2017, after reading the article by *Greuel - Purkert*. The editors gladly accepted this proposal and contacted a number of students and colleagues who had worked with Brieskorn or were influenced by his work. Many of them agreed to contribute and the result is the present volume collecting the refereed papers that were submitted.

We are most grateful to the authors for their positive response and by their scientific contribution to the volume. Thanks also to the many referees for their readiness, their careful and sometimes very laborious work, and their keen judgments. We also like to thank David Massey and the Journal of Singularities for accepting our proposal for a special volume in honor of Brieskorn.

An essential feature of singularity theory is that it combines methods from different branches of mathematics, from algebraic topology over complex analysis to algebra, algebraic geometry, and Lie theory. We tried to group the papers according to related subjects. This is of course not perfect, as some papers would also fit in at least one other category, but we hope that the grouping corresponds to the main area to which each paper is ascribed.

0. The first three papers are of historical nature.
The articles by *Brieskorn* and *Hirzebruch* are reproduced reports from their talks given at the workshop "Singularitäten" in 1996 at the Mathematisches Forschungsinstitut Oberwolfach (MFO), and which never appeared elsewhere. The editors would like to thank Heidrun Brieskorn, the Hirzebruch family, and the MFO for permission to reproduce their reports in this volume.
The article by *Greuel - Purkert* describes Brieskorn's mathematical and evocative work and his life from a personal point of view.

I. Complex analytic methods lie at the heart of singularity theory. A fundamental contribution was the analytic description of the monodromy by Brieskorn. *Brasselet - Sebastiani* give a sketch of Brieskorn's fundamental manuscripta paper from 1970, explaining some central ideas in the style of that time.
Brieskorn introduced in that paper certain important concepts like the Gauß-Manin connection, a connection on a certain vector bundle, in the local situation.
Hamm - Lê look at connections in general and realize that line bundles with connection allow a much more complete theory than vector bundles with connection.
By definition connections involve differential forms. These form the subject of different papers: *Barlet* studies meromorphic differential forms with a good

pull-back property, *Dimca - Greuel* look a differential 1-forms on curves, connecting them with several geometric invariants (and offer an interesting conjecture relating the Milnor and the Tjurina number), while *Schulze - Tozzo* look at a generalization of K. Saito's free divisors, passing from divisors to complete intersections.
There is a bridge from differential forms to foliations and *Campillo - Olivares* look at the relation between foliations on a surface and their singular set.
A new central subject in Brieskorn's paper is the so-called Brieskorn lattice, the importance of which has only been realized much later. *Sabbah* looks at it in the global context for a tame function and *M. Saito* treats the uniqueness of sections of the Brieskorn module. In a long and fundamental paper *Gauss - Hertling* use the Brieskorn lattice and other invariants to determine an isolated hypersurface singularity up to right equivalence.
The Gauß-Manin connection allows also to study the eigenvalues of the monodromy. These are related to the Bernstein-Sato polynomial, too, which is studied by *Artal - Cassou-Noguès - Luengo - Melle Hernandez* in their paper.

II. There is a group of papers which deal with topology or real algebraic objects.
Classical homology theory is a basic tool, in the presence of singularities modifications of it are useful (e.g. intersection homology). *Kreck* generalizes in a different direction, comparing singular homology with bordism theory.
Traditionally singularity theory deals with complex singularities but it is natural to consider real ones, too. *Leviant - Shustin* study morsifications of these in the case of real plane curve singularities with some of their branches complex conjugate. *Oka* has discovered that several results which hold for complex polynomials hold also for "mixed" polynomials, in the present paper he focuses on the fundamental group. A classical question which refers to certain semi-algebraic objects has been taken up by *Vassiliev*, considering the following question: when does the volume of a space obtained by cutting a bounded domain with a half-space depend locally algebraically on the defining inequality of the latter?

III. Finally there are some papers which belong to the algebraic resp. algebro-geometric context, in quite different respects.
Recall that singularity theory started with isolated singularities of a holomorphic function, that is with the local case.
There are different analogues in the global case: *Damon* studies the global Milnor fibre in case of matrices (also matrices which are symmetric or skew-symmetric). *Libgober - Settepanella* look at a certain type of hyperplane arrangements.
Brieskorn was fascinated by the appearance of finite subgroups of $Sl_2(\mathbb{C})$ in singularity theory; *Ebeling* studies the MacKay correspondence for certain finite subgroups of $Sl_3(\mathbb{C})$.
Apart from singular homology the fundamental group has been from the beginning an object of study in singularity theory, especially the fundamental group of the complement of a discriminant. *Lönne* deals with a conjecture,

which was already formulated by Brieskorn in this context in 1972.
When discussing singularities one can expect more precise results by restricting to more special situations, for instance surfaces: *Némethi* looks at a class of normal surface singularities which look quite special but allow more comprehensive results, *Stevens* considers Kulikov singularities - these arise from families of curves.
A surprise is the title of the paper by *Goldman - Salman - Yomdin*: it refers to neuroscience. It turns out that questions from algebraic geometry are basic here, the paper deals with Prony systems of polynomials which are important in this context.
Varchenko treats a question from Lie theory: how to find common eigenvectors of Gaudin operators. In fact he passes from $\mathbb{C}$ to $\mathbb{F}_p$!
Deformation theory is an important branch of singularity theory; *Laudal* studies deformations of thick points - in fact non-commutative deformations.

The articles in this special volume confirm that singularity theory is nowadays a widely branched and still active subject. But it is worth while to keep common roots in mind, and Brieskorn's work plays a fundamental role here.

The three editors knew Egbert Brieskorn from the very beginning of their scientific career and profited a lot from his ideas, stimulation and encouragement. Brieskorn has been the teacher of two of us (Greuel and Hamm) and he influenced also the career of Lê, in particular by initiating the long-lasting collaboration with Hamm. It is our great pleasure to express with this special volume our gratitude for his many years of support, for his great contributions to singularity theory and his visionary leadership in the field that has influenced a generation of mathematicians.

Gert-Martin Greuel
Helmut A. Hamm
Lê Dũng Tráng

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Special volume in honor of the life and mathematics of Egbert Brieskorn
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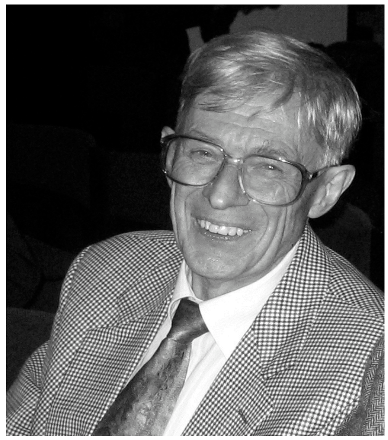
LIFE AND WORK OF EGBERT BRIESKORN (1936 – 2013)[1]

GERT-MARTIN GREUEL AND WALTER PURKERT



Brieskorn 2007

Egbert Brieskorn died on July 11, 2013, a few days after his 77th birthday. He was an impressive personality who left a lasting impression on anyone who knew him, be it in or out of mathematics. Brieskorn was a great mathematician, but his interests, knowledge, and activities went far beyond mathematics. In the following article, which is strongly influenced by the authors' many years of personal ties with Brieskorn, we try to give a deeper insight into the life and work of Brieskorn. In doing so, we highlight both his personal commitment to peace and the environment as well as his long–standing exploration of the life and work of Felix Hausdorff and the publication of Hausdorff 's Collected Works. The focus of the article, however, is on the presentation of his remarkable and influential mathematical work.

The first author (GMG) has spent significant parts of his scientific career as a graduate and doctoral student with Brieskorn in Göttingen and later as his assistant in Bonn. He describes in the first two parts, partly from the memory of personal cooperation, aspects of Brieskorn's life and of his political and social commitment. In addition, in the section on Brieskorn's mathematical work, he explains in detail the main scientific results of his publications. The second author (WP) worked together with Brieskorn for many years, mainly in connection with the Hausdorff project; the corresponding section on the Hausdorff project was written by him.

We thank Wolfgang and Bettina Ebeling, Helmut Hamm, Thomas Peternell, Anna Pratoussevitch and Wolfgang Soergel for useful information and especially Brieskorn's wife Heidrun Brieskorn for the release of material from Brieskorn's estate. We also thank Andrew Ranicki for encouraging us to translate the article into English and special thanks to him and Ida Thompson for checking the translation.

2010 *Mathematics Subject Classification.* 01A61, 14B05, 14B07, 14D05, 14F45, 14H20, 14J17, 14J70, 17B22, 32S05, 32S25, 32S40, 32S55, 57R55.

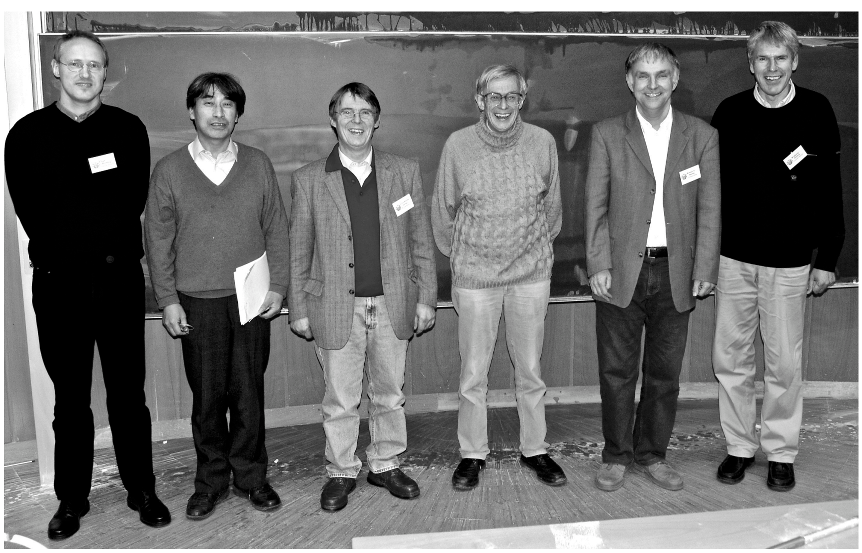
[1] Translation of the German article "Leben und Werk von Egbert Brieskorn (1936 – 2013)", Jahresber. Dtsch. Math.–Ver. 118, No. 3, 143-178 (2016).

Stations of his life

Brieskorn was born on July 7, 1936 in Rostock, Germany, the son of a mill–construction engineer, and grew up with his sister and his mother in Siegerland. Little is known about his youth and the source of his enthusiasm for mathematics. But from the chapter *Childhood and Education* from the Simons Foundation film about Brieskorn [27] we know that his mother supported his childlike curiosity and that his father promoted his technical interest. He also had a good maths teacher in grammar school, who provided him with mathematical literature beyond the subject matter of the curriculum. He was particularly interested in geometric constructions (and less, for example, in a work by Gauss).

Even though his technical interest initially prevailed, his interest in mathematics was already strong before his studies. At the examination for admission to the Evangelische Studienwerk Villigst, the funding organization of the Protestant Church for gifted students, the examiner said to him: 'Mr Brieskorn, your talent and your enthusiasm for mathematics are extraordinary but do not forget that there are other things besides mathematics in life.' Egbert Brieskorn told this episode to the first–named author of these lines with a slightly ironic undertone much later, when in fact other things than mathematics determined his life and work. The head of the Evangelische Studienwerk recognized that his original desire to study electrical engineering was not right for his nature and convinced him to study something theoretical.

Brieskorn therefore began to study mathematics and physics in Munich in October 1956. After five semesters he followed the advice of Karl Stein and moved to Bonn for the summer semester of 1959, in order to understand the theorem of Hirzebruch-Riemann-Roch, which he described as 'my first love in mathematics' [25]. Friedrich Hirzebruch, who himself had only come to Bonn in 1956, deeply impressed the young student Brieskorn with his friendly, open personality and his clear style of presentation. Brieskorn became Hirzebruch's student and received his doctorate in 1963 with the thesis "Differentialtopologische und analytische Klassifizierung gewisser algebraischer Mannigfaltigkeiten" (Differential topological and analytical classification of certain algebraic manifolds). Hirzebruch later described Brieskorn as his most talented student and Brieskorn highly revered his teacher Hirzebruch as a mathematician and human being all his life.



Brieskorn (3. from r.) and his students (from l.) Claus Hertling, Kyoji Saito, Gert-Martin Greuel, Helmut Hamm, Wolfgang Ebeling, 2004

In 1968, Brieskorn habilitated in Bonn with the thesis *Singularitäten komplexer Räume* (Singularities of complex spaces) and was appointed full professor in Göttingen in 1969, where he remained until 1973. Because of his wife Heidrun, whom he married in 1973 and who got a job as a violist at the Cologne Radio Symphony Orchestra (today the WDR Symphony Orchestra Cologne), he moved in 1973 to Bonn, first to the Sonderforschungsbereich Theoretische Mathematik and from 1975 to a position as a full professor, where he worked until his retirement in 2001.

Mathematics and political–social engagement

Although Brieskorn has always been socially engaged, e.g. by working in a steel rolling mill of the "Dortmunder Hörder Hüttenunion" in the context of a work semester of the Evangelischen Studienwerk, for him, during his studies and many years as a professor, mathematics was the most important thing in his life, as he writes himself (curriculum vitae work semester). He was completely excited by the fascinating beauty and clarity and the high standards of mathematics. But he saw mathematical phenomena everywhere, also in small things, and was fascinated by them. This enthusiasm for mathematics and his own field of research, the theory of singularities, was passed on to the students.

The following sections show the development of the relation of mathematics and political–social engagement of Brieskorn from the perspective and memory of the first-named author. In the late summer of 1969 he had invited me to tea in his apartment in Göttingen to discuss a topic for my diploma thesis. Before he started, he discovered a refractory caustic in his teacup, which he classified as a "simple singularity", and then interpreted the spiral chocolate trail in the biscuits as a dynamical system. He enthusiastically explained to me exotic spheres and how they can be described by real-analytic equations as the boundaries of certain isolated hypersurface singularities. I was infected, and when he then suggested that I generalize the results of his unpublished work, *Die Monodromie der isolierten Singularitäten von Hyperflächen* (The monodromy of isolated singularities of hypersurfaces) [12], to isolated singularities of complete intersections, I immediately accepted.

I became a student of Brieskorn on the recommendation of Hans Grauert, whose beginner course "Differential and integral calculus" I had attended in Göttingen in 1966/67. After returning from a one–year study visit at ETH Zurich, I received a telephone call from Brieskorn asking if I would be interested in writing a thesis with him. Grauert recommended me because we were in the same "fraternity" (i.e. the Evangelische Studienwerk). So it happened that I became Brieskorn's first graduate student.

Brieskorn had come to Göttingen in the summer semester 1969, first as a substitute professor and from July 1969 as a full professor. In the winter semester 1969/70 he was on leave for a research stay at the IHES in Bures-sur-Yvette. I had missed his first lecture in the summer of 1969 in Göttingen on 2-dimensional schemes and over winter I learned sheaf theory and hypercohomology in the reading room of the Mathematical Institute in the Bunsenstrasse with the help of Godement's *Topologie Algébrique et Théory des Faisceaux*. Except for occasional whispering and the unmistakable sound of the heavy breathing of Carl Ludwig Siegel, who disappeared into the back of the reading room to consult the older works, one was undisturbed and could discuss with fellow students in the adjoining discussion room.

After returning from France, Brieskorn gave the following lectures: "Differential Topology", then the beginner lecture "Calculus I and II", then "Analysis on Manifolds", "Qualitative Theory of Dynamical Systems", "Algebraic Topology II", and before he moved to Bonn in the summer semester 1973 "Simple Singularities". Brieskorn's lectures differed fundamentally from those of Grauert. While Grauert only told what he wrote and proved on the blackboard, Brieskorn often presented larger connections and mentioned cross-connections without proving them. Obviously, both approaches have their advantages, and the students in Göttingen greatly appreciated both Brieskorn's and Grauert's lectures.

Brieskorn often made an extraordinary effort in the preparation of his lectures, to show mathematical and historical backgrounds or side branches of the material he treated. This can be clearly seen in his textbooks. It can be said that his quest for perfection, which even increased over the years, was characteristic of him. He also expected perfection from his students, which he greatly promoted, both mathematically, among others with weekly working meetings or with recommendations, but also in the personal sphere. For example, he let his first doctoral student Helmut Hamm live in his apartment during his stay in France free of charge.

The Göttingen period from 1969 to 1973 caused a remarkable change in Brieskorn's views and attitudes. In addition to mathematics, political and social issues gained importance for him. It was the time shortly after the violent student protests of the '68 generation, in which the students of Göttingen, primarily the theologians, followed by the mathematicians, were very active (for example in a left-wing action group "Basisgruppe Mathematik"). Brieskorn was very positive about some of the demands of the students and rather critical of others. He objected to scientifically unfounded hierarchical structures, was committed to greater co-determination of research assistants and students, and he sympathized with ideas of the reform universities in Bremen and Osnabrück. However, the scientific quality of the study always came first.

Even more important than student reform ideas for Brieskorn, however, were his commitment against the Vietnam War and, in turn, his commitment to supporting oppressed peoples. From [26] we know that during his stay at the Massachusetts Institute of Technology (MIT) he participated in a major demonstration against the Vietnam War, together with Michael Artin and other MIT colleagues in New York, at which Martin Luther King spoke.

From the beginning he was involved in Göttingen in the Committee for Scientific Cooperation with Cuba (KoWiZuKu), which was founded in 1970 and whose first Secretary General was the mathematician Klaus Krickeberg. The fact that he was always very precise in these activities is shown by an episode when, together with me and my wife in Göttingen, he stuck up posters announcing a lecture by the Cuban health minister at the university. Brieskorn meticulously ensured that everything was correct, and no posters were glued to distribution boxes as this could lead to heat problems.

Later in Bonn Brieskorn was involved from the outset in the peace movement, which was formed in protest against the 1979 "NATO double–track decision" and thus against the deployment of medium-range missiles in the then FRG. Brieskorn was one of the initial signatories of the "Mainzer Appell", the final declaration of the congress "Responsibility for peace – scientists warn against new nuclear armaments" in June 1983 in Mainz, in which the mathematician Stephen Smale also participated (see [29]). Especially physicists, but also many mathematicians such as Brieskorn, organized themselves in the "Scientist Initiative Responsibility for Peace" (today: "Scientist Initiative for Peace and Sustainability").

Brieskorn's political views in the Bonn era can certainly be classified as left of social-democratic ideas, but over the years they have evolved into radical ecological convictions, which he supported together with his wife Heidrun. They have lived since 1982 in a house at an isolated location on the edge of the forest in Eitorf an der Sieg where both, together with a small group, have devoted themselves intensively to nature conservation and more specifically to the conservation of species. The species in question were initially indigenous bats, for which they controlled winter quarters in old mine tunnels and secured and built new winter quarters. In order to distinguish the different species, Heidrun Brieskorn made many sound recordings, which Egbert Brieskorn subjected to self-written programs using a Fourier analysis.

Both devoted even more time and effort to protect and preserve the living conditions of a very rare species of butterfly, the large blue (Latin: *Maculinea*). This went so far that Brieskorn and his wife persuaded the community to change their development plans and they themselves bought up grassland to maintain the habitat of these butterflies. They founded the Maculinea Foundation NRW so that the work to preserve the butterfly species can continue on a permanent basis. For their commitment they were jointly awarded in 2013 the decoration "Bundesverdienstkreuz am Bande" ("Cross of the Order of Merit") of the Federal Republic of Germany.

In addition to the volunteer work in nature conservation, the last 20 years of Brieskorn's life were determined by his collaboration in the edition project "Felix Hausdorff – Collected Works" of the North Rhine-Westphalian Academy of Sciences and the Arts. Brieskorn himself wrote shortly before his 75th birthday in June 2011, knowing that he might not live much longer, in a letter to 'my dear former students and my students' how it came about and how much he was concerned with the biography of Hausdorff: "One of the tasks developed from the fact that the Mathematical Institute in Bonn in January 1992 wanted to celebrate the 50th anniversary of the death of Felix Hausdorff. Since no colleague wanted to give a lecture about his life, I took over this task at that time, not knowing what I had let myself in for. I have spent 20 years searching for archives and sources of all kinds for traces of this extremely remarkable man and mathematician. I learned a lot while sacrificing a lot of time. I made three attempts to write his biography, and with the third version, I think, I am on the right track. About 262 pages are written so far, but not even half of the path of his life is described. This biography is to appear in the first volume of a total of nine-volume edition of Felix Hausdorff's works. Six volumes are printed, and at least one or perhaps two of the missing volumes will be printed this year. The bad thing is that the first volume in which my biography of Hausdorff is to be released cannot possibly be completed on time."

The following section reports on Brieskorn's extensive work and his research on this project, which goes far beyond the usual.

The Hausdorff project

The work, which Egbert Brieskorn mentions in his letter, was a labour of love for him for more than 20 years, in particular the research into the life and work of Felix Hausdorff (1868-1942) and the publishing of Hausdorff's collected works.

Felix Hausdorff founded general topology as a freestanding area of mathematics and, in addition, made fundamental contributions to general and descriptive set theory, measure theory and analysis. His contributions to the theory of Lie algebras, probability theory and actuarial mathematics were also important to subsequent developments. Hausdorff was also (for a mathematician somewhat uniquely) a notable writer and philosophical author. From 1897 until 1913, under the pseudonym of Paul Mongré, he published a volume of aphorisms, an epistemological book, a volume of poetry, a play which has been performed more than 300 times in over 30 cities, as well as 17 essays in leading literary journals. In the twenties and early thirties, he was internationally recognised and respected as the leading representative of the Mathematical Institute of Bonn. As a Jew living under the dictatorship of the National Socialist Party, he and his wife took their own lives on 26th January 1942, when deportation to a concentration camp was imminent.

Egbert Brieskorn's involvement began in 1979, when the suggestion came from the students of the Mathematical Institute to honour Felix Hausdorff through a memorial plaque in the institute. Brieskorn, who had already been involved in the peace movement for years and who had talked in depth about questions concerning the social responsibility of scientists, supported the plan from the very beginning, gave his own financial support and organised a collection of donations amongst the teaching staff. In 1980, on the anniversary of Hausdorff's death on 26th January, the marble plaque was unveiled at the old institute on Wegelerstrasse. It was there for over 30 years, until it was recently transferred to the new institute building. On the occasion of the unveiling, an article appeared about Hausdorff's tragic fate penned by the historian Herbert Mehrtens. Egbert Brieskorn wrote an introduction for the article, the closing sentences of which are quoted here; he wrote: "No form of inhumanity and oppression must leave us indifferent, even when the victims are distant und unfamiliar to us. The thought of a person like Hausdorff, whom we all recognise for his great scientific achievements, can also help us to sharpen our conscience and sense of responsibility. Without a growing feeling of responsibility, scientists will not be in the position to make their contribution to a more humane society. For this task we also need to come to terms with the past."

In November 1980, Egbert Brieskorn was successful in procuring Hausdorff's literary estate for the university library in Bonn, consisting of some 26,000 pages in size and since 1964 in the private ownership of Prof. Günther Bergmann in Münster. The proceeds benefited Hausdorff's daughter Lenore König, who lived in poor conditions in Bonn in a retirement home. As the contract was signed, Egbert Brieskorn wrote to Günther Bergmann on 15th November: "The arrangements, which are now being made, seem to me to be very good. It is true that I am no historian, but I do have a particular interest in the history of mathematics and occasionally oversee dissertations with historical aspects. I hope that one day there will be a mathematical historian who will work on Hausdorff. Then the literary estate in our university library will be very important." Whilst writing these lines, he had surely not thought that it would be he himself who would 10 years later tackle the project of writing Hausdorff's biography.

26th January 1992 was the 50th anniversary of Hausdorff's death. On this occasion, Egbert Brieskorn organised a memorial colloquium of the Mathematical Institute of the University of Bonn, the result of which was the volume *Felix Hausdorff zum Gedächtnis – Aspekte seines Werkes* (In Memory of Felix Hausdorff – Aspects of His Work), edited by Brieskorn himself and published by Vieweg-Verlag. Furthermore, he arranged an exhibition about the life and work of Felix Hausdorff, which was met with a lively interest not just amongst the Mathematical Institute, but also the university and the general public of Bonn. For the exhibition, he published a comprehensive catalogue with an initial biographical sketch of Hausdorff. On 1st February 1992, there was an hour–long radio programme for the series "Mosaik" (Mosaic) on WDR, about Hausdorff and the exhibition in Bonn under the title "Auf dünner Schneide tanzt mein Glück" (My Happiness Dances on the Edge of a Knife), which is the first line of Hausdorff's poem "Wiederkunft" (Return) from the poetry volume "Ekstasen" (Ecstasy). Egbert Brieskorn and the WDR culture editor Friedrich Riehl conceived the programme together, and made the exhibition and its subject well known far beyond Bonn. Even the media coverage of the exhibition was significant.

The preparation for all of these activities began in 1989 with numerous conversations, which Egbert Brieskorn held with Hausdorff's daughter, with contact with Hausdorff's niece Else Pappenheim and with further contemporary witnesses, as well as with the collection of material for the biography, in particular through researching in archives. During this research, he showed thoroughness and the mind of a detective, which any professional historian would hold in high esteem. In his literary estate there are dozens of thick folders, documenting all of these endeavours and in successful instances, though that was not always the case, recording their results.

In this spirit and in the run-up to the memorial colloquium, the mathematicians from Bonn, Egbert Brieskorn, Friedrich Hirzebruch and Stefan Hildebrandt, discussed the possibilities and necessary steps for setting in motion an edition of the works of Hausdorff, under the consultation of external experts. Friedrich Hirzebruch suggested creating a Hausdorff Commission at the North Rhine-Westphalian Academy of Science, which was already overseeing a number of edition projects, to plan and then supervise a similar project. The academy agreed and the commission took up its work under the leadership of academy member Reinhold Remmert in 1991. As the first step, a careful indexing and cataloguing of the Hausdorff literary estate was envisaged. This was carried out from October 1993 until the end of 1995. The result was an inventory of 550 pages with short descriptions of the content of each individual fascicle. After this vital step towards the success of the edition project was completed, and with the inclusion of the literary estate, the final project was within sight.

In order to create a diligently commented edition with the inclusion of selected parts from the literary estate in the work, many things needed to be considered and done. One must establish editorial principles, develop a volume structure, find suitable personnel and, crucially, applications must be made in order to finance the project. Egbert Brieskorn was the guiding spirit throughout all of these discussions and activities. Particularly difficult was the winning over of suitable philosophers and literary academics as editors of the volumes dedicated to that side of Hausdorff's creations. For this purpose, he took part in a philosophical conference about Jewish Nietzscheanism in Greifswald, in order to come into contact with appropriate experts, and in Bonn organised an interdisciplinary conference for academics in the humanities, with the support of the MPI for Mathematics, about Hausdorff's philosophical and literary work.

The application for the financing of the project, which he submitted to the DFG in 1996 together with Friedrich Hirzebruch and Erhard Scholz, was finally approved and in November of 1996 the working team "the Hausdorff–Edition" at the Mathematical Institute took up work under his leadership. In 2002, the North Rhine-Westphalian Academy of Sciences and Arts took over the Hausdorff–Edition as one of their academy–projects. Some of the originally employed editors were not able to work on the project due to various reasons, meaning that new employees frequently needed to be found. In the end 14 mathematicians, four mathematical historians, two literary academics, one philosopher and one astronomer from four countries collaborated on the editing.

Egbert Brieskorn was particularly concerned during the editing with tracking the four noticeable threads, not immediately obvious, which lead from Hausdorff the mathematician to Mongré the writer and philosopher. For this reason he always emphasized the inter–disciplinary character of the project, holding pioneering lectures at three large editorial conferences exactly in this respect and organising a series of discussion circles, which involved mathematicians, philosophers and literary academics. In order to obtain an impression of his intentions, we quote here the beginning of his programmatic lecture at one of these conferences in February 2003 at Schloss Rauischholzhausen: " 'It is not always determined that the concept must lie within the lines or indeed only between the lines, perhaps it is somewhere else entirely, far, far away! Perhaps the author sounds his bell, and somewhere a string with the same vibration and the same tone colour answers – and it is not the actual bell, but the sound of the string that expresses the original concept.' We should always keep in mind the sense of this aphorism from Paul Mongré during our round-table discussion about this author, about the author Mongré and about the mathematician Felix Hausdorff. Each one of us will hear something different from the variety of motifs and the richness of tone colour. What someone hears is highly dependent on the hearing experience, which one has made in the course of his or her life. If literary academics, philosophers, historians and mathematicians also listen to each other, we can hope here and there to hear an original concept."

As leader of the working group, Egbert Brieskorn allowed a wide latitude to the co–ordinator of the edition project and the editors and employees of the individual volumes. When problems arose, he helped with advice. He did however reject some of the submissions when they did not satisfy his high demands. In these cases, decisive improvements could always be made. In the meantime, eight of the ten planned volumes were released with Springer: volume IA *General Set Theory* (2013), volume II *Basics of Set Theory* (2002), volume III *Descriptive Set Theory and Topology* (2008), volume IV *Analysis, Algebra and Number Theory* (2001), volume V *Astronomy, Optics and Probability Theory* (2006), volume VII *Philosophical Works* (2004), volume VIII *Literary Works* (2010), and volume IX *Correspondence* (2012). Egbert Brieskorn's voice is perceptible in all of the volumes, even when he did not work explicitly on every volume himself.

He did, however, take on the most difficult part of the project himself: volume IB, the biography. Here, next to the mathematical work of Hausdorff, were also further fields of interest and references to his life from very different areas: philosophy, in particular Kant, Schopenhauer, Nietzsche and Hausdorff's relationship to the Nietzsche archive, epistemology, in particular Hausdorff's language criticism and his thoughts on literary figures like Dehmel, Hartleben and Wedekind, music, particularly Hausdorff's relationship to Wagner and his relationship to Reger, and graphic art, particularly Hausdorff's friendship with Max Klinger. In this volume were also the family history in terms of the Jewish story and the history of anti–Semitism, up until Hausdorff's tragic end under the Nazi dictatorship.

In 2007 Egbert Brieskorn, Erhard Scholz and the co-author of this obituary had the opportunity to present the Hausdorff edition project in the Séminaire d'Histoire des Mathématiques de l'Institut Henri Poincaré in Paris. There Brieskorn gave the introductory speech, in which he said the following about his own work retrospectively (he spoke French of course): 'With regard to my own portion of the project, I must confess now that I am no historian and that I am not led predominately by a historical interest. On the contrary, my personal interest sprang originally from two motives. One of the motives was shame about the terrible guilt that Germany, through the persecution and murder of Jews in Europe, has brought upon itself. The other motive was very personal: In the eighties I met Felix Hausdorff's daughter Lenore König, who at that time was living in a retirement home in Bonn. She gave me an initial introduction to the life and personality of her father. Through this I later felt the personal obligation to better understand the life of this unique person. When the university of Bonn prepared a memorial event for Hausdorff's fiftieth anniversary of death, and later as the plan took shape for an edition of his works, I saw the biography of Felix Hausdorff as my personal task. At the beginning, I underestimated this task concerning the difficulties and also in respect to what this task meant to me personally. This work has greatly changed my own life and way of thinking: I have – or at least I hope – learnt something from Hausdorff.'

Egbert Brieskorn worked intensively on the transcript of the biography in his last years, also during his severe illness. Three weeks before his death he sent the last sub–chapter that he had still been able to complete to the working group in Bonn, it was a particularly difficult chapter about the relationship between Hausdorff and the philosopher, mystic and anarchist Gustav Landauer. To this day there are 546 pages of the biography written by Brieskorn and ready for printing. When he sensed that he could not manage any more, he proposed a meeting, in order to explain how he had imagined the rest of the biography. He suggested 12th July 2013 as the date for the meeting. On the evening of 11th July he passed away. It is a duty for the working group in Bonn to bring the volume to completion as well as we can. There are over 100 folders with material for the biography, which he had collected over more than 20 years through often painstakingly detailed work. They were given to us by Ms Brieskorn, and are of invaluable help. Volume IB is now finished and will appear at Springer in spring 2018, in the year of Hausdorff's 150th birthday.

Brieskorn's mathematical work

The mathematical work of Brieskorn is largely determined by the development of the "singularity theory" of complex hypersurfaces. In particular, his early work had a great influence on the development of singularity theory and it is no exaggeration to call Brieskorn, along with Vladimir Igorevich Arnold, John W. Milnor and René Thom, one of the fathers of singularity theory. In his review lecture [18] Brieskorn writes:

"Singularities exist in all areas of mathematics and in many applications, and the pair of opposites 'regular – singular' is one of the most common in a whole series of such opposite pairs in mathematics. What is meant by singularities is shown by the analysis of the many different definitions of singular objects. Such an analysis leads to a few basic meanings: a singularity within an entity is a place of uniqueness, peculiarity, degeneration, indefiniteness or infinity. All these meanings are closely related."

I use the term "singularity theory" here in the sense of exploring systems of finitely many differentiable, analytic or algebraic functions near a point in which the Jacobian matrix of the functions does not have maximum rank. This implies, according to the implicit function theorem, that the zero set of the functions is not a differentiable, analytic or algebraic manifold at a singular point. Here, singularities of vector fields or differential forms are included.

The term singularity theory was, to my knowledge, introduced by V.I. Arnold, although it is not really a closed theory with more or less uniform methods. On the contrary, a characteristic of this field is the variety of different methods used, and the relationships to many other mathematical disciplines. These include algebraic geometry, complex analysis, commutative algebra, combinatorics, representational theory, Lie groups, topology, differential topology, dynamical systems, symplectic geometry, and others. Brieskorn was particularly fascinated by the complexity and the manifold interactions of singularity theory with other mathematical and non-mathematical fields, such as geometric optics, and, as we shall see, he has contributed significantly to the study of some of these interactions. However, he has never been able to make friends with the term singularity "theory".

Almost all of Brieskorn's mathematical works either deal directly with singularities of complex hypersurfaces, or they are motivated by the study of these singularities. His work shows, in addition to originality and depth, a wide range of questions and methods, which are typical of the entire area.

In the following review of Brieskorn's work, I also try to highlight important results on which Brieskorn's works are based, as well as the developments that resulted from his work. A short description of the scientific work of Brieskorn can be found in [37].

Dissertation

Singularities do not play any role in the first two publications of Brieskorn that are parts of his dissertation [3], which he wrote in 1962 as a student of Friedrich Hirzebruch and which he published in [4] and [5].

The question there is to what extent does the differentiable structure of a Kähler manifold already determines its biholomorphic structure in the case of the complex quadric Q_n or the holomorphic $\mathbb{P}^n$ bundles over $\mathbb{P}^1$. This problem had been studied and solved by F. Hirzebruch and K. Kodaira in 1957 for the complex projective space $\mathbb{P}^n$.

The following main result of the first paper is an exact analogue of the mentioned theorem of Hirzebruch and Kodaira. Brieskorn had already announced the result in 1961 in the Notices of the AMS:

Let X be a n-dimensional Kähler manifold that is diffeomorphic to the n-dimensional complex projective quadric Q_n. Then:

(i) *If n is odd, then X is biholomorphic to Q_n.*
(ii) *If n is even and $n \neq 2$, then the 1st Chern class c_1 of X satisfies: $c_1 = \pm ng$, where g is the positive generator of $H^2(X,\mathbb{Z}) \cong \mathbb{Z}$; if $c_1 = +ng$, then X is biholomorphic to Q_n.*

Brieskorn asks if the assumption that X is Kähler can be omitted and if there are any Kähler manifolds with $c_1 = -ng$ that are diffeomorphic to Q_n. Both problems seem to be open to this day.

As a corollary Brieskorn proves statements about the deformation behavior of Q_n. He considers a family of complex manifolds $V_t,\ t \in \mathbb{C},\ |t|$ sufficiently small, and proves:

(i) *If $V_0 \cong Q_n$, then $V_t \cong Q_n$ for $t \neq 0$,*
(ii) *If V_0 is Kähler and $V_t \cong Q_n$ for $t \neq 0$ and $n \geq 3$, then $V_0 \cong Q_n$.*

Again, he asks if in (ii) the assumption that V_0 Kähler can be omitted.

That this is indeed the case was proved by J.M. Hwang in 1995, after the same question of "non-deformability" of $\mathbb{P}^n$ instead of Q_n, had previously been answered positively by Y.-T. Siu.

For $n = 2$ the first statement does not hold, because on the differential manifold Q_2 there are, according to Hirzebruch, infinitely many different complex structures, the so-called Hirzebruch Σ–surfaces Σ_{2m}.

The Σ–surfaces are total spaces of holomorphic fiber bundles over $\mathbb{P}^1$ with fiber $\mathbb{P}^1$. In the second part of his dissertation, Brieskorn examines holomorphic fiber bundles over $\mathbb{P}^1$ with fiber $\mathbb{P}^n$, which he calls *Σ–manifolds* in accordance with the Hirzebruch Σ–surfaces. Taking advantage of Grothendieck's splitting theorem for vector bundles over $\mathbb{P}^1$, Brieskorn classifies the Σ–manifolds up to biholomorphic and birational equivalence and up to diffeomorphism. As a result, as in the case of Σ–surfaces, there is an infinite number of different complex structures on every differentiable Σ–manifold. In addition, he proves that Σ–manifolds deform into Σ–manifolds and that in a Kähler family of Σ–surfaces, they specialize in a Σ–surface (similar to (ii) above).

The questions and the methods of proof in his dissertation come from algebraic and analytic geometry, as well as from algebraic topology. These methods, including sheaf theory, which came from France, were quite new at that time and began to slowly gain acceptance in Germany, especially in the generation of young mathematicians. In an exemplary way they were embodied by Brieskorn's teacher Friedrich Hirzebruch. In addition to Hirzebruch, Hans Grauert and Reinhold Remmert also had great influence on the development of modern analytic and algebraic geometry in Germany, in particular with the development of the theory of general complex spaces, whose structure might contain nilpotent elements. Brieskorn also thanked Reinhold Remmert and Antonius van de Ven in his dissertation. He met both of them while working as a research assistant and employee in Erlangen in 1962, showing that he had an extremely inspiring environment for modern mathematics. He was decisively influenced by the spirit of optimism that prevailed in Germany at that time, and especially by the support of his teacher Hirzebruch.

Deformation theory

Friedrich Hirzebruch's book *Neue topologische Methoden in der algebraischen Geometrie*, published as early as 1956 in the Springer series "Ergebnisse der Mathematik und ihrer Grenzgebiete", had just been published in the second, extended edition. The theorem of Hirzebruch–Riemann–Roch, which was proved there, was one of the foundations of Brieskorn's dissertation. Another basis was the deformation theory of analytic structures developed by K. Kodaira and D. C. Spencer.

The Hirzebruch–Riemann–Roch theorem was a great generalization of Riemann–Roch's classical theorem to complex vector bundles on arbitrary complex projective manifolds, rather than divisors on Riemann surfaces, using the methods of sheaf theory that were prevailing at the time.

As Brieskorn writes in his CV, this theorem was the reason why he moved from Munich to Bonn and Hirzebruch. In 1963, M. Atiyah and I. Singer generalized the Hirzebruch–Riemann–Roch theorem to elliptic differential operators on a complex manifold. This covers significant theorems of differential geometry and has important applications in theoretical physics, for which they jointly received the Abel Prize in 2004. The theorem was further extended in a functorial way to proper morphisms of quasi-projective schemes by Grothendieck in 1967. Modifications continue to this day, e.g. with the "Quantum Riemann–Roch" in the Gromov–Witten theory.

The deformation theory of complex manifolds developed by Kodaira and Spencer in [45], in particular the infinitesimal deformation theory, and Kuranishi's theorem on the existence of a semi–universal deformation, were later developed further and are now among the most important methods in complex analysis as well as algebraic and arithmetic geometry. I would like to mention only the existence of a semi–universal deformation for compact complex spaces, proven by Hans Grauert and Adrien Douady.

Even more important for the theory of singularities and for Brieskorn's later work was the theorem by Grauert on the existence of a semi-universal deformation of isolated singularities of complex spaces. Brieskorn himself was involved in the development of the proof by Grauert. And that came as follows.

Brieskorn had started his professorship in Göttingen in July 1969, but had been on leave from September 1969 to February 1970 for a research stay at the IHES in Bures-sur-Yvette. From there he brought an interesting problem for the research seminar in Göttingen, jointly led by Brieskorn and Grauert in the summer and winter semester 1970/71: to prove the existence of a semi–universal deformation of isolated singularities of complex spaces. This was to be based on the papers of M. Schlessinger *Functors of Artin Rings* and of G. N. Tyurina *Locally semi - universal flat deformations of isolated singularities of complex spaces*. Tyurina's work was published in 1969 in Russian, but only in 1971 in English translation. The author of these lines, who himself attended the seminar, believes that Brieskorn himself had brought along both these papers. They were not yet known in Göttingen and, in particular, he had procured an English translation of Tyurina's work.

Schlessinger had specified in his work conditions for the existence of a formal semi–universal deformation, the "Schlessinger conditions", while Tyurina had proved the existence for normal isolated singularities, with the additional condition that the second Ext group of the holomorphic 1–forms on the singularity vanishes. Tyurina did not seem to have known Schlessinger's work, and Brieskorn's idea was that the combination of Schlessinger's and Tyurina's approaches should provide evidence of a proof for any isolated singularity without Tyurina's additional condition. He had even more precise ideas on how both works should be brought together to a proof, and he distributed the lectures accordingly to the participants of the seminary. The last lecture was assigned to Grauert with Brieskorn's comment 'and you then prove the general proposition without any assumption'. Grauert answered only 'but I need the Christmas holidays', which caused general joy.

The participation in the seminar was quite demanding, especially for someone who had just started working on his diploma thesis, but at the same time enormously stimulating and enriching. All participants were aware that they were involved in the emergence of a significant result and waited anxiously for Grauert's lecture, which was to take place in early 1971. Grauert then began his lecture with the remark that he unfortunately could not present the proof and he was not sure whether one might need an assumption like that of Tyurina. However, he could recount an interesting generalization of the Weierstrass theorem, the division with remainder by an ideal (a result found independently by Hironaka in connection with the resolution of singularities). This general division theorem then provided the crucial tool for demonstrating the existence of a semi-universal deformation of isolated singularities without any assumption that Grauert completed in the summer of 1971 (published in [34]).

The episode shows Brieskorn's infallible grasp of interesting and important mathematical problems, which can be seen throughout his work and in the selection of topics for diploma and doctoral theses.

Quotient singularities and simultaneous resolution

The years after graduation were among the most fertile in Brieskorn's scientific life. At some point in 1963, Hirzebruch suggested that Brieskorn study and generalize the work *On analytic surfaces with double points* by Michael Atiyah [2]. In this work, Atiyah had shown, among other things, that a family $f : X \to S$ of compact complex surfaces over a smooth 1–dimensional manifold S, whose general fiber is smooth and whose finitely many special fibers only have singularities of the type A_1, has a simultaneous resolution. Here, a simultaneous resolution of a general holomorphic mapping $f{:}X \to S$ is a commutative diagram of holomorphic maps

$$\begin{tikzcd} Y \arrow[r, "\psi"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ T \arrow[r, "\varphi"] & S \end{tikzcd},$$

where φ is a branched covering and g is a non–singular, proper surjective mapping that induces for all fibers $X_s{=}f^{-1}(s)$ of f a resolution $\psi|Y_t$: $Y_t \to X_s,\ \varphi(t) = s$.

Because of the local monodromy around the singular fibers of f, the base change φ is necessary, i.e. a simultaneous resolution of f over S itself is in general not possible. The local "geometric monodromy" about a singular fiber X_{s_0} can be seen as follows: restrict f to a small closed path γ with start and end point s around $s_0 \in S$, so that γ does not go through a singular value of f and simply circles around s_0, you get a locally trivial fiber bundle over γ, that is in general not trivial. By means of a path lifting (for example by using an Ehresmann connection) one obtains a nontrivial diffeomorphism of the fiber X_s, the "geometric monodromy", which induces a nontrivial isomorphism of the homology of X_s. The base change φ eliminates the local monodromy: since all the fibers of g are non-singular, the monodromy of g over T is trivial.

Hirzebruch had proposed to Brieskorn to generalize the work of Atiyah to families of surfaces with singularities of the types A_k, D_k, E_6, E_7, E_8. It turned out to be a wonderful idea, and it started a highly interesting story with many actors and great discoveries. In the end, not only the simultaneous resolution of the families of surfaces with ADE singularities was achieved, but also the discovery of exotic spheres as neighbourhood boundaries of singularities. Brieskorn writes in [27] "I shall be grateful for it to my teacher until the day that I die".

Since the problem is local, it suffices to investigate a map f from $X = \mathbb{C}^3$ to $S = \mathbb{C}$ (or small neighborhoods of zeros) of the form $s = f(x, y, z)$. Here $f(x, y, z) = 0$ is the equation of an ADE singularity, i.e. f is a polynomial of the following list:

$$\begin{array}{lll} A_k: & x^{k+1} + y^2 + z^2, & k \geq 1 \\ D_k: & x^{k-1} + xy^2 + z^2, & k \geq 4 \\ E_6: & x^4 + y^3 + z^2 & \\ E_7: & x^3y + y^3 + z^2, & \\ E_8: & x^5 + y^3 + z^2\ . & \end{array}$$

These polynomials already appeared in the works of Hermann Amandus Schwarz and Felix Klein in the 19th century (see [43]). Since Klein's time, they have appeared in ever new, different contexts (see [36] for an overview) and have fascinated mathematicians to this day. Depending on the context in which they appear, they are also called "simple surface singularities", "rational double points", "Du–Val singularities" or "Kleinian singularities".

The context in which the ADE singularities appear in Klein's work is particularly interesting to us. Klein classified the finite subgroups G from $\mathrm{SL}(2,\mathbb{C})$ up to conjugation and obtained the following groups:

C_{k+1} :	the cyclic group of order $k+1$
D_{k-2} :	the binary dihedral group of order $4(k-2)$
T :	the binary tetrahedral group of order 24
O :	the binary octahedral group of the order 48
I :	the binary icosahedral group of the order 120

These groups are (complex) "reflection groups", i.e. groups generated by complex reflections (finite–order automorphisms that fix a hyperplane), and Klein proved that the ring $\mathbb{C}[z_1,z_2]^G$ of G–invariant polynomials in $\mathbb{C}[z_1,z_2]$ is generated by three invariant polynomials X,Y,Z, which satisfy exactly one relation $f(X,Y,Z)=0$. Klein determined the relations and found that these are given for the groups C_{k+1},D_{k-2},T,O,I by the polynomials A_k,D_k,E_6,E_7,E_8.

Because of this result, the ADE singularities are called "2–dimensional quotient singularities". More generally, let $G\subset \mathrm{GL}(2,\mathbb{C})$ be a finite subgroup acting on $\mathbb{C}^2$ by matrix multiplication from the right and on $\mathbb{C}[z_1,z_2]$ by $(gf)(z_1,z_2)=f((z_1,z_2)g))$ for $f\in\mathbb{C}[z_1,z_2]$ and $g\in G$. The invariant ring $\mathbb{C}[z_1,z_2]^G$ is a finitely generated $\mathbb{C}$–algebra, i.e. there are finitely many invariant polynomials $X_1,\dots,X_n\in\mathbb{C}[z_1,z_2]^G$ with $X_i(0)=0$, and finitely many relations $f_j(X_1,\dots,X_n)=0$ with $f_j\in\mathbb{C}[x_1,\dots,x_n], j=1,\dots,k$, so that the canonical map

$$\mathbb{C}[x_1,\dots,x_n]/\langle f_1,\dots,f_k\rangle\to\mathbb{C}[z_1,z_2]^G, x_i\mapsto X_i,$$

is an isomorphism. The bijection

$$\mathbb{C}^2/G\to X:=\{x\in\mathbb{C}^n|f_1(x)=\cdots=f_k(x)=0\},$$

makes the orbit space of G in a canonical way a normal analytic set in $\mathbb{C}^n$. The space germ $(\mathbb{C}^2/G,0)=(X,0)$ is called "quotient singularity". Naturally, Klein did not yet have this interpretation of the ADE singularity as quotient singularity. It is mainly due to Du Val ([30]).

The fact that ADE singularities are quotient singularities and, of course, the explicit equations were essential for Brieskorn's proof of the simultaneous resolution of these singularities. The accomplishment of this proof was not straightforward, but was interrupted by other great discoveries of Brieskorn. In particular, the case of the E_8 singularity has caused greater difficulties, which is also reflected in the fact that Brieskorn published the proof for the A_k,D_k,E_6 and E_7 singularities in 1966 in [6] but for E_8 only in 1968 in [10].

To determine the base change for a simultaneous resolution, one has to analyze the local monodromy. Since the ADE singularities have weighted homogeneous (or quasihomogeneous) equations, the geometric monodromy is analytically computable, and it is of finite order. In the case of the quotient singularities of the type A_k,D_k,E_k, the monodromy operation on the middle homology of the fiber is a Coxeter element of the reflection group, i.e. the product of the generators in a chamber of G. So it makes sense to consider a base change $s=t^d$, where d is the order of the Coxeter element, the Coxeter number. The fiber product of $X\to S$ and the base change $T\to S$ then has the equation

$$f(x,y,z)-t^d=0,$$

where $f(x,y,z)=0$ is the equation of an ADE singularity. In the A_1 case of Atiyah, we have the equation

$$x^2+y^2+z^2-t^2=0,$$

which after coordinate change has the form

$$z_1z_2 - z_3z_4 = 0.$$

This is the equation of a 3-dimensional singular quadric Q_3 in $\mathbb{C}^4$, i.e. the cone over a non–singular quadric in $\mathbb{P}^3$. By blowing up the vertex, one obtains a non–singular variety Y over T, whose exceptional divisor is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, which can be blown down in two ways to $\mathbb{P}^1$. The resulting varieties Y_1 and Y_2 are two different simultaneous resolutions of the given family. They are so–called “small resolutions” of the 3-dimensional quadric Q_3; small, since the exceptional set is a (rational) curve. The natural bijective equivalence between Y_1 and Y_2 is called (Atiyah–) flop. Flops and flips play a fundamental role in the so–called minimal models of an algebraic variety, flips in the construction itself, which is known only up to dimension 3, while various minimal models are connected by a sequence of flops.

Since for A_k, E_6, and E_8 the equation $f(x, y, z) - t^d = 0$ is the form

$$x^a + y^b + z^c + t^d = 0,$$

Brieskorn tried to construct small modifications for these singularities by attempting to map them to other varieties for which a small resolution was already known. For the quadratic cone Q_3, this meant writing the equation $f(x, y, z) - t^d$ in the form $\phi_1\phi_2 - \phi_3\phi_4$. Using such methods, Brieskorn succeeded in constructing simultaneous resolutions of the A_k, D_k, E_6, and E_7 singularities in 1964 (published in [6]). It also turned out that a simultaneous resolution of the map from a 3–manifold to a 1–manifold is only possible for the ADE singularities. This left only the case of the E_8 singularity.

However, the E_8 singularity proved to be extremely stubborn and Brieskorn failed to construct a simultaneous resolution for them. The various attempts to do so led him to surprising results about the topology and differential topology of singularities, which I will discuss in the next section.

The simultaneous resolution of the “icosahedron singularity” E_8 was found by Brieskorn in September 1966 (published in [10]) using very classical algebraic geometry, as he writes himself. For example, he used a paper by Max Noether from 1889 on rational dual planes and properties of exceptional curves on rational surfaces, which arise from the blowing up 8 points on a plane cubic. Brieskorn found that there are about 700 million simultaneous resolutions of E_8, exactly $2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$, the order of the Weyl group of type E_8. The divisor class group of the local ring of the singularity

$$x^2 + y^3 + z^5 + t^{30} = 0$$

has the structure of the lattice of weights of the root system of E_8. Brieskorn constructed the small resolutions of this singularity using curves with E_8 as a dual graph and thus the simultaneous resolutions of the surface singularity E_8. The various simultaneous resolutions correspond to the Weyl chambers, with the blow up of some ideal class in each chamber resulting in a simultaneous resolution.

Investigations of the simultaneous resolution of the ADE singularities as quotient singularities of $\mathbb{C}^2$ by a finite subgroup of $\mathrm{SL}(2, \mathbb{C})$ led Brieskorn to examine general quotient singularities $\mathbb{C}^2/G$ in [11], where G is any finite subgroup of $\mathrm{GL}(2, \mathbb{C})$. He classified these singularities using results from Mumford, Hirzebruch and above all from Prill by listing all small subgroups $G \subset \mathrm{GL}(2, \mathbb{C})$ (i.e. no element of G has 1 as eigenvalue with multiplicity 1). He determined the resolution graph of $\mathbb{C}^2/G$, weighted by the intersection multiplicities of the exceptional curves. Brieskorn showed that this resolution graph determines the singularity up to analytic isomorphism, and from this fact he deduced the remarkable result about the uniqueness of the 2–dimensional icosahedron singularity:

The ring $\mathbb{C}\{x,y,z\}/\langle x^2+y^3+z^5\rangle$ *(and its completion) is the only non–regular factorial* 2*–dimensional analytic local ring.*

In dimension 3 there are infinitely many factorial as well as non–factorial local rings of isolated hypersurface singularities (from dimension 4 on one has always factoriality).

Brieskorn's work on simultaneous resolution and on quotient singularities played an important role in the further development of the deformation theory of rational surface singularities. I mention here only Oswald Riemenschneider, Jonathan Wahl and in connection with the program of "minimal models" of Shigefumi Mori, János Kollár, Miles Reid and Vyacheslav Shokurov.

Topology of singularities and exotic spheres

In September 1965 Brieskorn took up a C.L.E. Moore Instructorship at MIT in Cambridge/Massachusetts. The problem of the simultaneous resolution of E_8 had not been solved at that time and Brieskorn was looking for solutions in discussions with Heisuke Hironaka at the 1965 Arbeitstagung in Bonn and with Michael Artin and David Mumford at MIT. Brieskorn tried to compute the divisor class group of

$$x^2+y^3+z^5+t^{30}=0,$$

but Mumford suggested to examine first the simpler equation $x^2+y^3+z^5+t^2=0$, that is, the equation of the 3–dimensional E_8 singularity. For practice, Brieskorn started with the 3–dimensional A_2 singularity

$$z_0^3+z_1^2+z_2^2+z_3^2=0$$

and found that it is factorial. He then re-examined the 3–dimensional E_8 singularity and, through a rather tedious explicit resolution, he showed that it is also factorial and that the second cohomology group of the singularity boundary vanishes. Since Brieskorn had expected a non-trivial divisor class group, he turned back to A_2 to better understand their topology.

Then, in September 1965, he made the unexpected discovery that *the boundary of the* 3*–dimensional* A_2*–singularity is homeomorphic to the* 5*–dimensional sphere.* The neighbourhood boundary of a hypersurface singularity in $\mathbb{C}^{n+1}$ is the intersection with a sufficiently small real sphere in the $\mathbb{R}^{2n+2}=\mathbb{C}^{n+1}$ around the singular point. For an isolated singularity, this is a compact $(2n-1)$–dimensional real analytic manifold. The singularity itself, i.e. the set of zeros of the defining equation is, according to Milnor, topologically the cone over the neighbourhood boundary with the singular point as the vertex of the cone. It follows that the 3-dimensional A_2–singularity is topologically a manifold. This discovery came as a complete surprise, because in [32], David Mumford had shown that isolated singularities of algebraic surfaces can never be topologically trivial, unless the singularity is analytically non–singular. Brieskorn then published in [7], that *all odd* $k\geq 3$ *the singularities*

$$z_0^3+z_1^2+\cdots+z_k^2=0,$$

are topological manifolds, so Mumford's theorem is a special phenomenon in dimension two.

The developments in 1965/66, which then led to the discovery of the exotic spheres as neighbourhood boundaries of singularities, are still fascinating in retrospect, above all because of the interaction of the ideas of several participants, which came about through happy circumstances. Hirzebruch reported this discovery in the seminar Bourbaki [41] and later at the 1996 singularity conference in Oberwolfach on the occasion of Brieskorn's 60th birthday; a short version can be found in [42].

Hirzebruch reported on a conference in Rome on Brieskorn's simultaneous resolution of the singularities of types A_k, D_k, E_6, and E_7 when he received a letter from Brieskorn there on 28.09.1965, in which he wrote:

"I have made the somewhat confusing discovery in recent days that there may be 3–dimensional normal singularities that are topologically trivial. I discussed this example with Mumford this afternoon, and he has not found a mistake until this evening; here it is: $X = \{x \in \mathbb{C}^4 | x_1^2 + x_2^2 + x_3^2 + x_4^3 = 0\}$."

This result of Brieskorn was quite exciting at that time and stimulated Hirzebruch, Milnor and others to further study the topology of isolated singularities. Of course, there was no e-mail at this time, but an extensive correspondence between Brieskorn, Hirzebruch, Jänich, Milnor and Nash. Hirzebruch wrote to Brieskorn in March 1966 that he found a close connection between the work of Klaus Jänich, who was also a student of Hirzebruch, on the classification of special $O(n)$–manifolds and the neighbourhood boundary of singularities investigated by Brieskorn. Jänich had studied the operation of a compact Lie group G on a differentiable manifold X without boundary. For special operations, the orbit space X/G is a canonically differentiable manifold with boundary. Motivated by Brieskorn's work, Hirzebruch considered the neighbourhood boundary $\Sigma = \Sigma(k+1, 2, \dots, 2)$ of the A_k–singularities in the $\mathbb{C}^{n+1}$, which was given by the following equations:

$$\begin{aligned} z_0^{k+1} + z_1^2 + \cdots + z_n^2 &= 0 \ , \\ |z_0|^2 + |z_1|^2 + \cdots + |z_n|^2 &= 1 \ . \end{aligned}$$

He proved that the orthogonal group $O(n)$ operates on Σ in a special way in the sense of Jänich, with orbit space the 2-dimensional disk D^2, and that Σ is a homology sphere for even k .

Even more exciting was his discovery that Σ is an exotic 9–sphere for $n = 5$ and $k = 2$, i.e. Σ is homeomorphic but not diffeomorphic to the standard sphere S^9. The boundary $\Sigma(3, 2, 2, 2, 2, 2)$ of the 5–dimensional A_2–singularity turned out to be the 9–dimensional exotic Kervaire–sphere, constructed by Michel Kervaire by the so-called "plumbing" of two copies of the tangential disk bundle of the 5–sphere.

The letter from Hirzebruch to Brieskorn dated 24.03.1966, in which he describes his discovery, was answered by Brieskorn on 29.03.1966 in the following words:

'Klaus Jänich and I had not noticed anything about this connection of our work, and I was delighted how you brought things together.'

While in Kervaire's construction the exotic sphere bounds a parallelizable manifold, Σ is the boundary of a singularity, and at first it remained mysterious where the parallelizable manifold could be found in the singularity image. Exotic spheres were first discovered by John Milnor in [48] and those of a fixed dimension form an abelian group Θ_n, with the connected sum as group operation. The important subgroup bP_{n+1} consisting of those spheres that bound a parallelizable manifold was introduced by Kervaire and Milnor in 1963. They proved that the group Θ_n is finite for $n \geq 5$ and that bP_{4k+2} is either 0 or $\mathbb{Z}/2\mathbb{Z}$ and that the second case occurs exactly when the generator of bP_{4k+2} is the $(4k+1)-$dimensional exotic Kervaire sphere.

Milnor had been stimulated by Brieskorn's example of the neighbourhood boundary $\Sigma(3, 2, 2, 2)$ as a topological manifold to study the neighbourhood boundaries of other singularities and explained his reflections in a letter to John Nash in April 1966. Brieskorn quoted in [27] from this letter:

'Dear John,

I enjoyed talking to you last week. The Brieskorn example is fascinating. After staring at it a while I think I know which manifolds of this type are spheres, but the statement is complicated and the proof doesn't exist yet. Let $\Sigma(p_1, \dots, p_n)$ *be the locus*

$$z_1^{p_1} + \cdots + z_n^{p_m} = 0 \ , \ |z_1|^2 + \cdots + |z_n|^2 = 1 \ .\text{'}$$

Then Milnor continues with a concrete guess which of these manifolds are topological spheres. Brieskorn further mentions that the letter contains on the edge a small sketch of about 1 cm, which he would not have understood at the time.

Milnor's sketch, which I copied from [27] (see Fig. 1), shows the image of the Milnor fibration and thus the parallelizable manifold you are looking for. This sketch later became an icon in the theory of singularities and decorated almost every lecture on the topology of singularities.

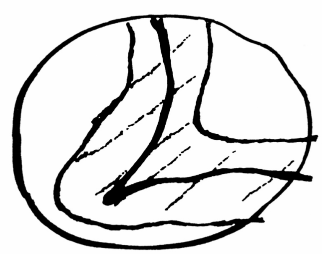


Fig. 1

Choose a sufficiently small sphere B_ε of radius ε around the isolated singular point of the hypersurface $f(z_1, \ldots, z_n) = 0$ and then a small disc D_δ of radius δ ($<< \varepsilon$) in the complex plane around 0. Then Milnor's sketch shows $X = f^{-1}(D_\delta) \cap B_\varepsilon$. $X \setminus f^{-1}(0)$ is a differentiable fiber bundle whose non–singular fiber $X_s = f^{-1}(s), s \neq 0$, is called "Milnor fibre". X_s is a $(n-2)$–connected parallelizable manifold whose boundary is diffeomorphic to the boundary Σ of X_0. Thus, the Milnor fibre is the parallelizable manifold bounded by the exotic Kervaire–sphere, which Hirzebruch and Brieskorn were looking for.

Brieskorn then succeeded in [8] to fully prove Milnor's conjecture within 14 days. At the same time he showed by an explicit calculation:

The neighbourhood boundary $\Sigma(2, 2, 2, 3, 5)$ of the icosahedron singularity is Milnor's exotic 7–sphere, the creator of the group $bP_8 = \Theta_7$ of order 28. All the different 28 exotic differentiable structures on S^7 are given by the boundary $\Sigma(2, 2, 2, 3, 6k - 1), k = 1, \ldots, 28$, hence by simple real analytic equations. In addition, he showed that every odd dimensional sphere bounding a parallelizable manifold is diffeomorphic to the neighbourhood boundary $\Sigma(a_1, \ldots, a_m)$.

This was considered a sensation. While Milnor's first construction of an exotic sphere was indeed very specific, Brieskorn's construction was quite natural and anything but "exotic".

That Brieskorn was able to prove Milnor's conjecture so quickly is also due to a lucky circumstance. Looking through the newly published journals in MIT's library, he came across the work [49] by Frédéric Pham. Pham, motivated by the singularities of Feynman integrals in theoretical physics, examined in this paper exactly the singularities

$$X_1^{a_1} + \cdots + X_n^{a_n} = 0,$$

which Milnor had also considered in his letter to Nash. Pham calculated for these singularities the homotopy type of the Milnor fiber and the monodromy of the Milnor fibration. Brieskorn used these results and Hirzebruch's calculation of the signature of the Milnor fibre to prove the above mentioned results. Since then, these singularities are also called "Brieskorn Singularities" or "Brieskorn–Pham Singularities".

Brieskorn's discovery of the exotic differentiable structures on the neighbourhood boundary of singularities have led to many applications in the work of other mathematicians about the differential topology of manifolds. In this context, there is only one work by Brieskorn himself, the construction of exotic Hopf manifolds, together with Antonius van de Ven in [9].

Brieskorn describes the two years in Boston and Cambridge as the two best of his mathematical life.

Picard–Lefschetz monodromy and Gauss–Manin connection

For an isolated singularity, given by $f \in \mathbb{C}\{x_0, \ldots, x_n\}$, $f(0) = 0$, consider Milnor's construction $f{:}X = f^{-1}(S) \cap B \to S$ with $B = B_\varepsilon$ and $S = D_\delta$ from the previous section. The non–singular Milnor fibre $X_s = f^{-1}(s)$ is a deformation of $X_0 = f^{-1}(0)$, the simplest deformation since it is given by f itself. The Milnor fibre is the singularity-theoretic explanation for the fact that Brieskorn's exotic spheres bound parallelizable manifolds. However, it does not yet explain how these parallelizable manifolds can be constructed by plumbing. This requires a somewhat more complicated deformation, a so-called morsification. The idea dates back to the two volumes *Théorie of the fonctions algébriques de deux variables indépendantes* by Picard-Simart published in 1897 and 1906, and to the monograph *L'analysis situs et la géométry algébrique* by Lefschetz. It was later developed into the local Picard–Lefschetz theory, to which Brieskorn contributed in 1970 in the appendix to [12].

There Brieskorn considers a deformation

$$f_a(x) = f(x) - \sum_{i=0}^{n} a_i x_i \ ,$$

where $a = (a_0, \ldots, a_n)$ is chosen to be sufficiently general. If $\mu = \mu(f)$ denotes the "Milnor number" of f, i.e. the vector space dimension of the Milnor algebra

$$\mathbb{C}\{x_0, \ldots, x_n\}/\langle \frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_n} \rangle,$$

then, in the neighbourhood of $0 \in \mathbb{C}^{n+1}$, there are exactly μ points z_r such that f_a has an ordinary double point in z_r, i.e. has a singularity of type A_1. A deformation with μ ordinary double points near 0 is called a "morsification" of f, an idea that goes back to René Thom. The map $f_a : X^a = f_a^{-1}(S) \cap B \to S$ is singular exactly in the points $z_1, \ldots, z_\mu$ and outside the fibers through these points a differentiable fiber bundle with fiber $X_s^a = f_a^{-1}(s),\ s \neq f_a(z_r)$, diffeomorphic to the Milnor fibre X_s.

Brieskorn now considers Milnor's construction for the ordinary double points z_r of f_a, i.e. $f_a^r : f_a^{-1}(D_r) \cap B_r \to D_r$, where $B_r \subset B$ is a sufficiently small ball around z_r of radius ρ and $D_r \subset S$ is a small disc of radius $\delta << \rho$ around $f_a(z_r)$. Then, for suitable coordinates, $y_0, \ldots, y_n$ in neighbourhood of z_r

$$f_a^r(y_0, \ldots, y_n) = f_a(z_r) + y_0^2 + \cdots y_n^2 \ .$$

It follows that the Milnor fibre F_a^r of f_a^r has the n–dimensional sphere

$$S_r^n = \{y | y \text{ real } , y_0^2 + \cdots + y_n^2 = \rho\}$$

as a deformation retract. $S_r^n \subset F_a^r$ defines a homology class d_r in $H_n(F_a^r, \mathbb{Z})$, $r = 1, \ldots, \mu$, and these are the "vanishing cycles" already considered by Lefschetz, as they contract to the singular point z_r when ρ goes to 0. By choosing appropriate paths γ_r in D from a boundary point of D_r to the non-critical value s, one can transport d_r into the Milnor fibre X_s^a and gets a homology class e_r in $H_n(X_s^a, \mathbb{Z})$. Brieskorn then shows

$$H_n(X_s^a, \mathbb{Z}) = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_\mu.$$

This procedure provides the desired plumbing construction of the Milnor fibre for the ADE–singularities as follows. By choosing an Ehresmann connection for the differentiable fiber bundle $X^a \smallsetminus \underset{r}{\cup} f_a^{-1}(f_a(z_r)) \to D \smallsetminus \{f_a(z_r) | r = 1, \ldots, \mu\}$ the vanishing cycles S_n^r themselves can be transported via γ_r to embedded n–spheres into the Milnor fiber X_s^a, which are also called vanishing cycles there. These vanishing cycles have tubular neighbourhoods in the Milnor fiber that are isomorphic to their tangent disc bundle. For the ADE–singularities, the vanishing cycles can be chosen in such a way that the Milnor fiber can be realized directly with the plumbing construction of these disc bundles as a parallelizable manifold.

However, Brieskorn's main goal in the paper [12] was not to construct vanishing cycles by means of a morsification, but to compute the algebraic monodromy of an isolated hypersurface singularity $f \in \mathbb{C}\{x_0, \ldots, x_n\}$ with $f{:}X \to S$ as at the beginning of this section. The geometric monodromy is a diffeomorphism of the Milnor fiber X_s to itself given by lifting a single closed path γ around 0 in S with start and end point s to the total space of the fiber bundle

$$X' := X \smallsetminus X_0 \to S \smallsetminus \{0\} =: S' \ .$$

The geometric monodromy induces the integral monodromy $H^n(X_s, \mathbb{Z}) \xrightarrow{\cong} H^n(X_s, \mathbb{Z})$ on the middle cohomology group of the Milnor fibre, the local Picard–Lefschetz monodromy of f, whose characteristic polynomial Δ_f largely determines, according to Milnor, the topology of the boundary Σ of f.

Brieskorn gives in this paper an algebraic description of the complex local Picard–Lefschetz monodromy

$$h_f : H^n(X_s, \mathbb{C}) \xrightarrow{\cong} H^n(X_s, \mathbb{C})$$

and derives from that an algorithm for computing the characteristic polynomial Δ_f.

Brieskorn uses holomorphic differential forms to compute the complex monodromy. First, the cohomology groups $H^p(X_s, \mathbb{C}), s \in S'$, are the fibers of a holomorphic vector bundle whose sheaf of holomorphic sections is canonically isomorphic to

$$R^n f_* \mathbb{C}_{X'} \otimes_{\mathbb{C}_{S'}} \mathcal{O}_{S'}.$$

Here $R^n f_* \mathbb{C}_{X'}$ is the n–th direct image sheaf of the constant sheaf $\mathbb{C}_{X'}$. Since the cohomology of the Stein manifold X_s can be calculated using holomorphic differential forms, Brieskorn considers the complex of relative holomorphic differential forms of X over S,

$$\Omega^\bullet_{X/S} = \Omega^\bullet_X / df \wedge \Omega^{\bullet-1} X,$$

with the differential $\Omega^p_{X/S} \to \Omega^{p+1}_{X/S}$ induced by the outer derivative on the complex $\Omega^\bullet_X$ of holomorphic differential forms on the manifold X. One now has a canonical isomorphism

$$R^n f_* \mathbb{C}_{X'} \otimes_{\mathbb{C}_{S'}} \mathcal{O}_{S'} \cong H^n(f_* \Omega^\bullet_{X'/S'})$$

and the right-hand side, the n–th cohomology sheaf of the image sheaf complex $f_* \Omega^\bullet_{X'/S'}$, has with

$$\mathcal{H}^n(X/S) := H^n(f_* \Omega^\bullet_{X/S})$$

a continuation to all of S. Brieskorn shows that $\mathcal{H}^n(X/S)$ is coherent on S and that the stalk satisfies

$$\mathcal{H}^n(X/S)_0 = H^n(\Omega^\bullet_{X/S,0}) =: H,$$

which depends only on the singularity of f in 0.

$\mathcal{H}^n(X/S)$ has as $\mathcal{O}_S$–sheaf the rank $\mu = \mu(f) = \dim_{\mathbb{C}} H^n(X_s, \mathbb{C})$, $s \in S'$, and Brieskorn conjectured that it is locally free, which was shortly thereafter proved by Marcos Sebastiani in [50]. Brieskorn defines on H the (meromorphic) *local Gauss–Manin connection* by the formula

$$\nabla_f \omega = \frac{d\omega}{df} \ .$$

This means that for a representative $\widetilde{\omega} \in \Omega^n_{X,0}$ of ω we have an equation $d\widetilde{\omega} = df \wedge \psi$ where $\frac{d\omega}{df}$ denotes the class of ψ in $\Omega^n_{X/S,0}/d\Omega^{n-1}_{X/S,0}$. That this is well defined follows from the so–called De Rham lemma, in principle a statement about the exactness of the Koszul complex for the regular sequence $\frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_n}$. For k with $f^k \in \langle \frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_n} \rangle$ we have $f^k \frac{d\omega}{df} \in H$ and Brieskorn shows:

∇_f is a singular (meromorphic) first–order differential operator on H whose monodromy (by analytic continuation of a fundamental system of solutions along a closed path in S') is canonical isomorphic to the local Picard–Lefschetz monodromy.

In addition, Brieskorn proves that ∇_f is "regular–singular", i.e. it can be transformed by a meromorphic transformation into a differential operator with a pole of 1st order. From this, Brieskorn derives an algorithm for the computation of the characteristic polynomial Δ_f of the monodromy of ∇_f.

Since Δ_f is an integer polynomial, which is algebraic in a certain sense (as Brieskorn shows), it can be deduced from the regularity of ∇_f together with the solution of the 7th Hilbert problem by Gelfand and Schneider (1934), that *the eigenvalues of the monodromy are roots of unity $e^{2\pi i \mu_j}$ with rational μ_j*. The statement of this theorem is also referred to as "monodromy theorem" that had already been proved by Pierre Deligne in 1970 for global algebraic morphisms by other methods. Brieskorn's proof is considered particularly elegant.

The Manuscripta paper on the local Gauss–Manin connection led to significant developments, among others by Brieskorn's students Kyoji Saito, John Peter Scherk, Wolfgang Ebeling, Claus Hertling and the author of these lines. I myself was a student in Götingen and Brieskorn's first diploma student when he finished the paper. I was given the task to generalize the results to complete intersections. The main difficulty was a generalization of the lemma of De Rham. With the help of cohomological methods, which I used at the suggestion of Jean–Pierre Serre during his visit to Göttingen, the proof was successful and was a main result of my diploma thesis, which was completed in 1971. Later, the "generalized De Rham–Lemma" was further generalized by Kyoji Saito as well as Wolfgang Ebeling and Sabir Gusein–Zade. The local Gauss–Manin connection, along with Malgrange's index theorem for regular–singular differential operators, was also the key to the proof of an algebraic formula for the Milnor number of isolated complete intersection singularities in [35] (announced in the joint work [17]), which was independently derived by Lê Dũng Tráng using topological methods.

Besides the module $H = H^n(\Omega^\bullet_{X/S,0})$ Brieskorn introduced the two modules

$$\begin{array}{rcl} H' : & = & df \wedge \Omega^n_{X,0}/df \wedge d\Omega^{n-1}_{X,0} \quad \text{and} \\ H'' : & = & \Omega^{n+1}_{X,0}/df \wedge d\Omega^{n-1}_{X,0}, \end{array}$$

which are also free $\mathcal{O}_{S,0}$ modules of rank $\mu(f)$, and the Gauss–Manin connection is then identified with a map $\nabla_f : H' \to H''$, $[df \wedge \omega] \to [d\omega]$. The meaning of this ad hoc definition was not clear at first, but it later turned out to be fundamental. H' and H'' are today referred to as the "Brieskorn–lattice" and especially H'' plays an important role in the study of the mixed Hodge structure of isolated singularities and in Kyoji Saito's "higher–residue pairings". In addition to the already mentioned students of Brieskorn important works in this context are due to Morihiko Saito, Daniel Barlet, Claude Sabbah, Mathias Schulze and Christian Sevenheck, to name but a few.

Simple singularities and simple Lie groups

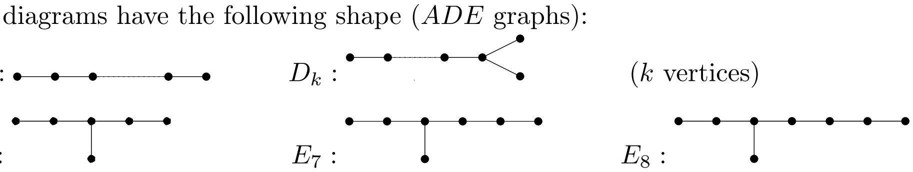
Brieskorn's work on simultaneous resolution of simple singularities led to one of his most important findings, the relationship between ADE singularities and simple Lie groups. He reported on this discovery to the International Congress of Mathematicians in Nice in 1970 and published the result in the short note [13].

Alexander Grothendieck had read Brieskorn's work on simultaneous resolution and was led to a conjecture which he told Brieskorn. While Brieskorn had studied 1–parametric deformations of the ADE singularities given by the defining polynomial, Grothendieck proposed to look at the semiuniversal deformation of these singularities. He suggested that this is determined by the adjoint quotient map of the simple Lie algebra of type A, D or E and that a simultaneous resolution of the semiuniversal deformation of the singularities of the corresponding type is given with the hep of the Springer–resolution of the nilpotent variety. Grothendieck himself had studied simultaneous resolutions of singularities of adjoint quotient maps and had come across the suspected connection through Brieskorn's work.

Let G be a simple complex (algebraic) Lie group, i.e. a complex algebraic manifold with regular group action that is simple as a group. If G is simply connected, then G is uniquely determined by its Lie algebra $\mathfrak{g}$ up to isomorphism. The simple Lie groups correspond to the simple Lie algebras and these are classified by their fundamental root system. The root systems, in turn, are described by their "Dynkin–diagram" (also Coxeter–Dynkin–Witt diagram) and determine G up to isomorphism. The classification of all Dynkin–diagrams resulting from the simple Lie groups provides four infinite series $A_k(k \geq 1), B_k(k \geq 2), C_k(k \geq 3), D_k(g \geq 4)$ and the five exceptional cases E_6, E_7, E_8, F_4 and G_2. The Dynkin–diagrams of type A_k, D_k, E_6, E_7, E_8 are characterized by the fact that they are homogeneous, i.e. their root systems have equal roots. These diagrams have the following shape (ADE graphs):

A_k : D_k : (k vertices)

E_6 : E_7 : E_8 :



The name ADE singularity for the quotient singularities of the finite subgroups of $\mathrm{SL}(2,\mathbb{C})$ comes from the relation to the simple Lie groups of type A_k, D_k or E_k. At the ICM 1970 in Nice, Brieskorn presented the construction of the ADE singularities and their seminuniversal deformation directly with the help of the corresponding Lie group as follows.

Considering the operation of the simple complex Lie group G on itself by conjugation, we call $x \in G$ "regular" if the orbit of x, that is its conjugate class in G, has the maximum dimension. If d is this dimension, then the next smaller orbit dimension is $d-2$ and elements of this orbit dimension are called "subregular". G has exactly one regular orbit, containing $1 \in G$ in its closure, and the closure of this orbit is called $\mathrm{Uni}(G)$, since it is the variety of the unipotent elements of the group. The complement of the regular orbit in $\mathrm{Uni}(G)$ has codimension 2 and is itself the closure of exactly one subregular orbit. If $x \in \mathrm{Uni}(G)$ is an arbitrary element, then consider a small slice $X \subset G$ through x transversal to the orbit of X and a regular projection $\pi : (G,x) \to (X,x)$ of complex space germs. The space germ (X,x) has in (G,x) complementary dimension to the orbit of x and is smooth if x is a regular element. Only slices through non-regular orbits produce singularities. If x is subregular, $X \cap \mathrm{Uni}(G)$ has dimension two and an isolated singularity in x.

Let $x = x_s x_n$ be the Jordan–decomposition of $x \in G$ into a semisimple and unipotent part. Assigning to x the conjugate class of x_s yields a morphism $\Phi : G \to T/W$, where T is a maximal torus in G and W is the Weyl group. Φ is the adjoint quotient map. Each fiber of Φ is the union of finitely many conjugation classes, T/W is a k–dimensional complex manifold (k = number of vertices of the Dynkin diagram) and Φ maps $\mathrm{Uni}(G)$ to $1 \in T/W$. With these notations Brieskorn proved the following in [13].

Let G be a simply connected complex Lie group of the type A_k, D_k, E_6, E_7, E_8 and $x \in G$ a subregular unipotent element. Then:

(1) $(X \cap \ Uni\ (G), x)$ *is isomorphic to an* ADE *singularity of the same type as* G.
(2) *The adjoint quotient map germ in* x *factorises as* $\Psi \circ \pi$,

$$\Phi : (G,x) \xrightarrow{\pi} (X,x) \xrightarrow{\Psi} (T/W, 1),$$

where Ψ *is the semi–universal deformation of the corresponding quotient singularity.*
(3) *Let*

$$\begin{tikzcd} \Gamma \arrow[r] \arrow[d] & G \arrow[d] \\ T \arrow[r] & T/W \end{tikzcd}$$

be the simultaneous resolution of the adjoint quotient map of Grothendieck, with $\Gamma = \{(x,B) | x \in G, B$ *Borel–subgroup containing* $x\}$ *and* Y *the preimage of the transversal slice* X *in* Γ. *Then*

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & X \arrow[d] \\ T \arrow[r] & T/W \end{tikzcd}$$

is a simultaneous resolution of the semiuniversal deformation of the quotient singularity of type A_k, D_k, E_k.

Brieskorn's proof makes essential use of the fact that Φ is given by weighted-homogeneous polynomials and that the ADE singularities are characterized by their weights. Incidentally, the Weyl group W is equal to the monodromy group of the singularity. The proof sketched by Brieskorn in [13] was completely worked out by Peter Slodowy. Slodowy later extended the construction to all simple Lie groups, including the inhomogeneous root systems B_k, C_k, F_4 and G_2, even over fields of arbitrary characteristic [51].

The relationship between Lie groups and singularities shown by Brieskorn led to further investigations. An entirely different construction of the ADE singularities with the help of the simple algebraic groups of type A, D and E is due to Friedrich Knop [44], although the singularities there are realized in different dimensions. A clarification of the occurrence of the polyhedral groups in Brieskorn's construction, and thus a direct relationship between the simple Lie groups and the finite Klein groups, was achieved by Peter Kronheimer [46], using differential geometric methods. Brieskorn had still written at the end of [13]:

'Thus we see that there is a relation between exotic spheres, the icosahedron and E_8. *But I still do not see why the regular polyhedra come in.'*

Slodowy also investigated more complicated singularities and associated them with Kac–Moody Lie algebras. Shortly before his death in 2002, he succeeded in constructing all the simply elliptic singularities with the aid of the adjoint quotient map of the infinite–dimensional loop group. The work was completed by Stefan Helmke in [39]. Slodowy had searched for this result for many years and he reported this to Brieskorn, a few days before his death when he was already badly marked by his illness, but still full of passion and enthusiasm. Brieskorn was deeply touched that the continuation of his ideas could bring consolation and joy even in the hardest hour.

Generalized braid groups, Milnor lattice and Lorentzian space–forms

The construction of the semi-universal deformation of an ADE singularity with the help of the adjoint operation of the corresponding simple Lie group led Brieskorn to investigate operations of generalized braid groups and thus to turn to the investigation of discrete structures of isolated singularities.

Let W be a finite reflection group operating linearly on the real finite–dimensional Euclidean vector space E' and let $D' \subset E'$ be the union of the reflection hyperplanes H'_s, where s is an element of the set of the reflections Σ in W. Brieskorn considers the complexification E of E' resp. H_s of H'_s and the union $D \subset E$ of the $H_s, s \in \Sigma$. The operation of W on E' extends canonically to E and maps $E_{reg} = E \setminus D$ to itself. E_{reg}/W is the space of regular orbits of the finite complex reflection group W, whose fundamental group was calculated by Brieskorn in [14] (see also cite EB1973). He shows:

The fundamental group $\Pi_1(E_{reg}/W)$ has a presentation with generators $g_s, s \in \Sigma$, and relations

$$g_s g_t g_s \cdots = g_t g_s g_t \cdots ,$$

with m_{st} factors on both sides.

Here (m_{st}) is the Coxeter–matrix of W with m_{st} = order of st, and the g_s are given by an explicit geometric construction.

For the symmetric group $W{=}S_n$ $(= A_{n-1})$, the corresponding fundamental group is the braid group B_n introduced in 1925 by Emil Artin, the father of Michael Artin, as proved by Fox and Neuwirth in 1962 and independently by Arnold in 1968. The finite irreducible reflection groups are classified and fall into the types $A_k, B_k, D_k, E_6, E_7, E_8, F_4, G_2, H_3, H_4$ and $I_2(m), m = 5$ or $m \geq 7$. The fundamental groups of the regular orbits of these complex reflection groups are therefore generalizations of the braid groups.

The connection with singularities comes from the fact that for W of type A_k, D_k, E_6, E_7, E_8 the space E_{reg}/W is the complement of the discriminant in the base space of the semiunversal deformation of the simple singularity of the same type. This follows from Brieskorn's construction in [13].

These generalized braid groups were baptized "Artin groups" by Brieskorn and Kyoji Saito in honor of Artin in [15], and they examined them from a combinatorial point of view. Among other things, they solve the word and conjugation problems for these groups and determine the center. These results were obtained at about the same time by Pierre Deligne in [28]. Deligne proved that the spaces E_{reg}/W considered above are Eilenberg–MacLane spaces, as was conjectured by Brieskorn in [14].

In the works mentioned below, Brieskorn studies discrete invariants of special classes of singularities. Let $(X_0, x) \subset (\mathbb{C}^{n+1}, x)$ be an isolated hypersurface singularity and $F{:}X \to S, F(x) = 0$, a suitable representative of the semiuniversal deformation of (X_0, x). If D is the discriminant of F, that is, the set of points $s \in S$ for which the fiber is not smooth, then, with $S' = S \setminus D$, the restriction $F : X' = F^{-1}(S') \to S'$ is a differentiable fiber bundle with fiber X_s, diffeomorphic to the Milnor fiber of (X_0, x). Since X_s has the homotopy type of a bouquet of n–dimensional spheres, the middle homology group $H_n(X_s, \mathbb{Z})$ is free of rank μ, the Milnor number of (X_0, x). If n is even, $H_n(X_s, \mathbb{Z})$ carries an integral symmetric quadratic form, the intersection form $<, >$, and the integer lattice

$$L = H_n(X_s, \mathbb{Z})$$

is called the "Milnor lattice" of the singularity.

If one selects a generic complex line near 0 in the affine space containing S, then the intersection with S is a small disc Δ intersecting the discriminant D in μ different points $c_1, \dots, c_\mu$. The restriction of F over Δ is a morsification of (X_0, x), as described above. For

$s \in \Delta' = \Delta \smallsetminus \{c_1, \dots, c_\mu\}$ and a choice of paths γ_i in Δ' from $s \in \Delta'$ to points near the $c_i, i = 1, \dots, \mu$, one obtains so–called "vanishing cycles" $e_i \in H_n(X_s, \mathbb{Z})$ with $\langle e_i, e_i \rangle = -2$. The set of vanishing cycles is denoted $\Delta^* \subset L$.

By a proper choice of the paths γ_i the $e_1, \dots, e_\mu$ form a basis of the lattice L, which is then called a "distinguished basis". The set of all distinguished bases of L is denoted B^*.

For every basis $B \in B^*$, the matrix of scalar products of the basis elements describes the bilinear form on L, which is characterized by a graph D_B. The vertices $\{1, \dots, \mu\}$ of D_B correspond to the basis elements $e_1, \dots, e_\mu$ and two vertices i, j are connected by $|\langle e_i, e_j \rangle|$ edges, each with the sign ± 1 of $\langle e_i, e_j \rangle \in \mathbb{Z}$. D_B is called (Coxeter–) Dynkin–diagram of B and the set of all Dynkin–diagrams is denoted by D^*.

On B^* and thus on D^* exists a natural operation of the classical braid group B_μ with μ strands, which can be described by elementary operations at the level of the paths. Brieskorn points out at various places that understanding this operation should be essential for an understanding of the semiuniversal deformation.

Another invariant is the "monodromy group" of (X_0, x). By definition, this is the image under the canonical homomorphism of the fundamental group of the complement of the discriminant in the automorphism group of the lattice L. It is already generated by the automorphisms that belong to a distinguished basis.

An overview of these invariants and the relationships between them is given by Brieskorn in the survey article [21], and he stresses their importance for the understanding of the geometry of the semiuniversal deformation. Important work on these invariants are the fundamental works of Andrei Gabrielov [33] and Sabir Gusein–Zade [38] as well as the lecture notes of Wolfgang Ebeling [31].

A first step is to understand the deformation relations between singularities of a fixed modality class, for, if one singularity deforms into another, this induces an inclusion of the corresponding Milnor lattices. The classification of isolated hypersurface singularities in terms of their modality (i.e., the number of independent parameters (moduli) of isomorphism classes in a neighbourhood of the origin of the semiuniversal deformation) was initiated by V.I. Arnold in [1] and is one of the starting points of singularity theory with far–reaching results. The adjacencies (deformation relations) between the ADE singularities were already determined by Arnold. In [19] Brieskorn calculates all possible adjacencies within Arnold's list of unimodular singularities, which is the next more complicated class in Arnold's hierarchy, after the ADE singularities. This work is refined in [22], where Brieskorn gives a very detailed description of the Milnor lattice of the 14 exceptional unimodular singularities.

The deformation relations within the unimodular singularities have been linked by Brieskorn with a theory that seems to be far away from the theory of singularities, namely the theory of partial compactifications of bounded symmetric domains. If $\mathcal{F}(L)$ denotes the isotropy–flag–complex of L, a building in the sense of Tits, then the monodromy group Γ operates on $\mathcal{F}(L)$ and the 1–dimensional simplicial complex $\mathcal{F}(L)/\Gamma$ is finite. For the simplest exceptional singularities E_{12}, Z_{11}, Q_{10}, Brieskorn proves that the Baily–Borel compactification of $\mathcal{F}(L)/\Gamma$ can be identified with the $\mathbb{C}^*$–quotient of the punctured negatively graded part of the base space of the semiuniversal deformation. Independent of Brieskorn, Eduard Looijenga proved these results for all triangular singularities $T_{p,q,r}$ in [47] and later, in a more general context, he constructed important new compactifications of locally symmetric varieties.

In the work [23] Brieskorn gives an overview of the operation of the braid group on the set B^* of distinguished bases of an isolated singularity. He also introduces the concept of an automorphic set, which unifies many aspects of the braid group operation, and which was later taken up and generalized by several authors. The following quote from the introduction shows again Brieskorn's joy in the unity of mathematics, which is expressed in the interplay of many different areas of mathematics.

"The beauty of braids is that they make ties between so many different parts of mathematics, combinatorial theory, number theory, group theory, algebra, topology, geometry and analysis, and, last but not least, singularities."

This brings me almost to the end of the review of Brieskorn's mathematical work. Still to mention is the textbook *Plane algebraic curves* written together with Horst Knörrer, whose latest reprint appeared in English translation in 2012. And of course his two textbooks *Lineare Algebra und Analytische Geometrie I, II* [2] (Linear Algebra and Analytic Geometry I, II), which are also worth reading because of the historical remarks by Erhard Scholz.

There is also a mathematical work together with his students Anna Pratoussevitch and Frank Rothenhäusler from the year 2003 [26], which I would like to mention. The origins of this work date back to at least 1992, when Brieskorn's student Thomas Fischer discovered a polyhedron that in a sense generalizes the classical dodecahedron. Brieskorn gave a report on this discovery in 1996 in Oberwolfach [24]. The polyhedron has a very similar combinatorial structure to the dodecahedron, but with an axis of symmetry of order 7 instead of 5. Let Γ be a discrete subgroup of the Lie group $\widetilde{SU}(1,1)$, which operates by left translations. The quotient $\Gamma\backslash\widetilde{SU}(1,1)$ is a "Lorentzian space–form" and is described by a fundamental domain F for Γ. For co-compact Γ, according to results of Dolgachev, $\Gamma\backslash\widetilde{SU}(1,1)$ is the boundary of a quasi-homogeneous surface singularity. The quasi-homogenous singularities of Arnold's series E_k, Z_k and Q_k are of this type.

In this work, the authors describe the fundamental domains for the corresponding groups Γ as polyhedra with total geodesic faces in the 3–dimensional Lorentz space, each series showing a regular characteristic combinatorial pattern associated with the classical polyhedra.

E_{12}–polyhedron

Brieskorn describes in the movie *Science Lives: Egbert Brieskorn*, see [27], the great joy he felt when Fischer discovered the E_{12} polyhedron and as Pratoussevitch could expand this to the infinite series E_k, Z_k and Q_k. In the sequence "Melencolia" of the film he explains that the correct beginning of this infinite series should be the polyhedron in Dürer's famous etching *Melencolia I*. In the same sequence of the movie, he also discusses the importance of intuition, especially in teaching students, in contrast to a purely analytical and structural approach. Fischer's discovery pleased Brieskorn so much that he himself calculated and drew a graphic representation of it and called it "Opus 2". He commissioned an artist to make a brass 3D–sculpture of the E_{12} polyhedron, which he donated to his teacher Friedrich Hirzebruch on his 75th birthday. As far as

[2] A third volume *Linear Algebra and Analytical Geometry III* was only partially completed by Brieskorn. These parts with valuable historical information and cross–links are being re-typed and will be freely available at https://imaginary.org/ soon.

I know, he made only two or three copies of which he gave me a copy, which made me extremely happy. A picture of this specimen, which closes the circle from the beginnings of the Platonic solids and quotient singularities to the Lorentzian space–forms, may be a fitting conclusion to this review of Brieskorn's mathematical work.

Everyone who knew Egbert Brieskorn valued his extensive mathematical knowledge, his immensely broad general education, his keen intelligence, his straightforwardness and absolute intellectual honesty, his kind attention and helpfulness, and his prudent advice. And everyone who got to know him better knows that he was a kind–hearted person. The history of science will preserve his work and his name, and those who know and appreciate him will not forget him.

Habilitations

Brieskorn supervised 24 Ph.D. dissertations, which can be found in the "Mathematics Genealogy Project". Of his doctoral students, seven have completed their habilitation:

1975: HAMM, HELMUT AREND: Zur analytischen und algebraischen Beschreibung der Picard-Lefschetz-Monodromie. Göttingen.

1980: GREUEL, GERT-MARTIN: Kohomologische Methoden in der Theorie isolierter Singularitäten. Bonn.

1984: SLODOWY, PETER: Singularitäten, Kac-Moody-Lie-Algebren. assoziierte Gruppen und Verallgemeinerungen. Bonn.

1985: KNÖRRER, HORST: Geometrische Aspekte integrabler Hamiltonscher Systeme. Bonn.

1986: EBELING, WOLFGANG: Die Monodromiegruppen der isolierten Singularitäten vollständiger Durchschnitte.Bonn.

1986: SCHOLZ, ERHARD: Symmetrie – Gruppe – Dualität. Studien zur Beziehung zwischen theoretischer Mathematik und Anwendungen in Kristallographie und Baustatik im 19. Jahrhundert. Wuppertal.

2000: HERTLING, CLAUS: Frobenius-Mannigfaltigkeiten, Gauß-Manin- Zusammenhänge und Modulräume von Hyperflächensingularitäten. Bonn.

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SINGULARITIES AND POLYHEDRA[1]

EGBERT BRIESKORN

I reported about work of my students Thomas Fischer, Alexandra Kaess, Ute Neuschäfer, Frank Rothenhäusler and Stefan Scheidt. This work describes the neighbourhood boundaries of quasi–homogeneous surface singularities in a new way. It is known that these neighbourhood boundaries are quotients G/Γ of a 3–dimensional Lie group G and a discrete subgroup Γ. For example, for the quotient singularities $\mathbf{C}^2/\Gamma$ the group G is Spin(3)$=S^3$, the group of unit quaternions, and Γ could for example be one of the three binary polyhedral groups (binary tetrahedral $\mathbb{T}$, binary octahedral $\mathbb{O}$, binary icosahedral $\mathbb{I}$). This gives the three singularities E_6, E_7, E_8. For the next set of examples, the simply–elliptic singularities $\widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8$, the group G is the Heisenberg group, and Γ is a congruence subgroup of the lattice of its integral matrices. In most cases however, G is $\mathrm{SU}(1,1)$ or some covering of it, and Γ comes from a Fuchsian group $\overline{\Gamma} \subset \mathrm{PSU}(1,1)$ acting on the hyperbolic plane $\mathbb{H} = \{x \in \mathbb{C} |\, |z| < 1\}$. All of this is well known.

Now I describe a very original construction discovered by Thomas Fischer in his 1992 PhD–thesis:

Let $\overline{\Gamma} \subset \mathrm{PSU}(1,1)$ be discrete with compact quotient $\mathbb{H}/\overline{\Gamma}$. Assume that $\overline{\Gamma}$ has at least one point in $\mathbb{H}$ with nontrivial isotropy subgroup. *Choose* such a point $o \in \mathbb{H}$. Let p be the order of its isotropy group $\{\overline{\gamma} \in \overline{\Gamma} \,|\, \overline{\gamma}(o) = o\}$. Let $\Gamma \subset \mathrm{SU}(1,1)$ be the inverse image of $\overline{\Gamma}$. For many singularities, the neighbourhood boundary is of the form $\mathrm{SU}(1,1)/\Gamma$ with a suitable $\overline{\Gamma}$. For example, for the 14 quasihomogeneous exceptional 1–modular singularities $E_{12}, E_{13}, E_{14}, Z_{11}, Z_{12}, Z_{13}, Q_{10}, Q_{11}, Q_{12}, W_{12}, W_{13}, S_{11}, U_{12}$ the group Γ is the group of orientation–preserving automorphisms of $\mathbb{H}$ in the group $\sum(p,q,r)$ generated by the reflections in the sides of a hyperbolic triangle with angles $\pi/p, \pi/q, \pi/r$. In this case, the choice of $o \in \mathbb{H}$ amounts to choosing one of the integers in the so–called Dolgachev triple (p,q,r). We shall indicate this by underlining this number, e.g. $(2,3,\underline{7})$. Fischer's construction:

$$\mathrm{SU}(1,1) = \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \,\big|\, a\bar{a} - b\bar{b} = 1 \right\} = \left\{ x \in \mathbb{R}^4 \;\big|\; x_0^2 + x_1^2 - x_3^2 - x_4^2 = 1 \right\} =: \mathbb{S}$$

is a 3–dimensional pseudosphere with Minkowski–metric with signature $(+,-,-)$. Up to a factor $-1/8$, this agrees with the Killing metric. The construction will be done in $\mathbb{R}^4$ with $\langle x, x\rangle = x_0^2 + x_1^2 - x_3^2 - x_4^2$. Let C^+ be the positive cone $C^+ = \{x \in \mathbb{R}^4 \,|\langle x, x\rangle > 0\}$ and $\pi : C^+ \to \mathbb{S}$ be the retraction by central projection $\pi(x) := x/\sqrt{\langle x, x\rangle}$. For any $g \in \mathbb{S}$, let H_g be the halfspace $H_g := \{x \in \mathbb{R}^4 | \langle x, g\rangle \leq 1\}$. Its boundary ∂H_g is the affine tangent space $\partial H_g = T_g(\mathbb{S})$. For any $z \in \overline{\Gamma}(o)$ in the chosen special orbit $\overline{\Gamma}(o) \subset \mathbb{H}$, let L_z be the coset $L_z = \{\gamma \in \Gamma \mid \gamma(o) = z\}$. It has the cardinality $2p$. Let $Q_z \in \mathbb{R}^4$ be defined by

$$Q_z := \bigcap_{g \in L_z} H_g \ .$$

[1] Tagungsbericht 27/1996, Singularitäten 14.07.-20.07.1996, Mathematisches Forschungsinstitut Oberwolfach (MFO)..

Q_z is a 4–dimensional prism, the product of $\mathbb{R}^2$ with a plane $2p$–gon. Consider

$$P := \bigcup_{z \in \overline{\Gamma}(o)} Q_z$$

and $\partial_+ P := \partial P \cap C^+$.

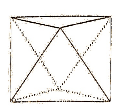
$\partial_+ P$ is the support of a 3–dimensional polyhedral complex and $\pi : \partial_+ P \to \mathbb{S}$ is a homeomorphism, which transfers the polyhedral structure to $\mathbb{S}$. The following definition and theorem of Fischer analyzes this structure:

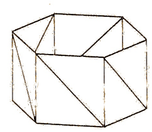
Definition: $F_g = C^+ \cap \partial H_g \cap (Q_{g(o)} \smallsetminus \bigcup_{\substack{z \in \Gamma(o) \\ z \neq g(o)}} Q_z)$.

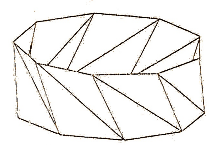
Theorem:

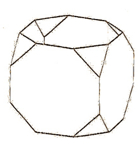
(1) F_g is a compact polyhedron in the Minkowski–3–space ∂H_g
(2) $\{F_g\}_{g \in \Gamma}$ is the set of 3–dimensional faces of a 3–dimensional polyhedral complex with support $\partial_+ P$.
(3) Γ operates simply transitively on $\{F_g | g \in \Gamma\}$.
(4) $\{\pi(F_g)\}$ is a tesselation of $\mathbb{S}$ by totally geodesic polyhedra in this Minkowski–pseudosphere. Γ acts simply transitively on the set of these $\pi(F_g)$, so each of them can serve as a fundamental domain.
(5) Hence $\mathbb{S}/\Gamma$ is obtained from F_G by pairing faces and identifying them in a specified way given by Γ and the construction.

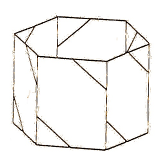
Fischer calculated the examples $(2,3,\underline{7}), (2,3,\underline{8}), (2,3,\underline{9})$. These fit in very well with the classical cases $E_6 = (2,3,\underline{3}), E_7 = (2,3,\underline{4})$ and $E_8 = (2,3,\underline{5})$. I myself added an analysis of the cases $\widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8$. The following pictures show the resulting 9 fundamental domains:

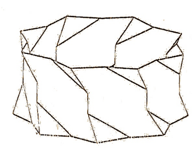

(2,3,3)

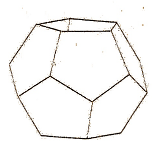

$\widetilde{E}_6$

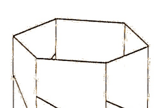

(2,3,9)

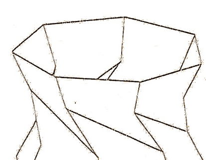

(2,3,4)


$\widetilde{E}_7$

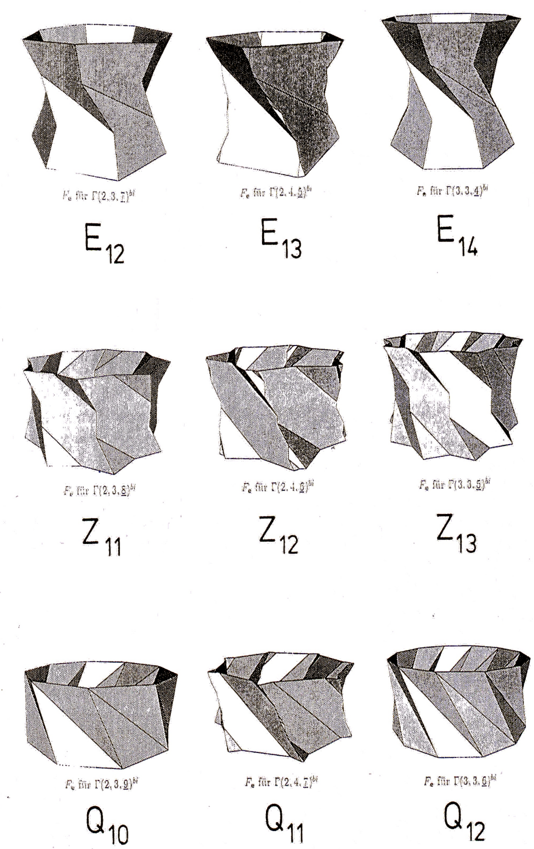

(2,3,8)


(2,3,5)


$\widetilde{E}_8$


(2,3,7)

The other four students worked out all 14 exceptional $(p, q, , \underline{r})$ with the exception of $r = 2$. As a result, a pattern seems to emerge. The following shows a sample of their pictures:



I presented some conjectures on the series–patterns. Work in progress by Ludwig Balke may lead to a new and original way of looking at symmetry–breaking.

The following pages show the handwritten notes of Brieskorn from the "Vortragsbuch" of the singularities workshop 1996 in Oberwolfach.

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Singularities and Polyhedra.

Egbert Brieskorn, Bonn

I reported about work of my students Thomas Fischer, Alexandra Kaess, Ute Neuschäfer, Frank Rothenhäusler and Stefan Scheidt. This work describes the neighbourhood boundaries of quasi-homogeneous surface singularities in a new way. It is known that these neighbourhood boundaries are quotients G/Γ of a 3-dimensional Liegroup and a discrete subgroup Γ. For example, for the quotient singularities $\mathbb{C}^2/\Gamma$ the group G is $\mathrm{Spin}(3) = S^3$, the group of unit quaternions, and Γ could for example be one of the three binary polyhedral groups (binary tetrahedral $\mathbb{T}$, binary octahedral $\mathbb{O}$, binary icosahedral $\mathbb{I}$. ~~As an~~ This gives the three singularities E_6, E_7, E_8. For the next set of examples, the simply-elliptic singularities $\tilde{E}_6, \tilde{E}_7, \tilde{E}_8$ the group G is the Heisenberg group, and Γ is a congruence subgroup of the lattice of its integral matrices. In most cases however, G is $SU(1,1)$ or some covering of it, and Γ comes from a Fuchsian group $\bar{\Gamma} \subset PSU(1,1)$ acting on the hyperbolic plane $\mathbb{H} = \{z \in \mathbb{C} \mid |z| < 1\}$. All this is well known.

Now I describe a very original construction discovered by Thomas Fischer in his 1992 Ph.D. thesis.

Let $\bar{\Gamma} \subset PSU(1,1)$ be discrete with compact quotient $\mathbb{H}/\bar{\Gamma}$. Assume, that $\bar{\Gamma}$ has at least one point in $\mathbb{H}$ with nontrivial isotropy subgroup. Choose such a point $0 \in \mathbb{H}$.
Let p be the order of its isotropy group $\{\bar{\gamma} \in \bar{\Gamma} \mid \bar{\gamma}(0) = 0\}$.
Let $\Gamma \subset SU(1,1)$ be the inverse image of $\bar{\Gamma}$.
For many singularities, the neighbourhood boundary is of the form $SU(1,1)/\Gamma$ with a suitable $\bar{\Gamma}$.
For example, for the 14 quasihomogeneous exceptional 1-modular singularities $E_{12}, E_{13}, E_{14}, Z_{11}, Z_{12}, Z_{13}, Q_{10}, Q_{11}, Q_{12},$

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$W_{12}, W_{13}, S_{11}, S_{12}, U_{12}$ the group Γ is the group of orientation-preserving automorphisms of $\mathbb{H}$ in the Group $\Sigma(p,q,r)$ generated by the reflections in the sides of a hyperbolic triangle with ~~At~~ angles $\pi/p, \pi/q, \pi/r$. In this case, the choice of $0 \in \mathbb{H}$ amounts to choosing one of the integers in the so called Dolgacev triple (p,q,r) We shall indicate this by underlining this number, e.g. $(2,3,\underline{7})$.

Fischer construction:

$$SU(1,1) = \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \; a\bar{a} - b\bar{b} \right\} = \{ x \in \mathbb{R}^4 \mid x_0^2 + x_1^2 - x_3^2 - x_4^2 = 1 \} =: \mathbb{S}$$

is a 3-dimensional pseudosphere with Minkowski-metric $\langle \, , \rangle$ with signature $(+,-,-)$. Up to a factor $-\frac{1}{8}$, this agrees with the Killing metric. The construction will be done in $\mathbb{R}^4$ with $\langle x, x \rangle = x_0^2 + x_1^2 - x_3^2 - x_4^2$.

Let C^+ be the positive cone $C^+ = \{ x \in \mathbb{R}^4 \mid \langle x, x \rangle > 0 \}$ and $\pi : C^+ \to \mathbb{S}$ be the retraction by central projection $\pi(x) = x / \sqrt{\langle x, x \rangle}$.

For any $g \in \mathbb{S}$, let H_g be the halfspace $H_g = \{ x \in \mathbb{R}^4 \mid \langle x, g \rangle \leq 1 \}$.

Its boundary ∂H_g is the ~~to~~ affine tangentspace $\partial H_g = T_g(\mathbb{S})$.

For any ~~$g \in \Gamma \in \mathbb{S}$, let~~ $z \in \Gamma(0)$ in the chosen special orbit $\Gamma(0) \subset \mathbb{H}$, let L_z be the coset $L_z = \{ \gamma \in \Gamma \mid \gamma(0) = z \}$. It has cardinality $2p$.

Let $Q_z \subset \mathbb{R}^4$ be defined by

$$Q_z = \bigcap_{g \in L_z} H_g .$$

Q_z is ~~the~~ a 4-dimensional prism, the product of $\mathbb{R}^2$ with a plane $2p$-gon. Conside

$$P = \bigcup_{z \in \Gamma(0)} Q_z$$

and $\partial_+ P = \partial P \cap C^+$.

$\partial_+ P$ is the support of a 3-dimensional polyhedral complex and $\pi : \partial_+ P \to \mathbb{S}$ is a homeomorphism, which transfers the polyhedral structure to $\mathbb{S}$. The following Definition and Theorem of Fischer analyzes this structure.

<u>Definition:</u> $$F_g = C^+ \cap \partial H_g \cap \Big(Q_{g(0)} \setminus \bigcup_{\substack{z \in \Gamma(0) \\ z \neq g(0)}} \mathring{Q}_z \Big)$$

<u>Theorem:</u>

(i) F_g is a compact polyhedron in Minkowski-3-space ∂H_g

(ii) $\{F_g\}_{g \in \Gamma}$ is the set of 3-dim. faces of a 3-dim. polyhedral complex with support $\partial_+ P$.

(iii) Γ operates simply transitively on $\{F_g \mid g \in \Gamma\}$.

(iv) $\{\pi(F_g)\}$ is a tesselation of $\$$ by totally geodisic polyhedra in this Minkowski-pseudosphere. Γ acts simply transitively on the set of these $\pi(F_g)$, so each of them can serve as a fundamental domain.

(v) Hence $\$/\Gamma$ is obtained from F_g by pairing faces and identifying them in a specified way given by Γ and the construction.

Fischer calculated the examples $(2,3,\underline{7})$, $(2,3,\underline{8})$, $(2,3,\underline{9})$. These fit in very well with the classical cases $E_6 = (2,3,\underline{3})$, $E_7 = (2,3,\underline{4})$ and $E_8 = (2,3,\underline{5})$. I myself added an analysis of the cases $\tilde{E}_6, \tilde{E}_7, \tilde{E}_8$. The next page shows the resulting 9 fundamental domains. The other four students worked out all 14 exceptional $(p,q,\underline{r})$ with the exception of $r=2$. As a result, a pattern seems to emerge. The second following page shows a sample of their pictures. I presented some conjectures on the series-pattons. Work in progress by Ludwig Bahlke may lead to a new and original way of looking at symmetry-breaking

Egbert Brieskorn

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SINGULARITIES AND EXOTIC SPHERES[1]

FRIEDRICH HIRZEBRUCH

In: Conference Report 27/1996, Singularities 14.07.-20.07.1996, Mathematical Research Institute Oberwolfach, "Brieskorn–Day". 16.07.1996, Lecture on the occasion of the 60th birthday of Egbert Brieskorn, short version.

Report on the academic year 1965/66. Brieskorn is C.L.E. Moore Instructor at M.I.T., Jänich is at Cornell University, then at IAS in Princeton. I am in Bonn. There is an extensive correspondence. From 30.09.–07.10.1965 I'm at a conference in Rome (report on Brieskorn's simultaneous resolutions). Brieskorn's letter from 28.09.1965 reaches me there: *"I have made the somewhat confusing discovery in recent days that there may be 3 - dimensional normal singularities that are topologically trivial. I discussed the example with Mumford this afternoon, and he had not found a mistake by this evening: here it is:* $X = \{x \in \mathbf{C}^4 \mid x_1^2 + x_2^2 + x_3^2 + x_4^3 = 0\}$*.".* Proof by resolution and calculation of all invariants of the neighbourhood boundary. In the Proc. Nat. Aca. Sci. USA appears the more general example $x_1^2 + \cdots + x_k^2 + x_{k+1}^3 = 0$ (k odd).

Report on the extensive correspondence that follows, about Brieskorn's discovery of the work of Pham, which allows him to prove Milnor's assertion in a letter to Nash - Milnor to Nash on 13.04.1966: *"The Brieskorn example is fascinating. After starting at if for a while, I think I know which manifolds of this type are spheres but the statement is complicated and a proof does not exist. Let* $\sum(p_1, \ldots, p_n)$ *be the locus* $z_1^{p_1} + \cdots + z_n^{p_n} = 0, |z_1|^2 + \cdots + |z_n|^2 = 1$ *where* $p_j \geq 2$ *..."* Then Milnor gives the condition a) or b) for the exponents. – Gradually it becomes clear to all parties that for the determination of the differentiable structure the calculation of the signature of $z_1^{p_1} + \cdots + z_n^{p_n} = 1 (n \geq 3,\ n \text{ odd})$ is required. There are several letters from Brieskorn to me and vice versa. Brieskorn writes his paper for the Inventiones Vol. 2 (1966). In this context, he also studied $(2, 3, 5, 30), 30 =$ Coxeter number of E_8, and he finally accomplished the small resolutions of this singularity in curves according to the E_8–tree and thus the simultaneous resolution of the surface families $x_1^2 + x_2^3 + x_3^5 + t^{30} = 0$ (parameter t) and the remaining case of his paper in Math. Ann. of 1966 (about which I reported in Rome). Understanding was achieved within the framework of the root systems and the Weyl group (Brieskorn's letter to Mrs. Tjurina dated 13.09.1966) – Jänich had studied $O(n)$–manifolds $W^{2n-1}(d)$ (two orbit types with isotropy groups $O(n-2), O(n-1)$ and orbit space D^2, S^i), and classified them as well as the knot manifolds $M^{2n+1}(k)$ on which $O(n)$ operates (three orbit types $O(n-2), O(n-1), O(n)$ with orbit space $D^4, S^3 - k, k$ (k the knot)). I bring the two located in the USA together by a report from March 1966, e.g. $W^{2n-1}(d)$ is $\sum(2, \ldots, 2, d)$ and M^{2n+1} (torus knot $3, 5$) is $\sum(2, \ldots, 2, 3, 5)$. Brieskorn writes on 29.03.1966: *"Klaus Jänich and I had not noticed anything about this connection of our work, and I was completely overjoyed, how you brought things together."*

I had the same joy here in Oberwolfach, to be able to tell about it.

[1] Translated from the german article in: Tagungsbericht 27/1996, Singularitäten 14.07.-20.07.1996, Mathematisches Forschungsinstitut Oberwolfach.

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ON THE b-EXPONENTS OF GENERIC ISOLATED PLANE CURVE SINGULARITIES

E. ARTAL BARTOLO[1], PI. CASSOU-NOGUÈS[2], I. LUENGO[3], AND A. MELLE-HERNÁNDEZ[3]

Dedicated to the memory of Egbert Brieskorn with great admiration

Abstract. In 1982, Tamaki Yano proposed a conjecture predicting how is the set of b-exponents of an irreducible plane curve singularity germ which is generic in its equisingularity class. In 1986, Pi. Cassou-Noguès proved the conjecture for the one Puiseux pair case in [9]. In [1] the authors proved the conjecture for two Puiseux pairs germs whose complex algebraic monodromy has distinct eigenvalues. A natural problem induced by Yano's conjecture is, for a generic equisingular deformation of an isolated plane curve singularity germ to study how the set of b-exponents depends on the topology of the singularity. The natural generalization suggested by Yano's approach holds in suitable examples (for the case of isolated singularites which are Newton non-degenerated, commode and whose set of spectral numbers are all distincts). Morevover we show with an example that this natural generalization is not correct. We restrict to germs whose complex algebraic monodromy has distinct eigenvalues such that the embedded resolution graph has vertices of valency at most 3 and we discuss some examples with multiple eigenvalues.

Introduction

Let $f:(\mathbb{C}^n,0)\to(\mathbb{C},0)$ be a germ of a complex analytic function whose zero locus

$$(f^{-1}(0),0)\subset(\mathbb{C}^n,0)$$

defines an isolated hypersurface singularity germ, that is the Minor number of f at 0,

$$\mu(f,0):=\dim_{\mathbb{C}}\frac{\mathbb{C}\{z_1,\dots,z_n\}}{\left(\frac{\partial f}{\partial z_1},\dots,\frac{\partial f}{\partial z_n}\right)}$$

is finite. A *Milnor fibration* was constructed in [19] as follows. Set $B_\varepsilon=\{z\in\mathbb{C}^n:|z|<\epsilon\}$ and $S_\epsilon=\{z\in\mathbb{C}^n:|z|=\epsilon\}$, one can choose ϵ_0 such that for all $0<\epsilon\le\epsilon_0$, $f^{-1}(0)$ is transverse to S_ϵ. For $0<\eta\ll\epsilon_0$ and $D_\eta=\{t\in\mathbb{C}:|t|<\eta\}$, let $X(t)=f^{-1}(t)\cap B_{\epsilon_0/2}$ and $X=f^{-1}(D_\eta)\cap B_{\epsilon_0/2}$. By Milnor, for such suitable ϵ and η, the mapping $X\setminus f^{-1}(0)\to D_\eta\setminus\{0\}$ is a C^∞-locally trivial fibration whose general fibre $F_{f,0}$, called *Milnor fibre*, has the homotopy type of a bouquet of exactly $\mu(f,0)$ of $(n-1)$-dimensional spheres.

The geometric monodromy $h_{F_{f,0}}:F_{f,0}\to F_{f,0}$ of the Milnor fibration is the monodromy transformation of the Milnor fibration over the loop $c\exp(2\pi t),t\in[0,1]$ and c small enough. The geometric monodromy induces the complex algebraic monodromy $h^{a,j}:H^j(F_{f,0},\mathbb{C})\to H^j(F_{f,0},\mathbb{C})$

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[2]Partially supported by MTM2016-76868-C2-1-P and MTM2016-76868-C2-2-P.
[3]Partially supported by the grant MTM2016-76868-C2-1-P and Grupo Singular UCM.

whose eigenvalues are roots of unity. Since the Milnor fibre is a connected bouquet of $(n-1)$-spheres, the only interesting algebraic monodromy is $h^{a,n-1} : H^{n-1}(F_{f,0}, \mathbb{C}) \to H^{n-1}(F_{f,0}, \mathbb{C})$, where $\dim_{\mathbb{C}} H^{n-1}(F_{f,0}, \mathbb{C}) = \mu(f, 0)$.

Let $\mathcal{O}$ be the ring of germs of holomorphic functions on $(\mathbb{C}^n, 0)$, let $\mathcal{D}$ be the ring of germs of holomorphic differential operators of finite order with coefficients in $\mathcal{O}$. Let s be an indeterminate commuting with the elements of $\mathcal{D}$ and set $\mathcal{D}[s] = \mathcal{D} \otimes_{\mathbb{C}} \mathbb{C}[s]$.

Given a holomorphic germ $f \in \mathcal{O}$, one considers $\mathcal{O}\left[\frac{1}{f}, s\right] \cdot f^s$ as a free $\mathcal{O}\left[\frac{1}{f}, s\right]$-module of rank 1 with the natural $\mathcal{D}[s]$-module structure. Then, there exits a non-zero polynomial $B(s) \in \mathbb{C}[s]$ and some differential operator $P = P(x, \frac{\partial}{\partial x}, s) \in \mathcal{D}[s]$, holomorphic in $x_1, \ldots, x_n$ and polynomial in $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$, which satisfy the following functional equation in $\mathcal{O}\left[\frac{1}{f}, s\right] f^s$:

$$P(s, x, D) \cdot f(x)^{s+1} = B(s) \cdot f(x)^s. \tag{1}$$

The monic generator $b_{f,0}(s)$ of the ideal of such polynomials $B(s)$ is called the *Bernstein-Sato polynomial* (or b-function or Bernstein polynomial) of f at 0. The same result holds if we replace $\mathcal{O}$ by the ring of polynomials in a field $\mathbb{K}$ of zero characteristic with the obvious corrections, see e.g. [12, Section 10, Theorem 3.3].

This result was first obtained for f polynomial by Bernstein in [3] and in general by Björk [4]. One can prove that $b_{f,0}(s)$ is divisible by $s+1$, and we also consider the *reduced Bernstein-Sato polynomial*

$$\tilde{b}_{f,0}(s) := \frac{b_{f,0}(s)}{s+1}.$$

In the case where f defines an isolated singularity, one can consider the nowadays called *Brieskorn lattice* $H_0^{''} := \Omega^n / df \wedge d\Omega^{n-2}$ introduced by Brieskorn in [8], and its saturation

$$\tilde{H}_0^{''} = \sum_{k \geq 0} (\partial_t t)^k H_0^{''}.$$

Malgrange [18] showed that the reduced Bernstein polynomial $\tilde{b}_{f,0}(s)$ is the minimal polynomial of the endomorphism $-\partial_t t$ on the vector space $F := \tilde{H}_0^{''} / \partial_t^{-1} \tilde{H}_0^{''}$, whose dimension equals the Milnor number $\mu(f, 0)$ of f at 0. Following Malgrange [18], the set of b-*exponents* are the μ roots $\{\tilde{\beta}_1, \ldots, \tilde{\beta}_\mu\}$ of the characteristic polynomial of the endomorphism $-\partial_t t$. Recall also that $\exp(-2i\pi \partial_t t)$ can be identified with the (complex) *algebraic monodromy* of the corresponding Milnor fibre $F_{f,0}$ of the singularity at the origin.

Kashiwara [15] expressed these ideas using differential operators and considered

$$\mathcal{M} := \mathcal{D}[s] f^s / \mathcal{D}[s] f^{s+1},$$

where s defines an endomorphism of $\mathcal{D}(s) f^s$ by multiplication. This morphism keeps invariant $\tilde{\mathcal{M}} := (s+1)\mathcal{M}$ and defines a linear endomorphism of $(\Omega^n \otimes_{\mathcal{D}} \tilde{\mathcal{M}})_0$ which is naturally identified with F and under this identification $-\partial_t t$ becomes the endomorphism defined by the multiplication by s.

In [18], Malgrange proved that the set $R_{f,0}$ of roots of the Bernstein-Sato polynomial is contained in $\mathbb{Q}_{<0}$, see also Kashiwara [15], who also restricts the set of candidate roots. The number $-\alpha_{f,0} := \max R_{f,0}$ is the opposite of the *log canonical threshold* of the singularity and Saito [21, Theorem 0.4] proved that

$$R_{f,0} \subset [\alpha_{f,0} - n, -\alpha_{f,0}]. \tag{2}$$

Also Saito in [20] showed that the local moduli of μ-constant deformation is determined by the *Brieskorn lattice* if the μ-constant stratum is smooth, as in the case of germs of plane curves where he gave in [20, p. 30] a more simple formula describing the reduced Bernstein-Sato. There are many papers devoted to study Bernstein-Sato polynomial but it would be worthwhile to refer to the existence of a relative Bernstein-Sato polynomial in [5], by Briançon et al., and for results on the computation of the roots of Bernstein-Sato polynomial for functions with isolated singularity, even if the methods used in [6] are different. In [7], Briançon et al. gave a multiple of the Bernstein-Sato polynomial for any two variables function with isolated singularities. Some general properties of μ-constant deformations are also given by Varchenko in [24].

There is another set which is important too, the set of *exponents of the monodromy* (or spectral numbers, up to the shift by one, in the terminology of Varchenko [25]). This notion was first introduced by Steenbrink [22].

Let $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$ be a germ of a holomorphic function with isolated singularity. In [22] Steenbrink constructed a mixed Hodge structure on $H^{n-1}(F_{f,0}, \mathbb{C})$. Let

$$H^{n-1}(F_{f,0}, \mathbb{C})_\lambda = \ker(T_s - \lambda : H^{n-1}(F_{f,0}, \mathbb{C}) \longrightarrow H^{n-1}(F_{f,0}, \mathbb{C}));$$

where T_u, T_s are, respectively, the unipotent and semi-simple factors of the Jordan decomposition of the monodromy h^{n-1}.

The set $\operatorname{Spec}(f)$ of spectral numbers are μ rational numbers

$$0 < \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_\mu < n$$

which are defined by the following condition:

$$\#\{j : \exp(-2\pi i \alpha_j) = \lambda, \lfloor \alpha_j \rfloor = n - p - 1\} = \dim_{\mathbb{C}} \operatorname{Gr}_F^p H^{n-1}(F_{f,0}, \mathbb{C})_\lambda, \qquad \lambda \neq 1$$
$$\#\{j : \alpha_j = n - p\} = \dim_{\mathbb{C}} \operatorname{Gr}_F^p H^{n-1}(F_{f,0}, \mathbb{C})_1.$$

The set $\operatorname{Spec}(f)$ of spectral numbers is symmetric, that is $\alpha_i + \alpha_{\mu-(i-1)} = n$. It is known that this set is constant under μ-constant deformation of f, see [25].

As it is well-known, neither the Bernstein-Sato polynomial nor the b-exponents are constant along μ-constant deformation. Given an equisingular type, a generic set of b-exponents or a generic Bernstein-Sato polynomial are expected. In [27], Yano proposed a formula (see next section) for the generic b-exponents for irreducible germs of curves (combined with the Jordan form of the monodromy, this also yields to a formula for the generic Bernstein polynomial). This formula was proved for one-Puiseux pair germs by the second named author in [10] and reproved by M. Saito in [20].

In [1], the conjecture was proved for irreducible singularities with two Puiseux pairs and monodromy without multiple eigenvalues. In this paper, we discuss how to extend the formula for reducible germs of singularities. There is a natural interpretation of Yano's formula in terms of the resolution graph of the singularity, see (5). We are going to prove in this paper that this formula holds for singularities with vertices of valency at most 3 (and at most two vertices of valency 3) and monodromy without multiple eigenvalues (distinct from 1) (in fact, the correct hypothesis may be distinct exponents of the monodromy, besides 1).

The restriction on the number 3-valency vertices comes from technical reason but it is most probably avoidable; for example, the second named author proved it in [11] for singularities with non-degenerate and commode Newton polygon (and distinct exponents for the monodromy). The other two conditions seem to be more important, since we will give examples where it does not hold in at least two cases: germs where the vertices have valencies at most 3 but there are multiple exponents, and germs with vertices with valency greater than 3. We will discuss also other examples and we will introduce the needed results about improper integrals.

1. Extended Yano's problem

Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be a germ of a non-zero holomorphic function such that its zero locus defines an isolated singularity germ.

Extended Yano's Problem ([27]). *For a generic equisingular deformation of an isolated plane curve singularity germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ and Milnor number μ, to study how the set of b-exponents $\{\tilde{\beta}_1, \ldots, \tilde{\beta}_\mu\}$ depends on the topology of f.*

The local Bernstein-Sato polynomial $b_{f,0}(s)$ of a singularity germ is a powerful analytic invariant, but it is, in general, extremely hard to compute, even in the case of irreducible plane curve singularities. It is well-known that the Bernstein-Sato polynomial varies in families in the (non-singular) μ-constant stratum $\Sigma_{\mu(f,0)}$ of f at 0. Since, for plane curves this stratum is irreducible, it is conceivable that a *generic* Bernstein-Sato polynomial exists, i.e., the Bernstein-Sato polynomial of a germ f with the same topology as f, depends on f, but there is a *generic* Bernstein-Sato polynomial $b^{\mathrm{gen}}_{\Sigma_{\mu(f,0)}}(s)$: for every μ-constant deformation of such an f, there is a Zariski dense open set $\mathcal{U}$ on which the Bernstein-Sato polynomial of any germ in $\mathcal{U}$ equals $b^{\mathrm{gen}}_{\Sigma_{\mu(f,0)}}(s)$.

1.1. **The original Yano's conjecture: the irreducible case.**

Let f be an irreducible germ of plane curve. In 1982, Tamaki Yano [27] made a conjecture concerning the b-exponents of such germs. Let $(n, b_1, b_2, \ldots, b_g)$ be the characteristic sequence of f, see e.g. [26, Section 3.1]. Recall that this means that $f(x, y) = 0$ has as root (say over x) a Puiseux expansion

$$x = \cdots + a_1 y^{\frac{b_1}{n}} + \cdots + a_g y^{\frac{b_g}{n}} + \ldots$$

with exactly g characteristic monomials. Denote $b_0 := n$ and define recursively

$$e^{(k)} := \begin{cases} n & \text{if } k = 0, \\ \gcd(e^{(k-1)}, b_k) & \text{if } 1 \leq k \leq g. \end{cases}$$

We define the following numbers for $1 \leq k \leq g$:

$$R_k := \frac{1}{e^{(k)}} \left(b_k e^{(k-1)} + \sum_{j=0}^{k-2} b_{j+1} \left(e^{(j)} - e^{(j+1)} \right) \right), \qquad r_k := \frac{b_k + n}{e^{(k)}}.$$

Note that R_k admits the following recursive formula:

$$R_k := \begin{cases} n & \text{if } k = 0, \\ \frac{e^{(k-1)}}{e^{(k)}} \left(R_{k-1} + b_k - b_{k-1} \right) & \text{if } 1 \leq k \leq g. \end{cases}$$

We end with the following definitions $R'_0 := n$, $r'_0 := 2$ and for $1 \leq k \leq g$:

$$R'_k := \frac{R_k e^{(k)}}{e^{(k-1)}}, \quad r'_k := \left\lfloor r_k e^{(k)} / e^{(k-1)} \right\rfloor + 1.$$

Yano defined the following polynomial with fractional powers in t

$$R(n, b_1, \ldots, b_g; t) := t + \sum_{k=1}^{g} t^{\frac{r_k}{R_k}} \frac{1 - t}{1 - t^{\frac{1}{R_k}}} - \sum_{k=0}^{g} t^{\frac{r'_k}{R'_k}} \frac{1 - t}{1 - t^{\frac{1}{R'_k}}}, \tag{3}$$

and he proved that $R(n, b_1, \ldots, b_g; t)$ has non-negative coefficients.

Yano's Conjecture ([27]). *For almost all irreducible plane curve singularity germs $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ with characteristic sequence $(n, b_1, b_2, \ldots, b_g)$, the b-exponents $\{\tilde{\beta}_1, \ldots, \tilde{\beta}_\mu\}$ are given by the generating series*

$$\sum_{i=1}^{\mu} t^{\tilde{\beta}_i} = R(n, b_1, \ldots, b_g; t).$$

For almost all means for an open dense subset in the μ-constant strata in a deformation space.

Yano's conjecture holds for $g = 1$ as it was proved by Pi. Cassou-Noguès in [10] making explicitly a relation between two variables improper integrals and the Bernstein-Sato polynomial of f, see also [9].

In [1], the authors, with the same ideas, were interested in the case $g = 2$. For $g = 2$, the characteristic sequence (n, b_1, b_2) can be written as $(n_1 n_2, mn_2, mn_2+q)$ where $n_1, m, n_2, q \in \mathbb{Z}_{>0}$ satisfying

$$\gcd(n_1, m) = \gcd(n_2, q) = 1.$$

In [1] we solve Yano's conjecture for the case

$$\gcd(q, n_1) = 1 \text{ or } \gcd(q, m) = 1. \tag{4}$$

The above condition is equivalent to ask for the algebraic monodromy to have distinct eigenvalues. In that case, the μ b-exponents are all distinct and they coincide with the opposite of roots of the reduced Bernstein-Sato polynomial (which turns out to be of degree μ).

To encode the topology of a germ of an irreducible plane curve singularity

$$(C = f^{-1}\{0\}, 0) \subset (\mathbb{C}^2, 0)$$

several sets of invariants can be used: Puiseux characteristic exponents, Puiseux pairs, Newton pairs, (minimal) embedded resolution graph, Eisenbud-Neumann splice diagram, semigroup $\Gamma_{(C,0)} \subset \mathbb{N}$ generated by all the possible intersection multiplicities $i(\{h = 0\}, C)$ at 0 for all $h \in \mathcal{O}_{(\mathbb{C}^2,0)}$, etc.

Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be a germ of a non-zero holomorphic function f. Let B be an open ball centered at the origin. Let $\pi : X \to B$ be an embedded resolution of $(f^{-1}\{0\}, 0)$. We denote by $E_i, i \in J$, the irreducible components of $\pi^{-1}(f^{-1}\{0\})_{\text{red}}$. For every $i \in J$, let N_i and $\nu_i - 1$ be the multiplicities of E_i in the divisor of respectively $f \circ \pi$ and $\pi^*(dx \wedge dy)$ on X. One has that N_i and ν_i belong to $\mathbb{N}^*$ and if E_i is an irreducible component of the strict transform of $f^{-1}\{0\}$ then $\nu_i = 1$. Denote also $\mathring{E}_i := E_i \setminus (\cup_{j \neq i} E_j)$ for $i \in J$. Then one has the following interpretation of the $R(n, b_1, \dots, b_g; t)$

$$R(n, b_1, \dots, b_g; t) = t - \sum_{i \in J,\, E_i \neq \tilde{C}} \chi(\mathring{E}_i) t^{\nu_i / N_i} \frac{1 - t}{1 - t^{1/N_i}}$$

where $\tilde{C}$ is the unique strict transform of $f^{-1}\{0\}$. For a vertex i of the minimal embedded resolution graph its valency δ_i is the number of adjacent vertices to it. A vertex is called a *rupture vertex* if its valency is at least 3. Most of the vertices in the resolution graph have valency 2 and since the corresponding exceptional divisors E_i are rational curves $\chi(\mathring{E}_i) = 0$. Furthermore in this case the valency of the vertex are either 1, 2 or 3.

The shape of the minimal embedded resolution graph in this case is the same as the Eisenbud-Neumann splice diagram (cf. [14, page 49]). If the germ $(C, 0)$ has g Newton pairs $\{(p_k, q_k)\}_{k=1}^g$ with $\gcd(p_k, q_k) = 1$ and $p_k \geq 2$ and $q_k \geq 1$ (and by convention, $q_1 > p_1$), define the integers $\{a_k\}_{k=1}^g$ by $a_1 := q_1$ and $a_{k+1} := q_{k+1} + p_{k+1} p_k a_k$ for $k \geq 1$. Then its Eisenbud-Neumann splice diagram decorated by the following splice data $\{(p_k, a_k)\}_{k=1}^g$ and has the following shape:

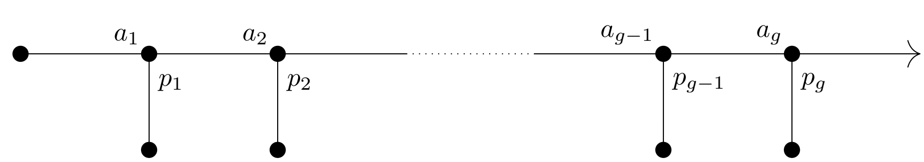


FIGURE 1.

The g rupture components $\tilde{E}_1,\dots,\tilde{E}_g$, ordered from the left to the right of the resolution graph are the same as in the splice diagram and their numerical data can be computed inductively from the

$$\begin{array}{ll} \tilde{N}_k := a_k \cdot p_k \cdot p_{k+1} \cdot \ldots \cdot p_g & \text{for } 1 \leq k \leq g; \\ \tilde{\nu}_k := p_k \tilde{\nu}_{k-1} + q_k & \text{where } \tilde{\nu}_0 = 1, \end{array}$$

The numerical data associated to the components $g+1$ components of valency 1 $E_0, E_1, \dots, E_g$, here E_0 is the most left hand side vertex corresponding to the first blow-up and its numerical data is equal to $(N_0, \nu_0) = (n, 2)$ with $n = p_1 p_2 \cdots p_g$. The numerical data associated to other valency one components can be also computed from

$$\begin{array}{ll} N_k = a_k \cdot p_{k+1} \cdot \ldots \cdot p_g & \text{for } 1 \leq k \leq g; \\ \nu_k = \tilde{\nu}_{k-1} + \lceil \frac{q_k}{p_k} \rceil & \text{for } 1 \leq k \leq g \end{array}$$

1.2. **Yano's conjecture for isolated germs of plane curves.**

A natural extension of the Yano conjecture for isolated plane curve singularity germ could be the following conjecture

Extended Yano's Conjecture. *For almost all isolated plane curve singularity germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ with isolated singularity and Milnor number μ, the b-exponents $\{\tilde{\beta}_1, \dots, \tilde{\beta}_\mu\}$ are given by the generating series*

$$\sum_{i=1}^{\mu} t^{\tilde{\beta}_i} = t + \sum_i (\delta_i - 2) \left(t^{\nu_i/N_i} \frac{1-t}{1-t^{1/N_i}} \right), \tag{5}$$

showing how b-exponents depends on the topology of f.

Example 1.1. Let $f(x, y) = y^4 - x^6$ be a germ with two $\mathbb{A}_2$-singularities having intersection number equals 6. The minimal embeded resolution graph has 3 exceptional divisors E_1, E_2, E_3 with numerical data (N, ν, δ) given respectively by equals $(4, 2, 1)$, $(6, 3, 1)$ and $(12, 5, 4)$. Then (5) equals

$$t + 2\left(t^{5/12} \frac{(1-t)}{(1-t^{1/12})}\right) - \left(t^{2/4} \frac{(1-t)}{(1-t^{1/4})} + t^{3/6} \frac{(1-t)}{(1-t^{1/6})}\right)$$

equals

$$t + t^{4/3} + t^{5/4} + t^{7/6} + 2t^{13/12} + 2t^{11/12} + t^{5/6} + t^{3/4} + t^{2/3} + 2t^{7/12} + 2t^{5/12}.$$

Using `Singular` [13] inside [23], a μ-constant versal deformation of f is given by

$$g(x, y, a, b) := f + ax^3y^2 + bx^4y^2$$

and the Bernstein-Sato polynomial of g for random values of a and b is equal to

$$-17/12, -4/3, -5/4, -7/6, -13/12, -1, -11/12, -5/6, -3/4, -2/3, -7/12, -5/12,$$

so that they do not coincide.

This can be confirmed using `checkRoot` for $s = -17/12$ of [16] in `Singular` [13], where the base field is $\mathbb{C}(a, b)$. Moreover, it can be proved that for general a, b the Tjurina number equals the expected value for Hertling-Stahlke bound, i.e., 14; using [17] the values of Tjurina number are constant in these μ-constant strata.

The previous example shows that the proposed conjecture may not hold when there are vertices with valency greater than 3. Based on the irreducible case we want to study the conjecture for the case where valencies are at most 3.

Modified extended Yano's Conjecture. *Let Σ_μ be the μ-constant stratum of a germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ of isolated singularity, such that no eigenvalue $\zeta \neq 1$ of the monodromy is mutiple (in particular the valency of the vertices of the resolution graph is at most 3). Then the μ b-exponents $\{\tilde{\beta}_1, \ldots, \tilde{\beta}_\mu\}$ of a generic element of Σ_μ are given by the generating series (5)*

Most probably, the hypothesis on the monodromy can be replaced *no repeated non-integral exponent of the monodromy* as the result in [11] for non-degenerate Newton polynomial germs suggests; some examples in the last section go in the same direction. The condition on the valency seems to be more essential, due to Example 1.1.

1.3. Singularities with non-degenerated principal part and commode.

Assume that the power series f has non-degenerated principal part and denote its Newton polygon at 0 by Γ_f , with ℓ facets and commode (Γ_f meets with $x = 0$ at $(0, \tau_0)$ and with $y = 0$ at $(\sigma_0, 0)$). We also assume that the set $\operatorname{Spec}(f)$ of spectral numbers are distinct.

Assume that $f_i(x, y) = 1$, with $f_i(x, y) = \frac{c_i x + d_i y}{n_i}$, is the equation of the facet F_i of Γ_f so that $\gcd(c_i, d_i, n_i) = 1$, $1 \leq i \leq \ell$.

Set

$$\mathcal{N} = \{q \in \mathbb{Q} : \sigma_0 q \in \mathbb{N} \, or \, \tau_0 q \in \mathbb{N}\}.$$

Let b_f be the monic polynomial such that its roots are the rational numbers $\sigma_{i,k} := -\frac{c_i + d_i + k}{n_i}$: with $0 \leq k < n_i$ and for all facet F_i such that $\sigma_{i,k} \notin \mathcal{N}$.

Theorem 1.2 ([11, Theorem 1])**.** *For almost all germs of plane curves which have Γ_f as Newton polygon at the origin and all non-integral elements in $\operatorname{Spec}(f)$ are distinct then f admits b_f as Bernstein-Sato polynomial.*

Note that Example 1.1 does not satisfy the hypotheses of the above theorem. The minimal embeded resolution graph of germs in Theorem 1.2 has all exceptional divisors of valencies exactly $1, 2$ and 3. There are at most 2 divisors with valency 1 and ℓ divisors of valency 3. For all $1 \leq i \leq \ell$, let E_i be the corresponding divisor has numerical data $(N_i, \nu_i, \delta_i) = (n_i, c_i + d_i, 3)$. So that the roots in this case appear as in the EN-diadram of the germ. So that a generic equisingular deformation of f admits b_f as Bernstein-Sato polynomial.

If two spectral numbers are congruent $\bmod \mathbb{Z}$, their difference is ± 1, and they correspond to a 2-Jordan block of the monodromy, so we can recover the b-exponents from the Bernstein-Sato polynomial.

Proposition 1.3. *If the germ f is Newton non-degenerated with respect to its Newton polygon, commode and all the spectral numbers are distinct then for a generic equisingular deformation of f the b-exponents are given by (5).*

2. Improper integrals

Most of the results in this section come from [1]. We start with 1-variable improper integrals.

Proposition 2.1. *Let $f : [0, 1] \times \mathbb{C} \to \mathbb{C}$ be an analytic function. Then the function*

$$s \mapsto \int_0^1 f(t, s) t^s \frac{dt}{t}$$

is holomorphic on $\Re s > 0$ and admits a meromorphic continuation to $\mathbb{C}$ with poles contained in $\mathbb{Z}_{\leq 0}$. Moreover, if $f(t, s)$ is algebraic whenever t is algebraic and s rational, then, the residues are algebraic.

If the function f is independent of s, then the above function will be denoted by $G_f(s)$. Let us consider now the 2-variable case.

Proposition 2.2. *Let $f \in \mathbb{R}[x,y]$ such that $f > 0$ in $[0,1]^2$ and let $a_1, b_1, a_2, b_2 \in \mathbb{Z}_{\geq 0}$ (by convention $\frac{b_i}{a_i} = +\infty$ if $a_i = 0$). The function*

$$s \mapsto \int_0^1 \int_0^1 f(x,y)^s x^{a_1 s + b_1} y^{a_2 s + b_2} \frac{dx}{x} \frac{dy}{y}.$$

is holomorphic in $\Re s > \max\left(-\frac{b_1}{a_1}, -\frac{b_2}{a_2}\right)$ and admits a meromorphic continuation on $\mathbb{C}$, where the set of poles is a subset of $S = \left\{-\frac{b_1+\nu_1}{a_1},\ \nu_1 \in \mathbb{Z}_{\geq 0}\right\} \cup \left\{-\frac{b_2+\nu_2}{a_2},\ \nu_2 \in \mathbb{Z}_{\geq 0}\right\}$.

We can be more explicit on those poles.

Proposition 2.3. *With the hypotheses of Proposition 2.2, let $\alpha \in S$.*

(P1) *If $\alpha = -\frac{b_1+\nu_1}{a_1}$ for some $\nu_1 \in \mathbb{Z}_{\geq 0}$ and $\alpha \neq -\frac{b_2+\nu_2}{a_2}$ $\forall \nu_2 \in \mathbb{Z}_{\geq 0}$, then the pole is of order at most one and its residue equals*

$$\frac{1}{\nu_1! a_1} G_{h_{\nu_1,\alpha,x}}(a_2\alpha + b_2), \quad h_{\nu_1,\alpha,x}(y) := \frac{\partial^{\nu_1} f^\alpha}{\partial x^{\nu_1}}(0,y).$$

(P2) *If $\alpha = -\frac{b_2+\nu_2}{a_2}$ for some $\nu_2 \in \mathbb{Z}_{\geq 0}$ and $\alpha \neq -\frac{b_1+\nu_1}{a_1}$ $\forall \nu_1 \in \mathbb{Z}_{\geq 0}$, then the pole is of order at most one and its residue equals*

$$\frac{1}{\nu_2! a_2} G_{h_{\nu_2,\alpha,y}}(a_1\alpha + b_1), \quad h_{\nu_2,\alpha,y}(x) := \frac{\partial^{\nu_2} f^\alpha}{\partial y^{\nu_2}}(x,0).$$

(P3) *If $\alpha = -\frac{b_1+\nu_1}{a_1} = -\frac{b_2+\nu_2}{a_2}$ for some $\nu_1, \nu_2 \in \mathbb{Z}_{\geq} 0$, then the pole is of order at most 2 and the coefficient of $(s-\alpha)^{-2}$ in the Laurent expansion is*

$$\frac{1}{\nu_1! \nu_2! a_1 a_2} \frac{\partial^{\nu_1+\nu_2} f^\alpha}{\partial x^{\nu_1} \partial y^{\nu_2}}(0,0).$$

(P4) *If in the previous situation the pole is of order at most one, then the continuation of the functions $G_{h_{\nu_1,\alpha,x}}$ and $G_{h_{\nu_2,\alpha,y}}$ are holomorphic at $a_2\alpha + b_2$ and $a_1\alpha + b_1$, respectively and its residue equals*

$$\frac{1}{\nu_1! a_1} G_{h_{\nu_1,\alpha,x}}(a_2\alpha + b_2) + \frac{1}{\nu_2! a_2} G_{h_{\nu_2,\alpha,y}}(a_1\alpha + b_1).$$

The last result does not appear in [1] but it can be deduced easily. The following lemma is useful for the residue computations.

Lemma 2.4. *Let $p \in \mathbb{N}$ and $c \in \mathbb{R}_{>0}$. Given $s_1, s_2 \in \mathbb{C}$ such that $-\alpha = s_1 + s_2 > 0$ then*

$$G_{(y^p+c)^\alpha}(ps_1) + G_{(1+cx^p)^\alpha}(ps_2) = \frac{c^{-s_2}}{p} \boldsymbol{B}(s_1, s_2) \tag{6}$$

where $\boldsymbol{B}$ is the beta function*.*

In [1], we proceeded as follows. For a fixed equisingularity type, we consider *generic* polynomial representatives f with real algebraic coefficients, in some field $\mathbb{K}$, and such that for a suitable semi-algebraic compact domain $\mathcal{D}$, we had $f > 0$ in $\mathcal{D} \setminus \{(0,0)\}$ (the origin is in the boundary of $\mathcal{D}$). For a special choice of coordinates and a *weight* function g we consider the following integrals

$$\mathcal{I}(f, g, \beta_1, \beta_2, \beta_3)(s) := \int_{\mathcal{D}} f(x,y)^s x^{\beta_1} y^{\beta_2} g(x,y)^{\beta_3} \frac{dx}{x} \frac{dy}{y} \tag{7}$$

where $\beta_1, \beta_2, \beta_3 + 1 \in \mathbb{Z}_{>0}$. These integrals are holomorphic in a semiplane of $\mathbb{C}$ and admitted a meromorphic continuation (see Example 4.3 for an idea of the proof). The knowledge of the residues allowed us to prove the following theorem.

Theorem 2.5. *Let $f \in \mathbb{K}[x,y]$ be as above. Let α be a pole of $\mathcal{I}(f,\beta_1,\beta_2,\beta_3)(s)$ with transcendental residue, and such that $\alpha+1$ is not a pole of $\mathcal{I}(f,\beta'_1,\beta'_2,\beta'_3)(s)$ for any $(\beta'_1,\beta'_2,\beta'_3)$. Then α is a root of the Bernstein-Sato polynomial $b_f(s)$ of f.*

3. Partial proof of the conjecture

We are going to prove the modified extended conjecture when the number of rupture vertices is small.

Theorem 3.1. *The extended Yano's conjecture holds for germs of plane curve singularities with no multiple eigenvalues of the monodromy (except maybe* 1*), and such that there are at most two rupture vertices and their valency is at most* 3*.*

Sketch of the proof. As we have seen in Example 1.1, the valency condition and the non-existence of multiple values distinct from 1 seem to be essential. The condition of 1 or 2 branching vertices is only technical.

There are three types of such singularities.

- (S1) The resolution graph is linear.
- (S2) The germ is the product of two irreducible germs with one-Puiseux pair (m,n) and intersection number $> mn$, and eventually two smooth branches with intersection numbers m,n with the singular branches.
- (S3) The resolution graph coincides with the one of a two-Puiseux pair irreducible (which is part of the germ).

The case (S1) is a consequence of [11, Theorem 1]. The case (S2) is represented by the μ-constant versal deformation of $f = x^\epsilon y^\eta((y^m-x^n)^2 - x^u y^v)$, where $\epsilon,\eta \in \{0,1\}$ and u,v depend on the intersection number of the two singular branches. We omit the cases where there are multiple eigenvalues distinct from 1. We follow the strategy in [1]. The presence of x,y does not affect this strategy as we explain later for (S3). If there are more than 2 branches, 1 is a multiple eigenvalue of the monodromy. Nevertheless, the only point where this condition is needed is for Varchenko's lower semicontinuity [24] and only eigenvalues distinct from 1 cannot be multiple for this result.

Let us finish with (S3). Let us consider the improper integral $\mathcal{I}(f,g,\beta_1,\beta_2,\beta_3)$ of (7), studied in [1], where $\beta_1,\beta_2,\beta_3+1 \in \mathbb{Z}_{>0}$, f,g are real polynomials positive on $[0,1]^2 \setminus \{(0,0)\}$, f is a 2-Puiseux-pair germ singularity for which the Newton polygone is of type $(y^m \pm x^n)^p$, g is a 1-Puiseux pair singularity with Newton polygone $y^m \pm x^n$ and maximal contact with f. For (S3) we replace f by $x^\epsilon y^\eta f g^\gamma$, $\epsilon,\eta,\gamma \in \{0,1\}$. We repeat the process as in [1]. □

4. Computations on examples with multiple eigenvalues

Example 4.1. Let us consider $f(x,y) = y^5 + x^2y^2 + x^5$; its μ-constant miniversal deformation is a singleton, so its Bernstein-Sato polynomial coincides with the generic one. This singularity does not satisfy [11, Theorem 1] since the exponents $\pm\frac{1}{10}, \pm\frac{3}{10}$ appear twice ($\pm\frac{1}{2}$ appear only once). Using `Singular`, the Bernstein polynomial is

$$\left(s+\frac{1}{2}\right)^2\left(s+\frac{7}{10}\right)\left(s+\frac{9}{10}\right)(s+1)\left(s+\frac{11}{10}\right)\left(s+\frac{13}{10}\right).$$

The extended conjecture is satisfied even though we are not in the hypotheses of the modified one.

Example 4.2. Let us consider $f(x,y) = y^5+x^2y^2+x^7$; its μ-constant versal deformation is also a singleton, so its Bernstein polynomial coincides with the generic one. This singularity *does* satisfy

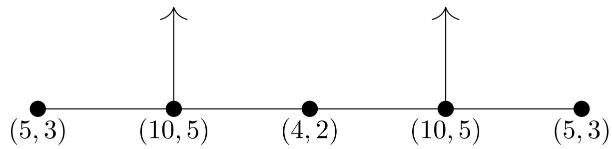
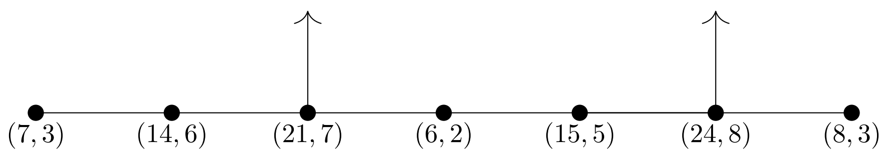


FIGURE 2. Resolution graph of $y^5 + x^2y^2 + x^5$ with (N, ν)-data.

[11, Theorem 1] since $\pm\frac{1}{2}$ appear as exponents of the monodromy, even though $\exp\left(2i\pi\frac{\pm 1}{2}\right) = -1$ is a double eigenvalue. Using `Singular`, we can confirm the expected Bernstein-Sato polynomial.

Example 4.3. Let us consider $f(x, y) = x^3y^3 + x^7 + y^8$; a μ-constant versal deformation is given by $f_{t,s}(x, y) := x^3y^3 + x^7 + tx^6y + sxy^7 + y^8$. As in the previous example the hypotheses of [11, Theorem 1] are satisfied and hence the extended conjecture holds; note that there are multiple eigenvalues for the monodromy but the exponents of the monodromy are distinct.



Yano's candidates start at $\frac{1}{3} = \frac{7}{21} = \frac{8}{24}$. The particular Bernstein-Sato polynomials may depend on s, t; let us study some jumps using improper integrals. Choose $t, s \in \mathbb{R}_{\geq 0}$; note that $f_{t,s} > 0$ in $[0, 1]^2 \setminus \{(0, 0)\}$. Let us denote, for $\beta_1, \beta_2 \in \mathbb{Z}_{\geq 1}$:

$$\mathcal{I}_{\beta_1,\beta_2} = \int_{[0,1]^2} f_{t,s}(x, y)^s x^{\beta_1} y^{\beta_2} \frac{dx}{x}\frac{dy}{y}$$

Let us decompose this square in two domains:

$$\{(x, y) \in [0, 1]^2 \mid x^{\frac{4}{3}} \leq y \leq 1\}, \quad \{(x, y) \in [0, 1]^2 \mid 0 \leq y \leq x^{\frac{4}{3}}\}.$$

Integrating on each subdomain we decompose $\mathcal{I}_{\beta_1,\beta_2} = \mathcal{I}_{1,\beta_1,\beta_2} + \mathcal{I}_{2,\beta_1,\beta_2}$.

Let us consider the change of variables $x \mapsto xy^3$, $y \mapsto y^4$:

$$x \mapsto xy^3, \quad y \mapsto y^4 \Longrightarrow \mathcal{I}_{1,\beta_1,\beta_2} = 4\int_{[0,1]^2} \tilde{f}_{t,s}(x, y)^s x^{\beta_1} y^{3\beta_1+4\beta_2+21s} \frac{dx}{x}\frac{dy}{y}$$

where

$$\tilde{f}_{t,s}(x, y) := tx^6y + sxy^{10} + x^7 + x^3 + y^{11}.$$

In the same way is $x \mapsto x^3$, $y \mapsto x^4y$;

$$x \mapsto x^3, \quad y \mapsto x^4y \Longrightarrow \mathcal{I}_{2,\beta_1,\beta_2} = 3\int_{]0,1]^2} f^*_{t,s}(x, y)^s x^{3\beta_1+4\beta_2+21s} y^{\beta_2} \frac{dx}{x}\frac{dy}{y}.$$

where

$$f^*_{t,s}(x, y) := txy + sx^{10}y^7 + x^{11}y^8 + y^3 + 1.$$

Note that I_{2,β_1,β_2} satisfies the hypotheses of Proposition 2.2, which was the goal of these changes of variables. Since it is not the case for I_{1,β_1,β_2}, let us perform a decomposition of the square as

$$\{(x, y) \in [0, 1]^2 \mid 0 \leq y \leq x^{\frac{3}{11}}\}, \quad \{(x, y) \in [0, 1]^2 \mid x^{\frac{3}{11}} \leq y \leq 1\},$$

and denote the corresponding integral decomposition as $I_{1,\beta_1,\beta_2} = I_{1,1,\beta_1,\beta_2} + I_{1,2,\beta_1,\beta_2}$. Suitable changes of variables yield:

$$x \mapsto x^{11},\ y \mapsto x^3 y \Longrightarrow \mathcal{I}_{1,1,\beta_1,\beta_2} = 44\int_{[0,1]^2} \hat{f}_{t,s}(x,y)^s x^{4(5\beta_1+3\beta_2+24s)} y^{3\beta_1+4\beta_2+21s} \frac{dx}{x}\frac{dy}{y},$$

where

$$\hat{f}_{t,s}(x,y) := tx^{36}y + sx^8y^{10} + x^{44} + y^{11} + 1,$$

and

$$x \mapsto xy^{11}, \quad y \mapsto y^3 \Longrightarrow \mathcal{I}_{1,2,\beta_1,\beta_2} = 12\int_{[0,1]^2} \check{f}_{t,s}(x,y)^s x^{\beta_1} y^{4(5\beta_1+3\beta_2+24s)} \frac{dx}{x}\frac{dy}{y},$$

where

$$\check{f}_{t,s}(x,y) := tx^6y^{36} + sxy^8 + x^7y^{44} + x^3 + 1.$$

The candidate pole $-\frac{8}{21}$ can be pole only for $\beta_1 = \beta_2 = 1$, and in this case the residue is

$$\begin{gathered}
\frac{44}{21}\int_0^1 \frac{\partial \hat{f}^{-\frac{8}{21}}}{\partial y}(x,0)x^{-\frac{32}{7}}\frac{dx}{x} + \frac{3}{21}\int_0^1 \frac{\partial f^{*-\frac{8}{21}}}{\partial x}(0,y)y\frac{dy}{y} = \\
-\frac{8\cdot 44t}{21^2}\int_0^1 (1+x^{44})^{-\frac{29}{21}} x^{\frac{220}{7}}\frac{dx}{x} - \frac{3\cdot 8t}{21^2}\int_0^1 (1+y^3)^{-\frac{29}{21}} y^2 \frac{dy}{y} = \\
-\frac{8t}{21^2}\int_0^1 (1+u)^{-\frac{29}{21}} u^{\frac{5}{7}}\frac{du}{u} - \frac{8t}{21^2}\int_0^1 (1+u)^{-\frac{29}{21}} u^{\frac{2}{3}}\frac{du}{u} = -\frac{8t}{21^2}\boldsymbol{B}\left(\frac{5}{7},\frac{2}{3}\right).
\end{gathered}$$

Hence, for $t \neq 0$ (and algebraic), we have that $-\frac{8}{21}$ is a root of the Bernstein-Sato polynomial. Note that we can prove that $-\frac{29}{21}$ is a pole of $\mathcal{I}_{7,2}$ with transcendental residue for any (algebraic) value of t,s. In particular, $-\frac{29}{21}$ is a root of the Bernstein polynomial if $t=0$ and s is algebraic after Theorem 2.5. Note that $-\frac{8}{21}$ and $-\frac{29}{21}$ cannot be simultaneously roots of the Bernstein-Sato polynomial, since $\exp\left(-2i\pi\frac{8}{21}\right) = \exp\left(-2i\pi\frac{29}{21}\right)$ is a simple eigenvalue of the monodromy. These results are confirmed by `Singular` and `checkRoot`. We have then proved that there is a function f_0 in the μ-constant stratum such that $-\frac{8}{21}$ is not a root of Bernstein-Sato polynomial for f_0, compare with [2]

Example 4.4. Let us consider $f_\pm(x,y) := (x^4-y^3)^2 + x^6y^2$ which corresponds to the case (S3). A μ-constant versal deformation is given by $f_{\mathbf{t}}(x,y) = f_\pm(x,y) + t_1x^8y + t_2x^9$. Let $\mathcal{D} := \{(x,y)\in[0,1]^2 \mid 0 \leq y \leq x^{\frac{4}{3}}\}$ and for $t_1,t_2 \in \mathbb{R}_{\geq 0}$, consider

$$\mathcal{I}_{\beta_1,\beta_2,\beta_3} := \int_{\mathcal{D}} f_{\mathbf{t}}(x,y)^s x^{\beta_1} y^{\beta_2}(x^4-y^3)^{\beta_3}\frac{dx}{x}\frac{dy}{y}$$

for $\beta_1,\beta_2,\beta_3+1 \in \mathbb{Z}_{>0}$. In order to check that it is holomorphic with meromorphic continuation, we perform a first change of variable:

$$x \mapsto x^3, y \mapsto x^4(1-y) \Longrightarrow \mathcal{I}_{\beta_1,\beta_2,\beta_3} = 3\int_{[0,1]^2} \tilde{f}_{\mathbf{t}}(x,y)^s x^{3\beta_1+4\beta_2+12\beta_3+24s} y^{\beta_3+1} q(y)\frac{dx}{x}\frac{dy}{y}$$

where $q(y) := (1-y)^{\beta_2-1}(3-3y+y^2)^{\beta_3}$ and

$$\tilde{f}_{\mathbf{t}}(x,y) = y^2(3-3y+y^2)^2 + x^2(1-y)^2 + t_1x^4(1-y) + t_2x^3.$$

Decomposing the square in two triangles with the diagonal line, we can decompose

$$\mathcal{I}_{\beta_1,\beta_2,\beta_3} = \mathcal{I}_{1,\beta_1,\beta_2,\beta_3} + \mathcal{I}_{2,\beta_1,\beta_2,\beta_3};$$

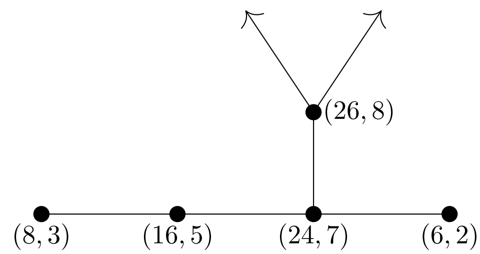


FIGURE 3.

with the following changes of variables we obtain

$$x \mapsto x, y \mapsto xy \Longrightarrow \mathcal{I}_{1,\beta_1,\beta_2,\beta_3}{=}3\int_{[0,1]^2} \hat{f}_{\mathbf{t}}(x,y)^s x^{3\beta_1+4\beta_2+13\beta_3+1+26s} y^{\beta_3+1} q(xy) \frac{dx}{x}\frac{dy}{y}$$

and $x \mapsto xy,\ y \mapsto y \Longrightarrow$:

$$\mathcal{I}_{2,\beta_1,\beta_2,\beta_3}{=}3\int_{[0,1]^2} \check{f}_{\mathbf{t}}(x,y)^s x^{3\beta_1+4\beta_2+12\beta_3+24s} y^{3\beta_1+4\beta_2+13\beta_3+1+26s} q(y) \frac{dx}{x}\frac{dy}{y},$$

where

$$\hat{f}_{\mathbf{t}}(x,y) = y^2(3-3xy+x^2y^2)^2 + (1-xy)^2 + t_1x^2(1-xy) + t_2x,$$
$$\tilde{f}_{\mathbf{t}}(x,y) = (3-3y+y^2)^2 + x^2(1-y)^2 + t_1x^4y^2(1-y) + t_2x^3y.$$

Example 4.5. A μ-constant miniversal deformation for $f(x,y) = (y^2-x^3)^2 + x^{12}$ is constant. It does not satisfy the hypotheses of the modified extended conjecture, since there are multiple eigenvalues (and multiple exponents of the monodromy) but, nevertheless, the extended conjecture holds.

Example 4.6. Let $f(x,y) := x(y^3-x^2)(y^2-x^{10})$, with μ-constant miniversal deformation $f_t(x,y) := f(x,y) + ty^7$. This example has multiple eigenvalues (besides 1) and it is a counterexample for the extended conjecture. It is not hard to prove that $\frac{19}{13}$ is not a Yano's candidate while $-\frac{19}{13}$ is a root of the Bernstein polynomial as it can be checked with `checkRoot` in `Singular` (working over $\mathbb{C}(t)$ instead of randomly evaluating t).

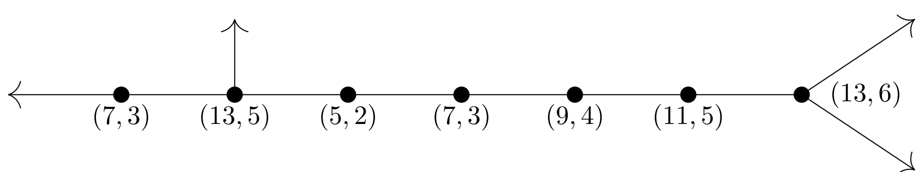


FIGURE 4. Resolution graph for Example 4.6

Example 4.7. Let $f(x,y) := y^{10} - x^3y^5 - x^{12}$. A μ-constant versal deformation is given by

$$f_{\mathbf{t}}(x,y) := f(x,y) + t_1x^7y^3 + t_2xy^9 + t_3x^9y^2 + t_4x^8y^3 + t_5x^{11}y$$
$$+t_6x^{10}y^2 + t_7x^9y^3 + t_8x^{11}y^2 + t_9x^{10}y^3 + t_{10}x^{11}y^3.$$

Using random values we can prove that $-\frac{19}{15}$ and $-\frac{4}{15}$ are both roots of the Bernstein polynomial, but only $\frac{4}{15}$ is a Yano's candidate.

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THE SHEAF $\alpha_X^\bullet$

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ABSTRACT. We introduce, in a reduced complex space, a "new coherent sub-sheaf" of the sheaf $\omega_X^\bullet$ which has the "universal pull-back property" for any holomorphic map, and which is, in general, bigger than the usual sheaf of holomorphic differential forms $\Omega_X^\bullet$/torsion. We show that the meromorphic differential forms which are sections of this sheaf satisfy integral dependence equations over the symmetric algebra of the sheaf $\Omega_X^\bullet$/torsion. This sheaf $\alpha_X^\bullet$ is closely related to the normalized Nash transform.

We also show that these $q-$meromorphic differential forms are locally square-integrable on **any** $q-$dimensional cycle in X and that the corresponding functions obtained by integration on an analytic family of $q-$cycles are locally bounded and continuous on the complement of a closed analytic subset.

INTRODUCTION

In this article, we discuss the following question: given a reduced complex space X, the normalization of X consists in building a proper modification $\nu : \tilde{X} \to X$ such that meromorphic locally bounded functions on X becomes holomorphic after pull-back to $\tilde{X}$. Moreover this process gives a desingularization process for curves, that is to say for X of pure dimension 1.

It seems then natural to define an analogous process for meromorphic locally bounded differential forms. The main trouble is to define what means "locally bounded" for a meromorphic differential form of positive degree on a reduced complex space. To define this notion is the purpose of this paper. Of course, this does not lead to a simple proof of a desingularization process for a reduced complex space, but we will show that the natural process associated to "normalization of meromorphic differential forms" is simply the classical **normalized Nash transform**, and it is an old (an probably very difficult) conjecture that this process leads to a desingularization. We hope that the introduction of this "new sheaf" $\alpha_X^\bullet$ will be useful in that direction.

But in fact, the main reason to introduce this sheaf is the look for the **"universal pull-back property"** which means to define a coherent sheaf of meromorphic differential forms which admits a natural pull-back for **any** holomorphic map between reduced complex spaces and which is "maximal" with this property. Note that if we only consider complex manifolds the sheaf $\Omega_X^\bullet$ has this property, but we will show that this is no longer maximal when X admits singularities.

Our main result is the theorem 4.1.1 (and its precise formulation 4.1.2) giving the "universal pull-back property" for these sheaves. We obtain also two other results which may be useful:

- The fact that for any section α of the sheaf α_X^q the form $\alpha \wedge \bar{\alpha}$ is locally integrable on **any** holomorphic cycle of dimension q and also the local boundness and the "generic" continuity of such an integral when the $q-$cycle moves in an analytic family (see theorem 5.1.7);

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- The existence of a local integral dependence equation for a section of α_X^q over the symmetric algebra of the sheaf Ω_X^q/torsion (see proposition 5.2.1).

We conclude this article by computing some simple examples showing that the sheaf $\alpha_X^\bullet$ may be different from other classical sheaves of meromorphic differential forms which are used on singular complex spaces.

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1. Universal pull-back for $\Omega_X^\bullet$/torsion

It is well known that the sheaves $\Omega_X^\bullet$ of holomorphic differential forms on complex spaces have a functorial pull-back. To begin we shall prove that the sheaves $\Omega_X^\bullet$/torsion still have this "universal pull-back" property on reduced complex spaces.

Proposition 1.0.1. *Let X be a reduced complex space and consider a torsion holomorphic $p-$form α on X (meaning that α vanishes at smooth points in X). Let Z be an analytic subset in X. Then the $p-$holomorphic form induced by α on Z is again a torsion form on Z.*

Proof. Without any lost of generality we can assume that Z is irreducible. Let S be the singular set of X. If Z is not contained in S the result is obvious. Also if the dimension of Z is less that p the conclusion is again obvious. So let dim $Z = p + q$ with $q \geq 0$ and let Z' be the dense open set of smooth points x in Z for which the multiplicity of x in X is minimal. It is enough to show that the restriction of α to Z' vanishes. As the problem is local on Z', we can assume that we have an open neighbourhood X' of x_0 in X and a local parametrisation $\pi : X' \to U$ on a polydisc U of $\mathbb{C}^n$ with the following properties:

i) $\pi(x_0) = 0$.
ii) $U = V \times W$ where V and W are polydiscs with center 0 respectively in $\mathbb{C}^{p+q}$ and $\mathbb{C}^{n-p-q}$.
iii) $Z'' := Z' \cap X' = \pi^{-1}(V \times \{0\})$ set theoretically and $\pi : Z'' \to V \times \{0\}$ is an isomorphism.

Define the analytic family of $(p + q)-$cycles $(Z_w)_{w\in W}$ in X' parametrized by W by letting $Z_w := \pi^*(V \times \{w\})$, where the pull-back by π is taken in the sense of cycles[1]. Then, if k is the degree of π (which is the multiplicity in X of each point in Z'') we have $Z_0 = k.Z''$ as a cycle in X'. Remark that for w generic in W the intersection of the cycle Z_w with the ramification set of π has no interior point in Z_w which is a reduced cycle. So the restriction of the holomorphic form α to Z_w for w generic is a torsion form.

Now choose a non-negative continuous function with compact support ρ on X', a holomorphic $q-$form β on X' and define the function on W

$$\varphi : W \to \mathbb{R}^+, \quad w \mapsto \varphi(w) := \int_{Z_w} \rho.(\alpha \wedge \beta) \wedge \overline{(\alpha \wedge \beta)}.$$

It is a continuous function (see [B-M 1] ch.IV) and it vanishes for w generic in W as α generically vanishes on Z_w for such a w. Then it vanishes for $w = 0$ and this shows that the restriction of α to an open dense subset of Z' vanishes. $\square$

Corollary 1.0.2. *Consider a holomorphic map $f : X \to Y$ where X and Y are reduced complex spaces. Then, if α is a $p-$holomorphic form on Y which is a torsion form, the $p-$holomorphic form $f^*(\alpha)$ is a torsion form on X.*

[1] This means that if $f : U \to \mathrm{Sym}^k(X')$ is the holomorphic map classifying the fibers of π, the cycle Z_w is the cycle-graph of the analytic family of $k-$tuples in X' defined by the restriction of f to $V \times \{w\}$; see [B-M 1] ch.IV.

Proof. It is enough to consider the case where X is a connected complex manifold. Let X' be the open dense subset of X where f has maximal rank. On X' the map f is locally a submersion on a locally closed complex sub-manifold of Y and the previous proposition applies to show that the pull-back of α on this locally closed sub-manifold vanishes. So the holomorphic form $f^*(\alpha)$ vanishes on X'. Then it is a torsion form on X. $\square$

Definition 1.0.3. *Let $f : X \to Y$ a holomorphic map between two reduced complex spaces. We have a natural graded pull-back $\mathcal{O}_X-$morphism*

$$f^* : f^*(\Omega^\bullet_Y/\text{torsion}) \to \Omega^\bullet_X/\text{torsion} \tag{*}$$

We shall denote $f^{**}(\Omega^\bullet_Y)$ the image of this graded sheaf morphism.

We shall also denote $f^{**}(\mathcal{G})$ for any sub-sheaf $\mathcal{G}$ of $\Omega^\bullet_Y/$torsion its image by the morphism f^* above (or also when $\mathcal{G}$ is a subs-sheaf of $\Omega^\bullet_Y$).

So, by definition, $f^{**}(\Omega^\bullet_Y)$ (and more generally $f^{**}(\mathcal{G})$) is a sub-sheaf of the sheaf $\Omega^\bullet_X/$torsion, so it has no $\mathcal{O}_X-$torsion.

Lemma 1.0.4. *Let $f : X \to Y$ and $g : Y \to Z$ two holomorphic maps between reduced complex spaces. Then we have equality of the sub-sheaves $f^{**}(g^{**}(\mathcal{H}))$ and $(g\circ f)^{**}(\mathcal{H})$ for any sub-sheaf $\mathcal{H}$ of the sheaf $\Omega^\bullet_Z/$torsion.*

Proof. The pull-back by g gives a morphism

$$g^*(\Omega^\bullet_Z/\text{torsion}) \to \Omega^\bullet_Y/\text{torsion}$$

with image $g^{**}(\Omega^\bullet_Z/\text{torsion})$ and the pull-back by f gives a morphism

$$f^*(g^*(\Omega^\bullet_Z/\text{torsion})) \to f^*(\Omega^\bullet_Y/\text{torsion})$$

which, by right-exactness of the tensor product, is surjective on $f^*(g^{**}(\Omega^\bullet_Z/\text{torsion}))$. Then we have the following commutative diagram

$$\begin{tikzcd}
f^*(g^*(\Omega^\bullet_Z/\text{torsion})) \arrow[r, "\alpha"] \arrow[d, "\simeq"'] & f^*(g^{**}(\Omega^\bullet_Z/\text{torsion}) \arrow[r, "u"] \arrow[dr, "\beta"] & f^*(\Omega^\bullet_Y/\text{torsion}) . \arrow[d, "v"] \\
(g\circ f)^*(\Omega^\bullet/\text{torsion}) \arrow[rr, "\gamma"] & & \Omega^\bullet_X/\text{torsion}
\end{tikzcd}$$

Here α is surjective and the image of β is the sub-sheaf $f^{**}(g^{**}(\Omega_Z))$ by definition. Also the image of γ is in $(g\circ f)^{**}(\Omega^\bullet_Z)$ by definition. Now the commutativity of the diagram allows to conclude. $\square$

Conclusion.

- The usual pull-back for holomorphic differential forms induced a natural pull-back for the sheaf $\Omega^\bullet_X/$torsion by any holomorphic map between reduced complex spaces. The previous lemma shows that this pull-back is functorial.

2. Normalization of a coherent sheaf

2.1. **Definition.** Let $\mathcal{F}$ be a coherent sheaf on a reduced complex space X and let $pr : F \to X$ be the associated linear bundle over X. Recall that, if on the open set U in X we have a presentation

$$\mathcal{O}_X^m \xrightarrow{M} \mathcal{O}_X^n \to \mathcal{F} \to 0$$

where M is a matrix with holomorphic entries, then $F_{|U}$ is given as the kernel

$$Ker[{}^tM : U\times\mathbb{C}^n \to U\times\mathbb{C}^m].$$

Then a section of $\mathcal{F}$ over an open set U in X is a holomorphic map over U, $F_{|U} \to U \times \mathbb{C}$, which is linear on the fibres of $pr_{|U}$; and conversely if $f : F_{|U} \to U \times \mathbb{C}$ is a holomorphic map which is linear on the fibres of $pr_{|U}$, let $g : V \times W \to \mathbb{C}$ be a holomorphic function on a neighbourghood of $(x_0, 0) \in U \times \mathbb{C}^n$ inducing f on $F \cap (V \times W)$. Write $g = \sum_{\nu=0}^{+\infty} \gamma_\nu$ be the Taylor expansion of g in homogeneous polynomials in the $\mathbb{C}^n$ −variables. Then γ_1, the homogeneous part of degree 1 in the $\mathbb{C}^n$ −variables, induces f on $X \cap (V \times \mathbb{C}^n)$. And γ_1 is a holomorphic function which is linear on fibres.

For the notion of linear bundle see [F.76], [A-M.86] or [B-M 2].

The symetric algebra of a linear bundle. We define the symetric algebra

$$S_\bullet(\mathcal{F}) := \oplus_{h=0}^{+\infty} S_h(\mathcal{F}),$$

where $S_h(\mathcal{F})$ is the sheaf of holomorphic functions on F which are homogeneous of degree h along the fibres of F. If $\sigma_1, \dots, \sigma_N$ is a local generator of $\mathcal{F}$ near a point $x_0 \in X$ then, for $\alpha \in \mathbb{N}^N$ such that $|\alpha| = \sum_{j=1}^N \alpha_j = h$, the $\sigma^\alpha := \sigma_1^{\alpha_1} \dots \sigma_N^{\alpha_N}$ for all such α generate locally $S_h(\mathcal{F})$ near x_0.

Note that if F is, on an open set $U \subset X$, the kernel of ${}^tM : U \times \mathbb{C}^n \to U \times \mathbb{C}^m$, the linear bundle $S_h(F)$ associated to the coherent sheaf $S_h(\mathcal{F})$ is defined as the kernel of the holomorphic map, linear on the fibers:

$$S_h({}^tM) : U \times S_h(\mathbb{C}^n) \to U \times S_h(\mathbb{C}^m).$$

As the complex space F is not reduced in general, the vanishing of a holomorphic function homogeneous of degree h on the fibres of F is not given, in general, by generic vanishing on F of such a function. But, when we assume that X is reduced and $F_{X \setminus S}$ is a vector bundle, the vanishing of a section of $S_h(\mathcal{F})$ on an open set $U \setminus S$ is just pointwise vanishing.

Recall that the exterior algebra of a coherent sheaf may be defined in the same way using the kernel of the map $\Lambda^q({}^tM) : U \times \Lambda^q(\mathbb{C}^n) \to U \times \Lambda^q(\mathbb{C}^m)$ on the "linear bundle side", or directly as a quotient of the tensor product $\mathcal{F}^{\otimes q}$.

Proposition 2.1.1. *Let X be a reduced complex space and let $\mathcal{F}$ be a coherent sheaf on X. Let $S \subset X$ be a closed analytic subset with no interior point in X such that on $X \setminus S$ the sheaf $\mathcal{F}$ is locally free. Then there exists a modification $\tau : \tilde{X} \to X$ with the following properties :*

i) The center of τ is contained in S.
ii) The sheaf $\tau^*(\mathcal{F})\big/\mathrm{torsion}$ is locally free on $\tilde{X}$.
iii) The reduced complex space $\tilde{X}$ is normal.
iv) For any holomorphic map $f : Y \to X$ from a normal complex space Y such that $f^{-1}(S)$ has no interior point in Y and such that the coherent sheaf $f^*(\mathcal{F})\big/\mathrm{torsion}$ is locally free, there exists an unique holomorphic lifting $\tilde{f} : Y \to \tilde{X}$ such that $\tau \circ \tilde{f} = f$. And in this situation we have

$$\tilde{f}^*(\tau^*(\mathcal{F})\big/\mathrm{torsion}) = f^*(\mathcal{F})\big/\mathrm{torsion}.$$

Proof. Note that, without any lost of generality, we may assume that $\mathcal{F}$ has no torsion. Consider first an open set U in X such that on U the coherent sheaf $\mathcal{F}$ has a presentation

$$\mathcal{O}_X^m \xrightarrow{M} \mathcal{O}_X^n \to \mathcal{F} \to 0.$$

Let $n-p$ be the generic rank of the holomorphic matrix M. Then the linear bundle L associated to $\mathcal{F}_{|U}$ is the kernel of the holomorphic map, linear on the fibres

$$\mathrm{id}_X \times {}^tM : U \times \mathbb{C}^n \to U \times \mathbb{C}^m\,.$$

Then we have a holomorphic map $g : U \setminus S \to Gr(p,n)$ which sends the $p-$dimensional vector sub-space $Ker^t M_x$ to the corresponding point in $Gr(p,n)$, the grassmannian of $p-$vector sub-spaces in $\mathbb{C}^n$. Consider then the closed analytic subset

$$Z := \{(x,P) \in U \times Gr(p,n) \ / \ P \subset Ker^t M(x)\}.$$

Over $U \setminus S$ the set Z coincides with the graph of the holomorphic map g. Then define $\tilde{X}_U$ as the normalization of the union of the irreducible components of Z which dominate an irreducible component of U. The projection map $\tau : \tilde{X}_U \to U$ is clearly a (proper) modification of U with center contained in S.

Let $V \to Gr(p,n)$ be the universal bundle of $Gr(,p,n)$ and let $\mathcal{U}$ the associated coherent sheaf. Let $p_2 : \tilde{X}_U \to Gr(p,n)$ the composition of the normalization with the projection on $Gr(p,n)$. Then let us show that there is a natural isomorphism $\tau^*(\mathcal{F})\big/\text{torsion} \to p_2^*(\mathcal{U})$. For that purpose it is equivalent to prove that there is a natural holomorphic map, linear on the fibres

$$p_2^*(V) \to \tau^*(F)$$

of linear bundles from the pull-back on $\tilde{X}_U$ of the tautological rank $p-$vector bundle V on $Gr(p,n)$ to the linear bundle $\tau^*(F)$ associated to $\tau^*(\mathcal{F})$. But this map is obvious as the fiber of $p_2^*(V)$ at $\tilde{x} \in \tilde{X}_U$ is, by definition, a $p-$vector subspace of the fibre of $\tau^*(F)$ at $\tilde{x}$. Moreover, this map is an isomorphism on $U \setminus S$ by construction, so it is injective. The corresponding morphism of coherent sheaves $\tau^*(\mathcal{F}) \to p_2^*(\mathcal{U})$ is then surjective and its kernel is supported by $\tau^{-1}(S)$. This implies that it induces an isomorphism $\tau^*(\mathcal{F})\big/\text{torsion} \simeq p_2^*(\mathcal{U})$.

To complete the proof of the assertions $i)$ to $iv)$, it is enough now to prove that the property $iv)$ holds for the modification $\tau : \tilde{X}_U \to U$ because this will imply the globalisation of this construction, thanks to the patching of these local pieces via the "universal property".

So let $f : Y \to U$ be a holomorphic map from a normal complex space Y such that $f^{-1}(S)$ has no interior point in Y and such that $f^*(\mathcal{F})\big/\text{torsion}$ is locally free on Y. Then by right exactness of the tensor product we have on Y the exact sequence

$$\mathcal{O}_Y^m \overset{f^*(M)}{\longrightarrow} \mathcal{O}_Y^n \to f^*(\mathcal{F}) \to 0.$$

This implies that the rank p vector bundle G associated to the locally free sheaf $f^*(\mathcal{F})\big/\text{torsion}$ is a sub-vector bundle of the linear bundle $f^*(F)$ which is the kernel of the holomorphic map, linear in the fibres

$$\mathrm{id}_Y \times f^*({}^tM) : Y \times \mathbb{C}^n \to Y \times \mathbb{C}^m \,.$$

This induces a holomorphic map $\tilde{g} : Y \to Gr(p,n)$ which sends $y \in Y$ to the fibre at y of $G \subset Y \times \mathbb{C}^n$. As G and $f^*(F)$ are isomorphic over $Y \setminus f^{-1}(S)$ which is a dense open set by assumption, this proves the uniqueness of $\tilde{g}$ and then of the map $\tilde{f} := (f, \tilde{g}) : Y \to \tilde{X}_U$. $\square$

Definition 2.1.2. *In the situation of the previous proposition we shall call the modification* $\tau : \tilde{X} \to X$ *the* **normalization of the coherent sheaf $\mathcal{F}$ on X**.

We shall say that a holomorphic map $f : Y \to X$ is **normalizing for the coherent sheaf** *$\mathcal{F}$ on X which is locally free outside the closed analytic subset S with no interior point in X, when it satisfies the following conditions:*

i) The complex space Y is normal.
ii) The closed analytic subset $f^{-1}(S)$ has no interior point in Y.
iii) The sheaf $f^*(\mathcal{F})\big/$torsion is locally free on Y.

Thanks to the universal property of the normalization $\tau : \tilde{X} \to X$ of $\mathcal{F}$, the holomorphic map f is normalizing for $\mathcal{F}$ if and only if the map f factorizes through the modification τ.

Remark. The normalization of a coherent sheaf $\mathcal{F}$ on a reduced complex space X is always a locally projective modification, as , by construction, it is locally contained in a product of an open set in X by a grassmannian.

Note that the proposition 2.1.1 is consequence of rather elementary results and do not use the desingularization theorem of H. Hironaka. But thanks to Hironaka, for any X and any coherent sheaf $\mathcal{F}$ on X there always exists a proper modification $\tau : \hat{X} \to X$ which is smooth and normalizes the sheaf $\mathcal{F}$: it is enough to apply the desingularization theorem to the normalization of $\mathcal{F}$ constructed above. Moreover, we may always assume that such a "normalizing" desingularization is a projective modification of X. This remark will be used in the next section.

Note that, in general, a desingularization of X is not necessarily normalizing for the sheaf Ω^1_X, see for instance the case of S_3 in example 6.2.

For a pure dimensional reduced complex space X the **Nash transform** (resp. the normalized Nash transform) is simply the previous results applied to the coherent sheaf Ω^1_X. Note that the corresponding linear bundle on X is the Zariski tangent linear bundle on X. See section 5.

Lemma 2.1.3. *In the situation of the proposition 2.1.1, consider an integer $q \geq 1$ and the coherent sheaf $\Lambda^q(\mathcal{F})$ and its normalization $\tau_q : \tilde{X}_q \to X$. Then we have a natural holomorphic map*

$$\varphi_q : \tilde{X} \to \tilde{X}_q$$

satisfying the following properties

(1) *φ_q is a modification with center contained in S and $\tau_q \circ \varphi_q = \tau$.*

(2) *We have a natural isomorphism of locally free sheaves on $\tilde{X}$*

$$e_q : \Lambda^q(\tau^*(\mathcal{F})\big/\text{torsion})) \to \varphi_q^*(\tau_q^*(\Lambda^q(\mathcal{F})\big/\text{torsion}).$$

Proof. Note that we may assume without any lost of generality that $\mathcal{F}$ has no torsion. As the sheaf $\tau^*(\mathcal{F})\big/\text{torsion}$ is locally free on $\tilde{X}$ the sheaf $\Lambda^q(\tau^*(\mathcal{F})\big/\text{torsion})$ is also locally free on $\tilde{X}$.

The natural surjective morphism

$$\Lambda^q(\tau^*(\mathcal{G})) \to \tau^*(\Lambda^q(\mathcal{G})), \quad \tau^*(g_1) \wedge \cdots \wedge \tau^*(g_q) \mapsto \tau^*(g_1 \wedge \cdots \wedge g_q)$$

for any coherent sheaf $\mathcal{G}$ induces an isomorphism

$$\Lambda^q(\tau^*(\mathcal{F})\big/\text{torsion}) \to \tau^*(\Lambda^q(\mathcal{F})\big/\text{torsion}) \tag{@}$$

because the kernel must be a torsion sub-sheaf of $\Lambda^q(\tau^*(\mathcal{F})\big/\text{torsion})$ which is locally free. Then the universal property of the normalization of the sheaf $\Lambda^q(\mathcal{F})$ gives the holomorphic map $\varphi_q : \tilde{X} \to \tilde{X}_q$ such that $\tau_q \circ \varphi_q = \tau$, and the isomorphism (@) allows to obtain the isomorphism e_q. $\square$

Consequence. If the holomorphic map $f : Y \to X$ is normalizing for the coherent sheaf $\mathcal{F}$ it is normalizing for the sheaf $\Lambda^q(\mathcal{F})$ for any integer $q \geq 0$.

This will be useful for instance for $\mathcal{F} = \Omega^1_X$, because a normalizing map for Ω^1_X is then normalizing for each $\Omega^q_X \quad \forall q \geq 1$.

2.2. **Locally bounded sections.** Let X be a reduced complex space, $\mathcal{F}$ a coherent sheaf on X which is locally free outside the closed analytic subset $S \subset X$ with no interior point in X. Consider the linear bundle on X, $pr : F \to X$, associated to $\mathcal{F}$. For any open set U in X a section $\sigma \in \Gamma(U, \mathcal{F})$ corresponds to a holomorphic function $f : F_U \to \mathbb{C}$ which is linear on the fibres of F.

Definition 2.2.1. *We shall say that* $\sigma \in \Gamma(U \setminus S, \mathcal{F})$ *is a* **locally bounded section** *of* $\mathcal{F}$ *near the point* $s_0 \in U$ *when there exist an open neighbourhood* U_0 *of* s_0 *in* U*, sections* $\sigma_1, \dots, \sigma_N$ *sections of* $\mathcal{F}$ *on* U_0 *and continuous bounded functions* $\rho_1, \dots, \rho_N$ *on* $U_0 \setminus S$ *such that the function* f *on* $F_{U_0 \setminus S}$ *corresponding to* σ *is given by*

$$f = \sum_{j=1}^{N} \rho_j(x).f_j(x,v) \quad \forall (x,v) \in F_{U_0 \setminus S}$$

where, for each $j \in [1, N]$*,* $f_j : F_{U_0} \to \mathbb{C}$ *is the holomorphic function linear on the fibres of* F *which corresponds to* $\sigma_j \in \Gamma(U_0, \mathcal{F})$

Remark that, by definition of S, $F_{U_0 \setminus S}$ is a reduced complex space: it is a holomorphic vector bundle on a reduced complex space. So the equality above is a "pointwise equality".

Of course, if σ is the restriction to $U_0 \setminus S$ of a section $\sigma \in \Gamma(U_0, \mathcal{F})$, it is locally bounded near each point in U_0: take $\sigma_1 = \sigma$ and $\rho_1 \equiv 1$!

Note that the function $f : F_{U_0 \setminus S} \to \mathbb{C}$ corresponding to a locally bounded section

$$\sigma \in \Gamma(U_0 \setminus S, \mathcal{F})$$

is locally bounded near each point of $F_{U_0 \cap S}$ which belongs to the irreducible components of F_{U_0} which surject onto an irreducible component of U_0. So, in general, such a f is not a locally bounded function on F_{U_0} but only on the closure in F_{U_0} of $F_{U_0 \setminus S}$.

Lemma 2.2.2. *Let* $S \subset X$ *be a closed analytic subset with no interior point in* X *containing the singular set in* X *and assume that the coherent sheaf* $\mathcal{F}$ *is locally free on* $X \setminus S$*. Let*

$$\sigma \in \Gamma(U_0 \setminus S, \mathcal{F})$$

and $f : F_{U_0 \setminus S} \to \mathbb{C}$ *the corresponding holomorphic function linear on the fibers of* F*.Then the fonction* f *is bounded in a neighbourhood of the point* $\{s_0\} \times \{0\}$ *in the closure of* $F_{U_0 \setminus S}$ *in* F[2] *if and only if the section* σ *is locally bounded near* s_0 *as a section of* $\mathcal{F}$ *on* $U_0 \setminus S$ *in the sense of the definition 2.2.1.*

Proof. Let first consider a section σ of $\mathcal{F}$ which is locally bounded near s_0 in the sense of the definition 2.2.1. Then we can find holomorphic sections $\sigma_1, \dots, \sigma_N$ on an open neighbourhood U_0 of s_0 in U and continuous bounded functions $\rho_1, \dots, \rho_N$ on $U_0 \setminus S$, such that $\sigma = \sum_{n=1}^{N} \rho_n.\sigma_n$ on $U_0 \setminus S$. Then, if $f_1, \dots, f_N$ are the holomorphic functions (linear on the fibres) on $F_{|U_0}$ corresponding to $\sigma_1, \dots, \sigma_N$, we have $f = \sum_{j=1}^{N} \rho_j.f_j$ on $F_{|U_0 \setminus S}$. This implies that the function f is locally bounded near points in the intersection of $pr^{-1}(s_0)$ with the closure of $F_{U_0 \setminus S}$. In particular near $\{s_0\} \times \{0\}$.

Conversely, if f is locally bounded on the intersection with $F_{|U_0 \setminus S}$ of a neighbourghood of $\{s_0\} \times \{0\}$ in the closure of $F_{U_0 \setminus S}$, remark that, as an obvious consequence of its homogeneity on the fibres of $\overline{F_{U_0 \setminus S}}$, it is locally bounded in a neighbourhood of each point of $pr^{-1}(s_0) \cap \overline{F_{U_0 \setminus S}}$.

Consider now a modification $\tau : \tilde{X} \to X$ with center contained in S such that $\tilde{X}$ is normal and such that the strict transform $\tilde{\tau} : \tilde{F} \to F$ of F is a holomorphic vector bundle. Then the function $f \circ \tilde{\tau}$ is locally bounded near $\tau^{-1}((s_0) \times \{0\})$ in $\tilde{F}$. As $\tilde{F}$ is a holomorphic vector bundle over the normal complex space $\tilde{X}$, it is a normal complex space and then $f \circ \tilde{\tau}$ extends to $\tilde{F}_{|\tau^{-1}(U_0)}$ to a holomorphic function $\tilde{f}$ which is linear on the fibres. If $\sigma_1, \dots, \sigma_N$ are sections of $\mathcal{F}$ on an open

[2]Note that if G is an irreducible component of F_V which is contained in $pr^{-1}(V \cap S)$ then G does not meet the open set where f is defined. So we obtain the same condition on σ if we replace $\mathcal{F}$ by $\mathcal{F}/$torsion.

neighbourhood U_0 of s_0 in U which generate $\mathcal{F}$ at each point of U_0, their pull-back by τ generate the coherent sheaf on $\tilde{X}$ associated to $\tilde{F}$ at each point of $\tau^{-1}(U_0)$. Near each such point we can write $\tilde{f} = \sum_{j=1}^{N} c_j \otimes \tau^{-1}(\sigma_j)$ where $c_1, \dots, c_N$ are local holomorphic functions on $\tilde{X}$. Using a continuous partition of unity along the compact fibre $\tau^{-1}(s_0)$ we obtain that f can be written as $\sum_{j=1}^{N} \rho_j.\sigma_j$ on $U_0 \setminus S$ where $\rho_1, \dots, \rho_N$ are bounded continuous functions on $U_0 \setminus S$. $\square$

Corollary 2.2.3. *Let X be a reduced complex space and let $\mathcal{F}$ be a coherent sheaf on X. Let $S \subset X$ be a closed analytic subset with no interior point in X such that $\mathcal{F}$ is locally free on $X \setminus S$. Note $j : X \setminus S \to X$ the inclusion. Let Y be a normal complex space and consider a (proper) modification $\tau : Y \to X$ normalizing the sheaf $\mathcal{F}$. Then the sheaf $\tau_*(\tau^*(\mathcal{F})\big/\mathrm{torsion})$ is the sub-sheaf of the sheaf $j_*j^*(\mathcal{F})$ of sections which are locally bounded along S.*
So this sheaf is independent of the choice of such a τ.

Proof. First consider a section of $\theta \in \tau_*(\tau^*(\mathcal{F})\big/\mathrm{torsion})$. It can be written locally on Y as a sum $\sum_{j=1}^{N} g_j.\tau^*(\sigma_j)$ where $\sigma_1, \dots, \sigma_N$ generate locally $\mathcal{F}$ and where $g_1, \dots, g_N$ are local holomorphic functions on Y. Then using a continuous partition of unity along the fibres of τ we see that θ satisfies the definition 2.2.1.

Conversely, if η is a section of the sheaf $j_*j^*(\mathcal{F})$ which is locally bounded along S, its lifting gives a holomorphic function on $\tau^*(F)$ on the complement of $\tau^{-1}(S)$, which is linear on the fibres and locally bounded near the points of $\tau^*(F)$ which are in the closure of the restriction of $\tau^*(F)$ to $Y \setminus \tau^{-1}(S)$. But this closure is a vector bundle, by our assumption on τ. As a vector bundle on a normal complex space is a normal complex space, the Riemann extension theorem holds, and this holomorphic function extends holomorphically to this vector bundle. Then it is a section of the sheaf $\tau_*(\tau^*(\mathcal{F})\big/\mathrm{torsion})$ concluding the proof. $\square$

Proposition 2.2.4. *Let $S \subset X$ be a closed analytic subset with no interior point in X containing the singular set in X and assume that the coherent sheaf $\mathcal{F}$ is locally free on $X \setminus S$. Consider a holomorphic function f on $F_{|U\setminus S}$ which is linear on the fibres of F and which is locally bounded along $pr^{-1}(S) \cap \overline{pr^{-1}(U \setminus S)}$ corresponding to a locally bounded section σ of $\mathcal{F}$ on $U \setminus S$. Then for each point s_0 in S there exists an open neighbourhood U_0 of s_0 in X, an integer $h \geq 1$ and sections $s_1, \dots, s_h$ on U_0 respectively of the sheaves $S_1(\mathcal{F}), \dots, S_h(\mathcal{F})$ such that the equality of sections of $S_h(\mathcal{F})$:*

$$\sigma^h + \sum_{a=1}^{h} s_a.\sigma^{h-a} = 0$$

is satisfied on the open set $U_0 \setminus S$.

proof. We keep the notations of the proof of the previous lemma 2.2.2. As the function f is locally bounded on F_1, the conic bundle over X which is the union of the irreducible components of F near the point $\{s_0\} \times \{0\}$ which dominate an irreducible component of X at s_0, there exist an open neighbourhood U_0 of s_0 in X, an integer $h \geq 1$ and holomorphic functions $\tilde{s}_1, \dots, \tilde{s}_h$ on an open neighbourhood W of $F_1 \cap pr^{-1}(U_0 \times \{0\})$ such that $\sigma^h + \sum_{a=1}^{h} \tilde{s}_a.\sigma^{h-a} = 0$ on $W \cap pr^{-1}(U_0 \setminus S)$. Taking the homogeneous degree h parts of the expansions of this equality in the fibres of $pr : F \to X$ leads to sections $s_1, \dots, s_h$ of the sheaves $S_a(\mathcal{F})$, where s_a is the homogeneous degree a part of $\tilde{s}_a$[3] concluding the proof. $\square$

[3]which is in fact well-defined only modulo torsion in $S_a(\mathcal{F})(U_0)$, but this torsion is concentrated on S, so is irrelevant for the desired equality on $U_0 \setminus S$.

3. Definition of the sheaf $\alpha_X^\bullet$

It will be convenient to use the following definition in the sequel.

Definition 3.0.1. *Let X be a reduced complex space. We say that a modification $\tau : \tilde{X} \to X$ is a* **special desingularization of** *X when the following properties are satisfied:*

i) $\tilde{X}$ is a complex manifold.
ii) The modification τ is projective.
iii) The sheaf $\tau^*(\Omega_X^1)\big/\mathrm{torsion}$ is locally free on $\tilde{X}$.

We have already remark that, thanks to Hironaka and to the fact that the normalization of the sheaf Ω_X^1 is a projective modification of X, for any modification $\theta : Y \to X$ there exists a special desingularization $\tau : \tilde{X} \to X$ which factors through θ.

The following result is the key of the definition of the sheaf $\alpha_X^\bullet$ on a reduced complex space X.

Theorem 3.0.2. *Let X be a reduced complex space and let S be a closed analytic subset with no interior point in X containing the singular set of X. Let α be a section on X of the sheaf ω_X^p. The following properties are equivalent for α:*

- *There exists locally on X a normalizing modification for the sheaf Ω_X^1*[4]
  $\tau : \tilde{X} \to X$ such that α extends to a section on X of the sub-sheaf $\tau_*\tau^{**}(\Omega_X^p)$ of ω_X^p. (A)
- *There exists,* **locally on** *X, a finite collection $(\rho_j)_{j\in J}$ of continuous functions on $X \setminus S$ which are bounded near S and holomorphic $p-$forms $(\omega_j)_{j\in J}$ in $\Omega_X^p\big/\mathrm{torsion}$ such that $\alpha = \sum_{j\in J}\ \rho_j.\omega_j$ as a $(p,0)$ currents on X.* (B)

Note that under the second property stated in the theorem, the $(p,0)-$current on X associated to the form $\sum_{j\in J}\ \rho_j.\omega_j$ on $X \setminus S$ is defined by

$$\mathscr{C}_c^\infty(X)^{n-p,n} \ni \varphi \mapsto \int_X\ \varphi \wedge (\sum_{j\in J}\ \rho_j.\omega_j)$$

and this integral is absolutely convergent as the functions ρ_j are locally bounded near each point in S. It defines a $(p,0)-$current on X with order 0. The assumption that α is a section of the sheaf ω_X^p implies that this current is $\bar{\partial}-$closed on X.

Proof. Let us begin by the implication $(A) \Rightarrow (B)$. By definition, a section $\alpha \in \omega_X^p$ is in the sub-sheaf $\tau_*\tau^{**}(\Omega_X^\bullet)$ if, locally on $\tilde{X}$, it can be written as a linear combination of pull-back of holomorphic forms on X with holomorphic coefficients in $\mathcal{O}_{\tilde{X}}$. Using the properness of the modification τ and a $\mathscr{C}^\infty$ partition of the unity on $\tilde{X}$ we obtain the first part of (B) because τ induces an isomorphism $\tilde{X} \setminus \tau^{-1}(S) \to X \setminus S$ by hypothesis. The last property in (B), that is to say the fact that the current defined on X by the right hand-side coincides with α, is consequence of the fact that both are sections of the sheaf ω_X^p and are equal on $X \setminus S$.

To prove the implication $(B) \Rightarrow (A)$ consider the pull-back to $\tilde{X} \setminus \tau^{-1}(S)$ of the form $\sum_{j\in J}\ \rho_j.\omega_j$. We obtain a holomorphic section on $\tilde{X} \setminus \tau^{-1}(S)$ of the locally free sheaf

$$\tau^*(\Omega_X^p)\big/\mathrm{torsion}$$

[4]In fact normalizing for the sheaf Ω_X^p would be enough; see lemma 2.1.3.

which has locally bounded coefficients along $\tau^{-1}(S)$ when we compute it in a local trivialisation near a point of $\tau^{-1}(S)$. So, by normality of $\tilde{X}$, it extends to a holomorphic section $\tilde{\alpha}$ of $\tau^*(\Omega_X^p)\big/\text{torsion}$ and then defines a section of $\tau_*(\tilde{\alpha})$ of the sheaf $\tau_*(\tau^*(\Omega_X^p)\big/\text{torsion})$. Note that the pull-back of holomorphic forms gives an injective morphism $\tau^*(\Omega_X^p)\big/\text{torsion} \to \Omega_{\tilde{X}}^p\big/\text{torsion}$ with image $\tau^{**}(\Omega_X^p)$. So $\tau^*(\alpha)$ defines a holomorphic form on $\tilde{X}$ and the direct image of this form and α coincide on $X \setminus S$, and then on X as sections of the sheaf ω_X^p because this sheaf has no non-zero section supported in S. $\square$

Remarks.

(1) The condition (B) does not depend on the choice of the modification τ normalizing the sheaf Ω_X^1.
(2) Let $L_X^\bullet$ be the direct image of the sheaf $\Omega_Y^\bullet$ where $\tau : Y \to X$ is a desingularization of X. Using a special desingularization of X in the proof above we obtain that the form α is in the coherent sub-sheaf $L_X^p \subset \omega_X^p$, so it gives the inclusion $\alpha_X^\bullet \subset L_X^\bullet$.

Corollary 3.0.3. *Let $\tau : \tilde{X} \to X$ be any proper modification of X which is normalizing the sheaf Ω_X^1. The graduate sub-sheaf $\alpha_X^\bullet := \tau_*(\tau^{**}(\Omega_X^\bullet))$ of the sheaf $L_X^\bullet$ is independent of the choice of the modification of X normalizing Ω_X^1.* $\square$

Corollary 3.0.4. *Let X be a pure dimensional reduced complex space and let*

$$X := \cup_{i\in I} X_i$$

be its decomposition in irreducible components. Then the sheaf $\alpha_X^\bullet$ has a natural injection in the locally finite direct sum of the direct images in X of the sheaves $\alpha_{X_i}^\bullet$ for $i \in I$.

Proof. This is an easy consequence of the fact that a section of the sheaf $L_X^\bullet$ is a section of $\alpha_X^\bullet$ if and only if it satisfies the condition (B) in the previous theorem, because we have an isomorphism $L_X^\bullet \simeq \oplus_{i\in I}\,(j_i)_*(L_{X_i}^\bullet)$, where $j_i : X_i \to X$ is the inclusion. $\square$

Note that when X is not irreducible the injective map $\alpha_X^\bullet \hookrightarrow \oplus_{i\in I}\,(j_i)_*(\alpha_{X_i}^\bullet)$ is not an isomorphism, in general, because the injective map

$$\Omega_X^\bullet/\text{torsion} \hookrightarrow \oplus_{i\in I}\,(j_i)_*(\Omega_{X_i}^\bullet/\text{torsion})$$

is not an isomorphism, in general.

But, for each $i \in I$, and any point $x \in X_i$, the "restriction" map

$$\alpha_{X,x}^\bullet \to \alpha_{X_i,x}^\bullet$$

is surjective because each restriction map $\Omega_{X,x}^\bullet/\text{torsion} \to \Omega_{X_i,x}^\bullet/\text{torsion}$ is surjective.

4. Universal pull-back for $\alpha_X^\bullet$

4.1. **Statement of the theorem.** The main result of this paragraph is the following theorem.

Theorem 4.1.1. *For any holomorphic map $f : X \to Y$ between reduced complex spaces, there exists a functorial*[5] *graduate $\mathcal{O}_X-$morphism*

$$\hat{f}^* : f^*\alpha_Y^\bullet \to \alpha_X^\bullet$$

which is compatible with the usual pull-back of the sheaf $\Omega_Y^\bullet/\text{torsion}$.
For any holomorphic maps $f : X \to Y$ and $g : Y \to Z$ between reduced complex spaces we have

$$\hat{f}^*(\hat{g}^*(\alpha)) = \widehat{g\circ f}^*(\alpha) \qquad \forall \alpha \in \alpha_Z^\bullet. \tag{1}$$

[5] We shall make this precise in the theorem 4.1.2 below.

Let now give a precise formulation of this result. For that purpose let $\mathcal{C}$ be the category of reduced complex spaces with morphisms all holomorphic maps. We may enrich this category, using the universal pull-back property for the graded sheaf $\Omega^\bullet_X/\text{torsion}$:

Let $\mathcal{C}_{diff}$ be the category whose objects are pairs $(X, \Omega^\bullet_X/\text{torsion})$ where X is an object in $\mathcal{C}$ and where the morphisms are given by pairs (f, f^*) where $f : X \to Y$ is a morphism in $\mathcal{C}$ and $f^* : f^*(\Omega^\bullet_Y/\text{torsion}) \to \Omega^\bullet_X/\text{torsion}$ is the graded pull-back by f of holomorphic forms modulo torsion (see section 1). Of course the forget-full functor $G_0 : \mathcal{C}_{diff} \to \mathcal{C}$ obtained by $(X, \Omega^\bullet_X/\text{torsion}) \mapsto X, (f, f^*) \mapsto f$ is an equivalence of category.

Then the precise content of the theorem above is the following result.

Theorem 4.1.2. [Precise formulation] *There exists a category $\mathcal{C}_{b-diff}$ whose objects are pairs $(X, \alpha^\bullet_X)$ where X is in $\mathcal{C}$ and where the graded coherent sheaf $\alpha^\bullet_X$ has been defined in section 3 for any object X in $\mathcal{C}$. The morphisms are given by pairs $(f, \hat{f}^*)$ for each $f : X \to Y$ a morphism in $\mathcal{C}$ where $\hat{f}^* : f^*(\alpha^\bullet_Y) \to \alpha^\bullet_X$ is the graded $\mathcal{O}_X-$linear sheaf map defined by f. Moreover, the following properties holds:*

(1) For each $X \in \mathcal{C}$ we have a graded $\mathcal{O}_X-$linear injection

$$\eta_X : \Omega^\bullet_X/\text{torsion} \to \alpha^\bullet_X.$$

(2) For any morphism $f : X \to Y$ in $\mathcal{C}$ we have a commutative diagram of graded $\mathcal{O}_X-$linear maps of sheaves

$$\begin{array}{ccc} f^*(\Omega^\bullet_Y/\text{torsion}) & \xrightarrow{f^*} & \Omega^\bullet_X/\text{torsion} \\ \downarrow{\scriptstyle f^*(\eta_Y)} & & \downarrow{\scriptstyle \eta_X} \\ f^*(\alpha^\bullet_Y) & \xrightarrow{\hat{f}^*} & \alpha^\bullet_X \end{array} \tag{2}$$

where $\hat{f}^*$ is the graded $\mathcal{O}_X-$linear map of coherent sheaves on X associated to the holomorphic map f.

Of course the interest of this result comes from the fact that the sheaf $\alpha^\bullet_X$ is, in general, strictly bigger that the sheaf $\Omega^\bullet_X/\text{torsion}$; see section 6.

For any holomorphic map $f : X \to Y$ between reduced complex spaces a pull-back morphism $f^\sharp : f^*(L^\bullet_Y) \to L^\bullet_X$ is defined in [K. 00]. But this pull-back is not functorial on these sheaves: let $\tau : \tilde{X} \to X$ be a desingularization of $X \in \mathcal{C}$ and let $x \in X$ be a point such that $\tau^{-1}(x)$ has dimension ≥ 1. Let ω be a holomorphic form near $\tau^{-1}(x)$ in $\tilde{X}$ which does not induce a torsion form on an irreducible component Γ of $\tau^{-1}(x)$. Then, because the map $\tau_{|\Gamma} : \Gamma \to X$ factorizes by the constant map to $\{x\}$ the functoriality of the pull-back of ω on Γ would imply that the pull-back has to be zero. But this map factorizes also by the inclusion of Γ in $\tilde{X}$ and τ. As the pull-back by τ is injective (by definition of $L^\bullet_X$), this gives a contradiction. Such an example is given in section 6.3.

4.2. **The proof.**

Preliminaries. Consider the following situation: let Z be a connected complex manifold and consider a proper holomorphic map $\pi : Z \to X$ which is surjective on a reduced (irreducible) complex space X. Let $q := \dim Z - \dim X$ and let k be the number of connected components of the generic fibre of π. Assume that we have a kähler form ω on Z.

Claim. After a suitable normalization of ω, the smooth $(q,q)-$form $w := \frac{1}{k}.\omega^{\wedge q}$ is $d-$closed and satisfies the condition $\pi_*(w) = 1$ as a $d-$closed $(0,0)-$current on X.

Proof. Consider the Stein factorization $\pi_0 : Z \to Y, \theta : Y \to X$ of π; the reduced complex space Y is irreducible. We have a meromorphic fibre-map $Y - -- > \mathcal{C}_q(Z)$ for π_0 (see [B-M 1] ch.IV Th. 9.1.1) and this implies, thanks to the irreducibility of Y, that the generic fibres of π_0 are in the same connected component of the space of $q-$cycles in Z. So the volume computed by $\omega^{\wedge q}$ of the connected components of the generic fibres of π is constant, and we may normalized ω in order that this volume is equal to 1. Then the $d-$closed $(0,0)-$current $\pi_*(w)$ on X is equal to 1 on a dense Zariski open set in X. This implies our claim.

Assume now that the complex manifold Z has finitely many connected components $Z_1, \dots, Z_r$ such that the restriction of π is surjective on each Z_j and such that each Z_j has a kähler form ω_j. We can normalize each ω_j in order that the integral of the form $w_j := \frac{1}{k_j}.\omega^{\wedge q_j}$ is equal to $1/r.k_j$ on each connected component of the generic fibres of $\pi_j := \pi_{|Z_j}$ and then the smooth form $w := \sum_{j=1}^r \ w_j$ satisfies again the condition $\pi_*(w) = 1$ as a $(0,0)-$current on X.

In this situation we shall say that the smooth form w on Z satisfies the condition (@).

The proof of the theorem 4.1.2 will use the following proposition.

Proposition 4.2.1. *Let $X = \cup_{i\in I} \ X_i$ be the decomposition of a reduced complex space X as the union of its irreducible components. Let $Z := \cup_{j\in J} \ Z_j$ be a disjoint union of connected complex kähler manifolds. Assume that we have a map $\theta : J \to I$ which is surjective and has finite fibres. Let $\pi : Z \to X$ be a proper holomorphic map normalizing the sheaf Ω_X^1, such that for each $j \in J$ it induces a surjective map*

$$\pi_j : Z_j \to X_{\theta(j)}$$

and let $q_j := \dim Z_j - \dim X_{\theta(j)}$. For each $j \in J$ let w_j be a smooth $(q_j, q_j)-$form on Z_j which is $d-$closed and satisfies the condition (@) relative to the restriction of π to Z_j (see preliminaries above). Let $w := \sum_{j\in J} w_j$.
Let β be a section on Z of the sheaf $\pi^{**}(\Omega_X^p)$. Then we have:

1. *The $\bar\partial-$closed $(p,0)-$current $\pi_*(\beta \wedge w)$ on X is independent of the choices of the forms w_j, assuming that they are $d-$closed and satisfy the condition (@).*
2. *The section $\pi_*(\beta \wedge w)$ on X of the sheaf ω_X^p is a section of the sub-sheaf α_X^p.*
3. *If there exists a section α of the sheaf $\Omega_X^p/\text{torsion}$ such that $\beta = \pi^{**}(\alpha)$ on Z, then $\alpha = \pi_*(\beta \wedge w)$ as a section on X of the sheaf ω_X^p.*

Remarks.

1. It is enough to prove assertion 1) and 3) of the proposition above for each map $\pi_j, j \in J$ because the sheaf ω_X^p is a sub-sheaf of the direct sum of the sheaves $\omega_{X_i}^p, i \in I$ and the restriction of β to Z_j is a section of the sheaf $\pi_j^{**}(\Omega_{X_{\theta(j)}}^p)$ for each $j \in J$.
   This is not the case for the assertion 2) of the proposition: the sheaf α_X^p is a sub-sheaf of the direct sum of the sheaves $\alpha_{X_i}^p, i \in I$ but, in general, strictly smaller than this direct sum. Note also that the condition on β to be a section of the sheaf $\pi^{**}(\Omega_X^p)$ is stronger than the condition on each $\beta_j := \beta_{|Z_j}, j \in J$ to be a section of the sheaf $\pi_j^{**}(\Omega_{X_{\theta(j)}}^p)$.
2. In general, a section $\beta \in \Gamma(Z, \pi^{**}(\Omega_X^p))$ is not equal to some $\pi^{**}(\alpha)$ where α is in $\Gamma(X, \Omega_X^p)$ even in the case where $\pi : Z \to X$ is a special desingularization of X. $\square$

Proof. Thanks to the previous remark, we may assume that X is irreducible to prove assertions 1) and 3) of the proposition.

In the case $q_j = 0$ the map π_j is generically finite and w_j is a locally constant function on Z with a prescribed value on each Z_j. So there is no choice for w_j and the first assertion of the proposition is trivial. As the second assertion is also clear in this case (the sheaf ω_X^p has no torsion on X by definition), we shall assume $q_j \geq 1$ in the sequel.

The fact that the current $\pi_*(\beta \wedge w)$ is $\bar{\partial}-$closed on X is consequence of the fact that on each Z_j the smooth $(p + q_j, q_j)$ form $\beta \wedge w_j$ is $\bar{\partial}-$closed and of the holomorphy of π. Let w' be a smooth form on Z which is $d-$closed and satisfies the condition (@). We want to show that $\pi_*(\beta \wedge (w - w'))$ vanishes as a section of the sheaf ω_X^p. Let X' be the open and dense subset of smooth points in X for which the Stein factorization of each $\pi_j : Z_j \to X$ is a covering of degree k_j. Remember that, as we assume that X is irreducible here, the set I is reduced to one point and so J is a finite set. On this open set X' it is enough to prove that for each $j \in J$ the current $(\pi_j)_*(\beta \wedge (w_j - w'_j))$ vanishes. So we can fix j and replace locally X' by one sheet of the corresponding finite covering and make the proof in this case. That is to say that we may assume that Z is smooth and connected and that $\pi : Z \to X$ has connected fibers on X.

In this case the generic fibres of π are irreducible and of dimension q. For any $x \in X'$ there exists an open neighbourhood $V(x)$ of $\pi^{-1}(x)$ which is a deformation retract of $\pi^{-1}(x)$. Then we have an isomorphism $H^{2q}(V(x), \mathbb{C}) \to \mathbb{C}$ which is given by integration on $\pi^{-1}(x)$. But w and w' have the same integral on $\pi^{-1}(x)$ by the property (@). So there exists a $(2q-1)$ smooth form θ on $V(x)$ such that $d\theta = w - w'$ by de Rham's theorem.

Consider now a small open neighbourhood U of x in X' such that $\pi^{-1}(U) \subset V(x)$. Let $x_1, \dots, x_n$ be a local coordinate system on U. Then the sheaf $\pi^*(\Omega_X^p)$ is a free sheaf of $\mathcal{O}_Z-$modules on $\pi^{-1}(U)$ with basis $\pi^*(dx^L)$ where L runs in all ordered sub-sets of cardinal p in $[1, n]$. If we write $\beta = \sum_{|L|=p} g_L.\pi^*(dx^L)$ on U the holomorphic functions g_L on $\pi^{-1}(U)$ are constant along the fibres of π and so there exists holomorphic functions $f_L, |L| = p$, with $g_L = \pi^*(f_L)$ (recall that U is a smooth open set in X). This means that there exists a holomorphic $p-$form α on U such that $\beta = \pi^*(\alpha)$ on $\pi^{-1}(U)$.
Let $\psi \in \mathscr{C}_c^\infty(U)^{(n-p,n)}$. By definition of the direct image we have

$$\langle \pi_*(\beta \wedge d\theta), \psi \rangle = \int_{\pi^{-1}(U)} \beta \wedge d\theta \wedge \pi^*(\psi).$$

But it follows from the equality $\beta = \pi^*(\alpha)$ on $\pi^{-1}(U)$ that the form

$$\beta \wedge \pi^*(\psi) = \pi^*(\alpha \wedge \psi)$$

is $d-$closed as $\alpha \wedge \psi$ is $d-$closed on U (its degree is $2n$). So by Stokes formula the integral

$$\int_{\pi^{-1}(U)} \beta \wedge d\theta \wedge \pi^*(\psi) = \pm \int_{\pi^{-1}(U)} d\big(\beta \wedge \theta \wedge \pi^*(\psi)\big)$$

vanishes. This implies that the section $\pi_*(\beta \wedge (w - w'))$ of the sheaf ω_X^p vanishes on the open dense subset X', so everywhere on X as the sheaf ω_X^p has no torsion.

Assertion 3) of the proposition is clear, because the equality is obvious at the generic points in X.

Let us prove assertion 2). We no longer assume that I has a unique point.
Let $\tau : \tilde{X} \to X$ be a special desingularization of X, so $\tilde{X}$ is the disjoint union of special

desingularizations $\tau_i : \tilde{X}_i \to X_i$ for each $i \in I$, and consider the commutative diagram

$$\begin{tikzcd} \tilde{X} \times_{X,str} Z \arrow[r, "\tilde{\tau}"] \arrow[d, "\tilde{\pi}"] & Z \arrow[d, "\pi"] \\ \tilde{X} \arrow[r, "\tau"] & X \end{tikzcd}$$

where $\tilde{X} \times_{X,str} Z$ is the strict transform, so the union of irreducible components of $\tilde{X} \times_X Z$ which dominate some $\tilde{X}_i$.

Note that the map $\tau \circ \tilde{\pi}$ is normalizing for the sheaf Ω^1_X because it is the case for τ (and also for π). Then the $p-$form $\tilde{\tau}^{**}(\beta)$ gives, for each such component, a section of the sheaf $(\pi \circ \tilde{\tau})^{**}(\Omega^p_X)$ and as the $d-$closed form $\tilde{\tau}^*(w)$ satisfies the condition (@) for the map $\tilde{\pi}$, the $\bar{\partial}-$closed current $\tilde{\pi}_*(\tilde{\tau}^{**}(\beta) \wedge \tilde{\tau}^*(w))$ is in fact a $p-$holomorphic form on $\tilde{X}$ thanks to Dolbeault-Grothendieck's lemma. This already proved that $\alpha := \pi_*(\beta \wedge w)$ is a section of the sheaf L^p_X, because $\tau^{**}(\pi_*(\beta \wedge w)) = \tilde{\pi}_*(\tilde{\tau}^{**}(\beta) \wedge \tilde{\tau}^*(w)))$ at the generic points of $\tilde{X}$, so everywhere on $\tilde{X}$.

Now the map $\eta : \Omega^p_{\tilde{X}} \to \Omega^p_{\tilde{X}}$ given by $\gamma \mapsto \tilde{\pi}_*(\tilde{\pi}^{**}(\gamma) \wedge \tilde{\tau}^*(w))$ is the identity map, thanks to the assertion 3). So, if $\tilde{\pi}^{**}(\gamma)$ gives a section of the image of the sub-sheaf $\tilde{\pi}^{**}\big(\tau^{**}(\Omega^p_X)\big)$ of the sheaf $\Omega^p_{\tilde{X} \times_{X,str} Z}\big/\text{torsion}$, γ will be a section of the image of the sub-sheaf $\tau^{**}(\Omega^p_X)$ because the map $\tilde{\pi}^* : \tilde{\pi}^*(\Omega^p_{\tilde{X}}) \to \Omega^p_{(\tilde{X} \times_{X,str} Z)}$ is injective.

Apply this to $\gamma := \tau^{**}(\alpha) = \tilde{\pi}_*\big(\tilde{\tau}^{**}(\beta) \wedge \tilde{\tau}^*(w)\big)$ which is a section of $\Omega^p_{\tilde{X}}$ as we already proved that α is a section in L^p_X; we obtain that $\tau^{**}(\alpha)$ is a section of the sheaf $\tau^{**}(\Omega^p_X)$ because, as the diagram above commutes, $\tilde{\tau}^{**}(\beta)$ is a section of the sheaf $\tilde{\tau}^{**}\big(\pi^{**}(\Omega^p_X)\big) = \tilde{\pi}^{**}\big(\tau^{**}(\Omega^p_X)\big)$ thanks to the lemma 1.0.4. $\square$

Remark. If Z is not assumed to be smooth in the previous proposition, replacing Z by a projective desingularization $\sigma : \tilde{Z} \to Z$ (as before, this means that $\tilde{Z}$ is the disjoint union of projective desingularizations $\sigma_j : \tilde{Z}_j \to Z_j$ for $j \in J$), the proposition applies to the proper map $\pi \circ \sigma$ and to $\tilde{\beta} := \sigma^*(\beta)$ which is a section of the sheaf $(\pi \circ \sigma)^{**}(\Omega^p_X)$. Then the result is still true. $\square$

Proof of theorem 4.1.1. The first step in proving the theorem will be the construction of $\hat{f}^*(\alpha) \in \alpha^\bullet_X$ when α is a section of the sheaf $\alpha^\bullet_Y$. So let α be a section on Y of the sheaf α^p_Y. Let $\tau : \tilde{Y} \to Y$ be a special desingularization of Y. Consider the following commutative diagram

$$\begin{tikzcd} Z \arrow[r, "\theta"] \arrow[dr, "\pi_1"'] & \tilde{X} \arrow[r, "\tilde{f}"] \arrow[d, "\pi"] & \tilde{Y} \arrow[d, "\tau"] \\ & X \arrow[r, "f"] & Y \end{tikzcd}$$

where $\tilde{X} \subset X \times_Y \tilde{Y}$ is the strict transform of X, that is to say the union of irreducible components of $X \times_Y \tilde{Y}$ which dominate an irreducible component of X, and where π and $\tilde{f}$ are induced by the natural projections of $X \times_Y \tilde{Y}$. Then let Z be a special desingularization of X such that π_1 factorizes by π (see the remark following the definition 3.0.1).

Now the problem is local on X and Y and we may assume that $X, \tilde{X}, Y, \tilde{Y}$ and Z are kähler. So we may assume that we have on Z a smooth $d-$closed form w which satisfies the condition (@) for the proper map π_1 (we use a special desingularization to reach the precise situation of the proposition 4.2.1; see the remark above and the remark following the definition 2.1.2).

Let β be the section of $\tau^{**}(\Omega^p_Y)$ defined by α; then the form $(\tilde{f}\circ\theta)^{**}(\beta)$ is a section of $\pi_1^{**}(\Omega^p_X)$ because if we write locally on $\tilde{Y}$

$$\beta := \sum_l \ g_l.\tau^{**}(\omega_l)$$

where ω_l are local sections of Ω^p_Y and g_l are holomorphic functions on $\tilde{Y}$, we obtain

$$(\tilde{f}\circ\theta)^{**}(\beta) = \sum_l \ (\tilde{f}\circ\theta)^*(g_l).(\tilde{f}\circ\theta)^{**}(\tau^{**}(\omega_l))$$

and the equality $(\tilde{f}\circ\theta)^{**}(\tau^{**}(\omega_l)) = \pi_1^{**}(f^{**}(\omega_l))$ due to the commutativity of the diagram and the lemma 1.0.4 shows that $(\tilde{f}\circ\theta)^{**}(\beta)$ is a section of the sheaf $\pi_1^{**}(\Omega^p_X)$. So we can apply the proposition 4.2.1 and obtain that $(\pi_1)_*((\tilde{f}\circ\theta)^{**}(\beta)\wedge w)$ is a section of the sheaf α^p_X. This will give the definition of $\hat{f}^*(\alpha)$ when we shall have proved that it is independent of the choice of the special desingularization $\tau : \tilde{Y}\to Y$.

Note that the proposition 4.2.1 already gives the independence of the choice of w (assumed $d-$closed and satisfying (@)) in this construction.

The proposition 4.2.1 gives also that for α a section of $\Omega^p_Y/\text{torsion}$ $\hat{f}^*(\alpha)$ is a section of $\Omega^p_X/\text{torsion}$ and coincides with the usual pull-back $f^*(\alpha)$ (see section 1).

Remark now that, as the sheaf α^p_X has no torsion on X, to prove the independence of $\hat{f}^*(\alpha)$ on the choice of the special desingularization τ, it is enough to prove it at the generic points of X. Moreover, this problem is local on X and so we may assume that X is smooth and connected.

In our construction, we sum the various direct images $(\pi_j)_*(\tilde{f}^*(\beta)\wedge w_j)$ when j describes the various connected components of the desingularization of $\tilde{X}$. Each such component is sent by $\tilde{f}$ in a connected component of $\tilde{Y}$ and then it is enough to show the invariance of the current $(\pi_j)_*(\tilde{f}^*(\beta)\wedge w_j)$ if we change only one connected component of $\tilde{Y}$ in the given special desingularization, and also if we consider only the corresponding connected components of the special desingularization of $\tilde{X}$. So, in fact, it is enough to prove the following special case of our problem:

Assume that X is smooth and connected and that Y is irreducible. Let $\tau : \tilde{Y}\to Y$ be a special desingularization of Y and let $\theta : \tilde{\tilde{Y}}\to\tilde{Y}$ be a proper smooth modification of $\tilde{Y}$. So our new special desingularization of Y will be $\tau\circ\theta : \tilde{\tilde{Y}}\to Y$.

Now we shall consider the following diagram, where $\tilde{X}$ is a special desingularization of an irreducible component of the strict transform $X\times_Y\tilde{Y}$ and $\tilde{\tilde{X}}$ is a special desingularization of the strict transform of $\tilde{X}\times_{\tilde{Y}}\tilde{\tilde{Y}}$:

$$\begin{tikzcd}
\tilde{\tilde{X}} \arrow[r, "\tilde{\tilde{f}}"] \arrow[d, "\tilde{\theta}"] & \tilde{\tilde{Y}} \arrow[d, "\theta"] \\
\tilde{X} \arrow[r, "\tilde{f}"] \arrow[d, "\tilde{\tau}"] & \tilde{Y} \arrow[d, "\tau"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

Let q the dimension of the generic fibres of $\tilde{\tau}$ and k the number of connected components of its generic fibres. Let ω be a kähler form of $\tilde{Y}$ normalized in order that the form $\tilde{f}^*(\omega^{\wedge q})$ satisfies the condition (@) for the map $\tilde{\tau}$. Let $\tilde{q}$ be the dimension of the generic fibre of $\tilde{\theta}$ and let $\tilde{\omega}$ a kähler form on $\tilde{\tilde{Y}}$ normalized in order that the form $\tilde{\tilde{f}}^*(\tilde{\omega}^{\wedge\tilde{q}})$ satisfies the condition (@)

for the map $\tilde{\theta}$. Now consider the $(q+\tilde{q}, q+\tilde{q})-$smooth form $w := \tilde{\tilde{f}}^*(\theta^*(\omega^{\wedge q}) \wedge \tilde{\omega}^{\tilde{q}})$ on $\tilde{\tilde{X}}$ which is $d-$closed. It satisfies the condition (@) for the map $\tilde{\tau} \circ \tilde{\theta}$.

So the definition of $\hat{f}^*(\alpha)$ using the special desingularization $\tau \circ \theta$ is given by

$$(\tilde{\tau} \circ \tilde{\theta})_*\big((\theta \circ \tilde{\tilde{f}})^*(\beta) \wedge w\big),$$

But, as $\tilde{f}^{**}(\beta)$ is a section of the sheaf $\Omega^p_{\tilde{X}}/\text{torsion}$, we have the equality

$$\tilde{\theta}_*(\tilde{\theta}^{**}(\tilde{f}^{**}(\beta)) \wedge \tilde{\tilde{f}}^*(\tilde{\omega}^{\tilde{q}})) = \tilde{f}^{**}(\beta)$$

and the conclusion follows from the fact that

$$(\tilde{\tau} \circ \tilde{\theta})_*\big((\theta \circ \tilde{\tilde{f}})^{**}(\beta) \wedge w\big) = \tilde{\tau}_*[\tilde{\theta}_*\big(\tilde{\theta}^{**}(\tilde{f}^{**}(\beta)) \wedge \tilde{\tilde{f}}^*(\tilde{\omega}^{\tilde{q}})\big) \wedge \tilde{f}^*(\omega^q))].$$

The compatibility of this construction with the pull-back of holomorphic forms modulo torsion which is given by the last assertion of the proposition 4.2.1 obviously gives that the injective $\mathcal{O}_X-$linear morphism

$$\eta_X : \Omega^\bullet_X/\text{torsion} \to \alpha^\bullet_X$$

for each $X \in \mathcal{C}$ gives the commutative diagram (2) of the precise formulation 4.1.2 of the theorem for each morphism $f : X \to Y$ in $\mathcal{C}$.

Now we have to prove the functoriality of $\hat{f}^*$. Then consider a holomorphic maps $f : X \to Y$ and $g : Y \to Z$. We want to prove the formula (1) of the theorem.
Consider the commutative diagram

$$\begin{tikzcd}
\tilde{\tilde{X}} \arrow[r, "\tilde{\tilde{f}}"] \arrow[d, "\tilde{\theta}"] & \tilde{\tilde{Y}} \arrow[d, "\theta"] & \\
\tilde{X} \arrow[r, "\tilde{f}"] \arrow[d, "\tau_2"] & \tilde{Y} \arrow[r, "\tilde{g}"] \arrow[d, "\tau_1"] & \tilde{Z} \arrow[d, "\tau"] \\
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z
\end{tikzcd}$$

where $\tau : \tilde{Z} \to Z$ is a special desingularization, where $\tilde{g} : \tilde{Y} \to \tilde{Z}$ is the strict transform of g by τ, where $\tilde{f} : \tilde{X} \to \tilde{Y}$ is the strict transform of f by τ_1, where
$\tilde{\tilde{f}} : \tilde{\tilde{X}} \to \tilde{\tilde{Y}}$ is the strict transform of $\tilde{f}$ by $\theta : \tilde{\tilde{Y}} \to \tilde{Y}$ which is a special desingularization of $\tilde{Y}$.
Let α be a section of α^p_Z, note $\beta := \tau^{**}(\alpha) \in \tau^{**}(\Omega^p_Z)$[6] and let w_1 and w_2 be smooth $d-$closed forms satisfying the condition (@) of the proposition 4.2.1 for the maps τ_1 and $\tilde{\theta}$ respectively.
We have

$$\hat{g}^*(\alpha) = (\tau_1)_*(\tilde{g}^{**}(\beta) \wedge w_1)$$

but we have also, because $\tilde{g}^{**}(\beta)$ is a section of $\tau_1^{**}(\Omega^p_Y)$

$$\hat{g}^*(\alpha) = (\tau_1 \circ \theta)_*(\theta^{**}(\tilde{g}^{**}(\beta) \wedge \theta^*(w_1))).$$

Then we obtain

$$\hat{f}^*(\hat{g}^*(\alpha)) = (\tau_2 \circ \tilde{\theta})_*\big(\tilde{\tilde{f}}^{**}(\theta^{**}(\tilde{g}^{**}(\beta))) \wedge \tilde{\tilde{f}}^*(w_1) \wedge w_2\big).$$

[6]See the simple lemma 4.2.3 below.

As the square

$$\begin{tikzcd} \tilde{X} \arrow[r, "\tilde{g}\circ\tilde{f}"] \arrow[d, "\tau_2"'] & \tilde{Z} \arrow[d, "\tau"] \\ X \arrow[r, "g\circ f"] & Z \end{tikzcd}$$

is also the strict transform of $g \circ f$ by τ we have

$$\widehat{g\circ f}^*(\alpha) = (\tau_2)_*\big((\tilde{g}\circ\tilde{f})^{**}(\beta)\wedge\tilde{f}^*(w_1)\big).$$

Then the conclusion follows from the equality

$$\tilde{\theta}_*\big(\tilde{\tilde{f}}^{**}(\theta^{**}(\tilde{g}^{**}(\beta)))\wedge\tilde{\tilde{f}}^{**}(\theta^*(w_1))\wedge w_2\big) = \tilde{f}^{**}(\tilde{g}^{**}(\beta))\wedge\tilde{f}^*(w_1)$$

obtained by the comparaison of both hand-sides at the generic points of $\tilde{X}$. $\square$

Our next result shows that the sheaf $\alpha_X^\bullet$ is "maximal" in order to construct the pull-back via the method of the proposition 4.2.1.

Proposition 4.2.2. *Let $\pi : Z \to X$ be a proper surjective holomorphic map between irreducible complex spaces. Put $q := \dim Z - \dim X$. Let $\beta \in \alpha_Z^p$ be equal to $\hat{\pi}^*(\alpha)$ for a section α of the sheaf α_X^p. Let also w be a smooth $(q,q)-$form on Z which is $d-$closed and satisfies the condition (@) of the proposition 4.2.1 for the map π. Then the $(p,0)-$current $\pi_*(\beta\wedge w)$ on X (which is $\bar{\partial}-$closed and independent of the choice of w satisfying $dw = 0$ and (@); see proposition 4.2.1) is equal to the image in ω_X^p of the section α of the sheaf α_X^p.*

Using the "pull-back" theorem 4.1.1 the proof of the result above will follow from this simple lemma.

Lemma 4.2.3. *Let X be a reduced complex space and $\tau : \tilde{X} \to X$ a desingularization of X. Then the image of the pull-back $\hat{\tau}^* : \tau^*(\alpha_X^\bullet) \to \alpha_{\tilde{X}}^\bullet = \Omega_{\tilde{X}}^\bullet$ is the subsheaf $\tau^{**}(\Omega_X^\bullet)$ of $\Omega_{\tilde{X}}^\bullet$.*

Proof. By definition, a section of this image is locally on $\tilde{X}$ a $\mathcal{O}_{\tilde{X}}-$linear combination of holomorphic forms on $\tilde{X}$ which are locally $\mathcal{O}_{\tilde{X}}-$linear combinations of pull-back by τ of holomorphic forms on X. So the conclusion is clear. $\square$

Remark. As a consequence of the previous lemma, if τ is a special desingularization of X we have $\tau_*\hat{\tau}^*$ is the identity on the sheaf $\alpha_X^\bullet$.

Proof of the proposition 4.2.2. Let $\tilde{\pi} : \tilde{Z} \to \tilde{X}$ be the strict transform of π by τ, and denote by $\tilde{\tau} : \tilde{Z} \to Z$ the corresponding projection on Z which is a modification. So we have the following commutative diagram

$$\begin{tikzcd} \tilde{Z} \arrow[r, "\tilde{\pi}"] \arrow[d, "\tilde{\tau}"'] & \tilde{X} \arrow[d, "\tau"] \\ Z \arrow[r, "\pi"] & X \end{tikzcd}$$

The $(q,q)-$form $\tilde{\tau}^*(w)$ is smooth and $d-$closed in $\tilde{Z}$ and satisfies the condition (@) of the proposition 4.2.1 for the proper surjective holomorphic map $\tau\circ\tilde{\pi}$. As we can write $\beta = \hat{\pi}^*(\alpha)$ where α is a section of α_X^p, we have, by functoriality of the pull-back for the sheaf $\alpha_Y^\bullet$ and the equality $\tau\circ\tilde{\pi} = \pi\circ\tilde{\tau}$

$$\hat{\tilde{\tau}}^*(\beta) = \hat{\tilde{\pi}}^*(\hat{\tau}^*(\alpha)).$$

But, thanks to the previous lemma, we have $\hat{\tau}^*(\alpha)$ which is a section of $\tau^{**}(\Omega_X^p)$ and using the smoothness of $\tilde{X}$ we have $\widehat{\tilde{\pi}}^* = \tilde{\pi}^{**}$. Then we obtain, using the lemma 1.0.4, the fact that $\hat{\tilde{\tau}}^*(\beta)$ is a section of the sheaf $(\tau \circ \tilde{\pi})^{**}(\Omega_X^p)$. Then the proposition 4.2.1 applies to the map $\tau \circ \tilde{\pi} : \tilde{Z} \to X$ with the form $\tilde{\tau}^*(w)$ and the section $\hat{\tilde{\tau}}^*(\beta)$ of the sheaf $(\tau \circ \tilde{\pi})^{**}(\Omega_X^p)$ and gives that the $(p,0)-$current on X given by $\sigma := (\tau \circ \tilde{\pi})_*(\hat{\tilde{\tau}}^*(\beta) \wedge \tilde{\tau}^*(w))$ is $\bar{\partial}-$closed on X and is a section of the sheaf α_X^p.

But the $(p+q,q)-$current $\tilde{\tau}_*(\hat{\tilde{\tau}}^*(\beta) \wedge \tilde{\tau}^*(w))$ is equal to $\beta \wedge w$ at least over the generic points in X, the $(p,0)-$current $\pi_*(\beta \wedge w)$ is $\bar{\partial}-$closed in X and generically equal to σ and α. So α and σ are equal as sections of the sheaf α_X^p. $\square$

5. Integration on cycles

5.1. **Integrals.**

Notations. Let V be a complex manifold and h be a continuous hermitian form on V. So h is a real continuous positive definite $(1,1)-$ differential form on V. If ω is a continuous $(p,p)-$form on V, we shall consider ω as a continuous sesqui-linear form on $\Lambda^p(T_V)$ and we shall write

$$\|\omega\|_K \leq C.h^{\wedge p}$$

where K is a subset in V and $C > 0$ a constant, if for any point $x \in K$ and any $v_1, \dots, v_p \in T_{V,x}$ the inequality

$$|\omega(x)[v_1 \wedge \dots \wedge v_p]| \leq C.h^{\wedge p}(x)[v_1 \wedge \dots \wedge v_p]$$

holds. For instance, if $\alpha, \beta \in \Omega_V^p$ we shall write $\|\alpha \wedge \bar{\beta}\|_K \leq C_K.h^{\wedge p}$ when for any $x \in K$ and any $v_1, \dots, v_p \in T_{V,x}$ we have

$$|\alpha(x)[v_1 \wedge \dots \wedge v_p]|.|\beta(x)[v_1 \wedge \dots \wedge v_p]| \leq C_K.h^{\wedge p}(x)[v_1 \wedge \dots \wedge v_p]. \tag{1}$$

Remark. If $f : W \to V$ is a holomorphic map and if (1) holds then we shall have

$$\|f^*(\alpha) \wedge \overline{f^*(\beta)}\|_{f^{-1}(K)} \leq C_K.f^*(h)^{\wedge p} \tag{2}$$

but, in general, $f^*(h)$ is still positive but no longer definite on W.

Conversely if (2) holds on a set L in W then (1) is satisfied on $f(L)$.

Proposition 5.1.1. *Let X be a reduced complex space, let S be the singular set in X and let h be a continuous hermitian metric on X. Let U be a relatively compact open set in X. For all $\alpha, \beta \in \alpha_X^p$ there exists a constant $C_U > 0$ such that the following inequality holds at each point in $\bar{U} \setminus S$*

$$\|\alpha \wedge \bar{\beta}\|_{\bar{U} \setminus S} \leq C_U.h^{\wedge p}_{\bar{U} \setminus S}.$$

Proof. Remark that the problem is local on the compact set $\bar{U} \cap S$ because near smooth points in X the assertion obviously holds. Let $\tau : \tilde{X} \to X$ be a special desingularization of X. Then we shall show that for each point $y \in \tau^{-1}(\bar{U} \cap S)$ there exists an open neighbourghood W of y in $\tilde{X}$ and a positive constant C_W such that the inequality

$$\|\tau^{**}(\alpha) \wedge \overline{\tau^{**}(\beta)}\|_W \leq C_W.\tau^{**}(h)^{\wedge p}$$

holds: if y is a point in $\tilde{X}$ we can write in an open neighbourghood W of y

$$\alpha = \sum_{|I|=p} g_I.\tau^{**}(dx^I) \quad \text{and} \quad \beta = \sum_{|I|=p} h_I.\tau^{**}(dx^I)$$

where $x_1, \dots, x_N$ are local coordinates in a closed embedding of an open set $U \subset\subset X$ in $\mathbb{C}^N$ near $\tau(y)$. Our estimates is consequence of the facts that the holomorphic functions g_I and h_I are locally bounded and that for any (I, J) there is a constant $c_U^{I,J} > 0$ with

$$\|dx^I \wedge \overline{dx^J})\|_U \leq c_U^{I,J}.h^{\wedge p}$$

because we can assume that h is induced by a continuous hermitain form on $\mathbb{C}^N$.

Now the properness of τ allows to find a a constant C_U such that the inequality

$$\|\tau^{**}(\alpha) \wedge \overline{\tau^{**}(\beta)}\|_K \leq C_U.\tau^{**}(h)^{\wedge p}$$

holds on the compact set $K := \tau^{-1}(\bar{U})$. This allows to conclude thanks to the remark above. $\square$

Corollary 5.1.2. *Let X be a complex space of pure dimension n, and let α, β be sections on X of the sheaf L_X^n. Then, if ρ is a continuous compactly supported function on X the integral*

$$\int_{X \setminus S} \rho.\alpha \wedge \bar{\beta}$$

is absolutely convergent for any closed analytic subset S containing the singular set in X and its value does not depends on the choice of S.
Now fix a continuous hermitian metric h on X and a compact set K in X. If α and β are sections of the sheaf α_X^n, there is constant $C > 0$ depending on α, β, h and K such that for any $\rho \in \mathscr{C}_K^0(X)$ we have

$$\Big| \int_{X \setminus S} \rho.\alpha \wedge \bar{\beta} \Big| \ \leq \ C.\int_X |\rho|.h^{\wedge n} \leq C.||\rho||.\int_{\mathrm{Supp}\,\rho} h^{\wedge n}. \tag{3}$$

Proof. The first part is consequence of the fact that $\tau^{**}(\alpha)$ and $\tau^{**}(\beta)$ are holomorphic $n-$forms on $\tilde{X}$. The estimates when α, β are sections of α_X^n is a direct consequence of the previous proposition. $\square$

Remarks.

(1) Of course, in the second part of the corollary we may replace ρ by the characteristic function of an open subset $V \subset K$ in order to obtain, with the same constant C independent on the choice of V, the estimate

$$| \int_{V \setminus S} \alpha \wedge \bar{\beta}| \ \leq \ C.\int_V h^{\wedge n}. \tag{3 bis}$$

(2) Note that the estimations (3) or ($3bis$) do not hold in general when α and β are sections in the sheaf L_X^n. For instance let

$$X := \{(x, y, z) \in \mathbb{C}^3 \ / \ x.y = z^2\} \quad \text{and} \quad \alpha = \beta = \frac{dx \wedge dy}{z}.$$

They are sections of the sheaf L_X^2 but not sections of the sheaf α_X^2 (see the example with $k = 2$ in the paragraph 6.2); let $K := \{|x| \leq 1\} \cap \{|y| \leq 1\}$ in X and let h be the metric induced on X by the standard kähler form on $\mathbb{C}^3$. Then we have

$$\int_{V(r)} \alpha \wedge \bar{\alpha} = \gamma.r^2$$

where $V(r) := \{|x| \leq r\} \cap \{|y| \leq r\} \cap X$ and $\int_{V(r)} h^{\wedge 2} = \delta.r^4$ for any $r \in]0, 1[$, showing that the estimate ($3bis$) cannot hold.

Definition 5.1.3. *For α, β sections of the sheaf L_X^n the common values of the absolutely convergent integrals $\int_{X\setminus S}\ \rho.\alpha\wedge\bar{\beta}$ will be denoted simply by $\int_X\ \rho.\alpha\wedge\bar{\beta}$.*

Lemma 5.1.4. *Let $f : Y \to X$ a proper generically finite and surjective holomorphic map between two complex spaces of pure dimension n; let k be the generic degree of π. Let α, β be sections on X of the sheaf L_X^n and $\rho \in \mathscr{C}_c^0(X)$. Then the holomorphic $n-$forms $f^{**}(\alpha)$ and $f^{**}(\beta)$ are well defined on a dense Zariski open set in Y and extend as sections on Y of the sheaf L_Y^n. We have the equality*

$$\int_X\ \rho.\alpha\wedge\bar{\beta} = k.\int_Y\ f^*(\rho).f^{**}(\alpha)\wedge\overline{f^{**}(\beta)}.$$

Proof. Remark that it is enough to prove the lemma for $\alpha = \beta$. Let $\tau : \tilde{X} \to X$ be a desingularization of X. As $\tau^{**}(\alpha)$ is an holomorphic $n-$form on $\tilde{X}$ the form α is locally L^2 on X. Let H_ε be an open $\varepsilon-$neighbourhood of H a closed analytic subset in X such that the map $f : Y \setminus f^{-1}(H) \to X \setminus H$ is a finite covering between two complex manifolds. Then the usual change of variable gives, if ρ is in $\mathscr{C}_c^0(X)$

$$\int_{X\setminus H_\varepsilon}\ \rho.\alpha\wedge\overline{\alpha} = k.\int_{Y\setminus f^{-1}(H_\varepsilon)}\ f^*(\rho).f^{**}(\alpha)\wedge\overline{f^{**}(\alpha)}.$$

Letting ε goes to 0 shows that $f^{**}(\alpha)$ is locally L^2 on any desingularization of Y and so $f^{**}(\alpha)$ is a section of the sheaf L_Y^n. The conclusion follows easily. $\square$

Definition 5.1.5. *Let X be a complex space and let $Y \subset X$ be an irreducible $p-$dimensional analytic subset in X. We shall denote $j : Y \to X$ the the inclusion. Let α, β be sections of the sheaf α_X^p on X and ρ be a continuous function with compact support in X. We define the number $\int_Y\ \rho.\alpha\wedge\overline{\beta}$ as the integral*

$$\int_Y\ j^*(\rho).\hat{j}^*(\alpha)\wedge\overline{\hat{j}^*(\beta)}.$$

Note that this definition makes sense because the pull-back $\hat{j}^* : j^*(\alpha_X^p) \to \alpha_Y^p$ is well defined and because the inclusion $\alpha_Y^p \subset L_Y^p$ allows to use the definition 5.1.3. Remark that this definition only depends on the irreducible analytic subset Y of X. So we may extend by additivity the definition of the integral

$$\int_Y\ \rho.\alpha\wedge\overline{\beta}$$

to any $p-$dimensional cycle Y in X.

The next lemma shows that the change of variable holds for such a integral.

Lemma 5.1.6. *Let $f : X \to Y$ be a holomorphic map and let α, β be sections on Y of the sheaf α_Y^p. Let ρ be a continuous compactly supported function on Y. Let Z be a $p-$cycle in X and assume that the cycle $f_*(Z)$ is defined in Y[7]. Then the restriction to $|Z|$ of the continuous function $f^*(\rho)$ has compact support and the integral $\int_Z\ f^*(\rho).\hat{f}^*(\alpha)\wedge\overline{\hat{f}^*(\beta)}$ is well defined and we have*

$$\int_Z\ f^*(\rho).\hat{f}^*(\alpha)\wedge\overline{\hat{f}^*(\beta)} = \int_{f_*(Z)}\ \rho.\alpha\wedge\bar{\beta}.$$

[7]This means that the restriction of f to $|Z|$ is proper; see [B-M 1] chapter IV.

PROOF. First remark that any irreducible component Γ of Z which has an image of dimension at most equal to $p-1$ does not contribute to the right hand-side and also to the left hand-side because the forms $\hat{f}^*(\alpha)$ and $\hat{f}^*(\beta)$ vanish on such a irreducible component:

Let $g : \Gamma \to f(\Gamma)$ be the map induced by f; by functoriality of the pull-back $\hat{g}^*$ factorizes through $\alpha^p_{f(\Gamma)}$ which is zero.

Then the result is in fact a local statement near each point of the support of the cycle $f_*(Z)$. And because of our previous remark and the fact that closed analytic subsets with no interior point can be neglected in the integrals, it is enough to prove the result when Z is smooth and when f induces an isomorphism of Z on $f(Z)$. In this case, which is not trivial because Z and $f(Z)$ can be contained in the singular sets of X and Y, the functorial property of the pull-back and the fact that for a complex manifold V we have $\alpha^p_V = \Omega^p_V$ allow to conclude. $\square$

Theorem 5.1.7. *Let X be a reduced complex space and let $(Y_t)_{t\in T}$ be an analytic family of $p-$cycles in X parametrized by a reduced complex space T. Fix a compact set K in X and let α, β be sections of the sheaf α^p_X on X. Let ρ be a continuous function with a compact support in K and define the function $\varphi : T \to \mathbb{C}$ by*

$$\varphi(t) := \int_{Y_t} \rho.\alpha \wedge \overline{\beta}.$$

Then φ is locally bounded and for any given hermitian metric h on X and any compact set L in T there exists a constant C depending only on K, α, β, h and L (but not on the choice of ρ) such that the following estimate holds for each $t \in L$:

$$|\varphi(t)| \ \leq \ C. \int_{Y_t} |\rho|.h^{\wedge p} \ \leq \ C.||\rho||. \int_{Y_t \cap \mathrm{Supp}\,\rho} h^{\wedge p}. \tag{E}$$

Moreover for each point $t_0 \in T$ there exists an open neighbourhood T_0 of t_0 in T and a closed analytic subset $\Theta_0 \subset T_0$ with no interior point in T_0 such that φ is continuous on $T_0 \setminus \Theta_0$.

PROOF. We shall cut this proof in several steps.

STEP 1. Let $\nu : \tilde{T} \to T$ the normalization of T. The family $(Y_{\nu(\tilde{t})})_{\tilde{t}\in\tilde{T}}$ is an analytic family of $p-$cycles in X parametrized by $\tilde{T}$, and if the theorem is proved for this family it implies the result for the initial family, because the function is constant on the fibres of the normalization map.

So we shall assume that T is normal in the sequel.

STEP 2. If the generic cycle Y_t is not reduced and irreducible, the normality of T allows to write the family $(Y_t)_{t\in T}$ as a finite sum of analytic families of $p-$cycles in X parametrized by T such that the sum of these families is our initial family and such that the generic cycle in each family is reduced and irreducible (see ch. IV theorem 3.4.1 of [B-M 1]). So it is enough to prove the theorem for such a family.

So we shall assume that for t generic in T the cycle Y_t is reduced and irreducible.

STEP 3. Let $G \subset T \times X$ the cycle-graph of our analytic family. It is a reduced and irreducible cycle and the projection $\pi : G \to T$ is (by definition) a geometrically flat map, that is to say that there exists an analytic family of cycles $(Z_t)_{t\in T}$ in G such that for each $t \in T$ we have $|Z_t| = \pi^{-1}(t)$ and such that the generic cycle Z_t is reduced and irreducible. Of course, here we have $Z_t := \{t\} \times Y_t$ for each $t \in T$.

Note $pr : G \to X$ the projection and define on G the sections of the sheaf α^p_G by letting $\alpha_1 := \hat{pr}^*(\alpha)$ and $\beta_1 := \hat{pr}^*(\beta)$. Then, it is enough to prove the theorem for the function $t \mapsto \int_{Z_t} \tilde{\rho}.\alpha_1 \wedge \bar{\beta}_1$ where $\tilde{\rho} := pr^*(\rho)$ thanks to the change variable theorem proved in lemma

5.1.6. Remark that pr induces an isomorphism of $|Z_t|$ onto $|Y_t|$ for each $t \in T$ and also that the continuous function $\tilde{\rho}$ on G has a $\pi-$proper support.

STEP 4. Let $\tau : \tilde{G} \to G$ be a special desingularization of G. Define the subset

$$\Theta := \{t \in T \ / \ \exists y \in K \ \dim_y (pr \circ \tau)^{-1}(t) \geq p+1\}.$$

This is a locally closed analytic subset[8] in T with no interior point. For a given $t_0 \in T$, fix an open neighbourhood T_0 of t_0, small enough in order that $\Theta_0 := \Theta \cap T_0$ is a closed analytic subset. The map

$$q : \tilde{G} \cap (pr \circ \tau)^{-1}(T_0) \setminus (pr \circ \tau)^{-1}(\Theta) \to T_0 \setminus \Theta_0$$

is $p-$equidimensional on a normal basis, so it is geometrically flat and we have an analytic family $(\tilde{Z}_t)_{t \in T_0 \setminus \Theta_0}$ of fibres of q which are $p-$cycles in $\tilde{G}$, and for t generic in $T_0 \setminus \Theta_0$ the cycle $\tilde{Z}_t$ is irreducible.

Note that the pull-back of α_1 and β_1 on $\tilde{G} \cap (pr \circ \tau)^{-1}(T_0)$ are holomorphic $p-$forms. So, by the usual result of the continuity of integration of a continuous form on a continuous family of cycles (see [B-M 1] ch. IV prop. 2.3.1), we conclude using the lemma 5.1.6 that the function φ is continuous on $T_0 \setminus \Theta_0$.

STEP 5. The local boundness on T of the function φ is given by the corollary 5.1.2 which gives the estimate (E) by integration. $\square$

REMARKS.

(1) In the case of a proper family of compact cycles in X, it is easy, using results of [B-M 1] chapter IV, to prove that the function φ becomes continuous after a suitable modification of the complex space T.

(2) Already in the case of the normalization map, if α is a locally bounded meromorphic function on X, the function $x \mapsto |\alpha(x)|^2$ is not continuous on X in general.

5.2. **Normalized Nash transform.** Let us begin by two examples.

TWO EXAMPLES.

(1) We shall show in section 6.2 that for $k \geq 2$ and $k-1 \geq q \geq k/2$ the form

$$\omega_q := z^q.(dx/x - dy/y)$$

is a section of the sheaf $\alpha^1_{S_k}$ where

$$S_k := \{(x,y,z) \in \mathbb{C}^3 \ / \ x.y = z^k\}$$

which are not sections of the sheaf $\Omega^1_{S_k}/\text{torsion}$.
But as we have $dx/x + dy/y = k.dz/z$ on S_k we obtain the equality

$$\omega_q^2 = k^2.z^{2q-2}.(dz)^2 - 4z^{2q-k}.dx.dy;$$

so ω_q^2 is equal, for $q \geq k/2$, modulo torsion to a section of $S^2(\Omega^1_{S_k})$, the piece of degree 2 in the symmetric algebra of the sheaf $\Omega^1_{S_k}$.

(2) We shall show in section 6.4 that on $X := \{(x,y,u,v) \in \mathbb{C}^4 \ / \ x.y = u.v\}$ the form $a := u.dv \wedge dx/x$ is a section of the sheaf α^2_X which is not in $\Omega^2_X/\text{torsion}$. But using the following identities on X:

$$u.dv \wedge dx/x + u.dv \wedge dy/y = dv \wedge du$$
$$u.dv \wedge dy/y + v.du \wedge dy/y = dx \wedge dy$$

[8] See, for instance, the lemma 2.1.8 in [B.15].

we obtain that

$$a^2 + a.(du \wedge dv + dx \wedge dy) - (dv \wedge dx).(du \wedge dy) = 0$$

which is a homogeneous integral dependence equation for a on the symmetric algebra of the sheaf $\Omega^2_X/\mathrm{torsion}$.

The next proposition will show that these examples are special cases of a general phenomenon.

Proposition 5.2.1. *Let X be a normal complex space. Then for each integer q the sheaf α^q_X is the sub-sheaf of meromorphic sections of the sheaf $\Omega^q_X/\mathrm{torsion}$ which satisfy a homogeneous integral dependence equation over the sheaf $S^\bullet(\Omega^q_X)$, the symmetric algebra of the sheaf $\Omega^q_X/\mathrm{torsion}$.*

PROOF. This is a special case of the proposition 2.2.4. $\square$

NOTATION. For integers $n < N$ we shall denote $Gr(n, N)$ the grassmannian manifold of sub-vector spaces in $\mathbb{C}^N$ of dimension n.

Let X be a reduced complex space pure of dimension n and let S its singular locus. Assuming that X is embedded in an open set U in $\mathbb{C}^N$ we have a holomorphic map

$$\theta : X \setminus S \to Gr(n, N)$$

sending each point $x \in X \setminus S$ to the $n-$dimensional vector sub-space of $\mathbb{C}^N$ which directs the tangent space at x to X. This map is holomorphic on $X \setminus S$ and meromorphic along S: assuming that X is locally defined by $\{f = 0\}$ in an open set in $\mathbb{C}^N$ the analytic subset $G \subset \tilde{G} := \{(x, P) \in X \times Gr(n, N) \ / \ P \subset Ker[df_x]\}$, which is the union of the irreducible components of $\tilde{G}$ which contain an irreducible component of the graph of the map θ, is a proper modification of X which is the closure of the graph of the map θ.

We shall note $\mathcal{N} : \hat{X} \to X$ the projection on X of the normalization of G. We shall call the (local) **normalized Nash transform** of X this modification.

Let $\pi : \mathcal{U} \to Gr(n, N)$ the universal $n-$vector bundle of $Gr(n, N)$ and let $\mathcal{L}^q$ be the sheaf of section of the dual vector bundle to $\Lambda^q(\mathcal{U})$. Let $pr : \hat{X} \to Gr(n, N)$ be the projection.

Proposition 5.2.2. *For each integer q there is a canonical isomorphism*

$$c^q : \mathcal{N}^*(\alpha^q_X)/\mathrm{torsion} \to pr^*(\mathcal{L}^q).$$

PROOF. This proposition is an easy consequence of Corollary 2.2.3 and Lemma 2.1.3. $\square$

As a consequence of this proposition we obtain that for a normal complex space we have $\alpha^q_X \simeq \mathcal{N}_*(\mathcal{L}^q)$ for any integer $q \geq 0$.

Lemma 5.2.3. *Let X be a reduced complex space and let $\tau : \tilde{X} \to X$ be any (proper) modification. Then we have a natural inclusion $\alpha^\bullet_X \hookrightarrow \tau_*(\alpha^\bullet_{\tilde{X}})$.*

PROOF. Consider a special desingularization $\theta : \tilde{\tilde{X}} \to \tilde{X}$ and remark that $\pi := \tau \circ \theta$ is a desingularization of X. Then we have

$$\alpha^\bullet_X = \pi_*(\pi^{**}(\Omega^\bullet_X) = \tau_*(\theta_*\big(\theta^{**}(\tau^{**}(\Omega^\bullet_X))\big).$$

Now the equality $\alpha^\bullet_{\tilde{X}} = \theta_*(\theta^{**}(\Omega^\bullet_{\tilde{X}}))$ and the inclusion $\tau^{**}(\Omega^\bullet_X) \subset \Omega^\bullet_{\tilde{X}}/\mathrm{torsion}$ give

$$\theta^{**}(\tau^{**}(\Omega^\bullet_X) \subset \theta^{**}(\Omega^\bullet_{\tilde{X}})$$
$$\theta_*\big(\theta^{**}(\tau^{**}(\Omega^\bullet_X)\big) \subset \alpha^\bullet_{\tilde{X}} \quad \text{and then}$$
$$\alpha^\bullet_X \subset \tau_*(\alpha^\bullet_{\tilde{X}})$$

concluding the proof. $\square$

Remark. This shows that when we consider a sequence of successive modifications over a reduced complex space X, the sequence of coherent sub-sheaves $(\tau_\nu)_*(\alpha_{X_\nu}^\bullet)$ is locally stationary on X. For instance, this is the case for iterated normalized Nash transforms over a given X.

6. Some examples

6.1. **Computation of $\omega_X^\bullet$ for hypersurfaces.** We shall need the following elementary lemma.

Lemma 6.1.1. *Let U be an open polydisc in $\mathbb{C}^n$ and D an open disc in $\mathbb{C}$. Let $X \subset U \times D$ be a reduced multiform graph of degree k in $U \times D$ with canonical equation $P \in \mathcal{O}(U)[z]$, which is a monic degree k polynomial in z. Then we have the inclusion*

$$\Gamma(X, \omega_X^q) \subset \sum_{j=0}^{k-1} \frac{z^j}{P'(z)}.\Gamma(U, \Omega_U^q)$$

with equality for $q = n$.

Proof. First will shall prove the following formula, where $(j,h) \in [0, k-1]^2$:

$$\det_{j,h}\big[\mathrm{Trace}_{X/U}(\frac{z^{j+h}}{P'(z)})\big] = (-1)^{k.(k-1)/2}.$$

Assume, without loss of generality, that D is centered at the origin with radius R. Then for $r > R$ we have, thanks to Cauchy's formula

$$\mathrm{Trace}_{X/U}(\frac{z^m}{P'(z)}) = \frac{1}{2i\pi}.\int_{|z|=r} \frac{z^m.dz}{P(z)}.$$

Then for $m \leq k-2$ put $z = r.e^{i.\theta}$ we obtain

$$\mathrm{Trace}_{X/U}(\frac{z^m}{P'(z)}) = \frac{1}{2\pi}.\int_0^{2\pi} \frac{r^{m+1-k}.e^{i.(m+1-k)}.d\theta}{1+O(1/r)}$$

and letting $r \to +\infty$ gives 0. For $m = k-1$ the same computation gives

$$\mathrm{Trace}_{X/U}(\frac{z^{k-1}}{P'(z)}) = \frac{1}{2\pi}.\int_0^{2\pi} \frac{d\theta}{1+Q((1/r).e^{-i.\theta})}$$

Where Q is a polynomial without constant term.

So we obtain that $\mathrm{Trace}_{X/U}(\frac{z^{k-1}}{P'(z)}) = 1$. This is enough to get the formula above.

To prove the inclusion

$$\Gamma(X, \omega_X^q) \subset \sum_{j=0}^{k-1} \frac{z^j}{P'(z)}.\Gamma(U, \Omega_U^q),$$

take $\alpha \in \omega_X^q$ and write

$$\alpha = \sum_{|H|=q} g_H.dt^H$$

where g_H are degree $\leq k-1$ polynomials in z with meromorphic functions on U as coefficients. As we have $P'(z).dz = -\sum_{h=1}^n \frac{\partial P}{\partial t_h}.dt_h$ on X, this is possible. Now for any $f \in \mathcal{O}(X)$ we have

$\mathrm{Trace}_{X/U}[f.\alpha] \in \Omega^q(U)$ and this implies that for any $H \subset [1, n], \mathrm{Trace}_{X/U}[f.g_H]$ is in $\mathcal{O}(U)$. Let g be a meromorphic function on X and assume that we write

$$g = \sum_{j=0}^{k-1} a_j . \frac{z^j}{P'(z)}$$

where $a_j, j \in [0, k-1]$ is a meromorphic function on U. This is always possible for the g_H as we can see in what follows. Let $m_p := \mathrm{Trace}_{X/U}[z^p.g]$ for $p \in [0, k-1]$. Then we have the linear system in the $(a_j), j \in [0, k-1]$:

$$\sum_{j=0}^{k-1} a_j.\mathrm{Trace}_{X/U}[\frac{z^{p+j}}{P'(z)}] = m_p \quad \forall p \in [0, k-1].$$

But the determinant of this linear system is $(-1)^{k.(k-1)/2}$, so this implies, if we assume that the functions m_p are holomorphic on U, that the functions a_j for $j \in [0, k-1]$, are holomorphic in U and so that g is in $\frac{1}{P'(z)}.\mathcal{O}(X)$. Then our inclusion is proved, as $\mathcal{O}(X) = \sum_{j=0}^{k-1} \mathcal{O}(U).z^j$.
Note that in the situation above, the condition in order that $\alpha = \sum_{j=0}^{k-1} \frac{z^j}{P'(z)}.\Omega^q(U)$ will be in $\omega^q(X)$ is that for any $j \in [0, k-1]$ the $(q+1)-$forms

$$\mathrm{Trace}_{X/U}[z^j.dz \wedge \alpha]$$

are holomorphic in U for all $j \in [0, k-1]$. This is consequence of the fact that for any $\beta \in \Omega^p(X)$ the $(p+q)-$form $\mathrm{Trace}_{X/U}[\alpha \wedge \beta]$ must be holomorphic (see [B. 78] for this characterization of the sheaf $\omega_X^\bullet$). For $q = n$ this extra condition is empty, so the equality occurs. $\square$

Remark. For a general reduced multiform graph $X \subset U \times B$ where B is now a polydisc in $\mathbb{C}^p$, for any linear form l in $\mathbb{C}^p$ which separates generically the fibres of the projection $\pi : X \to U$, the map $id_U \times l : U \times B \to U \times \mathbb{C}$ is proper and generically injective on X. If we define $Y_l := (id_U \times l)(X)$, we are in the situation of the lemma above, and, as the direct image by π induces an injective sheaf map $\pi_* : \omega_X^\bullet \to \pi^*(\omega_{Y_l}^\bullet)$, we obtain the inclusion

$$\pi_*\omega_X^\bullet \subset \sum_{j=0}^{k-1} \frac{l(x)^j}{P_l'}.\Omega_U^\bullet$$

for any such l, where P_l is the canonical equation for Y_l (see [B-M 1] chapter II). Note that the canonical equation P_l is obtained from the canonical equation of the reduced multiform graph X by the evaluation at l (with $z = l(x)$); see *loc. cit.* $\square$

Note that, if X is a reduced complex space of pure dimension n, a section $\alpha \in \omega_X^n$ is in L_X^n iff $\alpha \wedge \bar{\alpha}$ is locally integrable on X. The analogous characterization, for $p < n$, involves local integrability of $\alpha \wedge \bar{\alpha}$ on all $p-$dimensional irreducible analytic subset $Y \subset X$ not contained in the singular set of X; so it may be useful as a necessary condition but very difficult to check as a sufficient condition.

Preliminary remark. Let $\tau : \tilde{X} \to X$ be a desingularization of a reduced complex space X. Note S the singular set in X and assume that the center of τ is contained in S.

- Let $\alpha \in \omega_X^p$. To check if α is in L_X^p is equivalent to check if $\tau^*(\alpha)$, as a section of $\Omega_{\tilde{X}}^p$ on $\tau^{-1}(X \setminus S)$, extends to a section of $\Omega_{\tilde{X}}^p$ on $\tilde{X}$.
- For $\alpha \in L_X^p$ to check if α is a section of α_X^p is equivalent to check if $\tau^{**}(\alpha)$ extends to a section of $\tau^{**}(\Omega_X^p)$ **when τ is a special desingularization of X**. But this not true, in general, for an arbitrary desingularization of τ.

- But for any desingularization, it is a necessary condition in order that $\alpha \in L_X^p$ belongs to α_X^p that $\tau^{**}(\alpha)$ is a section of $\tau^{**}(\Omega_X^p)$ on $\tilde{X}$.

So, in order to have a complete description of the sheaf $\alpha_X^\bullet$, we shall use a special desingularization of X.

6.2. The case $X := \{(x,y,z) \in \mathbb{C}^3 \ / \ x.y = z^k\}, k \geq 2$.

Notation. After blow-up (x,y,z) in $\mathbb{C}^3$ the homogeneous coordinates in $\mathbb{P}_2$ will be (α, β, γ). The symetry between x and y allows to consider only the chart $\{\alpha \neq 0\}$ on which we put $b := \beta/\alpha, c := \gamma/\alpha$ and the chart $\{\gamma \neq 0\}$ on which we put $a := \alpha/\gamma, b := \beta/\gamma$.

Our first example will be the normal complex spaces, where $k \in \mathbb{N}, k \geq 2$

$$X := S_k := \{(x,y,z) \in \mathbb{C}^3 \ / \ x.y = z^k\}.$$

Note that S_0 and S_1 are smooth complex surfaces.

Lemma 6.2.1. *For any $k \geq 2$ the normal complex space S_k is nearly smooth*[9]*. So we have $L_{S_k}^\bullet = \omega_{S_k}^\bullet$ for any k.*

Proof. Let ζ be a $k-$th primitive root of 1. Then S_k is isomorphic to the quotient of $\mathbb{C}^2$ by the action of the automorphism $\theta(u,v) = (\zeta.u, \zeta^{-1}.v)$. The quotient map is given by $q(u,v) = (u^k, v^k, u.v) \in \mathbb{C}^3$. $\square$

Now compute the sheaf ω_X^h for $h \in [0,2]$. We have $\omega_X^0 = \mathcal{O}_X$ as X is normal, and $\omega_X^2 = \mathcal{O}_X.\frac{dx\wedge dy}{z^{k-1}}$. A rather easy computation shows that the quotient $\omega_X^1\big/\Omega_X^1$ is generated on $\mathcal{O}_X$ by the image of $x.dy\big/z^{k-1} = -y.dx/z^{k-1} + k.dz$ which is annihilated in this quotient by x, y and z^{k-1}.

Lemma 6.2.2. *For any $k \geq 2$, the sheaf $\alpha_{S_k}^2$ coincides with $\Omega_{S_k}^2\big/$torsion.*

Proof. Remark that for $k = 0, 1$ the lemma is obvious as S_k is smooth. We shall prove the lemma by induction on $k \geq 2$.
We have to consider the case $k = 2$ first because it appears that the computation is special in this case (see the denominator $k - 2$ in the computation for $k \geq 3$).

For $k = 2$ after blowing-up the origin we have a smooth manifold:
Claim. This a special desingularization of S_2.
Proof. In the chart $\{\alpha \neq 0\}$ we have $y = x.b, \quad z = x.c, \quad b = c^2$ so (x,c) is a coordinate system in this chart and the sheaf $\tau^*(\Omega_{S_2}^1)\big/$torsion is generated by dx and $x.dc$, so it is free.
In the chart $\{\gamma \neq 0\}$ we have $x = z.a, \quad y = z.b, \quad a.b = 1$ and so (z,a) is a coordinate system with $a \neq 0$. Then the sheaf $\tau^*(\Omega_{S_2}^1)\big/$torsion is generated by dz and $z.da$ which is also free. By symetry in x and y, the proof of the claim is complete.

Let us come back to the computation of $\alpha_{S_2}^2$.

In the chart $\{\alpha \neq 0\}$, we have

$$\frac{dx \wedge dy}{z} = 2.dx \wedge dc$$

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[9]See [B-M. 17]

which is holomorphic but not in $\tau^*(\Omega^2_{S_2}) \simeq \mathcal{O}_{\tilde{X}}.x.dx \wedge dc$.

In the chart $\{\gamma \neq 0\}$, we have $x = z.a, \quad y = z.b, \quad a.b = 1$ and

$$\frac{dx \wedge dy}{z} = -2dz \wedge da/a$$

which is holomorphic but not in $\tau^{**}(\Omega^2_{S_2}) \simeq \mathcal{O}_{\tilde{X}}.z.dz \wedge da$.

The assertion is proved for $k = 2$.

Consider now the case $k = 3$. Then the blowing-up the origin gives a smooth manifold:

Claim. This desingularization of S_3 is not special.

Proof. In the chart $\{\alpha \neq 0\}$ we have $y = x.b, \quad z = x.c, \quad b = x.c^3 \quad$ so (x, c) is a coordinate system in this chart and the sheaf $\tau^*(\Omega^1_{S_3})\big/$torsion is generated by dx and $x.dc$ so it is free. But in the chart In the chart $\{\gamma \neq 0\}$ we have $x = z.a, y = z.b$ and $a.b = z$ and (a, b) is a coordinate system. As $x = a^2.b$ and $y = a.b^2$, the sheaf $\tau^*(\Omega^1_{S_2})\big/$torsion is generated by $d(a^2.b), d(a.b^2), d(a.b)$ and it is not locally free near the point $a = b = 0$.

But blowing-up the point $a = b = 0$ in the second chart make the pull-back of the sheaf $\tilde{\tau}^*(\Omega^1_{S_3})\big/$torsion locally free, where $\tilde{\tau}$ is the composition of τ and the blow-up of the point $a = b = 0$ in the second chart:

In the chart $a = \theta.b$ of this second blow-up the coordinate system is given by (b, θ) so $x = \theta^2.b^3, y = \theta.b^3, z = \theta.b^2$. Then the sheaf $\tilde{\tau}^*(\Omega^1_{S_3})\big/$torsion is generated by dx, dy, dz. An easy computation shows that $dx = -\theta.dy + 3\theta.b.dz$ so sheaf $\tilde{\tau}^*(\Omega^1_{S_3})\big/$torsion is free in this chart. The other chart is obtained by exchanging a and b.

Consider now the section $\frac{dx \wedge dy}{z}$ of $\omega^3_{S_3}$. Its pull-back by $\tilde{\tau}$ is given by

$$\frac{d(\theta^2.b^3) \wedge d(\theta.b^3)}{\theta.b^2} = -3\theta.b^3.db \wedge d\theta$$

and the generator of the sheaf $\tilde{\tau}^*(\Omega^2_{S_3})\big/$torsion is given by

$$\tilde{\tau}^*(dy \wedge dz) = \theta.b^4.db \wedge d\theta.$$

So we conclude that neither $\frac{dx \wedge dy}{z}$ nor $\frac{dx \wedge dy}{z^2}$ are in $\alpha^2_{S_3}$.

As the assertion is proved for $k = 2, 3$ we may assume that, for $k \geq 4$ the equality is proved for S_{k-2}. Then let $\tilde{X} \to X := S_k$ be the blow-up of S_k at the singular point $x = y = z = 0$. In the chart $\{\gamma \neq 0\}$ of $\tilde{X}$ we have the relations

$$x = a.z, \quad y = b.z \quad a.b = z^{k-2}$$

and we find a copy of S_{k-2}. For $k \geq 4$ we have

$$dx \wedge dy = \frac{k}{k-2}.z^2.da \wedge db = k.\frac{a.b}{k-2}.\frac{da \wedge db}{z^{k-4}},$$
$$dx \wedge dz = \frac{a}{k-2}.\frac{da \wedge db}{z^{k-4}}, \quad dy \wedge dz = \frac{b}{k-2}.\frac{da \wedge db}{z^{k-4}}.$$

So in this chart

$$\tau^{**}(\Omega^2_{S_k}/\text{torsion}) = \mathcal{O}_{S_{k-2}}.\big(a, b\big).\frac{da \wedge db}{z^{k-4}}$$

and, as a consequence of the fact that z^{k-q-2} is not in the ideal $\big(a, b\big).\mathcal{O}_{S_{k-2}}$ for $q \geq 1$, for each $q \geq 1$ the 2−form $dx \wedge dy/z^q$ is not a section of the sheaf $\tau^{**}(\Omega^2_{S_k}/\text{torsion})$ near the origin

$a = b = z = 0$ in this chart. So the sheaf $\alpha^2_{S_k}$ is equal to $\Omega^2_{S_k}\big/$torsion. $\square$

Lemma 6.2.3. *For all $k \geq 0$ the vector space $L^1_{S_k}\big/\alpha^1_{S_k}$ has dimension $p = [(k-1)/2]$ the integral part of $(k-1)/2$. A basis is given by the $1-$forms $x.dy/z^q$ for q in $[[k/2]+1, k-1]$, for $k \geq 2$.*

Proof. We shall begin by a simple remark.

Assume that $k \geq 2$ and let $p := [(k-1)/2]$. Then for any $q \in [1, p]$ the form $x.dy/z^q$ satisfies an integral dependence equation on $\Omega^1_{S_k}$. We have

$$x.dy/z^q + y.dx/z^q = d(z^k)/z^q = k.z^{k-q-1}.dz$$

and also

$$(x.dy/z^q).(y.dx/z^q) = z^{k-2q}.(dx).(dy).$$

This implies that $x.dy/z^q$ is solution of the integral dependence equation

$$X^2 - (k.z^{k-q-1}.dz).X + z^{k-2q}.(dx).(dy) = 0$$

in $S_2(\Omega^1_{S_k})\big/$torsion. So these sections of the sheaf ω^1_X are in fact sections of the sheaf α^1_X.

Now remark also that with the weights $x \to k, y \to k, z \to 2$ the form $x.dy/z^q$ has weight $2(k-q)$. Then they have different quasi-homogeneities, so they are linearly independent over $\mathbb{C}$. Let now prove that for $k-1 > q > p$ the form $x.dy/z^q$ is not in $\alpha^1_{S_k}$ by induction on $k \geq 0$. As the assertion is empty for $k = 0, 1$ assume $k \geq 2$ and the assertion proved for $k-2$.

We have seen that after blowing-up the singular point in S_k for any $k \geq 2$ we find only one singular point of the type S_{k-2} in the chart $\{\gamma \neq 0\}$ and that the form $x.dy/z^q$ is given by the following computation in this chart $\{\gamma \neq 0\}$:

$$x = z.a, \quad y = z.b, \quad a.b = z^{k-2} \quad x.dy/z^q = a.db/z^{q-2} + z^{k-q-1}.dz.$$

But on S_{k-2} we know, by the induction hypothesis, that the form $a.db/z^{q-2}$ is not a section of $\alpha^1_{S_{k-2}}$ for $q - 2 > [\frac{k-3}{2}] = p-1$. So only the case $q = p+1$ is left.

Assume first that $k = 2p+1$. In the last chart $\{\gamma \neq 0\}$ in the desingularization process of S_{2p+1} by blowing up the unique singular point at each step, we reach the following relations:

$$x = u^p.v^{p+1}, y = u^{p+1}.v^p, z = u.v \quad x.dy/z^{p+1} = (p+1).u^{p-1}.v^p.du + p.u^p.v^{p-1}.dv$$

where $(u, v) \in \mathbb{C}^2$ is a local coordinate system.

But, as we have seen for $k = 3$ this desingularization is not special. So we have to blow up the origin one more time and chek that we obtain now a special desingularization of S_{2k+1}. In the chart $u = \theta.v$ we obtain $x = \theta^p.v^{2p+1}, y = \theta^{p+1}.v^{2p+1}, z = \theta.v^2$ which gives

$$\begin{aligned}
dx &= \theta^{p-1}.v^{2p}.(p.v.d\theta + (2p+1).\theta.dv) := \theta^{p-1}.v^{2p}.A\\
dy &= \theta^p.v^{2p}.((p+1).v.d\theta + (2p+1).\theta.dv) := \theta^p.v^{2p}.B\\
dz &= v.(v.d\theta + 2\theta.dv) := v.C
\end{aligned}$$

Now remark that $B = -A + (2p+1).C$ which implies that

$$dy = -\theta.dx + (2p+1).\theta^p.v^{2p-1}.dz$$

and so the pull-back of $\Omega^1_{S_{2p+1}}$ is locally free after this last blow-up.
Now the pull-back of the form $x.dy/z^{p+1}$ is given by

$$\theta^{p-1}.v^{2p-1}.B = \theta^{p-1}.v^{2p-1}.(-A + (2p+1).C)$$

and it is now easy to see that this does not belong to the sub-sheaf generated by dx and dz.

Now assume that $k = 2p$ with $p \geq 2$ then in the last chart $\{\gamma \neq 0\}$ we shall have, with coordinates (z, u) with $u \neq 0$

$$x = z^p.u, \quad y = z^p/u.$$

We again have to check that this is a special desingularization of S_{2p}. But as $u \neq 0$ in this chart, (dx, dz) generate the pull-back of $\Omega^1_{S_{2p}}$.

Now, as $x.dy/z^{p+1} = u.dy/z$ to see if this form belongs to sub-sheaf generated by (dx, dz) is equivalent to see if $z^{p-1}.du$ is a section of this sub-sheaf. This is clearly not the case as $z^{p-1}.du = dx/z - p.z^{p-2}.u.dz$. □

6.3. **The case** $X := \{(x, y, z) \in \mathbb{C}^3 \ / \ x^3 + y^3 + z^3 = 0\}$**.** Now consider

$$X := \{(x, y, z) \in \mathbb{C}^3 \ / \ x^3 + y^3 + z^3 = 0\}.$$

The lemma 6.1.1 gives the inclusion

$$\omega^1_X \subset \frac{1}{z^2}.\Omega^1_{\mathbb{C}^2} + \frac{1}{z}.\Omega^1_{\mathbb{C}^2} + \Omega^1_{\mathbb{C}^2}$$

where x, y are the coordinates on $\mathbb{C}^2$. An easy computation shows that the forms

$$\alpha := (x.dy - y.dx)/z^2$$

and $z.\alpha$ generate ω^1_X/Ω^1_X.

Let $\tau : \tilde{X} \to X$ the blowing-up at the origin of X.

Claim. This is a special desingularization:

In the chart $\{\alpha \neq 0\}$ we have

$$y = u.x, \quad z = v.x, \quad u^3 + v^3 + 1 = 0.$$

Then we can choose (x, u) or (x, v) as local coordinates when $v \neq 0$ or $u \neq 0$. The sheaf $\tau^*(\Omega^1_X)$ is generated by dx and $x.du$ when $v \neq 0$ and so is free on this open set. So the sheaf $\tau^*(\Omega^1_X)\big/$torsion is locally free on this blow-up, proving the claim.

In the chart $\{\gamma \neq 0\}$ let $a := \alpha/\gamma$ and $b := \beta/\gamma$; then we have the relations

$$x = z.a, \quad y = z.b, \quad a^3 + b^3 + 1 = 0$$

and then we can choose (z, a) or (z, b) as local coordinates. Then we have

$$\alpha = a.db - b.da = db/a^2 = -da/b^2.$$

In the chart $\{\alpha \neq 0\}$ we have

$$y = u.x, z = v.x, u^3 + v^3 + 1 = 0.$$

Then we can choose (x, u) or (x, v) as local coordinates and $\alpha = du/v^2 = -dv/u^2$. This shows that $\omega^1_X = L^1_X$. But α does not vanish on the exceptional divisor, so α is not a section of α^1_X.

But, in the first chart,

$$z.\alpha = z.a.db - z.b.da = a.dy - a.b.dz - b.dx + a.b.dz = a.dy - b.dx \in \tau^{**}(\Omega^1_X)$$

and in the second chart

$$x.\alpha = x.du/v^2 = dy/v^2 - u.dx/v^2 = -dz/u^2 + v.dx/u^2$$

also belong to $\tau^{**}(\Omega^1_X)$.

Then $x.\alpha, y.\alpha$ and $z.\alpha$ are sections of α^1_X and the quotient $L^1_X\big/\alpha^1_X$ is a vector space of dimension 1 with basis α. □

Note that $x.y.z.\alpha$ is not a section of $\Omega_X^1/\text{torsion}$ because if we assume that $x.y.z.\alpha$ is a section of $\Omega_X^1/\text{torsion}$, we can write

$$x.y.(x.dy - y.dx) = z.\big[\lambda.dx + \mu.dy + \nu.dz + \rho.df + \sigma.f\big]$$

in $\mathbb{C}^3$, where λ, μ, ν where homogeneous of degree 2, ρ is a complex number and where

$$\sigma := u.dx + v.dy + w.dz$$

with u, v, w complex numbers. This gives, for instance $-x.y^2 = z.\lambda + 3z.\rho.x^2 + u.f$ which is impossible.

So the vector space α_X^1/Ω_X^1 has dimension at least 2. The complete determination of the quotient α_X^1/Ω_X^1 is a non-trivial exercise left to the reader.

Lemma 6.3.1. *For* $X := \{(x, y, z) \in \mathbb{C}^3 \ / \ x^3 + y^3 + z^3 = 0\}$ *we have*

$$\dim_{\mathbb{C}} \alpha_X^2/\Omega_X^2 = 2, \quad \dim_{\mathbb{C}} L_X^2/\alpha_X^2 = 3 \quad \dim_{\mathbb{C}} \omega_X^2/L_X^2 = 1.$$

Proof. After blowing-up (x, y, z) in $\mathbb{C}^3$ we consider the chart $\{\gamma \neq 0\}$ as above. We have

$$\omega := \frac{dx \wedge dy}{z^2} = -\frac{dz}{z} \wedge \frac{db}{a^2} = \frac{dz}{z} \wedge \frac{da}{b^2}.$$

Then $x.\omega, y.\omega, z\omega$ are holomorphic in this chart, as we have $x = z.a$ and $y = z.b$ and this chart is enough as $dx \wedge dy/z^2 = dy \wedge dz\big/x^2 = dz \wedge dx\big/y^2$ so $x.\omega, y.\omega, z.\omega$ belongs to L_X^2.

But this is not the case for ω. So $\dim \omega^2/L_X^2 = 1$.

The sheaf $\tau^{**}(\Omega_X^2/\text{torsion})$ in this chart is generated by

$$z.(da/a^2) \wedge dz = -z.(db/b^2) \wedge dz.$$

Then it is equal to $z.\Omega_{\tilde{X}}^2$ in this chart. So a section in L_X^2 is in α_X^2 if and only if it belongs to $(x.L_X^2) \cap (y.L_X^2) \cap (z.L_X^2)$. This intersection is generated by $x.y.\omega, y.z.\omega, z.x.\omega$ as a $\mathcal{O}_X-$module. The vector space L_X^2/α_X^2 is generated by $x.\omega, y.\omega, z.\omega$ because $x^2.\omega, y^2.\omega, z^2.\omega$ are in $\Omega_X^2 \subset \alpha_X^2$. We let to the reader the proof that they give a basis of L_X^2/α_X^2.

Let us prove that $x.y.z.\omega$ is not in $\Omega_X^2/\text{torsion}$.

Assume that $x.y.z.\omega \in \Omega_X^2/\text{torsion}$. Then we can write on $\mathbb{C}^3$:

$$x.y.dx \wedge dy - z\big[\lambda.dx \wedge dy + \mu.dy \wedge dz + \nu.dz \wedge dx + (a.dx + b.dy + c.dz) \wedge df\big] = 0$$

where we can assume that λ, μ, ν are linear forms on $\mathbb{C}^3$ and a, b, c are complex number, using the homogeneity of the situation. The coefficient of $dx \wedge dy$ in this identity is equal to

$$x.y - z.\lambda - a.y^2 + b.x^2,$$

which cannot be identically zero. Contradiction.

As it is easy to see that $x.y.\omega = y.z.\omega = z.x.\omega$ and $x.y.z.\omega$ are linearly independent over $\mathbb{C}$ (different homogeneities) we conclude that $\dim \alpha_X^2/\Omega_X^2 = 2$. $\square$

Remark. We have on X

$$\omega := \frac{dx \wedge dy}{z^2} = \frac{dy \wedge dz}{x^2} = \frac{dz \wedge dx}{y^2}$$

so

$$(x.y.\omega)^2 = \frac{x^2.y^2.(dx \wedge dy)^2}{z^4} = \frac{x^2.dx \wedge dy}{z^2}.\frac{y^2.(dx \wedge dy)}{z^2} = (dz \wedge dy).(dx \wedge dz),$$

because on X we have $x^2.dx \wedge dy = -z^2.dz \wedge dy$ and $y^2.dx \wedge dy = -z^2.dx \wedge dz$. This gives an integral dependence relation for $x.y.\omega$ in the symetric algebra of $\Omega_X^2/\text{torsion}$.

6.4. **The case $X := \{(x,y,u,v) \in \mathbb{C}^4 \ / \ x.y = u.v\}$.** Let us begin by the verification that blowing-up the origin gives a special desingularization for X.

Write X as $\{(x_1,x_2,x_3,x_4) \in \mathbb{C}^4 \ / \ x_1^2+x_2^2+x_3^2+x_4^2 = 0\}$ and look at the chart $\alpha \neq 0$. So we have $x_2 = b.x_1, x_3 = c.x_1, x_4 = e.x_1$ with the relation $1+b^2+c^2+e^2 = 0$ and coordinates (x_1,b,c) on the subset $e \neq 0$. The the sheaf $\tau^*(\Omega^1_X)$ is generated by $dx_1, x_1.db, x_1.dc$ because for $e \neq 0$ we have $x_1.de = -e^{-1}[c.x_1.dc + b.x_1.db]$. So modulo its torsion, the sheaf $\tau^*(\Omega^1_X)$ is locally free.

Lemma 6.4.1. *The sheaf L^3_X is equal to ω^3_X and is given by $\mathcal{O}_X.\omega$ where we define*

$$\omega := \frac{dy \wedge du \wedge dv}{y}.$$

Moreover, ω does not belong to α^3_X.

PROOF. On X we have $x.dy + y.dx = u.dv + v.du$

$$\omega = -\frac{dx \wedge du \wedge dv}{x} = \frac{du \wedge dx \wedge dy}{u} = \frac{dv \wedge dx \wedge dy}{v}.$$

To see that $\omega^3_X = \mathcal{O}_X.\omega$ it is enough (X is a hypersurface !) to see that

$$\omega \wedge df/f = dx \wedge dy \wedge du \wedge dv/f$$

where $f := x.y - u.v$. This is clear.

Using the symetries between the coordinates, it is enough to see that $\tau^*(\omega)$ is holomorphic in the first chart of the strict transform $\tilde{X}$ of X by the blow-up at the origin in $\mathbb{C}^4$ to show that ω is a section of L^3_X. Let $y = \lambda.x, u = \mu.x, v = \nu.x$. Then

$$\tau^*(\omega) = -\frac{dx}{x} \wedge x.d\mu \wedge x.d\nu = -x.dx \wedge d\mu \wedge d\nu$$

where x, μ, ν are the coordinates for $\tilde{X}$ in this chart (and $\lambda = \mu.\nu$). So $\omega \in L^3_X$.

To see that ω is not in α^3_X it is enough to see that ω does not belongs to $\tau^{**}(\Omega^3_X)$ in the first chart above. An easy computation show that $\tau^{**}(\Omega^3_X)$ is generated by

$$\pi^{**}(dx \wedge du \wedge dv) = x^2.dx \wedge d\mu \wedge d\nu$$

and so $\omega = -x.dx \wedge d\mu \wedge d\nu$ does not belong to $\tau^{**}(\Omega^3_X)$. $\square$

Lemma 6.4.2. *The meromorphic form $w := u.dv \wedge dx/x$ is a section of α^2_X but it is not a section of $\Omega^2_X/$*torsion *and its differential is not a section of α^3_X.*

PROOF. As

$$u.dv \wedge dx/x + v.du \wedge dx/x = -dx \wedge dy$$

is holomorphic on X, u and v play the same role for this form modulo holomorphic forms. Also $u.dv \wedge (dx/x + dy/y) = dv \wedge du$ so x and y play also the same role modulo holomorphic forms on X. So it is enough to see that in the first chart of the strict transform $\tilde{X}$ of X by the blow-up at the origin in $\mathbb{C}^4$ the form $\tau^{**}(w)$ is a section of $\tau^{**}(\Omega^2_X)$ to prove that w is a section of α^2_X. Using the same coordinates as above we obtain

$$\tau^{**}(w) = \mu.x.d(\nu.x) \wedge dx/x = \mu.x.d\nu \wedge dx = \mu.dv \wedge dx$$

which is a section of $\tau^{**}(\Omega^2_X)$.

To see that w is not a section of $\Omega_X^2/\mathrm{torsion}$ assume the contrary. Then, by symmetry[10] $w' := v.du \wedge dx/x$ is also a section of $\Omega_X^2/\mathrm{torsion}$ and the differential of $w - w'$ must be a section of $\Omega_X^3/\mathrm{torsion}$. But we have already seen that $2.\omega = -d(w - w')$ is not a section of α_X^3. Contradiction. $\square$

Note that an integral dependence relation on the symmetric algebra of the sheaf $\Omega_X^2/\mathrm{torsion}$ for w is given in the second example of the begining of the section 5.1.

Lemma 6.4.3. *We have* $\Omega_X^1/\mathrm{torsion} = \alpha_X^1 = L_X^1 = \omega_X^1$.

Proof. Write $X := \{x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0\} \subset \mathbb{C}^4$. Then thanks to the lemma 6.1.1 we have:

$$\omega_X^1 \subset \Omega_{\mathbb{C}^3}^1 + \frac{1}{x_4}.\Omega_{\mathbb{C}^3}^1.$$

To prove that $\Omega_X^1/\mathrm{torsion} = \omega_X^1$ it is enough to consider a section in ω_X^1, let

$$w := (a.dx_1 + b.dx_2 + c.dx_3)$$

and put $v := w/x_4$ and to show that v is a section of $\Omega_X^1/\mathrm{torsion}$. But then

$$\mathrm{Trace}_\pi(v \wedge dx_4) = w \wedge \mathrm{Trace}_\pi(dx_4/x_4)$$

must be a holomorphic form on $\mathbb{C}^3$, where $\pi : X \to \mathbb{C}^3$ is the projection which makes X a branched covering of degree 2. This condition implies $df \wedge w$ is in $f.\Omega_{\mathbb{C}^3}^2$ where $f := x_1^2 + x_2^2 + x_3^2$. As the sheaf $\Omega_{S_2}^1$ has no torsion[11], this implies that $w = u.df + f.\xi$ where $u \in \mathcal{O}_{\mathbb{C}^3}$ and $\xi \in \Omega_{\mathbb{C}^3}^1$. But $f = -x_4$ on X, so this gives $v = -\xi - u.dx_4$ on X and v is in Ω_X^1. $\square$

Lemma 6.4.4. *We have* $\omega_X^2 = \Omega_X^2/\mathrm{torsion} \oplus \mathbb{C}\,.\eta$ *where*

$$\eta := \frac{x_1.dx_2 \wedge dx_3 + x_2.dx_3 \wedge dx_1 + x_3.dx_1 \wedge dx_2}{x_4}.$$

Proof. Write $\omega := (a.dx_1 \wedge dx_2 + b.dx_2 \wedge dx_3 + c.dx_3 \wedge dx_1)/x_4$ where a, b, c are holomorphic on $\mathbb{C}^3$. Then ω is in ω_X^2 if and only if $\mathrm{Trace}_\pi(dx_4 \wedge \omega)$ is a section of $\Omega_{\mathbb{C}^3}^3$. This is satisfyed if and only if $a.x_3 + b.x_1 + c.x_2$ is a multiple of $\xi := x_1^2 + x_2^2 + x_3^2$ in $\mathcal{O}_{\mathbb{C}^3}$. This gives the relation $(a - g.x_3).x_3 + (b - g.x_1).x_1 + (c - g.x_2).x_2 = 0$. And, as x_1, x_2, x_3 is a regular sequence, this implies

$$a = g.x_3 + \lambda.x_1 + \mu.x_2, \quad b = g.x_1 + \lambda'.x_2 - \lambda.x_3, \quad c = g.x_2 - \lambda'.x_1 - \mu.x_3$$

where λ, λ', μ are in $\mathcal{O}_{\mathbb{C}^3}$. This shows that ω_X^2 is generated as a $\mathcal{O}_X-$module by Ω_X^2 and η. Note that we already know that η is not a section of $\Omega_X^2/\mathrm{torsion}$ as we have shown that ω_X^2 is not equal to $\Omega_X^2/\mathrm{torsion}$

Claim. For $i = 1, 2, 3, 4 \quad x_i.\eta$ is in $\Omega_X^2/\mathrm{torsion}$:

for instance:

$$\begin{aligned}
\frac{x_1.\eta}{x_4} &= \frac{x_1}{x_4}.(x_1.dx_2 \wedge dx_3 + x_2.dx_3 \wedge dx_1 + x_3.dx_1 \wedge dx_2)\\
&= \frac{1}{x_4}.(-(x_2^2 + x_3^2 + x_4^2).dx_2 \wedge dx_3 + (x_2.dx_3 - x_3.dx_2) \wedge x_1.dx_1)\\
&= \frac{1}{x_4}.(-(x_2^2 + x_3^2 + x_4^2).dx_2 \wedge dx_3 + (x_2.dx_3 - x_3.dx_2) \wedge (-x_2.dx_2 - x_3.dx_3 - x_4.dx_4))\\
&= -x_4.dx_2 \wedge dx_3 - x_2.dx_3 \wedge dx_4 + x_3.dx_2 \wedge dx_4 \in \Omega_X^2
\end{aligned}$$

[10] or using $(u.dv + v.du) \wedge dx/x = dy \wedge dx$.

[11] This is easy to see using the fact that $S_2 = \{f = 0\}$ is the quotient of $\mathbb{C}^2$ by ± 1.

proving our claim. $\square$

6.5. **The case $X := \{(x,y,z,t) \in \mathbb{C}^4 \ / \ x.y.z = t^3\}$.** Remark first that the form $\omega_1 := y.z.dx/t^2$ is in ω^1_X because we have, with the notation $f := x.y.z - t^3$:

$$\omega_1 \wedge df = \frac{z.t^3.dx \wedge dy + y.t^3.dx \wedge dz + 3t^2.y.z.dx \wedge dt}{t^2} \in \Omega^2_{\mathbb{C}^4} \quad \text{modulo}(f/t^2).\Omega^2_{\mathbb{C}^4}$$

which allows to conclude as t is not a zero divisor in X (see [B.78]).

Consider now the following sections of ω^1_X:

$$u := t.\omega_1 \quad v := t.\omega_2 \quad w := t.\omega_3$$

where ω_2 and ω_3 are deduced from ω_1 respectively by

$$x \to y, y \to z, z \to x \qquad \text{and} \qquad x \to z, y \to x, z \to y.$$

Then we have in the symmetric algebra of Ω^1_X:

$$u + v + w = 3t.dt \quad u.v + v.w + w.u = t.(z.dx.dy + x.dy.dz + z.dx.dy) \quad u.v.w = t^3.dx.dy.dz.$$

This shows that u, v, w satisfy the following integral dependence relation over the symmetric algebra of Ω^1_X:

$$\Theta^3 - 3t.dt.\Theta^2 + t.(z.dx.dy + x.dy.dz + z.dx.dy).\Theta - t^3.dx.dy.dz = 0.$$

Note that, because the coefficient of Θ does not belong to (t^2), we do not obtain an integral dependence relation over the symmetric algebra of Ω^1_X for Θ/t so for the forms $\omega_i, i = 1, 2, 3$! In fact they are not sections of the sheaf α^1_X (for instance the restriction of ω_1 to the surface $S_3 \simeq \{z = 1\} \cap X$ is not in $\alpha^1_{S_3}$ (see sub-section 6.2).

Let us now verify that $t.u$ is not a section of Ω^1_X/torsion. Assume that we can write

$$y.z.dx = t.\big(\lambda.dx + \mu.dy + \nu.dz + \theta.dt\big) \quad \text{modulo } f.\Omega^1_X + \mathcal{O}_X.df$$

then, by homogeneity, we may assume that λ, μ, ν are homogeneous of degree 2 and

$$y.z.dx = t.\big(\lambda.dx + \mu.dy + \nu.dz + \theta.dt\big) + \sigma.df$$

where σ is a constant. This implies

$$y.z.(1 - \sigma) - t.\lambda = 0, \quad t.\mu + \sigma.x.z = 0$$

which is already enough to obtain a contradiction, as these equations imply $\sigma = 1$ and $\sigma = 0$ respectively. $\square$

Remark. Using the map $((x,y,z) \mapsto (x + y, x + j.y, x + j^2.y, -z)$ which sends the previous $Y := \{x^3 + y^3 + z^3 = 0\}$ to $X = \{x.y.z = t^3\}$ allows to find an integral equation over the symmetric algebra of Ω^1_Y of the section

$$\frac{(x^2 + y^2 - x.y).d(x + y)}{z}$$

of α^1_Y.

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BRIESKORN AND THE MONODROMY

JEAN-PAUL BRASSELET AND MARCOS SEBASTIANI

To Egbert Brieskorn, in memory.

1. Introduction

Brieskorn's paper "Die Monodromie der isolierten Singularitäten von Hyperflächen," published in 1970 in Manuscripta Mathematica, gave a new insight to the theory of monodromy and Gauß-Manin connections. The paper, written in the framework of isolated hypersurface singularities, has been generalized for isolated complete intersection singularities by G.-M. Greuel in 1975 [10]. In the following times and also more recently, a long list of authors, among them P. Deligne [7], W. Ebeling [8], H. Hamm [12], Lê D. T. [20], B. Malgrange [24], D.Siersma [37] etc. provided generalizations and developments of the monodromy theory. The regularity of the Gauß-Manin connection, proved by Brieskorn in the framework of isolated hypersurface singularities has been proved and developped in various situations by many authors, among them G.-M. Greuel [10], C. Hertling [15], F. Pham [28], K. Saito [29], M. Saito [30], J. Scherk and J.H.M. Steenbrink [31], M. Schulze [32], A. Varchenko [38], etc.

There are many surveys concerning the various aspects of monodromy and including developments of the theory. In particular, Ebeling's survey [8] shows very well the importance of Brieskorn's article as well as developments and generalisations of the Brieskorn's results. Siersma's survey [37] deals with the non-isolated case, and presents new results in this framework.

The present paper, based on ideas of the second author [34, 35, 36], does not pretend any originality. It is not devoted to specialists, but to "beginners". The aim of the paper is to introduce monodromy theory and provide some elementary view about the Brieskorn paper. Our aim is not to replace the reading of this very important Brieskorn article, but hopefully to encourage one to read it.

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2. Connections and monodromy

2.1. **Definitions and notations.** Let $f : (\mathbb{C}^{n+1}, 0) \to \mathbb{C}$ be an analytic function defined in a neighbourhood of the origin 0 in $\mathbb{C}^{n+1}$ and such that $f(0) = 0$. We denote by $(z_0, \dots, z_n)$ the local coordinates of $\mathbb{C}^{n+1}$ at 0. Let us assume that f admits an isolated singularity at 0, that is the partial derivatives $(\partial f/\partial z_i)(z)$ have a common zero at the origin and there is no other singularity in a neighbourhood of 0.

One denotes by $\mathcal{O}$ the local ring of $\mathbb{C}^{n+1}$ at 0 and by I the ideal of $\mathcal{O}$

$$I = \left\langle \frac{\partial f}{\partial z_0}, \dots, \frac{\partial f}{\partial z_n} \right\rangle.$$

The Milnor number of the singularity is defined by:

$$\mu = \dim_{\mathbb{C}} \mathcal{O}/I,$$

denoted by $b_{f,0}$ in Brieskorn [5].

Let us fix some (classical) notations. Let ε and η be such that $0 < \eta < \varepsilon$ and denote:

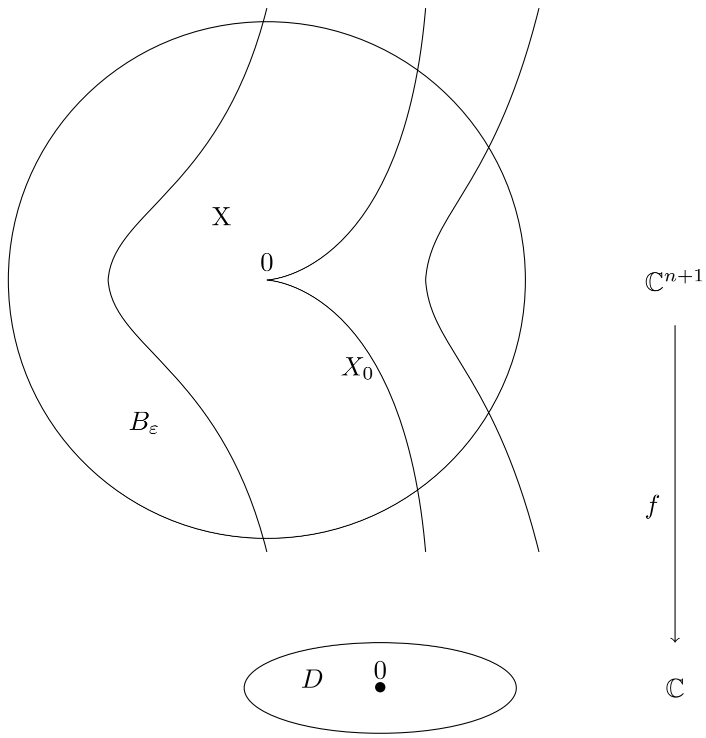
$B_\varepsilon \subset \mathbb{C}^{n+1}$ the ball defined by $\|z\| < \varepsilon$, $z \in \mathbb{C}^{n+1}$,

$D \subset \mathbb{C}$ the disk defined by $|t| < \eta$, $t \in \mathbb{C}$, and $D' = D \setminus \{0\}$,

$X = B_\varepsilon \cap f^{-1}(D) = \{z \in \mathbb{C}^{n+1} ; \|z\| < \varepsilon \text{ and } |f(z)| < \eta\}$,

$X' = X \setminus f^{-1}(0)$ and $X_t = X \cap f^{-1}(t)$ for all $t \in D$.

The following classical picture illustrates the situation.



The fundamental theorem, due to Milnor is the following:

Theorem 2.1 (Milnor)**.** If ε and η are small enough, then:
(i) The map $f : X \setminus f^{-1}(0) \to D \setminus \{0\}$ is a $\mathcal{C}^\infty$ differentiable fibration, locally trivial and whose fibres have the homotopy type of a bouquet of μ spheres with dimension n.
(ii) There exists $\varepsilon_1 < \varepsilon$ such that the intersection of X_t with the sphere $S_r \subset \mathbb{C}^{n+1}$ centered at 0 and with radius r is transverse for all $|t| \leq \eta$ and $\varepsilon_1 \leq r \leq \varepsilon$.

In the following, we intend to make explicit the action of the fundamental group $\pi_1(D')$ on the cohomology of the fibre $H^n(X_t; \mathbb{C})$ for $t \in D'$. We will use some results on connections.

2.2. **Connections.** Let $\pi : E \to B$ a locally trivial fibre bundle, where the fibre F and basis B are locally compact. We assume that F has the homotopy type of a finite complex. One define a (complex) vector fibre bundle $H^n(\pi)$ with basis B in the following way:

The total space is the set of pairs (t, α) where $t \in B$ and $\alpha \in H^n(F_t; \mathbb{C})$, where $F_t = \pi^{-1}(t)$. The projection of $H^n(\pi)$ on B sends (t, α) to t. The vectorial structure of the fibres is clear. The topology of $H^n(\pi)$ is defined by the way of local charts: for every open subset $U \subset B$ which is a trivialization domain of $\pi : E \to B$, one has a homeomorphism $\psi|_U : E|_U \to U \times F$ and then for every $t \in U$, an identification $\psi_t : F_t \xrightarrow{\sim} F$. For all $t \in B$, one has an isomorphism $\psi_t^* : H^*(F;\mathbb{C}) \xrightarrow{\sim} H^*(F_t;\mathbb{C})$. The local chart of $H^n(\pi)$ over U is defined by the bijection

$$\Psi_U : H^n(\pi)|_U \to U \times H^n(F;\mathbb{C})$$

such that $\Psi_U(t,\alpha) = (t, (\psi_t^*)^{-1}(\alpha))$. The vector bundle $H^n(\pi)$ is then well defined.

The charts Ψ_U are given by locally constant maps: if $V = U \cap U'$ is connected, the transition map $V \to \mathrm{Aut}(H^n(F))$ defined by the local charts, is constant. The group $\mathrm{Aut}(H^n(F))$ is a discrete group, that allows to introduce on $H^n(\pi)$ a locally flat connection ∇ (by the way of the parallel transport, see for example [19]).

Definition 2.2. The horizontal sections of the bundle $H^n(\pi)$ are sections for which the covariant derivative for the connection ∇ vanishes, that is sections which, locally, are transformed by each Ψ_U into constant sections of the trivial bundle $U \times H^n(F;\mathbb{C})$.

One notices that, if B is a complex analytic manifold, then $H^n(\pi)$ is a holomorphic vector bundle over B and ∇ is a locally flat holomorphic connection.

2.2.1. *Monodromy.* The parallel transport defines, for all $t_0 \in B$, an action of $\pi_1(B,t_0)$ on $H^n(\pi)_{t_0}$. A practical way to determine this action is the following:

Let $\lambda : [0,1] \to B$ a loop at t_0 and let $\alpha \in H^n(F_{t_0};\mathbb{C})$. One considers a subdivision

$$0 = \tau_0 < \tau_1 < \cdots < \tau_q = 1$$

of $[0,1]$ sufficiently fine so that, for all $i = 1,\dots,q-1$, there exists a horizontal section v_i of $H^n(\pi)$ defined in an open subset of B containing $\lambda([\tau_i,\tau_{i+1}])$ and such that:

$$v_0(\lambda(0)) = \alpha \qquad v_{i-1}(\lambda(\tau_i)) = v_i(\lambda(\tau_i)), \quad i = 1,\dots,q-1.$$

The homotopy class of λ in $\pi_1(B,t_0)$ acting on $\alpha \in H^n(F_{t_0};\mathbb{C})$ provides an element

$$v_{q-1}(\lambda(1)) \in H^n(F_{t_0};\mathbb{C}).$$

One has:

$$\begin{array}{ccc} \pi_1(B,t_0) \times H^n(F_{t_0};\mathbb{C}) & \mapsto & H^n(F_{t_0};\mathbb{C}) \\ \lambda \quad , \quad \alpha & \rightsquigarrow & v_{q-1}(\lambda(1)) \end{array}$$

and the result is independent of the performed choices.

This action of the fundamental group on the cohomology of the fibre is called *monodromy* of the fibre bundle $\pi : E \to B$. We can also define it as the holonomy of the bundle $H^n(\pi)$.

2.3. **Application to the Brieskorn-Milnor bundle.** With the notations of section 1, let us denote $\pi = f|_{X'} : X' \to D'$. Then π is the projection of a locally trivial bundle to which the construction of section 2 applies.

One obtains a complex vector bundle $H^n(\pi)$ of rank μ. That is a complex analytic bundle on a non-compact Riemann surface, then, following Grauert [9], an analytically trivial fibre bundle. That implies that $H^n(\pi)$ admits a system of μ holomorphic sections s_j over $D' = D \setminus \{0\}$, linearly independent at each point. In general, they are not horizontal sections. In fact, one can choose them horizontal when the monodromy is identity, and according to A'Campo [1], that implies that the singularity is quadratic and n is odd.

In the case of the Milnor bundle, the connection ∇ defined on $H^n(\pi)$ is called Gauß-Manin connection. We have seen that it defines an action of $\pi_1(D',t_0)$ on $H^n(\pi)_{t_0}$ and that action coincides with the action of $\pi_1(D',t_0) = \mathbb{Z}$ on $H^n(X_{t_0};\mathbb{C}) = \mathbb{C}^\mu$ determined by the Milnor fibration.

In other words, the local solutions of $\nabla(s) = 0$ give a locally constant sheaf of $\mathbb{C}$-vector spaces of dimension μ and the action of $\pi_1(D')$ on a fibre of this sheaf is the monodromy of the singularity.

In order to compute this monodromy, we need to determine the solutions of $\nabla(s) = 0$. That is the reason for which, in section 4.2, we will have to extend ∇ at 0. But, in a first step, we will show that the horizontal sections of $H^n(\pi)$ can be characterized as solutions of a differential equation the monodromy of which coincide with the monodromy of the singularity.

Let U, open subset in D', and $s_1, \dots, s_\mu$ a basis of holomorphic sections of $H^n(\pi)$. Every holomorphic section s of $H^n(\pi)$ over U, can be written as $s = \sum_{j=1}^{\mu} \phi_j s_j$ where the functions $\phi_j : U \to \mathbb{C}$ are holomorphic.

Let us still denote by ∇ the covariant derivative $\nabla_{\frac{\partial}{\partial t}}$ determined by the connection ∇, relatively to the vector field $\frac{\partial}{\partial t}$ of D'. For every $j = 1, \dots, \mu$, then $\nabla(s_j)$ is written

$$\nabla(s_j) = \sum_{k=1}^{\mu} a_{kj} s_k$$

where the a_{kj} are holomorphic functions defined in D'. Then we have

$$\nabla(s) = \sum_j \phi'_j s_j + \sum_j \phi_j \sum_k a_{kj} s_k = \sum_k (\phi'_k + \sum_j a_{kj} \phi_j) s_k.$$

Let us denote $\Phi = (\phi_1, \dots, \phi_\mu)^t$ (column vector) and denote by A the matrix $((a_{kj}))$. One has:

Lemma 2.3. A holomorphic section $s = \sum_{j=1}^{\mu} \phi_j s_j$ of $H^n(\pi)$ over U is a horizontal section if and only if the differential equation

$$\Phi' + A\Phi = 0 \tag{2.4}$$

is satisfied.

The monodromy of the singularity can then be interpreted in the following way:

For initial values given at $t_0 \in D'$, one can define locally solutions of (2.4) which generate the μ-dimensional vector space of solutions of (2.4) over a neighbourhood of t_0. In the same way as before, for every loop $\lambda : [0,1] \to D'$ at t_0, one considers a subdivision $0 = \tau_0 < \tau_1 < \cdots < \tau_q = 1$ of $[0,1]$ sufficiently fine so that, for all $0 \le i \le q-1$, then $\lambda([\tau_i, \tau_{i+1}])$ is contained in an open subset of B, trivialization of $H^n(\pi)$. Then, one can follow, by analytic extension, the μ solutions of (2.4), which are given at t_0, along the loop λ. One obtains in every point of λ a system of μ linearly independent solutions of (2.4). The matrix giving the "new" sections, obtained in that way at the point t_0, in terms of the "old" ones is a monodromy matrix of the singularity.

The monodromy of the solutions of the differential equation (2.4) is then equivalent to the monodromy of the singularity.

Computing the monodromy of the singularity is then equivalent to solving the differential equation (2.4). In order to do that, we need to:
(i) construct a basis of holomorphic sections of $H^n(\pi)$,
(ii) compute the matrix A, given the function f.

That is the aim of the following section.

3. Construction of analytic sections of $H^n(\pi)$

Let us denote by ω a differential form of degree n over X. The restriction of ω to each fibre X_t, for $t \neq 0$, denoted by $\omega|_{X_t}$, has maximum degree and is a closed differential form. We show now that the section $s_\omega : D' \to H^n(\pi)$ defined by

$$s_\omega(t) = [\omega|_{X_t}] \in H^n(X_t; \mathbb{C})$$

is a holomorphic section of $H^n(\pi)$ and we compute $\nabla(s_\omega)$.

The main part of this section comes from [33] and [34].

3.1. **Leray coboundary.** Let X be a complex analytic manifold with (complex) dimension $n+1$ and W a complex analytic submanifold of X with (complex) codimension 1. The long exact sequence in cohomology with compact supports and with coefficients in $\mathbb{C}$ is written:

$$\cdots \longrightarrow H^p_c(X \setminus W) \xrightarrow{i^*} H^p_c(X) \longrightarrow H^p_c(W) \xrightarrow{\delta} H^{p+1}_c(X \setminus W) \longrightarrow \cdots$$

where i^* is induced by the inclusion $X \setminus W \subset X$ and δ is the classical coboundary operator.

By Poincaré duality, applied to $X \setminus W, X$ and W, one obtains the exact sequence:

$$\cdots \longrightarrow H_{q+1}(X \setminus W) \xrightarrow{i_*} H_{q+1}(X) \longrightarrow H_{q-1}(W) \xrightarrow{\partial} H_q(X \setminus W) \longrightarrow \cdots$$

with $p + q + 1 = 2n + 2$. The map ∂, dual of the coboundary δ, is called *Leray boundary.*

Applying the functor $\mathrm{Hom}(\cdot; \mathbb{C})$, one deduces from the second exact sequence, the following long exact sequence:

$$\cdots \longrightarrow H^q(X \setminus W) \xrightarrow{r} H^{q-1}(W) \longrightarrow H^{q+1}(X) \longrightarrow H^{q+1}(X \setminus W) \longrightarrow \cdots .$$

where $H^{q+1}(X) \longrightarrow H^{q+1}(X \setminus W)$ is induced by the inclusion of $X \setminus W$ into X and where the map r, dual of ∂, is called *Leray coboundary.*

3.2. **Residue - Leray-Norguet Theorem.**

Definition 3.1. Let us consider ω a closed holomorphic form in $X \setminus W$, one says that ω admits a *pole of order less or equal to* 1 on W if, for all $x \in W$ and for all holomorphic function g defined in a neighbourhood U_x of x and vanishing on $U_x \cap W$, then $g\,\omega$ admits a holomorphic extension in U_x.

If ω admits a pole of order less or equal to 1 on W and if U is the domain of a system of local coordinates $z_1, \dots, z_{n+1}$ such that $W \cap U$ is defined by the equation $z_1 = 0$, then the coefficients of ω in this coordinate system are holomorphic functions of $z_2, \dots, z_{n+1}$ and meromorphic with a pole of order ≤ 1 in the coordinate z_1.

As ω is closed, one has: $d(z_1\omega) = dz_1 \wedge \omega$ on $U \setminus W$ and, as $z_1\omega$ is holomorphic on U, then $dz_1 \wedge \omega$ is also holomorphic on U. That implies that ω is of the form

$$\omega = \frac{dz_1}{z_1} \wedge \varphi + \eta \tag{3.2}$$

where φ and η are holomorphic on U.

Lemma 3.3 ([33])**.** The restriction of φ to $U \cap W$, denoted by $\varphi|_{U\cap W}$, depends only on ω.

Then there exists a well determined holomorphic form on W, called *residue* of ω and denoted by $\mathrm{res}_W(\omega)$, characterized by the fact to be locally the restriction of a holomorphic form φ which verifies equation (3.2).

Lemma 3.4 ([33])**.** The form $\mathrm{res}_W(\omega)$ is a closed form on W.

We can now state the Leray-Norguet Theorem:

Theorem 3.5 ([21] and [27])**.** Let ω be a closed holomorphic q-form on $X \setminus W$ with a pole of order ≤ 1 on W, then

$$r([\omega]) = 2i\pi\,[\mathrm{res}_W(\omega)]\,. \tag{3.6}$$

Corollary 3.7. Under the same hypothesis as in theorem 3.5, one has, for all $(q-1)$-dimensional cycle ξ on W:

$$\int_{\partial(\xi)} \omega = 2i\pi \int_{\xi} \mathrm{res}_W(\omega). \tag{3.8}$$

3.3. **Return to the Brieskorn bundle.** Under the hypothesis of section 1, let us denote by ω a holomorphic form of degree n on X. For all $t \in D'$, the form $\omega|_{X_t}$ is closed. We show now the following theorem:

Theorem 3.9 (Brieskorn)**.** Let s_ω the section of $H^n(\pi)$ defined by $s_\omega(t) = [\omega|_{X_t}]$, one has
(i) s_ω is a holomorphic section of $H^n(\pi)$,
(ii) if $d\omega = df \wedge \varphi$, then $\nabla(s_\omega) = s_\varphi$.

To show the theorem, one proves a preliminary result: Let us consider a holomorphic form α of degree $n+1$ on X. For all $t \in D'$, the form $\alpha/(f-t)$ is a closed holomorphic form on $X \setminus X_t$. According to Lemma 3.3, the form $\mathrm{res}_t\left(\dfrac{\alpha}{f-t}\right) = \mathrm{res}_{X_t}\left(\dfrac{\alpha}{f-t}\right)$ is a closed holomorphic form of degree n on X_t; moreover, the map

$$t \rightsquigarrow \left[\mathrm{res}_t\left(\frac{\alpha}{f-t}\right)\right] \in H^n(X_t : \mathbb{C})$$

defines a section of the bundle $H^n(\pi)$.

Lemma 3.10. a) The map $t \rightsquigarrow \left[\mathrm{res}_t\left(\frac{\alpha}{f-t}\right)\right]$ defines a holomorphic section of $H^n(\pi)$.
b) One has

$$\nabla\left(\left[\mathrm{res}_t\left(\frac{\alpha}{f-t}\right)\right]\right) = \frac{1}{2i\pi}\, r_t\left[\frac{\alpha}{(f-t)^2}\right]$$

where $r_t : H^{n+1}(X \setminus X_t) \longrightarrow H^n(X_t)$ is the Leray coboundary.

Proof. Let t_0 be a point in D' and U a neighbourhood of t_0 in D' which is a trivialization domain of the bundle $\pi : X' \to D'$. For every homology class $\xi_{t_0} \in H_n(X_{t_0})$, there exists a class $\xi \in H_n(\pi^{-1}(U))$ whose restriction to X_{t_0} is ξ_{t_0}. We denote by $\xi_t \in H_n(X_t)$ the restriction of ξ to X_t, for $t \in U$.

In order to prove a) of the Lemma, we show that the map

$$t \rightsquigarrow \langle \mathrm{res}_t\left(\frac{\alpha}{f-t}\right), \xi_t\rangle$$

is holomorphic. According to (3.6), one has:

$$\left\langle \mathrm{res}_t\left(\frac{\alpha}{f-t}\right), \xi_t\right\rangle = \frac{1}{2i\pi}\left\langle r_t\left[\frac{\alpha}{f-t}\right], \xi_t\right\rangle = \frac{1}{2i\pi}\left\langle \frac{\alpha}{f-t}, \partial_t \xi_t\right\rangle$$

because r_t is the dual of the Leray boundary $\partial_t : H_n(X_t) \to H_{n+1}(X \setminus X_t)$.

Let

$$j_t : H_{n+1}(X \setminus \pi^{-1}(U)) \to H_{n+1}(X \setminus X_t)$$

be the morphism induced by the inclusion $X_t \subset \pi^{-1}(U)$; there exists a class

$$z \in H_{n+1}(X \setminus \pi^{-1}(U))$$

such that for all t, one has $j_t(z) = \partial_t(\xi_t)$. In fact, let us assume that U is a closed disk, centered at t_0, then one has a commutative diagram.

$$\begin{tikzcd}
H_c^n(\pi^{-1}(U)) \arrow[rr, "\delta"] \arrow[dr, "P'", "\sim"'] \arrow[dd, "i_t"'] & & H_c^{n+1}(X \setminus \pi^{-1}(U)) \arrow[r, "P", "\sim"'] \arrow[dd] & H_{n+1}(X \setminus \pi^{-1}(U)) \arrow[dd, "j_t"] \\
& H_n(\pi^{-1}(U)) \arrow[dd] & & \\
H_c^n(X_t) \arrow[rr, "\delta_t"] \arrow[dr, "P'_t", "\sim"'] & & H_c^{n+1}(X \setminus X_t) \arrow[r, "P_t", "\sim"'] & H_{n+1}(X \setminus X_t) \\
& H_n(X_t) \arrow[urr, "\partial_t"'] & &
\end{tikzcd}$$

in which the vertical arrows are induced by the inclusion $X_t \subset \pi^{-1}(U)$ and the morphisms P, P', P_t and P'_t are Poincaré duality isomorphisms. Let us denote $\zeta_t = (P'_t)^{-1}(\xi_t) \in H_c^n(X_t)$ and $\zeta = (P')^{-1}(\xi) \in H_c^n(\pi^{-1}(U))$, then one has $i_t(\zeta) = \zeta|_{X_t} = \zeta_t$. The class $z = P\delta(\zeta)$ satisfies, for all $t \in U$, the equality $j_t(z) = \partial_t(\xi_t)$. One has:

$$\left\langle \operatorname{res}_t\left(\frac{\alpha}{f-t}\right), \xi_t \right\rangle = \frac{1}{2i\pi}\int_z \frac{\alpha}{f-t}, \tag{3.11}$$

that is a holomorphic function in t. In fact, the cycle z on which we take integration is fixed (*i.e.* independent of t) and situated in $X \setminus \pi^{-1}(U)$, out of the singularities of $\dfrac{\alpha}{f-t}$. That proves a).

In order to show b), firstly we observe that if s denotes a holomorphic section of $H^n(\pi)$, then one has

$$\langle \nabla(s)(t), \xi_t \rangle = \frac{d}{dt}\langle s(t), \xi_t \rangle. \tag{3.12}$$

In fact, in U, the section s can be written as $s(t) = \sum \varphi_i(t) s_i(t)$ where the sections s_i are a basis of horizontal sections of $H^n(\pi)$. As the classes ξ_t are restriction of the same class ξ in $H_n(\pi{-}1(U))$, then $\langle s_i(t), \xi_t\rangle$ is constant.

That implies:

$$\langle \nabla(s)(t), \xi_t \rangle = \sum \varphi_i'(t) \left\langle s_i(t), \xi_t \right\rangle = \frac{d}{dt}\left\langle s(t), \xi_t \right\rangle.$$

Using the computations performed in the proof of a), one obtains, for the section s of $H^n(\pi)$ defined by $s(t) = \left[\operatorname{res}_t\left(\frac{\alpha}{f-t}\right)\right]$:

$$\left\langle \nabla\left[\operatorname{res}_t\left(\frac{\alpha}{f-t}\right)\right], \xi_t \right\rangle = \frac{d}{dt}\left\langle \operatorname{res}_t\left(\frac{\alpha}{f-t}\right), \xi_t \right\rangle = \frac{1}{2i\pi}\frac{d}{dt}\int_z \frac{\alpha}{f-t} =$$
$$\frac{1}{2i\pi}\int_z \frac{\alpha}{(f-t)^2} = \frac{1}{2i\pi}\left\langle r_t\left[\frac{\alpha}{(f-t)^2}\right], \xi_t \right\rangle.$$

That proves b) of the Lemma. $\square$

Proof of Theorem 3.9. Let ω be a holomorphic form of degree n on X, the lemma 3.10 can be applied to the form $\alpha = df \wedge \omega$. In particular, the section

$$t \rightsquigarrow \left[\operatorname{res}_t \frac{df \wedge \omega}{f-t}\right]$$

of $H^n(\pi)$ is holomorphic. But, by definition of the residue (formula (3.2)), one has:

$$\left[\operatorname{res}_t \frac{df \wedge \omega}{f-t}\right] = \left[\operatorname{res}_t \frac{d(f-t) \wedge \omega}{f-t}\right] = [\omega|_{X_t}] = s_\omega(t). \tag{3.13}$$

That proves (i) of the theorem.

According to (3.13) and (b) of lemma 3.10, one has:

$$\nabla(s_\omega)(t) = \frac{1}{2i\pi}\, r_t \left[\frac{df \wedge \omega}{(f-t)^2}\right] = \frac{1}{2i\pi}\, r_t \left[\frac{d\omega}{f-t} - d\left(\frac{\omega}{f-t}\right)\right]$$

and, as the class of $d\left(\frac{\omega}{f-t}\right)$ is zero, one has:

$$\nabla(s_\omega)(t) = \frac{1}{2i\pi}\, r_t \left[\frac{d\omega}{f-t)}\right] = \frac{1}{2i\pi}\, r_t \left[\frac{df \wedge \varphi}{f-t}\right] = [\varphi|_{X_t}] = s_\varphi(t),$$

that is (ii) of the theorem. □

4. Brieskorn's results and the Gauss-Manin connection

In this section, one constructs a complex of sheaves, whose cohomology sheaf, restricted to D', is isomorphic to the sheaf of germs of holomorphic sections of $H^n(\pi)$. That allows us to extend the connection ∇ into a differential operator which is singular at the origin.

4.1. **Relative de Rham complex.** Given a manifold Y, we denote by Ω^*_Y the complex of sheaves of germs of holomorphic forms on Y. We know that, if Y is a Stein manifold, then $H^*(Y;\mathbb{C})$ is the cohomology of Ω^*_Y. That applies in particular for all points t in D to the fibre $X_t = f^{-1}(t) \cap B_\varepsilon$ of $f|_X : X \to D$.

To study the monodromy, that is the action of the parallel transport along a loop in D' on a fibre, we construct a complex of differential forms which, when restricted to a fibre X_t, is $\Omega^*_{X_t}$. That will be the relative de Rham complex of $f|_X : X \to D$, denoted by $\Omega^*_{X/D}$ and defined by:

$$\Omega^p_{X/D} = \Omega^p_X/(df \wedge \Omega^{p-1}_X).$$

We verify that $\Omega^*_{X/D}$ is a complex, because one has:

$$d(df \wedge \omega) = -df \wedge d\omega \in df \wedge \Omega^*_X.$$

We want to study the germs, in D, of differential forms defined along the fibres of the function $f|_X : X \to D$. In other words, we want to consider, for every open subset U in D, the sections of the sheaf $\Omega^p_{X/D}$ over $f^{-1}(U)$. They are, by definition, the sections of the sheaf $f_*\Omega^p_{X/D}$ over U.

Now, it is natural to define the relative de Rham cohomology sheaves of $f|_X : X \to D$ by:

$$\mathcal{H}^p(X/D) = H^p(f_*\Omega^*_{X/D}).$$

Theorem 4.1. [Brieskorn [5, Satz 1.5]] The sheaf $\mathcal{H}^n(X/D)$ is an analytic coherent sheaf on D.

We denote by $\mathcal{H}^n$ the sheaf of germs of holomorphic sections of $H^n(\pi)$ and by $\mathcal{O}_{D'}$ the structural sheaf of D', *i.e.* the sheaf of germs of holomorphic sections on D'. Brieskorn shows the following result:

Theorem 4.2. [Brieskorn [5]] The correspondence $\omega \rightsquigarrow s_\omega$ induces an isomorphism of $\mathcal{O}_{D'}$-modules:

$$\Psi : \mathcal{H}^n(X/D)|_{D'} \to \mathcal{H}^n. \tag{4.3}$$

Here, we will verify only that Ψ is well defined. In fact, an element ω of $\mathcal{H}^n(X/D)|_{D'}$ can be represented by a section of $f_*(\Omega^n_{X/D})$ on an open subset U in D', or, that is equivalent to say, a section of $\Omega^n_{X/D}$ on $f^{-1}(U)$.

For every disk U in D', the inverse image $f^{-1}(U)$ is a Stein manifold. As the sheaves Ω^n_X and $\Omega^n_{X/D}$ are coherent (see [5]), the obtained section can be lifted into a section of Ω^n_X on $f^{-1}(U)$, that is a holomorphic differential form of degree n on $f^{-1}(U)$. We still denote it by ω.

By theorem 3.9, one obtains a holomorphic section s_ω of $H^n(\pi)$ whose germ at the point t is $s_\omega(t)$. That defines Ψ.

4.2. **Gauß-Manin connection.** The previous construction provides an extension of the sheaf $\mathcal{H}^n$ into a sheaf $\mathcal{H}^n(X/D)$ which is defined over all of D. The isomorphism of theorem 4.2 allows us to identify the homomorphism $\nabla : \mathcal{H}^n \to \mathcal{H}^n$, defined at section 2, with a $\mathbb{C}$-linear homomorphism:

$$\nabla : \mathcal{H}^n(X/D)|_{D'} \to \mathcal{H}^n(X/D)|_{D'}.$$

According to section 3, the local solutions of $\nabla(s) = 0$ give a locally constant sheaf of $\mathbb{C}$-vector spaces of dimension μ and the action of $\pi_1(D', t_0)$ on the fibre at t_0 of this sheaf is the monodromy of the singularity.

To compute the monodromy, we will extend ∇ into a singular differential operator ∇_f defined on $\mathcal{H}^n(X/D)_0$ and will prove that its monodromy is equivalent to the one of the singularity.

We will admit the following theorem which provides an interpretation of $\mathcal{H}^n(X/D)_0$:

Theorem 4.4 (Brieskorn [5])**.** Let $\Omega^*_{X/D,0}$ considered as a complex of $\mathcal{O}_{D,0}$-modules. One has a canonical isomorphism:

$$\mathcal{H}^n(X/D)_0 \to H^n(\Omega^*_{X/D,0}) \tag{4.5}$$

induced by the restriction $\Omega^*_{X/D} \to \Omega^*_{X/D,0}$.

More precisely, let U be a neighbourhood of 0 in D and ω be a holomorphic form of degree n on $f^{-1}(U)$. That one represents a cycle of $\Gamma(f^{-1}(U), \Omega^n_{X/D})$ that gives a section of $\mathcal{H}^n(X/D)$ over U. The isomorphism of the theorem sends the value of this section at 0 to the class, in $H^n(\Omega^*_{X/D,0})$, of the cycle represented by ω.

The differential operator ∇_f will be defined on

$$E = H^n(\Omega^*_{X/D,0}) = \frac{\{\omega \in \Omega^n_{X,0} : \exists \eta \in \Omega^n_{X,0},\ d\omega = df \wedge \eta\}}{df \wedge \Omega^{n-1}_{X,0} + d\Omega^{n-1}_{X,0}}. \tag{4.6}$$

As ∇_f is a singular operator, it will take values, not in E, but in a $\mathcal{O}_{D,0}$-module F containing E as sub-$\mathcal{O}_{D,0}$-module. That will be

$$F = \Omega^n_{X/D,0}/d\Omega^{n-1}_{X/D,0} = \Omega^n_{X,0}/df \wedge \Omega^{n-1}_{X,0} + d\Omega^{n-1}_{X,0}. \tag{4.7}$$

We can now define ∇_f:

An element $\overline{\omega}$ in E is represented by a holomorphic form ω of degree n defined in a neighbourhood of 0 in X and such that $d\omega = df \wedge \varphi$ where φ is holomorphic in a neighbourhood of 0. We define $\nabla_f : E \to F$ by:

$$\nabla_f(\overline{\omega}) = \overline{\varphi} \tag{4.8}$$

where $\overline{\varphi}$ is the class of φ in F.

One verifies easily that ∇_f is a *differential operator with polar singularity* in the following sense:

i) ∇_f is $\mathbb{C}$-linear,

ii) $\nabla_f(\overline{g(t)\,\omega}) = g'(t)\,\overline{\omega} + g(t)\,\nabla_f(\overline{\omega})$,

iii) there exists a positive integer k such that $t^k\,\nabla_f(E) \subset E$.

In order to verify (iii), let us recall that, for k large enough, f^k belongs to the ideal generated by $(\frac{\partial f}{\partial z_0}, \dots, \frac{\partial f}{\partial z_n})$ in the local ring of $\mathbb{C}^{n+1}$ at origin. Then for every $(n+1)$-holomorphic form α, there is η such that $f^k\alpha = df \wedge \eta$. For all elements $\overline{\varphi}$ in F, represented by a holomorphic n-form φ, one has:

$$\begin{aligned} d(f^k\varphi) &= f^k d\varphi + kf^{k-1}df \wedge \varphi \\ &= df \wedge \eta' \end{aligned}$$

that shows that $t^k\,\overline{\varphi} \in E$, then (iii).

We observe that this shows more, namely:

Lemma 4.9. *F/E is torsion.*

The result of Sebastiani [36] is the following:

Theorem 4.10. *$\mathcal{H}^n(X/D)_0$ is a free $\mathcal{O}_{D,0}$-module.*

We know (theorem 4.1) that $\mathcal{H}^n(X/D)$ is coherent, that implies that $\mathcal{H}^n(X/D)$ is locally free of rank μ at the point 0. Then we can show that the monodromy of ∇_f is equivalent to the monodromy of the singularity of f at the origin. More precisely:

Theorem 4.11. *Let $\overline{\omega}$ an element in E represented by a holomorphic form ω of degree n on X and such that $\nabla_f(\overline{\omega}) = \overline{\varphi}$, where $\overline{\varphi}$ is the class in F of a holomorphic form φ on X, then $\nabla(s_\omega) = s_\varphi$.*

According to the previous observation, if U denotes an open disk centered at 0, one can find holomorphic forms $\omega_1, \dots, \omega_\mu$ defined on $f^{-1}(U)$, such that $d\omega_j = df \wedge \varphi_j$ with φ_j holomorphic in $f^{-1}(U)$ and such that the sections $\widetilde{\omega}_1, \dots, \widetilde{\omega}_\mu$ of $\mathcal{H}^n(X/D)|_U$, induced by $\omega_1, \dots, \omega_\mu$ generate the sheaf.

Each ω_j represents an element $\overline{\omega}_j$ in E and one has:

$$\nabla_f(\overline{\omega}_j) = \overline{\varphi}_j. \tag{4.12}$$

As F/E is torsion, $\overline{\varphi}_j$ can be written:

$$\overline{\varphi}_j = \sum_{k=1}^{\mu} a_{kj}\,\overline{\omega}_k \tag{4.13}$$

where the a_{kj} are germs of meromorphic functions at the origin in D. If U is small enough, one can assume that the a_{kj} are holomorphic in D'. In the same way as above, let us denote by A the matrix of a_{kj}. The system of differential equations associated to ∇_f in the basis $\overline{\omega}_1, \dots, \overline{\omega}_\mu$ of E and determined by (4.12) is written:

$$\Phi' + A\Phi = 0. \tag{4.14}$$

Let V be an open subset in D' contained in U. The system (4.14) is the same as the one associated to ∇ in the basis $s_{\omega_1}, \dots, s_{\omega_\mu}$ of $\mathcal{H}^n|_V$. In fact, according to theorem 3.9, one has:

$$\nabla(s_{\omega_j}) = s_{\varphi_j} \tag{4.15}$$

and, according to (4.13):

$$s_{\varphi_j} = \sum_{k=1}^{\mu} a_{kj}\; s_{\omega_k} \tag{4.16}$$

If $\Phi = (g_1, \ldots, g_\mu)$ is a solution of (4.14) on V, let us denote $s = \sum_{j=1}^{\mu} g_j \, s_{\omega_j}$; then one has $\nabla(s) = 0$ and s is a horizontal section of $H^n(\pi)$ over V (see lemma 2.3).

One deduces that the monodromy of ∇_f is the same as the one of ∇ and, according to what we have seen above, the monodromy of solutions of (4.14) coincides with the monodromy of the singularity.

Let us denote by K the field of fractions of $\mathcal{O}_{D,0}$, *i.e.* the field of germs of meromorphic functions on D at 0. As F/E is torsion, then ∇_f can be extended into a connection, still denoted by ∇, on the K-vector space:

$$\mathcal{E} = E \otimes_{\mathcal{O}} K = F \otimes_{\mathcal{O}} K.$$

In the following section, we show that the connection ∇ is regular.

5. Regularity of the Gauss-Manin connection

5.1. **Recall of the theory of differential equations.** Let us denote, as before, K the field of germs of meromorphic functions on D and ∇ a connection on a K-vector space $\mathcal{E}$. Let us denote by $(e_1, \ldots, e_\mu)$ a basis for $\mathcal{E}$, one defines the $a_{kj} \in K$ by

$$\nabla(e_j) = \sum_{k=1}^{\mu} a_{kj} \, e_k.$$

A computation (already made, see lemma 2.3), shows that the horizontal sections for the connection ∇ are characterized by a differential system. More precisely, if $\Phi = (g_1, \ldots, g_\mu)^t$ are the components of $s \in \mathcal{E}$ in the basis $(e_1, \ldots, e_\mu)$ and if $A = ((a_{kj}))$ is the matrix of the a_{kj}, one obtains the differential system:

$$\Phi' + A\Phi = 0 \tag{5.1}$$

whose solutions are the horizontal sections of ∇.

Definition 5.2. One calls *fundamental matrix* $Y(t)$ of (5.1), every matrix $\mu \times \mu$ whose columns are solutions of (5.1) and such that $\det Y(t) \neq 0$.

One knows, by the general theory ([6, p.111], [13, p.70]), that every linear system of differential equations of the type $\Phi' + A(t)\Phi = 0$ where $A(t)$ is a matrix of analytic functions over $0 < |t| < a$, admits fundamental matrices of the form

$$Y(t) = Z(t) t^R \tag{5.3}$$

where $Z(t)$ is a matrix of analytic functions for $0 < |t| < a$ and R a constant matrix.

Then, one can provide the theorem of the classical theory:

Theorem 5.4. The following conditions are equivalent:
(a) By a change of variables of the type $Y = MZ$, where M is an invertible matrix with meromorphic coefficients, equation (5.1) can be transformed into an equation in which the matrix A admits at most a simple pole at the origin.
(b) There exists a fundamental matrix of (5.1) in which $Z(t)$ admits at most a pole at the origin.
(c) In every angular sector $0 \le \arg t \le \beta$ of the universal covering of D', the horizontal sections of ∇ have low growing, that means that in one (or all) basis of $\mathcal{E}$, the components g_j verify an estimation of the type $|g_j(t)| \le C_{\alpha,\beta} \, t^{-N}$.

Definition 5.5. One says that the connection ∇ is *regular* (or *with regular singular points*) if one of the previous conditions is satisfied.

5.2. **Regularity of the Gauß-Manin connection.** Brieskorn proved in [5] the regularity of the Gauß-Manin connection of an isolated hypersurface singularity, using results of Griffiths. The general theorem can be proved by analytic methods (Nilsson [26], Griffiths [11], Malgrange [24]), or arithmetic ones (Katz [17]), or algebraic ones (Deligne [7]). We will adopt the proof by Malgrange [24].

Theorem 5.6. The Gauß-Manin connection is regular.

Let $p : S \to D'$ the universal covering of D'. Let us consider a family of cycles

$$\gamma(u) \in H_n(X_{p(u)}; \mathbb{C})$$

depending continuously on $u \in S$, *i.e.* if u' is near u, then $\gamma(u')$ is image of $\gamma(u)$ by the canonical isomorphism:

$$H_n(X_{p(u')}; \mathbb{C}) \simeq H_n(X_{p(u)}; \mathbb{C}).$$

By abuse of notation, we will denote $\gamma(t)$ instead of $\gamma(u)$, when $p(u) = t$, providing if necessary the argument of t.

Considering, for $\omega \in \Gamma(X; \Omega^n_X)$, the function on S (multiform function on D') defined by $I(t) = \int_{\gamma(t)} \omega$. In a first step, we show that the integrals $I(t)$ verify a regular differential system if and only if (5.1) is regular, then we will show that these integrals verify (c) of the Theorem 5.4.

It results from Theorem 3.9 and from (3.12) that I is holomorphic and one has:

$$\frac{d}{dt} \int_{\gamma(t)} \omega = \int_{\gamma(t)} \nabla(\overline{\omega}).$$

Taking D smaller if necessary, one can find $\omega_1, \ldots, \omega_\mu$ in $\Gamma(X; \Omega^n_X)$ such that $\overline{\omega}_1, \ldots, \overline{\omega}_\mu$ is a basis of $\mathcal{E} = F \otimes_{\mathcal{O}} K$. In this basis, the matrix of the connection is the matrix $A = ((a_{kj}))$ such that:

$$\nabla(\overline{\omega}_j) = \sum_{k=1}^{\mu} a_{kj} \overline{\omega}_k$$

(see 4.12 and 4.13). The equation associated to the Gauß-Manin connection is the equation (4.14): $\Phi' + A\Phi = 0$.

Let us denote

$$I_j(t) = \int_{\gamma(t)} \overline{\omega}_j, \tag{5.7}$$

one has:

$$\frac{dI_j}{dt} = \int_{\gamma(t)} \nabla(\overline{\omega}_j) = \sum_{k=1}^{\mu} a_{kj} \int_{\gamma(t)} \overline{\omega}_k = \sum_{k=1}^{\mu} a_{kj} I_k.$$

In another words, $I = I_1, \ldots, I_\mu$ is solution of the system

$$I' - A^t I = 0 \tag{5.8}$$

dual of (5.1).

Lemma 5.9. The system (5.1) is regular if and only if (5.8) is regular.

Proof. Let Y a fundamental matrix for (5.1); derivating the equality $Y \cdot Y^{-1} = id$ and replacing $Y'(t)$ by $-A(t)Y(t)$, we show that $(Y^{-1})^t$ is a fundamental matrix for (5.8), in other words one has $((Y^{-1})^t)' - A^t(Y^{-1})^t = 0$. That proves the lemma. ∎

Now to prove regularity of the Gauß-Manin connection, it suffices to prove the following result: "When $t \to 0$, with $\alpha \leq \arg t \leq \beta$, the $I_j(t)$ have slow growing."

In fact, Malgrange proves a more precise result based on the following Lemma:

Lemma 5.10. Let $\omega \in \Gamma(X; \Omega^n_X)$, one has:

$$\lim_{t\to 0,\, \arg t = 0} \int_{\gamma(t)} \omega = 0.$$

Proof. Let us choose a strictly positive real number t_0 and denote $T = f^{-1}([0, t_0]) \cap X$. Then T is a semi-analytic set and is contractible (because T can be contracted in a neighbourhood of X_0 and X_0 is contractible). Following Łojaciewicz [22], one can find a semi-analytic triangulation K of T such that X_0 and X_{t_0} are sub-complexes and such that 0 is a vertex.

Let Γ a cycle in X_{t_0} representing $\gamma(t_0)$; as T is contractible, there is a chain Δ in K such that $\partial\Delta = \Gamma$.

Let us recall the result by Herrera [14]: for every chain with integer coefficients $\Lambda = \sum a_j \sigma_j$ where the σ_j are oriented simplices in K, we define:

$$\int_\Lambda \omega = \sum a_j \int_{\sigma_j} \omega \tag{5.11}$$

where $\int_{\sigma_j} \omega = 0$ if $\deg \omega \neq \dim \sigma_j$ and, if $\deg \omega = \dim \sigma_j$, then $\int_{\sigma_j} \omega = \int_{\mathring{\sigma}_j} \omega = \lim_C \int_C \omega$, where C describes the family of compact subsets situated in the interior $\mathring{\sigma}_j$ of σ_j. Following Herrera [14], the integral (5.11) converges and one has

$$\int_{\partial\Lambda} \omega = \int_\Lambda d\omega.$$

Then, the integral $I(t_0)$ is written:

$$I(t_0) = \int_\Gamma \omega = \int_\Delta d\omega.$$

Let us fix $t \in]0, t_0]$ and consider a subdivision $\widetilde{K}$ of K such that X_t and $f^{-1}(]0, t])$ are sub-complexes of $\widetilde{K}$. Denoting by τ_j the oriented simplices of $\widetilde{K}$, one can consider Δ as a chain $\widetilde{\Delta} = \sum n_j \tau_j$ in $\widetilde{K}$. One can write:

$$\widetilde{\Delta} = \Delta'_t + \Delta''_t$$

where $\Delta'_t = \sum m_j \tau_j$ with $m_j = n_j$ if $\tau_j \subset f^{-1}([0, t])$ and $m_j = 0$ otherwise, and where Δ''_t is a chain in $\widetilde{K}$ whose support is contained in $f^{-1}([t, t_0])$. Moreover, one has:

$$\partial\Delta''_t = \partial\widetilde{\Delta} - \partial\Delta'_t.$$

On the one hand, the cycle $\partial\widetilde{\Delta}$ represents $\gamma(t_0)$ in X_{t_0} (in fact we have $\widetilde{\partial\Delta} = \partial\widetilde{\Delta}$). On the other hand the support of $\partial\Delta'_t$ is a cycle of X_t homologous, in $f^{-1}([t, t_0])$, to $\gamma(t_0)$. Then $\partial\Delta'_t$ represents $\gamma(t)$ in X_t. One has:

$$I(t) = \int_{\partial\Delta'_t} \omega = \int_{\Delta'_t} d\omega.$$

The chain Δ is written $\Delta = \sum a_j \sigma_j$ in the triangulation K. We show now the formula:

$$\int_{\Delta'_t} d\omega = \sum a_j \int_{\sigma_j \cap f^{-1}([0,t])} d\omega \tag{5.12}$$

which makes sense, according to Herrera [14], because $\sigma_j \cap f^{-1}([0,t])$ is a semi-analytic set. To prove the lemma, it suffices to show that

$$\lim_{t\to 0}\int_{\sigma_j\cap f^{-1}([0,t])} d\omega = 0. \tag{5.13}$$

If σ_j is in X_0, that is trivial. In fact, as 0 is a vertex in K, then $\mathring{\sigma}_j \subset X_0 \setminus \{0\}$ and $d\omega|_{\mathring{\sigma}_j} = 0$.

If σ_j is not in X_0, then $\mathring{\sigma}_j \cap X_0$ is the empty set. For every compact subset C in $\mathring{\sigma}_j$, one can find a sufficiently small t so that $\mathring{\sigma}_j \cap f^{-1}([0,t]) \subset \mathring{\sigma}_j \setminus C$. One has:

$$\int_{\sigma_j} d\omega = \int_{\mathring{\sigma}_j} d\omega = \int_{\mathring{\sigma}_j\cap f^{-1}([0,t])} d\omega + \int_{\mathring{\sigma}_j\setminus f^{-1}([0,t])} d\omega$$

where the second member of the sum tends to $\int_{\mathring{\sigma}_j} d\omega$ when t tends to 0. That shows (5.13).

Let us prove (5.12). We write all simplices σ_j in K as a sum $\sum \tau_{jk}$ of simplices in $\widetilde{K}$. More precisely, σ_j can be written as

$$\widetilde{\sigma}_j = \sum_{k\in I}\tau_{jk} + \sum_{k\in J}\tau_{jk}$$

where, for $k \in I$, one has $\tau_{jk} \subset \sigma_j \cap f^{-1}([0,t])$ and, for $k \in J$, then τ_{jk} is not contained in $\sigma_j \cap f^{-1}([0,t])$. With the previous notations, $\widetilde{\Delta}$ can be written as:

$$\widetilde{\Delta} = \Delta'_t + \Delta''_t$$

where $\Delta'_t = \sum_j a_j \sum_{k\in I}\tau_{jk}$. One has:

$$\int_{\Delta'_t} d\omega = \sum_j a_j \sum_{k\in I}\int_{\tau_{jk}} d\omega = \sum_j a_j \int_{\sigma_j\cap f^{-1}([0,t])} d\omega.$$

That ends the proof of the lemma. □

Proof of Theorem 5.6. In order to prove the theorem 5.6, it suffices now to prove the following: "With the hypothesis of Lemma 5.10, for $\alpha \le \arg t \le \beta$, then $I(t)$ remains bounded when t tends to 0."

It is sufficient to prove the result for the integrals of type $I_j(t)$ because $I(t)$ is linear combination of $I_j(t)$ with coefficients in $\mathcal{O}_{D,0}$.

From the equation $\frac{dI_j}{dt} = \sum_k a_{kj} I_k$, one deduces that there exists a constant C and an integer $k > 0$ such that:

$$\left|\frac{dI_j}{dt}\right| \le \frac{C}{k+1}\frac{1}{|t|^{k+1}}\sup(|I_1|,\dots,|I_\mu|).$$

Passing to polar coordinates (in (r,θ)) and integrating in r, one deduces that, when $\arg t$ is bounded one has:

$$|I_j(t)| \le C'\, e^{C|t|^{-k}}.$$

From Lemma 5.10 and from the Phragmen-Lindelöf Theorem [6, p. 162] one obtains the result for $|\beta - \alpha| < \frac{\pi}{k}$. The general case α and β can be deduced immediately. □

5.3. **Development of the integral $I(t)$.** Firstly let us recall the classical results of monodromy theory [18].

Let $t_0 \in D'$, with, for instance $\arg t_0 = 0$. Let us denote by h the endomorphism of $H_n(X_{t_0};\mathbb{C})$ induced by action of the generator of $\pi_1(D', t_0)$ represented by the loop $\lambda \rightsquigarrow e^{2i\pi\lambda}t_0$ with $\lambda \in [0,1]$.

Theorem 5.14. (a) The eigenvalues of h are roots of unity.
(b) If $h = S\cdot U$ with S semi-simple and U unipotent, and $[S,U] = 0$, then one has $(U-I)^{n+1} = 0$.

That implies that, in the Jordan decomposition of the matrix of h, the submatrices corresponding to the eigenvalues of h have at most rank $n+1$.

Let us choose $\gamma_1, \dots, \gamma_\mu$ such that the set $\gamma_1(t_0), \dots, \gamma_\mu(t_0)$ is a basis for $H_n(X_{t_0};\mathbb{C})$ and such that $\int_{\gamma_k(t_0)} \omega_j = \delta_{jk}$. Let us denote

$$I_{jk}(t) = \int_{\gamma_k(t)} \omega_j.$$

The set $I_{1k}, \dots I_{\mu k}$ is a basis of solutions of the equation

$$\frac{dI_j}{dt} = \sum a_{kj} I_k.$$

From theorem 5.6 and from the classical theory of systems of differential equations with regular singular points [13, p. 73], one obtains that the matrix $I = (I_{jk})$ is of the type:

$$I(t) = J(t) t^C = J(t) \cdot e^{C \log(t)}$$

where $J \in GL(\mu, K)$ and $C \in \operatorname{End}(\mathbb{C}^\mu)$.

The action of h on I is translated by the substitution $\log t \rightsquigarrow \log t + 2i\pi$; then, in the basis $\gamma_j(t_0)$, h is expressed by the multiplication by $\exp(2i\pi C)$. Writing C in Jordan form, we obtain the following result:

Proposition 5.15. Let $\omega \in \Gamma(X; \Omega^n_X)$, and let γ defined as above, one has a converging development in D':

$$\int_{\gamma(t)} \omega = \sum_{\alpha, q} C_{\alpha,q}(\omega)\, t^\alpha\, (\log t)^q \tag{5.16}$$

where $\exp(2i\pi\alpha)$ belongs to the set of eigenvalues of h (so that $\alpha \in \mathbb{Q}$) and

$$\alpha > 0 \qquad \text{and} \qquad 0 \le q \le n+1.$$

Moreover, as J is meromorphic, then the set of α has lower bound [13]. One deduces from the lemma 5.10 that one has:

$$C_{\alpha,q}(\omega) \neq 0 \text{ implies } \alpha > 0.$$

On the other hand, let λ be an eigenvalue for h, then, for a certain $p \ge 1$ and for a suitable choice of $\gamma(t_0)$, one has $(h-\lambda)^p \gamma(t_0) = 0$ and $(h-\lambda)^{p-1}\gamma(t_0) \neq 0$.

Lemma 5.17. There are $\eta \in \Omega^n_X$ and $\alpha > 0$ such that $\exp 2i\pi\alpha = \lambda$ and $C_{\alpha,p-1}(\eta) = 0$.

Proof. If that would not be the case , writing $\widetilde{\gamma}(t_0) = (h-\lambda)^{p-1}\gamma(t_0)$, one would have $\int_{\widetilde{\gamma}(t_0)} \eta = 0$, for all $\eta \in \Omega^n_X$. But, as X_{t_0} and X are Stein manifolds, the differential forms $\eta|_{X_{t_0}}$ generate $H^n(X_{t_0};\mathbb{C})$. That would imply $\widetilde{\gamma}(t_0) = 0$, that is contradictory with hypothesis. □

6. Relation between monodromy and Bernstein polynomials

6.1. **Bernstein polynomials.** Let s be an indeterminate and consider the set of finite summations

$$\sum_{k,\ell} a_{k,\ell}(x)\, s^k\, (f(x))^{s-k}$$

where $a_{k,\ell}$ are germs of analytic functions at the origin in $\mathbb{C}^{n+1}$. With obvious relations $f(x)f(x)^{s-k-1} = f(x)^{s-k}$ and also obvious composition laws, that is a $\mathcal{O}_{X,0}$-algebra.

Let us now consider the differential operators $P(x, s, \frac{\partial}{\partial x})$ with analytic coefficients in x and polynomials in s:

$$P(x, s, \frac{\partial}{\partial x}) = \sum b_{k\alpha}(x)\, s^k \left(\frac{\partial}{\partial x}\right)^\alpha.$$

These operators act on the previous ring, writing

$$\frac{\partial}{\partial x_i} f^{s-k} = (s-k)\,\frac{\partial f}{\partial x_i} f^{s-k-1}.$$

Giving to s integer values, the previous operations are compatible with the classical operations on meromorphic functions. We can now provide the theorem proved by I.N. Bernstein [2] when f is a polynomial and extended by J.E. Björk [3] when f is a germ of an analytic function with any singularity at the origin:

Theorem 6.1. There exists a polynomial $B(s) \neq 0$ and a differential operator $P(x, s, \frac{\partial}{\partial x})$ such that:

$$P\left(x, s, \frac{\partial}{\partial x}\right) f^s = B(s) f^{s-1}. \tag{6.2}$$

It is clear that the set of polynomials $B(s)$ such that one has a relation of type (6.2) is an ideal. We will denote by $b(s)$ and will call *Bernstein polynomial* of f the generator of this ideal whose highest degree term is equal to 1.

One has $P(x, 0, \frac{\partial}{\partial x}) = b(0) f^{-1}$, that implies $b(0) = 0$. We will denote

$$b(s) = s\widetilde{b}(s).$$

The Malgrange's result is the following:

Theorem 6.3 (Malgrange [23])**.** Let λ be an eigenvalue of h whose multiplicity in the minimal polynomial of h equals p, then there are rational numbers $\nu_1, \dots, \nu_p \in \mathbb{Q}$ with the following properties:
(a) $\exp(2i\pi\nu_j) = \lambda$ for $j = 1, \dots, p$,
(b) the polynomial $(s+\nu_1)\cdots(s+\nu_p)$ divides $\widetilde{b}$.

We will restrict ourselves to prove the theorem in the case $\lambda \neq 1$. In fact, Malgrange shows that all roots of the Bernstein polynomial can be obtained in the previous way, thus they are rational numbers. In a more precise way, let $\Phi' + A\Phi$ the equivalent form of (5.1) for which tA is holomorphic at 0; then $b(s) = s\widetilde{b}(s)$ where $\widetilde{b}(s)$ is the minimal polynomial of $(tA)(0)$. Many authors extended and generalized these results, let us quote the work of Kashiwara [16] in relation with $\mathcal{D}$-modules.

Example 6.4. Let us consider the polynomial $f = z_1^2 + \cdots z_{n+1}^2$; choosing $P = \sum \frac{\partial^2}{\partial z_i^2}$, one finds $\widetilde{b}(s) = s + \frac{n-1}{2}$. But $H_n(X_1; \mathbb{C})$ has dimension 1 on $\mathbb{C}$ and we have $h = (-1)^{n-1}$.

6.2. **Periods of integrals.** Let α be an $(n+1)$-holomorphic form on X, there is $\omega \in \Omega_X^n$ such that $d\omega = \alpha$. The differential form $\dfrac{\alpha}{f-t}$ is closed and holomorphic in $X - X_t$ and it admits a pole with order 1 along X_t. We denote

$$\frac{\alpha}{df}(t) = \mathrm{res}_t\left(\frac{\alpha}{f-t}\right)$$

and

$$\int_{\gamma(t)} \frac{\alpha}{df} = \int_{\gamma(t)} \frac{\alpha}{df}(t).$$

This integral does not depend on the homology class of $\gamma(t)$ in $H_n(X_t; \mathbb{C})$, moreover one has:

Lemma 6.5.

$$\int_{\gamma(t)} \frac{\alpha}{df} = \frac{d}{dt}\int_{\gamma(t)} \omega. \tag{6.6}$$

Proof. According to Theorem 3.5, one has:

$$\int_{\gamma(t)} \frac{\alpha}{df} = \frac{1}{2i\pi} \int_{\delta\gamma(t)} \frac{\alpha}{f-t}$$

where δ is the Leray boundary. We have:

$$\int_{\gamma(t)} \frac{\alpha}{df} = \frac{1}{2i\pi} \int_{\delta\gamma(t)} \frac{d\omega}{f-t} = \frac{1}{2i\pi} \int_{\zeta} \frac{d\omega}{f-t}$$

where, for t in a small enough open subset U, ζ is a fixed cycle in $H_{n+1}(X \setminus \pi^{-1}(U))$ (see the comments after (3.11)).

From the relation

$$\frac{df \wedge \omega}{(f-t)^2} = -d\left(\frac{\omega}{f-t}\right) + \frac{d\omega}{f-t},$$

one obtains

$$\int_{\zeta} \frac{d\omega}{f-t} = \int_{\zeta} \frac{df \wedge \omega}{(f-t)^2} = \frac{d}{dt} \int_{\zeta} \frac{df \wedge \omega}{f-t}.$$

Using again results of section 3.3, one can write:

$$\frac{1}{2i\pi} \int_{\zeta} \frac{df \wedge \omega}{f-t} = \frac{1}{2i\pi} \int_{\delta\gamma(t)} \frac{df \wedge \omega}{f-t} = \int_{\gamma(t)} \operatorname{res}_t \left(\frac{df \wedge \omega}{f-t}\right) = \int_{\gamma(t)} \omega | X_t = \int_{\gamma(t)} \omega.$$

That proves the Lemma. □

Given the converging development of $\int_{\gamma(t)} \omega$ (see (5.16)), the integral admits a converging development

$$\int_{\gamma(t)} \frac{\alpha}{df} = \sum_{\beta,q} d_{\beta,q}(\alpha)\, t^{\beta}\, (\log t)^q \tag{6.7}$$

where $\beta \in \mathbb{Q}_{>-1}$, and $\exp(2i\pi(\beta+1)) = \lambda$ is an eigenvalue of h whose multiplicity in the minimal polynomial is p, and $p-1 \geq q \geq 0$. Moreover, there exists an $(n+1)$-holomorphic form α and a rational number β with $d_{\beta,q}(\alpha) \neq 0$. In fact, according to Lemma 5.17, if $\eta \in \Omega^n_X$ and if $\alpha = df \wedge \eta$, one has:

$$\int_{\gamma(t)} \eta = \int_{\gamma(t)} \frac{\alpha}{df} \tag{6.8}$$

which is not zero.

6.3. **Proof of Theorem 6.3.**

A) Proof in the case $\lambda \neq 1$. Let λ be an eigenvalue for h, with multiplicity p in the minimal polynomial of h. For $1 \leq k \leq p$, one defines ν_k as the infimum of β such that there exists $q \leq k-1$ and $\alpha \in \Omega^{n+1}_X$ such that $d_{\beta,q}(\alpha) \neq 0$ and $\exp(2i\pi(\beta+1)) = \lambda$.

In order to show (b) of Theorem 6.3, in the case $\lambda \neq 1$, it is sufficient to show that the polynomial $(s+\nu_1)\cdots(s+\nu_p)$ divides $b(s)$. We will proceed in three steps:

1) Let us consider a fixed point $\tau \in [0,1]$ such that $\tau < \eta$. We consider a $\mathcal{C}^\infty$ singular cycle in X_τ which represents $\gamma(\tau)$, in other words, $\gamma(\tau) = \sum n_i s_i$ where the s_i are applications $s_i : \Delta_n \to X_\tau$, with Δ_n standard simplex in $\mathbb{R}^{n+1}$.

Considering a trivialization of the bundle $\pi : X' \to D'$, restricted to $]0,\tau]$, one defines applications $\widetilde{s}_i$ such that the following diagram commutes:

$$\begin{tikzcd}
\Delta_n \times]0,\tau] \arrow[r, "\widetilde{s}_i"] \arrow[dr, "p_2"'] & X \arrow[d, "\pi"] \\
&]0,\tau]
\end{tikzcd}$$

and such that $\widetilde{s}_i|_{\{1\}} = s_i$. Here, p_2 is obviously the second projection.

Let us denote

$$\Gamma(t,\tau) = \sum n_i \widetilde{s}_i|_{\Delta_n \times]0,\tau]}\, f^{s-1}\,\alpha.$$

For every $s \in \mathbb{C}$, one has (choosing t_0 such that $\arg t_0 = 0$):

$$\int_{\Gamma(t_0,\tau)} f^{s-1}\,\alpha = \sum n_i \int_{\widetilde{s}_i|_{\Delta_n \times]0,\tau]}} f^{s-1}\,\alpha$$

We can assume that each $\widetilde{s}_i|_{\Delta_n \times]0,\tau]}$ is contained in an open subset U_i in which $\alpha|_{U_i} = df \wedge \eta_i$. In that case,

$$\int_{\widetilde{s}_i|_{\Delta_n \times]0,\tau]}} f^{s-1}\,\alpha = \int_{t_0}^{\tau} t^{s-1} dt \int_{\widetilde{s}_i|_{\Delta_n \times \{t\}}} \eta_i.$$

In fact, one knows that there exists $\omega \in \Omega_X^n$ such that $d\omega = \alpha$. Then

$$\int_{\widetilde{s}_i|_{\Delta_n \times]0,\tau]}} f^{s-1}\,\alpha = \int_{t_0}^{\tau} t^{s-1} (d\omega)^{\sharp}.$$

where $(d\omega)^{\sharp}$ is the result of integration of $d\omega$ along the fibres of p_2.

By Stokes, one obtains (for $0 < t \leq t_0$):

$$\int_{t}^{t_0} (d\omega)^{\sharp} = \int_{\widetilde{s}_i|_{\Delta_n \times [t,t_0]}} d\omega = \int_{\widetilde{s}_i|_{\Delta_n \times \{t_0\}}} \omega - \int_{\widetilde{s}_i|_{\Delta_n \times \{t\}}} \omega.$$

Then, by derivation and using (6.6) and (6.8), one has:

$$(d\omega)^{\sharp} = \frac{d}{dt} \int_{\widetilde{s}_i|_{\Delta_n \times \{t\}}} \omega = \int_{\widetilde{s}_i|_{\Delta_n \times \{t\}}} \frac{\alpha}{df} = \int_{s_i|_{\Delta_n \times \{t\}}} \eta_i.$$

On the one hand, by construction of the $\widetilde{s}_i$, the cycle $\gamma(t)$ is homologous to $\sum n_i \widetilde{s}_i|_{\Delta_n \times \{t\}}$. On the other hand, $\widetilde{s}_i|_{\Delta_n \times \{t\}}$ is contained in an open subset V_i (contained in U_i) and such that:

$$\eta_i|_{V_i \cap X_t} = \mathrm{res}_t \left(\frac{\alpha}{f-t} \right) \Big|_{V_i}$$

because , in V_i, one has $\dfrac{\alpha}{f-t} = \dfrac{df}{f-t} \wedge \eta_i$.

One obtains that $\sum n_i \displaystyle\int_{s_i|_{\Delta_n \times \{t\}}} = \int_{\gamma(t)} \frac{\alpha}{df}$ and:

$$\int_{\Gamma(t_0,\tau)} f^{s-1}\,\alpha = \int_{t_0}^{\tau} t^{s-1} dt \int_{\gamma(t)} \frac{\alpha}{df}.$$

2) The previous computation allows us to write:

$$b(s) \int_{t_0}^{\tau} t^{s-1} dt \int_{\gamma(t)} \frac{\alpha}{df} = b(s) \int_{\Gamma(t_0,\tau)} f^{s-1}\,\alpha = \int_{\Gamma(t_0,\tau)} [P(x,s,\frac{\partial}{\partial x}) f^s]\alpha.$$

Let us denote by P^* the adjoint operator of P, acting on Ω_X^{n+1}. It is defined, in local coordinates in the following way:

If $P = \sum a_\nu(s,x) D^\nu$ with $D^\nu = \left(\frac{\partial}{\partial z_1}\right)^{\nu_1} \cdots \left(\frac{\partial}{\partial z_{n+1}}\right)^{\nu_{n+1}}$ and if $\alpha = g dz_1 \wedge \cdots \wedge dz_{n+1}$, then $P^*\alpha = \left(\sum (-1)^{|\nu|} D^\nu(a_\nu g)\right) dz_1 \wedge \cdots \wedge dz_{n+1}$. The operator P^* satisfies:

$$[P(x,s,\frac{\partial}{\partial x}) f^s]\alpha = f^s(P^*\alpha) + d(f^s \alpha_p)$$

with $P^*\alpha \in \Omega_X^{n+1}[s]$ and $\alpha_p \in \Omega_X^n[s]$.

By Stokes and by construction of the $\widetilde{s}_i$ (see above) one obtains

$$\int_{\Gamma(t_0,\tau)} [P(x,s,\frac{\partial}{\partial x}) f^s]\alpha = \int_{\Gamma(t_0,\tau)} f^s(P^*\alpha) + \int_{\gamma(\tau)-\gamma(t_0)} f^s \alpha_p.$$

The same argument as in the first step of the proof shows that

$$\int_{\Gamma(t_0,\tau)} f^s(P^*\alpha) = \int_{t_0}^{\tau} t^s dt \int_{\gamma(t)} \frac{P^*\alpha}{df},$$

and then

$$b(s) \int_{t_0}^{\tau} t^{s-1} dt \int_{\gamma(t)} \frac{\alpha}{df} = \int_{t_0}^{\tau} t^s dt \int_{\gamma(t)} \frac{P^*\alpha}{df} + \int_{\gamma(1)} \alpha_p - t_0^s \int_{\gamma(t_0)} \alpha_p.$$

According to Lemma 5.10, for sufficiently large $\mathcal{R}e(s)$, one can consider the limit for t_0 tending to 0 in the previous equality. One obtains:

$$b(s) \int_0^{\tau} t^{s-1} dt \int_{\gamma(t)} \frac{\alpha}{df} = \int_0^{\tau} t^s dt \int_{\gamma(t)} \frac{P^*\alpha}{df} + \int_{\gamma(1)} \alpha_p. \tag{6.9}$$

3) Let us assume that $\nu_1 = \nu_2 = \ldots = \nu_k < \nu_{k+1}$, and let us choose $\alpha \in \Omega_X^{n+1}$ such that $d_{\nu_1,k-1}(\alpha) \neq 0$ (in (6.7)).

Using the development (6.7)) of $\int_{\gamma(t)} \frac{\alpha}{df}$ and the formula $\int_0^1 t^{\nu+s-1} (\log t)^k dt = \frac{d^k}{ds^k}\left(\frac{1}{s+\nu}\right)$, the integral $\int_0^\tau t^{s-1} dt \int_{\gamma(t)} \frac{\alpha}{df}$ can be extended, for $\mathcal{R}e(s) > 1$, into a meromorphic function of $s \in \mathbb{C}$, with a pole of order k at $-\nu_1$.

In the same way, $\int_0^\tau t^s dt \int_{\gamma(t)} \frac{P^*\alpha}{df}$ admits a development of type (6.7) and can be extended into a meromorphic function of $s \in \mathbb{C}$, without pole at $-\nu_1$.

Finally $\int_{\gamma(\tau)} \alpha_p$ is a polynomial in s.

Equality (6.9) implies that $(-\nu_1)$ is a root of order k of $b(s)$. One works in the same way for $\nu_{k+1}, \ldots, \nu_p$, that implies the result. □

B) The case $\lambda = 1$. In the case $\lambda = 1$, the proof is similar but requires more carefulness. The previous method proves only that $(s+\nu_1)\cdots(s+\nu_p)$ divides $b = s\widetilde{b}$ and there is a risk to "lose" some root of $\widetilde{b}$ (see [23]). □

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FOLIATIONS BY CURVES UNIQUELY DETERMINED BY MINIMAL SUBSCHEMES OF ITS SINGULARITIES

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Dedicated to the memory of Professor Egbert Brieskorn

Abstract. It is well-known that a foliation by curves of degree greater than or equal to two, with isolated singularities, in the complex projective space of dimension greater than or equal to two, is uniquely determined by the scheme of its singular points. The main result in this paper is that the set of foliations which are uniquely determined by a subscheme (of the minimal possible degree) of its singular points, contains a nonempty Zariski-open subset. Our results hold in the projective space defined over any algebraically closed ground field.

1. Introduction and statement of the results

Let $\mathbb{P}^n = \mathbb{P}^n_{\mathbf{K}}$ be the projective space of dimension $n \geq 2$ over an algebraically closed ground field $\mathbf{K}$ and let $\mathcal{O}_{\mathbb{P}^n}, \Theta_{\mathbb{P}^n}$ and $\mathcal{H}$ denote its structure, tangent and hyperplane sheaves. For an $\mathcal{O}_{\mathbb{P}^n}$-sheaf $\mathcal{E}$, we will write $\mathcal{E}(d)$ for $\mathcal{E} \otimes \mathcal{H}^{\otimes d}$, if $d \geq 0$ and $\mathcal{E} \otimes (\mathcal{H}^*)^{\otimes |d|}$, if $d < 0$.

Let

$$\mathbf{E} = \mathbf{E}(n, r-1) = \mathrm{H}^0(\mathbb{P}^n, \Theta_{\mathbb{P}^n}(r-1)), \text{ and } \mathbf{e} = \mathbf{e}(n, r-1) = \dim_{\mathbf{K}}\mathbf{E}. \tag{1.1}$$

A foliation by curves with singularities (or simply a *foliation* in the sequel) of degree r on $\mathbb{P}^n$ is the class $[s] \in \mathbb{P}\mathbf{E}$ of a global section $s \in \mathbf{E}$. We denote the scheme of zeroes of $[s]$ by $([s])_0$. We say $[s]$ has isolated singularities if $\dim\, ([s])_0 = 0$ and we say is non-degenerate, if it has isolated singularities and $([s])_0$ is reduced.

It is known that a foliation $[s]$ of degree $r \geq 2$ in $\mathbb{P}^n$ with isolated singularities is uniquely determined by $([s])_0$, in the sense that $([s'])_0 \supseteq ([s])_0$ for some $[s']$ of degree r, implies that $[s'] = [s]$ (that is, $s' = \lambda \cdot s$, for some $\lambda \in \mathbf{K}^*$): For $\mathbf{K} = \mathbb{C}$, the result was first established for $[s]$ non-degenerate, in [10] and the general statement was later obtained in [6]. For an algebraically closed ground field $\mathbf{K}$, it was established for $n = 2$ in [5], and the general version was finally established in [1].

Let $[s]$ be a foliation of degree $r > 2$ in $\mathbb{P}^n$, with isolated singularities. At least if $\mathbf{K} = \mathbb{C}$, there always exist *proper* subschemes $Z \subset ([s])_0$ such that $[s]$ is uniquely determined by Z in the sense that $([s'])_0 \supseteq Z$ for some $[s']$ of degree r, implies that $[s'] = [s]$. This is the content of Proposition 1.1 below. Given $n \geq 2$, and $r \geq 2$, the degree of such subschemes Z is bounded from below by a certain integer $m(n, r-1)$ which we compute in Lemma 1.2 below. The main result of the paper, Theorem 1.3 below, is that the set of foliations $[s]$ (with isolated singularities or not) which are uniquely determined by a $Z \subset ([s])_0$ having this *minimal* degree contains a nonempty Zariski open subset of $\mathbb{P}\mathbf{E}$.

Our main reference is [3]. Our notation comes from there.

Let $U \subset \mathbb{P}^n$ be an open affine that trivializes $\Theta_{\mathbb{P}^n}(r-1)$, and let $p \in U$. The restriction of a section $s \in \mathbf{E}$ to U is an affine vector field $\hat{s} = (s^1, \ldots, s^n)$. The multiplicity $\mu(s, p)$ of s at p, is

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the intersection multiplicity at p of the hypersurfaces $s^j = 0$, i.e., the vector-codimension in the local ring $\mathcal{O}_{\mathbb{P}^n,p}$ of the ideal generated by $\{s^j\}_{j=1}^n$:

$$\mu(s,p) = \dim_{\mathbf{K}} \mathcal{O}_{\mathbb{P}^n,p}/(s^1,...,s^n)\cdot\mathcal{O}_{\mathbb{P}^n,p}. \tag{1.2}$$

It is clear that $\mu(s,p) = \mu(\lambda s,p)$, for every $\lambda \in \mathbf{K}^*$, so that $\mu([s],p) = \mu(s,p)$ is well-defined and, moreover, that p is a singularity of $[s]$ if and only if $\mu(s,p) \neq 0$ and that $\mu(s,p)$ is non-zero and finite if and only if p is an isolated singularity of $[s]$. Moreover, $[s]$ is non-degenerate if and only if $\mu([s],p) = 1$, for every $p \in ([s])_0$.

It follows from the Euler sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(r-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(r)^{\oplus(n+1)} \xrightarrow{\Pi_*} \Theta_{\mathbb{P}^n}(r-1) \longrightarrow 0, \tag{1.3}$$

that $\mathbf{e} = (n+1)\binom{n+r}{n} - \binom{n+r-1}{n}$ and that a foliation $[s]$ with isolated singularities has

$$\deg([s])_0 = c_n(\Theta_{\mathbb{P}^n}(r-1)) = \sum_{j=0}^{n} r^j$$

zeroes, counting multiplicities.

The subsets $\mathcal{U}_{nd} \subset \mathcal{U}_0$ of foliations which are non-degenerate resp. have isolated singularities are both non-empty Zariski-open in $\mathbb{P}\mathbf{E}$.

The sheaf of ideals of a closed subscheme $Z \subset \mathbb{P}^n$ will be denoted by I_Z. For a zero-dimensional subscheme $Y \subset \mathbb{P}^n$, the space of sections $\mathrm{H}^0(\mathbb{P}^n, \Theta_{\mathbb{P}^n} \otimes I_Y(r-1))$ that vanish on Y will be denoted by

$$\mathbf{E}_Y = \mathrm{H}^0(\mathbb{P}^n, \Theta_{\mathbb{P}^n} \otimes I_Y(r-1)), \text{ with } \mathbf{e}_Y = \dim_{\mathbf{K}} \mathbf{E}_Y. \tag{1.4}$$

If Y has degree y and it is reduced, we may consider it as a point in the symmetric product $S^y\mathbb{P}^n$.

Our first result generalizes [7, Corollary 3.3]:

Proposition 1.1. *Let $n \geq 2$ and $r \geq 2$ be integers, let $[s]$ be a foliation with isolated singularities of degree r in the complex projective space $\mathbb{P}^n$, and let $s_1 \in H^0(\mathbb{P}^n, \Theta_{\mathbb{P}^n}(r-1))$ be a section that vanishes at a subscheme $Z' \subset ([s])_0$ whose degree satisfies $\deg Z' \geq \deg([s])_0 - (n(r-1)-r)$. Then $s_1 = \lambda s$ for some $\lambda \in \mathbb{C}^*$ and hence $[s_1] = [s]$.*

Lemma 1.2. *Let $n \geq 2$ and $r \geq 2$, be integers, let $\mathbf{e}$ be given by (1.1). Let $\omega = \omega(n, r-1) = [\frac{\mathbf{e}-1}{n}]$ be the integral part of the number between brackets and let $0 \leq \rho \leq n-1$ be the unique integer such that $\mathbf{e} - 1 = n\cdot\omega + \rho$.*

Let Y be a zero-dimensional closed subscheme of $\mathbb{P}^n$ of degree y, and assume that

$$\mathbf{e}_Y = h^0(\mathbb{P}^n, \Theta_{\mathbb{P}^n} \otimes I_Y(r-1)) = 1.$$

Then $y \geq \omega$, if $\rho = 0$ and $y \geq \omega + 1$, if $1 \leq \rho \leq n-1$. In consequence, the minimal possible degree $m(n, r-1)$ of a zero-dimensional subscheme $Y \subset ([s])_0$ such that $\mathbf{e}_Y = 1$ is

$$m(n,r-1) = \begin{cases} \omega & \text{if } \rho = 0 \text{ , and} \\ \omega+1 & \text{if } 1 \leq \rho \leq n-1. \end{cases}$$

Theorem 1.3. *Let $n \geq 2$ and $r \geq 2$, be integers. Let $\omega = [\frac{\mathbf{e}-1}{n}]$ and $0 \leq \rho \leq n-1$ be as in Lemma 1.2, and $\mathbf{e}_Y$, as in (1.4).*

(a) *If $\rho = 0$, then the subset*

$$\mathbf{B}_\omega = \{[s] \in \mathbb{P}\mathbf{E} \mid \exists Y \in S^\omega\mathbb{P}^n \text{ with } Y \subset ([s])_0 \text{ and } \mathbb{P}\mathbf{E}_Y = \{[s]\}\}$$

contains a nonempty Zariski-open subset V_ω of $\mathbb{P}\mathbf{E}$.

(b) *If $1 \leq \rho \leq n-1$, then the subset*

$$\mathbf{B}_{\omega+1} = \{[s] \in \mathbb{P}\mathbf{E} \mid \exists Y^1 \in S^{\omega+1}\mathbb{P}^n \text{ with } Y^1 \subset ([s])_0 \text{ and } \mathbb{P}\mathbf{E}_{Y^1} = \{[s]\}\}$$

contains a nonempty Zariski-open subset $V_{\omega+1}$ of $\mathbb{P}\mathbf{E}$.

It follows in particular that for $\mathcal{U} = \mathcal{U}_{nd}$ or $\mathcal{U}_0$ and $V = V_\omega$ or $V_{\omega+1}$ (depending on (a) or (b) above, resp.), the subsets $\mathcal{U} \bigcap V$ are nonempty Zariski-open subsets of $\mathbb{P}\mathbf{E}$.

2. The proofs

Proof of Proposition 1.1. Consider $X = \mathbb{P}^n, E = \Theta_{\mathbb{P}^n}(r-1), Z = (s)_0$ and a fixed divisor L of degree ℓ on $\mathbb{P}^n$. It is then clear that $\det E = \mathcal{O}_{\mathbb{P}^n}(nr+1)$ and that F in the complete linear system $|\det E - L|$ has degree $nr + 1 - \ell = r + 1$ if and only if $\ell = (n-1)r$. Hence, the linear system $|K_{\mathbb{P}^n} + L| = \mathbb{P}\mathrm{H}^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(\ell - (n+1))) = \mathbb{P}\mathrm{H}^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(n(r-1) - r - 1)))$ is $(k-1)$−very ample [14, Definition 1.1] if and only if $k = n(r-1) - r$. It follows from [14, Theorem 1.2] that any $F \in |\det E - L| = \mathbb{P}\mathrm{H}^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r+1))$ that passes through a subscheme $Z' \subset (s)_0$ of degree $\deg Z' \geq \deg(s)_0 - (n(r-1) - r)$ necessarily passes through all of $(s)_0$. Now, for an s_1 as in the statement, each of its components satisfy the conditions of the F above, and hence $(s_1)_0 \supseteq (s)_0$. This, together with [6, Theorem 3.5], gives the desired conclusion. ☐

Proof of Lemma 1.2. Let $Y \in S^y\mathbb{P}^n$ be a zero-dimensional closed subscheme of $\mathbb{P}^n$ of degree y, with sheaf of ideals I_Y. We have a short exact sequence of sheaves

$$0 \longrightarrow I_Y \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_Y \longrightarrow 0. \tag{2.1}$$

It follows that $\mathrm{h}^2(\mathbb{P}^n, I_Y(j)) = 0$, for $n = 2$ and $j \geq -2$ from [9, Lemma 2.4] and for $n > 2$ and every j, from Serre's computations.

Now, consider the short exact sequence obtained by tensoring (1.3) with the sheaf I_Y above and its associated long exact cohomology sequence. Using an appropriate twist of (2.1), it follows easily that

$$\begin{aligned} \mathrm{h}^0(\mathbb{P}^n, \Theta_{\mathbb{P}^n} \otimes I_Y(r-1)) - \mathrm{h}^1(\mathbb{P}^n, \Theta_{\mathbb{P}^n} \otimes I_Y(r-1)) \\ = \mathrm{h}^0(\mathbb{P}^n, \Theta_{\mathbb{P}^n}(r-1)) - n \cdot y, \text{ or} \end{aligned} \tag{2.2}$$

$$\mathbf{e}_Y - \mathbf{e}^1_Y = \mathbf{e} - n \cdot y, \text{ where } \mathbf{e}^1_Y = \mathrm{h}^1(\mathbb{P}^n, \Theta_{\mathbb{P}^n} \otimes I_Y(r-1)).$$

For a closed point $p \in \mathbb{P}^n$, the (linear) space $\mathbf{E}_p$ has codimension n in $\mathbb{P}\mathbf{E}$. Hence, the term $\mathbf{e}^1_Y = \mathrm{h}^1(\mathbb{P}^n, \Theta_{\mathbb{P}^n} \otimes I_Y(r-1))$ in (2.2) is equal to the number of dependent conditions imposed by the points of Y in $\mathbf{E}$.

Now assume $\mathbf{e}_Y = 1$.

It then follows from (2.2) that $n \cdot y - \mathbf{e}^1_Y = \mathbf{e} - \mathbf{e}_Y = \mathbf{e} - 1 \leq n \cdot y$, and hence, that $\omega = [\frac{\mathbf{e}-1}{n}] \leq \frac{\mathbf{e}-1}{n} \leq y$, which is the first assertion. In the same vein, it is easy to see that if $\mathbf{e}_Y = 1$ and $y = \omega$, then $\mathbf{e}^1_Y = 0$ if and only if $\rho = 0$.

On the other hand, if $1 \leq \rho \leq n-1$, then $\mathbf{e} - 1 = \omega \cdot n + \rho = \mathbf{e} - \mathbf{e}_Y = y \cdot n - \mathbf{e}^1_Y$, hence $(y - \omega) \cdot n = (\mathbf{e}^1_Y - \rho)$: This equation cannot hold for $y = \omega$ and hence $y - \omega \geq 1$, which is the second assertion. We close by recalling that if $y = \omega + 1$ and $\mathbf{e}_Y = 1$, then $\mathbf{e}^1_Y = n - \rho > 0$, as is easy to see. ☐

Proof of Theorem 1.3 (a). This is a straight-forward consequence of [3, Theorem 0.1]. The proof is included to fix our further notation.

For $y \leq \omega = \omega(n, r-1)$, let $N_{y,k} = \{Y \in S^y\mathbb{P}^n \mid \mathbf{e}_Y = \mathbf{e} - n \cdot y + k\}$. It follows that $N_y = N_{y,0} \subset S^y\mathbb{P}^n$ is open (because it is the subvariety of $Y \in S^y\mathbb{P}^n$ where $\mathbf{e}_Y$ attains its minimum value) and nonempty (because of [3, Theorem 0.1(a)]). It is hence dense in $S^y\mathbb{P}^n$ and $\dim N_y = n \cdot y$.

Let

$$S^y\mathbb{P}^n \xleftarrow{\hat{\Pi}_1} S^y\mathbb{P}^n \times \mathbb{P}\mathbf{E} \xrightarrow{\hat{\Pi}_2} \mathbb{P}\mathbf{E}.$$

be the product variety with canonical projections. Let A be the universal family of foliations of degree $r-1$ from [10], and consider the variety

$$\begin{aligned} Z_y &= \{(Y,[s]) \in S^y\mathbb{P}^n \times \mathbb{P}\mathbf{E} \,|\, A(p,[s]) = [s](p) = 0, \text{ for every } p \in Y\} \\ &= \{(Y,[s]) \in S^y\mathbb{P}^n \times \mathbb{P}\mathbf{E} \,|\, p \in ([s])_0, \text{ for every } p \in Y\} \\ &= \{(Y,[s]) \in S^y\mathbb{P}^n \times \mathbb{P}\mathbf{E} \,|\, Y \subset ([s])_0\} \\ &= \{(Y,[s]) \in S^y\mathbb{P}^n \times \mathbb{P}\mathbf{E} \,|\, [s] \in \mathbb{P}\mathbf{E}_Y\} \subset S^y\mathbb{P}^n \times \mathbb{P}\mathbf{E}, \end{aligned} \tag{2.3}$$

with restrictions $\Pi_1 = \hat{\Pi}_1|_{Z_y} : Z_y \longrightarrow S^y\mathbb{P}^n$ and $\Pi_2 = \hat{\Pi}_2|_{Z_y} : Z_y \longrightarrow \mathbb{P}\mathbf{E}$. Let

$$Z_{y,0} = {\Pi_1}^{-1}(N_y) = \{(Y,[s]) \in Z_y \,|\, Y \in N_y\}.$$

$Z_{y,0}$ is open in Z_y. It is moreover irreducible and has the same dimension

$$n \cdot y + \mathbf{e} - n \cdot y - 1 = \mathbf{e} - 1$$

as $\mathbb{P}\mathbf{E}$ does, because all fibers ${\Pi_1}^{-1}(Y)$ are irreducible and have the same dimension

$$\dim \mathbb{P}\mathbf{E}_Y = \mathbf{e} - n \cdot y - 1$$

(which is equal to zero, if $y = \omega$ and $\rho = 0$). Now consider the restrictions

$$\mathbf{\Pi}_1 = \Pi_1|_{Z_{y,0}} : Z_{y,0} \longrightarrow N_y \subset S^y\mathbb{P}^n$$

and $\mathbf{\Pi}_2 = \Pi_2|_{Z_{y,0}} : Z_{y,0} \longrightarrow \mathbb{P}\mathbf{E}$ and recall that, set-theoretically,

$$\mathbf{\Pi}_2(Z_{y,0}) = \{[s] \in \mathbb{P}\mathbf{E} \,|\, \exists Y \in N_y \text{ such that } Y \subset ([s])_0\}.$$

$\mathbf{\Pi}_2$ is a regular map between irreducible varieties of the same dimension which *we claim to be dominant* (the closure of its image $\overline{\mathbf{\Pi}_2(Z_{y,0})}$ is $\mathbb{P}\mathbf{E}$ or $B = \mathbf{\Pi}_2(Z_{y,0})$ is contained in no hypersurface). Assuming this for a moment, we may finish the proof applying [11, Proposition 6.4.1] which shows the existence of a subset

$$V_y \subset B \tag{2.4}$$

which is open and dense in $\overline{B} = \mathbb{P}\mathbf{E}$ and the desired conclusion follows taking $y = \omega$ in (2.4).

We prove that $\mathbf{\Pi}_2$ is dominant by contradiction: If it were not, then we may assume that $\overline{B} = C$ is an irreducible hypersurface and there exists a nonempty subset $V' \subset B$ open and dense in C such that $\dim {\mathbf{\Pi}_2}^{-1}([s_0]) = \dim Z_{y,0} - \dim C = 1$, for every $[s_0] \in V'$. Since

$${\mathbf{\Pi}_2}^{-1}([s_0]) = \{(Y,[s_0]) \,|\, Y \in N_y \text{ and } Y \subset ([s_0])_0\}, \tag{2.5}$$

the condition $\dim {\mathbf{\Pi}_2}^{-1}([s_0]) = 1$ implies that some $p \in Y$ is a non-isolated singularity of $[s_0]$. Since $V' = C \cap W$ for some non-empty open set $W \subset \mathbb{P}\mathbf{E}$, it follows that ${\mathbf{\Pi}_2}^{-1}(V')$ is a non-empty open subset of the irreducible $Z_{y,0}$, hence it is dense. Consider a $Y \in N_y$ from (2.5). Then ${\mathbf{\Pi}_2}^{-1}(V') \cap (Y \times \mathbb{P}\mathbf{E}_Y) \neq \emptyset$ is therefore open and dense. This implies that

$$\{[s] \in \mathbb{P}\mathbf{E}_Y \,|\, \exists p \in Y \text{ with } \dim_p([s])_0 = 1\}$$

is non-empty and open, therefore is dense in $\mathbb{P}\mathbf{E}_Y$.

On the other hand, for $Y \in N_y$, the subspace $\mathbf{E}_Y$ is the transversal intersection $\bigcap_{p\in Y} \mathbf{E}_p$ of the linear subspaces $\mathbf{E}_p \subset \mathbf{E}$ and $\mathbf{E}_p^1 = \{s \in \mathbf{E} \,|\, \mu([s],p) = 1\}$ is open and dense in $\mathbf{E}_p$ (by [10, Lemma 1.2]), so that

$$\bigcap_{p\in Y} \mathbf{E}_p^1 \subset \mathbf{E}_Y \text{ is also open and dense in } \mathbf{E}_Y,$$

and

$$\mathbb{P}(\bigcap_{p\in Y}\mathbf{E}^1_p) \subset \mathbb{P}\mathbf{E}_Y \text{ is open and dense.} \tag{2.6}$$

These two open sets of $\mathbb{P}\mathbf{E}_Y$ should have non-empty intersection, which is clearly absurd. This finishes the proof of Theorem 1.3 (a). □

To prepare for the proof of Theorem 1.3 (b), we keep the previously introduced notation. We still consider $y = \omega$ and the unique ρ with $1 \le \rho \le n-1$ such that $\mathbf{e} - 1 = n\cdot\omega + \rho$. It follows that $\mathbf{e}_Y = \rho + 1$ if $Y \in N_\omega$. For each such Y, let $\mathbf{s} = \{s_0, \ldots, s_\rho\}$ be a $\mathbf{K}$-basis of $\mathbf{E}_Y$: These sections define a vector-bundle map

$$\phi = \phi_{\mathbf{s}} : \mathbf{T} \longrightarrow \Theta_{\mathbb{P}^n}(r-1)$$

from the trivial vector bundle $\mathbf{T}$ of rank $\rho+1$. In an open affine $U \subset \mathbb{P}^n$ that trivializes both $\mathbf{T}$ and $\Theta_{\mathbb{P}^n}(r-1)$, ϕ is represented by the matrix

$$A = A_{\mathbf{s}} = [\hat{s_0}, \ldots, \hat{s_\rho}] \in M_{n\times(\rho+1)}(\mathcal{O}_{\mathbb{P}^n}(U)) \tag{2.7}$$

whose column $\hat{s}_i = (s_i^1, \ldots, s_i^n) \in \mathcal{O}_{\mathbb{P}^n}(U)^{\oplus n}$ is the restriction of s_i to U, for $i = 0, \ldots, \rho$.

On the other hand, let $\mathbf{M} = M_{n\times(\rho+1)}(\mathbf{K})$ be the affine variety of matrices with n rows and $(\rho+1)$ columns with coefficients in $\mathbf{K}$. It is well-known (see [4]) that the subvariety $\mathbf{M}_\rho$ of matrices $A \in \mathbf{M}$ with $\mathrm{rk}A \le \rho$ is irreducible and has codimension $n-\rho$ in $\mathbf{M}$. This means that the ideal $\mathbf{I}_\rho$ of $\mathbf{M}_\rho$ is generated by some $n-\rho$ (maximal) minors

$$A^J \subset A \text{ of size } |J| = \rho+1, \tag{2.8}$$

and $\mathbf{M}_\rho$ is (arithmetically) Cohen-Macaulay. The matrix $A_{\mathbf{s}}$ corresponds to a morphism $f : U \longrightarrow \mathbf{M};\ x \mapsto A_{\mathbf{s}}(x)$, and $U_{\rho+1} = f^{-1}(\mathbf{M}_\rho)$ is independent of the trivialization chosen. This allows to define the degeneracy locus $D_{\rho+1}(\mathbf{s})$ of the collection of sections $\mathbf{s}$ by

$$U_{\rho+1} = D_{\rho+1}(\mathbf{s})\bigcap U$$

(see [2, II, §4]). It is clear form this construction that

$$D_{\rho+1}(\mathbf{s}) = \{p \in \mathbb{P}^n \,|\, (s_0 \wedge \cdots \wedge s_\rho)(p) = 0\}, \text{ and that } \mathrm{codim} D_{\rho+1}(\mathbf{s}) \le n-\rho. \tag{2.9}$$

Similarly, we have

$$D_\rho(\mathbf{s}) = \{p \in \mathbb{P}^n \,|\, (s_{i_1} \wedge \cdots \wedge s_{i_\rho})(p) = 0, \text{ for every } \{i_1, \ldots, i_\rho\} \subset \{0, \ldots, \rho\}\}.$$

Our interest in these degeneracy loci comes from the following facts:

Lemma 2.1. *Let $n \ge 2$ and $r \ge 2$, and assume that $\mathbf{e} - 1 \equiv \rho$ mod n, with $1 \le \rho \le n-1$. Let $Y \in N_\omega$ and let $\mathbf{s} = \{s_0, \ldots, s_\rho\}$ be a $\mathbf{K}$-basis of $\mathbf{E}_Y$. Then $D_{\rho+1}(\mathbf{s})$ is the locus of singular points $p \in \mathbb{P}^n$ of sections $s = s_\lambda \in \mathbf{E}_Y$, and, moreover, $D_{\rho+1}(\mathbf{s}) \setminus D_\rho(\mathbf{s})$ is the locus of points $p \in \mathbb{P}^n$ such that there exists a unique $[s] \in \mathbb{P}\mathbf{E}_Y$ that vanishes both at Y and at p.*

Proof. Recall that

$$\begin{aligned} U_{\rho+1} =& D_{\rho+1}(\mathbf{s})\bigcap U = \{x \in U \,|\, \mathrm{rk}A_{\mathbf{s}}(x) \le \rho\} \\ =& \{x \in U \,|\, A_{\mathbf{s}}(x)\cdot\vec{\lambda} = \vec{0}, \text{ for some } 0 \ne \vec{\lambda} = (\lambda_0, \ldots, \lambda_\rho)^t \in \mathbf{K}^{\rho+1}\} \\ =& \{x \in U \,|\, \text{ there exists } \vec{\lambda} \ne \vec{0} \in \mathbf{K}^{\rho+1} \text{ such that } \hat{s}_\lambda(x) = (\sum_{i=0}^{\rho} \lambda_i \hat{s}_i)(x) = \vec{0}\} \\ =& \{x \in U \,|\, \hat{s_\lambda}(x) = 0, \text{ for some } s_\lambda \in \mathbf{E}_Y\}, \end{aligned}$$

which proves the first statement.

For the second one, recall that $(D_{\rho+1}(\mathbf{s})\setminus D_\rho(\mathbf{s}))\bigcap U$ is the set of $x\in U$ such that $\mathrm{rk}A_{\mathbf{s}}(x)=\rho$, so that there exists a unique $\vec{0}\neq\vec{\lambda}$ (modulo scalar multiplication) such that $s_\lambda(x)=0$ for $s_\lambda=\sum_{i=0}^{\rho}\lambda_i s_i\in\mathbf{E}_Y$. □

Remark 2.2. Under the conditions of Lemma 2.1:

(1) We have $D_{\rho+1}(\mathbf{s})\setminus D_\rho(\mathbf{s})\neq\emptyset$: Indeed, if $(D_{\rho+1}(\mathbf{s})\setminus D_\rho(\mathbf{s}))\bigcap U=\emptyset$ for every such U, then the sections in $\mathbf{s}$ are linearly dependent in all $\mathbb{P}^n$ and hence, they form no basis of $\mathbf{E}_Y$, which is absurd.

(2) It is easy to see that, for $\tau=\rho$ and $\rho+1$, we have $D_\tau(\mathbf{s})=D_\tau(\mathbf{s}')$, for any other $\mathbf{K}$-basis $\mathbf{s}'$ of $\mathbf{E}_Y$. This allows us to define $D_\tau(Y)$ to be $D_\tau(\mathbf{s})$, for some $\mathbf{K}$-basis $\mathbf{s}$ of $\mathbf{E}_Y$, and

$$C_Y=D_{\rho+1}(Y)\setminus D_\rho(Y).$$

Hence, C_Y is nonempty for every $Y\in N_\omega$ and it follows from Lemma 2.1 that, for every $Y^1=Y\times\{p\}\in N_\omega\times C_Y\subset S^{\omega+1}\mathbb{P}^n$, there exists a unique $[s]\in\mathbb{P}\mathbf{E}_{Y^1}$. It follows moreover that

$$\dim\, C_Y=\,\dim\, D_{\rho+1}(Y),\ \text{for every } Y\in N_\omega.$$

We have the following refinement of [2, II§4 Proposition 4.1]:

Proposition 2.3. *Let $n\geq 2$ and $r\geq 2$ be integers such that*

$$\mathbf{e}-1\equiv\rho\ \mathit{mod}\ n,\quad\text{with}\quad 1\leq\rho\leq n-1$$

and let $Y\in N_\omega$. Then, $D_{\rho+1}(Y)$ has the expected codimension $n-\rho$ and hence it is a complete intersection. In consequence, it is not only (arithmetically) Cohen-Macaulay, but also equidimensional of dimension ρ. In particular, C_Y is equidimensional of dimension ρ.

Proof. Let $\mathbf{s}=\{s_0,\dots,s_\rho\}$ be a $\mathbf{K}$-basis of $\mathbf{E}_Y$. On the one hand, it follows from (2.6) that we may assume that $\mu([s_i],p)=1$, for every $p\in Y$ and every $i=0,\dots,\rho$. On the other hand, consider the matrix $A_{\mathbf{s}}(x)$ from (2.7), with x in some such $U\subset\mathbb{P}^n$. For $J=\{j_1<\cdots<j_{\rho+1}\}$, let

$$A_{\mathbf{s}}(x)^J=\begin{pmatrix} s_0^{j_1}(x) & \cdots & s_\rho^{j_1}(x)\\ \vdots & \ddots & \vdots\\ s_0^{j_{\rho+1}}(x) & \cdots & s_\rho^{j_{\rho+1}}(x)\end{pmatrix}$$

be a (maximal) minor of $A_{\mathbf{s}}(x)$ from (2.8). We have already seen in (2.9) that

$$\mathrm{codim}D_{\rho+1}(Y)\leq n-\rho.$$

If $\rho=n-1$, then it is clear that $D_n(Y)$ is a hypersurface, so we can assume that $\rho<n-1$: Assume that $\mathrm{codim}D_{\rho+1}(Y)$ is strictly smaller than $n-\rho$, say, equal to $n-\rho-1$, then one of these (maximal) minors $A_{\mathbf{s}}(x)^J$ has determinant *identically* equal to zero and hence, at least one of its rows is linearly dependent to the others. This implies that, for every $s_i\in\mathbf{s}$, no $p\in Y\cap U$ is an isolated singularity of s_i, because $(s_{i,p}^1,\dots,s_{i,p}^n)\subset\mathcal{O}_{\mathbb{P}^n,p}$ is not a regular sequence. This contradiction shows that $\mathrm{codim}D_{\rho+1}(Y)=n-\rho$. The Cohen-Macaulay and equidimensionality properties of $D_{\rho+1}(Y)$ follow from [8] (Proposition 18.13 and Corollary 18.14, respectively). The last statement is clear from Remark 2.2. □

Proof of Theorem 1.3 (b). In analogy with (2.3), let

$$Z_{\omega+1}=\{(Y^1,[s])\in S^{\omega+1}\mathbb{P}^n\times\mathbb{P}\mathbf{E}\,|\,Y^1\subset([s])_0\}\subset S^{\omega+1}\mathbb{P}^n\times\mathbb{P}\mathbf{E},$$

with restrictions $\Pi_1 : Z_{\omega+1} \longrightarrow S^{\omega+1}\mathbb{P}^n$ and $\Pi_2 : Z_{\omega+1} \longrightarrow \mathbb{P}\mathbf{E}$. Consider

$$
\begin{aligned}
\widetilde{N}_{\omega+1} &= \{Y^1 = Y \times \{p\} \mid Y \in N_\omega \text{ and } p \in D_{\rho+1}(Y)\} \subset S^{\omega+1}\mathbb{P}^n, \\
\widetilde{N}'_{\omega+1} &= \{Y^1 = Y \times \{p\} \mid Y \in N_\omega \text{ and } p \in D_{\rho}(Y)\} \subset \widetilde{N}_{\omega+1}, \text{ and} \\
M_{\omega+1} &= \{Y^1 = Y \times \{p\} \mid Y \in N_\omega \text{ and } p \in C_Y\} \subset \widetilde{N}_{\omega+1}.
\end{aligned} \tag{2.10}
$$

Let

$$\widetilde{Z}_{\omega+1} = \Pi_1^{-1}(\widetilde{N}_{\omega+1}) \subset Z_{\omega+1}$$

and let $\widetilde{\Pi}_1 : \widetilde{Z}_{\omega+1} \longrightarrow \widetilde{N}_{\omega+1}$ and $\widetilde{\Pi}_2 : \widetilde{Z}_{\omega+1} \longrightarrow \mathbb{P}\mathbf{E}$ be the restrictions to $\widetilde{Z}_{\omega+1}$ of the projections Π_1 and Π_2 above, respectively.

$\widetilde{N}_{\omega+1}$ is a nonempty quasiprojective subvariety, possibly reducible (for $D_{\rho+1}(Y)$ may have components), singular (for $D_{\rho+1}(Y)$ is singular along $D_\rho(Y)$), but equidimensional of dimension equal to $n \cdot \omega + \rho = \mathbf{e} - 1$.

We claim that, set-theoretically:

$$\widetilde{\Pi}_2(\widetilde{Z}_{\omega+1}) = \mathbf{\Pi}_2(Z_{\omega,0}) \supset V_\omega \neq \emptyset,$$

and in consequence, $\widetilde{\Pi}_2$ is dominant (because $\overline{V_\omega} = \mathbb{P}\mathbf{E}$).

It only remains to prove the equality in the claim and this goes as follows: $[s] \in \mathbf{\Pi}_2(Z_{\omega,0})$ if and only if $[s] = \mathbf{\Pi}_2(Y, [s])$, for some $Y \in N_\omega$ and $Y \subset ([s])_0 \subset D_{\rho+1}(Y)$ by Lemma 2.1. In particular, any $q \in ([s])_0 \setminus Y$ lies in $D_{\rho+1}(Y)$ so that $[s] = \widetilde{\Pi}_2(Y \times \{q\}, [s])$, with $Y^1 = Y \times \{q\} \in \widetilde{N}_{\omega+1}$. The reciprocal inclusion is trivial.

We claim moreover that there exist

$$[s] \in V_\omega \text{ such that } [s] = \widetilde{\Pi}_2(Y \times \{p\}, [s]), \text{ with } p \in C_Y \ (Y \in N_\omega). \tag{2.11}$$

Otherwise, for every $[s] \in V_\omega$, $[s] = \widetilde{\Pi}_2(Y \times \{q\}, [s])$, for some $q \in D_\rho(Y)$ and the restriction $\mathrm{r}\widetilde{\Pi}_2$ of $\widetilde{\Pi}_2$ to $\widetilde{\Pi}_1^{-1}(\widetilde{N}'_{\omega+1})$:

$$\widetilde{Z}_{\omega+1} \supset \widetilde{\Pi}_1^{-1}(\widetilde{N}'_{\omega+1}) \xrightarrow{\mathrm{r}\widetilde{\Pi}_2} \mathbb{P}\mathbf{E}$$

is dominant. This is absurd, for $\dim \widetilde{\Pi}_1^{-1}(\widetilde{N}'_{\omega+1}) < \mathbf{e} - 1$, by Remark 2.2, and (2.11) follows.

Finally by the moment, we claim that

$$\mu([s_0], q) = 1, \text{ for every } [s_0] \text{ and } q \in Y \times \{p\} \in M_{\omega+1} \text{ satisfying } (2.11): \tag{2.12}$$

It follows from Lemma 2.1 that $[s_0]$ is the unique foliation that vanishes at every

$$q \in Y^1 = Y \times \{p\},$$

so that $\{[s_0]\} = \mathbb{P}\mathbf{E}_{Y^1} \subset \mathbb{P}\mathbf{E}$ is zero-dimensional. By [10, Lemma 1.2],

$$\{[s] \in \mathbb{P}\mathbf{E}_{Y^1} \mid \mu([s_0], q) > 1, \text{ for some } q \in Y^1\}$$

is a proper closed subset of $\mathbb{P}\mathbf{E}_{Y^1}$, hence is empty, and the conclusion follows.

Now, let

$$\widetilde{Z}^1_{\omega+1} = \widetilde{\Pi}_1^{-1}(M_{\omega+1}) \subset \widetilde{Z}_{\omega+1}$$

(which is non-empty by (2.11)) and let $\widetilde{\mathbf{\Pi}}_{\mathbf{1}} : \widetilde{Z}^1_{\omega+1} \longrightarrow M_{\omega+1}$ and $\widetilde{\mathbf{\Pi}}_{\mathbf{2}} : \widetilde{Z}^1_{\omega+1} \longrightarrow \mathbb{P}\mathbf{E}$ be the restrictions to $\widetilde{Z}^1_{\omega+1}$ of the projections $\widetilde{\Pi}_1$ and $\widetilde{\Pi}_2$.

The quasiprojective variety $M_{\omega+1}$ is non-empty, possibly reducible but equidimensional of dimension equal to $\mathbf{e}-1$, by Proposition 2.3. Moreover, $\widetilde{\mathbf{\Pi}}_{\mathbf{1}}$ is surjective and for every $Y^1 \in M_{\omega+1}$, the fibre $\widetilde{\mathbf{\Pi}}_{\mathbf{1}}^{-1}(Y^1)$ has dimension zero (it is a closed point $(Y^1, [s])$ because $[s]$ is unique, by

Lemma 2.1). This, together with [13, Theorem 1.26], shows that for every irreducible component $M^c_{\omega+1}$ of $M_{\omega+1}$,

$$\widetilde{Z}^{1,c}_{\omega+1} = \widetilde{\mathbf{\Pi}}_{\mathbf{1}}^{-1}(M^c_{\omega+1}) \subset \widetilde{Z}^1_{\omega+1}$$

is irreducible and has dimension $\mathbf{e}-1$. It is hence a component of $\widetilde{Z}^1_{\omega+1}$.

Now consider a component $\widetilde{Z}^{1,c}_{\omega+1}$ containing a $(Y \times \{p\}, [s])$ satisfying (2.11) and let

$$\widetilde{\mathbf{\Pi}}^{\mathbf{c}}_{\mathbf{2}} : \widetilde{Z}^{1,c}_{\omega+1} \longrightarrow \mathbb{P}\mathbf{E} \tag{2.13}$$

be the restriction to $\widetilde{Z}^1_{\omega+1}$ of the map $\widetilde{\mathbf{\Pi}}_{\mathbf{2}}$. As with part (a), the proof of Theorem 1.3 (b) follows if we prove that $\widetilde{\mathbf{\Pi}}^{\mathbf{c}}_{\mathbf{2}}$ is dominant for, in this situation [11, Proposition 6.4.1] gives the existence of the Zariski-open set we are seeking for: $V_{\omega+1} \subset \widetilde{\mathbf{\Pi}}^{\mathbf{c}}_{\mathbf{2}}(\widetilde{Z}^{1,c}_{\omega+1})$, open and dense in $\mathbb{P}\mathbf{E}$, the closure of $\widetilde{\mathbf{\Pi}}^{\mathbf{c}}_{\mathbf{2}}(\widetilde{Z}^{1,c}_{\omega+1})$.

We prove that $\widetilde{\mathbf{\Pi}}^{\mathbf{c}}_{\mathbf{2}}$ is dominant modifying the previous proof that $\mathbf{\Pi}_2 : Z_{y,0} \longrightarrow \mathbb{P}\mathbf{E}$ is dominant, this time letting $B = \widetilde{\mathbf{\Pi}}^{\mathbf{c}}_{\mathbf{2}}(\widetilde{Z}^{1,c}_{\omega+1})$: If it were not, then we may assume that $\overline{B} = C$ is an irreducible hypersurface and there exists a nonempty subset $V' \subset B$ open and dense in C such that $\dim\,(\widetilde{\mathbf{\Pi}}^{\mathbf{c}}_{\mathbf{2}})^{-1}([s_0]) = \dim\,\widetilde{Z}^{1,c}_{\omega+1} - \dim\,C = 1$, for every $[s_0] \in V'$. Since

$$(\widetilde{\mathbf{\Pi}}^{\mathbf{c}}_{\mathbf{2}})^{-1}([s_0]) = \{(Y^1, [s_0]) \mid Y^1 \in M_{\omega+1} \text{ and } Y^1 \subset ([s_0])_0\}, \tag{2.14}$$

the condition $\dim\,(\widetilde{\mathbf{\Pi}}^{\mathbf{c}}_{\mathbf{2}})^{-1}([s_0]) = 1$ implies that some $p \in Y^1$ is a non-isolated singularity of $[s_0]$, for every $[s_0] \in V'$.

But on the other hand, these $[s_0]$ certainly do satisfy (2.11) for some $Y^1 = Y \times \{p\} \in M_{\omega+1}$ and hence, they also must satisfy (2.12). This contradiction shows that (2.13) is dominant and the proof of Theorem 1.3 (b) has been completed.

□

3. Closing remarks

For $n = 2$ and $r \geq 2$, let $M_r = r(r+5)/2$. It is easy to see that

$$m(2, r-1) = M_r - (t-1), \quad \text{either if} \quad r = 2t \text{ or } 2t+1\ .$$

Recall form [7, Theorem 3.5] that for *every* non-degenerate foliation $[s]$ of degree r in $\mathbb{P}^2$, there exists a subscheme $Z \subset ([s])_0$ of degree M_r which determines $[s]$ uniquely (although $\mathbf{K} = \mathbb{C}$ in [7], the attentive reader will notice that the results therein hold for an algebraically closed ground field $\mathbf{K}$).

For small values of r, we have the following values:

r	$m(2, r-1)$	M_r	$c_2(\Theta_{\mathbb{P}^2}(r-1))$
2	7	7	7
3	12	12	13
4	17	18	21
5	24	25	31
6	31	33	43
7	40	42	57

We conclude that for a non-degenerate foliation $[s]$ of degree 2 in $\mathbb{P}^2$ no *proper* subscheme of $([s])_0$ may uniquely determine $[s]$ and that every non-degenerate foliation $[s]$ of degree 3 in $\mathbb{P}^2$ has a minimal subscheme which uniquely determines it.

At this point, we can moreover prove (see [12]) that for any such $Z \subset ([s])_0$, there exists a subscheme of degree $M_r - 1$ of Z which still determines $[s]$ uniquely. The conclusion is that every non-degenerate foliation $[s]$ of degrees 4 and 5 in $\mathbb{P}^2$ has a minimal subscheme which uniquely determines it.

The question wether every non-degenerate foliation $[s]$ of degree $r \geq 6$ in $\mathbb{P}^2$ has a minimal subscheme which uniquely determines it remains open.

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ON 1-FORMS ON ISOLATED COMPLETE INTERSECTION CURVE SINGULARITIES

ALEXANDRU DIMCA[1] AND GERT-MARTIN GREUEL

To the memory of Egbert Brieskorn

ABSTRACT. We collect some classical results about holomorphic 1-forms of a reduced complex curve singularity. They are used to study the pull-back of holomorphic 1-forms on an isolated complete intersection curve singularity under the normalization morphism. We wonder whether the Milnor number μ and the Tjurina number τ of any isolated plane curve singularity satisfy the inequality $3\mu < 4\tau$.

1. INTRODUCTION

Consider a reduced complex curve singularity $(X,0) \subset (\mathbb{C}^N,0)$, defined by an ideal $I \subset \mathcal{O}_{\mathbb{C}^N,0}$, with $r = r(X,0)$ branches. Let $\nu : (\bar{X},\bar{0}) \to X,0)$ be the normalization, where $(\bar{X},\bar{0})$ is the multi-germ consisting of r smooth branches. We set

$$\begin{array}{lcl}
\mathcal{O} & := & \mathcal{O}_{X,0} = \mathcal{O}_{\mathbb{C}^N,0}/I, \text{ the local ring of the germ } (X,0);\\
\bar{\mathcal{O}} & := & \nu_*\mathcal{O}_{\bar{X},\bar{0}}, \text{ the direct image of the local ring of the multi-germ } (\bar{X},\bar{0});\\
\Omega & := & \Omega^1_{\mathbb{C}^N,0}/I\Omega^1_{\mathbb{C}^N,0} + \mathcal{O}_{\mathbb{C}^N,0}dI, \text{ the holomorphic 1-forms on } (X,0);\\
\bar{\Omega} & := & \nu_*\Omega^1_{\bar{X},\bar{0}}, \text{ the direct image of the holomorphic 1-forms on } (\bar{X},\bar{0});\\
\omega & := & \omega_{X,0} = Ext^{N-1}_{\mathcal{O}_{\mathbb{C}^N,0}}(\mathcal{O},\Omega^N_{\mathbb{C}^N,0}), \text{ the dualizing module of } (X,0);\\
T\Omega & := & H^0_{\{0\}}(\Omega), \text{ the torsion submodule of the } \mathcal{O}\text{-module } \Omega.
\end{array}$$

Let $d : \mathcal{O} \to \Omega$ be the exterior derivation. We have the following maps

$$d\mathcal{O} \to \Omega \to \bar{\Omega} \to \omega,$$

where $d\mathcal{O} \to \Omega$ is the inclusion, $\Omega \to \bar{\Omega}$ is given by the pull-back of forms under the morphism ν, and $\bar{\Omega} \to \omega$ is the inclusion, if we identify the dualizing module ω with the module of Rosenlicht's regular differential forms as explained in [BG80]. Then the maps $d\mathcal{O} \to \Omega$ and $\bar{\Omega} \to \omega$ are clearly injective and $T\Omega$ is the kernel of the map $\Omega \to \bar{\Omega}$ (cf. [BG80]). We write $\bar{\Omega}/\Omega$ for the cokernel of the map $\Omega \to \bar{\Omega}$ and similarly for the other maps. These objects give rise to the following numerical invariants:

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$$
\begin{array}{lcl}
m & := & mt(X,0),\ \text{the multiplicity of } (X,0);\\
\delta & := & \delta(X,0) = \dim_{\mathbb{C}}(\bar{\mathcal{O}}/\mathcal{O}),\ \text{the delta-invariant of } (X,0);\\
\mu & := & \mu(X,0) = \dim_{\mathbb{C}}(\omega/d\mathcal{O}),\ \text{the Milnor number of } (X,0);\\
\lambda & := & \lambda(X,0) = \dim_{\mathbb{C}}(\omega/\Omega);\\
\tau' & := & \tau'(X,0) = \dim_{\mathbb{C}}(T\Omega);\\
\tau & := & \tau(X,0) = \dim_{\mathbb{C}}(T^1_{X,0}),\ \text{the Tjurina number of } (X,0).
\end{array}
$$

Here $T^1_{X,0}$ is the tangent space of the base space of the semiuniversal deformation of $(X,0)$. If $(X,0)$ is a plane curve singularity with $I = \langle f \rangle$, then $\mu = \dim_{\mathbb{C}}(\mathcal{O}/J_f)$ (the classical Milnor number, cf. [BG80, M68]) and $\tau = \dim_{\mathbb{C}}(\mathcal{O}/\langle f \rangle + J_f)$, where J_f is the Jacobian ideal generated by the partials of f.

The aim of this note is to prove the following.

Theorem 1.1. *Let $(X,0)$ be a reduced complete intersection curve singularity. Then the following hold.*

(1) $\tau = \tau' = \lambda \geq \delta + m - r$,
(2) $\tau - \delta = \dim_{\mathbb{C}}(\bar{\Omega}/\Omega)$. *In particular, one has the equality*

$$\dim_{\mathbb{C}}(\bar{\Omega}/\Omega) = \delta - r + 1$$

if and only if the singularity $(X,0)$ is weighted homogeneous.
(3) $\tau > \mu/2$ *if $(X,0)$ is not smooth.*

In the second section we recall a number of classical results on isolated complete intersection singularities (due to the second author with several co-authors, and written partly in German), which are somewhat scattered in the literature and apparently not well known. We collect them here with reference to the original sources. In the third section we give a quick proof of Theorem 1.1 using the results quoted before and discuss its relations with similar results by Delphine Pol, see Remark 3.1. In the final section we discuss the possible values of the quotient $\rho(X,0) = \mu(X,0)/\tau(X,0)$ and ask whether $\rho(X,0) < 4/3$ for any plane curve singularity.

We would like to thank Mathias Schulze for a useful remark, see Remark 3.1.

2. The classical results

We start by recalling the following general result.

Theorem 2.1. *([BG80])*
For a reduced curve singularity the following holds.

(1) $\mu = 2\delta - r + 1$,
(2) $\mu \geq \lambda \geq \delta + m - r$,
(3) $\dim_{\mathbb{C}}(\Omega/d\mathcal{O}) = \mu + \tau' - \lambda$,
(4) $\dim_{\mathbb{C}}(\omega/\bar{\Omega}) = \delta$,
(5) *If $(X,0)$ is smoothable (e.g. if it is a complete intersection) then $\tau' \geq \lambda$, with equality iff $\mu = \dim_{\mathbb{C}}(\Omega/d\mathcal{O})$.*

Proof. All these claims are proved in [BG80]. Indeed, (1) is Proposition 1.2.1, (2) is Lemma 6.1.2, (3) appears in the proof of Theorem 6.1.3, (4) in the proof of Proposition 1.2.1, while (5) is Corollary 6.1.4 together with Corollary 6.1.6 of [BG80]. □

When $(X,0)$ is a complete intersection, we have the following additional properties. Some of these results are also reproduced in Looijenga's book [L84], see in particular Section 8.C. In the case of plane curves, the reader can also consult the introductory book [W04], in particular Section 11.6.

Theorem 2.2. *([Gr75] [Gr80]), [GMP85])*
Let $(X, 0)$ be a reduced complete intersection curve singularity. Then

(1) $\mu = \dim_{\mathbb{C}}(\Omega/d\mathcal{O})$, $d\mathcal{O} \cap T\Omega = 0$,
(2) $\tau = \tau' \leq \mu$,
(3) $\tau = \mu$ *iff* $(X, 0)$ *is quasihomogeneous.*

Proof. Indeed, (1) is Proposition 5.1, resp. Lemma 4.5 in [Gr75] (for arbitrary positive dimensional isolated complete intersection singularities, resp. for complete intersections with arbitrary singularities, suitably modified), (2) is Satz 3.1(2a) in [Gr80]. The claim (3) is Corollary 2.2 in [GMP85] (where also a generalization to Gorenstein curves is proved), while the plane curve case goes back to K. Saito [KS71]. □

3. The proof of Theorem 1.1

The sequence

$$0 \to T\Omega \to \Omega/d\mathcal{O} \to \omega/d\mathcal{O} \to \omega/\Omega \to 0$$

is exact by Theorem 2.2 (1) with $\dim_{\mathbb{C}}(\Omega/d\mathcal{O}) = \mu = \dim_{\mathbb{C}}(\omega/d\mathcal{O})$. Hence

$$\tau' = \dim_{\mathbb{C}}(T\Omega) = \dim_{\mathbb{C}}(\omega/\Omega) = \lambda.$$

Claim (1) follows now from Theorem 2.2 (2) and Theorem 2.1 (2). The claim (2) is a consequence of the exact sequence

$$0 \to \bar{\Omega}/\Omega \to \omega/\Omega \to \omega/\bar{\Omega} \to 0$$

together with (1), Theorem 2.1 (4) and the definition of λ. Using (1) and Theorem 2.1 (1) we get

$$\tau \geq \delta + m - r = (\mu + r - 1)/2 + m - r = \mu/2 + (m-1)/2 + (m-r)/2 > \mu/2,$$

since $m \geq r$ and $m > 1$ if $(X, 0)$ is not smooth.

Remark 3.1. It was drawn to our attention by Mathias Schulze that an alternative proof of the equality in Theorem 1.1 (2) can be obtained from [Pol14, Proposition 3.31]. Assume that $(X, 0)$ is irreducible for simplicity. Let $f_1 = \cdots = f_n = 0$ be the equations for the germ $(X, 0)$ in $(\mathbb{C}^N, 0)$, with $N = n + 1$ and $f_i \in \mathcal{O}_{\mathbb{C}^N,0}$, for $i = 1, ..., n$. Then τ' is the codimension of the Jacobian ideal J_X in $\mathcal{O}$, where J_X is the ideal of $\mathcal{O}$, spanned by all the $n \times n$-minors in the Jacobian matrix $(\partial f_i/\partial x_j)_{i=1,n; j=0,n}$, see [Gr75, Proposition 1.11(iii)]. Delphine Pol shows that one has the following equality

$$J_X = g \cdot \frac{\nu^*(\Omega)}{dt},$$

in the local ring $\bar{\mathcal{O}} = \mathbb{C}\{t\}$, where g is a generator of the conductor ideal C_X. Note that the codimension of J_X in $\bar{\mathcal{O}}$ is clearly by the above discussion $\tau + \delta$. Since g, regarded as an element of $\bar{\mathcal{O}} = \mathbb{C}\{t\}$, has order μ, it follows that the codimension of $g \cdot \frac{\nu^*(\Omega^1_{X,0})}{dt}$ in $\bar{\mathcal{O}}$ is given by

$$\mu + \dim_{\mathbb{C}}(\bar{\Omega}/\Omega).$$

The claim follows from these formulas. Note that both the proofs given, and the literature used, by Delphine Pol are quite different from ours.

Remark 3.2. (1) The equality $\tau = \tau'$ holds more generally if $(X, 0)$ is Gorenstein, which follows from local duality. For an arbitrary reduced curve singularity $(X, 0)$ the relation between τ and τ' is unclear. It is an old and still open question if for a non smooth $(X, 0)$ we have always $\tau > 0$ (i.e. $(X, 0)$ is not rigid) and $\tau' > 0$ (Berger's conjecture).

(2) For a plane curve singularity $(X, 0)$ the expression $\tau - \delta$ appears also as the codimension of the extended tangent space to the orbit of the parametrization $(\bar{X}, \bar{0}) \to (\mathbb{C}^2, 0)$ of $(X, 0)$ by the action of the right-left group $\mathcal{A}$ of Mather ([GLS07, Proposition II.2.30(5)]).

4. A remark on the quotient μ/τ

Assume in this section that we are in the case of plane curve singularities, and we write $f_1 = f$ to simplify our notation. Let $M(f) = \mathcal{O}_{\mathbb{C}^2,0}/J_f$ be the Milnor algebra of the singularity $(X, 0)$, where J_f denotes the Jacobian ideal of f. Let $\langle f \rangle$ denote the principal ideal spanned by f in $M(f)$ and $\ker m_f$ denote the kernel of the morphism $m_f : M(f) \to M(f)$ given by the multiplication by f. Then we know that $\langle f \rangle \subset \ker m_f$, see [BrS74]. Moreover, one has $\dim_{\mathbb{C}}(\langle f \rangle) = \mu - \tau$ and $\dim_{\mathbb{C}}(\ker m_f) = \tau$. Using this approach, Yongqiang Liu has shown in [Li17] that

$$\tau \geq \frac{1}{2}\mu.$$

He asked there which values can take the quotient

$$\rho := \rho(X, 0) = \mu(X, 0)/\tau(X, 0).$$

The obvious inequality $\tau \leq \mu$ and Theorem 1.1 (3) show that

$$1 \leq \rho(X, 0) < 2$$

when $(X, 0)$ is non smooth. It also shows that the inclusion of ideals $\langle f \rangle \subset \ker m_f$ is strict when $(X, 0)$ is not a smooth germ.

To construct singularities $(X, 0)$ with a large quotient $\rho(X, 0)$ is not easy, since the Tjurina number $\tau(X, 0)$ is difficult to compute in general, e.g. since it is not a topological invariant it cannot be expressed in terms of Puiseux pairs.

Example 4.1. There is a sequence of isolated plane curve singularities $(X_m, 0)$ such that the sequence of rational numbers $\rho(X_m, 0)$ is strictly increasing with limit $4/3$. Moreover, the singularities can be chosen to be all either irreducible, or consisting of smooth branches with distinct tangents.

In the irreducible case, consider the sequence of singularities

$$(X_m, 0) : f = x^{2m+1} + x^m y^{m+1} + y^{2m} = 0.$$

Then the associated projective plane curve of degree $d = 2m + 1$

$$C : x^{2m+1} + x^m y^{m+1} + y^{2m} z = 0$$

is free with exponents $(d_1, d_2) = (m, m)$, see [DSt17, Theorem 1.1]. This implies that

$$\tau = \tau(X_m, 0) = \tau(C) = (d - 1)^2 - d_1 d_2 = 3m^2,$$

see [DSt17, Equation (2.2)]. Since clearly $(X_m, 0)$ is a semi-weighted homogeneous singularity, it follows that $\mu = \mu(X_m, 0) = 2m(2m - 1)$, and hence the claim follows in this case.

In the case of singularities consisting of smooth branches with distinct tangents, consider the sequence

$$(X_m, 0) : f = x^{2m+1} + y^{2m+1} + x^{m+1} y^{m+1}.$$

Again $(X_m, 0)$ is a semi-weighted homogeneous singularity, and from that we get

$$\mu = \mu(X_m, 0) = 4m^2.$$

To determine the Tjurina number, note that the monomials $x^a y^b$ for $0 \leq a, b \leq 2m-1$ form a basis for the Milnor algebra $M(f)$. The Euler formula implies that the monomial $x^{m+1}y^{m+1}$ belongs to the ideal $(f) \subset M(f)$. To get a basis for the Tjurina algebra $T(f) = M(f)/(f)$ of f, we have to discard from the above basis all the multiples of $x^{m+1}y^{m+1}$, namely $(m-1)^2$ elements. It follows that $\tau = \tau(X_m, 0) = 4m^2 - (m-1)^2$, which yields the claim in this case as well.

Question 4.2. Is it true that

$$\rho(X,0) = \mu(X,0)/\tau(X,0) < \frac{4}{3}$$

for any isolated plane curve singularity?

The answer to this question is positive for semi-quasi-homogeneous singularities $(X, 0)$; see the recent preprint [AB18].

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μ-CONSTANT MONODROMY GROUPS AND TORELLI RESULTS FOR THE QUADRANGLE SINGULARITIES AND THE BIMODAL SERIES

FALKO GAUSS AND CLAUS HERTLING

To the memory of Egbert Brieskorn

Abstract. This paper is a sequel to [He11] and [GH17]. In [He11] a notion of marking of isolated hypersurface singularities was defined, and a moduli space M_μ^{mar} for marked singularities in one μ-homotopy class of isolated hypersurface singularities was established. It is an analogue of a Teichmüller space. It comes together with a μ-constant monodromy group $G^{mar} \subset G_{\mathbb{Z}}$. Here $G_{\mathbb{Z}}$ is the group of automorphisms of a Milnor lattice which respect the Seifert form.

It was conjectured that M_μ^{mar} is connected. This is equivalent to $G^{mar} = G_{\mathbb{Z}}$. Also Torelli type conjectures were formulated. In [He11] and [GH17] $M_\mu^{mar}, G_{\mathbb{Z}}$ and G^{mar} were determined and all conjectures were proved for the simple, the unimodal and the exceptional bimodal singularities. In this paper the quadrangle singularities and the bimodal series are treated. The Torelli type conjectures are true. But the conjecture that $G^{mar} = G_{\mathbb{Z}}$ and M_μ^{mar} is connected does not hold for certain subseries of the bimodal series.

Contents

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1. Introduction

We dedicate this paper to the memory of Egbert Brieskorn. It has its roots in work which the second author, Claus Hertling, had done as a student of Brieskorn in Bonn in the early 90's.

1.1. **Reminiscences of the second author.** Prof. Dr. Egbert Brieskorn accepted me as a diploma student in the spring of 1989. On March 10 and 13, 1989, he gave two full days (Friday + Monday) of lectures for his new diploma students (including me) and doctoral students. I still have his handwritten manuscript of 52 pages. There he introduced us to isolated hypersurface singularities. He talked about the Jacobi algebra, the universal unfolding with its discriminant, the Milnor fibration, its monodromy, local systems and integrable connections and systems of regular singular linear differential equations in general, his own work on the Gauss-Manin connection and especially the Brieskorn lattice, and the mixed Hodge structure which it induces. He strongly recommended to read [AGV88], [SaM89] and [SS85]. He proposed to me to work on the moduli of singularities using the Gauss-Manin connection.

I followed his advice in my diploma thesis and my doctoral thesis and beyond the doctoral thesis. The subject developed into a long-going project of mine, which I took up again and again. The present paper is in some sense a final step of it.

In the doctoral thesis [He93], I formulated the global Torelli type conjecture that an isolated hypersurface singularity is determined up to right equivalence by its Brieskorn lattice together with the Milnor lattice and the Seifert form (conjecture 1.1 (b) reformulates this conjecture). I proved it in the doctoral thesis for all unimodal singularities, the exceptional bimodal singularities, the bimodal quadrangle singularities, and the bimodal series $E_{3,p}$.

For the other seven bimodal series, I made in the spring 1993, some months after finishing the doctoral thesis, long calculations (120 pages) which led to a proof of this Torelli type conjecture for all series except the three bimodal subseries $S^\sharp_{1,10r}, S_{1,10r}, Z_{1,14r}$. At that time I thought that I would never review and publish these results. The paper [He95] recapitulated the main results of the doctoral thesis and of these calculations for the eight bimodal series, but it did not at all give all details (only 2.5 pages are devoted to the bimodal series).

Later I constructed a classifying space D_{BL} for Brieskorn lattices [He99] and a moduli space $M_\mu(f_0)$ of the right equivalence classes of all singularities in the μ-homotopy class of a reference singularity f_0 [He02]. More recently, in [He11], I defined the notion of a *marked singularity*, I constructed a classifying space $M_\mu^{mar}(f_0)$ for marked singularities, and I formulated a Torelli type conjecture for marked singualarities, which is stronger than the Torelli type conjecture in the doctoral thesis for unmarked singularities.

The three papers [He11], [GH17] and the present paper prove the Torelli conjecture for marked singularities for all singularities with modality 0, 1 and 2. The present paper deals with the bimodal quadrangle singularities and the eight bimodal series. It comprises the calculations from the spring 1993 and adds a lot more arguments and calculations, which are necessary for the marked version.

It is satisfying, that the Torelli type conjectures hold for all singularities with modality 0, 1 and 2. For each family, the interplay between the variations of the Brieskorn lattices and the automorphism group of the Milnor lattice with Seifert form is fascinating and takes the best possible shape. I believe that Brieskorn would have liked these positive results and the many techniques used for their proofs. I thank him for proposing to me in March 1989 to work on the moduli of singularities using the Gauss-Manin connection. It was a good advice.

1.2. **Notions, conjectures and results.** In this paper, a *singularity* is a holomorphic function germ $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ with an isolated singularity at 0. Then its *Milnor lattice* $Ml(f) \cong \mathbb{Z}^\mu$ is the $\mathbb{Z}$-lattice $H_n(f^{-1}(\tau), \mathbb{Z})$ for some small $\tau \in \mathbb{R}_{>0}$ for a suitable representative of f. Its *Seifert* *form* is called $L : Ml(f) \times Ml(f) \to \mathbb{Z}$. Its *monodromy* is called $M_h : Ml(f) \to Ml(f)$. The automorphism group of the Milnor lattice with the Seifert form is $G_{\mathbb{Z}}(f) := \mathrm{Aut}(Ml(f), L)$. It will play a predominant role in this paper.

This paper is a sequel to [He11] and [GH17]. In [He11], a *strongly marked singularity* (f, ρ) and a *marked singularity* $(f, \pm\rho)$ are defined. Here one has to fix first a reference singularity f_0. Then f is in the μ-homotopy class of f_0, i.e. a μ-constant family of singularities exists which contains f_0 and f. And $\rho : (Ml(f), L(f)) \to (Ml(f_0), L(f_0))$ is a chosen isomorphism. Two singularities f_1 and f_2 are right equivalent if a coordinate change φ with $f_1 = f_2 \circ \varphi$ exists. Two strongly marked singularities (f_1, ρ_1) and (f_2, ρ_2) are right equivalent if a coordinate change φ with $f_1 = f_2 \circ \varphi$ and $\rho_1 = \rho_2 \circ (\varphi)_{hom}$ exists, where $(\varphi)_{hom} : Ml(f_1) \to Ml(f_2)$ is the induced isomorphism.

In [He02] a moduli space $M_\mu(f_0)$ for the right equivalence classes of all singularities in the μ-homotopy class of a reference singularity f_0 was constructed as an analytic geometric quotient. In [He11], this construction was enhanced to the construction of moduli spaces $M_\mu^{mar}(f_0)$ and $M_\mu^{smar}(f_0)$ of marked and strongly marked singularities. Here $M_\mu^{smar}(f_0)$ is Hausdorff and an analytic space only if assumption (8.1) or assumption (8.2) holds.

Assumption (8.1):	Any singularity in the μ-homotopy class of f_0 has multiplicity ≥ 3.
Assumption (8.2):	Any singularity in the μ-homotopy class of f_0 has multiplicity 2.

We expect that one of them holds for any μ-homotopy class of singularities. This would be an implication of the Zariski multiplicity conjecture. But that is not proved in general.

But $M_\mu^{mar}(f_0)$ is Hausdorff and an analytic space, independently of these assumptions. Locally it is isomorphic to the μ-constant stratum $S_\mu(f)$ of a singularity in the base space of a universal unfolding of that singularity. The group $G_{\mathbb{Z}}(f_0)$ acts properly discontinuously on $M_\mu^{mar}(f_0)$. The quotient is $M_\mu^{mar}(f_0)/G_{\mathbb{Z}} \cong M_\mu(f_0)$. Therefore a neighborhood of $[f]$ in $M_\mu(f_0)$ is isomorphic to the quotient of $S_\mu(f)$ by a finite group. $M_\mu^{mar}(f_0)$ can be considered as a *Teichmüller space* for singularities, in analogy to the Teichmüller spaces for closed complex curves. It can also be considered as a global μ-constant stratum, simultaneously for all singularities in one μ-homotopy class.

The papers [He11], [GH17] and this paper determine $M_\mu^{mar}(f_0)$ for all singularities with modality 0, 1 and 2. The second column of the following table (1.1) gives their isomorphism classes.

Singularity family	$M_\mu^{mar}(f_0)$	$D_{BL}(f_0)$
ADE-singularities	point	point
simple elliptic sing.	$\mathbb{H}$	$\mathbb{H}$
hyperbolic sing.	$\mathbb{C}$	$\mathbb{C}$
exc. unimodal sing.	$\mathbb{C}$	$\mathbb{C}$
exc. bimodal sing.	$\mathbb{C}^2$	$\mathbb{C}^2$
quadrangle sing.	$(\mathbb{H} - \text{(a discrete set)}) \times \mathbb{C}$	$\mathbb{H} \times \mathbb{C}$
the 8 series, for $m \not| p$	$\mathbb{C}^* \times \mathbb{C}$	$\mathbb{C}^{N_{BL}}$
the 8 subseries with $m|p$	∞ many copies of $\mathbb{C}^* \times \mathbb{C}$	$\mathbb{H} \times \mathbb{C}^{N_{BL}}$

(1.1)

Here the eight series and the respective numbers m are given in the following table (1.2). Here $p \in \mathbb{Z}_{\geq 1}$.

$$\begin{array}{lllllllll} \text{series} & W_{1,p}^\sharp & S_{1,p}^\sharp & U_{1,p} & E_{3,p} & Z_{1,p} & Q_{2,p} & W_{1,p} & S_{1,p} \\ m & 12 & 10 & 9 & 18 & 14 & 12 & 12 & 10 \end{array} \tag{1.2}$$

One sees that $M_\mu^{mar}(f_0)$ is simply connected for all singularities with modality 0 and 1 and for the exceptional bimodal singularities. For the quadrangle singularities and the series with $m \not| p$, it is connected, but not simply connected. And for the subseries with $m|p$, it is not even connected, but has infinitely many components. This last result is a counterexample to conjecture 3.2 (a) in [He11], which said that $M_\mu^{mar}(f_0)$ should be connected.

In [He11], also two subgroups $G^{smar}(f_0)$ and $G^{mar}(f_0)$ of $G_\mathbb{Z}(f_0)$ were defined. $G^{smar}(f_0)$ was defined as the subgroup which is generated by the transversal monodromies of all μ-constant families which contain f_0. Here the *transversal monodromy* of a μ-constant family $f_t, t \in T$, with $f_{t^0} = f_0$ is the representation $\pi_1(T, t^0) \to G_\mathbb{Z}(f_0)$ which comes from the local system $\bigcup_{t\in T} Ml(f_t)$. Then $G^{mar}(f_0)$ is the group generated by $G^{smar}(f_0)$ and $-\operatorname{id}$. A rough way to talk about this description is to say that the elements of $G^{smar}(f_0)$ are of geometric origin. $G^{mar}(f_0)$ can also be characterized as the subgroup of $G_\mathbb{Z}$ which maps the component $(M_\mu^{mar})^0$ of $M_\mu^{mar}(f_0)$, which contains $[(f_0, \pm\operatorname{id})]$, to itself. This last characterization gives

$$G_\mathbb{Z}(f_0)/G^{mar}(f_0) \stackrel{1:1}{\longleftrightarrow} \{\text{components of } M_\mu^{mar}(f_0)\}. \tag{1.3}$$

In view of this, $M_\mu^{mar}(f_0)$ is connected if and only if $G_\mathbb{Z}(f_0) = G^{mar}(f_0)$. By table (1.1), this holds for all singularities with modality 0, 1 or 2 except the eight subseries with $m|p$. Obviously, it is important to control $G_\mathbb{Z}(f_0)$. This was the major task in [He11] and [GH17] for the singularities considered there, and it takes approximately half of this paper for the singularities considered here, the bimodal series and the quadrangle singularities. The rough outcome in all cases is that the pair $(Ml(f_0), L)$ is surprisingly rigid and that $G_\mathbb{Z}(f_0)$ is surprisingly small. The next table (1.4) gives more information on $G_\mathbb{Z}(f_0)$ for all singularities with modality 0, 1 and 2. Here $M_h \in G_\mathbb{Z}$ is the classical monodromy. It commutes with all elements of $G_\mathbb{Z}$. The only families in table (1.4) where $\{\pm M_h^k \,|\, k \in \mathbb{Z}\}$ is not finite, are the hyperbolic singularities T_{pqr}.

Singularity family	$G_\mathbb{Z}(f_0)/\{\pm M_h^k \,\|\, k \in \mathbb{Z}\}$
ADE-singularities	$\{\text{id}\}$ or S_2 or S_3
simple elliptic sing.	a finite extension of $SL(2,\mathbb{Z})$
hyperbolic sing.	a finite group
exc. unimodal sing.	$\{\text{id}\}$ or S_2 or S_3
exc. bimodal sing.	$\{\text{id}\}$ or S_2 or S_3
quadrangle sing.	a triangle group
the 8 series, for $m \not| p$	a cyclic finite group
the 8 subseries with $m\|p$	an infinite Fuchsian group

(1.4)

[He11] treats the ADE-singularities and 22 of the 28 exceptional (unimodal and bimodal) singularities. [GH17] treats the other 6 exceptional singularities, the simple elliptic singularities and the hyperbolic singularities. The present paper treats the quadrangle singularities and the 8 series.

In the case of the eight subseries with $m|p$, $G^{mar}(f_0)$ is the finite subgroup of the infinite group $G_\mathbb{Z}(f_0)$ such that $G^{mar}(f_0)/\{\pm M_h^k \,|\, k \in \mathbb{Z}\}$ is the finite cyclic group which is generated by one elliptic element.

If the μ-homotopy class of f_0 contains at least one singularity with multiplicity two, then $-\operatorname{id} \in G^{smar}(f_0)$ and $G^{smar}(f_0) = G^{mar}(f_0)$. Conjecture 3.2 (b) in [He11] complements this.

It claims that $-\mathrm{id} \notin G^{smar}(f_0)$ if assumption (8.1) holds. This is true for all singularities with modality 0, 1 and 2. For the bimodal series and the quadrangle singularities, it is proved in this paper.

In [He99] the second author defined a classifying space $D_{BL}(f_0)$ for Brieskorn like lattices (i.e. for objects which are sufficiently similar to the Brieskorn lattice $H_0''(f_0)$, see section 7 before theorem 7.11 for details). The group $G_{\mathbb{Z}}(f_0)$ acts properly discontinuously on it. The elements of $D_{BL}(f_0)$ are *marked Brieskorn like lattices*, and the elements of $D_{BL}(f_0)/G_{\mathbb{Z}}(f_0)$ are isomorphism classes of Brieskorn like lattices. One obtains a holomorphic period map

$$BL : M_\mu^{mar}(f_0) \to D_{BL}(f_0). \tag{1.5}$$

By [He02, Theorem 12.8] it is $G_{\mathbb{Z}}(f_0)$-equivariant, and it is an immersion (this fact is an infinitesimal Torelli type result). Now the following Torelli type conjectures are natural. Part (a) is for marked singularities. Part (b) recasts the Torelli type conjecture in [He93]. Part (a) implies part (b).

Conjecture 1.1. *(a)* [He11, Conjecture 5.3] *The map BL is injective.*
(b) [He93, Kap. 2 d)] *The map $BL/G_{\mathbb{Z}}(f_0) : M_\mu(f_0) \to D_{BL}(f_0)/G_{\mathbb{Z}}(f_0)$ is injective.*

Theorem 1.2. *(*[He93][He11][GH17] *and the theorems 9.1 and 10.1 in this paper) Both Torelli type conjectures are true for all singularities with modality 0, 1 and 2.*

The proofs have in almost all cases two parts:

(1) A good control of an (often multivalued) period map $T \to D_{BL}(f_0)$, where T is the parameter space of a well chosen family of normal forms.
(2) A good control of $G_{\mathbb{Z}}(f)$ and its action on $M_\mu^{mar}(f_0)$ and $D_{BL}(f_0)$.

In all cases, (1) is less work than (2). For the ADE-singularities, (1) is empty as there T is a point, but (2) is not.

Part (b) of conjecture 1.1 was proved in [He93] for the unimodal and bimodal singularities except seven of the eight series. For the seven series, the second author had unpublished calculations shortly after [He93]. But for technical reasons, part (b) stayed open for the subseries $S^\sharp_{1,10r}, S_{1,10r}, Z_{1,14r}$. [He93] and these unpublished calculations give (1) and a part of (2).

In view of these old results, the major point in [He11], [GH17] and in this paper is (2). But also some refinement of (1) is needed in the case of the singularities in this paper. The refinement is used for a better control of the transversal monodromy of the family of normal forms.

Finally, the conjecture $G_{\mathbb{Z}}(f_0) = G^{mar}(f_0)$ is probably wrong in general as it is wrong for the subseries with $m|p$. But for all singularities with modality 0, 1 and 2 except the eight series, the Torelli result for marked singularities and (1.3) require $G_{\mathbb{Z}}(f_0) = G^{mar}(f_0)$ to be true, as BL is an immersion and there $\dim M_\mu^{mar}(f_0) = \mathrm{modality}(f_0) = \dim D_{BL}(f_0)$. And there $G_{\mathbb{Z}}(f_0) = G^{mar}(f_0)$ holds indeed. For the eight series, $\dim D_{BL}(f_0) > \dim M_\mu^{mar}(f_0)$, so there is enough space in D_{BL} for infinitely many copies of $(M_\mu^{mar}(f_0))^0$.

Open questions are now how to control the subgroup $G^{mar}(f_0) \subset G_{\mathbb{Z}}(f_0)$ in general, and how to attack the Torelli conjectures in greater generality. For the second question, we plan to thicken $M_\mu^{mar}(f_0)$ to a μ-dimensional F-manifold $M^{mar}(f_0)$ which is locally at each point of $M_\mu^{mar}(f_0)$ the base space of a universal unfolding. Then we will try to embed the Torelli type conjecture for M_μ^{mar} into a family of Torelli type conjectures for all the μ-homotopy strata of multigerms of singularities in $M^{mar}(f_0)$. We hope that this global point of view and the different geometry there with Stokes structures will give us new techniques. But this is a hope for the future.

1.3. **Structure of the paper.** **Section 2** is a collection of techniques which are useful to control the automorphisms of a pair (Λ, L) or a pair (Λ, M_h) where Λ is a $\mathbb{Z}$-lattice, L is a unimodular bilinear form and M_h is an automorphism of finite order. We define *Orlik blocks* and study their automorphisms (lemma 2.8 will be very useful), and we cite classical algebraic facts on unit roots ζ and the rings $\mathbb{Z}[\zeta]$. All this is needed for the control of $G_{\mathbb{Z}}(f_0)$ in the sections 5 and 6.

Section 3 discusses infinite Fuchsian groups which arise as subgroups of groups $GL(2, \mathbb{Z}[\zeta])$ with ζ a unit root. They are in fact arithmetic Fuchsian groups. But our treatment is essentially self-contained. Solutions of Pell equations with coefficients in $\mathbb{Z}[\zeta]$ play a role. For the quadrangle singularities, we need a precise analysis of some of these groups. They are certain triangle groups.

Section 4 recalls some classical notions and facts around singularities: Milnor fibration, Milnor lattice $Ml(f)$, monodromy M_h, Seifert form L, Coxeter-Dynkin diagram, Stokes matrix, Thom-Sebastiani type results, suspension, polarized mixed Hodge structure on $H^\infty_{\mathbb{C}}$, its polarizing form.

Section 5 is long. It studies $G_{\mathbb{Z}}(f_0)$ for the eight bimodal series. Theorem 5.1 states the results. We start with a distinguished basis of the Milnor lattice with Coxeter-Dynkin diagram in [Eb81]. We calculate the monodromy M_h and find 2 or 3 (3 only for $Z_{1,p}$) Orlik blocks whose direct sum is of index 1 or 2 in $Ml(f_0)$. Then $G_{\mathbb{Z}}(f_0)$ is studied using these Orlik blocks and their rigidity and the results from the sections 2 and 3. A lot of calculations are needed, the different series behave differently. The singularities in the families $Q_{2,p}, W_{1,6s-3}, S_{1,10}$ need special care.

Section 6 gives similar results for $G_{\mathbb{Z}}(f_0)$ for the quadrangle singularities. Theorem 6.1 states the results. Many, but not all, calculations and arguments in section 5 are also valid in section 6. Therefore this section is much shorter.

Section 7 gives a rather complete account on the Gauss-Manin connection and the Brieskorn lattice $H_0''(f)$ of a singularity f. It does not rewrite the proofs in [Br70] and other papers, but it cites almost all known results. A highlight is the treatment of the bilinear forms. The polarizing form of the polarized mixed Hodge structure is connected with the restriction of K. Saito's higher residue pairings to $H_0''(f)$ and with Pham's intersection form for Lefschetz thimbles. We need the Fourier-Laplace transform $FL(H_0''(f))$ for a Thom-Sebastiani formula for Brieskorn lattices. We need this in the special case of a suspension $f(z_0, ..., z_n) + z_{n+1}^2$ because we want to treat the suspensions in a more conceptual way than in [He93][He11][GH17].

Section 8 reviews the notions and results from [He11], the (strongly) marked singularities and their moduli spaces $M_\mu^{smar}(f_0)$ and $M_\mu^{mar}(f_0)$, the μ-constant monodromy groups $G^{smar}(f_0)$ and $G^{mar}(f_0)$, and the Torelli conjectures. Corollary 8.14 is an application of the Thom-Sebastiani result for $FL(H_0''(f))$ in section 7 and states that the marked Torelli conjecture for f_0 is equivalent to the marked Torelli conjecture for $f_0(z_0, ..., z_n) + \sum_{j=n+1}^m z_j^2$ for any fixed $m \geq n+1$. This allows us to consider in the sections 9 and 10 only the surface singularities.

Section 9 proves the marked Torelli conjecture for the bimodal series (theorem 9.1). It establishes the good control (1) of the multivalued period map $T \to D_{BL}(f_0)$ where $T = \mathbb{C}^* \times \mathbb{C}$ is the parameter space of normal forms in [AGV85]. Theorem 5.1 provides crucial information on $G_{\mathbb{Z}}(f_0)$.

Section 10 proves the marked Torelli conjecture for the quadrangle singularities (theorem 10.1). It starts with a careful choice of normal forms with parameter space $T = (\mathbb{C} - \{0, 1\}) \times \mathbb{C}$. It establishes the good control (1) of the multivalued period map $T \to D_{BL}(f_0)$. Theorem 6.1 provides crucial information on $G_{\mathbb{Z}}(f_0)$.

2. $\mathbb{Z}$-lattices with unimodal bilinear form and monodromy

This section provides tools for the study of the Milnor lattices with Seifert form and monodromy for the bimodal series and the quadrangle singularities, in the sections 5 and 6. These lattices turn out to be quite rigid and to have rather few automorphisms. This is important for the global

Torelli results in the sections 9 and 10. This section puts together elementary, but nontrivial observations about $\mathbb{Z}$-lattices with a unimodal bilinear form and an (induced) monodromy.

Let Λ be a $\mathbb{Z}$-lattice of rank $\mu \in \mathbb{Z}_{\geq 1}$, i.e. a free $\mathbb{Z}$-module of rank μ. Let $L : \Lambda \times \Lambda \to \mathbb{Z}$ be a unimodal bilinear form, i.e. for any basis $\delta_1, \dots, \delta_\mu$ we have $\det(L(\delta_i, \delta_j)_{i,j=1,\dots,\mu}) = \pm 1$. We do not suppose that L is symmetric or antisymmetric. Let $M_h : \Lambda \to \Lambda$ be the automorphism which is uniquely determined by

$$L(M_h(a), b) = -L(b, a) \text{for } a, b \in \Lambda. \tag{2.6}$$

We call L the *Seifert form* and M_h the *monodromy.* (2.6) implies

$$L(M_h(a), M_h(b)) = L(a, b), \tag{2.7}$$

i.e. L is M_h-invariant. We make the assumption that

$$M_h \text{ is finite,} \tag{2.8}$$

i.e. M_h is semisimple and its eigenvalues are unit roots. Then the characteristic polynomial p_Λ of M_h is a product of cyclotomic polynomials.

Notations 2.1. (a) For any subring $R \subset \mathbb{C}$ denote $\Lambda_R := \Lambda \otimes_{\mathbb{Z}} R$. For any monodromy invariant subspace $V \subset \Lambda_{\mathbb{C}}$ denote by $E(V) \subset S^1$ the set of eigenvalues of M_h on V and by p_V its characteristic polynomial. For $\lambda \in E(V)$ denote $V_\lambda := \ker(M_h - \lambda \,\mathrm{id} : V \to V) \subset V$. For any monodromy invariant sublattice $\Lambda^{(1)} \subset \Lambda$ write $E(\Lambda^{(1)}) := E(\Lambda^{(1)}_{\mathbb{C}})$ and $p_{\Lambda^{(1)}} := p_{\Lambda^{(1)}_{\mathbb{C}}}$ and $\Lambda^{(1)}_\lambda := (\Lambda^{(1)}_{\mathbb{C}})_\lambda$. For any product $p \in \mathbb{Z}[t]$ of cyclotomic polynomials with $p | p_{\Lambda^{(1)}}$ denote

$$\Lambda^{(1)}_{\mathbb{C},p} := \bigoplus_{\lambda :\, p(\lambda)=0} \Lambda^{(1)}_\lambda \text{and} \Lambda^{(1)}_p := \Lambda^{(1)}_{\mathbb{C},p} \cap \Lambda^{(1)}. \tag{2.9}$$

Then $\Lambda^{(1)}_p$ is a primitive and monodromy invariant sublattice of $\Lambda^{(1)}$.

(b) Recall that a sublattice $\Lambda^{(1)}$ of Λ is *primitive* (in Λ) if and only if $\Lambda/\Lambda^{(1)}$ has no torsion. Recall also that for any sublattice $\Lambda^{(2)} \subset \Lambda$ there is a unique primitive sublattice $\Lambda^{(3)}$ with $\Lambda^{(3)}_{\mathbb{Q}} = \Lambda^{(2)}_{\mathbb{Q}}$, that it is $\Lambda^{(3)} = \Lambda^{(2)}_{\mathbb{Q}} \cap \Lambda$ and that $[\Lambda^{(3)} : \Lambda^{(2)}] < \infty$.

(c) For $n \in \mathbb{Z}_{\geq 1}$, the cyclotomic polynomial Φ_n is

$$\Phi_n = \prod_{\lambda :\, \mathrm{ord}(\lambda)=n} (t - \lambda).$$

It is unitary, in $\mathbb{Z}[t]$ and irreducible in $\mathbb{Z}[t]$ and $\mathbb{Q}[t]$.

(d) We define the square root on $S^1 - \{-1\}$ by $\sqrt{e^{2\pi i\alpha}} := e^{\pi i \alpha}$ for $\alpha \in] - \frac{1}{2}, \frac{1}{2}[$.

Lemma 2.2. *(a) Let $\lambda \in E(\Lambda) - \{1\}$. Then the sesquilinear (i.e., linear$\times$semilinear) form $h_\lambda : \Lambda_\lambda \times \Lambda_\lambda \to \mathbb{C}$ with*

$$h_\lambda(a, b) := \sqrt{-\lambda} \cdot L(a, \overline{b}) \tag{2.10}$$

is hermitian, i.e. $h_\lambda(b, a) = \overline{h_\lambda(a, b)}$. Especially, $\sqrt{-\lambda} \cdot L(a, \overline{a}) \in \mathbb{R}$. Together, these forms define a hermitian form $h := \bigoplus_{\lambda \in E(\Lambda)-\{1\}} h_\lambda$.

(b) Let $V \subset \Lambda_{\mathbb{C}}$ be a monodromy invariant subspace with $1 \notin E(V)$. The following two properties are equivalent.

- (α) *$h|_V$ is positive definite.*
- (β) *The hermitian form on V defined by $(a, b) \mapsto L(a, \overline{b}) + L(\overline{b}, a)$ is positive definite.*

Proof: (a) For $a, b \in V_\lambda$

$$\begin{aligned}\sqrt{-\lambda}\cdot L(b,\overline{a}) &= -\sqrt{-\lambda}\cdot L(M_h(\overline{a}),b) = -\sqrt{-\lambda}\cdot\overline{\lambda}\cdot L(\overline{a},b) \\ &= \sqrt{-\overline{\lambda}}\cdot\overline{L(a,\overline{b})} = \overline{\sqrt{-\lambda}\cdot L(a,\overline{b})}.\end{aligned}$$

(b) Consider some $\lambda \in E(V)$. Observe $\sqrt{-\lambda}+\sqrt{-\overline{\lambda}} > 0$ and for $a, b \in V$

$$\begin{aligned}L(a,\overline{b}) + L(\overline{b},a) &= L(a,\overline{b}) + \overline{L(b,\overline{a})} \\ &= \sqrt{-\overline{\lambda}}\cdot h_\lambda(a,b) + \overline{\sqrt{-\overline{\lambda}}\cdot h_\lambda(b,a)} \\ &= (\sqrt{-\overline{\lambda}}+\sqrt{-\lambda})\cdot h_\lambda(a,b). \qquad \square\end{aligned}$$

Remarks 2.3. (i) The surface singularities considered in this paper do not have 1 as an eigenvalue of their monodromy. Therefore we do not treat the case $\lambda = 1$ here.

(ii) Part (b) of lemma 2.2 connects to the polarization of the polarized Hodge structure of these surface singularities and rewrites it in different ways. (β) is the classical way, with $-L-L^t$ on $\Lambda_\mathbb{R}$ as intersection form and $L+L^t$ as polarizing form. And (α) is the way used in the sections 3, 5 and 6.

In 1972 Orlik formulated the beautiful conjecture 2.5 below on the integral monodromy of quasihomogeneous singularities [Or72]. It is known to be true for the quasihomogeneous curve singularities [MW86] and for the quasihomogeneous singularities with modality ≤ 2 [He95]. But it is open for most other quasihomogeneous singularities.

A key observation for the treatment of the Milnor lattices of the bimodal series singularities and the quadrangle singularities is that they all have a structure close to Orlik's conjecture. The following definition gives the ingredients.

Definition 2.4. Let (Λ, L, M_h) be as above. An *Orlik block* is a primitive and monodromy invariant sublattice $\Lambda^{(1)} \subset \Lambda$ with $\Lambda^{(1)} \supsetneqq \{0\}$ and with a cyclic generator, i.e. a lattice vector $e^{(1)} \in \Lambda^{(1)}$ with

$$\Lambda^{(1)} = \bigoplus_{j=0}^{\deg p_{\Lambda^{(1)}}-1} \mathbb{Z}\cdot M_h^j(e^{(1)}). \tag{2.11}$$

Conjecture 2.5. [Or72, conjecture 3.1] *Let (Λ, M_h) be the Milnor lattice with monodromy of a quasihomogeneous singularity. Let $k := \max(\dim\Lambda_\lambda \,|\, \lambda \in E(\Lambda))$. Then a decomposition $\Lambda = \bigoplus_{j=1}^k \Lambda^{(k)}$ into Orlik blocks $\Lambda^{(1)}, \dots, \Lambda^{(k)}$ with $p_{\Lambda^{(j+1)}} | p_{\Lambda^{(j)}}$ for $0 \leq j < k$ exists.*

Remarks 2.6. (i) A cyclic monodromy module has only one Jordan block for each eigenvalue. In this paper M_h is semisimple. Therefore in an Orlik block, each eigenvalue has multiplicity one.

(ii) In Orlik's conjecture 2.5, the polynomials $p_{\Lambda^{(1)}}, \dots, p_{\Lambda^{(k)}}$ are unique. They are

$$p_{\Lambda^{(j)}} = \prod_{\lambda\in E(\Lambda):\, \dim\Lambda_\lambda \geq j} (t-\lambda) \text{ for } j = 1, \dots, k. \tag{2.12}$$

(iii) In the sections 5 and 6, we will work most often with two Orlik blocks $\Lambda^{(1)}$ and $\Lambda^{(2)}$ such that $\Lambda^{(1)} + \Lambda^{(2)} = \Lambda^{(1)} \oplus \Lambda^{(2)}$ and that it is either equal to Λ or has index 2 in Λ and such that $L(\Lambda^{(1)}, \Lambda^{(2)}) = L(\Lambda^{(2)}, \Lambda^{(1)}) = 0$.

(iv) In all cases in section 5 with $[\Lambda : \Lambda^{(1)} \oplus \Lambda^{(2)}] = 2$ except $S_{1,10}$, we will show

$$\mathrm{Aut}(\Lambda, L) = \mathrm{Aut}(\Lambda^{(1)} \oplus \Lambda^{(2)}, L). \tag{2.13}$$

In many of these cases, there is an element $\gamma_5\in\Lambda^{(1)}_{\Phi_2}-\{0\}$ which is mapped by any element g of $\mathrm{Aut}(\Lambda,L)\cup\mathrm{Aut}(\Lambda^{(1)}\oplus\Lambda^{(2)},L)$ to $\pm\gamma_5$ and such that

$$\Lambda^{(1)}\oplus\Lambda^{(2)}=\{a\in\Lambda\,|\,L(a,\gamma_5)\in 2\mathbb{Z}\}. \tag{2.14}$$

Then any $g\in\mathrm{Aut}(\Lambda,L)$ maps $\Lambda^{(1)}\oplus\Lambda^{(2)}$ to itself, so $\mathrm{Aut}(\Lambda,L)\subset\mathrm{Aut}(\Lambda^{(1)}\oplus\Lambda^{(2)},L)$.

If this inclusion $\subset$ holds, the following argument shows that $\mathrm{Aut}(\Lambda,L)$ is either equal to or a subgroup of index 2 in $\mathrm{Aut}(\Lambda^{(1)}\oplus\Lambda^{(2)},L)$. Unfortunately it looks hard to exclude the second case. Therefore in section 5 we show the equality (2.13) in a different (and more laborious) way.

Let $\Lambda^{(0)}\subset\Lambda_{\mathbb{Q}}$ be the unique lattice such that

$$L:\Lambda^{(0)}\times(\Lambda^{(1)}\oplus\Lambda^{(2)})\to\mathbb{Z}$$

is unimodal. Then $\Lambda^{(0)}\supset\Lambda\supset\Lambda^{(1)}\oplus\Lambda^{(2)}$ and $[\Lambda^{(0)}:\Lambda]=2$ and

$$\mathrm{Aut}(\Lambda^{(1)}\oplus\Lambda^{(2)},L)=\mathrm{Aut}(\Lambda^{(0)},L).$$

1st case, $\Lambda^{(0)}/(\Lambda^{(1)}\oplus\Lambda^{(2)})\cong\mathbb{Z}/4\mathbb{Z}$. Then Λ is the unique lattice between $\Lambda^{(0)}$ and $\Lambda^{(1)}\oplus\Lambda^{(2)}$ with $[\Lambda^{(0)}:\Lambda]=2$. Then any $g\in\mathrm{Aut}(\Lambda^{(1)}\oplus\Lambda^{(2)},L)$ respects Λ, so (2.13) holds.

2nd case, $\Lambda^{(0)}/(\Lambda^{(1)}\oplus\Lambda^{(2)})\cong\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$. Then there are three lattices between $\Lambda^{(0)}$ and $\Lambda^{(1)}\oplus\Lambda^{(2)}$ with index 2 in $\Lambda^{(0)}$, one for each subgroup of index 2 in $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$. One of them is Λ. Another one is $\{a\in\Lambda^{(0)}\,|\,L(a,\gamma_5)\in 2\mathbb{Z}\}$. No element of $\mathrm{Aut}(\Lambda^{(0)},L)$ maps Λ to this lattice. But it looks hard to exclude the possibility that half of the elements of $\mathrm{Aut}(\Lambda^{(0)},L)$ map Λ to the third lattice between $\Lambda^{(0)}$ and $\Lambda^{(1)}\oplus\Lambda^{(2)}$.

(v) If $\Lambda^{(1)}\subset\Lambda$ is an Orlik block with cyclic generator $e^{(1)}$ and if $p_{\Lambda^{(1)}}=p_1\cdot p_2$ with $\deg p_1\geq 1$ and $\deg p_2\geq 1$, then the sublattice $\Lambda^{(2)}:=\Lambda^{(1)}_{p_1}$ is also an Orlik block, and a cyclic generator is

$$e^{(2)}:=p_2(M_h)(e^{(1)}). \tag{2.15}$$

(vi) If $\Lambda^{(1)}\subset\Lambda$ is an Orlik block with generator $e^{(1)}$ and $\lambda\in E(\Lambda^{(1)})$ is an eigenvalue of the monodromy on $\Lambda^{(1)}$, then an eigenvector is

$$v(e^{(1)},\lambda):=\frac{p_{\Lambda^{(1)}}}{t-\lambda}(M_h)(e^{(1)}). \tag{2.16}$$

And then

$$\begin{aligned}
& L(v(e^{(1)},\lambda),v(e^{(1)},\overline{\lambda}))\\
=\ & L(v(e^{(1)},\lambda),\frac{p_{\Lambda^{(1)}}}{t-\overline{\lambda}}(M_h)(e^{(1)}))\\
=\ & L(\frac{p_{\Lambda^{(1)}}}{t-\overline{\lambda}}(M_h^{-1})v(e^{(1)},\lambda),e^{(1)})\\
=\ & \frac{p_{\Lambda^{(1)}}}{t-\overline{\lambda}}(\overline{\lambda})\cdot L(v(e^{(1)},\lambda),e^{(1)})\\
=\ & \frac{p_{\Lambda^{(1)}}}{t-\overline{\lambda}}(\overline{\lambda})\cdot L(\frac{p_{\Lambda^{(1)}}}{t-\lambda}(M_h)(e^{(1)}),e^{(1)}).
\end{aligned} \tag{2.17}$$

This calculation will be useful in section 5.

The following two lemmata concern automorphisms of sums of Orlik blocks (lemma 2.7) or of a single Orlik block (lemma 2.8). They will be useful tools in order to show the rigidity of the Milnor lattices in the sections 5 and 6.

Lemma 2.7. *Let (Λ,M_h) be as above (we will not need L here, only M_h). Let $\Lambda^{(1)},\dots,\Lambda^{(k)}\subset\Lambda$ be Orlik blocks with cyclic generators $e^{(1)},\dots,e^{(k)}$ and with*

$$\Lambda^{(1)}+\ldots+\Lambda^{(k)}=\Lambda^{(1)}\oplus\ldots\oplus\Lambda^{(k)}.$$

Consider an element

$$g\in \mathrm{Aut}(\Lambda^{(1)}\oplus\ldots\oplus\Lambda^{(k)},M_h).$$

Then there are unique polynomials $p_{ij}\in\mathbb{Z}[t]_{<\mathrm{rank}\,\Lambda^{(j)}}$ *for* $i,j=1,\ldots,k$ *with*

$$g(e^{(j)})=\sum_{i=1}^k p_{ij}(M_h)(e^{(i)}). \tag{2.18}$$

Suppose now that $p_0\in\mathbb{Z}[t]$ *divides* $\gcd(p_{\Lambda^{(1)}},\ldots,p_{\Lambda^{(k)}})$ *and that*

$$g=\mathrm{id}\ \text{ on } \Lambda^{(j)}_{p_{\Lambda^{(j)}}/p_0} \text{ for any } j, \tag{2.19}$$

so that g *acts nontrivial only on* $(\Lambda^{(1)}\oplus\ldots\oplus\Lambda^{(k)})_{p_0}$. *Then*

$$p_{ij}=\delta_{ij}+\frac{p_{\Lambda^{(i)}}}{p_0}\cdot q_{ij} \tag{2.20}$$

for suitable polynomials $q_{ij}\in\mathbb{Z}[t]_{<\deg p_0}$.

Suppose furthermore that a unit root ξ *satisfies* $p_0(\xi)=0$. *Then* g *with respect to the eigenvectors* $v(e^{(1)},\xi)\in\Lambda^{(1)}_\xi,\ldots,v(e^{(k)},\xi)\in\Lambda^{(k)}_\xi$ *(defined in (2.16)) is given by*

$$g(v(e^{(j)},\xi))=\sum_{i=1}^k(\delta_{ij}+\frac{p_{\Lambda^{(j)}}}{p_0}\cdot q_{ij})(\xi)\cdot v(e^{(i)},\xi) \tag{2.21}$$

Proof: Only the part after (2.18) is nontrivial. Suppose that p_0 and g are as stated above. By assumption

$$\begin{array}{rcl} g(e^{(j)})-e^{(j)} & \in & (\Lambda^{(1)}\oplus\ldots\oplus\Lambda^{(k)})_{p_0} \\ & \subset & \displaystyle\bigoplus_{i=1}^k \Lambda^{(i)}_{\mathbb{C},p_0}=\bigoplus_{i=1}^k \frac{p_{\Lambda^{(i)}}}{p_0}(M_h)(\Lambda^{(i)}_{\mathbb{C}}). \end{array}$$

Thus $p_{ij}-\delta_{ij}\in\frac{p_{\Lambda^{(i)}}}{p_0}\cdot\mathbb{C}[t]$, thus $p_{ij}-\delta_{ij}\in\frac{p_{\Lambda^{(i)}}}{p_0}\cdot\mathbb{Z}[t]_{<\deg p_0}$.

The following calculation proves (2.21).

$$\begin{aligned} g(v(e^{(j)},\xi)) &= g\left(\frac{p_{\Lambda^{(j)}}}{t-\xi}(M_h)(e^{(j)})\right)=\frac{p_{\Lambda^{(j)}}}{t-\xi}(M_h)\left(g(e^{(j)})\right) \\ &= \frac{p_{\Lambda^{(j)}}}{t-\xi}(M_h)\left(\sum_{i=1}^k\left(\delta_{ij}+\frac{p_{\Lambda^{(i)}}}{p_0}\cdot q_{ij}\right)(M_h)(e^{(i)})\right) \\ &= \sum_{i=1}^k\left(\left(\delta_{ij}+\frac{p_{\Lambda^{(i)}}}{p_0}\cdot q_{ij}\right)\cdot\frac{p_{\Lambda^{(j)}}}{t-\xi}\right)(M_h)(e^{(i)}) \\ &= \sum_{i=1}^k\left(\delta_{ij}+\frac{p_{\Lambda^{(j)}}}{p_0}\cdot q_{ij}\right)(M_h)(v(e^{(i)},\xi)) \\ &= \sum_{i=1}^k\left(\delta_{ij}+\frac{p_{\Lambda^{(j)}}}{p_0}\cdot q_{ij}\right)(\xi)\cdot v(e^{(i)},\xi). \end{aligned}$$

□

Let (Λ,L,M_h) be as above, and suppose that Λ is a single Orlik block. Because of (2.8) $\mathrm{Aut}(\Lambda,L,M_h)\supset\{\pm M_h^k\,|\,k\in\mathbb{Z}\}$. The paper [He18] solves the problem when equality

$\mathrm{Aut}(\Lambda, L, M_h) = \{\pm M_h^k \mid k \in \mathbb{Z}\}$ holds. It turns out that it depends only on the finite set

$$\mathrm{Ord} := \{\mathrm{ord}\,\lambda \mid \lambda \text{ eigenvalue of } M_h\} \subset \mathbb{Z}_{\geq 1} \tag{2.22}$$

of orders of the eigenvalues of the monodromy M_h. Though the necessary and sufficient conditions in theorem 1.2 in [He18] are involved. They use the directed graph with vertex set Ord and set of directed edges $\{(a,b) \in \mathrm{Ord}^2 \mid \frac{b}{a} \text{ is a power of a prime number}\}$. A simpler sufficient condition (which is sufficient for the cases in this paper) is given in the following lemma. There the graph is connected and has a root m_1, and an additional property holds for the prime number 2. The lemma is cited from [He11, lemma 8.2], but it goes back to arguments in [He98, ch. 6].

Lemma 2.8. *Let (Λ, L, M_h) be as above. Suppose that Λ is a single Orlik block. We make the following nontrivial assumption on the set* Ord*: There exist four sequences $(m_i)_{i=1,\dots,|\,\mathrm{Ord}\,|}$, $(j(i))_{i=2,\dots,|\,\mathrm{Ord}\,|}$, $(p_i)_{i=2,\dots,|\,\mathrm{Ord}\,|}$, $(k_i)_{i=2,\dots,|\,\mathrm{Ord}\,|}$ of numbers in $\mathbb{Z}_{\geq 1}$ and two numbers $i_1, i_2 \in \mathbb{Z}_{\geq 1}$ with $i_1 \leq i_2 \leq |\,\mathrm{Ord}\,|$ and with the properties:*

$\mathrm{Ord} = \{m_1, \dots, m_{|\,\mathrm{Ord}\,|}\}$,
p_i is a prime number, $p_i = 2$ for $i_1 + 1 \leq i \leq i_2$, $p_i \geq 3$ else,
$j(i) = i - 1$ for $i_1 + 1 \leq i \leq i_2$, $j(i) < i$ else,
$m_i = m_{j(i)}/p_i^{k_i}$.

Then

$$\mathrm{Aut}(\Lambda, L, M_h) = \{\pm M_h^k \mid k \in \mathbb{Z}\}. \tag{2.23}$$

We will need some basic facts for the unit roots $\zeta = e^{2\pi i/m}$ with $m \in \{10, 12, 14, 18\}$. The following theorem 2.9 collects some facts for general unit roots. Theorem 2.10 cites two classical results on orders in algebraic number fields. Lemma 2.11 puts together some specific properties for the unit roots of the orders $m \in \{10, 12, 14, 18\}$.

Theorem 2.9. *Fix $m \in \mathbb{Z}_{\geq 3}$ and define $\zeta := e^{2\pi i/m}$, $p_1 := \zeta + \overline{\zeta}$.*
(a)

$$\begin{aligned} \mathrm{Eiw}(\zeta) &:= \{\pm \zeta^k \mid k \in \mathbb{Z}\} \\ &= \{\text{unit roots in } \mathbb{Q}(\zeta)\} = \{\text{unit roots in } \mathbb{Z}[\zeta]\} \\ &= \{a \in \mathbb{Z}[\zeta] \mid |a| = 1\}. \end{aligned}$$

(b) $\mathbb{Z}[\zeta]$ is the ring of algebraic integers of $\mathbb{Q}(\zeta)$.
(c) $\mathbb{Z}[p_1]$ is the ring of algebraic integers of $\mathbb{Q}(p_1)$. And $\mathbb{Q}(p_1)$ is the maximal real subfield of $\mathbb{Q}(\zeta)$.
(d) $\mathbb{Q}(\zeta)$ has class field number 1 and thus $\mathbb{Z}[\zeta]$ is a principal ideal domain if and only if $m \in A_1 \cup A_2 \cup A_3$ where

$$\begin{aligned} A_1 &= \{1, 3, 5, \dots, 21\} \cup \{25, 27, 33, 35, 45\}, \\ A_2 &= \{2n \mid n \in A_1\}, \\ A_3 &= \{4n \mid n \in A_4\}, A_4 = \{1, 2, 3, \dots, 12\} \cup \{15, 21\}. \end{aligned}$$

(e) If $\mathbb{Q}(\zeta)$ has class field number 1, then $\mathbb{Q}(p_1)$ has class field number 1 *and thus $\mathbb{Z}[p_1]$ is a principal ideal domain.*
(f) $\zeta - 1 \in (\mathbb{Z}[\zeta])^*$ if $m \notin \{p^k \mid p \text{ a prime number}, \ k \in \mathbb{Z}_{\geq 1}\}$.
$\zeta + 1 \in (\mathbb{Z}[\zeta])^*$ if $m \notin \{2 \cdot p^k \mid p \text{ a prime number}, \ k \in \mathbb{Z}_{\geq 1}\}$.

Proof: (a) [Wa97] lemma 1.6 and exercise 2.3. (b) [Wa97] theorem 2.6. (c) [Wa97] proposition 2.16. (d) [Wa97] theorem 11.1. (e) [Wa97] theorem 4.10. (f) [Wa97] proposition 2.8. □

Theorem 2.10. *Let K be an algebraic number field of degree $n = s + 2t$ over $\mathbb{Q}$ with s real embeddings $\sigma_j : K \to \mathbb{R}$, $j = 1, \dots, s$, and $2t$ complex embeddings $\sigma_j : K \to \mathbb{C}$, $j = s+1, \dots, n$, with $\sigma_{s+t+j} = \overline{\sigma_{s+j}}$ for $j = 1, \dots, t$.*

(a) [BS66, Ch. 2, 3.1 Theorem 1] *Define $\underline{\sigma} := (\sigma_1, \dots, \sigma_{s+t}) : K \to \mathbb{R}^s \times \mathbb{C}^t \cong \mathbb{R}^n$. Any $\mathbb{Q}$-basis of K maps to an $\mathbb{R}$-basis of $\mathbb{R}^n$. Thus the image under $\underline{\sigma}$ of any order $\mathcal{O} \subset K$ is a lattice of rank n in $\mathbb{R}^n$.*

(b) (Dirichlet's unit theorem, [BS66, Ch. 2, 4.3 Theorem 5]*) Let $\mathcal{O} \subset K$ be an order. One can choose $r = s + t - 1$ units $a_1, \dots, a_r \in \mathcal{O}^*$ such that any unit has a unique representation $\xi \cdot a_1^{k_1} \cdot \ldots \cdot a_r^{k_r}$ with $k_1, \dots, k_r \in \mathbb{Z}$ and ξ a root of 1 in $\mathcal{O}$.*

Of course, $n = \varphi(m) = 2t$ in the case $\mathcal{O} = \mathbb{Z}[\zeta] \subset K = \mathbb{Q}(\zeta)$, and $n = \frac{\varphi(m)}{2} = s$ in the case $\mathcal{O} = \mathbb{Z}[p_1] \subset K = \mathbb{Q}[p_1]$, where $\zeta = e^{2\pi i/m}$ and $p_1 = \zeta + \overline{\zeta}$.

The unit roots of orders $m \in \{10, 12, 14, 18\}$ are most important in this paper. The next lemma collects specific properties of $\mathbb{Z}[\zeta]$ for these orders.

Lemma 2.11. *Fix $m \in \{10, 12, 14, 18\}$ and define $\zeta = e^{2\pi i/m}$ and $p_1 = \zeta + \overline{\zeta}$.*

$\mathbb{Z}[\zeta]$ and $\mathbb{Z}[p_1]$ are principal ideal domains (by theorem 2.9 (d)+(e)).

(a) $m = 10$: $\Phi_{10}(t) = t^4 - t^3 + t^2 - t + 1$,

$$\begin{aligned}
\mathbb{Z}[\zeta]^* &= \mathrm{Eiw}(\zeta) \cdot \mathbb{Z}[p_1]^* \supset \{\zeta - 1\}, \\
\mathbb{Z}[p_1]^* &= \{\pm 1\} \times \{p_1^k \,|\, k \in \mathbb{Z}\} \supset \{p_1 - 2, p_1 - 1, p_1, p_1 + 1\}, \\
p_1 &= \frac{\sqrt{5}+1}{2} > 0, p_3 := \zeta^3 + \overline{\zeta}^3 = \frac{-\sqrt{5}+1}{2} < 0, \\
\mathrm{Gal}(\mathbb{Q}(p_1) : \mathbb{Q}) &= \{\mathrm{id}, \varphi\}, \varphi : p_1 \mapsto p_3 \mapsto p_1, \\
(x - p_1)(x - p_3) &= x^2 - x - 1, p_1 + p_3 = 1,\ p_1 p_3 = -1,\ p_1^2 = p_1 + 1.
\end{aligned}$$

(b) $m = 12$: $\Phi_{12}(t) = t^4 - t^2 + 1$,

$$\begin{aligned}
\mathbb{Z}[\zeta]^* &= \mathrm{Eiw}(\zeta) \cdot \mathbb{Z}[p_1]^* \cup (\zeta + 1) \cdot \mathrm{Eiw}(\zeta) \cdot \mathbb{Z}[p_1]^* \\
&= \mathrm{Eiw}(\zeta) \cdot \{(\zeta + 1)^k \,|\, k \in \mathbb{Z}\} \supset \{\zeta - 1, \zeta + 1\}, \\
\mathbb{Z}[p_1]^* &= \{\pm 1\} \times \{p_1^k \,|\, k \in \mathbb{Z}\} \supset \{p_1 - 2, p_1 + 2\}, \\
p_1 &= \sqrt{3} > 0, p_5 := \zeta^5 + \overline{\zeta}^5 = -\sqrt{3} < 0, \\
\mathrm{Gal}(\mathbb{Q}(p_1) : \mathbb{Q}) &= \{\mathrm{id}, \varphi\}, \varphi : p_1 \mapsto p_5 \mapsto p_1, \\
(x - p_1)(x - p_5) &= x^2 - 3, p_1 + p_5 = 0,\ p_1 p_5 = -3,\ p_1^2 = 3.
\end{aligned}$$

(c) $m = 14$: $\Phi_{14}(t) = t^6 - t^5 + t^4 - t^3 + t^2 - t + 1$,

$$\begin{aligned}
\mathbb{Z}[\zeta]^* &= \mathrm{Eiw}(\zeta) \cdot \mathbb{Z}[p_1]^* \supset \{\zeta - 1\}, \\
\mathbb{Z}[p_1]^* &= \{\pm 1\} \times \{p_1^{k_1} p_3^{k_3} \,|\, k_1, k_3 \in \mathbb{Z}\} \\
&\supset \{p_1 - 2, p_1 - 1, p_1, p_1 + 1\}, \\
p_1 &> 0,\ p_3 := \zeta^3 + \overline{\zeta}^3 > 0,\ p_5 := \zeta^5 + \overline{\zeta}^5 < 0, \\
\mathrm{Gal}(\mathbb{Q}(p_1) : \mathbb{Q}) &= \{\mathrm{id}, \varphi, \varphi^2\}, \varphi : p_1 \mapsto p_3 \mapsto p_5 \mapsto p_1, \\
(x - p_1)(x - p_3)(x - p_5) &= x^3 - x^2 - 2x + 1, p_1 + p_3 + p_5 = 1, \\
p_1 p_3 p_5 &= -1, p_1 p_3 = p_1 - 1, p_1^2 = -p_5 + 2.
\end{aligned}$$

(d) $m = 18$: $\Phi_{18}(t) = t^6 - t^3 + 1$,

$$\begin{aligned}
\mathbb{Z}[\zeta]^* &= \mathrm{Eiw}(\zeta) \cdot \mathbb{Z}[p_1]^* \supset \{\zeta - 1\},\\
\mathbb{Z}[p_1]^* &= \{\pm 1\} \times \{p_1^{k_1} p_5^{k_5} \mid k_1, k_5 \in \mathbb{Z}\}\\
&\supset \{p_1 - 2, p_1, p_1 + 1\},\\
p_1 &> 0,\ p_5 := \zeta^5 + \overline{\zeta}^5 < 0,\ p_7 := \zeta^7 + \overline{\zeta}^7 < 0,\\
\mathrm{Gal}(\mathbb{Q}(p_1) : \mathbb{Q}) &= \{\mathrm{id}, \varphi, \varphi^2\}, \varphi : p_1 \mapsto p_5 \mapsto p_7 \mapsto p_1,\\
(x - p_1)(x - p_5)(x - p_7) &= x^3 - 3x - 1, p_1 + p_5 + p_7 = 0,\\
p_1 p_5 p_7 &= 1, p_1 p_5 = -p_5 - 1, p_1^2 = -p_7 + 2.
\end{aligned}$$

Proof: That the index $[\mathbb{Z}[\zeta]^* : \mathrm{Eiw}(\zeta) \cdot \mathbb{Z}[p_1]^*]$ is 1 for $m \in \{10, 14, 18\}$ and 2 for $m = 12$, follows from [Wa97, theorem 4.12 and corollary 4.13]. That $\mathbb{Z}[p_1]^*$ is as stated, follows for $m \in \{10, 14, 18\}$ from [Wa97, theorem 8.2 and lemma 8.1 (a)]. For $m = 12$ [Wa97, §8.1] is not so useful, but there the proof of $\mathbb{Z}[p_1]^* = \{\pm 1\} \cdot \{p_1^k \mid k \in \mathbb{Z}\}$ is easy. Everything else is elementary. □

Part (b) of the following lemma applies with $\Lambda = Ml(f)$ and $\Lambda^{(1)} = \widetilde{B}_1 \oplus B_2$ (see the theorems 5.1 and 6.1) to most of the Milnor lattices in the sections 5 and 6. We will need (2.24).

Lemma 2.12. *(a) Let $p = \prod_{i \in I} \Phi_{m_i}$ be a product of cyclotomic polynomials. Then $p(1) \equiv 1(2)$ if and only if all $m_i \in \mathbb{Z}_{\geq 1} - \{2^k \mid k \in \mathbb{Z}_{\geq 0}\}$.*

(b) Let (Λ, L, M_h) be as above (we will not need L here, only M_h). Let $\Lambda^{(1)} \subset \Lambda$ be an M_h-invariant sublattice with $[\Lambda : \Lambda^{(1)}] = 2$. Write

$$\begin{aligned}
p_\Lambda &= p_1 \cdot p_2 \text{with } p_j = \prod_{m \in J_j} \Phi_m\\
\text{and} \quad & J_1 \subset \mathbb{Z}_{\geq 1} - \{2^k \mid k \in \mathbb{Z}_{\geq 0}\}, J_2 \subset \{2^k \mid k \in \mathbb{Z}_{\geq 0}\}.
\end{aligned}$$

Then $J_2 \neq \emptyset$, $p_2 \neq 1$, and

$$\Lambda_p = \Lambda_p^{(1)} \quad \text{for any } p \text{ with } p | p_1, \tag{2.24}$$

$$[\Lambda_p : \Lambda_p^{(1)}] = 2 \quad \text{for any } p \text{ with } p_2 | p. \tag{2.25}$$

Proof: (a) Observe $\Phi_{2^k}(t) = t^{2^{k-1}} + 1$ for $k \geq 1$ and

$$t^{2^k \cdot q} - 1 = (t^{2^k} - 1)(t^{2^k(q-1)} + t^{2^k(q-2)} + \ldots + t^{2^k} + 1). \tag{2.26}$$

For odd $q > 1$, the second factor has at $t = 1$ the odd value q. Therefore $\Phi_m(1) \equiv 1(2)$ for any m with $2^k | m | 2^k \cdot q$ and $2^k \neq m$ with q odd.

(b) For an arbitrary element $\gamma \in \Lambda - \Lambda^{(1)}$,

$$\Lambda - \Lambda^{(1)} = \gamma + \Lambda^{(1)}.$$

This set is M_h-invariant because $\Lambda^{(1)}$ is M_h-invariant. Thus for any $k \in \mathbb{Z}_{\geq 1}$ $M_h^k(\gamma) \in \Lambda - \Lambda^{(1)}$. By part (a) $p_1(1) \equiv 1(2)$. Thus $p_1(M_h)(\gamma) \in \Lambda - \Lambda^{(1)}$ and

$$p_1(M_h)(\Lambda - \Lambda^{(1)}) \subset \Lambda - \Lambda^{(1)}.$$

On the other hand

$$\begin{aligned}
p_1(M_h)(\Lambda_{p_1}) &= \{0\} \subset \Lambda^{(1)}, \text{thus } \Lambda_{p_1} \subset \Lambda^{(1)}, \text{thus (2.24)}.\\
p_1(M_h)(\Lambda) &\subset \Lambda_{p_2}, \text{thus } \Lambda_{p_2} \cap (\Lambda - \Lambda^{(1)}) \neq \emptyset, \text{thus (2.25)}.
\end{aligned}$$

□

3. Some Fuchsian groups

Notations 3.1. For any $m \in \mathbb{Z}_{\geq 3}$ define $\zeta := e^{2\pi i/m}$ and $p_1 := \zeta + \overline{\zeta}$. The letter ξ will denote in this section a primitive m-th unit root. An element of $\mathbb{Q}(\zeta)$ will be written as a or $a(\zeta)$. Then $a(\xi)$ is the image $\varphi(a)$ for $\varphi \in \mathrm{Gal}(\mathbb{Q}(\zeta):\mathbb{Q})$ with $\varphi(\zeta) = \xi$.

Any element $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{C})$ acts on $\mathbb{P}^1\mathbb{C}$ by the linear transformation $z \mapsto \frac{az+b}{cz+d}$, which is an automorphism of $\mathbb{P}^1\mathbb{C}$. The *limit set* $L(\Gamma) \subset \mathbb{P}^1\mathbb{C}$ of a subgroup $\Gamma \subset GL(2,\mathbb{C})$ is [Le64, III 1B]

$$\begin{aligned} L(\Gamma) \quad = \quad & \{z \in \mathbb{P}^1\mathbb{C} \,|\, \exists\, z_0 \in \mathbb{P}^1\mathbb{C} \text{ and } \exists \text{ a sequence of different} \\ & \text{elements } \gamma_i \in \Gamma \text{ with } \gamma_i(z_0) \to z\}. \end{aligned}$$

A subgroup $\Gamma \subset GL(2,\mathbb{C})$ and the induced subgroup of $PGL(2,\mathbb{C})$ are called *Fuchsian* if Γ maps a certain circle $C \subset \mathbb{P}^1\mathbb{C}$ to itself and $L(\Gamma) \subset C$. By a theorem of Poincaré [Le64, III 3I], a subgroup $\Gamma \subset GL(2,\mathbb{C})$ is Fuchsian if it maps a certain circle $C \subset \mathbb{P}^1\mathbb{C}$ to itself and is discrete in $GL(2,\mathbb{C})$.

In the sections 5 and 6 we will encounter Fuchsian groups which arise in the following way.

Theorem 3.2. *Let $m \in \mathbb{Z}_{\geq 3}$, $\zeta := e^{2\pi i/m}$, $p_1 := \zeta + \overline{\zeta}$, and $w = w(\zeta) \in \mathbb{Q}(\zeta)$ with*

$$w(\zeta) > 0 \qquad (\text{thus } w(\zeta) = w(\overline{\zeta}) \in \mathbb{Q}(p_1)), \tag{3.1}$$

$$w(\xi) < 0 \qquad \text{for any primitive } m\text{-th unit root } \xi \notin \{\zeta, \overline{\zeta}\}. \tag{3.2}$$

Then the matrix group

$$\Gamma := \{A \in GL(2,\mathbb{Z}[\zeta]) \,|\, \begin{pmatrix} -1 & 0 \\ 0 & w \end{pmatrix} = A^t \begin{pmatrix} -1 & 0 \\ 0 & w \end{pmatrix} \overline{A}\} \tag{3.3}$$

is an infinite Fuchsian group. It preserves the circle

$$C = \{z \in \mathbb{C} \,|\, |z|^2 = w\}. \tag{3.4}$$

The map

$$\begin{aligned} \{(a,c,\delta) \in \mathbb{Z}[\zeta]^2 \times \mathrm{Eiw}(\zeta) \,|\, |a|^2 - 1 = w \cdot |c|^2\} &\to \Gamma \\ (a,c,\delta) \mapsto A := \begin{pmatrix} a & w\cdot\overline{c}\cdot\delta \\ c & \overline{a}\cdot\delta \end{pmatrix} & \end{aligned} \tag{3.5}$$

is a bijection (here $\mathrm{Eiw}(\zeta) = \{\pm\zeta^k \,|\, k \in \mathbb{Z}\}$, see theorem 2.9 (a)).

Proof: The matrix $\begin{pmatrix} -1 & 0 \\ 0 & w \end{pmatrix}$ defines an indefinite hermitian form on $\mathbb{C}^2$. The isotropic lines are $\mathbb{C} \cdot \begin{pmatrix} z \\ 1 \end{pmatrix}$ with $z \in C$. Therefore any matrix $A \in \Gamma$ maps C to itself.

The matrix equation which defines Γ can be spelled out as follows,

$$\begin{aligned} \begin{pmatrix} -1 & 0 \\ 0 & w \end{pmatrix} &= \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & w \end{pmatrix} \begin{pmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{pmatrix} \\ &= \begin{pmatrix} -a\overline{a} + wc\overline{c} & -a\overline{b} + wc\overline{d} \\ -\overline{a}b + w\overline{c}d & -b\overline{b} + wd\overline{d} \end{pmatrix}. \end{aligned} \tag{3.6}$$

The determinant $\delta = \det A = ad - bc$ is in $\mathbb{Z}[\zeta]$ and has absolute value 1, so it is in $\mathrm{Eiw}(\zeta)$ by theorem 2.9 (a). The equations above give

$$\begin{aligned} \overline{a}\delta &= \overline{a}(ad - bc) = (wc\overline{c} + 1)d - (w\overline{c}d)c = d, \\ w\overline{c}\delta &= w\overline{c}(ad - bc) = (\overline{a}b)a - (a\overline{a} - 1)b = b. \end{aligned} \tag{3.7}$$

This yields the bijection (3.5).

The defining equation

$$|a(\zeta)|^2 - 1 = w(\zeta) \cdot |c(\zeta)|^2 \tag{3.8}$$

for the pairs $(a(\zeta), c(\zeta)) \in \mathbb{Z}[\zeta]^2$ on the left hand side of (3.5) is in the case $(a, c) \in \mathbb{Z}[p_1]^2$ and $w(\zeta) \in \mathbb{Z}[p_1]$ a *Pell equation.* We obtain the inequalities

$$\begin{array}{rcl} 0 & \leq & |c(\zeta)|^2 = w(\zeta)^{-1}(|a(\zeta)|^2 - 1), \\ |a(\zeta)| & \geq & 1 \end{array} \tag{3.9}$$

and

$$\begin{array}{rcl} 0 & \leq & |c(\xi)|^2 = (-w(\xi))^{-1}(1 - |a(\xi)|^2) < (-w(\xi))^{-1}, \\ |a(\xi)| & \leq & 1 \text{for any primitive } m\text{-th unit root } \xi \notin \{\zeta, \overline{\zeta}\}. \end{array} \tag{3.10}$$

Γ maps C to itself. Therefore by Poincaré's theorem, it is a Fuchsian group if it is a discrete matrix group. This holds if the set

$$P_1 := \{a \in \mathbb{Z}[\zeta] \mid \exists\, c \in \mathbb{Z}[\zeta] \text{ with } |a|^2 - 1 = w \cdot |c|^2\}$$

intersects each compact set $K \subset \mathbb{C}$ in a finite set.

The embedding $\underline{\sigma} : \mathbb{Q}(\zeta) \to \mathbb{R}^{\varphi(n)}$ from theorem 2.10 (a) maps $\mathbb{Z}[\zeta]$ to a lattice in $\mathbb{R}^{\varphi(n)}$. Because of (3.10), it maps $P_1 \cap K$ to a subset of

$$\underline{\sigma}(\mathbb{Z}[\zeta]) \cap \Big(K \times \{z \in \mathbb{C} \mid |z| \leq 1\}^{\varphi(n)/2-1}\Big).$$

This is a finite set. Therefore Γ is a Fuchsian group.

The next lemma shows that the set P_1 and the group Γ contain infinitely many elements. $\square$

Lemma 3.3. *Let $m \in \mathbb{Z}_{\geq 3}$, ζ, p_1 and $w \in \mathbb{Q}(p_1)$ be as in theorem 3.2. Then the set*

$$P_2 := \{(a, c) \in \mathbb{Z}[p_1] \mid a^2 - 1 = w \cdot c^2\} \tag{3.11}$$

contains infinitely many pairs. If $w \in \mathbb{Z}[p_1]$, then P_2 contains pairs (a, c) with $w|(a-1)$.

Proof: If $\widetilde{w} = w \cdot u^2$ for some $u \in \mathbb{Z}[p_1] - \{0\}$ then a pair $(a, \widetilde{c}) \in \mathbb{Z}[p_1]^2$ with $a^2 - 1 = \widetilde{w} \cdot \widetilde{c}^2$ induces a pair $(a, c) = (a, \widetilde{c} \cdot u)$ in P_2. Therefore we can suppose $w \in \mathbb{Z}[p_1]$.

We will now construct infinitely many units in $\mathbb{Z}[\sqrt{w}, p_1]^* - \mathbb{Z}[p_1]^*$ and from them infinitely many pairs (a, c) in P_2.

The algebraic number field $\mathbb{Q}(\sqrt{w}, p_1)$ has degree $\varphi(m)$ over $\mathbb{Q}$ and two real embeddings and $\varphi(m) - 2$ complex embeddings, because of (3.1) and (3.2). By Dirichlet's unit theorem (theorem 2.10 (b)), the unit group $\mathbb{Z}[\sqrt{w}, p_1]^*$ of the order $\mathbb{Z}[\sqrt{w}, p_1]$ in $\mathbb{Q}(\sqrt{w}, p_1)$ contains a free abelian group of rank $2 + \frac{\varphi(m)-2}{2} - 1 = \frac{\varphi(m)}{2}$.

The unit group $\mathbb{Z}[p_1]^*$ contains only a free abelian group of rank $\frac{\varphi(m)}{2} - 1$. Therefore infinitely many units $a_1 + \sqrt{w}c_1 \in \mathbb{Z}[\sqrt{w}, p_1]^*$ with $a_1 \neq 0$ and $c_1 \neq 0$ exist. Then also $a_1 - \sqrt{w}c_1$,

$$\begin{array}{rcl} (a_1 + \sqrt{w}c_1)^2 & = & (a_1^2 + wc_1^2) + \sqrt{w}(2a_1c_1) =: a_2 + \sqrt{w}c_2, \\ \text{and} h & := & (a_1 + \sqrt{w}c_1)(a_1 - \sqrt{w}c_1) = a_1^2 - wc_1^2 \end{array}$$

are units, h being in $\mathbb{Z}[p_1]^*$. Then

$$(a_3, c_3) := (\frac{a_2}{h}, \frac{c_2}{h}) \in P_2 \tag{3.12}$$

because

$$\begin{array}{rcl} a_3^2 - wc_3^2 & = & h^{-2}(a_2^2 - wc_2^2) = h^{-2}(a_2 + \sqrt{w}c_2)(a_2 - \sqrt{w}c_2) \\ & = & h^{-2}(a_1 + \sqrt{w}c_1)^2(a_1 - \sqrt{w}c_1)^2 = 1. \end{array}$$

Only finitely many units $a_1+\sqrt{w}c_1$ can give the same pair (a_3,c_3). Therefore there are infinitely many pairs (a_3,c_3) in P_2.

For the last statement, suppose that $(a_4,c_4)\in P_2$ with $c_4\neq 0$. Then the pair

$$(a_5,c_5):=(a_4^2+wc_4^2,2a_4c_4)$$

is also in P_2,

$$\begin{aligned} a_5^2-wc_5^2 &= (a_5+\sqrt{w}c_5)(a_5-\sqrt{w}c_5) \\ &= (a_4+\sqrt{w}c_4)^2(a_4-\sqrt{w}c_4)^2=(a_4^2-wc_4^2)^2=1. \end{aligned}$$

And it satisfies $w|(a_5-1)$ because of

$$a_5-1=a_4^2+wc_4^2-1=2wc_4^2.$$

□

Remarks 3.4. (i) The equation $a^2-1=wc^2$ is for $w\in\mathbb{Z}[p_1]$ a *Pell equation*. A generalization of lemma 3.3 is theorem 3 in [Sch06].

(ii) The notion of an *arithmetic Fuchsian group* is defined in [Sh71, ch 9.2]. The group Γ in theorem 3.2 is in fact an arithmetic Fuchsian group. This would follow immediately from [Ta75, theorem 2], if it were clear a priori that Γ is a Fuchsian group of the first kind, i.e. a Fuchsian group with limit set $L(\Gamma)=C$. It follows with some work from a comparison of the data in theorem 3.2 with the data in [Sh71, ch. 9.2].

(iii) The five triangle groups below in theorem 3.6 are arithmetic triangle groups. They are in the list in [Ta77, theorem 3] of all 85 arithmetic triangle groups.

(iv) Theorem 3.2 and lemma 3.3 will be used in the steps 2 and 4 in the proof of theorem 5.1 on the groups $G_{\mathbb{Z}}$ for the bimodal series.

Remarks 3.5. (i) The triangle groups below in theorem 3.6 will arise in theorem 6.1 as quotients of the groups $G_{\mathbb{Z}}$ for the quadrangle singularities.

(ii) There the first six of the eight elements $w(\zeta)$ in table (5.72) in the case $r=0$ will be used. So here $W_{1,0}$ and $S_{1,0}$ are seen as 0-th members of the series $W_{1,p}^\sharp$ and $S_{1,p}^\sharp$, not the series $W_{1,p}$ and $S_{1,p}$.

(iii) Using the notations and formulas from lemma 2.11, the first six of the eight elements $w(\zeta)$ in table (5.72) in the case $r=0$ can be written as follows. In the case $U_{1,0}$ we change from $m=9$ to $m=18$, so below $\zeta=e^{2\pi i/18}$ for $E_{3,0}$ and $U_{1,0}$.

$$\begin{aligned}
W_{1,0} &: w(\zeta)=\frac{6}{(2-p_1)p_1}=\frac{1}{(2-p_1)(2+p_1)}\cdot 2p_1(p_1+2). \\
S_{1,0} &: w(\zeta)=\frac{-2}{(-p_3)(-p_3-1)}=1\cdot 2p_1^3. \\
U_{1,0} &: w(\zeta)=\frac{-3}{(2+p_7)(1-p_1)}=1\cdot p_1(p_1+2). \\
E_{3,0} &: w(\zeta)=\frac{3(2-p_1)}{(p_1+2)(p_1-1)}=(2-p_1)^2\cdot p_1(p_1+2). \\
Z_{1,0} &: w(\zeta)=\frac{1}{-p_5}=1\cdot(-p_5)^{-1}=1\cdot(p_1-1). \\
Q_{2,0} &: w(\zeta)=\frac{2-p_1}{p_1+1}=(2-p_1)\cdot\frac{1}{p_1+1}. \qquad (3.13)
\end{aligned}$$

(iv) In theorem 3.2 one can replace w by $\widetilde{w} := w \cdot u\overline{u}$ for any $u \in \mathbb{Z}[\zeta]^*$. The group Γ for w and the group $\widetilde{\Gamma}$ for $\widetilde{w}$ are isomorphic, and the triples in (3.5) are related by

$$(\widetilde{a}, \widetilde{c}, \widetilde{\delta}) = (a, c \cdot u^{-1}, \delta).$$

We can choose u such that $\widetilde{w}$ is simpler to work with than w. In the products for w in (iii), the left terms are of the form $u\overline{u}$ for a suitable unit $u \in \mathbb{Z}[\zeta]^*$. The right terms are $\widetilde{w}$. We will work with the terms $\widetilde{w}$ in theorem 3.6.

Theorem 3.6. *The image in $PGL(2,\mathbb{C})$ of the group Γ in theorem 3.2 for the following values of m and w*

$$\begin{array}{c|l|l|l|l|l} & W_{1,0} & S_{1,0} & E_{3,0}\ \&\ U_{1,0} & Z_{1,0} & Q_{2,0} \\ m & 12 & 10 & 18 & 14 & 12 \\ w & 2p_1(p_1+2) & 2p_1^3 & p_1(p_1+2) & (-p_5)^{-1} & (p_1+1)^{-1} \end{array} \tag{3.14}$$

is a Schwarzian triangle group of the following type:

$$\begin{array}{l|l|l|l|l} W_{1,0} & S_{1,0} & E_{3,0}\ \&\ U_{1,0} & Z_{1,0} & Q_{2,0} \\ (2,12,12) & (2,10,10) & (2,3,18) & (2,3,14) & (2,3,12) \end{array} \tag{3.15}$$

Proof: The proof has three steps. In step 1, we will present two matrices A_1 and A_2 in Γ whose images in $PGL(2,\mathbb{C})$ are elliptic and generate in each case a Schwarzian triangle group of the claimed type. We will prove this. In step 2, we will show that no matrix in Γ is closer to A_1 than A_2. This will be used in step 3 to prove that the images in $PGL(2,\mathbb{C})$ of A_1 and A_2 generate the image of Γ in $PGL(2,\mathbb{C})$. The steps 1 and 3 together give theorem 3.6.

Step 1: One checks easily with (3.5) that the following matrices A_1 and A_2 are in Γ.

$$A_1 = \begin{pmatrix} \zeta & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{for all 5 cases.} \tag{3.16}$$

$$\begin{aligned}
&W_{1,0}: && A_2 = \begin{pmatrix} p_1+2 & -2p_1(p_1+2) \\ 1 & -(p_1+2) \end{pmatrix}, \det A_2 = -1, \\
&S_{1,0}: && A_2 = \begin{pmatrix} (\zeta+1)p_1 & -2p_1^3\zeta \\ 1 & -(\zeta+1)p_1 \end{pmatrix}, \det A_2 = -\zeta, \\
&E_{3,0}\ \&\ U_{1,0}: && A_2 = \begin{pmatrix} p_1+1 & -p_1(p_1+2) \\ 1 & -(p_1+1) \end{pmatrix}, \det A_2 = -1, \\
&Z_{1,0}: && A_2 = p_1(1-\zeta^3) \cdot \begin{pmatrix} 1 & -(-p_5)^{-1} \\ 1 & -1 \end{pmatrix}, \det A_2 = \zeta^3, \\
&Q_{2,0}: && A_2 = \begin{pmatrix} \zeta+1 & -\zeta \\ p_1+1 & -(\zeta+1) \end{pmatrix}, \det A_2 = -\zeta.
\end{aligned} \tag{3.17}$$

A matrix $A \in GL(2,\mathbb{C})$ is elliptic if its eigenvalues λ_1 and λ_2 satisfy $\frac{\lambda_2}{\lambda_1} \in S^1$. Let $\begin{pmatrix} z_j \\ 1 \end{pmatrix}$ be an eigenvector with eigenvalue λ_j for $j = 1,2$ (possibly $z_1 = 0$ and $z_2 = \infty$). Then the linear transformation of A is a rotation around the fixed point z_1 with angle $\alpha(A) = \arg \frac{\lambda_2}{\lambda_1}$. For $A \in \Gamma$ elliptic we number the eigenvalues λ_1, λ_2 such that $|z_1| < |z_2|$, so then $|z_1|^2 < w$ and z_1 is in the interior of the circle C. One sees in all 5 cases

$$\lambda_1(A_1) = 1, \lambda_2(A_1) = \zeta, \alpha(A_1) = \frac{2\pi}{m}, \tag{3.18}$$

$$\mathrm{tr}(A_2) = 0, \alpha(A_2) = \pi. \tag{3.19}$$

The following table lists for the product A_1A_2 the eigenvalues λ_1, λ_2 and the angle $\alpha(A_1A_2)$.

$$\begin{array}{llll} & \lambda_1 & \lambda_2 & \alpha \\ W_{1,0} & \zeta^4 & \zeta^3 & \frac{-2\pi}{12} \\ S_{1,0} & \zeta^4 & \zeta^3 & \frac{-2\pi}{10} \\ E_{3,0} \ \&\ U_{1,0} & \zeta^8 & \zeta^2 & \frac{-2\pi}{3} \\ Z_{1,0} & e^{2\pi i/6}\zeta^2 & e^{-2\pi i/6}\zeta^2 & \frac{-2\pi}{3} \\ Q_{2,0} & \zeta^6 & \zeta^2 & \frac{-2\pi}{3} \end{array} \tag{3.20}$$

Therefore the images of A_1 and A_2 in $PGL(2,\mathbb{C})$ generate a Schwarzian triangle group of the type in table (3.15) [Le64, VII 1G].

Step 2: Write $A_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$ and write $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for any $A \in \Gamma$.

Claim 1: Any $A \in \Gamma$ with $c \neq 0$ satisfies $|a| \geq |a_2|$.

The proof consists in making the proof of theorem 3.2 more constructive.

First we look for candidates $f \in \mathbb{Z}[p_1]$ of $|a|^2$ which are compatible with the inequalities (3.9) and (3.10) and which satisfy $f < |a_2|^2$. Then we will show that these candidates are not compatible with the equality $|a|^2 = 1 + w \cdot |c|^2$.

Denote by

$$\underline{\sigma}^{\mathbb{R}} = (\sigma_1^{\mathbb{R}}, \ldots, \sigma_{\varphi(m)/2}^{\mathbb{R}}) : \mathbb{Q}(p_1) \to \mathbb{R}^{\varphi(m)/2}$$

the tuple of the embeddings $\sigma_j^{\mathbb{R}} : \mathbb{Q}(p_1) \to \mathbb{R}$. Then $\underline{\sigma}^{\mathbb{R}}(\mathbb{Z}[p_1])$ is a $\mathbb{Z}$-lattice in $\mathbb{R}^{\varphi(m)/2}$. The candidates are the numbers $f = f(p_1)$ in $\mathbb{Z}[p_1]$ with

$$\underline{\sigma}^{\mathbb{R}}(f) \quad \in \quad]1, |a_2|^2[\; \times \;]0,1[^{\varphi(m)/2-1}. \tag{3.21}$$

This follows from the inequalities (3.9) and (3.10). With sufficient numerical precision of the numbers p_j in lemma 2.11, it is easy to find these candidates. They are as follows.

$$\begin{array}{rcl} W_{1,0} & : & f(p_1) = \alpha \cdot 1 + \beta \cdot p_1, (\alpha, \beta) \in \{(2,1),(4,2),(6,3)\}. \\ S_{1,0} & : & f(p_1) = \alpha \cdot 1 + \beta \cdot p_1, (\alpha, \beta) \in \{(2,2),(2,3)\}. \\ E_{3,0} & \& & U_{1,0} \ : \ \emptyset. \\ Z_{1,0} & : & \emptyset. \\ Q_{2,0} & : & \emptyset. \end{array}$$

All these candidates will be excluded with the help of the condition

$$\mathrm{Norm}(|a|^2 - 1) = \mathrm{Norm}(w \cdot |c|^2) = \mathrm{Norm}(w) \cdot \mathrm{Norm}(|b|^2).$$

Here the norm is the norm in $\mathbb{Q}(p_1)$ and $\mathbb{Z}[p_1]$ with values in $\mathbb{Q}$ respectively $\mathbb{Z}$.

The case $W_{1,0}$: $\mathrm{Norm}(w) = -12$, $\mathrm{Norm}(1+p_1) = -2$, $\mathrm{Norm}(3+2p_1) = -3$, $\mathrm{Norm}(5+3p_1) = -2$.

The case $S_{1,0}$: $\mathrm{Norm}(w) = -4, \mathrm{Norm}(1+2p_1) = -1, \mathrm{Norm}(1+3p_1) = -5$.

Step 3: It is sufficient to show the following claim 2.

Claim 2: For any matrix $A_3 \in \Gamma$ with $c_3 \neq 0$, a number $k \in \mathbb{Z}$ exists such that the product

$$A_4 := A_3 \cdot A_1^{-k} A_2 A_1^k = \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix} \begin{pmatrix} a_2 & \zeta^{-k} b_2 \\ \zeta^k c_2 & d_2 \end{pmatrix} \tag{3.22}$$

satisfies

$$|c_4| < |c_3|, \qquad \text{here } c_4 = c_3 a_2 + \zeta^k d_3 c_2. \tag{3.23}$$

We can choose $k \in \mathbb{Z}$ such that

$$\beta := |\arg(c_3 a_2) - \arg(-\zeta^k d_3 c_2)| \leq \frac{\pi}{m}. \tag{3.24}$$

Observe

$$\frac{|\zeta^k d_3 c_2|^2}{|c_3 a_2|^2} = \frac{|a_3|^2 \frac{|a_2|^2-1}{w(\zeta)}}{\frac{|a_3|^2-1}{w(\zeta)}|a_2|^2} = \frac{1-|a_2|^{-2}}{1-|a_3|^{-2}}. \tag{3.25}$$

The trivial inequality $1-|a_3|^{-2} < 1$ and the inequality $|a_3| \geq |a_2|$ from step 2 give the inequalities

$$\left(1-|a_2|^{-2}\right)|c_3 a_2|^2 < |\zeta^k d_3 c_2|^2 \leq |c_3 a_2|^2. \tag{3.26}$$

Observe also

$$\sqrt{1-|a_2|^{-2}} < \cos\frac{\pi}{m}. \tag{3.27}$$

Therefore

$$\begin{aligned} |c_4| &= |c_3 a_2|^2 (\sin\beta)^2 + (|c_3 a_2|\cos\beta - |d_3 c_2|)^2 \\ &< |c_3 a_2|^2 (\sin\frac{\pi}{m})^2 + \left(1-\sqrt{1-|a_2|^{-2}}\right)^2 \cdot |c_3 a_2|^2 \\ &= |c_3|^2 \cdot |a_2|^2 \left((\sin\frac{\pi}{m})^2 + \left(1-\sqrt{1-|a_2|^{-2}}\right)^2\right) \\ &\overset{(*)}{<} |c_3|^2. \end{aligned} \tag{3.28}$$

$\overset{(*)}{<}$ follows in all 5 cases by an explicit calculation. □

4. Review on the topology of singularities

In this section, we recall some classical facts about the topology of singularities, and we fix some notations.

An *isolated hypersurface singularity* (short: *singularity*) is a holomorphic function germ $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ with an isolated singularity at 0. Its *Jacobi ideal* is

$$J(f) := \left(\frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_n}\right) \subset \mathcal{O}_{\mathbb{C}^{n+1},0}.$$

Its *Jacobi algebra* is $\mathcal{O}_{\mathbb{C}^{n+1},0}/J(f)$. Its *Milnor number* $\mu := \dim \mathcal{O}_{\mathbb{C}^{n+1},0}/J(f)$ is finite. For the following notions and facts compare [AGV88] and [Eb07]. A *good representative* of f has to be defined with some care [Mi68][AGV88][Eb07]. It is $f : X \to \Delta$ with $\Delta = \{\tau \in \mathbb{C} \mid |\tau| < \delta\}$ a small disk around 0 and $X = \{x \in \mathbb{C}^{n+1} \mid |x| < \varepsilon\} \cap f^{-1}(\Delta)$ for some sufficiently small $\varepsilon > 0$ (first choose ε, then δ). Then $f : X' \to \Delta'$ with $X' = X - f^{-1}(0)$ and $\Delta' = \Delta - \{0\}$ is a locally trivial C^∞-fibration, the *Milnor fibration*. Each fiber has the homotopy type of a bouquet of μ n-spheres [Mi68].

Therefore the (reduced for $n = 0$) middle homology groups are $H_n^{(red)}(f^{-1}(\tau), \mathbb{Z}) \cong \mathbb{Z}^\mu$ for $\tau \in \Delta'$. Each comes equipped with an intersection form I, which is a datum of one fiber, a monodromy M_h and a Seifert form L, which come from the Milnor fibration, see [AGV88, I.2.3] for their definitions. M_h is a quasiunipotent automorphism, I and L are bilinear forms with values in $\mathbb{Z}$, I is $(-1)^n$-symmetric, and L is unimodular. L determines M_h and I because of the formulas [AGV88, I.2.3]

$$L(M_h a, b) = (-1)^{n+1} L(b, a), \tag{4.1}$$

$$I(a, b) = -L(a, b) + (-1)^{n+1} L(b, a) = L((M - \mathrm{id})a, b). \tag{4.2}$$

(4.2) tells especially that $\ker(M_h - \mathrm{id})$ is the radical of I and that L is $(-1)^{n+1}$-symmetric on this radical. The semisimple part of M_h is called M_s, the unipotent part M_u, the nilpotent part $N = \log M_u$.

The Milnor lattices $H_n(f^{-1}(\tau), \mathbb{Z})$ for all Milnor fibrations $f : X' \to \Delta'$ and then all

$$\tau \in \mathbb{R}_{>0} \cap \Delta'$$

are canonically isomorphic, and the isomorphisms respect M_h, I and L. This follows from Lemma 2.2 in [LR73]. These lattices are identified and called *Milnor lattice* $Ml(f)$.

The group $G_\mathbb{Z}$ is

$$G_\mathbb{Z} = G_\mathbb{Z}(f) := \mathrm{Aut}(Ml(f), L) = \mathrm{Aut}(Ml(f), M_h, I, L), \tag{4.3}$$

the second equality is true because L determines M_h and I. A good control of this group for the bimodal series and the quadrangle singularities will be crucial in this paper. It is the task of the sections 5 and 6.

The Milnor lattice comes equipped with a set $\mathcal{B}$ of *distinguished bases*, certain tuples $\underline{\delta} = (\delta_1, \dots, \delta_\mu)$ of $\mathbb{Z}$-bases of the Milnor lattice. Each one is defined with a generic deformation of f which has μ A_1-singularities which have all different critical values. One chooses a *distinguished system of paths* in Δ from the critical values to $\delta \in \partial\Delta$ and pushes vanishing cycles along these paths to $H_n(f^{-1}(\delta), \mathbb{Z}) = Ml(f)$. See [AGV88] or [Eb07] for details. In all cases except the simple singularities, the set $\mathcal{B}$ is infinite. Each distinguished basis determines the monodromy by the formula

$$M_h = s_{\delta_1} \circ \ldots \circ s_{\delta_\mu} \tag{4.4}$$

where

$$\begin{aligned} s_\delta \quad &: \quad Ml(f) \to Ml(f), \\ s_\delta(b) \quad &:= \quad b - (-1)^{n(n+1)/2} \cdot I(\delta, b) \cdot \delta, \end{aligned} \tag{4.5}$$

is the *Picard-Lefschetz transformation* of a vanishing cycle δ, a reflection for even n and a symplectic transvection for odd n.

The matrix of the Seifert form with respect to a distinguished basis is lower triangular with $(-1)^{(n+1)(n+2)/2}$ on the diagonal. This motivates two definitions, the *normalized Seifert form*

$$L^{hnor} \quad := \quad (-1)^{(n+1)(n+2)/2} \cdot L, \tag{4.6}$$

and the *Stokes matrix* S of the distinguished basis with

$$S \quad := \quad (-1)^{(n+1)(n+2)/2} \cdot L(\underline{\delta}^t, \underline{\delta})^t = L^{hnor}(\underline{\delta}^t, \underline{\delta})^t. \tag{4.7}$$

S is an upper triangular matrix in $GL(\mu, \mathbb{Z})$ with 1's on the diagonal.

The *Coxeter-Dynkin diagram* (short: CDD) of a distinguished basis encodes S in a geometric way. It has μ vertices which are numbered from 1 to μ. Between two vertices i and j with $i < j$ one draws

no edge	if $S_{ij} = 0$,
$\lvert S_{ij}\rvert$ edges	if $S_{ij} < 0$,
S_{ij} dotted edges	if $S_{ij} > 0$.

Coxeter-Dynkin diagrams for the 8 bimodal series will be given in section 5, following [Eb81].

A result of Thom and Sebastiani compares the Milnor lattices and monodromies of the singularities $f = f(x_0, \dots, x_n), g = g(y_0, \dots, y_m)$ and $f + g = f(x_0, \dots, x_n) + g(x_{n+1}, \dots, x_{m+n+1})$. There are extensions by Deligne for the Seifert form and by Gabrielov for distinguished bases.

All results are in [AGV88, I.2.7]. They are restated here. There is a canonical isomorphism

$$\begin{aligned}\Phi : Ml(f+g) &\xrightarrow{\cong} Ml(f)\otimes Ml(g), &(4.8)\\ \text{with } M_h(f+g) &\cong M_h(f)\otimes M_h(g) &(4.9)\\ \text{and } L^{hnor}(f+g) &\cong L^{hnor}(f)\otimes L^{hnor}(g). &(4.10)\end{aligned}$$

If $\underline{\delta} = (\delta_1,\dots,\delta_{\mu(f)})$ and $\underline{\gamma} = (\gamma_1,\dots,\gamma_{\mu(g)})$ are distinguished bases of f and g with Stokes matrices $S(f)$ and $S(g)$, then

$$\Phi^{-1}(\delta_1\otimes\gamma_1,\dots,\delta_1\otimes\gamma_{\mu(g)},\delta_2\otimes\gamma_1,\dots,\delta_2\otimes\gamma_{\mu(g)},\dots,\delta_{\mu(f)}\otimes\gamma_1,\dots,\delta_{\mu(f)}\otimes\gamma_{\mu(g)})$$

is a distinguished basis of $Ml(f+g)$, that means, one takes the vanishing cycles $\Phi^{-1}(\delta_i\otimes\gamma_j)$ in the lexicographic order. Then by (4.7) and (4.10), the matrix

$$S(f+g) = S(f)\otimes S(g) \tag{4.11}$$

(where the tensor product is defined so that it fits to the lexicographic order) is the Stokes matrix of this distinguished basis.

In the special case $g = x_{n+1}^2$, the function germ $f+g = f(x_0,\dots,x_n)+x_{n+1}^2 \in \mathcal{O}_{\mathbb{C}^{n+2},0}$ is called *stabilization* or *suspension* of f. As there are only two isomorphisms $Ml(x_{n+1}^2)\to\mathbb{Z}$, and they differ by a sign, there are two equally canonical isomorphisms $Ml(f)\to Ml(f+x_{n+1}^2)$, and they differ just by a sign. Therefore automorphisms and bilinear forms on $Ml(f)$ can be identified with automorphisms and bilinear forms on $Ml(f+x_{n+1}^2)$. In this sense [AGV88, I.2.7]

$$\begin{aligned}L^{hnor}(f+x_{n+1}^2) &= L^{hnor}(f), &(4.12)\\ M(f+x_{n+1}^2) &= -M(f), &(4.13)\\ G_{\mathbb{Z}}(f+x_{n+1}^2) &= G_{\mathbb{Z}}(f). &(4.14)\end{aligned}$$

The image in $Ml(f+x_{n+1}^2)$ of a distinguished basis in $Ml(f)$ under either of the both isomorphisms $Ml(f)\to Ml(f+x_{n+1}^2)$ is again a distinguished basis, and it has the same Stokes matrix.

Denote by $H^\infty_{\mathbb{C}}$ the μ-dimensional vector space of global flat multi-valued sections in the flat cohomology bundle $\bigcup_{\tau\in\Delta'} H^n(f^{-1}(\tau),\mathbb{C})$ (reduced cohomology for $n=0$). It comes equipped with a $\mathbb{Z}$-lattice $H^\infty_{\mathbb{Z}}$, a real subspace $H^\infty_{\mathbb{R}}$, a monodromy which is also denoted by M_h, and the dual L^{nor} of the normalized Seifert form L^{hnor}. It is a unimodular form on $H^\infty_{\mathbb{Z}}$, and the analogue of (4.1),

$$L^{nor}(M_h a,b) = (-1)^{n+1}L^{nor}(b,a) \text{for } a,b\in H^\infty_{\mathbb{Z}}, \tag{4.15}$$

holds.

We apply the notations 2.1 (a) to $Ml(f)$ and to $H^\infty_{\mathbb{Z}}$ and extend them slightly:

$$\begin{aligned}Ml(f)_\lambda &:= \ker(M_h-\lambda\,\mathrm{id})^\mu : Ml(f)_{\mathbb{C}}\to Ml(f)_{\mathbb{C}}, &(4.16)\\ Ml(f)_{\neq 1} &:= \bigoplus_{\lambda\neq 1} Ml(f)_\lambda, Ml(f)_{\neq -1} := \bigoplus_{\lambda\neq -1} Ml(f)_\lambda,\\ Ml(f)_p &:= \bigoplus_{\lambda:\, p(\lambda)=0} Ml(f)_\lambda, Ml(f)_{p,\mathbb{Z}} := Ml(f)_p\cap Ml(f).\end{aligned}$$

H^∞_λ, $H^\infty_{\neq 1}$, $H^\infty_{\neq -1}$, H^∞_p and $H^\infty_{p,\mathbb{Z}}$ are defined analogously.

There are a natural Hodge filtration $F^\bullet_{St}$ on $H^\infty_{\mathbb{C}}$ and a weight filtration $W_\bullet$ on $H^\infty_{\mathbb{Q}}$ such that $(H^\infty_{\neq 1}, H^\infty_{\neq 1,\mathbb{Z}}, F^\bullet_{St}, W_\bullet, -N, S)$ is a polarized mixed Hodge structure of weight n and $(H^\infty_1, H^\infty_{1,\mathbb{Z}}, F^\bullet_{St}, W_\bullet, -N, S)$ is a polarized mixed Hodge structure of weight $n+1$ [He02, Theorem 10.30].

In the case of a singularity with semisimple monodromy, so $N=0$, the weight filtrations become trivial, and the polarized mixed Hodge structures are polarized pure Hodge structures. This holds for all bimodal singularities. Therefore we do not care here about the weight filtration. We will define the Hodge filtration using the Brieskorn lattice in theorem 7.7 (following Varchenko, Scherk&Steenbrink and M. Saito).

The pure Hodge structure of weight n on $H^\infty_{\neq 1}$ for any singularity with semisimple monodromy has the following properties. The Hodge filtration is M_s-invariant and satisfies

$$\begin{aligned} H^\infty_\lambda &= \bigoplus_{p\in\mathbb{Z}} H^{p,n-p}_\lambda \text{for } \lambda\neq 1, \qquad (4.17)\\ \text{where}\ \ H^{p,n-p}_\lambda &:= F^pH^\infty_\lambda \cap \overline{F^{n-p}H^\infty_{\overline{\lambda}}}, (\Rightarrow H^{n-p,p}_{\overline{\lambda}} = \overline{H^{p,n-p}_\lambda},)\\ \text{equivalently}\ \ H^\infty_\lambda &= F^pH^\infty_\lambda \oplus \overline{F^{n+1-p}H^\infty_{\overline{\lambda}}}. \end{aligned}$$

The polarizing form carries an isotropy and a positivity condition,

$$S(H^{p,n-p}_\lambda, H^{q,n-q}_{\overline{\lambda}}) = 0 \text{if } p+q\neq 0, \qquad (4.18)$$

$$i^{p-(n-p)}\cdot S(a,\overline{a}) > 0 \quad \text{for } a\in H^{p,n-p}_\lambda - \{0\}. \qquad (4.19)$$

The pure Hodge structure of weight $n+1$ on H^∞_1 has analogous properties, with n replaced by $n+1$.

The polarizing form $S: H^\infty_\mathbb{Q}\times H^\infty_\mathbb{Q}\to\mathbb{Q}$ is defined by [He02, 10.6].

$$S(a,b) := -L^{nor}(a,\nu b) \qquad (4.20)$$

where $\nu : H^\infty_\mathbb{Q}\to H^\infty_\mathbb{Q}$ is the M_h-invariant automorphism

$$\nu := \begin{cases} \frac{1}{M_h-\mathrm{id}} & \text{on } H^\infty_{\neq 1},\\ \frac{-N}{M_h-\mathrm{id}} & \text{on } H^\infty_1, \end{cases} \qquad (4.21)$$

S is nondegenerate and M_h-invariant. It is $(-1)^n$-symmetric on $H^\infty_{\neq 1}$ and $(-1)^{n+1}$-symmetric on H^∞_1. The restriction to $H^\infty_{\neq 1}$ is $(-1)^{n(n+1)/2}\cdot I^\vee$, where $I^\vee$ on $H^\infty_{\neq 1}$ is dual to I (which is nondegenerate on $Ml(f)_{\neq 1}$).

5. The group $G_\mathbb{Z}$ for the bimodal series singularities

The normal forms from [AGV85, §13] for the eight bimodal series will be listed below in section 9. The following table gives their names, the Milnor numbers, certain polynomials b_1, b_2 or, in the case of the series $Z_{1,p}$, polynomials b_1, b_2, b_3 such that b_1b_2 respectively $b_1b_2b_3$ is the characteristic polynomial of the surface singularities, and two important numbers m and r_I. In the series $p\in\mathbb{Z}_{\geq 1}$.

(5.1)

series	μ	b_1	b_2	b_3	m	r_I
$W^\sharp_{1,p}$	$15+p$	Φ_{12}	$(t^{12+p}-1)/\Phi_1$	–	12	1
$S^\sharp_{1,p}$	$14+p$	$\Phi_{10}\Phi_2$	$(t^{10+p}-1)/\Phi_1$	–	10	1
$U_{1,p}$	$14+p$	Φ_9	$(t^{9+p}-1)/\Phi_1$	–	9	1
$E_{3,p}$	$16+p$	$\Phi_{18}\Phi_2$	$t^{9+p}+1$	–	18	2
$Z_{1,p}$	$15+p$	$\Phi_{14}\Phi_2$	$t^{7+p}+1$	Φ_2	14	2
$Q_{2,p}$	$14+p$	$\Phi_{12}\Phi_4\Phi_3$	$t^{6+p}+1$	–	12	2
$W_{1,p}$	$15+p$	$\Phi_{12}\Phi_6\Phi_3\Phi_2$	$t^{6+p}+1$	–	12	2
$S_{1,p}$	$14+p$	$\Phi_{10}\Phi_5\Phi_2$	$t^{5+p}+1$	–	10	2

The following theorem on the group $G_{\mathbb{Z}}$ will be proved in two steps. Directly after the theorem, the arguments and properties which hold for all eight series will be given. Then in eight subsections, one for each series, the corresponding objects will be made explicit and some specific details will be given. For each series, denote $\zeta := e^{2\pi i/m} \in S^1 \subset \mathbb{C}$.

Theorem 5.1. *For any surface singularity f in any of the eight bimodal series, the following holds.*

(a) (See definition 2.3 for the notion Orlik block*) For all series except $Z_{1,p}$, there are Orlik blocks $B_1, B_2 \subset Ml(f)$, and for the series $Z_{1,p}$, there are Orlik blocks $B_1, B_2, B_3 \subset Ml(f)$ with the following properties. The characteristic polynomial p_{B_j} of the monodromy on B_j is b_j. The sum $\sum_{j\geq 1} B_j$ is a direct sum $\bigoplus_{j\geq 1} B_j$, and it is a sublattice of $Ml(f)$ of full rank μ and of index r_I. Define*

$$\widetilde{B}_1 := \begin{cases} B_1 & \text{for all series except } Z_{1,p}, \\ B_1 \oplus B_3 & \text{for the series } Z_{1,p}. \end{cases} \tag{5.2}$$

Then

$$L(\widetilde{B}_1, B_2) = 0 = L(B_2, \widetilde{B}_1) \qquad \text{for all series,} \tag{5.3}$$

$$G_{\mathbb{Z}} = \mathrm{Aut}(\bigoplus_{j\geq 1} B_j, L) \qquad \text{for all series except } S_{1,10}. \tag{5.4}$$

In the case $S_{1,10}$, a substitute for (5.4) is

$$g \in G_{\mathbb{Z}} \text{ with } g((B_1)_{\Phi_{10}}) = (B_1)_{\Phi_{10}} \Rightarrow g(B_j) = B_j \text{ for } j = 1, 2. \tag{5.5}$$

(b) $\Phi_m \not| b_2 \iff m \not| p$. In that case

$$G_{\mathbb{Z}} = \{(\pm M_h^{k_1}|_{\widetilde{B}_1}) \times (\pm M_h^{k_2}|_{B_2}) \,|\, k_1, k_2 \in \mathbb{Z}\}. \tag{5.6}$$

(c) In the case of the subseries with $m|p$, the eigenspace $Ml(f)_\zeta \subset Ml(f)_{\mathbb{C}}$ is 2-dimensional. The hermitian form h_ζ on it from lemma 2.2 (a) with $h_\zeta(a, b) := \sqrt{-\zeta} \cdot L(a, \overline{b})$ for $a, b \in Ml(f)_\zeta$ is nondegenerate and indefinite, so $\mathbb{P}(Ml(f)_\zeta) \cong \mathbb{P}^1$ contains a half-plane

$$\mathcal{H}_\zeta := \{\mathbb{C} \cdot a \,|\, a \in Ml(f)_\zeta \text{ with } h_\zeta(a, a) < 0\} \subset \mathbb{P}(Ml(f)_\zeta). \tag{5.7}$$

Therefore the group $\mathrm{Aut}(Ml(f)_\zeta, h_\zeta)/S^1 \cdot \mathrm{id}$ is isomorphic to $PSL(2, \mathbb{R})$. The homomorphism

$$\Psi : G_{\mathbb{Z}} \to \mathrm{Aut}(Ml(f)_\zeta, h_\zeta)/S^1 \cdot \mathrm{id}, g \mapsto g|_{Ml(f)_\zeta} \bmod S^1 \cdot \mathrm{id}, \tag{5.8}$$

is well-defined. $\Psi(G_{\mathbb{Z}})$ is an infinite Fuchsian group acting on the half-plane $\mathcal{H}_\zeta$. And

$$\ker \Psi = \{\pm M_h^k \,|\, k \in \mathbb{Z}\}. \tag{5.9}$$

Proof: Here we explain the common arguments of the proof, which hold for all eight series. We will announce definitions and properties of several objects. In the following eight subsections, one for each series, the objects will be defined, and their properties will be shown.

(a) For each of the eight series of surface singularities, a distinguished basis $e_1, \dots, e_\mu$ with the Coxeter-Dynkin diagram in the corresponding figure will be given in the subsections 5.1 to 5.8. The distinguished basis is the one in [Eb81, Tabelle 6 & Abb. 16], with a small change in the cases $W_{1,1}$ and $S_{1,1}$. They are exceptional in [Eb81]. With the actions of the braids $\alpha_1, \dots, \alpha_{\mu-1}$ (see [Eb07, 5.7] for these braids and their actions) and a sign change, we arrive at a new numbering of the same unnumbered diagram, such that $W_{1,1}$ and $S_{1,1}$ are no longer exceptional (i.e. the top vertex has the number $p + q + r + 3$ in the notation of [Eb81, Abb. 16] even for $W_{1,1}$ and $S_{1,1}$). We thank Wolfgang Ebeling for the explanation how to arrive at this numbering.

Recall that for a surface singularity (then $n = 2$) the reflection along a vanishing cycle δ is

$$s_\delta(b) = b + I(\delta, b) \cdot \delta \qquad \text{for any } b \in Ml(f).$$

The Coxeter-Dynkin diagram has between the vertices i and j with $i < j$ no edge if $S_{ij} = 0$, $|S_{ij}|$ edges if $S_{ij} < 0$ and S_{ij} dotted edges if $S_{ij} > 0$. Here for $i < j$

$$\begin{array}{lll} I(e_i, e_j) = I(e_j, e_i) & = -S_{ij}, & I(e_i, e_i) = -2, \\ L(e_i, e_j) = 0, & L(e_j, e_i) = S_{ij}, & L(e_i, e_i) = 1. \end{array} \tag{5.10}$$

The monodromy can be calculated fairly efficiently by hand (one should write down some intermediate steps) with the formula

$$M_h = s_{e_1} \circ \ldots \circ s_{e_\mu}. \tag{5.11}$$

The cyclic sublattices $B_j \subset Ml(f)$ are chosen by choosing the generating lattice vectors β_j with

$$B_j := \sum_{i \geq 0} \mathbb{Z} \cdot M_h^i(\beta_j). \tag{5.12}$$

The following table gives them.

(5.13)

series	β_1	β_2	β_3
$W^\sharp_{1,p}$	e_3	e_8	$-$
$S^\sharp_{1,p}$	e_8	e_9	$-$
$U_{1,p}$	e_8	e_{10}	$-$
$E_{3,p}$	e_3	e_{10}	$-$
$Z_{1,p}$	e_8	e_{11}	$e_3 - e_4 - e_9$
$Q_{2,p}$	e_8	e_{11}	$-$
$W_{1,p}$	$e_3 + e_9 + e_{11}$	e_{16}	$-$
$S_{1,p}$	$-e_8 + e_{13}$	e_{15}	$-$

We will write down the action of the powers of the monodromy,

$$\beta_j \mapsto M_h(\beta_j) \mapsto M_h^2(\beta_j) \mapsto \ldots \mapsto M_h^{\deg b_j}(\beta_j), \tag{5.14}$$

in the subsections. Verifying $b_j(M_h)(\beta_j) = 0$ will show that the characteristic polynomial of M_h on B_j is b_j. We will also write down nice generators of B_j. This will show that B_j is a primitive sublattice of $Ml(f)$, that $\sum_{j\geq 1} B_j = \bigoplus_{j \geq 1} B_j$ is a direct sum and that it is a sublattice of full rank and of index r_I in $Ml(f)$. In all cases except $W_{1,p}$ and $S_{1,p}$, the index r_I is obvious from the nice generators, in the two cases $W_{1,p}$ and $S_{1,p}$, it requires the calculation of a determinant.

The left and right L-orthogonality of $\widetilde{B}_1$ and B_2 in (5.3) will be proved now. e_μ is a cyclic generator for B_2 in all eight series. The nice generators for $\widetilde{B}_1$ show $\widetilde{B}_1 \subset \bigoplus_{j=1}^{\mu-2} \mathbb{Z} \cdot e_j$ for all cases except $W_{1,1}$ and $S_{1,1}$. This and $L(e_i, e_\mu) = 0$ for $i < \mu$ show $L(\widetilde{B}_1, e_\mu) = 0$, thus $L(\widetilde{B}_1, B_2) = 0$. From the CDD one sees easily $L(e_\mu, e_i) = 0$ for $i \leq \mu - 2$ for all cases except $W_{1,1}$ and $S_{1,1}$, thus $L(e_\mu, \widetilde{B}_1) = 0$ and $L(B_2, \widetilde{B}_1) = 0$. For the cases $W_{1,1}$ and $S_{1,1}$, $L(B_1, e_\mu) = 0 = L(e_\mu, B_1)$ and thus $L(B_1, B_2) = 0 = L(B_2, B_1)$ hold also.

(5.5) for $S_{1,10}$ will be shown in subsection 5.8. With respect to part (a), it rests to show (5.4). It is trivial for the 3 series with $r_I = 1$. It will be shown in subsection 5.6 for the series $Q_{2,p}$ and in subsection 5.7 for the subseries $W_{1,6s-3}$ ($s \in \mathbb{Z}_{\geq 1}$) of the series $W_{1,p}$. For all other series, it will be shown below. It requires a study of smaller Orlik blocks. $\Phi_2 | b_1$ holds in the series $S^\sharp_{1,p}$, $E_{3,p}$, $Z_{1,p}$, $W_{1,p}$ and $S_{1,p}$. In these cases define (see (2.16) for the notion $v(\beta_1, -1)$)

$$\gamma_1 := v(\beta_1, -1) := \frac{b_1}{\Phi_2}(M_h)(\beta_1) \tag{5.15}$$

and calculate $L(\gamma_1,\gamma_1)$ using (2.17): $L(\gamma_1,\gamma_1) = \frac{b_1}{\Phi_2}(-1)\cdot L(\gamma_1,\beta_1)$.

$$\begin{array}{lll} \text{series} & \gamma_1 & L(\gamma_1,\gamma_1) \\ \hline S^\sharp_{1,p} & \Phi_{10}(M_h)(e_8) = 2e_1+e_2-e_4-e_5-e_6+e_8 & 5 \\ E_{3,p} & \Phi_{18}(M_h)(e_3) = -e_2+2e_3+e_6-e_7+e_9 & 6 \\ Z_{1,p} & \Phi_{14}(M_h)(e_8) & \\ & = e_2+e_3-3e_4-e_6+e_7-3e_9-e_{10} & 21 \\ W_{1,p} & (\Phi_{12}\Phi_6\Phi_3)(M_h)(e_3+e_9+e_{11}) & \\ & = e_4-e_5+e_9+e_{11}-e_{13}-e_{15} & 6 \\ S_{1,p} & (\Phi_{10}\Phi_5)(M_h)(-e_8+e_{13}) & \\ & = -2e_1+e_7-e_8-e_9-e_{11}-e_{12}-e_{14} & 10 \end{array} \tag{5.16}$$

In the case of the series $Z_{1,p}$, define $\gamma_3 := \beta_3$ and calculate

$$L(\gamma_3,\gamma_3) = 3, L(\gamma_1,\gamma_3) = L(\gamma_3,\gamma_1) = 7. \tag{5.17}$$

$\Phi_2|b_2$ holds in certain subseries of the series $S^\sharp_{1,p}$, $E_{3,p}$, $Z_{1,p}$, $W_{1,p}$ and $S_{1,p}$. In these cases define

$$\gamma_2 := v(\beta_2,-1) := \frac{b_2}{\Phi_2}(M_h)(\beta_2) \tag{5.18}$$

and calculate $L(\gamma_2,\gamma_2)$ using (2.17): $L(\gamma_2,\gamma_2) = \frac{b_2}{\Phi_2}(-1)\cdot L(\gamma_2,\beta_2)$.

$$\begin{array}{lll} \text{series} & \text{Condition for } \Phi_2|b_2 & L(\gamma_2,\gamma_2) \\ \hline S^\sharp_{1,p} & p\equiv 0(2) & 5+\frac{p}{2} \\ E_{3,p} & p\equiv 0(2) & 18+2p \\ Z_{1,p} & p\equiv 0(2) & 14+2p \\ W_{1,p} & p\equiv 1(2) & 12+2p \\ S_{1,p} & p\equiv 0(2) & 10+2p \end{array} \tag{5.19}$$

In table (5.20), the first line for $S^\sharp_{1,p}$ is the case $p\equiv 0(4)$, the second line is the case $p\equiv 2(4)$.

$$\begin{array}{ll} \text{series} & \gamma_2 \\ \hline S^\sharp_{1,p} & -e_2+e_4+e_5+e_6-e_7+\sum_{j=1}^{2+p/4}(e_{7+2j}+e_{10+\frac{p}{2}+2j}) \\ & -e_4+e_5+\sum_{j=1}^{(6+p)/4}(-e_{8+2j}+e_{11+\frac{p}{2}+2j}) \\ E_{3,p} & -e_2+2e_5+e_6-e_7+e_9+2\sum_{j=1}^{4+p/2}e_{8+2j} \\ Z_{1,p} & -e_2+2e_5+e_6-e_7+e_{10}+2\sum_{j=1}^{3+p/2}e_{9+2j} \\ W_{1,p} & -2e_3+e_4+e_5+e_9+e_{11}+e_{13}+e_{15}+2\sum_{j=1}^{(1+p)/2}e_{14+2j} \\ S_{1,p} & 2(-e_1-e_2+e_4+e_5+e_6)-e_7-e_8 \\ & \quad +e_9+e_{11}+e_{12}+e_{14}-2\sum_{j=1}^{p/2}e_{14+2j} \end{array} \tag{5.20}$$

In the subseries of $E_{3,p}, W_{1,p}$ and $S_{1,p}$ with $\Phi_2|b_2$, one sees

$$\widetilde{\gamma}_2 := \frac{1}{2}(\gamma_1+\gamma_2) \overset{!}{\in} Ml(f). \tag{5.21}$$

In the subseries of $Z_{1,p}$ with $\Phi_2|b_2$, one sees

$$\widetilde{\gamma}_2 := \frac{1}{2}(\gamma_1+\gamma_2-3\gamma_3) \overset{!}{\in} Ml(f). \tag{5.22}$$

Together with $[Ml(f):B_1\oplus B_2]=2$ for these subseries, this shows

$$\begin{aligned} Ml(f)_{\Phi_2} &= \mathbb{Z}\gamma_1\oplus\mathbb{Z}\widetilde{\gamma}_2 \qquad \text{for } E_{3,2q}, W_{1,2q-1}, S_{1,2q}, & (5.23)\\ Ml(f)_{\Phi_2} &= \mathbb{Z}(\gamma_1-2\gamma_3)\oplus\mathbb{Z}\widetilde{\gamma}_2\oplus\mathbb{Z}\gamma_3 \qquad \text{for } Z_{1,2q}. & (5.24)\end{aligned}$$

For $S^\sharp_{1,2q}$, $Ml(f)=B_1\oplus B_2$ gives $Ml(f)_{\Phi_2}=\mathbb{Z}\gamma_1\oplus\mathbb{Z}\gamma_2$. The matrices of L for these bases of $Ml(f)_{\Phi_2}$ in these cases are

$$\begin{array}{ccc} S^\sharp_{1,2q} & E_{3,2q} & Z_{1,2q} \\ \begin{pmatrix}5 & 0\\ 0 & 5+q\end{pmatrix} & \begin{pmatrix}6 & 3\\ 3 & 6+q\end{pmatrix} & \begin{pmatrix}5 & 2 & 1\\ 2 & 5+q & -1\\ 1 & -1 & 3\end{pmatrix} \\ W_{1,2q-1} & S_{1,2q} & \\ \begin{pmatrix}6 & 3\\ 3 & 4+q\end{pmatrix} & \begin{pmatrix}10 & 5\\ 5 & 5+q\end{pmatrix} & \end{array} \tag{5.25}$$

These matrices are positive definite. The corresponding quadratic forms $(x_1\, x_2)(\text{matrix})\begin{pmatrix}x_1\\ x_2\end{pmatrix}$ respectively $(x_1\, x_2\, x_3)(\text{matrix})\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}$ are

$$\begin{array}{rl} 5x_1^2+(5+q)x_2^2 & \text{for } S^\sharp_{1,2q}\\ 3x_1^2+3(x_1+x_2)^2+(3+q)x_2^2 & \text{for } E_{3,2q}\\ (2x_1+x_2)^2+(x_1+x_3)^2 & \\ +(x_2-x_3)^2+(3+q)x_2^2+x_3^2 & \text{for } Z_{1,2q}\\ 3x_1^2+3(x_1+x_2)^2+(1+q)x_2^2 & \text{for } W_{1,2q-1}\\ 5x_1^2+5(x_1+x_2)^2+qx_2^2 & \text{for } S_{1,2q} \end{array} \tag{5.26}$$

This shows

$$\{a\in Ml(f)_{\Phi_2}\,|\,L(a,a)=L(\gamma_1,\gamma_1)\}=\{\pm\gamma_1\}. \tag{5.27}$$

for $W_{1,2q-1}$ with $q\neq 2$, for $S_{1,2q}$ with $q\neq 5$, and for all $S^\sharp_{1,2q}$ and $E_{3,2q}$. It shows for $Z_{1,2q}$

$$\begin{aligned} \{a\in Ml(f)_{\Phi_2}\,|\,L(a,a)=3\} &= \{\pm\gamma_3\}, & (5.28)\\ \{a\in Ml(f)_{\Phi_2}\,|\,L(a,a)=5\} &= \{\pm(\gamma_1-2\gamma_3)\}. & (5.29)\end{aligned}$$

All this implies

$$\begin{aligned} \mathrm{Aut}(Ml(f)_{\Phi_2},L) &= \{\pm\mathrm{id}\,|_{\mathbb{Z}\gamma_1}\}\times\{\pm\mathrm{id}\,|_{\mathbb{Z}\gamma_2}\} \qquad \text{for } S^\sharp_{1,2q}, \\ & \quad \text{for } E_{3,2q},\ \text{for } S_{1,2q} \text{ with } q\neq 5,\\ & \quad \text{and for } W_{1,2q-1} \text{ with } q\neq 2, & (5.30)\\ \mathrm{Aut}(Ml(f)_{\Phi_2},L) &= \{\pm\mathrm{id}\,|_{\mathbb{Z}\gamma_1\oplus\mathbb{Z}\gamma_3}\}\times\{\pm\mathrm{id}\,|_{\mathbb{Z}\gamma_2}\} \text{ for } Z_{1,2q}. & (5.31)\end{aligned}$$

In the cases $S^\sharp_{1,2q-1}$, $E_{3,2q-1}$, $Z_{1,2q-1}$, $W_{1,2q}$ and $S_{1,2q-1}$ with $\Phi_2 \not| b_2$,

$$Ml(f)_{\Phi_2}=(\widetilde{B}_1)_{\Phi_2}\text{and}\,\mathrm{Aut}(Ml(f)_{\Phi_2},L)=\{\pm\mathrm{id}\}. \tag{5.32}$$

Define

$$\gamma_4:=\begin{cases}\gamma_1 & \text{for } E_{3,p}, W_{1,p}, S_{1,p}\\ \gamma_1-3\gamma_3 & \text{for } Z_{1,p}.\end{cases} \tag{5.33}$$

Then for $E_{3,p}$, $W_{1,p}$ with $p \neq 3$, $S_{1,p}$ with $p \neq 10$, $Z_{1,p}$

$$g(\gamma_4) = \pm\gamma_4 \qquad \text{for } g \in G_{\mathbb{Z}}, \tag{5.34}$$

and for $E_{3,p}$, $W_{1,p}$ (including $p = 3$), $S_{1,p}$ (including $p = 10$), $Z_{1,p}$

$$\widetilde{B}_1 \oplus B_2 = \{a \in Ml(f) \mid L(a, \gamma_4) \equiv 0(2)\}. \tag{5.35}$$

Here $\subset$ in (5.35) follows from

$$L(B_2, \gamma_4) = 0 \qquad \text{and} \qquad L(\beta_1, \gamma_4) \equiv 0(2)$$

and in the case of $Z_{1,p}$ $L(\beta_3, \gamma_4) = 4$. Now $=$ in (5.35) follows from $L(Ml(f), \gamma_4) = \mathbb{Z}$ and $[Ml(f) : \widetilde{B}_1 \oplus B_2] = 2$. Together (5.34) and (5.35) show that any $g \in G_{\mathbb{Z}}$ respects $\widetilde{B}_1 \oplus B_2$, so

$$G_{\mathbb{Z}} \subset \operatorname{Aut}(\widetilde{B}_1 \oplus B_2, L) \tag{5.36}$$

for $E_{3,p}$, $W_{1,p}$ with $p \neq 3$, $S_{1,p}$ with $p \neq 10$ and $Z_{1,p}$. We claim that (5.34) and thus (5.36) hold also for $W_{1,3}$. That will be proved in the subsection 5.7.

It rests to show $\operatorname{Aut}(\widetilde{B}_1 \oplus B_2, L) \subset G_{\mathbb{Z}}$ for the series $E_{3,p}$, $Z_{1,p}$, $W_{1,p}$, $S_{1,p}$. We will extend the definition of $\widetilde{\gamma}_2$ in such a way to the cases with $\Phi_2 \not| b_2$ that $(\widetilde{B}_1 \oplus B_2) + \mathbb{Z} \cdot \widetilde{\gamma}_2 = Ml(f)$. And we will show $g(\widetilde{\gamma}_2) \in Ml(f)$ for any $g \in \operatorname{Aut}(\widetilde{B}_1 \oplus B_2, L)$. This implies $\operatorname{Aut}(\widetilde{B}_1 \oplus B_2, L) \subset G_{\mathbb{Z}}$. The proof of $g(\widetilde{\gamma}_2) \in Ml(f)$ requires a better control of $\operatorname{Aut}(\widetilde{B}_1 \oplus B_2, L)$.

Consider all eight series and define

$$b_4 := \frac{\gcd(b_1, b_2)}{\gcd(b_1, b_2, \Phi_m)} = \gcd(\frac{b_1}{\Phi_m}, b_2) \in \mathbb{Z}[t]. \tag{5.37}$$

Then

$$b_4 = \begin{cases} 1 & \text{for } W^\sharp_{1,p}, S^\sharp_{1,2q-1}, U_{1,p}, E_{3,2q-1}, Z_{1,2q-1}, \\ & Q_{2,p} \text{ with } p \not\equiv 0(4), W_{1,2q}, S_{1,2q-1}, \\ \Phi_2 & \text{for } S^\sharp_{1,2q}, E_{3,2q}, Z_{1,2q}, W_{1,2q-1} \text{ with } q \not\equiv 2(3), S_{1,2q}, \\ \Phi_4 & \text{for } Q_{2,4s}, \\ \Phi_6\Phi_2 & \text{for } W_{1,6s-3}. \end{cases} \tag{5.38}$$

We claim that in all cases except $S_{1,10}$, any $g \in G_{\mathbb{Z}} \cup \operatorname{Aut}(\widetilde{B}_1 \oplus B_2, L)$ maps $(\widetilde{B}_1)_{b_4}$ to $(\widetilde{B}_1)_{b_4}$ and $(B_2)_{b_4}$ to $(B_2)_{b_4}$. In the cases with $b_4 = 1$ this is an empty statement as then

$$(\widetilde{B}_1)_{b_4} = \{0\} = (B_2)_{b_4}.$$

In the cases $Q_{2,p}$ with $p \equiv 0(4)$ and $W_{1,6s-3}$, this will be shown in the subsections 5.6 and 5.7. In all other cases $b_4 = \Phi_2$ and $(B_2)_{b_4} = \mathbb{Z} \cdot \gamma_2$ and

$$(\widetilde{B}_1)_{b_4} = \begin{cases} \mathbb{Z} \cdot (\gamma_1 - 2\gamma_3) \oplus \mathbb{Z} \cdot \gamma_3 & \text{for } Z_{1,2q}, \\ \mathbb{Z} \cdot \gamma_1 & \text{else.} \end{cases} \tag{5.39}$$

Because $(\widetilde{B}_1 \oplus B_2)_{\Phi_2} \subset Ml(f)_{\Phi_2}$, (5.27)–(5.29) hold also with $(\widetilde{B}_1 \oplus B_2)_{\Phi_2}$ instead of $Ml(f)_{\Phi_2}$. They characterize $(\widetilde{B}_1)_{\Phi_2}$ within $Ml(f)_{\Phi_2}$ and within $(\widetilde{B}_1 \oplus B_2)_{\Phi_2}$. Thus any

$$g \in G_{\mathbb{Z}} \cup \operatorname{Aut}(\widetilde{B}_1 \oplus B_2, L)$$

maps $(\widetilde{B}_1)_{\Phi_2}$ to itself, and then it maps also the L-orthogonal sublattice $(B_2)_{\Phi_2}$ to itself.

For all eight series except $S_{1,10}$, this implies the following. For any $g \in G_{\mathbb{Z}} \cup \operatorname{Aut}(\widetilde{B}_1 \oplus B_2, L)$

$$g : \widetilde{B}_1 \to \widetilde{B}_1 \text{ and } B_2 \to B_2 \qquad \text{if } m \not| p, \tag{5.40}$$

$$\left.\begin{array}{l} g : (\widetilde{B}_1)_{b_1/\Phi_m} \to (\widetilde{B}_1)_{b_1/\Phi_m} \\ g : (B_2)_{b_2/\Phi_m} \to (B_2)_{b_2/\Phi_m} \end{array}\right\} \quad \begin{array}{c} \text{if } m|p \text{ and the} \\ \text{type is not } S_{1,10}. \end{array} \tag{5.41}$$

Now we want to apply lemma 2.8 to these Orlik blocks. One checks easily that all hypotheses are satisfied. Therefore

$$\begin{aligned} & \mathrm{Aut}(\widetilde{B}_1 \oplus B_2, L) \\ = \ & \{\pm M_h^k|_{\widetilde{B}_1} \,|\, k \in \mathbb{Z}\} \times \{\pm M_h^k|_{B_2} \,|\, k \in \mathbb{Z}\} \qquad \text{if } m \not| p, \end{aligned} \tag{5.42}$$

and if $m|\,p$ and the type is not $S_{1,10}$, then $\mathrm{Aut}(\widetilde{B}_1 \oplus B_2, L)$ projects to a subgroup of

$$\begin{aligned} & \mathrm{Aut}((\widetilde{B}_1)_{b_1/\Phi_m}, L) \times \mathrm{Aut}((B_2)_{b_2/\Phi_m}, L) \\ = \ & \{\pm M_h^k|_{(\widetilde{B}_1)_{b_1/\Phi_m}} \,|\, k \in \mathbb{Z}\} \times \{\pm M_h^k|_{(B_2)_{b_2/\Phi_m}} \,|\, k \in \mathbb{Z}\}. \end{aligned} \tag{5.43}$$

The group $\mathrm{Aut}(\widetilde{B}_1 \oplus B_2, L)$ for $m \not| p$ is generated by

$$M_h, -\mathrm{id}, \quad M_h|_{\widetilde{B}_1} \times \mathrm{id}|_{B_2}, \quad \text{and} \quad (-\mathrm{id}|_{\widetilde{B}_1}) \times \mathrm{id}|_{B_2},$$

and analogously for the group in (5.43) if $m|\,p$.

Now we extend the definition of γ_2. For $E_{3,2q-1}$, $Z_{1,2q-1}$ and $S_{1,2q-1}$ define it as follows:

$$\begin{aligned} \gamma_2 &:= e_2 - e_6 + e_7 + e_9 \text{for } E_{3,2q-1}, \\ \gamma_2 &:= e_2 - e_6 + e_7 + e_{10} \text{for } Z_{1,2q-1}, \\ \gamma_2 &:= 2(-e_1 - e_2 + \sum_{j\in\{4,5,6\}} e_j) - e_7 - e_8 + \sum_{j\in\{9,11,12,14\}} e_j \text{for } S_{1,2q-1}. \end{aligned} \tag{5.44}$$

(5.105), (5.110) and (5.162) show $\gamma_2 \in B_2$. For $W_{1,2q}$ (so $p = 2q$) define

$$\begin{aligned} \gamma_2 &:= (t^p(t+1)\Phi_{12} + \sum_{j=0}^{p-1} t^j)(M_h)(e_{16}) \\ &= (t^p(1+t-t^2-t^3+t^4+t^5) + \sum_{j=0}^{p-1} t^j)(M_h)(e_{16}) \\ &= -2e_2 + 2e_6 - 2e_7 + e_4 + e_5 + e_9 - e_{11} + e_{13} - e_{15}. \end{aligned} \tag{5.45}$$

Observe that in the case $12|p$, Φ_{12} divides $\sum_{j=0}^{p-1} t^j$ so that then $\gamma_2 \in \Phi_{12}(M_h)(B_2) = (B_2)_{b_2/\Phi_{12}}$. In all four cases $\frac{1}{2}(\gamma_4 + \gamma_2) \in Ml(f)$.

Now for the series $E_{3,p}, Z_{1,p}, W_{1,p}$ and $S_{1,p}$

$$\gamma_4 \in (B_1)_{\Phi_2}, \qquad \begin{cases} \gamma_2 \in B_2 & \text{if } m \not| p, \\ \gamma_2 \in (B_2)_{b_2/\Phi_m} & \text{if } m|\,p, \end{cases} \tag{5.46}$$

$$\widetilde{\gamma}_2 := \frac{1}{2}(\gamma_4 + \gamma_2) \overset{!}{\in} Ml(f), \tag{5.47}$$

$$Ml(f) = (\widetilde{B}_1 \oplus B_2) + \mathbb{Z}\widetilde{\gamma}_2, \tag{5.48}$$

$$\begin{aligned} (M_h|_{\widetilde{B}_1} \times \mathrm{id}|_{B_2})(\widetilde{\gamma}_2) &= ((-\mathrm{id}|_{\widetilde{B}_1}) \times \mathrm{id}|_{B_2})(\widetilde{\gamma}_2) \\ &= \frac{1}{2}(-\gamma_4 + \gamma_2) = -\gamma_4 + \widetilde{\gamma}_2 \in Ml(f). \end{aligned} \tag{5.49}$$

Therefore any $g \in \mathrm{Aut}(\widetilde{B}_1 \oplus B_2, L)$ maps $\widetilde{\gamma}_2$ to an element of $Ml(f)$. Thus it maps $Ml(f)$ to $Ml(f)$, thus $g \in G_{\mathbb{Z}}$. This finishes the proof of (5.4) and of part (a) for all series except $Q_{2,p}$ and $W_{1,6s-3}$ and $S_{1,10}$. For $Q_{2,p}$ and $W_{1,6s-3}$ and $S_{1,10}$ see the subsections 5.6, 5.7 and 5.8.

(b) This follows immediately from (5.4) and (5.42). The subsections 5.6 and 5.7 establish (5.4) and (5.42) also for the series $Q_{2,p}$ and $W_{1,6s-3}$.

(c) Now we consider the eight subseries with $m|p$. Write $p = m \cdot r$ with $r \in \mathbb{Z}_{\geq 1}$. Recall $\zeta = e^{2\pi i/m}$, and recall that $\mathbb{Z}[\zeta]$ is a principal ideal domain (lemma 2.11). In the following, ξ will be any primitive m-th unit root.

Formula (2.24) in lemma 2.12 (b) applies with $\Lambda = Ml(f), \Lambda^{(1)} = \widetilde{B}_1 \oplus B_2, p = \Phi_m$, and gives

$$Ml(f)_{\Phi_m} = (\widetilde{B}_1 \oplus B_2)_{\Phi_m} = (B_1 \oplus B_2)_{\Phi_m} = (B_1)_{\Phi_m} \oplus (B_2)_{\Phi_m}. \tag{5.50}$$

Therefore the space

$$Ml(f)_{\xi,\mathbb{Z}[\zeta]} \quad := \quad Ml(f)_\xi \cap Ml(f)_{\mathbb{Z}[\zeta]} \tag{5.51}$$

is a free $\mathbb{Z}[\zeta]$-module of rank 2 with basis $v_{1,\xi}, v_{2,\xi}$ with

$$v_{j,\xi} := v(\beta_j, \xi) = \frac{b_j}{t-\xi}(M_h)(\beta_j) \qquad \text{for } j = 1, 2 \tag{5.52}$$

(see (2.16) for the notion $v(\beta_j, \xi)$). Observe $v_{j,\overline{\xi}} = \overline{v_{j,\xi}}$.

The proof of part (c) will consist of four steps. Step 1 calculates the values of the hermitian form h_ξ from lemma 2.2 on a suitable $\mathbb{Z}[\zeta]$-basis of $Ml(f)_{\xi,\mathbb{Z}[\zeta]}$. Step 2 analyzes what this implies for automorphisms of the pair $(Ml(f)_{\xi,\mathbb{Z}[\zeta]}, L)$ and thus gives a first approximation to $\Psi(G_\mathbb{Z})$. Step 3 uses (5.5) for $S_{1,10}$ and (5.41) for all other singularities and the Orlik block structure of the blocks B_j to control the action of $g \in G_\mathbb{Z}$ on all eigenspaces simultaneously. It will prove (5.9). Step 4 combines the steps 2 and 3 with results from section 3 and shows that $\Psi(G_\mathbb{Z})$ is an infinite Fuchsian group.

Step 1: The form

$$h_\xi : Ml(f)_\xi \times Ml(f)_\xi \to \mathbb{C}, (a, b) \mapsto \sqrt{-\xi} \cdot L(a, \overline{b})$$

from lemma 2.2 is hermitian. In this step it will be calculated with respect to the $\mathbb{Z}[\zeta]$-basis $v_{1,\xi}, v_{2,\xi}$ of $Ml(f)_{\xi,\mathbb{Z}[\zeta]}$. For $i \neq j$

$$h_\xi(v_{i,\xi}, v_{j,\xi}) = \sqrt{-\xi} \cdot L(v_{i,\xi}, v_{j,\overline{\xi}}) = 0 \tag{5.53}$$

because of (5.3). $L(v_{j,\xi}, v_{j,\overline{\xi}})$ will be calculated with (2.17),

$$L(v_{j,\xi}, v_{j,\overline{\xi}}) = \frac{b_j}{t-\overline{\xi}}(\overline{\xi}) \cdot L(\frac{b_j}{t-\xi}(M_h)(\beta_j), \beta_j), \tag{5.54}$$

first for $j = 2$, then for $j = 1$.

One calculates for all eight subseries:

k	0	1	2	$\cdots$	$\deg b_2 - 1$	$\deg b_2$
$L(M_h^k(\beta_2), \beta_2)$	1	-1	0	$\cdots$	0	0 if $r_I = 1$, -1 if $r_I \geq 2$

For the three subseries with $r_I = 1$ (so $W^\sharp_{1,12r}, S^\sharp_{1,10r}, U_{1,9r}$)

$$\frac{b_2}{t-\xi} \quad = \quad \frac{t^{m+p}-1}{(t-\xi)\cdot \Phi_1} = \Phi_1^{-1} \cdot \sum_{j=0}^{m+p-1} \xi^{m+p-1-j} \cdot t^j, \tag{5.55}$$

$$\frac{b_2}{t-\overline{\xi}}(\overline{\xi}) \quad = \quad (\overline{\xi}-1)^{-1} \cdot (m+p) \cdot \xi = m(1+r)(\overline{\xi}-1)^{-1} \cdot \xi, \tag{5.56}$$

$$L(\frac{b_2}{t-\xi}(M_h)(\beta_2), \beta_2) \quad = \quad (\xi-1)^{-1} \cdot \overline{\xi} \cdot (1-\overline{\xi}) = \overline{\xi}^2, \tag{5.57}$$

$$h_\xi(v_{2,\xi}, v_{2,\xi}) \quad = \quad m(1+r) \cdot (1-\xi)^{-1} \cdot \sqrt{-\xi} > 0. \tag{5.58}$$

For the five subseries with $r_I = 2$

$$\begin{aligned}
\frac{b_2}{t-\xi} &= \frac{t^{m/2+p}+1}{t-\xi} = \sum_{j=0}^{m/2+p-1} \xi^{m/2+p-1-j}\cdot t^j, && (5.59)\\
\frac{b_2}{t-\overline{\xi}}(\overline{\xi}) &= (\frac{m}{2}+p)(-\xi) = \frac{m}{2}(1+2r)(-\xi), && (5.60)
\end{aligned}$$

$$\begin{aligned}
L(\frac{b_2}{t-\xi}(M_h)(\beta_2),\beta_2) &= -\overline{\xi}(1-\overline{\xi}), && (5.61)\\
h_\xi(v_{2,\xi},v_{2,\xi}) &= \frac{m}{2}(1+2r)\cdot(1-\overline{\xi})\cdot\sqrt{-\xi} > 0. && (5.62)
\end{aligned}$$

Now we turn to $h_\xi(v_{1,\xi},v_{1,\xi})$. One calculates for all eight series

k $L(M_h^k(\beta_1),\beta_1)$	0	1	2	3	4	5	6	7	8	9	10	11
for $W_{1,p}^\sharp$	1	-1	1	0	0	1						
for $S_{1,p}^\sharp$	1	-1	0	1	0							
for $U_{1,p}$	1	-1	0	0	1	0	-1	0	0			
for $E_{3,p}$	1	-1	1	0	1	0	1	0	1			
for $Z_{1,p}$	1	-1	0	0	1	0	0					
for $Q_{2,p}$	1	-1	0	1	0	0	0	0	0	0	-1	0
for $W_{1,p}$	3	-3	2	-1	0	1	-1	1	-1	0	1	-2
for $S_{1,p}$	2	-2	0	1	0	-1	1	0	-1	0		

and

$$\text{for } W_{1,p}^\sharp \qquad \frac{b_1}{t-\xi} = \frac{\Phi_{12}}{t-\xi} = t^3+\xi t^2+(\xi^2-1)t+(\xi^3-\xi),$$

$$\text{for } S_{1,p}^\sharp \qquad \frac{b_1}{t-\xi} = \frac{\Phi_{10}\Phi_2}{t-\xi} = \frac{t^5+1}{t-\xi} = t^4+\xi t^3+\xi^2t^2+\xi^3t+\xi^4,$$

$$\begin{aligned}
\text{for } U_{1,p} \qquad & \frac{b_1}{t-\xi} = \frac{\Phi_9}{t-\xi} = \frac{t^6+t^3+1}{t-\xi}\\
& = t^5+\xi t^4+\xi^2t^3+(\xi^3+1)t^2+(\xi^4+\xi)t+(\xi^5+\xi^2),
\end{aligned}$$

$$\begin{aligned}
\text{for } E_{3,p} \qquad & \frac{b_1}{t-\xi} = \frac{\Phi_{18}\Phi_2}{t-\xi} = \frac{t^7+t^6-t^4-t^3+t+1}{t-\xi} = t^6+(\xi+1)t^5\\
& +(\xi^2+\xi)t^4+(\xi^6+\xi^2)t^3+(\xi^7+\xi^6)t^2+(\xi^8+\xi^7)t+\xi^8,\\
\text{for } Z_{1,p} \qquad & \frac{b_1}{t-\xi} = \frac{t^7+1}{t-\xi} = t^6+\xi t^5+\xi^2t^4+\xi^3t^3+\xi^4t^2+\xi^5t+\xi^6,\\
\text{for } Q_{2,p} \qquad & \frac{b_1}{t-\xi} = \frac{\Phi_{12}\Phi_4\Phi_3}{t-\xi} = \frac{t^8+t^7+t^6+t^2+t+1}{t-\xi}\\
& = t^7+(\xi+1)t^6+(\xi^2+\xi+1)t^5+(\xi^3+\xi^2+\xi)t^4\\
& +(\xi^4+\xi^3+\xi^2)t^3+(\xi^5+\xi^4+\xi^3)t^2+(\xi^5+\xi^4)t+\xi^5,
\end{aligned}$$

$$\text{for } W_{1,p} \qquad \frac{b_1}{t-\xi} = \frac{\Phi_{12}\Phi_6\Phi_3\Phi_2}{t-\xi} = \frac{t^9+t^8+t^5+t^4+t+1}{t-\xi}$$
$$= t^8+(\xi+1)t^7+(\xi^2+\xi)t^6+(\xi^3+\xi^2)t^5+(\xi^3+\xi^2)t^4$$
$$+(\xi^3+\xi^2)t^3+(\xi^4+\xi^3)t^2+(\xi^5+\xi^4)t+\xi^5,$$
$$\text{for } S_{1,p} \qquad \frac{b_1}{t-\xi} = \frac{\Phi_{10}\Phi_5\Phi_2}{t-\xi} = \frac{\sum_{j=0}^9 t^j}{t-\xi}$$
$$= t^8+(\xi+1)t^7+(\xi^2+\xi+1)t^6+(\xi^3+\xi^2+\xi+1)t^5$$
$$+(\xi^4+\xi^3+\xi^2+\xi+1)t^4+(\xi^4+\xi^3+\xi^2+\xi)t^3$$
$$+(\xi^4+\xi^3+\xi^2)t^2+(\xi^4+\xi^3)t+\xi^4.$$

This table and this list give the following values.

	$\frac{b_1}{t-\overline{\xi}}(\overline{\xi})$	$L(\frac{b_1}{t-\xi}(M_h)(\beta_1),\beta_1)$
$W_{1,p}^\sharp$	$4\overline{\xi}^3-2\overline{\xi}=-2(\xi+\overline{\xi})\xi^2$	$\xi^3(1-\xi)$
$S_{1,p}^\sharp$	$5\overline{\xi}^4=-5\xi$	$-\xi(\xi^2+\overline{\xi}^2-1)$
$U_{1,p}$	$6\overline{\xi}^5+3\overline{\xi}^2=3\xi(\xi^3-1)$	$-\xi^6(\xi^2+\overline{\xi}^2)$
$E_{3,p}$	$3(\overline{\xi}^6+\overline{\xi}^5+\overline{\xi}^9+\overline{\xi}^8)=-3(\xi+1)(\xi^3+1)$	$\xi^2(\xi+\overline{\xi})(\xi^2+\overline{\xi}^2)$
$Z_{1,p}$	$7\overline{\xi}^6=-7\xi$	$\xi^2(\xi^4+\overline{\xi}^4+1)$
$Q_{2,p}$	$6(\overline{\xi}^7+\overline{\xi}^6+\overline{\xi}^5)=-6(\xi+\overline{\xi}+1)$	$\xi^2(\xi+1)=(1-\xi)^{-1}$
$W_{1,p}$	$4(\overline{\xi}^8+\overline{\xi}^7+\overline{\xi}^6+\overline{\xi}^5)=4\overline{\xi}^7(1+\xi)(\xi+\overline{\xi})$	$\xi^3(\xi-1)(\overline{\xi}-1)$
$S_{1,p}$	$5(\overline{\xi}^8+\overline{\xi}^7+\overline{\xi}^6+\overline{\xi}^5+\overline{\xi}^4)$	$-1+\xi+\xi^2-2\xi^3+\xi^4$

With $h_\xi(v_{1,\xi},v_{1,\xi}) = \sqrt{-\xi}\cdot L(v_{1,\xi},v_{1,\overline{\xi}})$ and (5.54) and the information on the rings $\mathbb{Z}[\zeta]$ in lemma 2.11, we obtain the following values.

	$h_\xi(v_{1,\xi},v_{1,\xi})$
$W_{1,p}^\sharp$	$(-2)(\xi+\overline{\xi})\cdot(1-\overline{\xi})\sqrt{-\xi}$
$S_{1,p}^\sharp$	$5(\xi^2+\overline{\xi}^2)(\xi^2+\overline{\xi}^2-1)\cdot(1-\xi)^{-1}\sqrt{-\xi}$
$U_{1,p}$	$3(\xi^4+\overline{\xi}^4+1)\cdot(1-\overline{\xi})\sqrt{-\xi}$
$E_{3,p}$	$(-3)(1+\xi)(1+\overline{\xi})(\xi+\overline{\xi}-1)\cdot(1-\xi)^{-1}\sqrt{-\xi}$
$Z_{1,p}$	$(-7)(\xi^2+\overline{\xi}^2)\cdot(1-\overline{\xi})\sqrt{-\xi}$
$Q_{2,p}$	$(-6)(\xi+\overline{\xi}+1)\cdot(1-\xi)^{-1}\sqrt{-\xi}$
$W_{1,p}$	$(-4)(\xi+\overline{\xi})\cdot(1-\overline{\xi})\sqrt{-\xi}$
$S_{1,p}$	$(-10)(\xi^2+\overline{\xi}^2)\cdot(1-\overline{\xi})\sqrt{-\xi}$

(5.63)

Here observe that as in (5.58) and (5.62) $(1-\overline{\xi})\sqrt{-\xi}>0$ and $(1-\xi)^{-1}\sqrt{-\xi}>0$. In each of the eight cases we find

$$h_\xi(v_{1,\xi},v_{1,\xi}) \;>\; 0 \qquad \text{for } \xi\notin\{\zeta,\overline{\zeta}\}, \tag{5.64}$$
$$h_\xi(v_{1,\xi},v_{1,\xi}) \;<\; 0 \qquad \text{for } \xi\in\{\zeta,\overline{\zeta}\}, \tag{5.65}$$

and

$$L(v_{1,\xi},\beta_1) = L(\frac{b_1}{t-\xi}(M_h)(\beta_1),\beta_1)\in\mathbb{Z}[\zeta]^*. \tag{5.66}$$

Step 2: Define for each of the eight series

$$b_5 := \frac{b_1}{\Phi_m}\in\mathbb{Z}[t] \qquad \text{unitary}. \tag{5.67}$$

Then

series	$W^\sharp_{1,p}$	$S^\sharp_{1,p}$	$U_{1,p}$	$E_{3,p}$	$Z_{1,p}$	$Q_{2,p}$	$W_{1,p}$	$S_{1,p}$
b_5	1	Φ_2	1	Φ_2	Φ_2	$\Phi_4\Phi_3$	$\Phi_6\Phi_3\Phi_2$	$\Phi_5\Phi_2$

and

$$b_5(\xi)/b_5(\overline{\xi}) \in \{\pm\xi^k \mid k\in\mathbb{Z}\}. \tag{5.68}$$

Define for each of the eight subseries with $m|p$

$$b_6 := \frac{b_2}{\Phi_m} \in \mathbb{Z}[t] \qquad \text{unitary} \tag{5.69}$$

and

$$\begin{aligned} w(\xi) &:= -\frac{h_\xi(v_{2,\xi},v_{2,\xi})}{h_\xi(v_{1,\xi},v_{1,\xi})} = -\frac{\frac{b_2}{t-\overline{\xi}}(\overline{\xi})\cdot L(v_{2,\xi},\beta_2)}{\frac{b_1}{t-\overline{\xi}}(\overline{\xi})\cdot L(v_{1,\xi},\beta_1)} \\ &= -\frac{b_6}{b_5}(\overline{\xi})\cdot\frac{L(v_{2,\xi},\beta_2)}{L(v_{1,\xi},\beta_1)}. \end{aligned} \tag{5.70}$$

Then

$$b_5(\overline{\xi})w(\xi) = b_6(\overline{\xi})\cdot\frac{L(v_{2,\xi},\beta_2)}{L(v_{1,\xi},\beta_1)} \in \mathbb{Z}[\zeta]. \tag{5.71}$$

It is in $\mathbb{Z}[\zeta]$ because of (5.66). The following table lists $w(\xi)$.

$$\begin{array}{l|l} & w(\xi) \\ \hline W^\sharp_{1,p} & (1+r)(+6)[(1-\xi)(1-\overline{\xi})(\xi+\overline{\xi})]^{-1} \\ S^\sharp_{1,p} & (1+r)(-2)[(\xi^2+\overline{\xi}^2)(\xi^2+\overline{\xi}^2-1)]^{-1} \\ U_{1,p} & (1+r)(-3)[(1-\xi)(1-\overline{\xi})(\xi^4+\overline{\xi}^4+1)]^{-1} \\ E_{3,p} & (1+2r)(+3)(1-\xi)(1-\overline{\xi})[(1+\xi)(1+\overline{\xi})(\xi+\overline{\xi}-1)]^{-1} \\ Z_{1,p} & (1+2r)(+1)[\xi^2+\overline{\xi}^2]^{-1} \\ Q_{2,p} & (1+2r)(+1)(1-\xi)(1-\overline{\xi})[\xi+\overline{\xi}+1]^{-1} \\ W_{1,p} & (1+2r)(+\frac{3}{2})[\xi+\overline{\xi}]^{-1} \\ S_{1,p} & (1+2r)(+\frac{1}{2})[\xi^2+\overline{\xi}^2]^{-1} \end{array} \tag{5.72}$$

The inequalities (5.58)(5.62)(5.64)(5.65) give

$$w(\xi)\begin{cases} <0 & \text{for } \xi\notin\{\zeta,\overline{\zeta}\}, \\ >0 & \text{for } \xi\in\{\zeta,\overline{\zeta}\}. \end{cases} \tag{5.73}$$

Using the $\mathbb{Z}[\zeta]$-basis $v_{1,\xi},v_{2,\xi}$ of $Ml(f)_{\xi,\mathbb{Z}[\zeta]}$, the automorphism group $\mathrm{Aut}(Ml((f)_{\xi,\mathbb{Z}[\zeta]},h_\xi)$ can be identified with the matrix group

$$\begin{aligned} \{A(\xi)\in GL(2,\mathbb{Z}[\zeta]) \quad & | \\ \begin{pmatrix}-1 & 0\\ 0 & w(\xi)\end{pmatrix} \quad &= \quad A(\xi)^t\cdot\begin{pmatrix}-1 & 0\\ 0 & w(\xi)\end{pmatrix}\cdot\overline{A(\xi)}\}. \end{aligned} \tag{5.74}$$

The isomorphism is $A(\xi)\mapsto g$ with

$$g(v_{1,\xi},v_{2,\xi}) \quad = \quad (v_{1,\xi},v_{2,\xi})\cdot A(\xi). \tag{5.75}$$

The inequalities (5.73) and theorem 3.2 tell that the matrix group in the case of $\xi=\zeta$ projects to an infinite Fuchsian group. Additionally, 3.2 tells that the elements of the matrix group for any ξ can be represented by triples $(a(\xi),c(\xi),\delta(\xi))\in\mathbb{Z}[\zeta]^2\times\{\pm\zeta^k\mid k\in\mathbb{Z}\}$ with

$$a(\xi)a(\overline{\xi})-1 = w(\xi)\cdot c(\xi)c(\overline{\xi}), \tag{5.76}$$

where

$$A(\xi) = \begin{pmatrix} a(\xi) & w(\xi)\cdot c(\overline{\xi})\cdot \delta(\xi) \\ c(\xi) & a(\overline{\xi})\cdot\delta(\xi)\end{pmatrix}. \tag{5.77}$$

This gives a first approximation of $\Psi(G_{\mathbb{Z}})$. It took into account only the eigenspace $Ml(f)_{\xi,\mathbb{Z}[\zeta]}$ and the pairing h_ξ which L and complex conjugation induce on it.

Step 3: Now (5.9) will be shown. We will use that the B_j are Orlik blocks and lemma 2.8 and (5.5) for $S_{1,10}$ and (5.43) for all other singularities.

Let $g\in\ker\Psi\subset G_{\mathbb{Z}}$, i.e. $g|_{Ml(f)_\zeta}\in\mathbb{C}^*\cdot\mathrm{id}$. Then $g|_{Ml(f)_\xi}\in\mathbb{C}^*\cdot\mathrm{id}$ for all ξ with $\Phi_m(\xi)=0$, and

$$g((B_j)_{\Phi_m}) = (B_j)_{\Phi_m} \text{ for } j=1,2. \tag{5.78}$$

Now $g(B_j)=B_j$ for $j=1,2$ follows in the case $S_{1,10}$ from (5.5). For all other singularities $g(B_j)=B_j$ for $j=1,2$ follows with (5.43) (and (5.32) for B_3 in the case $Z_{1,14r}$).

We want to apply lemma 2.8 to the Orlik blocks B_1 and B_2. One checks easily that all hypotheses are satisfied. In the case $Z_{1,14r}$ B_3 is glued to B_1 by (5.32). Therefore in all cases

$$g = (\varepsilon_1\cdot M_h^{k_1})|_{B_1}\times(\varepsilon_2\cdot M_h^{k_2})|_{B_2} \tag{5.79}$$

for some $\varepsilon_1,\varepsilon_2\in\{\pm1\}$ and $k_1,k_2\in\mathbb{Z}$. Now consider

$$\widetilde{g} \;:=\; \varepsilon_2\cdot M_h^{-k_2}\circ g. \tag{5.80}$$

It satisfies

$$\begin{aligned}\widetilde{g}|_{B_1} = \varepsilon_1\varepsilon_2\cdot M_h^{k_1-k_2}|_{B_1}, \widetilde{g}|_{B_2}=\mathrm{id}, \widetilde{g}|_{Ml(f)_\xi}\in\mathbb{C}^*\cdot\mathrm{id},\\ \text{thus } \widetilde{g}|_{Ml(f)_\xi}=\mathrm{id}, \widetilde{g}|_{Ml(f)_{\Phi_m}}=\mathrm{id}.\end{aligned} \tag{5.81}$$

Comparison with table (5.1) shows

$$\begin{aligned}\widetilde{g} &= \mathrm{id} \quad\text{ for the first 5 series in (5.1)},\\ \widetilde{g} &= \mathrm{id} \text{ or } \widetilde{g} = -M_h^{\frac{m}{2}(1+2r)} \text{ for the last 3 series in (5.1)}.\end{aligned}$$

In any case, $\widetilde{g}$ and g are in $\{\pm M_h^k\,|\,k\in\mathbb{Z}\}$, and thus $\ker\Psi=\{\pm M_h^k\,|\,k\in\mathbb{Z}\}$.

Step 4: By step 2, $\Psi(G_{\mathbb{Z}})$ is a subgroup of an infinite Fuchsian group and therefore itself a Fuchsian group. It rests to show that it is an infinite group. By step 3, the kernel of

$$\Psi: G_{\mathbb{Z}}\to\Psi(G_{\mathbb{Z}})$$

is $\{\pm M_h^k\,|\,k\in\mathbb{Z}\}$, so it is finite. Therefore it rests to show that $G_{\mathbb{Z}}$ is infinite. We will see that the subgroup of elements $g\in G_{\mathbb{Z}}$ with

$$\begin{aligned} g &= \mathrm{id} \text{ on any eigenspace } Ml(f)_\lambda \text{ with } \Phi_m(\lambda)\neq 0,\\ \text{i.e. } g &= \mathrm{id} \text{ on } (\widetilde{B}_1)_{b_5} \text{ and on } (B_2)_{b_6}.\end{aligned} \tag{5.82}$$

is infinite.

Consider an element $g\in G_{\mathbb{Z}}$ with (5.82). For all singularities except $S_{1,10}$ (5.4) holds. For $S_{1,10}$ (5.82) implies $g(\gamma_4)=\pm\gamma_4$, and then (5.36) gives $g\in\mathrm{Aut}(B_1\oplus B_2,L)$. In the case of the series $Z_{1,14r}$, the element g maps $B_1\oplus B_2$ to itself because $(B_1\oplus B_2)_{\mathbb{C}}$ contains $\ker\Phi_m(M_h)$. In any case, lemma 2.7 applies with $k=2,\Lambda^{(1)}=B_1,\Lambda^{(2)}=B_2,e^{(1)}=\beta_1,e^{(2)}=\beta_2,p_0=\Phi_m$. By (2.20) there are unique polynomials $p_{ij}\in\mathbb{Z}[t]_{<\deg b_i}$ for $i=1,2$ with

$$g(\beta_j) \;=\; p_{1j}(M_h)(\beta_1)+p_{2j}(M_h)(\beta_2) \tag{5.83}$$

and

$$\begin{aligned} p_{11} &= 1 + b_5 \cdot q_{11}, & p_{12} &= b_5 \cdot q_{12}, \\ p_{21} &= b_6 \cdot q_{21}, & p_{22} &= 1 + b_6 \cdot q_{22} \end{aligned} \tag{5.84}$$

for suitable polynomials $q_{ij} \in \mathbb{Z}[t]_{<\varphi(m)}$.

g restricts to an automorphism of the pair $(B_1 \oplus B_2)_{\Phi_m}, L)$. By (2.21), the matrix $A(\xi)$ from (5.75) in step 2 takes the form

$$A(\xi) = \begin{pmatrix} 1 + b_5(\xi) q_{11}(\xi) & b_6(\xi) q_{12}(\xi) \\ b_5(\xi) q_{21}(\xi) & 1 + b_6(\xi) q_{22}(\xi) \end{pmatrix}. \tag{5.85}$$

By step 2, this matrix $A(\xi)$ satisfies (5.76) and (5.77).

Vice versa, any polynomials $q_{ij} \in \mathbb{Z}[t]_{<\varphi(m)}$ for $i = 1, 2$ such that the matrix in (5.85) satisfies (5.76) and (5.77), give rise via (5.84) and (5.83) to an element $g \in G_{\mathbb{Z}}$ with (5.82).

We have to prove existence of infinitely many polynomials $q_{ij} \in \mathbb{Z}[t]_{<\varphi(m)}$ such that the matrix in (5.85) satisfies (5.76) and (5.77) and that $q_{12}(\xi) \neq 0$ and $q_{21}(\xi) \neq 0$. We start by defining

$$w_0(\xi) := w(\xi) b_5(\xi) b_5(\overline{\xi}) \in \mathbb{Z}[\zeta] \cap \mathbb{R} \tag{5.86}$$

and asking for infinitely many solutions $a(\xi), f(\xi) \in \mathbb{Z}[\zeta] \cap \mathbb{R}$ of the Pell equation

$$a(\xi)^2 - 1 = w_0(\xi) \cdot f(\xi)^2 \tag{5.87}$$

with the additional condition

$$w_0(\xi) \quad | \quad a(\xi) - 1. \tag{5.88}$$

Such solutions exist due to lemma 3.3. They give rise to the elements

$$q_{11}(\xi) := \frac{a(\xi) - 1}{b_5(\xi)}, \qquad q_{12}(\xi) := f(\xi) \cdot \frac{w(\xi) b_5(\overline{\xi})}{b_6(\xi)}, \tag{5.89}$$

$$q_{21}(\xi) := f(\xi), \qquad q_{22}(\xi) := \frac{a(\xi) - 1}{b_6(\xi)}. \tag{5.90}$$

Here observe

$$b_6(\xi) \,|\, w(\xi) b_5(\overline{\xi}) \,|\, w_0(\xi) \,|\, a(\xi) - 1,$$

see (5.71), (5.68) and (5.66). These elements come from unique polynomials $q_{ij} \in \mathbb{Z}[t]_{<\varphi(m)}$. These polynomials satisfy all desired properties.

5.1. **The series $W^\sharp_{1,p}$.** Here we only describe the case when $p = 2q$ is even. But one can easily obtain the odd case $p = 2q - 1$ from that via replacing each $e_{\alpha+q}$ by $e_{\alpha-1+q}$ in the following lists.

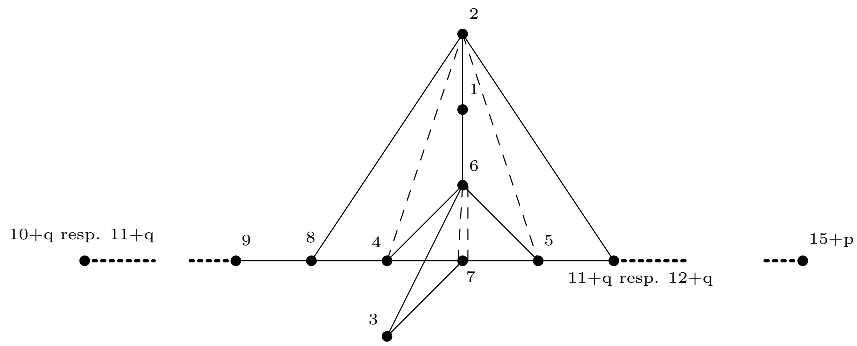


FIGURE 1. The CDD of a distinguished basis $e_1, \dots, e_\mu$ for $W^\sharp_{1,2q-1}$ resp. $W^\sharp_{1,2q}$ from [Eb81, Tabelle 6 & Abb. 16]

The monodromy acts on the distinguished basis $e_1, \dots, e_\mu$ with the CDD in figure 1 as follows:

$$\begin{array}{rcl}
e_1 & \mapsto & -e_1 - e_2 + e_3 + e_4 + e_5 + e_6, \\
e_2 & \mapsto & e_1 + e_2 + e_8 + e_{12+q}, \\
e_3 & \mapsto & -e_1 - e_3 - e_6 + e_7, \\
e_4 & \mapsto & e_2 - e_6 + e_7 + e_8, \\
e_5 & \mapsto & e_2 - e_6 + e_7 + e_{12+q}, \\
e_6 & \mapsto & e_1 - 2e_2 + e_3 + e_4 + e_5 + 3e_6 - 2e_7, \\
e_7 & \mapsto & -2e_2 + e_3 + e_4 + e_5 + 2e_6 - e_7, \\
e_{7+i} & \mapsto & e_{8+i} \qquad \text{for } 1 \leq i \leq 3+q, \\
e_{11+q} & \mapsto & -e_4 - e_8 - e_9 - \ldots - e_{11+q}, \\
e_{11+q+i} & \mapsto & e_{12+q+i} \qquad \text{for } 1 \leq i \leq 3+q, \\
e_{15+p} & \mapsto & -e_5 - e_{12+q} - \ldots - e_{15+p}.
\end{array}$$

By table (5.13) the generators of the Orlik blocks B_1 and B_2 are $\beta_1 := e_3$ and $\beta_2 := e_8$. The monodromy acts on them as follows:

$$\begin{array}{rcl}
e_3 & \mapsto & -e_1 - e_3 - e_6 + e_7 \mapsto e_1 + e_2 - e_4 - e_5 - e_6 \\
& \mapsto & -e_1 \mapsto e_1 + e_2 - e_3 - e_4 - e_5 - e_6,
\end{array} \tag{5.91}$$

$$\begin{array}{rcl}
e_8 & \mapsto & e_9 \mapsto \ldots \mapsto e_{11+q} \mapsto -e_4 - e_8 - e_9 - \ldots - e_{11+q} \\
& \mapsto & -e_2 + e_4 + e_6 - e_7 \mapsto -e_{12+q} \mapsto -e_{13+q} \mapsto \ldots \mapsto -e_{15+p} \\
& \mapsto & e_5 + e_{12+q} + \ldots + e_{15+p} \mapsto e_2 - e_5 - e_6 + e_7 \mapsto e_8.
\end{array} \tag{5.92}$$

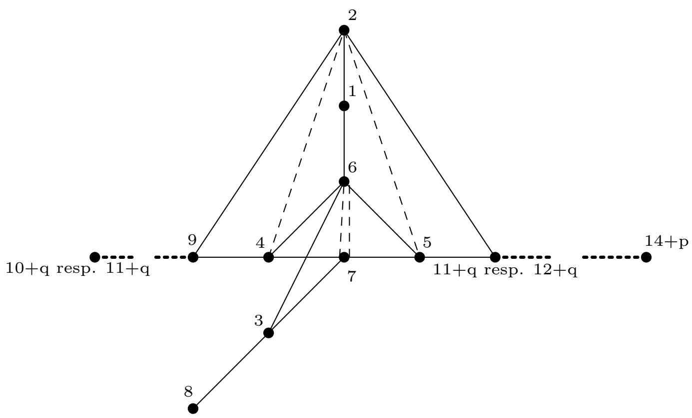


FIGURE 2. The CDD of a distinguished basis $e_1, \dots, e_\mu$ for $S^\sharp_{1,2q-1}$ resp. $S^\sharp_{1,2q}$ from [Eb81, Tabelle 6 & Abb. 16]

Thus the characteristic polynomial of M_h on B_j is b_j, and the blocks are

$$\begin{aligned} B_1 &= \langle e_3, e_1, e_6 - e_7, e_2 - e_4 - e_5 - e_6\rangle, && (5.93)\\ B_2 &= \langle e_8, e_9, \dots, e_{15+p}; e_4, e_5, -e_2 + e_6 - e_7\rangle. && (5.94)\end{aligned}$$

This shows that B_1 and B_2 are primitive sublattices with $B_1 + B_2 = B_1 \oplus B_2 = Ml(f)$, i.e. $r_I = 1$.

5.2. **The series $S^\sharp_{1,p}$.** Again we only describe the case when $p = 2q$ is even. But one can easily obtain the odd case $p = 2q-1$ from that via replacing each $e_{\alpha+q}$ by $e_{\alpha-1+q}$ in the following lists. The monodromy acts on the distinguished basis $e_1, \dots, e_\mu$ with the CDD in figure 2 as follows:

$$\begin{aligned}
e_1 &\mapsto -e_1 - e_2 + e_3 + e_4 + e_5 + e_6,\\
e_2 &\mapsto e_1 + e_2 + e_9 + e_{12+q},\\
e_3 &\mapsto -e_1 - e_6 + e_7 + e_8,\\
e_4 &\mapsto e_2 - e_6 + e_7 + e_9,\\
e_5 &\mapsto e_2 - e_6 + e_7 + e_{12+q},\\
e_6 &\mapsto e_1 - 2e_2 + e_3 + e_4 + e_5 + 3e_6 - 2e_7,\\
e_7 &\mapsto -2e_2 + e_3 + e_4 + e_5 + 2e_6 - e_7,\\
e_8 &\mapsto -e_3 - e_8,\\
e_{8+i} &\mapsto e_{9+i} \qquad \text{for } 1 \le i \le 2+q,\\
e_{11+q} &\mapsto -e_4 - e_9 - e_{10} - \ldots - e_{11+q},\\
e_{11+q+i} &\mapsto e_{12+q+i} \qquad \text{for } 1 \le i \le 2+q,\\
e_{14+p} &\mapsto -e_5 - e_{12+q} - e_{13+q} - \ldots - e_{14+p}.
\end{aligned}$$

By table (5.13) the generators of the Orlik blocks B_1 and B_2 are $\beta_1 := e_8$ and $\beta_2 := e_9$. The monodromy acts on them as follows:

$$\begin{aligned} e_8 &\mapsto -e_3 - e_8 \mapsto e_1 + e_3 + e_6 - e_7 \\ &\mapsto -e_1 - e_2 + e_3 + e_4 + e_5 + e_6 + e_8 \\ &\mapsto -e_3 - e_6 + e_7 \mapsto -e_8, \end{aligned} \tag{5.95}$$

$$\begin{aligned} e_9 &\mapsto e_{10} \mapsto \ldots \mapsto e_{11+q} \mapsto -e_4 - e_9 - e_{10} - \ldots - e_{11+q} \\ &\mapsto -e_2 + e_4 + e_6 - e_7 \mapsto -e_{12+q} \mapsto -e_{13+q} \mapsto \ldots \mapsto -e_{14+p} \\ &\mapsto e_5 + e_{12+q} + \ldots + e_{14+p} \mapsto e_2 - e_5 - e_6 + e_7 \mapsto e_9. \end{aligned} \tag{5.96}$$

Thus the characteristic polynomial of M_h on B_j is b_j, and the blocks are

$$B_1 = \langle e_8, e_3, e_6 - e_7, e_1, -e_2 + e_4 + e_5 + e_6 \rangle, \tag{5.97}$$

$$B_2 = \langle e_9, e_{10}, \ldots, e_{14+p}; e_4, e_5, -e_2 + e_6 - e_7 \rangle. \tag{5.98}$$

This shows that B_1 and B_2 are primitive sublattices with $B_1 + B_2 = B_1 \oplus B_2 = Ml(f)$ and $r_I = 1$.

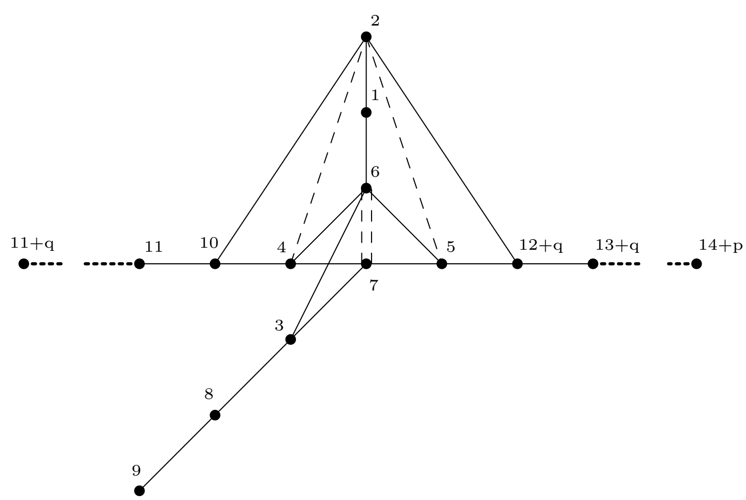


FIGURE 3. The CDD of a distinguished basis $e_1, \ldots, e_\mu$ for $U_{1,p}$ from [Eb81, Tabelle 6 & Abb. 16]

5.3. **The series $U_{1,p}$.** Here (and in all series except $W^\sharp_{1,p}$ and $S^\sharp_{1,p}$) the list of the monodromy action on the distinguished basis $e_1, \ldots, e_\mu$ with the CDD in figure 3 includes both cases $p = 2q$

and $p=2q-1$. It looks as follows:

$$\begin{aligned}
e_1 &\mapsto -e_1-e_2+e_3+e_4+e_5+e_6,\\
e_2 &\mapsto e_1+e_2+e_{10}+e_{12+q},\\
e_3 &\mapsto -e_1-e_6+e_7+e_8,\\
e_4 &\mapsto e_2-e_6+e_7+e_{10},\\
e_5 &\mapsto e_2-e_6+e_7+e_{12+q},\\
e_6 &\mapsto e_1-2e_2+e_3+e_4+e_5+3e_6-2e_7,\\
e_7 &\mapsto -2e_1+e_3+e_4+e_5+2e_6-e_7,\\
e_8 &\mapsto e_9,\\
e_9 &\mapsto -e_3-e_8-e_9,\\
e_{9+i} &\mapsto e_{10+i} \qquad \text{for } 1\leq i\leq 1+q,\\
e_{11+q} &\mapsto -e_4-e_{10}-e_{11}-\ldots-e_{11+q},\\
e_{11+q+i} &\mapsto e_{12+q+i} \qquad \text{for } 1\leq i\leq 2+p-q,\\
e_{14+p} &\mapsto -e_5-e_{12+q}-e_{13+q}-\ldots-e_{14+p}.
\end{aligned}$$

By table (5.13) the generators of the Orlik blocks B_1 and B_2 are $\beta_1 := e_8$ and $\beta_2 := e_{10}$. The monodromy acts on them as follows:

$$\begin{aligned}
e_8 &\mapsto e_9\mapsto -e_3-e_8-e_9\mapsto e_1+e_3+e_6-e_7\\
&\mapsto -e_1-e_2+e_3+e_4+e_5+e_6+e_8\\
&\mapsto -e_6+e_7+e_8+e_9\mapsto -e_1-e_3-e_6+e_7-e_8, \qquad (5.99)\\
e_{10} &\mapsto e_{11}\mapsto\ldots\mapsto e_{11+q}\mapsto -e_4-e_{10}-e_{11}-\ldots-e_{11+q}\\
&\mapsto -e_2+e_4+e_6-e_7\mapsto -e_{12+q}\mapsto -e_{13+q}\mapsto\ldots\mapsto -e_{14+p}\\
&\mapsto e_5+e_{12+q}+\ldots+e_{14+p}\mapsto e_2-e_5-e_6+e_7\mapsto e_{10}. \qquad (5.100)
\end{aligned}$$

Thus the characteristic polynomial of M_h on B_j is b_j, and the blocks are

$$B_1 = \langle e_1,e_3,e_8,e_9,e_6-e_7,-e_2+e_4+e_5+e_6\rangle, \qquad (5.101)$$
$$B_2 = \langle e_{10},e_{11},\ldots,e_{14+p};e_4,e_5,-e_2+e_6-e_7\rangle. \qquad (5.102)$$

Again B_1 and B_2 are primitive sublattices with $B_1+B_2=B_1\oplus B_2=Ml(f)$ and $r_I=1$.

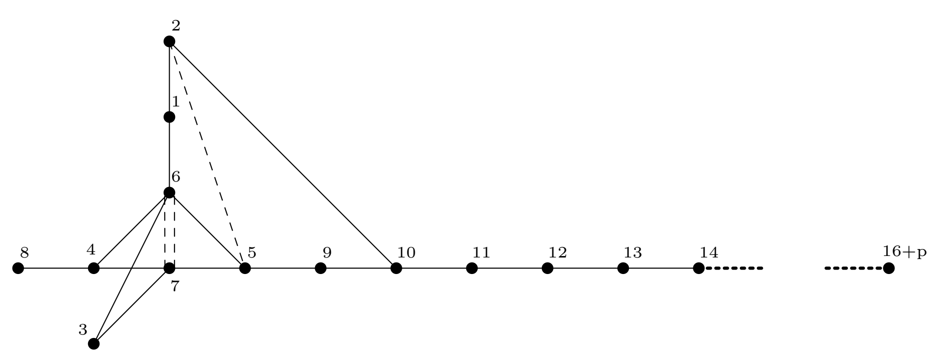


FIGURE 4. The CDD of a distinguished basis $e_1,\ldots,e_\mu$ for $E_{3,p}$ from [Eb81, Tabelle 6 & Abb. 16]

5.4. **The series $E_{3,p}$.** Here the monodromy acts on the distinguished basis $e_1, \ldots, e_\mu$ with the CDD in figure 4 as follows:

$$
\begin{array}{rcl}
e_1 & \mapsto & e_3 + e_4 + e_5 + e_6, \\
e_2 & \mapsto & e_9 + e_{10}, \\
e_3 & \mapsto & -e_1 - e_3 - e_6 + e_7, \\
e_4 & \mapsto & -e_1 - e_6 + e_7 + e_8, \\
e_5 & \mapsto & -e_1 - e_6 + e_7 + e_9, \\
e_6 & \mapsto & 2e_1 - e_2 + e_3 + e_4 + e_5 + 3e_6 - 2e_7, \\
e_7 & \mapsto & e_1 - e_2 + e_3 + e_4 + e_5 + 2e_6 - e_7, \\
e_8 & \mapsto & -e_4 - e_8, \\
e_9 & \mapsto & e_1 + e_2 + e_{10}, \\
e_{9+i} & \mapsto & e_{10+i} \qquad \text{for } 1 \le i \le 6 + p, \\
e_{16+p} & \mapsto & -e_5 - e_9 - e_{10} - \ldots - e_{16+p}.
\end{array}
$$

By table (5.13) the generators of the Orlik blocks B_1 and B_2 are $\beta_1 := e_3$ and $\beta_2 := e_{10}$. The monodromy acts on them as follows:

$$
\begin{array}{rcl}
e_3 & \mapsto & -e_1 - e_3 - e_6 + e_7 \mapsto -e_4 - e_5 - e_6 \\
 & \mapsto & e_2 - e_3 - e_4 - e_5 - e_6 - e_8 - e_9 \mapsto -e_5 - e_7 \\
 & \mapsto & e_2 - e_3 - e_4 - e_5 - e_6 - e_9 \mapsto -e_4 - e_5 - e_7 - e_8 \\
 & \mapsto & e_1 + e_2 - e_3 - e_5 - e_7 - e_9 \mapsto e_3 + e_6 - e_7 \\
 & \mapsto & -e_3,
\end{array} \tag{5.103}
$$

$$
\begin{array}{rcl}
e_{10} & \mapsto & e_{11} \mapsto \ldots \mapsto e_{16+p} \mapsto -e_5 - \sum_{i=9}^{16+p} e_i \\
 & \mapsto & -e_2 + e_5 + e_6 - e_7 \mapsto -e_{10}.
\end{array} \tag{5.104}
$$

Thus the characteristic polynomial of M_h on B_j is b_j, and the blocks are

$$B_1 = \langle e_1, e_3, e_4, e_8, e_6 - e_7, e_5 + e_6, e_2 - e_9 \rangle, \tag{5.105}$$

$$B_2 = \langle e_{10}, e_{11}, \ldots, e_{16+p}, e_5 + e_9, e_2 - e_6 + e_7 + e_9 \rangle. \tag{5.106}$$

This shows that B_1 and B_2 are primitive sublattices with $B_1 + B_2 = B_1 \oplus B_2$. Furthermore $B_1 \oplus B_2 \supset \{2e_2\}$ and $B_1 + B_2 + \mathbb{Z} \cdot e_2 = Ml(f)$. This shows $[Ml(f) : B_1 \oplus B_2] = 2 = r_I$.

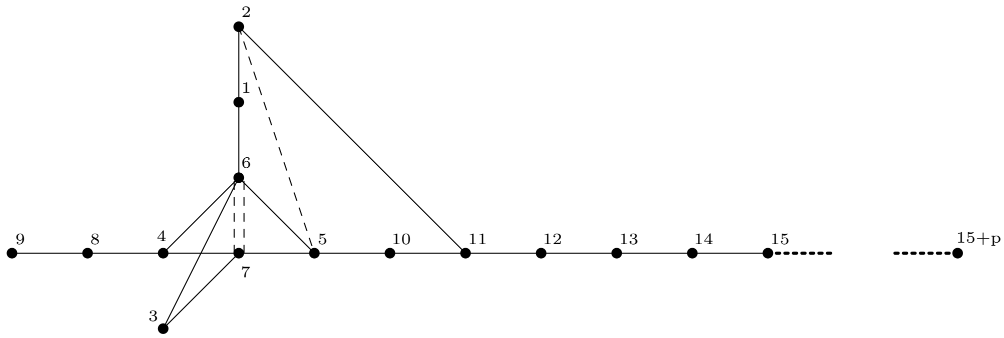


FIGURE 5. The CDD of a distinguished basis $e_1, \ldots, e_\mu$ for $Z_{1,p}$ from [Eb81, Tabelle 6 & Abb. 16]

5.5. **The series $Z_{1,p}$.** Here the monodromy acts on the distinguished basis $e_1, \ldots, e_\mu$ with the CDD in figure 5 as follows:

$$\begin{aligned}
e_1 &\mapsto e_3 + e_4 + e_5 + e_6, \\
e_2 &\mapsto e_{10} + e_{11}, \\
e_3 &\mapsto -e_1 - e_3 - e_6 + e_7, \\
e_4 &\mapsto -e_1 - e_6 + e_7 + e_8, \\
e_5 &\mapsto -e_1 - e_6 + e_7 + e_{10}, \\
e_6 &\mapsto 2e_1 - e_2 + e_3 + e_4 + e_5 + 3e_6 - 2e_7, \\
e_7 &\mapsto e_1 - e_2 + e_3 + e_4 + e_5 + 2e_6 - e_7, \\
e_8 &\mapsto e_9, \\
e_9 &\mapsto -e_4 - e_8 - e_9, \\
e_{10} &\mapsto e_1 + e_2 + e_{11}, \\
e_{11+i} &\mapsto e_{12+i} \qquad \text{for } 1 \leq i \leq 3 + p, \\
e_{15+p} &\mapsto -e_5 - e_{10} - e_{11} - \ldots - e_{15+p}.
\end{aligned}$$

Here there are three Orlik blocks B_1, B_2 and B_3. By table (5.13) their generators are

$$\beta_1 := e_8, \quad \beta_2 := e_{11}, \quad \text{and} \quad \beta_3 := e_3 + e_4 - e_9.$$

The monodromy acts on them as follows:

$$\begin{aligned}
e_8 &\mapsto e_9 \mapsto -e_4 - e_8 - e_9 \mapsto e_1 + e_4 + e_6 - e_7 \\
&\mapsto e_3 + e_4 + e_5 + e_6 + e_8 \\
&\mapsto -e_1 - e_2 + e_4 + e_5 + e_7 + e_8 + e_9 + e_{10} \\
&\mapsto -e_4 - e_6 + e_7 \mapsto -e_8, \qquad (5.107) \\
e_{11} &\mapsto e_{12} \mapsto \ldots \mapsto e_{15+p} \mapsto -e_5 - \sum_{i=10}^{15+p} e_i \\
&\mapsto -e_2 + e_5 + e_6 - e_7 \mapsto -e_{11}, \qquad (5.108) \\
e_3 - e_4 - e_9 &\mapsto -e_3 + e_4 + e_9. \qquad (5.109)
\end{aligned}$$

Thus the characteristic polynomial of M_h on B_j is b_j, and the blocks are

$$\begin{aligned} B_1 &= \langle e_8, e_9, e_4, e_1, e_6 - e_7, e_3 + e_5 + e_6, \\ &\quad -e_2 + e_5 + e_7 + e_{10}\rangle, \qquad (5.110) \\ B_2 &= \langle e_{11}, e_{12}, \dots, e_{15+p}; e_5 + e_{10}, -e_2 + e_5 + e_6 - e_7\rangle, \qquad (5.111) \\ B_3 &= \langle e_3 - e_4 - e_9\rangle. \qquad (5.112) \end{aligned}$$

This shows that B_1, B_2 and B_3 are primitive sublattices with $B_1 + B_2 + B_3 = B_1 \oplus B_2 \oplus B_3$. Furthermore $B_1 \oplus B_2 \oplus B_3 \supset \{2e_5\}$ and $B_1 + B_2 + B_3 + \mathbb{Z} \cdot e_5 = Ml(f)$. This shows $[Ml(f) : B_1 \oplus B_2 \oplus B_3] = 2 = r_I$.

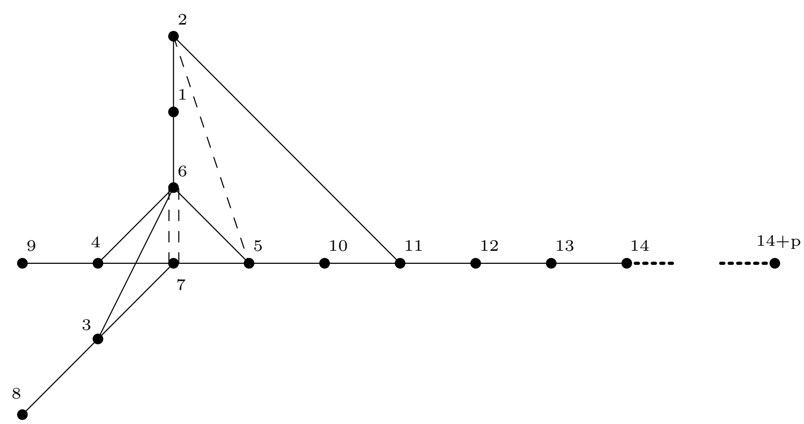


FIGURE 6. The CDD of a distinguished basis $e_1, \dots, e_\mu$ for $Q_{2,p}$ from [Eb81, Tabelle 6 & Abb. 16]

5.6. **The series $Q_{2,p}$.** Here the monodromy acts on the distinguished basis $e_1, \dots, e_\mu$ with the CDD in figure 6 as follows:

$$\begin{aligned} e_1 &\mapsto e_3 + e_4 + e_5 + e_6, \\ e_2 &\mapsto e_{10} + e_{11}, \\ e_3 &\mapsto -e_1 - e_6 + e_7 + e_8, \\ e_4 &\mapsto -e_1 - e_6 + e_7 + e_9, \\ e_5 &\mapsto -e_1 - e_6 + e_7 + e_{10}, \\ e_6 &\mapsto 2e_1 - e_2 + e_3 + e_4 + e_5 + 3e_6 - 2e_7, \\ e_7 &\mapsto e_1 - e_2 + e_3 + e_4 + e_5 + 2e_6 - e_7, \\ e_8 &\mapsto -e_3 - e_8, \\ e_9 &\mapsto -e_4 - e_9, \\ e_{10} &\mapsto e_1 + e_2 + e_{11}, \\ e_{10+i} &\mapsto e_{11+i} \qquad \text{for } 1 \le i \le 3 + p, \\ e_{14+p} &\mapsto -e_5 - e_{10} - e_{11} - \ldots - e_{14+p}. \end{aligned}$$

By table (5.13) the generators of the Orlik blocks B_1 and B_2 are $\beta_1 := e_8$ and $\beta_2 := e_{11}$. The monodromy acts on them as follows:

$$\begin{aligned} e_8 &\mapsto -e_3 - e_8 \mapsto e_1 + e_3 + e_6 - e_7 \mapsto e_3 + e_4 + e_5 + e_6 + e_8 \\ &\mapsto -e_1 - e_2 + e_4 + e_5 + e_7 + e_9 + e_{10} \mapsto -e_4 - e_6 + e_7 \\ &\mapsto -e_9 \mapsto e_4 + e_9 \mapsto -e_1 - e_4 - e_6 + e_7, \end{aligned} \tag{5.113}$$

$$\begin{aligned} e_{11} &\mapsto e_{12} \mapsto \ldots \mapsto e_{14+p} \mapsto -e_5 - \sum_{i=10}^{14+p} e_i \\ &\mapsto -e_2 + e_5 + e_6 - e_7 \mapsto -e_{11}. \end{aligned} \tag{5.114}$$

Thus the characteristic polynomial of M_h on B_j is b_j, and the blocks are

$$\begin{aligned} B_1 &= \langle e_8, e_3, e_9, e_4, e_1, e_6 - e_7, \\ &\quad e_5 + e_6, -e_2 + e_5 + e_7 + e_{10}\rangle, \end{aligned} \tag{5.115}$$

$$B_2 = \langle e_{11}, e_{12}, \ldots, e_{14+p}; e_5 + e_{10}, -e_2 + e_5 + e_6 - e_7\rangle. \tag{5.116}$$

This shows that B_1 and B_2 are primitive sublattices with $B_1 + B_2 = B_1 \oplus B_2$. Furthermore $B_1 \oplus B_2 \supset \{2e_5\}$ and $B_1 + B_2 + \mathbb{Z} \cdot e_5 = Ml(f)$. This shows $[Ml(f) : B_1 \oplus B_2] = 2 = r_I$.

The proof of (5.4) for $Q_{2,p}$ was postponed to this subsection and has to be given now. Recall the definition (5.37) of b_4 and recall $b_4 = \Phi_4$ for $Q_{2,4s}$ and $b_4 = 1$ for the other $Q_{2,p}$. The next aims are:

(i) For $Q_{2,4s}$: To show for any $g \in G_{\mathbb{Z}} \cup \mathrm{Aut}(B_1 \oplus B_2, L)$

$$g : (B_1)_{b_4} \to (B_1)_{b_4} \text{ and } (B_2)_{b_4} \to (B_2)_{b_4}. \tag{5.117}$$

(ii) For all $Q_{2,p}$: To find an element $\gamma_4 \in (B_1)_{\Phi_4}$ with

$$\begin{aligned} B_1 \oplus B_2 &= \{a \in Ml(f) \,|\, L(a, \gamma_4) \equiv 0(2)\} \end{aligned} \tag{5.118}$$

$$\begin{aligned} &= \{a \in Ml(f) \,|\, L(a, M_h(\gamma_4)) \equiv 0(2)\}, \end{aligned}$$

$$g(\gamma_4) \in \{\pm\gamma_4, \pm M_h(\gamma_4)\} \qquad \text{for any } g \in G_{\mathbb{Z}}. \tag{5.119}$$

(iii) For all $Q_{2,p}$: To find an element $\gamma_5 \in Ml(f)$ with

$$B_1 + B_2 + \mathbb{Z} \cdot \gamma_5 = Ml(f) \tag{5.120}$$

$$\text{and } g(\gamma_5) \in Ml(f) \text{ for any } g \in \mathrm{Aut}(B_1 \oplus B_2, L). \tag{5.121}$$

For all $Q_{2,p}$ define

$$\begin{aligned} \gamma_1 &:= \frac{b_1}{\Phi_4}(M_h)(\beta_1) = (\Phi_{12}\Phi_3)(M_h)(e_8) \\ &= (t^6 + t^5 - t^3 + t + 1)(M_h)(e_8) \\ &= -2e_3 - 2e_4 - e_5 - 2e_6 + e_7 - e_8 - e_9. \end{aligned} \tag{5.122}$$

Obviously $M_h^2(\gamma_1) = -\gamma_1$. By remark 2.6 (v), $(B_1)_{\Phi_4}$ is an Orlik block with cyclic generator γ_1, so $(B_1)_{\Phi_4} = \mathbb{Z} \cdot \gamma_1 \oplus \mathbb{Z} \cdot M_h(\gamma_1)$. Calculate

$$M_h(\gamma_1) = 2e_1 + e_2 - e_5 + e_6 - 2e_7 - e_8 - e_9 - e_{10}. \tag{5.123}$$

For $Q_{2,4s}$ define

$$\begin{aligned} \gamma_2 &:= \frac{b_2}{\Phi_4}(M_h)(\beta_2) = \frac{t^{6+4s}+1}{t^2+1}(M_h)(e_{11}) \\ &= (t^{4+4s} - t^{2+4s} + t^{4s} - \ldots - t^2 + 1)(M_h)(e_{11}) \\ &= -e_5 - e_{10} + (-1)\sum_{j=1}^{2+2s} e_{10+2j} + (-2)\sum_{j=1}^{1+s} e_{9+4j}. \end{aligned} \tag{5.124}$$

Obviously $M_h^2(\gamma_2) = -\gamma_2$. By remark 2.6 (v), $(B_2)_{\Phi_4}$ is an Orlik block with cyclic generator γ_2, so $(B_2)_{\Phi_4} = \mathbb{Z}\cdot\gamma_2 \oplus \mathbb{Z}\cdot M_h(\gamma_2)$. Calculate

$$M_h(\gamma_2) = -e_2 + e_5 + e_6 - e_7 + \sum_{j=1}^{2+2s}(-1)^{j+1}e_{10+2j}. \tag{5.125}$$

For $Q_{2,4s}$ define

$$\gamma_3 := \frac{1}{2}(\gamma_1 + M_h(\gamma_1) + \gamma_2 + M_h(\gamma_2)) \tag{5.126}$$

and observe

$$\begin{aligned} \gamma_3 &= e_1 - \sum_{j\in\{3,4,5,7,8,9,10\}} e_j - \sum_{j=1}^{1+s}(e_{9+4j} + e_{10+4j}) \\ &\overset{!}{\in} Ml(f). \end{aligned} \tag{5.127}$$

Together with $[Ml(f) : B_1 \oplus B_2] = 2$ this shows (5.120) and that $\gamma_1, M_h(\gamma_1), \gamma_3, M_h(\gamma_3)$ is a $\mathbb{Z}$-basis of $Ml(f)_{\Phi_4}$. We want to calculate the matrices of L with respect to the basis $\gamma_1, M_h(\gamma_1), \gamma_2, M_h(\gamma_2)$ of $(B_1 \oplus B_2)_{\Phi_4}$ and the basis $\gamma_1, M_h(\gamma_1), \gamma_3, M_h(\gamma_3)$ of $Ml(f)_{\Phi_4}$. Essentially we need to calculate only the values $L(\gamma_1, \gamma_1)$ and $L(\gamma_2, \gamma_2)$, because of (5.3) and because of the identities for any $a \in Ml(f)_{\Phi_4}$,

$$\begin{aligned} L(a, M_h(a)) &= L(M_h(a), M_h^2(a)) = -L(M_h(a), a) \\ &= L(a, a) = L(M_h(a), M_h(a)). \end{aligned} \tag{5.128}$$

Using $M_h^2(\gamma_j) = -\gamma_j$ and calculations similar to (2.17), we find

$$L(\gamma_1, \gamma_1) = L(\frac{b_1}{\Phi_4}(-M_h^{-1})(\gamma_1), e_8) = 3\cdot L(M_h(\gamma_1), e_8) = 3, \tag{5.129}$$

$$\begin{aligned} L(\gamma_2, \gamma_2) &= L(\frac{b_2}{\Phi_4}(M_h^{-1})(\gamma_2), e_{11}) \\ &= (3+2s)\cdot L(\gamma_2, e_{11}) = 3 + 2s, \end{aligned} \tag{5.130}$$

thus

$$L(\begin{pmatrix}\gamma_1\\ M_h(\gamma_1)\\ \gamma_2\\ M_h(\gamma_2)\end{pmatrix},\begin{pmatrix}\gamma_1\\ M_h(\gamma_1)\\ \gamma_2\\ M_h(\gamma_2)\end{pmatrix}^t)=\begin{pmatrix}3&3&0&0\\ -3&3&0&0\\ 0&0&3+2s&3+2s\\ 0&0&-(3+2s)&3+2s\end{pmatrix} \tag{5.131}$$

and

$$L(\begin{pmatrix}\gamma_1\\ M_h(\gamma_1)\\ \gamma_3\\ M_h(\gamma_3)\end{pmatrix},\begin{pmatrix}\gamma_1\\ M_h(\gamma_1)\\ \gamma_3\\ M_h(\gamma_3)\end{pmatrix}^t)=\begin{pmatrix}3&3&3&0\\ -3&3&0&3\\ 0&3&3+s&3+s\\ -3&0&-(3+s)&3+s\end{pmatrix}. \tag{5.132}$$

The quadratic form associated to the last matrix is

$$\frac{3}{2}\cdot\left[(x_1+x_3)^2+(x_1-x_4)^2+(x_2+x_3)^2+(x_2+x_4)^2\right] +s\cdot(x_3^2+x_4^2). \tag{5.133}$$

This shows (first for $Q_{2,4s}$, but in fact for all $Q_{2,p}$)

$$\{a\in Ml(f)_{\Phi_4}\,|\,L(a,a)=3\}=\{\pm\gamma_1,\pm M_h(\gamma_1)\}, \tag{5.134}$$

and because of $(B_1\oplus B_2)_{\Phi_4}\subset Ml(f)_{\Phi_4}$

$$\{a\in (B_1\oplus B_2)_{\Phi_4}\,|\,L(a,a)=3\}=\{\pm\gamma_1,\pm M_h(\gamma_1)\}, \tag{5.135}$$

This implies that any $g\in G_{\mathbb{Z}}\cup \mathrm{Aut}(B_1\oplus B_2,L)$ maps the set $\{\pm\gamma_1,\pm M_h(\gamma_1)\}$ to itself and thus $(B_1)_{\Phi_4}$ to itself and thus the L-orthogonal sublattice $(B_2)_{\Phi_4}$ to itself. This shows (5.117) and gives (i).

Define for all $Q_{2,p}$

$$\begin{aligned}\gamma_4 &:= \gamma_1+M_h(\gamma_1) \\ &= 2e_1+e_2-2e_3-2e_4-2e_5-e_6-e_7-2e_8-2e_9-e_{10}.\end{aligned} \tag{5.136}$$

Observe

$$M_h(\gamma_4) = -\gamma_1+M_h(\gamma_1) \tag{5.137}$$

$$= -2\gamma_1+\gamma_4. \tag{5.138}$$

(5.134) and (5.137) imply (5.119). (5.138) implies the second equality in (5.118). One calculates

$$L(e_8,\gamma_4)=0. \tag{5.139}$$

This shows $L(e_8,M_h(\gamma_4))\equiv 0(2)$ (in fact, it is $=-2$). The M_h-invariance of L and the fact that e_8 is a cyclic generator of the Orlik block B_1 give $B_1\subset\{a\in Ml(f)\,|\,L(a,\gamma_4)\equiv 0(2)\}$. As (5.3) implies $L(B_2,\gamma_4)=0$, so $B_1\oplus B_2\subset\{a\in Ml(f)\,|\,L(a,\gamma_4)\equiv 0(2)\}$. Now $r_I=2$ and for example $L(e_2,\gamma_4)=-1\not\equiv 0(2)$ show (5.118) and (ii). (ii) implies $G_{\mathbb{Z}}\subset \mathrm{Aut}(B_1\oplus B_2,L)$.

(iii) implies $\mathrm{Aut}(B_1\oplus B_2,L)\subset G_{\mathbb{Z}}$, but (iii) has still to be proved.

We continue as in the final part of the proof of part (a) for the other series. (i) holds. Lemma 2.8 can be applied. Therefore (5.42) and (5.43) hold for $Q_{2,p}$. The group $\mathrm{Aut}(B_1\oplus B_2,L)$ for $12\not|p$ is generated by $M_h,-\mathrm{id},M_h|_{B_1}\times \mathrm{id}|_{B_2}$ and $(-\mathrm{id})|_{B_1}\times\mathrm{id}|_{B_2}$, and analogously for the group in (5.43) if $12|p$.

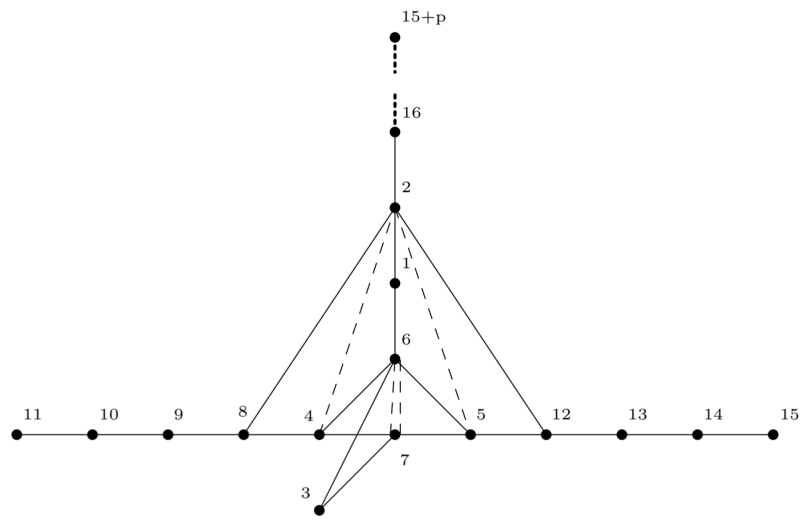


FIGURE 7. The CDD of a distinguished basis $e_1, \dots, e_\mu$ for $W_{1,p}$ from [Eb81, Tabelle 6 & Abb. 16]

For $Q_{2,4s}$ we define $\gamma_5 := \gamma_3$. It satisfies (5.120). If $12|4s$, it is in $(B_1)_{b_1/\Phi_m} + (B_3)_{b_2/\Phi_m}$, so we can work with the group in (5.43). If $12 \not| 4s$, we work with the group in (5.42). In both cases γ_5 satisfies (5.121), because of

$$(M_h|_{B_1} \times \mathrm{id}|_{B_2})(\gamma_5) = \gamma_5 - M_h(\gamma_1) \in Ml(f), \tag{5.140}$$

$$((-\mathrm{id})|_{B_1} \times \mathrm{id}|_{B_2})(\gamma_5) = \gamma_5 - (\gamma_1 + M_h(\gamma_1)) \in Ml(f). \tag{5.141}$$

For other $Q_{2,p}$, we choose a different (rather simple) γ_5,

$$\begin{aligned} \gamma_5 &:= e_{10} \\ &= \frac{1}{2}(-e_2 + e_6 - e_7 + e_{10}) - \frac{1}{2}(-e_2 + e_6 - e_7 - e_{10}), \\ \text{with} \quad & -e_2 + e_6 - e_7 + e_{10} \in B_1, -e_2 + e_6 - e_7 - e_{10} \in B_2. \end{aligned} \tag{5.142}$$

Then (5.120) holds. And

$$(M_h|_{B_1} \times \mathrm{id}|_{B_2})(\gamma_5) = e_1 + e_2 \in Ml(f), \tag{5.143}$$

$$((-\mathrm{id})|_{B_1} \times \mathrm{id}|_{B_2})(\gamma_5) = e_2 - e_6 + e_7 \in Ml(f). \tag{5.144}$$

In any case (5.120) and (5.121) and (iii) hold. Thus $\mathrm{Aut}(B_1 \oplus B_2, L) \subset G_{\mathbb{Z}}$, and (5.4) is proved for $Q_{2,p}$.

5.7. **The series $W_{1,p}$.** Here the monodromy acts on the distinguished basis $e_1, \ldots, e_\mu$ with the CDD in figure 7 as follows:

$$\begin{array}{rcl}
e_1 & \mapsto & -e_1 - e_2 + e_3 + e_4 + e_5 + e_6, \\
e_2 & \mapsto & 2e_1 + 2e_2 + e_8 + e_{12} + e_{16}, \\
e_3 & \mapsto & -e_1 - e_3 - e_6 + e_7, \\
e_4 & \mapsto & e_2 - e_6 + e_7 + e_8, \\
e_5 & \mapsto & e_2 - e_6 + e_7 + e_{12}, \\
e_6 & \mapsto & e_1 - 2e_2 + e_3 + e_4 + e_5 + 3e_6 - 2e_7, \\
e_7 & \mapsto & -2e_2 + e_3 + e_4 + e_5 + 2e_6 - e_7, \\
e_8 & \mapsto & e_9, \\
e_9 & \mapsto & e_{10}, \\
e_{10} & \mapsto & e_{11}, \\
e_{11} & \mapsto & -e_4 - e_8 - e_9 - e_{10} - e_{11}, \\
e_{12} & \mapsto & e_{13}, \\
e_{13} & \mapsto & e_{14}, \\
e_{14} & \mapsto & e_{15}, \\
e_{15} & \mapsto & -e_5 - e_{12} - e_{13} - e_{14} - e_{15}, \\
e_{15+i} & \mapsto & e_{16+i} \qquad \text{for } 1 \leq i \leq p-1, \\
e_{15+p} & \mapsto & -e_1 - e_2 - e_{16} - e_{17} - \ldots - e_{15+p}.
\end{array}$$

By table (5.13) the generators of the Orlik blocks B_1 and B_2 are $\beta_1 := e_3+e_9+e_{11}$ and $\beta_2 := e_{16}$. The monodromy acts on them as follows:

$$\begin{array}{rcl}
e_3 + e_9 + e_{11} & \mapsto & -e_1 - e_3 - e_4 - e_6 + e_7 - e_8 - e_9 - e_{11} \\
& \mapsto & e_1 - e_5 - e_7 + e_{11} \\
& \mapsto & -e_1 - e_4 - e_8 - e_9 - e_{10} - e_{11} - e_{12} \\
& \mapsto & e_1 - e_3 - e_5 - e_7 - e_{13} \\
& \mapsto & e_3 + e_6 - e_7 - e_{12} - e_{14} \\
& \mapsto & -e_3 - e_{13} - e_{15} \\
& \mapsto & e_1 + e_3 + e_5 + e_6 - e_7 + e_{12} + e_{13} + e_{15} \\
& \mapsto & -e_1 + e_4 + e_7 - e_{15} \\
& \mapsto & e_1 + e_5 + e_8 + e_{12} + e_{13} + e_{14} + e_{15} \\
& \mapsto & -e_1 + e_3 + e_4 + e_7 + e_9 \\
& \mapsto & -e_3 - e_6 + e_7 + e_8 + e_{10} \\
& \mapsto & e_3 + e_9 + e_{11},
\end{array} \tag{5.145}$$

$$\begin{aligned}
e_{16} &\mapsto e_{17} \mapsto \ldots \mapsto e_{14+p} \mapsto e_{15+p} \\
&\mapsto -e_1 - e_2 - \sum_{i=16}^{15+p} e_i \\
&\mapsto -e_3 - e_4 - e_5 - e_6 - e_8 - e_{12} \\
&\mapsto -e_4 - e_5 - e_7 - e_8 - e_9 - e_{12} - e_{13} \\
&\mapsto -e_3 - e_4 - e_5 - e_7 - e_8 - e_9 - e_{10} - e_{12} - e_{13} - e_{14} \\
&\mapsto e_1 - e_4 - e_5 + e_6 - 2e_7 - \sum_{i=8}^{15} e_i \\
&\mapsto -e_2 + e_4 + e_5 + 2e_6 - 2e_7 \qquad (5.146) \\
&\mapsto -e_{16}.
\end{aligned}$$

Thus the characteristic polynomial of M_h on B_j is b_j. Here the blocks B_1 and B_2 are generated by the first $\deg b_1$ respectively $\deg b_2$ of the elements above. Here $B_1 + B_2 = B_1 \oplus B_2$ and $[Ml(f) : B_1 \oplus B_2] = 2 = r_I$ follow by the calculation of the determinant which expresses these generators of B_1 and B_2 in the distinguished basis $e_1, \ldots, e_\mu$. Then it also follows that B_1 and B_2 are primitive sublattices.

The proof of (5.4) for $W_{1,6s-3}$ was postponed to this subsection and has to be given here. But the majority of the arguments was already given in the proof of part (a). It rests to prove the following two points:

(i) (5.34) holds for $W_{1,3}$.
(ii) In the case $W_{1,6s-3}$, any $g \in G_{\mathbb{Z}} \cup \mathrm{Aut}(B_1 \oplus B_2, L)$ maps $(B_1)_{b_4}$ to itself and $(B_2)_{b_4}$ to itself. Here $b_4 = \Phi_6\Phi_2$.

For the rest of this subsection we restrict to W_{6s-3}. Define for it

$$\begin{aligned}
\delta_1 &:= \frac{b_1}{\Phi_6\Phi_2}(M_h)(\beta_1) = (\Phi_{12}\Phi_3)(M_h)(e_3 + e_9 + e_{11}) \qquad (5.147) \\
&= \Phi_3(M_h)(e_9 - e_{13}) = e_9 + e_{10} + e_{11} - e_{13} - e_{14} - e_{15}, \\
\delta_2 &:= \frac{b_2}{\Phi_6\Phi_2}(M_h)(\beta_2) = \frac{t^{6+p}+1}{t^3+1}(M_h)(e_{16}) \qquad (5.148) \\
&= (t^{3+p} - t^p + \ldots - t^3 + 1)(M_h)(e_{16}) \\
&= e_1 + e_2 - \sum_{j \in \{3,4,5,7,8,9,10,12,13,14\}} e_j + \sum_{j=1}^{p} e_{15+j} + \sum_{j=0}^{p/3-1} (-1)^j e_{16+3j}.
\end{aligned}$$

δ_1 and δ_2 are cyclic generators of the Orlik blocks $(B_1)_{\Phi_6\Phi_2}$ and $(B_2)_{\Phi_6\Phi_2}$, see remark 2.6 (v). Thus $\delta_i, M_h(\delta_i)$ and $M_h^2(\delta_i)$ are a $\mathbb{Z}$-basis of $(B_i)_{\Phi_6\Phi_2}$. One calculates

$$\begin{aligned}
M_h(\delta_1) &= -e_4 - e_8 - e_9 + e_5 + e_{12} + e_{13}, && (5.149)\\
M_h^2(\delta_1) &= -e_8 - e_9 - e_{10} + e_{12} + e_{13} + e_{14}, && (5.150)\\
M_h(\delta_2) &= e_1 + e_3 + 2e_6 - 2e_7 - \sum_{j=9,10,11,13,14,15} e_j \\
&+ \sum_{j=0}^{p/3-1} (-1)^j e_{17+3j}, && (5.151)\\
M_h^2(\delta_2) &= -e_2 + 2e_4 + 2e_5 + 2e_6 - e_7 + e_8 + e_9 + e_{12} + e_{13} \\
&+ \sum_{j=0}^{p/3-1} (-1)^j e_{18+3j}. && (5.152)
\end{aligned}$$

We need to calculate the 6×6 matrix of values of L for the $\mathbb{Z}$-basis $\delta_1, M_h(\delta_1), M_h^2(\delta_1), \delta_2, M_h(\delta_2), M_h^2(\delta_2)$ of $(B_1 \oplus B_2)_{\Phi_6\Phi_2}$. Because of (5.3), it is block diagonal with two 3×3 blocks. Because L is M_h-invariant and because of the identities for any $a \in Ml(f)_{\Phi_6\Phi_2}$,

$$L(M_h(a), a) = -L(a, a), L(M_h^2(a), a) = -L(a, M_h(a)),$$
$$L(a, M_h^2(a)) = L(M_h(a), M_h^3(a)) = -L(M_h(a), a) = L(a, a),$$

each 3×3 matrix is determined by two values. The matrices are

$$L(M_h^i(\delta_1), M_h^j(\delta_1))_{i,j=0,1,2} = \begin{pmatrix} 2 & 2 & 2 \\ -2 & 2 & 2 \\ -2 & -2 & 2 \end{pmatrix}, \tag{5.153}$$

$$L(M_h^i(\delta_2), M_h^j(\delta_2))_{i,j=0,1,2} = \begin{pmatrix} 1+2s & 0 & 1+2s \\ -1-2s & 1+2s & 0 \\ 0 & -1-2s & 1+2s \end{pmatrix}. \tag{5.154}$$

Recall the definition $\widetilde{\gamma}_2 := \frac{1}{2}(\gamma_1 + \gamma_2)$ in (5.21), and recall

$$Ml(f)_{\Phi_2} = \mathbb{Z}\gamma_1 \oplus \mathbb{Z}\widetilde{\gamma}_2 \overset{2:1}{\supset} \mathbb{Z}\gamma_1 \oplus \mathbb{Z}\gamma_2 = (B_1 \oplus B_2)_{\Phi_2}. \tag{5.155}$$

Thus also

$$\begin{aligned}
Ml(f)_{\Phi_6\Phi_2} &= \langle \delta_1, M_h(\delta_1), M_h^2(\delta_1), \delta_2, M_h(\delta_2), \widetilde{\gamma}_2 \rangle \\
&\overset{2:1}{\supset} (B_1 \oplus B_2)_{\Phi_6\Phi_2}, && (5.156)
\end{aligned}$$

where

$$\widetilde{\gamma}_2 = \frac{1}{2}(\gamma_1 + \gamma_2) = \frac{1}{2}(\delta_1 - M_h(\delta_1) + M_h^2(\delta_1) + \delta_2 - M_h(\delta_2) + M_h^2(\delta_2)).$$

The matrix of L for the $\mathbb{Z}$-basis $\delta_1, M_h(\delta_1), M_h^2(\delta_1), \delta_2, M_h(\delta_2), \widetilde{\gamma}_2$ of $Ml(f)_{\Phi_6\Phi_2}$ is

$$\begin{pmatrix}
2 & 2 & 2 & 0 & 0 & 1 \\
-2 & 2 & 2 & 0 & 0 & -1 \\
-2 & -2 & 2 & 0 & 0 & 1 \\
0 & 0 & 0 & 1+2s & 0 & 1+2s \\
0 & 0 & 0 & -1-2s & 1+2s & -1-2s \\
1 & -1 & 1 & 1+2s & -1-2s & 3+3s
\end{pmatrix} \tag{5.157}$$

The associated quadratic form $(x_1 \dots x_6)(\text{matrix}) \begin{pmatrix} x_1 \\ \vdots \\ x_6 \end{pmatrix}$ is

$$\begin{aligned} & \frac{1}{2}\left[(2x_1+x_6)^2+(2x_2-x_6)^2+(2x_3+x_6)^2\right] \\ + \quad & \frac{1}{2}(1+2s)\left[(x_4-x_5+x_6)^2+(x_4+x_6)^2+(x_5-x_6)^2\right]. \end{aligned} \tag{5.158}$$

One finds

$$\{a \in Ml(f)_{\Phi_6\Phi_2} \mid L(a,a)=2\} = \{\pm M_h^j(\delta_1) \mid j=0,1,2\}, \tag{5.159}$$

and also

$$\{a \in (B_1 \oplus B_2)_{\Phi_6\Phi_2} \mid L(a,a)=2\} = \{\pm M_h^j(\delta_1) \mid j=0,1,2\}. \tag{5.160}$$

Thus any $g \in G_{\mathbb{Z}} \cup \mathrm{Aut}(B_1 \oplus B_2, L)$ maps δ_1 to an element of $\{\pm M_h^j(\delta_1) \mid j=0,1,2\}$. These are cyclic generators of the Orlik block $(B_1)_{\Phi_6\Phi_2}$. Thus any $g \in G_{\mathbb{Z}} \cup \mathrm{Aut}(B_1 \oplus B_2, L)$ maps $(B_1)_{\Phi_6\Phi_2}$ to itself. As $(B_2)_{\Phi_6\Phi_2}$ is the L-orthogonal sublattice within $Ml(f)_{\Phi_6\Phi_2}$, such a g maps also $(B_2)_{\Phi_6\Phi_2}$ to itself. This shows (ii) above. Especially such a g maps $(B_1)_{\Phi_2}$ to itself and its generator $\gamma_4 = \gamma_1$ to $\pm\gamma_4$. This shows (i) above.

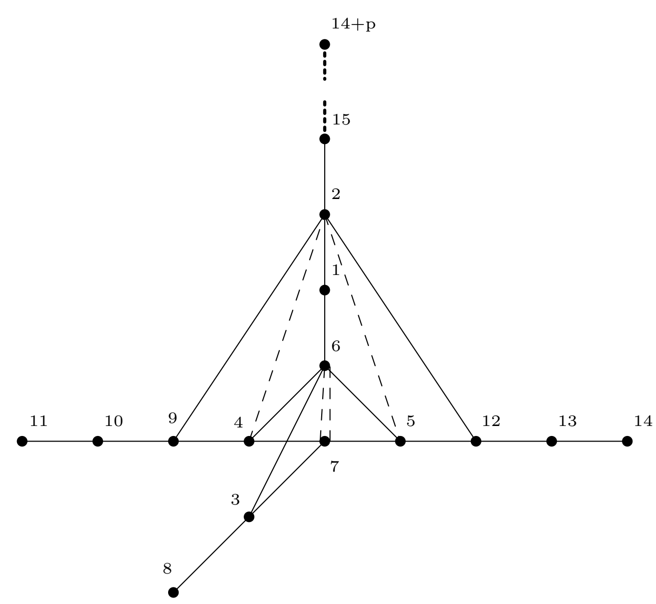


FIGURE 8. The CDD of a distinguished basis $e_1, \dots, e_\mu$ for $S_{1,p}$ from [Eb81, Tabelle 6 & Abb. 16]

5.8. **The series $S_{1,p}$.** Here the monodromy acts on the distinguished basis $e_1, \dots, e_\mu$ with the CDD in figure 8 as follows:

$$\begin{array}{rcl}
e_1 & \mapsto & -e_1 - e_2 + e_3 + e_4 + e_5 + e_6, \\
e_2 & \mapsto & 2e_1 + 2e_2 + e_9 + e_{12} + e_{15}, \\
e_3 & \mapsto & -e_1 - e_6 + e_7 + e_8, \\
e_4 & \mapsto & e_2 - e_6 + e_7 + e_9, \\
e_5 & \mapsto & e_2 - e_6 + e_7 + e_{12}, \\
e_6 & \mapsto & e_1 - 2e_2 + e_3 + e_4 + e_5 + 3e_6 - 2e_7, \\
e_7 & \mapsto & -2e_2 + e_3 + e_4 + e_5 + 2e_6 - e_7, \\
e_8 & \mapsto & -e_3 - e_8,
\end{array}$$

$$\begin{array}{rcl}
e_9 & \mapsto & e_{10}, \\
e_{10} & \mapsto & e_{11}, \\
e_{11} & \mapsto & -e_4 - e_9 - e_{10} - e_{11}, \\
e_{12} & \mapsto & e_{13}, \\
e_{13} & \mapsto & e_{14}, \\
e_{14} & \mapsto & -e_5 - e_{12} - e_{13} - e_{14}, \\
e_{14+i} & \mapsto & e_{15+i} \qquad \text{for } 1 \leq i \leq p-1, \\
e_{14+p} & \mapsto & -e_1 - e_2 - e_{15} - e_{16} - \ldots - e_{14+p}.
\end{array}$$

By table (5.13) the generators of the Orlik blocks B_1 and B_2 are $\beta_1 := -e_8 + e_{13}$ and $\beta_2 := e_{15}$. The monodromy acts on them as follows:

$$\begin{array}{rcl}
-e_8 + e_{13} & \mapsto & e_3 + e_8 + e_{14} \\
& \mapsto & -e_1 - e_3 - e_5 - e_6 + e_7 - e_{12} - e_{13} - e_{14} \\
& \mapsto & e_1 - e_3 - e_4 - e_7 - e_8 \\
& \mapsto & e_3 + e_6 - e_7 - e_9 \\
& \mapsto & e_8 - e_{10} \\
& \mapsto & -e_3 - e_8 - e_{11} \\
& \mapsto & e_1 + e_3 + e_4 + e_6 - e_7 + e_9 + e_{10} + e_{11} \\
& \mapsto & -e_1 + e_3 + e_5 + e_7 + e_8 \\
& \mapsto & -e_3 - e_6 + e_7 + e_{12} \\
& \mapsto & -e_8 + e_{13},
\end{array} \tag{5.161}$$

$$\begin{array}{rcl}
e_{15} & \mapsto & e_{16} \mapsto \ldots \mapsto e_{14+p} \mapsto -e_1 - e_2 - \sum_{i=15}^{14+p} e_i \\
& \mapsto & -e_3 - e_4 - e_5 - e_6 - e_9 - e_{12} \\
& \mapsto & -e_3 - e_4 - e_5 - e_7 - e_8 - e_9 - e_{10} - e_{12} - e_{13} \\
& \mapsto & e_1 - e_4 - e_5 + e_6 - 2e_7 - \sum_{j \in \{9,10,11,12,13,14\}} e_j \\
& \mapsto & -e_2 + e_4 + e_5 + 2e_6 - 2e_7 \mapsto -e_{15}.
\end{array} \tag{5.162}$$

Thus the characteristic polynomial of M_h on B_j is b_j. Here the blocks B_1 and B_2 are generated by the first $\deg b_1$ respectively $\deg b_2$ of the elements above. Here $B_1 + B_2 = B_1 \oplus B_2$ and $[Ml(f) : B_1 \oplus B_2] = 2 = r_I$ follow by the calculation of the determinant which expresses these generators of B_1 and B_2 in the distinguished basis $e_1, \dots, e_\mu$. Then it also follows that B_1 and B_2 are primitive sublattices.

The proof of (5.5) for $S_{1,10}$ was postponed to this section and has to be given here. From now on only $S_{1,10}$ is considered. (5.25) shows that $(Ml(f)_{\Phi_2}, L)$ is an A_2-lattice with roots $\{\pm\gamma_1, \pm\widetilde{\gamma}_2, \pm(\widetilde{\gamma}_2 - \gamma_1)\}$. Here γ_1 generates $(B_1)_{\Phi_2}$. We will show that $(B_1)_{\Phi_{10}}$ and $\pm\gamma_1$ satisfy the following special relationship:

$$\begin{aligned} & \Big[((B_1)_{\Phi_{10}} + \mathbb{Z}\cdot a)_{\mathbb{Q}} \cap Ml(f) : ((B_1)_{\Phi_{10}} + \mathbb{Z}\cdot a)\Big] \\ = & \left\{ \begin{array}{ll} 5 & \text{if } a = \pm\gamma_1, \\ 1 & \text{if } a \in \{\pm\widetilde{\gamma}_2, \pm(\widetilde{\gamma}_2 - \gamma_1)\}. \end{array} \right. \end{aligned} \tag{5.163}$$

If $a = \pm\gamma_1$, then

$$\begin{aligned} ((B_1)_{\Phi_{10}} + \mathbb{Z}\cdot a)_{\mathbb{Q}} \cap Ml(f) &= (B_1)_{\Phi_{10}\Phi_2} = \bigoplus_{j=0}^{4} \mathbb{Z}\cdot (t^j\Phi_5)(M_h)(\beta_1), \\ (B_1)_{\Phi_{10}} + \mathbb{Z}\cdot a &= (B_1)_{\Phi_{10}} + (B_1)_{\Phi_2} \\ = \bigoplus_{j=0}^{3} \mathbb{Z}\cdot (t^j\Phi_2\Phi_5)(M_h)(\beta_1) &\oplus \mathbb{Z}\cdot (\Phi_{10}\Phi_5)(M_h)(\beta_1), \end{aligned}$$

so the index is

$$\left[\bigoplus_{j=0}^{4} \mathbb{Z}\cdot t^j \ :\ \bigoplus_{j=0}^{3} \mathbb{Z}\cdot t^j\Phi_2 \oplus \mathbb{Z}\cdot\Phi_{10}\right] = 5.$$

Now recall that $(B_1)_{\Phi_{10}}$ is a primitive sublattice of $Ml(f)$ and that

$$B_1 \subset \bigoplus_{j=1}^{14} \mathbb{Z}\cdot e_j, \text{ so } (B_1)_{\Phi_{10}} \subset \bigoplus_{j=1}^{14} \mathbb{Z}\cdot e_j.$$

Observe that

$$\widetilde{\gamma}_2 \equiv \widetilde{\gamma}_2 - \gamma_1 \equiv -\sum_{j=15}^{24} e_j \mod \sum_{j=1}^{14} \mathbb{Z}\cdot e_j.$$

Because of the sum $-\sum_{j=15}^{24} e_j$ in $\widetilde{\gamma}_2$ and in $\widetilde{\gamma}_2 - \gamma_1$, the sublattices $(B_1)_{\Phi_{10}} \oplus \mathbb{Z}\cdot\widetilde{\gamma}_2$ and $(B_1)_{\Phi_{10}} \oplus \mathbb{Z}\cdot(\widetilde{\gamma}_2 - \gamma_1)$ are primitive in $Ml(f)$, so the index above is 1. This shows (5.163).

Now (5.5) is an easy consequence: Consider an element $g \in G_{\mathbb{Z}}$ with $g((B_1)_{\Phi_{10}}) = (B_1)_{\Phi_{10}}$. It must map γ_1 to some root of the A_2-lattice $(Ml(f)_{\Phi_{10}}, L)$. Because of (5.163), the image must be $\pm\gamma_1$, so $g((B_1)_{\Phi_2}) = (B_1)_{\Phi_2}$. Therefore $g((B_1)_{\Phi_{10}\Phi_2}) = (B_1)_{\Phi_{10}\Phi_2}$ and by its L-orthogonality also $g((B_2)_{\Phi_{10}\Phi_2}) = (B_2)_{\Phi_{10}\Phi_2}$.

For $S_{1,10}$ $b_1 = \Phi_{10}\Phi_5\Phi_2$ and $b_2 = \Phi_{30}\Phi_{10}\Phi_6\Phi_2$, so the eigenspaces with eigenvalues different from the roots of $\Phi_{10}\Phi_2$ are one-dimensional and are either in $(B_1)_{\mathbb{C}}$ or in $(B_2)_{\mathbb{C}}$. This implies (5.5) for $S_{1,10}$.

This finishes the proof of theorem 5.1. $\square$

6. The group $G_{\mathbb{Z}}$ for the quadrangle singularities

The normal forms from [AGV85, §13] for the six families of quadrangle singularities will be listed below in section 10. The quadrangle singularities can be seen as special 0-th members of the eight bimodal series, with the two series $W_{1,p}^\sharp$ and $W_{1,p}$ for $W_{1,0}$ and the two series $S_{1,p}^\sharp$ and $S_{1,p}$ for $S_{1,0}$.

The following table specializes the table (5.1) to the case $p = 0$. For $W_{1,0}$ and $S_{1,0}$, we have chosen the specialization of the cases $W_{1,p}^\sharp$ and $S_{1,p}^\sharp$, not $W_{1,p}$ and $S_{1,p}$. The reason is that the Orlik blocks in theorem 5.1 for $W_{1,p}^\sharp$ and $S_{1,p}^\sharp$ work also for $W_{1,0}$ and $S_{1,0}$, but those for $W_{1,p}$ and $S_{1,p}$ work not for $W_{1,0}$ and $S_{1,0}$. Again b_1b_2 respectively $b_1b_2b_3$ for $Z_{1,0}$ are the characteristic polynomials of the surface singularities.

family	μ	b_1	b_2	b_3	m	r_I
$W_{1,0}$	15	Φ_{12}	$\Phi_{12}\Phi_6\Phi_4\Phi_3\Phi_2$	–	12	1
$S_{1,0}$	14	$\Phi_{10}\Phi_2$	$\Phi_{10}\Phi_5\Phi_2$	–	10	1
$U_{1,0}$	14	Φ_9	$\Phi_9\Phi_3$	–	9	1
$E_{3,0}$	16	$\Phi_{18}\Phi_2$	$\Phi_{18}\Phi_6\Phi_2$	–	18	2
$Z_{1,0}$	15	$\Phi_{14}\Phi_2$	$\Phi_{14}\Phi_2$	Φ_2	14	2
$Q_{2,0}$	14	$\Phi_{12}\Phi_4\Phi_3$	$\Phi_{12}\Phi_4$	–	12	2

(6.1)

The following theorem on the group $G_{\mathbb{Z}}$ has a strong similarity with the analogous theorem 5.1 for the eight bimodal series. And luckily, also large parts of the proof of theorem 5.1 apply also to the case $p = 0$. We do not have (5.4) $G_{\mathbb{Z}} = \mathrm{Aut}(\bigoplus_{j\geq 1} B_j, L)$ for $E_{3,0}, Z_{1,0}, Q_{2,0}$. But we have an analogue of the substitute (5.5) for $S_{1,10}$, the formula (6.4). Contrary to theorem 5.1, we need and give a precise description of the induced Fuchsian group. The proof uses theorem 3.6. A part of the proof (a surjectivity) is postponed to section 10. For each family, denote $\zeta := e^{2\pi i/m} \in S^1 \subset \mathbb{C}$.

Theorem 6.1. *For any surface singularity f in any of the six families of quadrangle singularities, the following holds.*

(a) (See definition 2.3 for the notion Orlik block*) For all families except $Z_{1,0}$, there are Orlik blocks $B_1, B_2 \subset Ml(f)$, and for $Z_{1,0}$, there are Orlik blocks $B_1, B_2, B_3 \subset Ml(f)$ with the following properties. The characteristic polynomial p_{B_j} of the monodromy on B_j is b_j. The sum $\sum_{j\geq 1} B_j$ is a direct sum $\bigoplus_{j\geq 1} B_j$, and it is a sublattice of $Ml(f)$ of full rank μ and of index r_I. Define*

$$\widetilde{B}_1 := \begin{cases} B_1 & \text{for all cases except } Z_{1,0}, \\ B_1 \oplus B_3 & \text{for } Z_{1,0}. \end{cases} \tag{6.2}$$

Then

$$L(\widetilde{B}_1, B_2) = 0 = L(B_2, \widetilde{B}_1), \tag{6.3}$$

$$g \in G_{\mathbb{Z}} \text{ with } g((B_1)_{\Phi_m}) = (B_1)_{\Phi_m} \;\Rightarrow\; g(B_j) = B_j \text{ for } j \geq 1. \tag{6.4}$$

(b) The eigenspace $Ml(f)_\zeta \subset Ml(f)_{\mathbb{C}}$ is 2-dimensional. The hermitian form h_ζ on it from lemma 2.2 (a) with $h_\zeta(a,b) := \sqrt{-\zeta}\cdot L(a,\overline{b})$ for $a,b \in Ml(f)_\zeta$ is nondegenerate and indefinite, so $\mathbb{P}(Ml(f)_\zeta) \cong \mathbb{P}^1$ contains a half-plane

$$\mathcal{H}_\zeta := \{\mathbb{C}\cdot a \,|\, a \in Ml(f)_\zeta \text{ with } h_\zeta(a,a) < 0\} \subset \mathbb{P}(Ml(f)_\zeta). \tag{6.5}$$

Therefore the group $\mathrm{Aut}(Ml(f)_\zeta, h_\zeta)/S^1\cdot \mathrm{id}$ is isomorphic to $PSL(2,\mathbb{R})$. The homomorphism

$$\Psi : G_{\mathbb{Z}} \to \mathrm{Aut}(Ml(f)_\zeta, h_\zeta)/S^1\cdot\mathrm{id}, g \mapsto g|_{Ml(f)_\zeta} \mathrm{mod}\, S^1\cdot \mathrm{id}, \tag{6.6}$$

is well-defined. $\Psi(G_{\mathbb{Z}})$ *is an infinite Fuchsian group acting on the half-plane* $\mathcal{H}_\zeta$. *It is a triangle group of the same type as in theorem 3.6, so of the following type:*

$$\begin{array}{l|l|l|l|l} W_{1,0} & S_{1,0} & E_{3,0}\ \&\ U_{1,0} & Z_{1,0} & Q_{2,0} \\ (2,12,12) & (2,10,10) & (2,3,18) & (2,3,14) & (2,3,12) \end{array} \tag{6.7}$$

And

$$\ker\Psi \quad = \quad \{\pm M_h^k \,|\, k\in\mathbb{Z}\}. \tag{6.8}$$

Proof: (a) We choose again (as in section 5) for each of the six cases a distinguished basis with the Coxeter-Dynkin diagram in [Eb81, Tabelle 6 and Abb. 16].

The diagrams for $W_{1,p}^\sharp$ and $W_{1,p}$ specialize both to the same diagram for $W_{1,0}$. Though the description of the action of the monodromy on the distinguished basis for $W_{1,p}^\sharp$ in 5.1 specializes to $W_{1,0}$, but not the description for $W_{1,p}$ in 5.7. In the latter case $e_2\mapsto 2e_1+2e_2+e_8+e_{12}+e_{16}$, but e_{16} does not exist for $W_{1,0}$. Therefore we work with the specialization to $p=0$ of the formulas for $W_{1,p}^\sharp$ in subsection 5.1.

The same applies to $S_{1,0}$. There we work with the specialization to $p=0$ of the formulas for $S_{1,p}^\sharp$ in subsection 5.2.

The Orlik blocks B_1 and B_2 (and B_3 for $Z_{1,0}$) are defined as in the proof of theorem 5.1, there for $p>0$, now for $p=0$. By the same arguments, the sum $\sum_{j\geq 1}B_j$ is a direct sum $\bigoplus_{j\geq 1}B_j$ and a sublattice of $Ml(f)$ of full rank μ and index r_I, and (6.3) holds.

With respect to part (a), it rests to show (6.4). In the cases $W_{1,0}$ and $U_{1,0}$, it is trivial as $r_I=1$ and $b_1=\Phi_m$ and B_1 and B_2 are L-orthogonal.

In the cases $S_{1,0},E_{3,0},Z_{1,0}$ and $Q_{2,0}$, the proof will be similar to the proof of (5.5) for $S_{1,10}$ in subsection 5.8. First we treat $S_{1,0},E_{3,0}$ and $Z_{1,0}$ together, then we come to $Q_{2,0}$.

The following formulas in the proof of part (a) of theorem 5.1 specialize to the cases $S_{1,0},E_{3,0}$ and $Z_{1,0}$: (5.10)–(5.26), (5.28), (5.33), (5.35).

The quadratic forms in (5.26) give now the following variants of (5.27) and (5.29):

$$\begin{aligned} \{a\in Ml(f)_{\Phi_2}\,|\,L(a,a)=5\} &= \{\pm\gamma_1,\pm\gamma_2\}\text{for } S_{1,0}, \\ \{a\in Ml(f)_{\Phi_2}\,|\,L(a,a)=6\} &= \{\pm\gamma_1,\pm\widetilde{\gamma}_2,\pm(\widetilde{\gamma}_2-\gamma_1)\}\text{for } E_{3,0}, \\ \{a\in Ml(f)_{\Phi_2}\,|\,L(a,a)=5\} &= \{\pm(\gamma_1-3\gamma_2), \\ &\qquad \pm\widetilde{\gamma}_2,\pm(\widetilde{\gamma}_2-\gamma_2)\}\text{for } Z_{1,0}. \end{aligned} \tag{6.9}$$

The first element (up to sign) of each of these three sets generates in the corresponding case $(B_1)_{\Phi_2}$. We claim that $(B_1)_{\Phi_m}$ and this first element satisfy the following special relationship. For a in any of these three sets define

$$r(a) := [((B_1)_{\Phi_m}+\mathbb{Z}\cdot a)_{\mathbb{Q}}\cap Ml(f) : ((B_1)_{\Phi_m}+\mathbb{Z}\cdot a)]\in\mathbb{Z}_{\geq 1}. \tag{6.10}$$

Then we claim:

$$\begin{array}{cc|cc|cc|cc} & & S_{1,0} & & E_{3,0} & & Z_{1,0} & \\ \hline a & r(a) & \pm\gamma_1 & 5 & \pm\gamma_1 & 3 & \pm(\gamma_1-2\gamma_3) & 7 \\ a & r(a) & \pm\gamma_2 & 1 & \pm\widetilde{\gamma}_2,\pm(\widetilde{\gamma}_2-\gamma_1) & 1 & \pm\widetilde{\gamma}_2,\pm(\widetilde{\gamma}_2-\gamma_2) & 1 \end{array} \tag{6.11}$$

The proof is the same as the proof of (5.163) for $S_{1,10}$ in subsection 5.8. We use that for any unitary polynomial $p(t)\in\mathbb{Z}[t]$

$$\left[\bigoplus_{j=0}^{\deg p}\mathbb{Z}\cdot t^j : \bigoplus_{j=0}^{\deg p-1}\mathbb{Z}\cdot t^j\Phi_2\oplus\mathbb{Z}\cdot p(t)\right] = |p(-1)|, \tag{6.12}$$

and

$$\Phi_{10}(-1) = 5, \Phi_{18}(-1) = 3, \Phi_{14}(-1) = 7. \tag{6.13}$$

We also use

$$B_1 \subset \sum_{j=1}^{m_1} \mathbb{Z} \cdot e_j \qquad \text{with } m_1 := 8, 9, 10 \text{ for } S_{1,0}, E_{3,0}, Z_{1,0} \tag{6.14}$$

and that the elements in the second line of (6.11) are modulo $\sum_{j=1}^{m_1} \mathbb{Z} \cdot e_j$

$$\begin{aligned} S_{1,0} &: \quad \gamma_2 \equiv e_9 + e_{11} + e_{12} + e_{14}, \\ E_{3,0} &: \quad \widetilde{\gamma}_2 \equiv e_{10} + e_{12} + e_{14} + e_{16}, \widetilde{\gamma}_2 - \gamma_1 \equiv \widetilde{\gamma}_2, \\ Z_{1,0} &: \quad \widetilde{\gamma}_2 \equiv e_{11} + e_{13} + e_{15}, \widetilde{\gamma}_2 - \gamma_2 \equiv -\widetilde{\gamma}_2. \end{aligned} \tag{6.15}$$

Therefore $(B_1)_{\Phi_m} + \mathbb{Z} \cdot a$ for these elements a is primitive in $Ml(f)$, and thus $r(a) = 1$.

The derivation of (6.4) from (6.11) and (6.9) for $S_{1,0}, E_{3,0}$ and $Z_{1,0}$ is almost the same as the derivation of (5.5) from (5.163) for $S_{1,10}$ in subsection 5.8.

The only additional argument concerns $B_3 = \mathbb{Z} \cdot \gamma_3$ in the case $Z_{1,0}$. Because of (5.28) any $g \in G_{\mathbb{Z}}$ maps B_3 to itself. Because of $L(\gamma_1 - 2\gamma_3, \gamma_3) = 1 \neq 0$, B_3 and $(B_1)_{\Phi_2}$ are glued together: If $g = \varepsilon \cdot \mathrm{id}$ on $(B_1)_{\Phi_2}$ for some $\varepsilon \in \{\pm 1\}$, then $g = \varepsilon \cdot \mathrm{id}$ on B_3.

Now we come to $Q_{2,0}$. The formulas (5.113)–(5.116), (5.118)–(5.119), (5.122)–(5.133), (5.136)–(5.139) are also valid for $p = 0$ respectively $s = 0$. The quadratic form in (5.133) now gives the following variant of (5.134):

$$A := \{\gamma_1, \gamma_3, \gamma_1 - \gamma_3 + M_h(\gamma_3), \gamma_1 - M_h(\gamma_1) + M_h(\gamma_3)\}, \tag{6.16}$$

$$\{b \in Ml(f)_{\Phi_4} \mid L(b, b) = 3\} = \bigcup_{a \in A} \{\pm a, \pm M_h(a)\}, \tag{6.17}$$

so these are 16 elements which come in 4 sets of 4 elements such that each set is M_h-invariant. Recall that $M_h^2 = -\mathrm{id}$ on $Ml(f)_{\Phi_4}$. The set $\{\pm\gamma_1, \pm M_h(\gamma_1)\}$ generates $(B_1)_{\Phi_4}$.

We claim that $(B_1)_{\Phi_{12}}$ and this set satisfy the following special relationship. For $a \in A$ define the index

$$\begin{aligned} r(a) \quad := \quad & \big[((B_1)_{\Phi_{12}} + \mathbb{Z} \cdot a + \mathbb{Z} \cdot M_h(a))_{\mathbb{Q}} \cap Ml(f) \\ & : \ ((B_1)_{\Phi_{12}} + \mathbb{Z} \cdot a + \mathbb{Z} \cdot M_h(a))\big] \in \mathbb{Z}_{\geq 1}. \end{aligned} \tag{6.18}$$

Then we claim:

$$r(a) = \begin{cases} 9 & \text{for } a = \gamma_1, \\ 1 & \text{for } a \in \{\gamma_3, \gamma_1 - M_h(\gamma_1) + M_h(\gamma_3)\}, \\ 1 \text{ or } 2 & \text{for } a = \gamma_1 - \gamma_3 + M_h(\gamma_3). \end{cases} \tag{6.19}$$

$r(\gamma_1) = 9$ holds because of

$$\begin{aligned} & ((B_1)_{\Phi_{12}} + \mathbb{Z} \cdot \gamma_1 + \mathbb{Z} \cdot M_h(\gamma_1))_{\mathbb{Q}} \cap Ml(f) \\ = \quad & (B_1)_{\Phi_{12}\Phi_4} = \bigoplus_{j=0}^{5} \mathbb{Z} \cdot (t^j \Phi_3)(M_h)(\beta_1), \end{aligned} \tag{6.20}$$

$$\begin{aligned} & (B_1)_{\Phi_{12}} + \mathbb{Z} \cdot \gamma_1 + \mathbb{Z} \cdot M_h(\gamma_1) \\ = \quad & \bigoplus_{j=0}^{3} \mathbb{Z} \cdot (t^j \Phi_4 \Phi_3)(M_h)(\beta_1) \oplus \bigoplus_{j=0}^{1} \mathbb{Z} \cdot (t^j \Phi_{12} \Phi_3)(M_h)(\beta_1), \end{aligned} \tag{6.21}$$

and thus

$$r(\gamma_1) = \left[\bigoplus_{j=0}^{5}\mathbb{Z}\cdot t^j : \bigoplus_{j=0}^{3}\mathbb{Z}\cdot t^j\Phi_4 \oplus \bigoplus_{j=0}^{1}\mathbb{Z}\cdot t^j\Phi_{12}\right] = 3\cdot 3. \tag{6.22}$$

For $a \in A - \{\gamma_1\}$, $r(a) \in \{1, 2\}$ holds because of

$$B_1 \subset \sum_{j=1}^{10}\mathbb{Z}\cdot e_j, \tag{6.23}$$

and because the elements a and $M_h(a)$ for $a \in A - \{\gamma_1\}$ are modulo $\sum_{j=1}^{10}\mathbb{Z}\cdot e_j$

$$\begin{aligned}
\gamma_1 &\equiv -e_{13} - e_{14}, \qquad (6.24)\\
M_h(\gamma_1) &\equiv e_{12} + e_{13},\\
\gamma_1 - \gamma_3 + M_h(\gamma_3) &\equiv e_{12} + 2e_{13} + e_{14},\\
M_h(\gamma_1 - \gamma_3 + M_h(\gamma_3)) &\equiv -e_{12} + e_{14},\\
\gamma_1 - M_h(\gamma_1) + M_h(\gamma_3) &\equiv e_{12} + e_{13},\\
M_h(\gamma_1 - M_h(\gamma_1) + M_h(\gamma_3)) &\equiv e_{13} + e_{14}.
\end{aligned}$$

The derivation of (6.4) for $Q_{2,0}$ from (6.17) and (6.19) is a simple variant of the derivation of (5.5) from (5.163) for $S_{1,10}$ in subsection 5.8: Consider an element $g \in G_{\mathbb{Z}}$ with

$$g((B_1)_{\Phi_{12}}) = (B_1)_{\Phi_{12}}.$$

Because of (6.17), it maps the set $\{\pm\gamma_1, \pm M_h(\gamma_1)\}$ to one of the four sets on the right hand side of (6.17). Because of (6.19), the image must be the set $\{\pm\gamma_1, \pm M_h(\gamma_1)\}$ itself. As this set generates $(B_1)_{\Phi_4}$, g maps $(B_1)_{\Phi_4}$ to itself. Then g maps the sets $(B_1)_{\Phi_{12}\Phi_4}$, $B_1 = (B_1)_{\Phi_{12}\Phi_4\Phi_3}$ and $B_2 = (B_2)_{\Phi_{12}\Phi_4}$ to themselves. This finishes the proof of part (a).

(b) All the formulas and arguments in the proof of part (c) of theorem 5.1 for the cases $W^\sharp_{1,12r}, S^\sharp_{1,10r}, U_{1,9r}, E_{3,18r}, Z_{1,14r}$ and $Q_{2,12r}$ are also valid for $r = 0$.

In step 3 now (6.4) is used instead of (5.4), just as (5.5) for $S_{1,10}$. Therefore (6.7) holds and $\Psi(G_{\mathbb{Z}})$ is an infinite Fuchsian group.

By table (5.72), the remarks 3.5 and theorem 3.6, $\Psi(G_{\mathbb{Z}})$ is a subgroup of a triangle group of the same type as in theorem 3.6, for each case. The proof of theorem 10.1 will show that it is the full triangle group. □

7. Gauss-Manin connection and Brieskorn lattice

The Gauss-Manin connection of isolated hypersurface singularities had been considered first by Brieskorn in 1970 [Br70]. Since then it had been described by many people in many papers (K. Saito, Greuel, Pham, Varchenko, M. Saito, Hertling, and others). The following presentation will be short on the $\mathcal{D}$-module foundations. It will be very precise on the relations between the different pairings (more precise than anywhere in the literature). And it will emphasize the computational aspects. Other versions are in [AGV88], [He93], [He95], [Ku98] and [He02].

Throughout most of this section, we consider a fixed isolated hypersurface singularity $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$, its flat cohomology bundle $\bigcup_{\tau\in\Delta^*} H^n(f^{-1}(\tau), \mathbb{C})$, and the space $H^\infty_{\mathbb{C}}$ of global flat multi-valued sections (see section 4 for $H^\infty_{\mathbb{C}}$).

First we define the *elementary sections* $es(A, \alpha)$, the spaces C^α which they generate, and the V-filtration.

Any global flat multi-valued section $A \in H^\infty_\lambda$ and any choice of $\alpha \in \mathbb{Q}$ with $e^{-2\pi i\alpha} = \lambda$ leads to a holomorphic univalued section with specific growth condition at $0 \in \Delta$, the *elementary section* $es(A, \alpha)$ with

$$es(A,\alpha)(\tau) := e^{\log\tau(\alpha - \frac{N}{2\pi i})} \cdot A(\log\tau). \tag{7.1}$$

Recall that N is the nilpotent part of the monodromy M_h. Denote by C^α the $\mathbb{C}$-vector space of all elementary sections with fixed α and λ. The map

$$\psi_\alpha := es(.,\alpha) : H^\infty_\lambda \to C^\alpha \tag{7.2}$$

is an isomorphism. The space $V^{mod} := \bigoplus_{\alpha\in(-1,0]} \mathbb{C}\{\tau\}[\tau^{-1}] \cdot C^\alpha$ is the space of all germs at 0 of the sheaf of holomorphic sections on the flat cohomology bundle with moderate growth at 0. The *Kashiwara-Malgrange V-filtration* is given by the subspaces

$$V^\alpha := \bigoplus_{\beta\in[\alpha,\alpha+1)} \mathbb{C}\{\tau\} \cdot C^\beta, V^{>\alpha} := \bigoplus_{\beta\in(\alpha,\alpha+1]} \mathbb{C}\{\tau\} \cdot C^\beta. \tag{7.3}$$

It is a decreasing filtration by free $\mathbb{C}\{\tau\}$-modules of rank μ with $\mathrm{Gr}^\alpha_V = V^\alpha/V^{>\alpha} \cong C^\alpha$. And

$$\begin{aligned} \tau : C^\alpha \to C^{\alpha+1} \text{ bijective}, &\qquad \tau\cdot es(A,\alpha) = es(A,\alpha+1),\\ \partial_\tau : C^\alpha \to C^{\alpha-1} \text{ bijective} &\qquad \text{if } \alpha\neq 0,\\ \tau\partial_\tau - \alpha : C^\alpha \to C^\alpha \text{ nilpotent}, &\qquad (\tau\partial_\tau-\alpha)es(A,\alpha) = es(\frac{-N}{2\pi i}A,\alpha). \end{aligned} \tag{7.4}$$

Therefore $\partial_\tau^{-1} : V^{>-1} \to V^{>0}$ is an isomorphism, and $V^{>-1}$ is a free $\mathbb{C}\{\{\partial_\tau^{-1}\}\}$-module of rank μ.

With the polarizing form S (see (4.20)), we define a ∂_τ^{-1}-sesquilinear pairing K_f on $V^{>-1}$. Its restriction to the Brieskorn lattice will be the restriction of K. Saito's higher residue pairings to the Brieskorn lattice (which he defined on an extension of the Brieskorn lattice to a universal unfolding).

Lemma 7.1. *A unique pairing*

$$K_f : V^{>-1} \times V^{>-1} \to \mathbb{C}\{\{\partial_\tau^{-1}\}\} \tag{7.5}$$

with the properties in (7.6)–(7.9) exists. In (7.6) and (7.7) $A \in H^\infty_{e^{-2\pi i\alpha}}, B \in H^\infty_{e^{-2\pi i\beta}}$.

$$\begin{aligned} K_f(es(A,\alpha), es(B,\beta)) &= \frac{1}{(2\pi i)^n} S(A,B)\cdot\partial_\tau^{-1}, &&(7.6)\\ &\quad \text{for } \alpha,\beta\in(-1,0), \alpha+\beta=-1,\\ K_f(es(A,\alpha), es(B,\beta)) &= \frac{-1}{(2\pi i)^{n+1}} S(A,B)\cdot\partial_\tau^{-2}, &&(7.7)\\ &\quad \text{for } \alpha=\beta=0,\\ K_f : C^\alpha\times C^\beta \to 0 &\quad \text{for } \alpha,\beta\in\mathbb{R}_{>-1}, \alpha+\beta\notin\mathbb{Z}, &&(7.8)\\ \partial_\tau^{-1}\cdot K_f(a,b) &= K_f(\partial_\tau^{-1}a,b) = K_f(a,-\partial_\tau^{-1}b) &&(7.9)\\ &\quad \text{for } a,b\in V^{>-1}. \end{aligned}$$

It satisfies also (for $\alpha,\beta\in\mathbb{R}_{>-1}$*)*

$$\begin{aligned} K_f : C^\alpha\times C^\beta \to \mathbb{C}\cdot\partial_\tau^{-\alpha-\beta-2} &\qquad \text{if } \alpha+\beta\in\mathbb{Z}, &&(7.10)\\ K_f(\tau a,b) - K_f(a,\tau b) = [\tau, K_f(a,b)] &\qquad \text{for } a,b\in V^{>-1}, &&(7.11) \end{aligned}$$

where $[\tau,\partial_\tau^{-k}] = k\partial_\tau^{-k-1}$. *If one writes* $K_f(a,b) = \sum_{k\geq1} K_f^{(-k)}(a,b)\cdot\partial_\tau^{-k}$ *with* $K_f^{(k)}(a,b)\in\mathbb{C}$, *then* $K_f^{(-k)}$ *is* $(-1)^{k+n+1}$*-symmetric.*

Proof: It is clear that (7.6)–(7.9) define a unique ∂_τ^{-1}-sesquilinear pairing on $V^{>-1}$. Its ∂_τ^{-1}-sesquilinearity gives (7.10). One checks (7.11) with (7.4) and the infinitesimal N-invariance of S. The symmetry of the $K_f^{(k)}$ follows from the symmetry of S and the ∂_τ^{-1}-sesquilinearity of K_f. □

Remark 7.2. In the sections 9 and 10, we will prove the global Torelli conjecture for many families of marked bimodal surface singularities. We want to claim that it follows also for all suspensions of these families, and also for the curve singularities, if the surface singularities are themselves suspensions of curve singularities.

The Milnor lattices of f and $f+x_{n+1}^2$ are up to a sign uniquely isomorphic. The normalized Seifert form L^{hnor} and the group $G_{\mathbb{Z}}$ are the same.

But the Brieskorn lattices of f and $f+x_{n+1}^2$ are not isomorphic. In [He93], the second author had a lemma saying that they are sufficiently similar and vary in the same way in μ-constant families.

Stronger and more elegant is the specialization to $f+x_{n+1}^2$ of a Thom-Sebastiani formula. But that requires to look at a Fourier-Laplace transformation. In the present situation of sections of moderate growth, this can be done in a nice and explicit way. Lemma 7.3, definition 7.4 and theorem 7.5 do a good part of the work. Theorem 7.9 gives a Thom-Sebastiani formula for a Fourier-Laplace transform of the Brieskorn lattice. Theorem 7.7 states well known properties of the Brieskorn lattice.

The pairing in lemma 7.3 had been considered first by Pham [Ph85], see remark 7.6 (i).

Lemma 7.3. *Let $\gamma_{-\pi}: H^n(f^{-1}(z),\mathbb{C})\to H^n(f^{-1}(-z),\mathbb{C})$ (respectively γ_π) be the isomorphism by flat shift in mathematically negative (respectively positive) direction. Define a pairing*

$$\begin{aligned} P: H^n(f^{-1}(z),\mathbb{C})\times H^n(f^{-1}(-z),\mathbb{C}) &\to \mathbb{C}\text{for } z\neq 0 \\ \text{by}\qquad P(a,b) &:= \frac{1}{(2\pi i)^{n+1}}\cdot L^{nor}(a,\gamma_{-\pi}(b)). \end{aligned}\tag{7.12}$$

It is $(-1)^{n+1}$-symmetric and nondegenerate and takes values in $(2\pi i)^{-(n+1)}\cdot\mathbb{Z}$ on $H^n(f^{-1}(z),\mathbb{Z})\times H^n(f^{-1}(-z),\mathbb{Z})$. It is flat, i.e. it has constant values on pairs of flat sections in the cohomology bundle.

Proof: The only property which might not be immediately obvious, is the $(-1)^{n+1}$-symmetry. It compares the P in (7.12) with the P where in (7.12) z is replaced by $-z$. It follows from the flatness, from $M_h\gamma_{-\pi}=\gamma_\pi$ and (4.15): Let $a\in H^n(f^{-1}(z),\mathbb{Z}), b\in H^n(f^{-1}(-z),\mathbb{Z})$, then

$$\begin{aligned} (2\pi i)^{n+1}\cdot P(b,a) &= L^{nor}(b,\gamma_{-\pi}a)=(-1)^{n+1}L^{nor}(M_h\gamma_{-\pi}a,b)\\ &= (-1)^{n+1}L^{nor}(\gamma_\pi a,b)=(-1)^{n+1}L^{nor}(a,\gamma_{-\pi}b)\\ &= (2\pi i)^{n+1}\cdot(-1)^{n+1}\cdot P(a,b). \end{aligned}\tag{7.13}$$

□

Definition 7.4. [He02, (7.47)] For each $\alpha\in\mathbb{R}_{>0}$ define the automorphism

$$\begin{aligned} G^{(\alpha)} &: H^\infty_{e^{-2\pi i\alpha}}\to H^\infty_{e^{-2\pi i\alpha}},\\ G^{(\alpha)} &:= \sum_{k\geq 0}\frac{1}{k!}\Gamma^{(k)}(\alpha)\cdot\left(\frac{-N}{2\pi i}\right)^k = {}''\Gamma\left(\alpha\cdot\mathrm{id}+\frac{-N}{2\pi i}\right)''. \end{aligned}\tag{7.14}$$

Here Γ is the Gamma function, and $\Gamma^{(k)}$ is its k-th derivative. Define the automorphism

$$G:=\sum_{\alpha\in(0,1]}G^{(\alpha)}: H^\infty_{\mathbb{C}}\to H^\infty_{\mathbb{C}}.\tag{7.15}$$

The following theorem was first formulated in [He03, Proposition 7.7]. A detailed proof is in [BH17, Theorem 5.2]. The most difficult part is the proof of (7.21).

Theorem 7.5. *(a) Let τ and z both be coordinates on $\mathbb{C}$. For $\alpha > 0$ and $A \in H^\infty_{e^{-2\pi i\alpha}}$, the Fourier-Laplace transformation FL with*

$$FL(es(A,\alpha-1)(\tau))(z) := \int_0^{\infty\cdot z} e^{-\tau/z}\cdot es(A,\alpha-1)(\tau)d\tau \tag{7.16}$$

is well defined and maps the elementary section $es(A,\alpha-1)(\tau)$ in τ to the elementary section

$$FL(es(A,\alpha-1)(\tau))(z) = es(G^{(\alpha)}A,\alpha)(z) \tag{7.17}$$

in z.

(b) It extends to a well defined isomorphism

$$FL : \sum_{\alpha\in(-1,0]} \mathbb{C}\{\partial_\tau^{-1}\}\cdot C^\alpha_\tau \to V^{>0}_z. \tag{7.18}$$

Here the indices τ at C^α and z at $V^{>0}$ indicate that the coordinate τ respectively z has to be used. It satisfies for $a,b\in\sum_{\alpha\in(-1,0]}\mathbb{C}\{\partial_\tau^{-1}\}\cdot C^\alpha_\tau$

$$\begin{aligned} FL(\partial_\tau^{-1}a) &= z\cdot FL(a), & (7.19)\\ FL(\tau\cdot a) &= z^2\partial_z FL(a), & (7.20)\\ P(FL(a),FL(b)) &= \sum_{k\geq 1} c_k z^l \text{if } K_f(a,b) = \sum_{k\geq 1} c_k\partial_\tau^{-k}. & (7.21)\end{aligned}$$

Remarks 7.6. (i) Pham [Ph85] defined the pairing P in lemma 7.3 starting with an intersection form for Lefschetz thimbles. In our situation, $H_n(f^{-1}(z),\mathbb{Z})$ for $z\in\Delta^*$ is canonically isomorphic to the $\mathbb{Z}$-module generated by Lefschetz thimbles above the straight path from 0 to z. And it is easy to see that the pairing

$$\begin{gathered}(-1)^{n(n+1)/2}\cdot L^{hnor}(.,\gamma_{-\pi}) : \\ H_n(f^{-1}(z),\mathbb{Z})\times H_n(f^{-1}(-z),\mathbb{Z})\to\mathbb{Z}\end{gathered} \tag{7.22}$$

for $z\in\Delta^*$ is the intersection form for Lefschetz thimbles [He05]. This formula connects lemma 7.3 with Pham's definition.

(ii) Neither Pham nor K. Saito knew the formulas (7.6) and (7.7) for K_f with the polarizing form S. Pham had the version of (7.21) with K. Saito's higher residue pairings [SaK83] instead of K_f. He did not consider explicitly the automorphisms $G^{(\alpha)}$ and (7.17).

(iii) Because of (7.19), we have to consider on the left hand side of (7.18) and in (7.19)–(7.21) the subspace $\sum_{\alpha\in(-1,0]}\mathbb{C}\{\partial_\tau^{-1}\}\cdot C^\alpha_\tau$ of $V^{>-1}_\tau$. The convergence condition is stronger.

Now we come to the Brieskorn lattice. It is a free $\mathbb{C}\{\tau\}$-module $H_0''(f)\subset V^{>-1}$ of rank μ which had first been studied by Brieskorn [Br70]. The name *Brieskorn lattice* is due to [SaM89], the notation $H_0''(f)$ is from [Br70]. The Brieskorn lattice is generated by germs of sections $s[\omega]$ from holomorphic $(n+1)$-forms $\omega\in\Omega^{n+1}_X$: Integrating the Gelfand-Leray form $\frac{\omega}{df}|_{f^{-1}(\tau)}$ over cycles in $H_n(f^{-1}(\tau),\mathbb{C})$ gives a holomorphic section $s[\omega]$ in the cohomology bundle, whose germ $s[\omega]_0$ at 0 is in fact in $V^{>-1}$ (this was proved first by Malgrange). The following theorem collects well known properties of the Brieskorn lattice. Afterwards we make comments on their proofs. See also [He02].

Theorem 7.7. *Algebraic properties:*

$$\begin{aligned} H_0''(f) &\cong \Omega^{n+1}_{\mathbb{C}^{n+1},0}/df\wedge d\Omega^{n-1}_{\mathbb{C}^{n+1},0}, & (7.23)\\ \partial_\tau^{-1}: H_0''(f) &\xrightarrow{\cong} H_0'(f)\subset H_0''(f)\\ \text{with } H_0'(f) &\cong df\wedge\Omega^n_{\mathbb{C}^{n+1},0}/df\wedge d\Omega^{n-1}_{\mathbb{C}^{n+1},0},\\ \text{and } \partial_\tau: s[df\wedge\eta]_0 &\mapsto s[d\eta]_0. & (7.24)\end{aligned}$$

Compatibility with K_f: K_f is the restriction to $H_0''(f)$ of K. Saito's higher residue pairings. It satisfies

$$K_f: H_0''(f)\times H_0''(f)\to \partial_\tau^{-n-1}\cdot\mathbb{C}\{\{\partial_\tau^{-1}\}\}. \tag{7.25}$$

The leading part

$$K_f^{(-n-1)}: H_0''(f)/H_0'(f)\times H_0''(f)/H_0'(f)\to\mathbb{C} \tag{7.26}$$

is symmetric (lemma 7.2) and nondegenerate. It is Grothendieck's residue pairing on $\Omega^{n+1}_{\mathbb{C}^{n+1},0}/df\wedge\Omega^n_{\mathbb{C}^{n+1},0}$.

Relation to Steenbrink's Hodge filtration $F^\bullet H^\infty_\mathbb{C}$: For $\lambda=e^{-2\pi i\alpha}$ with $\alpha\in(-1,0]$,

$$F^p_{St}H^\infty_\lambda = \psi_\alpha^{-1}\Big(\partial_\tau^{n-p}\operatorname{Gr}_V^{n-p+\alpha} H_0''(f)\Big). \tag{7.27}$$

Define the unordered tuple $\operatorname{Sp}(f)=\sum_{i=1}^\mu(\alpha_i)=\sum_{\alpha\in\mathbb{Q}}d(\alpha)\cdot(\alpha)\in\mathbb{Z}_{\geq 0}[\mathbb{Q}]$ of spectral numbers $\alpha_1,\dots,\alpha_\mu\in\mathbb{Q}$ by

$$d(\alpha) := \dim\operatorname{Gr}_V^\alpha H_0''-\dim\operatorname{Gr}_V^\alpha H_0'. \tag{7.28}$$

Number them such that $\alpha_1\leq\dots\leq\alpha_\mu$. Then they satisfy the symmetry

$$\alpha_i+\alpha_{\mu+1-i}=n-1 \tag{7.29}$$

and

$$\begin{aligned} -1<\alpha_1\leq\dots\leq\alpha_\mu<n, & \qquad (7.30)\\ V^{>-1}\supset H_0''\supset V^{n-1},\\ 0=F^{n+1}H^\infty,\ F^0H^\infty_{\neq 1}=H^\infty_{\neq 1},\ F^1H^\infty_1=H^\infty_1. \end{aligned}$$

The algebraic properties had been proved by Brieskorn [Br70] with some help by Sebastiani. That K_f is the restriction to $H_0''(f)$ of K. Saito's higher residue pairings [SaK83] follows from (7.21) and Pham's identification of P with the Fourier-Laplace transform of K. Saito's higher residue pairings [Ph85]. See [He02] for an alternative reasoning. Then (7.25) and the properties of (7.26) follow from K. Saito's work.

Steenbrink defined the Hodge filtration $F^\bullet_{St}$ first using resolution of singularities [St77]. Then Varchenko [Va80-1] constructed a closely related Hodge filtration $F^\bullet_{Va}$ from the Brieskorn lattice $H_0''(f)$. Scherk and Steenbrink [SS85] (and also M. Saito) modified this construction to recover $F^\bullet_{St}$. This is (7.27). Then (7.29) and (7.30) follow from properties of the Hodge filtration. Though $V^{>-1}\supset H_0''$ was proved before by Malgrange.

Remark 7.8. The Fourier-Laplace transformation FL is defined on any sum of elementary sections with the stronger convergence condition in (7.17). Therefore it is not defined on arbitrary elements of H_0''. But because of (7.30),

$$H_0''=(H_0''\cap\bigoplus_{-1<\alpha<n-1}C^\alpha_\tau)\oplus V^{n-1}_\tau, \tag{7.31}$$

and the elements of the first summand are finite sums of elementary sections. Therefore the space

$$FL(H_0''\cap \bigoplus_{-1<\alpha<n-1} C_\tau^\alpha)\oplus V_z^n \tag{7.32}$$

is a well-defined free $\mathbb{C}\{z\}$-module of rank μ. For simplicity we call it $FL(H_0'')$, although that is not completely correct. It satisfies

$$z^2\partial_z : FL(H_0'') \to FL(H_0''), \tag{7.33}$$

$$\text{and } P : FL(H_0'')\times FL(H_0'') \to z^{n+1}\cdot\mathbb{C}\{z\}, \tag{7.34}$$

and the leading part of P is a symmetric and nondegenerate pairing on $FL(H_0'')/z\cdot FL(H_0'')$, all of this because of (7.19)–(7.21), (7.24)–(7.26). It thus satisfies all properties of a TERP-structure [He02, definition 2.12]. Because of the $\mathbb{Z}$-lattice $H_\mathbb{Z}^\infty$ and the $\mathbb{Z}$-lattice bundle in the cohomology, we can even call it a TEZP-structure. More precisely, we denote as TEZP structure the following tuple.

$$TEZP(f) := (H_\mathbb{Z}^\infty, L^{nor}, V_z^{mod}, P, FL(H_0''))(f). \tag{7.35}$$

Here V_z^{mod} comes equipped with the actions of z, ∂_z^{-1} and $z\partial_z$. We formulated theorem 7.5 and introduced $FL(H_0'')$ because of the following Thom-Sebastiani result.

Theorem 7.9. [SS85][BH17, Theorem 6.4] *Consider besides $f(x_0,\dots,x_n)$ a second singularity $g(x_{n+1},\dots,x_{n+m+1})$. Then*

$$TEZP(f+g)\cong TEZP(f)\otimes TEZP(g). \tag{7.36}$$

Remarks 7.10. (i) The isomorphism for the data $(H_\mathbb{Z}^\infty, L^{nor})$ is the classical Thom-Sebastiani result in (4.8) and (4.10). The isomorphism for P follows from its definition with L^{nor}. The isomorphism for V_z^{mod} is trivial. The isomorphism for H_0'' was essentially proved in [SS85, (8.7) Lemma]. Though Scherk and Steenbrink did not make the compatibility with the topological Thom-Sebastiani isomorphism between the cohomology bundles precise, and they avoided the use of the Fourier-Laplace transformation. They obtained a ∂_τ^{-1}-linear isomorphism

$$H_0''(f+g)\cong H_0''(f)\otimes H_0''(g).$$

(ii) They applied this isomorphism to obtain a Thom-Sebastiani formula for $F_{St}^\bullet$ in [SS85, Theorems (8.2) and (8.11)]. Though their Thom-Sebastiani formula is wrong if $N\neq 0$. In the application of the isomorphism, they had mixed ∂_τ^{-1}-linearity and τ-linearity and went with this isomorphism directly into the defining formula (7.27) of $F_{St}^\bullet$. But the true Thom-Sebastiani formula is quite close [BH17, Corollary 6.5]. One has to replace in [SS85, Theorems (8.2) and (8.11)] $F_{St}^\bullet$ by $G(F_{St}^\bullet)$. This follows immediately from (7.27) and (7.36). Of course, in the case $N=0$, the isomorphism G in definition 7.4 is just a rescaling, and then $G(F_{St}^\bullet)=F_{St}^\bullet$, so then their Thom-Sebastiani formula is correct.

(iii) As a corollary of theorem 7.9, we obtain for a suspension of f

$$TEZP(f+x_{n+1}^2)\cong TEZP(f)\otimes TEZP(x_{n+1}^2). \tag{7.37}$$

This allows us to consider in the sections 9 and 10 only the surface singularities. More generally, it implies the corollary 8.14. This corollary is the reason why we introduced $FL(H_0''(f))$. Formula (7.36) and this corollary are more elegant and general than the arguments with which suspensions were treated in [He93], [He95], [He11] and [GH17].

(iv) The Thom-Sebastiani formula for $F_{St}^\bullet$ expresses in the case of a suspension $F_{St}^\bullet(f+x_{n+1}^2)$ in terms of $F_{St}^\bullet(f)$. It is made explicit in [BH17, Theorem 4.6]. It can be seen as a square root of a Tate twist, because $F^\bullet_{St}(f)$ and $F^\bullet_{St}(f+x_{n+1}^2+x_{n+2}^2)$ are simply related by a Tate twist. f and $f+x_{n+1}^2+x_{n+2}^2$ have the same polarizing form S by (4.20) and (4.21), because $M_h(f)=M_h(f+x_{n+1}^2+x_{n+2}^2)$. But the polarizing form of $f+x_{n+1}^2$ is quite different, because of $M_h(f+x_{n+1}^2)=-M_h(f)$ and (4.20) and (4.21). The formula in [BH17, Theorem 4.6] which expresses $F^\bullet_{St}(f+x_{n+1}^2)$ in terms of $F^\bullet_{St}(f)$ involves the $G^{(\alpha)}$ from definition 7.4 and is compatible with the isotropy condition (4.18) and (the generalization in the case $N\neq 0$ of) the positivity condition (4.19).

Fix for a moment a reference singularity f_0. In [He99] a classifying space $D_{PMHS}(f_0)$ and a classifying space $D_{BL}(f_0)$ are constructed. D_{PMHS} is a classifying space for M_s-invariant Hodge filtrations $F^\bullet$ on $H^\infty_{\mathbb{C}}(f_0)$ such that $(H^\infty_{\neq 1}, H^\infty_{\neq 1,\mathbb{Z}}, F^\bullet, W, -N, S)$ and $(H^\infty_1, H^\infty_{1,\mathbb{Z}}, F^\bullet, W, -N, S)$ are polarized mixed Hodge structures of weight n respectively $n+1$ with the same Hodge numbers as $F^\bullet_{St}(f_0)$.

D_{BL} is a classifying space for subspaces $\mathcal{L}_0\subset V_\tau^{>-1}$ with the following properties:

- (α) $\mathcal{L}_0$ is a free $\mathbb{C}\{\tau\}$-module of rank μ.
- (β) $\mathcal{L}_0$ is a free $\mathbb{C}\{\{\partial_\tau^{-1}\}\}$-module of rank μ.
- (γ) The filtration $F^\bullet$ in $H^\infty_{\mathbb{C}}(f_0)$ which is constructed by formula (7.27) with $\mathcal{L}_0$ instead of $H_0''(f_0)$ is in D_{PMHS}.
- (δ) It satisfies $K_f(\mathcal{L}_0,\mathcal{L}_0)\subset \partial_\tau^{-n-1}\cdot\mathbb{C}\{\{\partial_\tau^{-1}\}\}$.

Theorem 7.11. *Fix a reference singularity $f_0(x_0,\dots,x_n)$.*

(a) [He99, ch. 2] *$D_{PMHS}(f_0)$ is a real homogeneous space and a complex manifold. It is a locally trivial bundle over a product D_{PHS} of classifying spaces for pure polarized Hodge structures. The fibers carry an affine algebraic structure and are isomorphic to $\mathbb{C}^{N_{PMHS}}$ for some $N_{PMHS}\in\mathbb{Z}_{\geq 0}$. The group $G_{\mathbb{Z}}(f_0)$ acts properly discontinuously on D_{PMHS}.*

(b) [He99, ch. 5] *$D_{BL}(f_0)$ is a complex manifold and a locally trivial bundle over D_{PMHS}. The fibers have a natural $\mathbb{C}^*$-action with negative weights and are affine algebraic manifolds and are isomorphic to $\mathbb{C}^{N_{BL}}$ for some $N_{BL}\in\mathbb{Z}_{\geq 0}$. The group $G_{\mathbb{Z}}(f_0)$ acts properly discontinuously on D_{BL}.*

(c) $D_{PMHS}(f_0)$ and $D_{PMHS}(f_0+x_{n+1}^2)$ are canonically isomorphic. $D_{BL}(f_0)$ and $D_{BL}(f_0+x_{n+1}^2)$ are canonically isomorphic.

Part (c) is not formulated in [He99]. The isomorphism $D_{BL}(f_0)\to D_{BL}(f_0+x_{n+1}^2)$ is given by the generalization of (7.36), namely the map

$$\mathcal{L}_0\mapsto FL^{-1}\Big(FL(\mathcal{L}_0)\otimes FL(H_0''(x_{n+1}^2))\Big). \tag{7.38}$$

The isomorphism $D_{PMHS}(f_0)\to D_{PMHS}(f_0+x_{n+1}^2)$ is obtained by applying $\mathrm{Gr}_V^\bullet$. It follows also from [BH17, Theorem 4.6].

In the sections 9 and 10, μ-constant families of singularities in two parameters will be studied. The following definition and theorem treat a more general situation. It had been considered especially in [Va80-2] [AGV88] [SaM91] [He93] [Ku98].

Definition 7.12. A holomorphic μ-constant family of singularities consists of a number $\mu\in\mathbb{Z}_{\geq 1}$, a connected complex manifold T, an open neighborhood $X\subset\mathbb{C}^{n+1}\times T$ of $\{0\}\times T$ and a holomorphic function $F:X\to\mathbb{C}$ such that $F_t:=F|_{X_t}$ with $X_t:=X\cap\mathbb{C}^{n+1}\times\{t\}$ for any $t\in T$ has an isolated singularity at 0 with Milnor number μ.

Theorem 7.13. *Consider a holomorphic μ-constant family as in definition 7.12.*

(a) The Milnor lattices $(Ml(F_t), L)$ with Seifert forms for $t \in T$ are locally canonically isomorphic. They glue to a local system $\bigcup_{t\in T} Ml(F_t)$ of free $\mathbb{Z}$-modules of rank μ.

(b) Therefore also the spaces $C^\alpha(F_t)$, $V^{mod}_\tau(F_t)$, $V^\alpha_\tau(F_t)$ are locally canonically isomorphic and glue to local systems.

(c) But the Brieskorn lattices $H_0''(F_t) \subset V_\tau^{>-1}(F_t)$ vary holomorphically. For $\omega \in \Omega^{n+1}_{X/T}$, $s[\omega]_0(t) := s[\omega|_{X_t}]_0 \in H_0''(F_t)$. Let ξ be a holomorphic vector field on T. Its canonical lifts to $\mathbb{C}\times T$ (with coordinate τ on $\mathbb{C}$) and X are also denoted ξ. The covariant derivative of $s[\omega]_0(t)$ by ξ is

$$\xi\ s[\omega]_0(t) = s[\mathrm{Lie}_\xi\, \omega]_0(t) + (-\partial_\tau)s[\xi(F)\cdot\omega]_0(t). \tag{7.39}$$

(d) All germs F_t have the same spectrum.

Remarks 7.14. (i) Part (a) is less trivial than one might expect, as it is not clear whether $\varepsilon(t)$ and $\delta(t)$ in the definition of a Milnor fibration $F_t : X(\varepsilon(t),\delta(t)) \to \Delta_{\delta(t)}$ can be chosen as continuous functions in t. But lemma 2.2 in [LR73] saves the situation. See [Va80-2] [He93] [Ku98] [He11] for details.

(ii) Part (b) follows from part (a). Formula (7.39) is well known, see e.g. [Va80-2] [AGV88] [He93] [Ku98]. Part (d) is proved in [Va82].

(iii) The bundle $\bigcup_{t\in T} H_0''(F_t) \subset \bigcup_{t\in T} V_\tau^{>-1}(F_t)$ can be seen as a germ along $\{0\}\times T$ on $(\mathbb{C},0)\times T$ of a holomorphic rank μ bundle.
$s[\omega]_0$ for $\omega\in\Omega^{n+1}_{X/T}$ is a holomorphic section in this bundle.
But in theorem 9.6 and theorem 10.6 we will be imprecise and consider $s[\omega]_0$ as a possibly multi-valued holomorphic map $s[\omega]_0 : T \to V_\tau^{>-1}(F_{t^0})$ for a reference singularity F_{t^0}.

(iv) $s[\omega]_0$ is a sum $s[\omega]_0 = \sum_{\alpha>-1} s(\omega,\alpha)$ of holomorphic families $s(\omega,\alpha)(t) \in C^\alpha(F_t)$, $t\in T$, of elementary sections. For each $t\in T$,

$$\alpha(s[\omega]_0(t)) := \alpha(\omega|_{X_t}) := \min(\alpha \,|\, s(\omega,\alpha)(t) \neq 0) \tag{7.40}$$

is the *order* of $s[\omega]_0(t)$, and $s(\omega,\alpha(\omega|_{X_t}))(t)$ is its *principal part*. The order is upper semicontinuous in t.

(v) A notation: $\omega_0 := dx_0\ldots dx_n$.

All bimodal series singularities in table (9.1) except $W^\sharp_{1,p}$ (see remark 9.5 for $W^\sharp_{1,p}$) are Newton nondegenerate. All quadrangle singularities in table (10.1) are semiquasihomogeneous. For such singularities there are useful results for the computation of the order $\alpha(\omega|_{X_t})$, which we describe in the following. We start with a definition of Kouchnirenko.

Definition 7.15. Let $f : (\mathbb{C}^{n+1},0)\to(\mathbb{C},0)$ be a singularity.
(a) [Ko76] Write $f = \sum_{i\in\mathbb{Z}^{n+1}_{\geq 0}} a_i x^i$ and define

$$\begin{aligned}
\mathrm{supp}(f) &:= \{i\in\mathbb{Z}^{n+1}_{\geq 0}\,|\, a_i\neq 0\}, \\
\Gamma_+(f) &:= \Big(\text{convex hull of } \bigcup_{i\in\mathrm{supp}(f)} (i+\mathbb{R}^{n+1}_{\geq 0})\Big) \subset \mathbb{R}^{n+1}, \\
\Gamma_{com}(f) &:= \{\sigma\,|\,\sigma \text{ is a compact face of } \Gamma_+(f)\}, \\
\Gamma_{com,n}(f) &:= \{\sigma\in\Gamma_{com}(f)\,|\,\dim\sigma = n\}, \\
l_\sigma &: \mathbb{R}^{n+1}\to\mathbb{R} \text{ for } \sigma\in\Gamma_{com,n}(f)
\end{aligned} \tag{7.41}$$

as the linear function with $\sigma\subset l_\sigma^{-1}(1)$.

(b) [SaM88][KV85] The Newton order $\nu : \mathbb{C}\{x_0,\dots,x_n\} \to \mathbb{Q}_{\geq 0} \cup \{\infty\}$ is

$$\nu(\sum_i b_i x^i) := \min(l_\sigma(i)\,|\,\text{all } i \text{ with } b_i \neq 0,\ \text{all } \sigma \in \Gamma_{com,n}(f)). \tag{7.42}$$

The Newton order $\nu : \Omega^{n+1}_{\mathbb{C}^{n+1},0} \to \mathbb{Q}_{>0} \cup \{\infty\}$ is

$$\nu((\sum_i b_i x^i)\cdot \omega_0) := \nu((\sum_i b_i x^i) x_0 \dots x_n). \tag{7.43}$$

The Newton order $\overline{\nu} : H_0''(f) \to \mathbb{Q}_{>0} \cup \{\infty\}$ is

$$\overline{\nu} := \max(\nu(\eta)\,|\,\eta \equiv \omega \mod df \wedge d\Omega^{n-1}_{\mathbb{C}^{n+1},0}). \tag{7.44}$$

(c) [Ko76] For $\sigma \in \Gamma_{com}(f)$ define $f_\sigma := \sum_{i\in\sigma} a_i x^i$. The singularity f is *Newton nondegenerate* if for each $\sigma \in \Gamma_{com}(f)$ the Jacobi ideal $J(f_\sigma)$ of f_σ has no zero in $(\mathbb{C}^*)^{n+1}$. It is *convenient* if f contains for each index $j \in \{0,\dots,n\}$ a monomial $x_j^{m_j}$ for some $m_j \geq 2$.

The following theorem was proved in 1983 by M. Saito [SaM88]. The proof shortly afterwards by Khovanskii and Varchenko [KV85] is completely different.

Theorem 7.16. *Let f be a Newton nondegenerate and convenient singularity. For any $\omega \in \Omega^{n+1}_{\mathbb{C}^{n+1},0}$, its order $\alpha(\omega)$ (defined in remark 7.14 (iv)) is $\alpha(\omega) = \overline{\nu}(\omega) - 1$.*

The following corollary is an easy consequence. It is proved in [He93, Satz 1.10].

Corollary 7.17. *Let f be a Newton nondegenerate and convenient singularity. Define*

$$s(f) := \min\left(\nu(\frac{\partial f}{\partial x_j}\cdot\omega_0) - 1\,|\,j \in \{0,\dots,n\}\right) > 0, \tag{7.45}$$

$$I(f) := \{i \in \mathbb{Z}^{n+1}_{\geq 0}\,|\,\nu(x^i\omega_0) - 1 < s(f)\}. \tag{7.46}$$

Then for $i \in I(f)$

$$\alpha(x^i\omega_0) = \nu(x^i\omega_0) - 1, \tag{7.47}$$

the numbers $\alpha(x^i\omega_0)$, $i \in I(f)$, are the spectral numbers in the interval $(-1, s(f))$, and

$$\alpha((\sum_i b_i x^i)\cdot\omega_0) = \begin{cases} \min(\alpha(x^i\omega_0)\,|\,i \in I(f), b_i \neq 0) \\ \qquad \text{if an } i \in I(f) \text{ with } b_i \neq 0 \text{ exists,} \\ \geq s(f) \qquad \text{else.} \end{cases} \tag{7.48}$$

Remarks 7.18. (i) We expect that theorem 7.16 holds also without the condition that f is convenient. This would be desirable as many normal forms of singularities are Newton nondegenerate, but not convenient.

(ii) A singularity is $(\mu+1)$-determined, i.e. $f + g \sim_{\mathcal{R}} f$ for any $g \in \mathbf{m}^{\mu+1}$, where $\mathbf{m}$ is the maximal ideal in $\mathbb{C}\{x\}$ [Ma68]. If f is Newton nondegenerate, then $f + \sum_{j=0}^n c_j x_j^{m_j}$ for arbitrary $m_j \geq \mu+1$ and sufficiently generic $c_j \in \mathbb{C}^*$ is Newton nondegenerate and convenient and right equivalent to f.

Furthermore, because of $\mathbf{m}^\mu \subset J(f)$ and the Artin approximation theorem, one can choose a coordinate change φ with $f + \sum_{j=0}^n c_j x_j^{m_j} = f\circ\varphi$ such that all $\varphi_j - x_j \in \mathbf{m}^{\min(m_k)-\mu}$. Unfortunately, this is not sufficient for a generalization of theorem 7.16 to the case where f is not convenient.

(iii) We claim that the calculations in the proof of theorem 9.6 can be carried out with almost no change (but with additional terms) for $f + \sum_{j=0}^n c_j x_j^{m_j}$ with large m_j and that they give essentially the same results. With this claim, we justify that we calculate in the proof of theorem

9.6 with the normal forms f in table (9.1) which are almost all not convenient, but that we apply theorem 7.16 and corollary 7.17.

(iv) Theorem 7.16 holds without the condition that f is convenient if f is semiquasihomogeneous. That is the case when there is only one compact face of dimension n.

Definition 7.19. (a) A singularity f is *semiquasihomogeneous* with weights $w_0,\dots,w_n\in\mathbb{Q}_{>0}$ if

$$f=\sum_{i\in\mathbb{Z}^{n+1}_{\geq 0}} a_i x^i \text{ with } \deg_w x^i\geq 1 \text{ for all } i \text{ with } a_i\neq 0, \tag{7.49}$$

and the quasihomogeneous polynomial

$$f_{qh}:=\sum_{i:\,\deg_w x^i=1} a_i x^i \tag{7.50}$$

has an isolated singularity at 0.

(b) A singularity f is *quasihomogeneous* if it is semiquasihomogeneous with $f=f_{qh}$.

A quasihomogeneous singularity f satisfies the Euler equation

$$f=\sum_{j=0}^n w_j x_j\frac{\partial f}{\partial x_j}. \tag{7.51}$$

This equation and (7.24) and elementary calculations in [Br70] imply part (a) of the following lemma.

Lemma 7.20. *(a) Let f be a quasihomogeneous singularity with weights $(w_0,\dots,w_n)$. If $\omega=x^i\omega_0$ is a monomial differential form then*

$$\begin{aligned}\text{either}\quad & s[\omega]_0=0\\ \text{or}\quad & \alpha(\omega)=\deg_w(x^ix_0\dots x_n)-1 \text{ and } s[\omega]_0=s(\omega,\alpha(\omega)).\end{aligned} \tag{7.52}$$

(b) Let f be a semiquasihomogeneous singularity with weights $(w_0,\dots,w_n)$ and $f\neq f_{qh}$. The 1-parameter family $f_{qh}+t\cdot(f-f_{qh})$ is a μ-constant family. If $\omega=x^i\omega_0$ is a monomial differential form then

$$\begin{aligned}\alpha(\omega) &\geq \deg_w(x^ix_0\dots x_n)-1, \\ s(\omega &\,,\ \deg_w(x^ix_0\dots x_n)-1)(t)=s[\omega]_0(0),\\ s(\omega,\alpha)(t) &= \sum_{k\geq 0}\frac{1}{k!}\cdot t^k\cdot(-\partial_\tau)^k s((f-f_{qh})^k\cdot\omega,\alpha+k)(0).\end{aligned} \tag{7.53}$$

The last expression is polynomial in t because $\alpha((f-f_{qh})^k\omega)>\alpha+k$ for large k.

Proof of part (b): In [AGV85, ch. 12] it is shown that $f_{qh}+t(f-f_{qh})$ is a μ-constant family. The other assertions follow with theorem 7.13 (c) and part (a) of lemma 7.20. □

8. Review on marked singularities, their moduli spaces, μ-constant monodromy groups and Torelli conjectures

This paper and the paper [GH17] complete the study of the data in the title of this section for the singularities of modality ≤ 2. These data were introduced in [He11]. Here we review them. We start with the notions *marked singularity* and *strongly marked singularity*.

Definition 8.1. Fix one reference singularity f_0.

(a) Then a strong marking for any singularity f in the μ-homotopy class of f_0 (i.e. there is a family of singularities with constant Milnor number and parameter space $[0,1]$ which connects f_0 and f) is an isomorphism $\rho : (Ml(f), L) \to (Ml(f_0), L)$.

(b) The pair (f, ρ) is a *strongly marked singularity.* Two strongly marked singularities (f_1, ρ_1) and (f_2, ρ_2) are right equivalent (notation: $\sim_{\mathcal{R}}$) if a coordinate change $\varphi : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{n+1}, 0)$ with

$$f_1 = f_2 \circ \varphi \text{and} \rho_1 = \rho_2 \circ \varphi_{hom}$$

exists, where $\varphi_{hom} : (Ml(f_1), L) \to (Ml(f_2), L)$ is the induced isomorphism.

(c) The notion of a marked singularity is slightly weaker. If f and ρ are as above, then the pair $(f, \pm\rho)$ is a *marked singularity* (writing $\pm\rho$, the set $\{\rho, -\rho\}$ is meant, neither ρ nor $-\rho$ is preferred).

(d) Two marked singularities $(f_1, \pm\rho_1)$ and $(f_2, \pm\rho_2)$ are right equivalent (notation: $\sim_{\mathcal{R}}$) if a coordinate change φ with

$$f_1 = f_2 \circ \varphi \text{and} \rho_1 = \varepsilon\rho_2 \circ \varphi_{hom} \qquad \text{for some } \varepsilon \in \{\pm 1\}$$

exists.

Remarks 8.2. (i) The notion of a marked singularity behaves better than the notion of a strongly marked singularity, because it is not known whether all μ-homotopy families of singularities satisfy one of the following two properties:

$$\begin{aligned} &\text{Assumption (8.1):} \quad && \text{Any singularity in the } \mu\text{-homotopy} \\ & && \text{class of } f_0 \text{ has multiplicity } \geq 3. \end{aligned} \tag{8.1}$$

$$\begin{aligned} &\text{Assumption (8.2):} \quad && \text{Any singularity in the } \mu\text{-homotopy} \\ & && \text{class of } f_0 \text{ has multiplicity } 2. \end{aligned} \tag{8.2}$$

We expect that always one of two assumptions holds. For curve singularities and singularities right equivalent to semiquasihomogeneous singularities and all singularities with modality ≤ 2 this is true, but in general it is not known. In a μ-homotopy family where neither of the two assumptions holds, strong marking behaves badly, see (ii).

(ii) If $\text{mult}(f) = 2$ then $(f, \rho) \sim_{\mathcal{R}} (f, -\rho)$, which is easy to see. If $\text{mult}(f) \geq 3$, then $(f, \rho) \not\sim_{\mathcal{R}} (f, -\rho)$, whose proof in [He11] is quite intricate. These properties imply that the moduli space for strongly marked singularities discussed below is not Hausdorff in the case of a μ-homotopy class which satisfies neither one of the assumptions (8.1) or (8.2).

In [He02] a moduli space $M_\mu(f_0)$ was constructed for the μ-homotopy class of any singularity f_0. As a set it is simply the set of right equivalence classes of singularities in the μ-homotopy class of f_0. But in [He02] it is constructed as an analytic geometric quotient, and it is shown that it is locally isomorphic to the μ-constant stratum of a singularity modulo the action of a finite group. The μ-constant stratum of a singularity is the germ $(S_\mu, 0) \subset (M, 0)$ within the germ of the base space of a universal unfolding F of f, such that for a suitable representative

$$S_\mu = \{t \in M \,|\, F_t \text{ has only one singularity } x_0 \text{ and } F_t(x_0) = 0\}. \tag{8.3}$$

It comes equipped with a canonical complex structure, and M_μ inherits a canonical complex structure, see the chapters 12 and 13 in [He02].

In [He11] analogous results for marked singularities were proved. A better property is that M_μ^{mar} is locally isomorphic to a μ-constant stratum without dividing out a finite group action. Therefore one can consider it as a *global μ-constant stratum* or as a *Teichmüller space for singularities.* The following theorem collects results from [He11, theorem 4.3].

Theorem 8.3. *Fix one reference singularity f_0. Define the sets*

$$\begin{aligned} M_\mu^{smar}(f_0) &:= \{\text{strongly marked } (f,\rho)\,| \\ &\quad f \text{ in the } \mu\text{-homotopy class of } f_0\}/\sim_{\mathcal{R}}, \end{aligned} \tag{8.4}$$

$$\begin{aligned} M_\mu^{mar}(f_0) &:= \{\text{marked } (f,\pm\rho)\,| \\ &\quad f \text{ in the } \mu\text{-homotopy class of } f_0\}/\sim_{\mathcal{R}}. \end{aligned} \tag{8.5}$$

(a) $M_\mu^{mar}(f_0)$ carries a natural canonical complex structure. It can be constructed with the underlying reduced complex structure as an analytic geometric quotient (see [He11, theorem 4.3] *for details).*

(b) The germ $(M_\mu^{mar}(f_0),[(f,\pm\rho)])$ with its canonical complex structure is isomorphic to the μ-constant stratum of f with its canonical complex structure (see [He02, chapter 12] *for the definition of that).*

(c) For any $\psi\in G_{\mathbb{Z}}(f_0)=:G_{\mathbb{Z}}$, the map

$$\psi_{mar}:M_\mu^{mar}\to M_\mu^{mar},[(f,\pm\rho)]\to[(f,\pm\psi\circ\rho)]$$

is an automorphism of M_μ^{mar}. The action

$$G_{\mathbb{Z}}\times M_\mu^{mar}\to M_\mu^{mar},(\psi,[(f,\pm\rho)]\mapsto\psi_{mar}([(f,\pm\rho)])$$

is a group action from the left.

(d) The action of $G_{\mathbb{Z}}$ on M_μ^{mar} is properly discontinuous. The quotient $M_\mu^{mar}/G_{\mathbb{Z}}$ is the moduli space M_μ for right equivalence classes in the μ-homotopy class of f_0, with its canonical complex structure. Especially, $[(f_1,\pm\rho_1)]$ and $[(f_2,\pm\rho_2)]$ are in one $G_{\mathbb{Z}}$-orbit if and only if f_1 and f_2 are right equivalent.

(e) If assumption (8.1) *or* (8.2) *holds then (a) to (d) are also true for M_μ^{smar} and ψ_{smar} with $\psi_{smar}([(f,\rho)]):=[(f,\psi\circ\rho)]$. If neither* (8.1) *nor* (8.2) *holds then the natural topology on M_μ^{smar} is not Hausdorff.*

We stick to the situation in theorem 8.3 and define two subgroups of $G_{\mathbb{Z}}(f_0)$. The definitions in [He11, definition 3.1] are different, they use μ-constant families. The following definitions are a part of theorem 4.4 in [He11].

Definition 8.4. Let $(M_\mu^{mar})^0$ be the topological component of M_μ^{mar} (with its reduced complex structure) which contains $[(f_0,\pm\,\mathrm{id})]$. Then

$$G^{mar}(f_0) \quad := \quad \{\psi\in G_{\mathbb{Z}}\,|\,\psi \text{ maps } (M_\mu^{mar})^0 \text{ to itself}\}\subset G_{\mathbb{Z}}(f_0). \tag{8.6}$$

If assumption (8.1) or (8.2) holds, $(M_\mu^{smar})^0$ and $G^{smar}(f_0)\subset G_{\mathbb{Z}}(f_0)$ are defined analogously.

The following theorem is also proved in [He11].

Theorem 8.5. *(a) In the situation above, the map*

$$\begin{aligned} G_{\mathbb{Z}}/G^{mar}(f_0) &\to \{\text{topological components of } M_\mu^{mar}\} \\ \psi\cdot G^{mar}(f_0) &\mapsto \text{the component } \psi_{mar}((M_\mu^{mar})^0) \end{aligned}$$

is a bijection.

(b) If assumption (8.1) *or* (8.2) *holds then (a) is also true for M_μ^{smar} and $G^{smar}(f_0)$.*

(c) $-\,\mathrm{id}\in G_{\mathbb{Z}}$ acts trivially on $M_\mu^{mar}(f_0)$. Suppose that assumption (8.2) *holds and that $f_0=g_0(x_0,\dots,x_{n-1})+x_n^2$. Then $-\,\mathrm{id}$ acts trivially on $M_\mu^{smar}(f_0)$ and*

$$\begin{aligned} M_\mu^{smar}(f_0) &= M_\mu^{mar}(f_0) = M_\mu^{mar}(g_0), \\ G^{smar}(f_0) &= G^{mar}(f_0) = G^{mar}(g_0). \end{aligned} \tag{8.7}$$

Suppose additionally that assumption (8.1) holds for g_0 (instead of f_0). Then $\{\pm \mathrm{id}\}$ acts freely on $M_\mu^{smar}(g_0)$, and the quotient map

$$M_\mu^{smar}(g_0) \stackrel{/\{\pm \mathrm{id}\}}{\longrightarrow} M_\mu^{mar}(g_0), [(f,\rho)] \mapsto [(f,\pm\rho)]$$

is a double covering.

The following conjecture was formulated as conjecture 3.2 in [He11].

Conjecture 8.6. [He11, Conjecture 3.2] *(a) Fix a singularity f_0. Then M_μ^{mar} is connected. Equivalently (in view of theorem 8.5 (a)): $G^{mar}(f_0) = G_{\mathbb{Z}}$.*

(b) If the μ-homotopy class of f_0 satisfies assumption (8.1), then $-\mathrm{id} \notin G^{smar}(f_0)$.

The study of the singularities with modality ≤ 2 in [He11][GH17] and this paper gives: Part (b) is true for all singularities with modality ≤ 2. Part (a) is true for almost all singularities with modality ≤ 2, but not for all. The exceptions are the subseries for $p = m \cdot r$ of the eight bimodal series. This is a part of theorem 9.1. Now we expect that part (a) will be wrong for many singularities.

Using the other definition of G^{mar} in [He11], part (a) says that up to $\pm\,\mathrm{id}$, any element of $G_{\mathbb{Z}}$ can be realized as transversal monodromy of a μ-constant family with parameter space S^1. As it is wrong for some singularities and probably for many more, part (a) of conjecture 8.6 has to be replaced now by the question whether the subgroup G^{mar} of $G_{\mathbb{Z}}$ can be described in a nice conceptual way.

In order to understand the stabilizers $\mathrm{Stab}_{G_{\mathbb{Z}}}([(f,\rho)])$ and $\mathrm{Stab}_{G_{\mathbb{Z}}}([(f,\pm\rho)])$ of points

$$[(f,\rho)] \in M_\mu^{smar}(f_0) \quad \text{and} \quad [(f,\pm\rho)] \in M_\mu^{mar}(f_0),$$

we have to look at the *symmetries* of a single singularity. These had been discussed in [He02, chapter 13.2]. The discussion had been taken up again in [He11].

Definition 8.7. Let $f_0 = f_0(x_0,\dots,x_n)$ be a reference singularity and let f be any singularity in the μ-homotopy class of f_0. If ρ is a marking, then $G_{\mathbb{Z}}(f) = \rho^{-1} \circ G_{\mathbb{Z}} \circ \rho$.

We define

$$\begin{aligned}
\mathcal{R} &:= \{\varphi : (\mathbb{C}^{n+1},0) \to (\mathbb{C}^{n+1},0) \ \text{ biholomorphic}\}, && (8.8)\\
\mathcal{R}^f &:= \{\varphi \in \mathcal{R} \,|\, f \circ \varphi = f\}, && (8.9)\\
R_f &:= j_1\mathcal{R}^f/(j_1\mathcal{R}^f)^0, && (8.10)\\
G_{\mathcal{R}}^{smar}(f) &:= \{\varphi_{hom} \,|\, \varphi \in \mathcal{R}^f\} \subset G_{\mathbb{Z}}(f), && (8.11)\\
G_{\mathcal{R}}^{mar}(f) &:= \{\pm\psi \,|\, \psi \in G_{\mathcal{R}}^{smar}(f)\}, && (8.12)\\
G_{\mathcal{R}}^{smar,gen}(f_0) &:= \bigcap_{[(f,\rho)] \in M_\mu^{smar}} \rho^{-1} \circ G_{\mathcal{R}}^{smar}(f) \circ \rho \subset G_{\mathbb{Z}}. && (8.13)
\end{aligned}$$

Again, the definition of $G_{\mathcal{R}}^{smar}$ is different from the definition in [He11, definition 3.1]. The characterization in (8.11) is [He11, theorem 3.3. (e)]. R_f is the finite group of components of the group $j_1\mathcal{R}^f$ of 1-jets of coordinate changes which leave f invariant. The following theorem collects results from several theorems in [He11].

Theorem 8.8. *Consider the data in definition 8.7.*

(a) If $\mathrm{mult}(f) \geq 3$ then $j_1\mathcal{R}^f = R_f$.

(b) The homomorphism $()_{hom} : \mathcal{R}^f \to G_{\mathbb{Z}}(f)$ factors through R_f. Its image is

$$(R_f)_{hom} = G_{\mathcal{R}}^{smar}(f) \subset G_{\mathbb{Z}}(f).$$

(c) The homomorphism $()_{hom} : R_f \to G_{\mathcal{R}}^{smar}(f)$ is an isomorphism.

(d)

$$-\mathrm{id} \notin G_{\mathcal{R}}^{smar}(f) \iff \mathrm{mult}\, f \geq 3. \tag{8.14}$$

Equivalently: $G_{\mathcal{R}}^{mar}(f) = G_{\mathcal{R}}^{smar}(f)$ *if* $\mathrm{mult}\, f = 2$, *and* $G_{\mathcal{R}}^{mar}(f) = G_{\mathcal{R}}^{smar}(f) \times \{\pm \mathrm{id}\}$ *if* $\mathrm{mult}\, f \geq 3$.

(e) $G_{\mathcal{R}}^{mar}(f) = G_{\mathcal{R}}^{mar}(f + x_{n+1}^2)$.

(f) $M_h \in G^{smar}(f)$. *If* f *is quasihomogeneous then* $M_h \in G_{\mathcal{R}}^{smar}(f)$.

(g) For any $[(f,\rho)] \in M_\mu^{smar}$

$$\mathrm{Stab}_{G_{\mathbb{Z}}}([(f,\rho)]) = \rho \circ G_{\mathcal{R}}^{smar}(f) \circ \rho^{-1}, \tag{8.15}$$

$$\mathrm{Stab}_{G_{\mathbb{Z}}}([(f,\pm\rho)]) = \rho \circ G_{\mathcal{R}}^{mar}(f) \circ \rho^{-1}. \tag{8.16}$$

((8.15) does not require assumption (8.1) or (8.2)). As $G_{\mathbb{Z}}$ *acts properly discontinuously on* $M_\mu^{mar}(f_0)$, $G_{\mathcal{R}}^{smar}(f)$ *and* $G_{\mathcal{R}}^{mar}(f)$ *are finite. (But this follows already from the finiteness of* R_f *and (b).)*

The group $G_{\mathcal{R}}^{smar,gen}(f_0)$ in (8.13) had not been considered in [He11]. Usually it is very small. It is useful because of the following elementary fact.

Lemma 8.9. *Let* T *be the parameter space of a* μ*-constant family as in definition 7.12. The transversal monodromy of it is the representation* $\pi_1(T,t^0) \to G_{\mathbb{Z}}(F_{t^0})$ *which comes from the local system* $\bigcup_{t\in T} Ml(F_t)$.

If its image is in $G_{\mathcal{R}}^{smar,gen}(F_{t^0})$, *then there is a natural map* $T \to M_\mu^{smar}(F_{t^0})$.

Proof: The trivial strong marking $+\mathrm{id}$ for F_{t^0} induces along any path strong markings of other singularities F_t. Two paths which meet at a point t, might not induce the same strong marking of F_t, but the two markings differ only by an element of $G_{\mathcal{R}}^{smar}(F_t)$. Therefore they induce the same right equivalence class of a marked singularity. □

Finally, we come to the Brieskorn lattices of marked singularities and Torelli problems. After fixing a reference singularity f_0, a marked singularity $(f,\pm\rho)$ comes equipped with a *marked Brieskorn lattice* $BL(f,\pm\rho)$. The classifying space $D_{BL}(f_0)$ in theorem 7.11 is a classifying space for marked Brieskorn lattices. Theorem 7.13 implies part (a) of the following theorem.

Theorem 8.10. *Fix one reference singularity* f_0.

(a) There is a natural holomorphic period map

$$BL : M_\mu^{mar}(f_0) \to D_{BL}(f_0). \tag{8.17}$$

It is $G_{\mathbb{Z}}$*-equivariant.*

(b) [He02, theorem 12.8] *It is an immersion, here the reduced complex structure on* $M_\mu^{mar}(f_0)$ *is considered.*

The second author conjectured part (b) of the following global Torelli conjecture in [He93], part (c) in [He02] and part (a) in [He11].

Conjecture 8.11. *Fix one reference singularity* f_0.

(a) The period map $BL : M_\mu^{mar} \to D_{BL}$ *is injective.*

(b) The period map $LBL : M_\mu = M_\mu^{mar}/G_{\mathbb{Z}} \to D_{BL}/G_{\mathbb{Z}}$ *is injective.*

(c) For any singularity f *in the* μ*-homotopy class of* f_0 *and any marking* ρ,

$$\mathrm{Stab}_{G_{\mathbb{Z}}}([(f,\pm\rho)]) = \mathrm{Stab}_{G_{\mathbb{Z}}}(BL([(f,\pm\rho)])) \tag{8.18}$$

(only $\subset$ *and the finiteness of both groups are clear).*

The second author has a long-going project on Torelli type conjectures. Already in [He93], part (b) was proved for all simple and unimodal singularities and almost all bimodal singularities (all except 3 subseries of the 8 bimodal series). This was possible without the general construction of M_μ and D_{BL}, which came later in [He02] and [He99]. In the concrete cases considered in [He93], it is easy to identify a posteriori the spaces M_μ and D_{BL}.

The following lemma from [He11] clarifies the logic between the parts (a), (b) and (c) of conjecture 8.11.

Lemma 8.12. *In conjecture 8.11, (a) $\iff$ (b) and (c).*

Part (a) of conjecture 8.11 was proved in [He11] for the simple and those 22 of the 28 exceptional unimodal and bimodal singularities, where all eigenvalues of the monodromy have multiplicity one. In [GH17] part (a) was proved for the remaining unimodal and the remaining exceptional bimodal singularities. In the sections 9 and 10, part (a) will be proved for the remaining bimodal singularities, namely the bimodal series singularities and the quadrangle singularities.

As part (b) had been proved for almost all singularities with modality ≤ 2, the main work in [GH17] and here is the good control of the group $G_{\mathbb{Z}}$. But that is surprisingly difficult. In the case of the bimodal singularities in this paper, also the control of the Gauss-Manin connection side had to be improved: We provide better information on the transversal monodromy of the studied families than in [He93]. Due to this improvement, also the annoying gap of 3 subseries of the 8 bimodal series, where part (b) was not proved in [He93], could be closed here.

Remark 8.13. In the sections 9 and 10, we will restrict to consider surface singularities, i.e. singularities in 3 variables. This is justified by the following corollary. It is an application for suspensions of the Thom-Sebastiani formula for the Fourier-Laplace transforms of Brieskorn lattices in theorem 7.9. This is elegant, but the preparations in section 7 were heavy. In the earlier papers [He93][He11][GH17], we had dealt with this problem in a less conceptual, but leaner way, sometimes with extra calculations for curve singularities.

Corollary 8.14. *Consider the μ-homotopy class of a reference singularity $f_0(x_0,\dots,x_n)$ which satisfies assumption (8.1) and such that for any $m \geq 1$ the μ-homotopy class of $f_0 + \sum_{j=n+1}^{n+m} x_j^2$ satisfies assumption (8.2).*

Fix a number $m \geq 1$. The global Torelli conjecture 8.11 (a) holds for f_0 if any only if it holds for the reference singularity $f_0 + \sum_{j=n+1}^{n+m} x_j^2$

Proof: By (8.7), $M_\mu^{mar}(f_0)$ and $M_\mu^{mar}(f_0 + \sum_{j=n+1}^{n+m} x_j^2)$ are canonically isomorphic. By theorem 7.11 (c), the classifying spaces $D_{BL}(f_0)$ and $D_{BL}(f_0 + \sum_{j=n+1}^{n+m} x_j^2)$ are canonically isomorphic. It rests to see that these isomorphisms are compatible with the period maps BL for f_0 and for $f_0 + \sum_{j=n+1}^{n+m} x_j^2$. This is also rather clear from the formula (7.37) for the TEZP-structure of a suspension. □

9. Period maps and Torelli results for the bimodal series and $G_{\mathbb{Z}} \supsetneqq G^{mar}$ for the subseries

In this section we will prove for the bimodal series the strong global Torelli conjecture 8.11 (a), the conjecture 8.6 (b) $-\mathrm{id} \notin G^{smar}$ and for the singularities with $m \not| p$ the conjecture 8.6 (a) $G_{\mathbb{Z}} = G^{mar}$. But for the singularities in the subseries with $m|p$, we will see $G_{\mathbb{Z}} \supsetneqq G^{mar}$, $|G_{\mathbb{Z}}| = \infty, |G^{mar}| < \infty$. Theorem 9.1 states these results in more detail.

The singularities in the eight bimodal series $W_{1,p}^\sharp$, $S_{1,p}^\sharp$, $U_{1,p}$, $E_{3,p}$, $Z_{1,p}$, $Q_{2,p}$, $W_{1,p}$ and $S_{1,p}$ have as surface singularities the normal forms in table (9.1) [AGV85, 15.1]. Here $p \geq 1$ and $q \geq 1$, and the parameters (t_1, t_2) are in $T := (\mathbb{C} - \{0\}) \times \mathbb{C}$.

$$\begin{array}{ll} W^\sharp_{1,2q-1} & (x^2+y^3)^2+(t_1+t_2y)xy^{4+q}+z^2 \\ W^\sharp_{1,2q} & (x^2+y^3)^2+(t_1+t_2y)x^2y^{3+q}+z^2 \\ S^\sharp_{1,2q-1} & x^2z+y^3z+yz^2+(t_1+t_2y)xy^{3+q} \\ S^\sharp_{1,2q} & x^2z+y^3z+yz^2+(t_1+t_2y)x^2y^{2+q} \\ U_{1,2q-1} & x^3+xz^2+xy^3+(t_1+t_2y)y^{1+q}z^2 \\ U_{1,2q} & x^3+xz^2+xy^3+(t_1+t_2y)y^{3+q}z \\ E_{3,p} & x^3+x^2y^3+(t_1+t_2y)y^{9+p}+z^2 \\ Z_{1,p} & x^3y+x^2y^3+(t_1+t_2y)y^{7+p}+z^2 \\ Q_{2,p} & x^3+yz^2+x^2y^2+(t_1+t_2y)y^{6+p} \\ W_{1,p} & x^4+x^2y^3+(t_1+t_2y)y^{6+p}+z^2 \\ S_{1,p} & x^2z+yz^2+x^2y^2+(t_1+t_2y)y^{5+p} \end{array} \tag{9.1}$$

Recall that table (5.1) lists for these singularities the Milnor number μ, the characteristic polynomials b_j, $j\geq 1$, of the monodromy on the Orlik blocks B_j in theorem 5.1, the order m of the monodromy on B_1 and the index $r_I=[Ml(f):\bigoplus_{j\geq 1}B_j]$. The order of the monodromy on B_2 is

$$m+r_I\cdot p=:m_2. \tag{9.2}$$

We will need the space $T^{cov}:=(\mathbb{C}-\{0\})\times\mathbb{C}$ and the m_2-fold covering

$$c_T:T^{cov}\to T,\qquad (\tau_1,t_2)\mapsto(\tau_1^{m_2},t_2). \tag{9.3}$$

For each 2-parameter family of singularities in table (9.1), we choose $f_0:=f_{(1,0)}$ as reference singularity. In the following, we will write M_μ^{mar}, $(M_\mu^{mar})^0$, $G_\mathbb{Z}$, G^{mar}, Ml, H^∞ and C^α for $M_\mu^{mar}(f_0)$,$(M_\mu^{mar}(f_0))^0$, $G_\mathbb{Z}(f_0)$, $G^{mar}(f_0)$, $Ml(f_0)$, $H^\infty(f_0)$ and $C^\alpha(f_0)$.

We denote by $M_T\in G_\mathbb{Z}$ the monodromy of the homology bundle $\bigcup_{(t_1,t_2)\in T}Ml(f_{(t_1,t_2)})\to T$ along the cycle $\{(e^{2\pi is},0)\,|\,s\in[0,1]\}$. We call M_T the *transversal monodromy.* By the other definition of G^{mar} in [He11], $M_T\in G^{mar}$. As always, $\zeta:=e^{2\pi i/m}$.

Theorem 9.1. *Consider a family of bimodal series singularities in table* (9.1).

(a) $M_T^{m_2}=\mathrm{id}$. Therefore the pull back to T^{cov} with c_T of the family of singularities over T has trivial transversal monodromy. Thus the strong marking $+\mathrm{id}$ for $f_{(1,0)}$ induces a well defined strong marking for each singularity of this family over T^{cov}. This gives a map $T^{cov}\to(M_\mu^{smar})^0$ and a map $T^{cov}\to(M_\mu^{mar})^0$.

(b) Both maps are isomorphisms. And $-\mathrm{id}\notin G^{smar}$, where G^{smar} is the group for the singularities of multiplicity ≥ 3, namely the curve singularities $W^\sharp_{1,p},E_{3,p},Z_{1,p},W_{1,p}$ and the surface singularities $S^\sharp_{1,p},U_{1,p},Q_{2,p},S_{1,p}$. So, conjecture 8.6 (b) is true.

(c) The period map $BL:M_\mu^{mar}\to D_{BL}$ is an embedding. So, the strong global Torelli conjecture 8.11 (a) is true.

(d) If $m\not|\,p$ then $G_\mathbb{Z}=G^{mar}$. So, here conjecture 8.6 (a) is true.

(e) In the case of the subseries with $m|p$, $G_\mathbb{Z}\supsetneqq G^{mar}$. So, here conjecture 8.6 (a) is wrong. More precisely, G^{mar} and $G_\mathbb{Z}$ are as follows. M_T has on the 2-dimensional $\mathbb{C}$-vector space Ml_ζ the eigenvalues 1 and $\overline{\zeta}$. Let $Ml_{\zeta,1}$ be the 1-dimensional eigenspace of M_T on Ml_ζ with eigenvalue 1. Then $|G_\mathbb{Z}|=\infty$ and $|G^{mar}|<\infty$ and

$$G^{mar}\quad=\quad\{g\in G_\mathbb{Z}\,|\,g(Ml_{\zeta,1})=Ml_{\zeta,1}\}. \tag{9.4}$$

$\Psi(G_{\mathbb{Z}})$ is an infinite Fuchsian group by theorem 5.1 (c). $\Psi(G^{mar})$ is the finite subgroup of elliptic elements which fix the point $[Ml_{\zeta,1}]\in\mathcal{H}_\zeta$ ($\mathcal{H}_\zeta$ was defined in (5.7)). And M_μ^{mar} consists of infinitely many copies of T^{cov}.

Theorem 9.1 will be proved in this section in several steps. It builds on two hard results. The first and more difficult one is theorem 5.1 on $G_{\mathbb{Z}}$. The second one is easier, but still rather technical. It is the calculation of the multi-valued period map $T\to D_{BL}$. The results are fixed in theorem 9.6.

But we prefer to present the nice geometry before the technical details. Therefore we will now explain everything what can be understood without going into the details of the Gauss-Manin connection and theorem 9.6. Afterwards we will come to the Gauss-Manin connection and theorem 9.6.

Define

$$\alpha_1:=\frac{-1}{m}<\beta_1:=\frac{-1}{m_2}<0<\alpha_2:=\frac{1}{m_2}<\beta_2:=\frac{1}{m} \tag{9.5}$$

and recall that $\psi_\alpha: H^\infty\to C^\alpha$, $A\mapsto es(A,\alpha)$, is an isomorphism. Therefore and because of table (5.1)

$$\begin{aligned}\dim C^{\beta_1}=\dim C^{\alpha_2}=1,\\ \dim C^{\alpha_1}=\dim C^{\beta_2}=\begin{cases}1 & \text{if } m \not| p,\\ 2 & \text{if } m|p.\end{cases}\end{aligned} \tag{9.6}$$

For the cases with $m\not| p$, define the 2-dimensional space

$$\begin{aligned} D_{BL}^{sub} &:= \{\mathbb{C}\cdot(v_1+v_2+v_4)\,|\,v_1\in C^{\alpha_1}-\{0\},v_2\in C^{\beta_1}-\{0\},v_4\in C^{\beta_2}\}\\ &= \{\mathbb{C}\cdot(v_1^0+\rho_1v_2^0+\rho_2v_4^0)\,|\,(\rho_1,\rho_2)\in(\mathbb{C}-\{0\})\times\mathbb{C}\}\\ &\quad\ \text{for some generators } v_1^0,v_2^0,v_4^0 \text{ of } C^{\alpha_1},C^{\beta_1},C^{\beta_2}\\ &\cong (\mathbb{C}-\{0\})\times\mathbb{C}.\end{aligned} \tag{9.7}$$

For the cases with $m|p$, the polarizing form S defines an indefinite hermitian form

$$((a,b)\mapsto S(a,\overline{b}))$$

on H_ζ^∞. This follows from the corresponding statement for h_ζ on Ml_ζ in theorem 5.1, from lemma 2.2 (b) and from the relation between Seifert form L and polarizing form S, see (4.20). Thus we get a half-plane

$$\begin{aligned}\mathcal{H}(C^{\alpha_1}) &:= \{\mathbb{C}\cdot v\,|\,v\in C^{\alpha_1} \text{ with } S(\psi_{\alpha_1}^{-1}(v),\overline{\psi_{\alpha_1}^{-1}(v)})<0\}\\ &\subset \mathbb{P}(C^\alpha).\end{aligned} \tag{9.8}$$

Now define for the cases with $m|p$ the 3-dimensional space

$$\begin{aligned} D_{BL}^{sub} &:= \{\mathbb{C}\cdot(v_1+v_2+v_4)\,|\,v_1\in C^{\alpha_1}-\{0\} \text{ with } [\mathbb{C}\cdot v_1]\in\mathcal{H}(C^{\alpha_1}),\\ &\qquad v_2\in C^{\beta_1}-\{0\},v_4\in\mathbb{C}\cdot\psi_{\beta_2}(\overline{\psi_{\alpha_1}^{-1}(v_1)})\subset C^{\beta_2}\}\\ &\cong \mathcal{H}(C^{\alpha_1})\times(\mathbb{C}-\{0\})\times\mathbb{C}.\end{aligned} \tag{9.9}$$

Theorem 9.2. *(a) D_{BL}^{sub} embeds canonically into D_{BL}.*

(b) For suitable $v_1^0\in C^{\alpha_1}-\{0\},v_2^0\in C^{\beta_1}-\{0\}$ and for $v_4^0:=\psi_{\beta_2}(\overline{\psi_{\alpha_1}^{-1}(v_1^0)})\in C^{\beta_2}-\{0\}$, the multi-valued period map $BL_T: T\to D_{BL}$ has its image in D_{BL}^{sub} and takes the form

$$(t_1,t_2)\mapsto\mathbb{C}\cdot\left(v_1^0+t_1^{1/m_2}\cdot v_2^0+\left(\frac{t_2}{t_1}+r(t_1)\right)v_4^0ß\right) \tag{9.10}$$

with

$$r(t_1) = \begin{cases} 0 & \text{in the cases } (r_I = 1 \text{ \& } p \geq 3), \\ & \text{the cases } (r_I = 2 \text{ \& } p \geq 2) \\ & \text{and the case } U_{1,2}, \\ c_T \cdot t_1 & \text{in the cases } (r_I = 2 \text{ \& } p = 1) \\ & \text{and the cases } W^\sharp_{1,2} \text{ and } S^\sharp_{1,2}, \\ c_T \cdot t_1^2 & \text{in the cases } (r_I = 1 \text{ \& } p = 1), \end{cases} \tag{9.11}$$

for a suitable constant $c_T \in \mathbb{C}$. In the cases with $m|p$, the transversal monodromy M_T has on C^{α_1} the eigenvalues 1 and $\overline{\zeta}$, and $\mathbb{C} \cdot v_1^0$ is the eigenspace with eigenvalue 1. The class $[\mathbb{C} \cdot v_1^0]$ is in $\mathcal{H}(C^{\alpha_1})$.

(c) The induced period map $BL_{T^{cov}} : T^{cov} \to D_{BL}^{sub}$ is an isomorphism if $m \not| p$ and an isomorphism to the fiber above $[\mathbb{C} \cdot v_1^0] \in \mathcal{H}(C^{\alpha_1})$ of the projection $D_{BL}^{sub} \to \mathcal{H}(C^{\alpha_1})$ if $m|p$.

(d) In the case of the subseries $U_{1,9r}$, G^{mar} contains an element g_3 such that $\Psi(g_3)$ is elliptic of order 18 (for all subseries with $p = m \cdot r$, $\Psi(M_T)$ is elliptic of order m, for $U_{1,9r}$ $m = 9$).

(e) $f_{(t_1,t_2)}$ and $f_{(\widetilde{t}_1,\widetilde{t}_2)}$ are right equivalent

$$\iff \begin{cases} \exists\ k \in \mathbb{Z} \text{ with } (\widetilde{t}_1, \widetilde{t}_2) = (\zeta^{r_I p k} \cdot t_1, \zeta^{(r_I p+2)k} \cdot t_2) \\ \qquad \text{for all 8 series except } U_{1,2q}, \\ \exists\ k \in \mathbb{Z} \text{ and } \varepsilon \in \{\pm 1\} \text{ with} \\ \qquad (\widetilde{t}_1, \widetilde{t}_2) = (\varepsilon\zeta^{r_I p k} \cdot t_1, \varepsilon\zeta^{(r_I p+2)k} \cdot t_2) \text{ for } U_{1,2q}. \end{cases} \tag{9.12}$$

The parts (a), (b) and (d) of theorem 9.2 will be proved after theorem 9.6.

Proof of theorem 9.2 (c) and (e):

(c) This follows immediately from (9.10).

(e) First we prove $\Leftarrow$. We give explicit coordinate changes. A case by case comparison with the normal forms in table (9.1) shows that the following equality (9.13) holds. Here $(\delta_1, \delta_2, \delta_3)$ are as in table (9.14), and $k \in \mathbb{Z}$.

$$f_{(t_1,t_2)}(x \cdot \zeta^{\delta_1 \cdot k}, y \cdot \zeta^{\delta_2 \cdot k}, z \cdot \zeta^{\delta_3 \cdot k}) = f_{(t_1 \cdot \zeta^{r_I p k}, t_2 \cdot \zeta^{(r_I p+2)k})}(x, y, z). \tag{9.13}$$

	δ_1	δ_2	δ_3
$W^\sharp_{1,p}$ and $W_{1,p}$	3	2	0
$S^\sharp_{1,p}$ and $S_{1,p}$	3	2	4
$U_{1,p}$	3	2	3
$E_{3,p}$	6	2	0
$Z_{1,0}$	4	2	0
$Q_{2,p}$	4	2	5

(9.14)

In the case $U_{1,2q}$ we have additionally

$$f_{(t_1,t_2)}(x, y, -z) \quad = \quad f_{(-t_1,-t_2)}(x, y, z). \tag{9.15}$$

This shows $\Leftarrow$.

Now we prove $\Rightarrow$. Let $f_{(t_1,t_2)}$ and $f_{(\widetilde{t}_1,\widetilde{t}_2)}$ be right equivalent. Then $BL_T(t_1, t_2)$ and $BL_T(\widetilde{t}_1, \widetilde{t}_2)$ are isomorphic, so a $g \in G_{\mathbb{Z}}$ with $g(BL_T(t_1, t_2)) = BL_T(\widetilde{t}_1, \widetilde{t}_2)$ exists. We claim that v_1^0, v_2^0 and v_4^0 are eigenvectors of g with some eigenvalues λ_1, λ_2 and $\overline{\lambda_1}$. For v_2^0 this is trivial as

$\dim C^{\beta_1} = 1$, for v_1^0 in the case $m \not| p$ also. In the case $m|p$, it follows for v_1^0 from (9.10). For v_4^0 use $v_4^0 = \psi_{\beta_2}(\overline{\psi_{\alpha_1}^{-1}(v_1^0)})$. We claim also

$$\lambda_1 \in \mathrm{Eiw}(\zeta), \lambda_2 \in \mathrm{Eiw}(e^{2\pi i/m_2}). \tag{9.16}$$

For λ_2 this is a consequence of the following three facts and of theorem 2.9 (a)&(b).

(i) The 1-dimensional eigenspace $Ml_{e^{2\pi i/m_2}}$ is already defined over $\mathbb{Q}(e^{2\pi i/m_2})$. Therefore $\lambda_2 \in \mathbb{Q}(e^{2\pi i/m_2})$.
(ii) $|\lambda_2| = 1$ because L pairs $Ml_{e^{2\pi i/m_2}}$ and $Ml_{e^{-2\pi i/m_2}}$.
(iii) λ_2 is an algebraic integer because $g \in G_{\mathbb{Z}}$.

If $m \not| p$, the same reasoning applies also to λ_1. Suppose for a moment $m|p$.

By part (b), the transversal monodromy M_T acts on C^{α_1} and on H_ζ^∞ with eigenvalues 1 and ζ, and the 1-dimensional eigenspaces with eigenvalue 1 are $\mathbb{C} \cdot v_1^0$ and $\mathbb{C} \cdot \psi_{\alpha_1}^{-1}(v_1^0)$. Therefore $\mathbb{C} \cdot \psi_{\alpha_1}^{-1}(v_1^0)$ is already defined over $\mathbb{Q}(\zeta)$, i.e. $\mathbb{C} \cdot \psi_{\alpha_1}^{-1}(v_1^0) \cap H_{\mathbb{Q}(\zeta)}^\infty$ is a 1-dimensional $\mathbb{Q}(\zeta)$-vector space. This implies (i) $\lambda_1 \in \mathbb{Q}(\zeta)$. (ii) $|\lambda_1| = 1$ holds because $v_1^0 \in \mathcal{H}(C^{\alpha_1})$. And (iii) ($\lambda_1$ is an algebraic integer) holds anyway. Again with theorem 2.9 (a)&(b) we conclude $\lambda_1 \in \mathrm{Eiw}(\zeta)$. Now (9.16) is proved in all cases.

The equality $g(BL_T(t_1, t_2)) = BL_T(\widetilde{t}_1, \widetilde{t}_2)$ becomes

$$\begin{aligned} & \mathbb{C} \cdot \left(\lambda_1 \cdot v_1^0 + \lambda_2 \cdot t_1^{1/m_2} \cdot v_2^0 + \overline{\lambda_1}\left(\frac{t_2}{t_1} + r(t_1)\right) \cdot v_4^0\right) \\ = \ & \mathbb{C} \cdot \left(v_1^0 + \widetilde{t}_1^{1/m_2} \cdot v_2^0 + \left(\frac{\widetilde{t}_2}{\widetilde{t}_1} + r(\widetilde{t}_1)\right) \cdot v_4^0\right), \end{aligned}$$

$$\text{so} \qquad \widetilde{t}_1^{1/m_2} = \lambda_2 \overline{\lambda_1} \cdot t_1^{1/m_2}, \frac{\widetilde{t}_2}{\widetilde{t}_1} + r(\widetilde{t}_1) = \overline{\lambda_1}^2 \left(\frac{t_2}{t_1} + r(t_1)\right),$$

$$\text{so} \qquad \widetilde{t}_1 = \lambda_2^{m_2} \overline{\lambda_1}^{m_2} \cdot t_1,$$

$$\text{and} \qquad \widetilde{t}_2 = \overline{\lambda_1}^2 \cdot \frac{\widetilde{t}_1}{t_1} \cdot t_2 + \widetilde{t}_1 \cdot (\overline{\lambda_1}^2 \cdot r(t_1) - r(\widetilde{t}_1)). \tag{9.17}$$

Because of (9.16), we can write λ_1 and λ_2 as follows, here $k, l \in \mathbb{Z}$ and $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$.

$$\begin{array}{lll} & \lambda_1 & \lambda_2 \\ \text{All cases with } m \equiv 0(2),\ m_2 \equiv 0(2) & \overline{\zeta}^k & e^{2\pi i l/m_2} \\ \text{The cases } W_{1,2q-1}^\sharp \text{ and } S_{1,2q-1}^\sharp & \varepsilon_2 \cdot \overline{\zeta}^k & \varepsilon_2 \cdot e^{2\pi i l/m_2} \\ \text{The cases } U_{1,2q-1} & \varepsilon_1 \cdot \overline{\zeta}^k & e^{2\pi i l/m_2} \\ \text{The cases } U_{1,2q} & \varepsilon_1 \cdot \overline{\zeta}^k & \varepsilon_2 \cdot e^{2\pi i l/m_2} \end{array} \tag{9.18}$$

One checks that (9.17) boils down to

$$\widetilde{t}_1 = \zeta^{r_I p k} \cdot t_1, \widetilde{t}_2 = \zeta^{(r_I p + 2)k} \cdot t_2, \tag{9.19}$$

in all cases except $U_{1,2q}$. In the cases $U_{1,2q}$, it boils down to

$$\widetilde{t}_1 = \varepsilon_1 \varepsilon_2 \cdot \zeta^{pk} \cdot t_1, \widetilde{t}_2 = \varepsilon_1 \varepsilon_2 \cdot \zeta^{(p+2)k}. \tag{9.20}$$

This finishes the proof of $\Rightarrow$ and the proof of theorem 9.2 (e). □

The statements in theorem 9.1 on the transversal monodromy ($M_T^{m_2} = \mathrm{id}$, M_T has the eigenvalues 1 and ζ on Ml_ζ) will be proved after theorem 9.6. The rest of theorem 9.1 will be proved now.

Proof of theorem 9.1 (without the statements on M_T):
(a) This is clear.
(b) Consider the maps

$$\begin{tikzcd} T^{cov} \arrow[r] \arrow[d, "\cong"'] & (M_\mu^{smar})^0 \arrow[d, "BL"] \\ D_{BL}^{sub} \arrow[r, hook] & D_{BL} \end{tikzcd} \tag{9.21}$$

As $T^{cov} \hookrightarrow D_{BL}^{sub} \hookrightarrow D_{BL}$ is an embedding, $T^{cov} \to (M_\mu^{smar})^0$ is an embedding.

Both spaces T^{cov} and $(M_\mu^{smar})^0$ are locally μ-constant strata of universal unfoldings and are therefore smooth of dimension 2. D_{BL}^{sub} is almost closed in D_{BL}. Its closure consists of itself and the space $\{\mathbb{C}\cdot(v_1+v_4)\,|\,v_1$ and v_4 as in (9.7) or (9.9)$\}$ (so $v_2=0$). No $g\in G_\mathbb{Z}$ maps a point of this space to a point of D_{BL}^{sub}. And T^{cov} contains representatives of any right equivalence class in the μ-homotopy family. Therefore the image of $(M_\mu^{smar})^0$ in D_{BL} cannot be bigger than D_{BL}^{sub}. Thus $T^{cov}\cong (M_\mu^{smar})^0$.

In the case of singularities of multiplicity 2, $M_\mu^{smar}\cong M_\mu^{mar}$ holds anyway by theorem 8.5 (c), and then also $(M_\mu^{smar})^0\cong (M_\mu^{mar})^0$ holds.

Consider the case of singularities of multiplicity ≥ 3. Then $-\mathrm{id}\in G_\mathbb{Z}$ acts nontrivially on M_μ^{smar} by theorem 8.5 (c). It acts trivially on D_{BL}. The map $(M_\mu^{smar})^0\to D_{BL}$ is an embedding. Therefore $-\mathrm{id}\in G_\mathbb{Z}$ does not act on $(M_\mu^{smar})^0$, therefore $-\mathrm{id}\notin G^{smar}$. Then $(M_\mu^{smar})^0\to (M_\mu^{mar})^0$ is an isomorphism by theorem 8.5 (c).

(c) for $m \not| p$ and (d): $(M_\mu^{mar})^0 \xrightarrow{\cong} T^{cov} \xrightarrow{\cong} D_{BL}^{sub}\hookrightarrow D_{BL}$ is an embedding. $G_\mathbb{Z}=G^{mar}$ would imply $M_\mu^{mar}=(M_\mu^{mar})^0$. Therefore it is sufficient to prove $G_\mathbb{Z}=G^{mar}$.

Let $g_1\in G_\mathbb{Z}$. It acts on D_{BL}^{sub}. By the proof of theorem 9.2 (e), the map

$$(M_\mu^{mar})^0/G^{mar}\to D_{BL}^{sub}/G_\mathbb{Z} \tag{9.22}$$

is an isomorphism. Therefore an element $g_2\in G^{mar}$ exists which acts in the same way on D_{BL}^{sub} as g_1. Consider $g_3:=g_1\circ g_2^{-1}$. It acts trivially on D_{BL}^{sub}. It has eigenvalues λ_1, λ_2 and $\overline{\lambda_1}$ on $C^{\alpha_1}, C^{\beta_1}$ and C^{β_2}. Therefore

$$\begin{aligned} &\mathbb{C}(v_1+v_2+v_4)=\mathbb{C}(\lambda_1\cdot v_1+\lambda_2\cdot v_2+\overline{\lambda_1}\cdot v_4)\\ &\qquad\qquad \text{for any } \mathbb{C}(v_1+v_2+v_4)\in D_{BL}^{sub},\\ &\text{thus } \lambda_2\overline{\lambda_1}=1, \overline{\lambda_1}^2=\mathrm{id}, \text{so } \lambda_1=\lambda_2\in\{\pm 1\},\\ &\text{and } g_3=\lambda_1\cdot\mathrm{id}\ \text{ on } Ml_\zeta\oplus Ml_{e^{2\pi i/m_2}}. \end{aligned} \tag{9.23}$$

$G_\mathbb{Z}$ was determined in theorem 5.1 (b). It contains very few automorphisms g_3 with (9.23). Formula (5.6) and table (5.1) show that the group $\{g\in G_\mathbb{Z}\,|\,g=\pm\mathrm{id}\ \text{ on } Ml_\zeta\oplus Ml_{e^{2\pi i/m_2}}\}$ is as follows:

$$\begin{aligned} &\{\pm\mathrm{id}\} && \text{in the cases } W^\sharp_{1,2q-1}, S^\sharp_{1,2q-1}, U_{1,2q}, E_{3,p}, Z_{1,p},\\ &\{\pm\mathrm{id}, \pm(\mathrm{id}\,|_{B_1}\times(-M_h^{m_2/2})|_{B_2})\} && \text{in the cases } W^\sharp_{1,2q}, S^\sharp_{1,2q}, U_{1,2q-1},\\ &\{\pm\mathrm{id}, \pm((-M_h^{m/2})|_{B_1}\times\mathrm{id}\,|_{B_2})\} && \text{in the cases } Q_{2,p}, W_{1,p}, S_{1,p}. \end{aligned} \tag{9.24}$$

Claim:

$$\{g\in G_\mathbb{Z}\,|\,g=\pm\mathrm{id}\ \text{ on } Ml_\zeta\oplus Ml_{e^{2\pi i/m_2}}\}=G^{mar}_\mathcal{R}. \tag{9.25}$$

This claim shows $g_3 \in G_{\mathcal{R}}^{mar}$ and $g_1 \in G^{mar}$, so that $G_{\mathbb{Z}} = G^{mar}$.

The inclusion $\supset$ in (9.25) holds because of the following: Any element of $G_{\mathcal{R}}^{mar} = G_{\mathcal{R}}^{mar}(f_{(1,0)})$ acts on D_{BL}^{sub} with $BL_T(1,0)$ as fixed point. The proof of theorem 9.2 (e) shows that it acts then trivially on D_{BL}^{sub}.

The group $G_{\mathcal{R}}^{mar}$ contains $\pm\,\mathrm{id}$. In order to prove equality in (9.25) for the cases in the second and third line of (9.24), it is sufficient to show that $G_{\mathcal{R}}^{mar}$ contains more elements than $\pm\,\mathrm{id}$. Equivalent is that $G_{\mathcal{R}}^{smar}(f)$ for a generic singularity f with multiplicity ≥ 3 contains one other element than $+\,\mathrm{id}$. The following table lists coordinate changes which give such an element.

$$\begin{array}{ll} W_{1,2q}^{\sharp} & (x,y)\mapsto(-x,y) \\ S_{1,2q}^{\sharp} & (x,y,z)\mapsto(-x,y,z) \\ U_{1,2q-1} & (x,y,z)\mapsto(x,y,-z) \\ Q_{2,p} & (x,y,z)\mapsto(x,y,-z) \\ W_{1,p} & (x,y)\mapsto(-x,y) \\ S_{1,p} & (x,y,z)\mapsto(-x,y,z) \end{array} \tag{9.26}$$

This proves the claim and finishes the proof of (c) for $m \not| p$ and (d).

(c) for $m|p$ and (e): First we prove (9.4).

$\Psi(M_T)$ is an elliptic element with fixed point $[Ml_{\zeta,1}] \in \mathcal{H}_\zeta$ and angle $\frac{2\pi}{m} = \arg(\frac{\zeta}{1})$. All elements of G^{mar}, including M_T, act on $\mathcal{H}(C^{\alpha_1})$ as elliptic elements with fixed point $[\mathbb{C}\cdot v_1^0]$, because all elements in G^{mar} act on $(M_\mu^{mar})^0$ and on its image $BL_{T^{cov}}((M_\mu^{mar})^0) \subset D_{BL}^{sub}$. Therefore all elements of G^{mar} act on $\mathcal{H}_\zeta$ as elliptic elements with fixed point $[Ml_{\zeta,1}]$. This shows $\subset$ in (9.4).

Now let $g_1 \in \{g\in G_{\mathbb{Z}}\,|\,g(Ml_{\zeta,1}) = Ml_{\zeta,1}\}$. It has an eigenvalue λ_1 on $Ml_{\zeta,1}$ and an eigenvalue λ_2 on the other eigenspace within Ml_ζ (which is the h_ζ-orthogonal subspace of Ml_ζ). By (9.16) λ_1 and $\lambda_2 \in \mathrm{Eiw}(\zeta)$. Therefore $\Psi(g_1)$ is an elliptic element with fixed point $[Ml_{\zeta,1}] \in \mathcal{H}_\zeta$ and angle $\arg\frac{\lambda_2}{\lambda_1}$.

In all cases except possibly $U_{1,9r}$, the product $g_2 = g_1\circ M_T^k$ for a suitable $k\in\mathbb{Z}$ acts trivially on $\mathcal{H}_\zeta$. In the cases $U_{1,9r}$, the product $g_2 = g_1\circ g_3^k$ for $g_3\in G^{mar}$ as in theorem 9.2 (d) does the same.

Formula (5.9) in theorem 5.1 (c) applies to g_2 and shows $g_2 \in \{\pm M_h^k\,|\,k\in\mathbb{Z}\}$. Therefore $g_2 \in G^{mar}$ and $g_1\in G^{mar}$. This shows $\supset$ in (9.4), so (9.4) is now proved.

Especially, $\Psi(G^{mar})$ and G^{mar} are finite. By theorem 5.1 (c), $\Psi(G_{\mathbb{Z}})$ and $G_{\mathbb{Z}}$ are infinite. Therefore $G_{\mathbb{Z}} \supsetneqq G^{mar}$.

By theorem 8.5 (a), M_μ^{mar} consists of infinitely many copies of $(M_\mu^{mar})^0$.

If two different copies would have intersecting images in D_{BL} under the period map BL, the images would coincide, and there would be a copy different from $(M_\mu^{mar})^0$ with the same image in D_{BL} as $(M_\mu^{mar})^0$. An element $g_3\in G_{\mathbb{Z}}$ which maps $(M_\mu^{mar})^0$ to this copy would be in $\{g\in G_{\mathbb{Z}}\,|\,g(Ml_{\zeta,1}) = Ml_{\zeta,1}\} - G^{mar} = \emptyset$, a contradiction. Therefore $BL : M_\mu^{mar}\to D_{BL}^{sub}$ is an embedding. □

Remarks 9.3. (i) The arithmetic triangle group of type $(2,3,14)$ for $Z_{1,0}$ in theorem 3.6 contains elliptic elements of order 3 although $\arg\zeta = \frac{2\pi}{14}$ and the matrices defining these elliptic elements are in $GL(2,\mathbb{Z}[\zeta])$. The eigenspaces in $M(2\times 1,\mathbb{C})$ of these matrices are not defined over $\mathbb{Q}(\zeta)$, but only over $\mathbb{Q}(e^{2\pi i/3},\zeta)$. This example shows that (9.16) in the case $m|p$ and the arguments proving it are nontrivial.

(ii) In 1993, the second author worked on the Torelli conjecture for the unmarked bimodal series singularities. He missed to consider M_T carefully and thus was not sure which elliptic elements fix $[\mathbb{C}\cdot v_1^0] \in \mathcal{H}(C^{\alpha_1})$. Therefore he could not prove the Torelli conjecture for the unmarked singularities in the subseries $S^\sharp_{1,10r}, S_{1,10r}$ and $Z_{1,14r}$. Now theorem 9.1 gives the marked and unmarked Torelli theorem for all bimodal series singularities.

Now we come to the spectral numbers and the classifying space D_{BL}.

Lemma 9.4. *Consider a family of bimodal series singularities in table (9.1).*

(a) The spectral numbers $\alpha_1, ..., \alpha_\mu$ with $\alpha_1 \leq ... \leq \alpha_\mu$ satisfy

$$\begin{aligned} \alpha_1 = \frac{-1}{m} < \alpha_2 = \frac{1}{m_2} < \alpha_3 \leq ... \leq \alpha_{\mu-2} \\ < \alpha_{\mu-1} = 1 - \frac{1}{m_2} < \alpha_\mu = 1 + \frac{1}{m} \end{aligned} \tag{9.27}$$

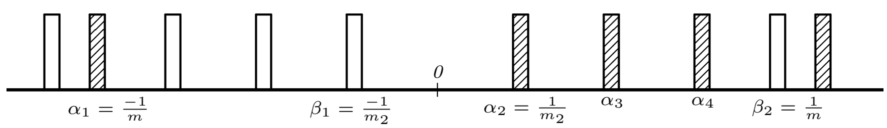
and are uniquely determined by this and the characteristic polynomial $\prod_{j\geq 1} b_j$ of the monodromy with b_j as in table (5.1).

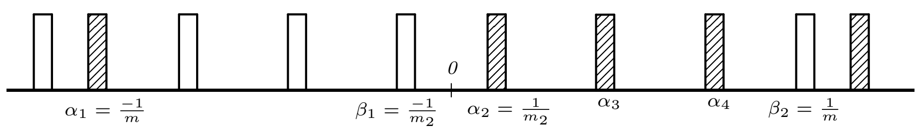
(b) Recall from (9.5) $\beta_1 = \frac{-1}{m_2} = -\alpha_2$ and $\beta_2 = \frac{1}{m} = -\alpha_1$. Then

$$\dim C^{\alpha_1} = \begin{cases} 1 & \text{if } m \not| p, \\ 2 & \text{if } m|p, \end{cases} \tag{9.28}$$

$$\dim C^\beta = \begin{cases} 1 & \text{for } \beta \in (\alpha_1, \beta_2) \cap \frac{1}{m_2}(\mathbb{Z} - \{0\}) \text{ if } r_I = 1, \\ & \text{and for } \beta \in (\alpha_1, \beta_2) \cap (\frac{1}{m_2} + \frac{2}{m_2}\mathbb{Z}) \text{ if } r_I = 2, \\ 0 & \text{for other } \beta \in (\alpha_1, \beta_2). \end{cases} \tag{9.29}$$

The following two pictures illustrate this for $2m < p < 3m$, the first for $r_I = 1$, the second for $r_I = 2$.





(c) Denote by $(*)$ the condition

$$(*): \qquad \beta \in (\alpha_1, 0) \text{ with } C^\beta \neq \{0\} (\text{then } \dim C^\beta = 1).$$

If $m \not| p$ the classfying space D_{BL} in [He99] is

$$\begin{aligned} D_{BL} &= \{\mathbb{C} \cdot (v_1 + \sum_{\beta:(*)} v_{(\beta)} + v_2) \mid \\ & \qquad v_1 \in C^{\alpha_1} - \{0\}, v_{(\beta)} \in C^\beta, v_2 \in C^{\beta_2}\} \\ &\cong \mathbb{C}^{N_{BL}} \qquad \text{with } N_{BL} := |\{\beta : (*)\}| + 1. \end{aligned} \tag{9.30}$$

In (9.8) $\mathcal{H}(C^{\alpha_1})$ was defined for $m|p$. If $m|p$ then D_{BL} is

$$\begin{aligned} D_{BL} &= \{\mathbb{C} \cdot (v_1 + \sum_{\beta:(*)} v_{(\beta)} + v_2) \mid \\ & \qquad v_1 \in C^{\alpha_1} - \{0\} \text{ with } [\mathbb{C} \cdot v_1] \in \mathcal{H}(C^{\alpha_1}), \\ & \qquad v_{(\beta)} \in C^\beta, v_2 \in \mathbb{C} \cdot \psi_{\beta_2}(\overline{\psi_{\alpha_1}^{-1}(v_1)}) \subset C^{\beta_2}\} \\ &\cong \mathcal{H}(C^{\alpha_1}) \times \mathbb{C}^{N_{BL}} \qquad \text{with } N_{BL} := |\{\beta : (*)\}| + 1. \end{aligned} \tag{9.31}$$

Proof: (a) The spectral numbers are well known [AGV88, 13.3.4, p. 389]. They also follow from corollary 7.17 and the proof of theorem 9.6.

(b) (9.28) follows from $\dim C^{\alpha_1} = \dim Ml_\zeta$ and $\Phi_m \not| b_2 \iff m \not| p$. (9.29) follows from the values of b_j in table (5.1).

(c) The spectral numbers and the numbers β with $C^\beta \neq \{0\}$ give for each $\mathcal{L}_0 \in D_{BL}$

$$\mathcal{L}_0 = \mathbb{C}\cdot\sigma_1 \oplus \mathcal{L}_0 \cap \bigoplus_{\beta:\,\alpha_2\leq\beta\leq\beta_2} C^\beta \oplus V^{>\beta_2} \tag{9.32}$$

where

$$\alpha(\sigma_1) = \alpha_1, \sigma_1 \in C^{\alpha_1} \oplus \bigoplus_{\beta:(*)} C^\beta \oplus C^{\beta_2}. \tag{9.33}$$

Here observe that for β with $\alpha_2 \leq \beta < \beta_2$ and $C^\beta \neq \{0\}$, the space C^β is one-dimensional and is generated by the principal part of a section in $\mathcal{L}_0$.

If $m \not| p$ then $\dim C^{\beta_2} = 1$ and C^{β_2} is not generated by the principal part of a section in $\mathcal{L}_0$. If $m|p$ then $\dim C^{\beta_2} = 2$ and the one-dimensional subspace

$$\{v \in C^{\beta_2} \,|\, K_f^{(-2)}(v, s(\sigma_1,\alpha_1)) = 0\} \subset C^{\beta_2}$$

is in $\mathcal{L}_0$, because then β_2 is a spectral number with multiplicity 1. And then the principal part $s(\sigma_1,\alpha_1)$ must be compatible with a polarized Hodge structure of weight 2 on $H^\infty_\zeta \oplus H^\infty_{\overline{\zeta}}$. This amounts to $[\mathbb{C}\cdot s(\sigma_1,\alpha_1)] \in \mathcal{H}(C^{\alpha_1})$. Especially then

$$C^{\beta_2} = \mathbb{C}\cdot\psi_{\beta_2}(\overline{\psi_{\alpha_1}^{-1} s(\sigma_1,\alpha_1)}) \oplus \{v \in C^{\beta_2} \,|\, K_f^{(-2)}(v, s(\sigma_1,\alpha_1)) = 0\}, \tag{9.34}$$

and σ_1 can be chosen with

$$\alpha(\sigma_1) = \alpha_1, \sigma_1 \in C^{\alpha_1} \oplus \bigoplus_{\beta:(*)} C^\beta \oplus \mathbb{C}\cdot\psi_{\beta_2}(\overline{\psi_{\alpha_1}^{-1} s(\sigma_1,\alpha_1)}). \tag{9.35}$$

σ_1 is (up to rescaling) uniquely determined by (9.33) if $m \not| p$ and by (9.35) if $m|p$. And it can be chosen freely with (9.33) respectively with (9.35) and $[\mathbb{C}\cdot s(\sigma_1,\alpha_1)] \in \mathcal{H}(C^{\alpha_1})$. The condition (δ) $K_f^{(-2)}(\mathcal{L}_0,\mathcal{L}_0) = 0$ on D_{BL} directly before theorem 7.11 implies that $\mathcal{L}_0 \cap \bigoplus_{\alpha_2\leq\beta\leq\beta_2} C^\beta$ is uniquely determined by σ_1. Therefore $\mathcal{L}_0$ is uniquely determined by σ_1. Therefore D_{BL} is as stated in (9.30) and (9.31). □

Remarks 9.5. (i) All the normal forms in table (9.1) except $W^\sharp_{1,p}$ are Newton nondegenerate. But also the normal form $f_p(x,y,\widetilde{z})$ for $W^\sharp_{1,p}$ in table (9.1) can be made easily Newton nondegenerate with the coordinate change $\widetilde{z} = z + i(x^2+y^3)$. Then

$$\begin{aligned} f_p(x,y,z+i(x^2+y^3)) &= z^2 + 2ix^2z + 2iy^3z \\ &\quad + \begin{cases} (t_1+t_2y)xy^{4+q} & \text{if } p = 2q-1, \\ (t_1+t_2y)x^2y^{3+q} & \text{if } p = 2q. \end{cases} \end{aligned} \tag{9.36}$$

(ii) The Newton boundaries of the normal forms in table (9.1) except for $W^\sharp_{1,p}$ and of the normal form in (9.36) for $W^\sharp_{1,p}$ have each two compact n-dimensional faces σ_1 and σ_2. The following table lists the corresponding linear forms l_{σ_j} and the value $s(f)$ from corollary 7.17. A linear form is encoded by the values $(l_{\sigma_j}(x), l_{\sigma_j}(y), l_{\sigma_j}(z))$.

$$\begin{array}{llll}
W^\sharp_{1,p} & \sigma_1 : \frac{1}{12}(3,2,6) & \sigma_2 : \frac{1}{12+p}(3,2,6+p) & \frac{5}{12+p} \\
S^\sharp_{1,p} & \sigma_1 : \frac{1}{10}(3,2,4) & \sigma_2 : \frac{1}{10+p}(3,2,4+p) & \frac{5}{10+p} \\
U_{1,p} & \sigma_1 : \frac{1}{9}(3,2,3) & \sigma_2 : \frac{1}{9+p}(3+p,2,3) & \frac{5}{9+p} \\
E_{3,p} & \sigma_1 : \frac{1}{18}(6,2,9) & \sigma_2 : \frac{1}{2(9+p)}(6+p,2,9+p) & \frac{4}{9} \\
Z_{1,p} & \sigma_1 : \frac{1}{14}(4,2,7) & \sigma_2 : \frac{1}{2(7+p)}(4+p,2,7+p) & \frac{3}{7} \\
Q_{2,p} & \sigma_1 : \frac{1}{12}(4,2,5) & \sigma_2 : \frac{1}{2(6+p)}(4+p,2,5+p) & \frac{1}{2} \\
W_{1,p} & \sigma_1 : \frac{1}{12}(3,2,6) & \sigma_2 : \frac{1}{2(6+p)}(3+p,2,6+p) & \frac{5}{12} \\
S_{1,p} & \sigma_1 : \frac{1}{10}(3,2,4) & \sigma_2 : \frac{1}{2(5+p)}(3+p,2,4+p) & \frac{1}{2}
\end{array} \tag{9.37}$$

Theorem 9.6. *Consider the normal form in (9.36) for $W^\sharp_{1,p}$ and the normal forms in table (9.1) for the other seven series. Recall the notation $\omega_0 := dxdydz$ from remark 7.14 (v). Define*

$$\begin{array}{lcl}
b_1 & := & s(\omega_0,\alpha_1)(1,0) \in C^{\alpha_1}, \\
b_2 & := & s(\omega_0,\beta_1)(1,0) \in C^{\beta_1}, \\
b_3 & := & s(y\omega_0,\alpha_2)(1,0) \in C^{\alpha_2}, \\
b_4 & := & s(y\omega_0,\beta_2)(1,0) \in C^{\beta_2}.
\end{array}$$

If $m|p$, choose $b_5 \in C^{\beta_2}$ with $\mathbb{C}\cdot b_5 = \{v \in C^{\beta_2} \mid K_f^{(-2)}(b_1,v) = 0\}$.

(a) All $b_j \neq 0$. And $K_f^{(-2)}(b_1+b_2,b_3+b_4) = 0$. If $m|p$ then $C^{\beta_2} = \mathbb{C}\cdot b_4 \oplus \mathbb{C}\cdot b_5$.

(b) We write $t = (t_1,t_2)$. Recall the notation $\alpha(s[\omega]_0(t)) = \min(\alpha \mid s(\omega,\alpha)(t) \neq 0)$ from remark 7.14 (iv).

$$\alpha(s[\omega_0]_0(t)) = \alpha_1, \tag{9.38}$$
$$s(\omega_0,\alpha_1)(t) = b_1, \tag{9.39}$$
$$s(\omega_0,\beta)(t) = 0 \text{for } \alpha_1 < \beta < \beta_1, \tag{9.40}$$
$$s(\omega_0,\beta_1)(t) = t_1^{1/m_2}\cdot b_2, \tag{9.41}$$
$$s(\omega_0,\alpha_2)(t) = \frac{t_2}{t_1}\cdot\frac{-1}{m_2}\cdot t_1^{-1/m_2}\cdot b_3 + s(\omega,\alpha_2)(t_1,0), \tag{9.42}$$
$$s(\omega_0,\beta_2)(t) \quad \begin{cases} = s(\omega_0,\beta_2)(t_1,0) & \text{if } m \not| p, \\ \in s(\omega_0,\beta_2)(t_1,0) + \mathbb{C}\cdot b_5 & \text{if } m|p, \end{cases} \tag{9.43}$$

with

	$s(\omega_0,\alpha_2)(t_1,0)$	$s(\omega_0,\beta_2)(t_1,0)$
($r_I = 2$ & $p \geq 2$) or ($r_I = 1$ & $p \geq 3$) or $U_{1,2}$	0	0
$W^\sharp_{1,1}, S^\sharp_{1,1}, U_{1,1}$	$c_1\cdot t_1^{2-1/m_2}\cdot b_3$	$c_2\cdot t_1^2\cdot b_4$
$W^\sharp_{1,2}, S^\sharp_{1,2}, E_{3,1}$ $Z_{1,1}, Q_{2,1}, W_{1,1}, S_{1,1}$	$c_1\cdot t_1^{1-1/m_2}\cdot b_3$	$c_2\cdot t_1\cdot b_4$

(9.44)

for some values $c_1, c_2 \in \mathbb{C}$.

$$\alpha(s[y\omega_0]_0(t) = \alpha_2, \tag{9.45}$$
$$s(y\omega_0,\alpha_2)(t) = t_1^{-1/m_2}\cdot b_3, \tag{9.46}$$
$$s(y\omega_0,\beta_2)(t) \quad \begin{cases} = b_4 & \text{if } m \not| p \text{ or } t_2 = 0, \\ \in b_4 + \mathbb{C}\cdot b_5 & \text{if } m|p, \end{cases} \tag{9.47}$$

$$s(\sigma,\beta_2)(t)\begin{cases} =0 & \text{if } m \not| p \\ \in \mathbb{C}\cdot b_5 & \text{if } m|p \end{cases} \tag{9.48}$$

for $\sigma \in H_0''(f_t)$ with $\alpha(\sigma) > \alpha_2$.

(c) In the five series with $r_I = 2$ (see table (5.1)) for $b \in \mathbb{Z}_{\geq 0}$

$$\alpha(s[y^{b+1}\omega_0]_0(t)) = \alpha_2 + \frac{2b}{m_2} = \frac{2b+1}{m_2}, \tag{9.49}$$

$$s(y^{b+1}\omega_0, \frac{2b+1}{m_2})(t) = t_1^{-(2b+1)/m_2}\cdot s(y^{b+1}\omega_0, \frac{2b+1}{m_2})(1,0). \tag{9.50}$$

Especially, if $p = mr$ then $\frac{2r+1}{m_2} = \frac{1}{m} = \beta_2$, b_5 can be chosen as $b_5 = s(y^{r+1}\omega_0, \beta_2)(1,0)$, and

$$s(y^{r+1}\omega_0,\beta_2)(t) = t_1^{-1/m}\cdot b_5. \tag{9.51}$$

(d) In the three subseries $W^\sharp_{1,12r}, S^\sharp_{1,10r}, U_{1,9r}$ (i.e. the subseries with $r_I = 1$ and $m|p$), b_5 can be chosen such that b_5 and ω in the following table (9.54) satisfy

$$\alpha(s[\omega]_0(t)) = \beta_2 = \frac{1}{m}, \tag{9.52}$$

$$s(\omega,\beta_2+1)(t) = t_1^{-1/m}\cdot b_5. \tag{9.53}$$

	ω
$W^\sharp_{1,12+24r}, S^\sharp_{1,10+20r}$	$xy^r\omega_0$
$U_{1,9+18r}$	$y^r z\omega_0$
$W^\sharp_{1,24r}, S^\sharp_{1,20r}, U_{1,18r}$	$y^{r+1}\omega_0$

(9.54)

Proof: (a) Observe $\nu(\omega_0) - 1 = \alpha_1 < s(f)$ and $\nu(y\omega_0) - 1 = \alpha_2 < s(f)$. This, theorem 7.16 and corollary 7.17 show (9.38), (9.45), $b_1 \neq 0$ and $b_3 \neq 0$. $b_2 \neq 0$ will be shown below. (9.40) (which will also be shown below) and $K_f^{(-2)}(H_0''(f_t), H_0''(f_t)) = 0$ give especially

$$0 = K_f^{(-2)}(s[\omega_0]_0(1,0), s[y\omega_0]_0(1,0)) = K_f^{(-2)}(b_1+b_2, b_3+b_4).$$

As $K_f^{(-2)}(b_2,b_3) \neq 0$, also $K_f^{(-2)}(b_1,b_4) \neq 0$ and $b_4 \neq 0$ and in the case $m|p$ $C^{\beta_2} = \mathbb{C}\cdot b_4 \oplus \mathbb{C}\cdot b_5$.

(b)–(d) We restrict to the series $E_{3,p}$. The calculations for the series $Z_{1,p}, Q_{2,p}, W_{1,p}$ and $S_{1,p}$ are very similar. The calculations for the series $W^\sharp_{1,p}, S^\sharp_{1,p}$ and $U_{1,p}$ are similar, but require more case discussions.

The two compact faces σ_1 and σ_2 (remark 9.5) of the Newton boundary give rise to the following two relations

$$\frac{1}{3}xf_x + \frac{1}{9}yf_y + \frac{1}{2}zf_z - \frac{p}{9}t_1y^{9+p} - \frac{p+1}{9}t_2y^{10+p} = f, \tag{9.55}$$

$$\frac{6+p}{2(9+p)}xf_x + \frac{2}{2(9+p)}yf_y + \frac{1}{2}zf_z - \frac{p}{2(9+p)}x^3 - \frac{1}{9+p}t_2y^{10+p} = f. \tag{9.56}$$

These relations and (7.24) give the following two values for $\partial_\tau \tau s[x^a y^b \omega_0]_0(t)$:

$$\begin{aligned}
& \partial_\tau \tau s[x^a y^b \omega_0]_0(t) \\
= \; & l_{\sigma_1}(a+1, b+1, 1) \cdot s[x^a y^b \omega_0]_0(t) && (9.57) \\
& -\frac{p}{9} t_1 \partial_\tau s[x^a y^{b+9+p} \omega_0]_0(t) - \frac{p+1}{9} t_2 \partial_\tau s[x^a y^{b+10+p} \omega_0]_0(t), \\
= \; & l_{\sigma_2}(a+1, b+1, 1) \cdot s[x^a y^b \omega_0]_0(t) && (9.58) \\
& -\frac{p}{2(9+p)} \partial_\tau s[x^{a+3} y^b \omega_0]_0(t) \; - \frac{1}{9+p} t_2 \partial_\tau s[x^a y^{b+10+p} \omega_0]_0(t).
\end{aligned}$$

This gives for any β with $\dim C^\beta \neq 0$

$$\begin{aligned}
& (\beta + 1 - l_{\sigma_1}(a+1, b+1, 1)) s(x^a y^b \omega_0, \beta)(t) \\
= \; & -\frac{p}{9} t_1 \partial_\tau s(x^a y^{b+9+p} \omega_0, \beta+1)(t) \\
& -\frac{p+1}{9} t_2 \partial_\tau s(x^a y^{b+10+p} \omega_0, \beta+1)(t), && (9.59) \\
& (\beta + 1 - l_{\sigma_2}(a+1, b+1, 1)) s(x^a y^b \omega_0, \beta)(t) \\
= \; & -\frac{p}{2(9+p)} \partial_\tau s(x^{a+3} y^b \omega_0, \beta+1)(t \\
& -\frac{1}{9+p} t_2 \partial_\tau s(x^a y^{b+10+p} \omega_0, \beta+1)(t). && (9.60)
\end{aligned}$$

Furthermore, (7.39) gives

$$\begin{aligned}
\partial_{t_1} s[x^a y^b \omega_0]_0(t) &= (-\partial_\tau) s[x^a y^{b+9+p} \omega_0]_0(t), && (9.61) \\
\partial_{t_2} s[x^a y^b \omega_0]_0(t) &= (-\partial_\tau) s[x^a y^{b+10+p} \omega_0]_0(t) \\
&= \partial_{t_1} s[x^a y^{b+1} \omega_0]_0(t). && (9.62)
\end{aligned}$$

(9.59)–(9.62) give

$$\begin{aligned}
& \Big(\frac{p}{9} t_1 \partial_{t_1} + \frac{p+1}{9} t_2 \partial_{t_2} - (\beta+1) + l_{\sigma_1}(a+1, b+1, 1)\Big) \\
& \qquad s(x^a y^b \omega_0, \beta)(t) = 0, && (9.63) \\
& \Big(\frac{1}{9+p} t_2 \partial_{t_2} - (\beta+1) + l_{\sigma_2}(a+1, b+1, 1)\Big) s(x^a y^b \omega_0, \beta)(t) \\
& \qquad = \frac{p}{2(9+p)} \partial_\tau s(x^{a+3} y^b \omega_0, \beta+1)(t). && (9.64)
\end{aligned}$$

(9.63) gives for $t_2 = 0$

$$s(x^a y^b \omega_0, \beta)(t_1, 0) = t_1^{\frac{9}{p}(\beta+1-l_{\sigma_1}(a+1,b+1,1))} \cdot s(x^a y^b \omega_0, \beta)(1, 0). \tag{9.65}$$

The following eight equations are special cases of (9.65).

$$\begin{aligned}
s(\omega_0, \alpha_1)(t_1, 0) &= b_1, && (9.66) \\
s(\omega_0, \beta_1)(t_1, 0) &= t_1^{1/m_2} \cdot b_2, && (9.67) \\
s(\omega_0, \alpha_2)(t_1, 0) &= t_1^{-1/m_2 + 1/p} \cdot s(\omega_0, \alpha_2)(1, 0), && (9.68) \\
s(\omega_0, \beta_2)(t_1, 0) &= t_1^{1/p} \cdot s(\omega_0, \beta_2)(1, 0), && (9.69)
\end{aligned}$$

$$\begin{aligned} s(y^{b+1}\omega_0, \frac{2b+1}{m_2})(t_1,0) &= t_1^{-(2b+1)/m_2}\cdot s(y^{b+1}\omega_0,\frac{2b+1}{m_2})(1,0), && (9.70)\\ s(y\omega_0,\alpha_2)(t_1,0) &= t_1^{-\alpha_2}\cdot b_3 = t_1^{-1/m_2}\cdot b_3, && (9.71)\\ s(y\omega_0,\beta_2)(t_1,0) &= b_4, && (9.72)\\ s(y^{r+1}\omega_0,\beta_2)(t_1,0) &= t_1^{-1/m_2}\cdot s(y^{r+1}\omega_0,\beta_2)(1,0)\text{if } p=18r. && (9.73)\end{aligned}$$

Claim: Fix some $b\in\mathbb{Z}_{\geq 0}$.

(i) $\nu(y^{b+1}\omega_0)=\alpha_2+\frac{b}{9+p}=\frac{2b+1}{m_2}$.

(ii) Any $(n+1)$-form $df\wedge d\eta$ which contains $y^{b+1}\omega_0$ as a summand, contains a summand $g\cdot\omega_0$ with g a monomial (times a nonzero scalar) with $\nu(g\cdot\omega_0)\leq\nu(y^{b+1}\omega_0)$.

(iii) $\overline{\nu}(y^{b+1}\omega_0)=\frac{2b+1}{m_2}$.

Proof of the claim: (i) Trivial. (iii) follows from (i) and (ii).

(ii) The only monomial differential $(n-1)$-forms η such that $df\wedge d\eta$ contains $f_y\cdot y^c\cdot\omega_0$ are $\eta_1=-xy^cdz$ and $\eta_2=y^czdx$, and

$$\begin{aligned} df\wedge d\eta_1 &= f_y\cdot y^c\cdot\omega_0 - f_x\cdot c\cdot xy^{c-1}\cdot\omega_0,\\ df\wedge d\eta_2 &= f_y\cdot y^c\cdot\omega_0 - f_z\cdot c\cdot y^{c-1}z\cdot\omega_0.\end{aligned}$$

These $(n+1)$-forms contain $(3-2c)x^2y^{c+2}\omega_0$ respectively $3x^2y^{c+2}\omega_0$, and

$$\nu(x^2y^{c+2}\omega_0)\leq\nu(y^{c+8+p}\omega_0).$$

($\square$)

The claim and theorem 7.16 imply

$$\begin{aligned} \alpha(s[y^{b+1}\omega_0]_0(t)) &= \frac{2b+1}{m_2}, && (9.74)\\ s(y^{b+1}\omega_0,\frac{2b+1}{m_2})(t) &\neq 0. && (9.75)\end{aligned}$$

Especially, $b_3\neq 0$, and if $p=18r$ also $s(y^{r+1}\omega_0,\beta_2)(t)\neq 0$. In this case $p=18r$, the vanishing

$$K_f^{(-2)}(s[\omega_0]_0(1,0),s[y^{r+1}\omega_0]_0(1,0))=0$$

gives $K_f^{(-2)}(b_1,s(y^{r+1}\omega_0,\beta_2)(1,0))=0$. Therefore in this case we can choose

$$b_5=s(y^{r+1}\omega_0,\beta_2)(1,0).$$

The elementary sections $s(y^{b+1}\omega_0,\frac{2b+1}{m_2})(t)$ are independent of t_2 because (9.62) gives

$$\partial_{t_2}s(y^{b+1}\omega_0,\frac{2b+1}{m_2})(t)=\partial_{t_1}s(y^{b+2}\omega_0,\frac{2b+1}{m_2})(t)=0.$$

Now part (c), i.e. (9.49)–(9.51), and (9.46) are proved.

(9.62) gives also

$$\begin{aligned} \partial_{t_2}s[\omega_0]_0(t) &= \partial_{t_1}s[y\omega_0]_0(t), && (9.76)\\ \text{so } s(\omega_0,\beta)(t) &= s(\omega_0,\beta)(t_1,0) \qquad \text{for } \alpha_1\leq\beta<\alpha_2. \end{aligned}$$

With (9.66) and (9.59) and (9.75) we obtain

$$s(\omega_0,\beta)(t_1,0)=\begin{cases} b_1 & \text{if } \beta=\alpha_1,\\ \frac{-p}{9(\beta-\alpha_1)}t_1\partial_\tau s(y^{9+p}\omega_0,\beta+1)(t_1,0)=0 & \text{if } \alpha_1<\beta<\beta_1,\\ \frac{-p}{9(\beta_1-\alpha_1)}t_1\partial_\tau s(y^{9+p}\omega_0,\beta_1+1)(t_1,0)\neq 0 & \text{if } \beta=\beta_1.\end{cases}$$

This gives $b_2 \neq 0$ and (together with (9.66) and (9.67)) (9.39)–(9.41).

The argument in the proof of part (a) with $K_f^{(-2)}(H_0''(f_t), H_0''(f_t)) = 0$ gives $b_4 \neq 0$ and (9.47) and (9.48).

It rests to show (9.42)–(9.44). From (9.76), (9.46) and (9.47) we obtain

$$\begin{aligned}
\partial_{t_2} s(\omega_0, \alpha_2)(t) &= \partial_{t_1} s(y\omega_0, \alpha_2)(t) = \partial_{t_1}(t_1^{-1/m_2} \cdot b_3), \\
\partial_{t_2} s(\omega_0, \beta_2)(t) &= \partial_{t_1} s(y\omega_0, \beta_2)(t) \begin{cases} = 0 & \text{if } m \not| p, \\ \in \mathbb{C} \cdot b_5 & \text{if } m|p, \end{cases}
\end{aligned}$$

which gives (9.42) and (9.43).

For (9.44) observe the following. The sections

$$\begin{aligned}
s(y\omega_0, \alpha_2)(t_1, 0) &= t_1^{-1/m_2} \cdot b_3, \\
s(y\omega_0, \beta_2)(t_1, 0) &= b_4, \\
\text{and in the case } m|p s(y^{r+1}\omega_0, \beta_2)(t_1, 0) &= t_1^{-1/m} \cdot b_5
\end{aligned}$$

are univalued nowhere vanishing sections in the bundles $\bigcup_{t_1 \in T} C^{\alpha_2}(t_1, 0)$ and $\bigcup_{t_1 \in T} C^{\beta_2}(t_1, 0)$, and they generate these bundles. Also $s(\omega_0, \alpha_2)(t_1, 0)$ and $s(\omega_0, \beta_2)(t_1, 0)$ are univalued sections in these bundles. (9.68) and (9.69) show for $p \geq 2$ that they are everywhere vanishing. For $p = 1$ they give the statement for $E_{3,1}$ in the last line of table (9.44). This finishes the proof of the parts (b) and (c) for the series $E_{3,p}$. □

Proof of $M_T^{m_2} = \mathrm{id}$:

By theorem 9.6, the following sections in the bundles $\bigcup_{t_1 \in T} C^{\beta}(t_1, 0)$ for β as in table (9.77) are univalued nowhere vanishing sections and generate these bundles (in the case $\beta = \alpha_1$ only if $m \not| p$).

section	b_1	$t_1^{1/m_2} \cdot b_2$	$t_1^{-1/m_2} \cdot b_3$	b_4	$t_1^{-1/m} \cdot b_5$ if $m\|p$
β	α_1	β_1	α_2	β_2	β_2
eigenvalue of M_T on $\mathbb{C} \cdot b_j$	1	$e^{-2\pi i/m_2}$	$e^{2\pi i/m_2}$	1	$e^{2\pi i/m}$

(9.77)

Therefore b_1 and b_4 are univalued, and b_2 and b_3 (and b_5 if $m|p$) are multivalued flat sections with eigenvalues of M_T as in the table. Thus $M_T^{m_2}$ is on $C^{\alpha_1}, C^{\beta_1}, C^{\alpha_2}, C^{\beta_2}, Ml_\zeta$ and $Ml_{e^{2\pi i/m_2}}$ the identity. We will show that it is the identity on all of Ml.

Consider firstly the case $m \not| p$. Then by (9.24) $M_T^{m_2}$ is in

$$\begin{aligned}
&\{\mathrm{id}\} \quad \text{in the cases } W^\sharp_{1,2q-1}, S^\sharp_{1,2q-1}, U_{1,2q}, E_{3,p}, Z_{1,p}, \\
&\{\mathrm{id}, \mathrm{id}|_{B_1} \times (-M_h^{m_2/2})|_{B_2}\} \text{in the cases } W^\sharp_{1,2q}, S^\sharp_{1,2q}, U_{1,2q-1}, \\
&\{\mathrm{id}, (-M_h^{m/2})|_{B_1} \times \mathrm{id}|_{B_2}\} \quad \text{in the cases } Q_{2,p}, W_{1,p}, S_{1,p}.
\end{aligned} \tag{9.78}$$

On the other hand, in the cases in the second and third line of (9.78), $m_2 = m + r_I p$ is even, and M_T itself is in $G_{\mathbb{Z}}$ which is given by (5.6) in theorem 5.1. Thus $M_T^{m_2} = \mathrm{id}$ also in the second and third line of (9.78).

Consider secondly the case $m|p$, so $p = mr$. By (5.9) in theorem 5.1, $M_T^{m_2} = \varepsilon \cdot M_h^k$ for some $\varepsilon \in \{\pm 1\}$ and some $k \in \mathbb{Z}$. Then $\varepsilon \cdot \zeta^k = 1$ and $\varepsilon \cdot e^{2\pi i k/m_2} = 1$. If $\varepsilon = 1$, then the two conditions boil down to $m|k$ and $m_2|k$, so to $m_2|k$. Then $M_T^{m_2} = \mathrm{id}$. If $\varepsilon = -1$, we will come below to a contradiction. Then the two conditions require m even and m_2 even.

For each eigenvalue λ of M_h on Ml with $\dim Ml_\lambda = 1$, an eigenvector in $Ml_{\lambda, \mathbb{Z}[\lambda]}$ exists. Then M_T has an eigenvalue in $\mathrm{Eiw}(\lambda)$ on this eigenvector, and $M_T^{m_2}$ has the eigenvalue 1 on this eigenvector. Here m_2 even is used. Therefore $M_T^{m_2} = \mathrm{id}$ on Ml_λ for each

$$\lambda \in \{\zeta, e^{2\pi i/m_2}\} \cup \{\widetilde{\lambda} \,|\, \dim Ml_{\widetilde{\lambda}} = 1\}.$$

Comparison with table (5.1) shows that no $k \in \mathbb{Z}$ with $-\lambda^k = 1$ for all these λ exists. This gives a contradiction. The case $\varepsilon = -1$ is impossible. $M_T^{m_2} = \mathrm{id}$ is proved in all cases. □

Proof that M_T has the eigenvalues 1 and $\overline{\zeta}$ on Ml_ζ and on C^{α_1}:

By table (9.77), M_T has on C^{β_2} and on $H^\infty_{e^{-2\pi i\beta_2}} = H^\infty_{\overline{\zeta}}$ the eigenvalues 1 and ζ. As Ml_ζ is dual to $H^\infty_{\overline{\zeta}}$ and H^∞_ζ is complex conjugate to $H^\infty_{\overline{\zeta}}$, M_t has on Ml_ζ, $H^\infty_\zeta = H^\infty_{e^{-2\pi i\alpha_1}}$ and C^{α_1} the eigenvalues 1 and $\overline{\zeta}$. □

Proof of theorem 9.2 (a)+(b)+(d):

(a) This follows immediately from (9.7), (9.9) and lemma (9.4) (c).

(b) All of this follows by carefully putting together the results in theorem 9.6. Here $v_1^0 = b_1$, $v_2^0 = b_2$, $v_4^0 \in \mathbb{C}^* \cdot b_4$ suitable, and the section in the brackets on the right hand side of (9.10) is

$$s[\omega_0]_0(t) \quad + \quad \left(\frac{1}{m}\frac{t_2}{t_1} + \begin{Bmatrix} 0 \\ -c_1 \cdot t_1^2 \\ -c_1 \cdot t_1 \end{Bmatrix}\right) \cdot s[y\omega_0]_0(t) \qquad \bmod \bigoplus_{\alpha_2<\beta<\beta_2} C^\beta \oplus \mathbb{C}\cdot b_5 \oplus V^{>\beta_2}. \tag{9.79}$$

The three cases in {...} correspond to the three lines in (9.44). The linear combination is chosen such that it has no part in C^{α_2}. This section and the fact $K_f^{(-2)}(H_0''(f_t), H_0''(f_t)) = 0$ determine $H_0''(f_t)$. By table (9.77), M_T has on $v_1^0 = b_4$ the eigenvalue 1.

(c) Consider the coordinate change

$$\varphi : (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0), (x, y, z) \mapsto (x, y, -z). \tag{9.80}$$

We treat the cases $U_{1,9+18r}$ and $U_{1,18r}$ separately.

The case $U_{1,9+18r}$: Then $\varphi \in G_{\mathcal{R}}^{smar,gen} \subset G^{smar}$, and

$$\varphi^*(\omega_0) = -\omega_0, \varphi^*(y^r z\omega_0) = y^r z\omega_0. \tag{9.81}$$

Now compare (9.39) and (9.54). φ induces an automorphism $(\varphi)_{coh}$ on C^{α_1} and C^{β_2} with

$$(\varphi)_{coh}(b_1) = -b_1, (\varphi)_{coh}(b_4) = -b_4, (\varphi)_{coh}(b_5) = b_5. \tag{9.82}$$

One can choose $g_3 = -M_T \circ (\varphi)_{hom} \in G^{mar}$.

The case $U_{1,18r}$: Because of (9.54) and (9.77), instead of (9.81) the identities

$$\varphi^*(\omega_0) = -\omega_0, \varphi^*(y^{r+1}\omega_0) = -y^{r+1}\omega_0 \tag{9.83}$$

are relevant. Now $(\varphi)_{coh}$ is because of (9.15) an isomorphism

$$H_0''(f_{(t_1,0)}) \to H_0''(f_{(-t_1,0)}), C^{\beta_2}(t_1, 0) \to C^{\beta_2}(-t_1, 0).$$

The composition

$$(-\,\mathrm{id}) \circ (\text{math. pos. flat shift from } C^{\beta_2}(-t_1, 0) \text{ to } C^{\beta_2}(t_1, 0)) \circ (\varphi)_{coh}$$

acts on $C^{\beta_2}(t_1,0)$ and has because of (9.76) the eigenvectors b_4 and b_5 with the eigenvalues 1 and $e^{\pi i/9}$:

$$\begin{array}{cccc}
b_4 & t_1^{-1/9}b_5 & & C^{\beta_2}(t_1,0) \\
\downarrow & \downarrow & (\varphi)_{coh} & \downarrow \\
-b_4 & -(e^{-\pi i}t_1)^{-1/9}b_5 & & C^{\beta_2}(-t_1,0) \\
\downarrow & \downarrow & \text{shift} & \downarrow \\
-b_4 & -e^{\pi i/9}t_1^{-1/9}b_5 & & C^{\beta_2}(t_1,0) \\
\downarrow & \downarrow & -\mathrm{id} & \downarrow \\
b_4 & e^{\pi i/9}t_1^{-1/9}b_5 & & C^{\beta_2}(t_1,0)
\end{array}$$

The corresponding composition

$$(-\mathrm{id})\circ(\text{math. pos. flat shift from } Ml(f_{(-t_1,0)}) \text{ to } Ml(f_{(t_1,0)}))\circ(\varphi)_{hom}$$

is in G^{mar} and can be chosen as g_3. □

10. Period maps and Torelli results for the quadrangle singularities

In this section we will prove for the quadrangle singularities the strong global Torelli conjecture 8.11 (a), the conjectures 8.6 (b) $-\mathrm{id}\notin G^{smar}$ and (a) $G_{\mathbb{Z}}=G^{mar}$. The Torelli conjecture for the unmarked singularities had been proved in [He93] (and the proof had been sketched in [He95]). The main new ingredient for the Torelli result for marked singularities is a much stronger control of the group $G_{\mathbb{Z}}$, in theorem 6.1. But we will also recall the old ingredients from [He93], the space D_{BL} and a period map for which we need calculations of the Gauss-Manin connection.

The six bimodal families of quadrangle singularities have as surface singularities the normal forms $f_{(t_1,t_2)}$ in table (10.1). These are not the normal forms in [AGV85, 15.1]. We will justify the normal forms and explain their properties after theorem 10.1. The parameters (t_1,t_2) are in $T^{(5)}:=(\mathbb{C}-\{0,1\})\times\mathbb{C}$. Table (10.1) lists additionally weights (w_x,w_y,w_z) such that $f_{(t_1,0)}$ is quasihomogeneous of weighted degree 1 and two numbers m_0 and m_∞ We set $m_1:=m_0$. Observe $w_y=\frac{2}{m}<w_x\le w_z$.

		(w_x,w_y,w_z)	m_0	m_∞
$W_{1,0}$	$x^4+(4t_1-2)x^2y^3+y^6+t_2x^2y^4+z^2$	$(\frac14,\frac16,\frac12)$	12	6
$S_{1,0}$	$x^2z+y^3z+yz^2+t_1x^2y^2+t_2x^2y^3$	$(\frac{3}{10},\frac{2}{10},\frac{4}{10})$	10	5
$U_{1,0}$	$xz(x-z)+y^3(x-t_1z)+t_2y^4z$	$(\frac13,\frac29,\frac13)$	9	9
$E_{3,0}$	$x(x-y^3)(x-t_1y^3)+t_2x^2y^4+z^2$	$(\frac13,\frac19,\frac12)$	9	9
$Z_{1,0}$	$xy(x-y^2)(x-t_1y^2)+t_2x^2y^4+z^2$	$(\frac27,\frac17,\frac12)$	7	7
$Q_{2,0}$	$x(x-y^2)(x-t_1y^2)+yz^2+t_2xz^2$	$(\frac13,\frac16,\frac{5}{12})$	6	6

(10.1)

Recall that table (6.1) lists for these singularities the Milnor number μ, the characteristic polynomials b_j, $j\ge1$, of the monodromy on the Orlik blocks B_j in theorem 5.1, the order m of the monodromy and the index r_I.

For each 2-parameter family in table (10.1), we choose $f_0:=f_{(i,0)}$ as reference singularity. And as in section 9, M_μ^{mar}, $(M_\mu^{mar})^0$, $G_{\mathbb{Z}}$, G^{mar}, Ml, H^∞ and C^α mean the objects for f_0. As always, $\zeta:=e^{2\pi i/m}$.

We will construct branched coverings $c^{(2)}$ and $c^{(6)}$ and unbranched coverings $c^{(1)}$ and $c^{(5)}$ as follows.

$$\begin{array}{ccc} T^{(3)} & \subset & T^{(4)} := \mathbb{H} \\ \downarrow c^{(1)} & & \downarrow c^{(2)} \\ T^{(1)} := \mathbb{C} - \{0,1\} & \subset & T^{(2)} := \mathbb{P}^1\mathbb{C} \end{array} \tag{10.2}$$

$$\begin{array}{ccc} T^{(7)} := T^{(3)} \times \mathbb{C} & \subset & T^{(8)} := T^{(4)} \times \mathbb{C} \\ \downarrow c^{(5)}:=c^{(1)}\times \mathrm{id} & & \downarrow c^{(6)}:=c^{(2)}\times \mathrm{id} \\ T^{(5)} = T^{(1)} \times \mathbb{C} & \subset & T^{(6)} := T^{(2)} \times \mathbb{C} \end{array}$$

Let $\Gamma \subset PGL(2,\mathbb{R})$ be a triangle group of type $(\frac{1}{m_0}, \frac{1}{m_1}, \frac{1}{m_\infty})$. The quotient $\mathbb{H}/\Gamma$ is an orbifold with three orbifold points of orders m_0, m_1 and m_∞. They are the images of the elliptic fixed points of Γ on $T^{(4)} = \mathbb{H}$ of orders m_0, m_1 and m_∞. As a manifold $\mathbb{H}/\Gamma \cong \mathbb{P}^1\mathbb{C}$. Choose coordinates on $\mathbb{H}/\Gamma$ such that 0 and 1 are orbifold points of order $m_0 = m_1$ and ∞ is an orbifold point of order m_∞. Denote by

$$c^{(2)} : T^{(4)} = \mathbb{H} \to T^{(2)} = \mathbb{P}^1\mathbb{C} \tag{10.3}$$

the quotient map. It is a branched covering. Denote

$$\begin{aligned} T^{(3)} &:= T^{(4)} - (c^{(2)})^{-1}(\{0,1,\infty\}), \\ \text{and} c^{(1)} &:= c^{(2)}|_{T^{(4)}} : T^{(3)} \to T^{(1)}. \end{aligned} \tag{10.4}$$

It is a covering.

Theorem 10.1. *Consider a bimodal family of quadrangle surface singularities in table* (10.1).
(a) There are canonical isomorphisms

$$T^{(7)} \to (M_\mu^{smar})^0 \to (M_\mu^{mar})^0. \tag{10.5}$$

(b) $-\mathrm{id} \notin G^{smar}$*, where* G^{smar} *is the group for the singularities of multiplicity* ≥ 3*, namely the curve singularities* $W_{1,0}, E_{3,0}, Z_{1,0}$ *and the surface singularities* $S_{1,0}, U_{1,0}, Q_{2,0}$*. So, conjecture 8.6 (b) is true.*

(c) $G_{\mathbb{Z}} = G^{mar}$*. So,* $M_\mu^{mar} = (M_\mu^{mar})^0$*, and conjecture 8.6 (a) is true.*

(d) The period map $BL : M_\mu^{mar} \to D_{BL}$ *is an embedding. So, the strong global Torelli conjecture 8.11 (a) is true.*

The Torelli result for unmarked singularities (the period map $M_\mu^{mar}/G_{\mathbb{Z}} \to D_{BL}/G_{\mathbb{Z}}$ is an embedding) was proved already in [He93], and also that there is a well defined period map $T^{(7)} \to D_{BL}$ and that it is an embedding. But we prefer to give an independent account and recover these results. The hardest part is in any case new. It is the precise control of $G_{\mathbb{Z}}$ in theorem 6.1.

First we discuss the normal forms in table (10.1) and the right equivalence classes in them.

Each bimodal family of quadrangle surface singularities contains a 1-parameter subfamily of quasihomogeneous singularities. The exceptional set of the minimal good resolution of such a singularity consists of 5 smooth rational curves. One, the central curve, intersects each of the other 4, the branches, in one point. The right equivalence class of one quasihomogeneous surface singularity is determined by the central curve with the 4 intersection points and the self intersection numbers of the 4 branches. Table (10.6) lists these self intersection numbers.

$$\begin{array}{llllll} W_{1,0} & S_{1,0} & U_{1,0} & E_{3,0} & Z_{1,0} & Q_{2,0} \\ (2,2,3,3) & (2,2,3,4) & (2,3,3,3) & (2,2,2,3) & (2,2,2,4) & (2,2,2,5) \end{array} \tag{10.6}$$

In table (10.1), the singularities with $t_2 = 0$ are quasihomogeneous. Their normal forms are not taken from [AGV85, 15.1], but from [Bi92, Anhang A2, p. 191]. They are chosen such that the cross ratio of the 4 intersection points on the central curve has j-invariant $j = \frac{4}{27}\frac{(t_1^2-t_1+1)^3}{t_1^2(1-t_1)^2}$. This fact implies that the families in table (10.1) contain representatives of all right equivalence classes in one μ-homotopy class.

From the weights (or the spectral numbers, see below theorem 10.6) one deduces easily that any monomial basis of the Jacobi algebra of one quasihomogeneous surface singularity $f_{t_1,0}$ contains precisely one monomial $p_{>1}$ of weighted degree > 1 and that

$$\deg_w p_{>1} = 1 + \frac{2}{m} = 1 + w_y.$$

[AGV85, 12.6 Theorem] says here that any semiquasihomogeneous singularity with quasihomogeneous part $f_{t_1,0}$ is right equivalent to $f_{t_1,0} + t_2 \cdot p_{>1}$ for some $t_2 \in \mathbb{C}$. In table (10.1) we have chosen the monomial $p_{>1}$ such that it is part of a monomial basis of the Jacobi algebra of $f_{t_1,0}$ for any $t_1 \in T^{(1)}$.

Remarks 10.2. It is nontrivial (and slightly surprising) that such a monomial $p_{>1}$ exists simultaneously for all $t_1 \in T^{(1)}$. In [He93][He95] the second author had overlooked this problem and had chosen in the four cases $S_{1,0}, E_{3,0}, Z_{1,0}, Q_{2,0}$ a monomial which does not work for special parameters $t_1 \in T^{(1)}$. The following table (10.8) lists for all 6 families all monomials $\widetilde{p}$ of weighted degree $1 + \frac{2}{m}$ and for each of them the function $q(t_1)$ with

$$\widetilde{p} \equiv q(t_1) \cdot p_{>1} \mod (\text{Jacobi ideal of } f_{t_1,0}), \tag{10.7}$$

where $p_{>1} = \frac{\partial f_{(t_1,t_2)}}{\partial t_2}$ is the monomial chosen in table (10.1).

$$\begin{array}{lllll} & p_{>1} & \widetilde{p} \ :\ q(t_1) & \widetilde{p} \ :\ q(t_1) & \widetilde{p} \ :\ q(t_1) \\ W_{1,0} & x^2y^4 & x^4y : 1-2t_1 & y^7 : 1-2t_1 & x^2yz : 0 \\ & & y^4z : 0 & yz^2 : 0 & \\ S_{1,0} & x^2y^3 & x^2yz : -t_1 & y^4z : -t_1 & y^2z^2 : t_1 \\ & & y^6 : 2t_1-1 & x^4 : 2t_1-1 & z^3 : t_1(2t_1-3) \\ U_{1,0} & y^4z & x^2yz : -t_1 & xyz^2 : -t_1 & xy^4 : t_1 \\ & & x^3y : t_1(t_1-2) & yz^3 : 1-2t_1 & \\ E_{3,0} & x^2y^4 & x^3y : \frac{t_1+1}{2} & xy^7 : \frac{t_1+1}{2t_1} & y^{10} : \frac{t_1^2-t_1+1}{t_1^2} \\ & & yz^2 : 0 & & \\ Z_{1,0} & x^2y^4 & x^3y^2 : \frac{t_1+1}{2} & xy^6 : \frac{t_1+1}{2t_1} & y^8 : \frac{t_1^2-t_1+1}{t_1^2} \\ & & x^4 : \frac{3}{2}t_1^2 - 2t_1 + \frac{3}{2} & yz^2 : 0 & \\ Q_{2,0} & xz^2 & x^2y^3 : \frac{1}{(1-t_1)^2} & x^3y : \frac{t_1+1}{2(1-t_1)^2} & xy^5 : \frac{t_1+1}{2t_1(1-t_1)^2} \\ & & y^7 : \frac{t_1^2-t_1+1}{t_1^2(1-t_1)^2} & y^2z^2 : 0 & \end{array} \tag{10.8}$$

Thus $p_{>1}$ could be replaced in the normal form in table (10.1) by any of the following monomials:

$$\begin{array}{llllll} W_{1,0} & S_{1,0} & U_{1,0} & E_{3,0} & Z_{1,0} & Q_{2,0} \\ - & x^2yz, y^4z, y^2z^2 & x^2yz, xyz^2, xy^4 & - & - & x^2y^3 \end{array} \tag{10.9}$$

We denote by G_3 and $G_2 \subset G_3$ the groups of automorphisms of $T^{(2)} = \mathbb{P}^1\mathbb{C}$

$$\begin{array}{lll} G_3 := & \{t_1 \mapsto t_1, 1-t_1, \frac{1}{t_1}, \frac{t_1}{t_1-1}, \frac{1}{1-t_1}, \frac{t_1-1}{t_1}\} & \cong S_3 \text{ as a group,} \\ G_2 := & \{t_1 \mapsto t_1, 1-t_1\} \subset G_3 & \cong S_2 \text{ as a group.} \end{array} \tag{10.10}$$

They act also on $T^{(1)} = \mathbb{C} - \{0,1\}$.

Theorem 10.3. *Consider a bimodal family of quadrangle surface singularities in table (10.1). A function*

$$\begin{aligned} \kappa : G_2 \times T^{(1)} \to \mathbb{C}^* \qquad & \text{for } W_{1,0}, S_{1,0}, \\ \kappa : G_3 \times T^{(1)} \to \mathbb{C}^* \qquad & \text{for } U_{1,0}, E_{3,0}, Z_{1,0}, Q_{2,0}, \end{aligned} \tag{10.11}$$

with the following properties exists.

$$f_{(t_1,t_2)} \sim_{\mathcal{R}} f_{(\widetilde{t}_1,\widetilde{t}_2)} \iff \exists\, g \in \left\{ \begin{array}{ll} G_2 & \text{for } W_{1,0}, S_{1,0}, \\ G_3 & \text{for } U_{1,0}, E_{3,0}, Z_{1,0}, Q_{2,0}, \end{array} \right.$$
$$\text{with } \widetilde{t}_1 = g(t_1), \widetilde{t}_2^{m_\infty} = \kappa(g,t_1)\cdot t_2^{m_\infty}, \tag{10.12}$$
$$\kappa(\mathrm{id}, t_1) = 1, \tag{10.13}$$
$$\kappa(g_2 g_1, t_1) = \kappa(g_1,t_1)\cdot \kappa(g_2, g_1(t_1)). \tag{10.14}$$

Table (10.15) lists $\kappa(g,t_1)$ for generators g of the group.

$$\begin{array}{lllllll} & W_{1,0} & S_{1,0} & U_{1,0} & E_{3,0} & Z_{1,0} & Q_{2,0} \\ t_1 \mapsto 1-t_1 & 1 & -1 & 1 & \left(\frac{1-t_1}{t_1}\right)^{18} & \left(\frac{1-t_1}{t_1}\right)^{14} & -1 \\ t_1 \mapsto t_1^{-1} & - & - & -t_1^{-3} & t_1^{-12} & t_1^{-10} & t_1^3 \end{array} \tag{10.15}$$

Proof: (10.13)–(10.15) are consistent (to check this is nontrivial only for $E_{3,0}$ and $Z_{1,0}$) and define a unique function κ as in (10.11). We will show now that it satisfies $\Leftarrow$ in (10.12). We postpone the proof of $\Rightarrow$ in (10.12) to the end of this section.

The equality

$$f_{(t_1,t_2)}(x\cdot e^{2\pi i w_x}, y\cdot e^{2\pi i w_y}, z\cdot e^{2\pi i w_z}) = f_{(t_1, t_2\cdot e^{2\pi i 2/m})} \tag{10.16}$$

gives $\Leftarrow$ in (10.12) for $g = \mathrm{id}$ and $\kappa(\mathrm{id},t_1) = 1$ (for $U_{1,0}$ $m = m_\infty = 9$, in the other cases $m_\infty = \frac{m}{2}$). We list now coordinate changes $(x,y,z) \mapsto \varphi^{(1)}(x,y,z)$ and $(x,y,z) \mapsto \varphi^{(2)}(x,y,z)$ with

$$\begin{aligned} f_{(t_1,t_2)}(\varphi^{(1)}(x,y,z)) &= f_{(1-t_1,0)} + t_2\cdot p^{(1)}(t_1,x,y,z) \\ & \quad \text{for all 6 cases,} \end{aligned} \tag{10.17}$$
$$\begin{aligned} f_{(t_1,t_2)}(\varphi^{(2)}(x,y,z)) &= f_{(t_1^{-1},0)}(x,y,z) + t_2\cdot p^{(2)}(t_1,x,y,z) \\ & \quad \text{for } U_{1,0}, E_{3,0}, Z_{1,0}, Q_{2,0} \end{aligned} \tag{10.18}$$

for certain quasihomogeneous polynomials $p^{(1)}$ and $p^{(2)}$ in x,y,z with

$$\deg_w p^{(1)} = \deg_w p^{(2)} = 1 + \frac{2}{m}.$$

$$\begin{array}{lll} & \varphi^{(1)}(x,y,z) & \varphi^{(2)}(x,y,z) \\ W_{1,0} & (x,-y,z) & - \\ S_{1,0} & (ix,y,-z-y^2) & - \\ U_{1,0} & (-x+z,-y,z) & (-z,t_1^{-1/3}y,-x) \\ E_{3,0} & (x-y^3,-y,z) & (x,t_1^{-1/3}y,z) \\ Z_{1,0} & (e^{-2\pi i/14}(x-y^2), i\cdot e^{-2\pi i/28}y,z) & (t_1^{1/7}x,t_1^{-3/7}y,z) \\ Q_{2,0} & (x-y^2,iy,e^{-2\pi i/8}z) & (x,t_1^{-1/2}y,t_1^{1/4}z) \end{array} \tag{10.19}$$

One can calculate $p^{(1)}$ and $p^{(2)}$ easily. The proof of [AGV85, 12.6 Lemma] implies here

$$\begin{aligned} & f_{(\widetilde{t}_1,0)} + t_2\cdot\widetilde{p} \ \sim_{\mathcal R}\ f_{(\widetilde{t}_1,\widetilde{t}_2)} \\ & \text{where } t_2\cdot\widetilde{p} \equiv \widetilde{t}_2\cdot p_{>1} \mod (\text{Jacobi ideal of } f_{(\widetilde{t}_1,0)}). \end{aligned} \tag{10.20}$$

With table (10.8) one finds $\widetilde{t}_2$ with (10.20) for $\widetilde{p}=p^{(1)}$ and for $\widetilde{p}=p^{(2)}$. Then one verifies table (10.15). □

Remarks 10.4. (i) For the quasihomogeneous singularities, (10.12) becomes

$$f_{(t_1,0)} \sim_{\mathcal R} f_{(\widetilde{t}_1,0)} \quad\iff\quad \exists\, g\in G_2 \text{ resp. } G_3 \text{ with } \widetilde{t}_1 = g(t_1).$$

This is proved in [Bi92, Satz 1.5.2] using the minimal good resolution. Our proof of $\Rightarrow$ in (10.12) for all singularities at the end of this section will be different.

(ii) The right equivalence classes in $T^{(5)}$ are the orbits of a group action on $T^{(5)}$ in the cases $W_{1,0}$ and $S_{1,0}$. There the group is a central extension of G_2 by a cyclic group of order $m_\infty = \frac{m}{2}$,

$$1\to \begin{pmatrix} \text{cyclic group} \\ \text{of order } m \end{pmatrix} \to (\text{group acting on } T^{(5)}) \to G_2 \to 1.$$

In the other cases $U_{1,0}, E_{3,0}, Z_{1,0}$ and $Q_{2,0}$, an m-th root of $\kappa(t_1\to t_1^{-1},.) : T^{(1)}\to\mathbb{C}^*$ is not uni-valued, but multi-valued. There one has only a groupoid acting on $T^{(5)}$, whose orbits are the right equivalence classes in $T^{(5)}$.

(iii) In any case, the space $M_\mu^{mar} = (M_\mu^{mar})^0 \cong T^{(7)}$ (by theorem 10.1) will be more canonical than $T^{(5)}$, and there the right equivalence classes are the orbits of the action of the group $G_{\mathbb Z} = G^{mar}$.

Now we come to the spectral numbers and the classifying space D_{BL}.

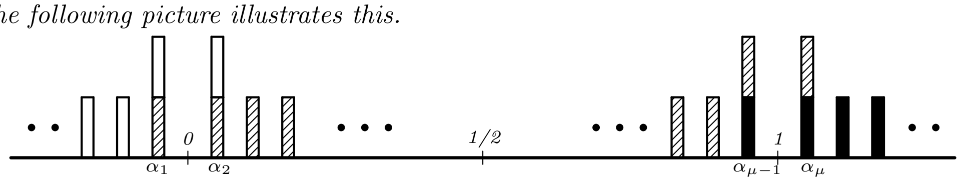
Lemma 10.5. *Consider a bimodal family of quadrangle surface singularities in table (10.1). Denote $\omega_0 := dxdydz$.*

(a) The spectral numbers $\alpha_1,\dots,\alpha_\mu$ with $\alpha_1\leq\dots\leq\alpha_\mu$ satisfy

$$\begin{aligned} \alpha_1 = \frac{-1}{m} < \alpha_2 = \frac{1}{m} < \alpha_3 \leq \dots \leq \alpha_{\mu-2} \\ < \alpha_{\mu-1} = 1-\frac{1}{m} < \alpha_\mu = 1+\frac{1}{m}, \end{aligned} \tag{10.21}$$

$$\dim C^{\alpha_1} = \dim C^{\alpha_2} = 2. \tag{10.22}$$

The following picture illustrates this.



We also have

$$\begin{aligned} V^{\alpha_1}(f_{(t_1,t_2)}) &\supset H_0''(f_{(t_1,t_2)} \supset V^{>\alpha_2}(f_{(t_1,t_2)}), \qquad (10.23)\\ H_0''(f_{(t_1,t_2)}) &= \mathbb{C}\cdot(s(\omega_0,\alpha_1)(t_1,t_2)+s(\omega_0,\alpha_2)(t_1,t_2))\\ &+ \mathbb{C}\cdot s(y\omega_0,\alpha_2)(t_1,t_2)+V^{>\alpha_2}(f_{(t_1,t_2)}). \qquad (10.24)\end{aligned}$$

(b) The polarizing form S defines an indefinite form $((a,b)\mapsto S(a,\overline{b}))$ on H^∞_ζ. We get a half-plane

$$\begin{aligned}\mathcal{H}(C^{\alpha_1}) &:= \{\mathbb{C}\cdot v\,|\,v\in C^{\alpha_1} \text{ with } S(\psi_{\alpha_1}^{-1}(v),\overline{\psi_{\alpha_1}^{-1}(v)})<0\} \qquad (10.25)\\ &\subset \mathbb{P}^1(C^{\alpha_1}).\end{aligned}$$

(c)

$$\begin{aligned} D_{BL} &= \{\mathbb{C}\cdot(v_1+v_2)\,|\,v_1\in C^{\alpha_1}-\{0\} \text{ with } [\mathbb{C}\cdot v_1]\in\mathcal{H}(C^{\alpha_1}),\\ &\qquad v_2\in\mathbb{C}\cdot\psi_{\alpha_2}(\overline{\psi_{\alpha_1}^{-1}(v_1)})\subset C^{\alpha_2}\} \qquad (10.26)\\ &\cong \mathcal{H}(C^{\alpha_1})\times\mathbb{C}.\end{aligned}$$

Proof: (a) The spectral numbers are well known [AGV88, 13.3.4, p. 389] and can be calculated in the semiquasihomogeneous cases for example with the generating series (here $m=2$, $(w_0,w_1,w_2)=(w_x,w_y,w_z)$)

$$\prod_{j=0}^{m}\frac{t-t^{w_j}}{t^{w_j}-1}=\sum_{i=1}^{\mu}t^{\alpha_i+1}. \qquad (10.27)$$

(10.22) and (10.23) are obvious. (10.24) follows from lemma 7.20 and

$$\deg_w(\omega_0)=\alpha_1+1,\ \deg_w(y\omega_0)=\alpha_2+1,\ \text{and}\ \deg_w(x^iy^jz^k\omega_0)>\alpha_2+1$$

for any other monomial $x^iy^jz^k$, because $w_y<w_x\leq w_z$.

(b) This follows as in section 9 before theorem 9.2. It follows also from the fact that $\mathrm{Gr}_V^\bullet H_0''(f_{(t_1,t_2)})$ and S induce as in (7.27) a polarized Hodge structure of weight 2 on $H^\infty(f_{(t_1,t_2)})$. Especially,

$$\begin{aligned} a_1(t_1,t_2) &:= \psi_{\alpha_1}^{-1}s(\omega_0,\alpha_1)(t_1,t_2)\in H^\infty(f_{(t_1,t_2)})_\zeta, \qquad (10.28)\\ a_2(t_1,t_2) &:= \psi_{\alpha_2}^{-1}s(y\omega_0,\alpha_2)(t_1,t_2)\in H^\infty(f_{(t_1,t_2)})_{\overline{\zeta}}\end{aligned}$$

satisfy

$$\begin{aligned} \text{on } H^\infty(f_{(t_1,t_2)})_\zeta: &\quad \mathbb{C}\cdot a_1 = H^{2,0}=F^2\subset H^\infty(f_{(t_1,t_2)})_\zeta \qquad (10.29)\\ &\quad = F^1 = H^{2,0}\oplus H^{1,1}=\mathbb{C}\cdot a_1\oplus\mathbb{C}\cdot\overline{a_2},\\ \text{on } H^\infty(f_{(t_1,t_2)})_{\overline{\zeta}}: &\quad \mathbb{C}\cdot a_2 = H^{1,1}=F^1\subset H^\infty(f_{(t_1,t_2)})_{\overline{\zeta}} \qquad (10.30)\\ &\quad = F^0 = H^{1,1}\oplus H^{0,2}=\mathbb{C}\cdot a_2\oplus\mathbb{C}\cdot\overline{a_1},\\ 0<i^{2-0}S(a_1,\overline{a_1}), &\quad 0<i^{1-1}S(\overline{a_2},a_2),\ 0=S(a_1,a_2). \qquad (10.31)\end{aligned}$$

(c) This follows as in lemma 9.4 (c) in the case $m|p$. □

The multi-valued period map $BL_{T^{(5)}}:T^{(5)}\to D_{BL}$ had been calculated in [He93]. We recall the result and sketch the proof. In part (e) of theorem 10.6 we add a formula for the case $S_{1,0}$ which will be useful for the determination of a transversal monodromy in theorem 10.7.

Theorem 10.6. *Consider a bimodal family of quadrangle surface singularities in table (10.1).*

(a) $s(\omega_0,\alpha_1)(t_1,t_2) = s(\omega_0,\alpha_1)(t_1,0) = s[\omega_0](t_1,0)$ *is independent of* t_2 *and satisfies the hypergeometric differential equation*

$$0 = \Big(t_1(1-t_1)\partial_{t_1}^2 + (c-(a+b+1)t_1)\partial_{t_1} - ab\Big)s[\omega_0](t_1,0) \tag{10.32}$$

with $(1-c, c-a-b, a-b) = (\frac{1}{m_0}, \frac{1}{m_1}, \frac{1}{m_\infty})$.

(b) The multi-valued period map

$$BL_{T^{(1)}} : T^{(1)} \to \mathcal{H}(C^{\alpha_1}), t_1 \mapsto \mathbb{C}\cdot s[\omega_0](t_1,0), \tag{10.33}$$

lifts to a uni-valued period map

$$BL_{T^{(3)}} : T^{(3)} \to \mathcal{H}(C^{\alpha_1}) \tag{10.34}$$

which is an open embedding and extends to an isomorphism

$$BL_{T^{(4)}} : T^{(4)} \to \mathcal{H}(C^{\alpha_1}). \tag{10.35}$$

(c)

$$s(\omega_0,\alpha_2)(t_1,t_2) = t_2\cdot(-\partial_\tau)s[p_{>1}\omega_0](t_1,0), \tag{10.36}$$

$$C^{\alpha_2} = \mathbb{C}\cdot s[y\omega_0](t_1,0) \oplus \mathbb{C}\cdot \partial_\tau s[p_{>1}\omega_0](t_1,0). \tag{10.37}$$

(d) The multi-valued period map

$$BL_{T^{(5)}} : T^{(5)} \to D_{BL} \tag{10.38}$$

is locally in $T^{(1)}$ *and* $\mathcal{H}(C^{\alpha_1})$ *an isomorphism of line bundles and lifts to an open embedding of line bundles*

$$BL_{T^{(7)}} : T^{(7)} \to D_{BL}. \tag{10.39}$$

(We do not know whether this extends to an isomorphism of line bundles $T^{(8)} \to D_{BL}$*, but we do not expect it.)*

(e) In the case of $S_{1,0}$

$$\partial_{t_1} s[x\omega_0](t_1,0) = \frac{2t_1-1}{5t_1(1-t_1)}\cdot s[x\omega_0](t_1,0). \tag{10.40}$$

Proof: (a) We just sketch the ansatz for the calculations which prove (10.32). $f_{(t_1,0)}$ and $\partial_{t_1} f_{(t_1,0)}$ are quasihomogeneous of weighted degree 1. List all monomials $d_1,\dots,d_l$ in x,y,z which turn up in $f_{(t_1,0)}^2$, $f_{(t_1,0)}\cdot\partial_{t_1} f_{(t_1,0)}$ and $(\partial_{t_1} f_{(t_1,0)})^2$, find $l-2$ independent linear combinations of $d_1\omega_0,\dots,d_l\omega_0$ in $df_{(t_1,0)}\wedge d\Omega^1_{\mathbb{C}^3}$, and determine an equation

$$\begin{aligned} &p_1\cdot(\partial_{t_1} f_{(t_1,0)})^2\cdot\omega_0 + p_2\cdot f_{(t_1,0)}\cdot\partial_{t_1} f_{(t_1,0)}\cdot\omega_0 + p_3\cdot f_{(t_1,0)}^2\cdot\omega_0 \\ &\equiv 0 \mod df_{(t_1,0)}\wedge d\Omega^1_{\mathbb{C}^3} \end{aligned} \tag{10.41}$$

with $p_1,p_2,p_3\in\mathbb{Q}[t_1]$. Then

$$\big(p_1\partial_{t_1}^2 - (\alpha_1+2)p_2\partial_{t_1} + (\alpha_1+2)(\alpha_1+1)p_3\big)\, s[\omega_0](t_1,0). \tag{10.42}$$

Because of corollary 8.14 one can work in the cases $W_{1,0}, E_{3,0}, Z_{1,0}$ with the curve singularities. There the number l of monomials is $l=5$. In the other cases, the surfaces singularities $S_{1,0}, U_{1,0}, Q_{2,0}$, it is $l=9$.

(b) The period map $BL_{T^{(1)}}$ is not constant because $s[\omega_0](t_1,0)$ and

$$\partial_{t_1} s[\omega_0](t_1,0) = (-\partial_\tau)s[\partial_{t_1} f_{(t_1,0)}\cdot\omega_0](t_1,0)$$

are linearly independent because $\partial_{t_1} f_{(t_1,0)}$ is not in the Jacobi ideal. Therefore the multi-valued coefficient functions $f_1(t_1)$ and $f_2(t_1)$ with

$$s[\omega_0](t_1,0) = f_1(t_1)\cdot v_1^0 + f_2(t_1)\cdot v_2^0 \tag{10.43}$$

for an arbitrary basis v_1^0, v_2^0 of C^{α_1} are linearly independent scalar solutions of the same hypergeometric differential equation. Their quotient $(t_1 \mapsto \frac{f_1(t_1)}{f_2(t_1)})$ is a Schwarzian function [Fo51, sec. 113+114], which maps the closure of the upper half-plane to a hyperbolic triangle with angles $\frac{\pi}{m_0}, \frac{\pi}{m_1}, \frac{\pi}{m_\infty}$. The vertices are the images of $0, 1, \infty$. Therefore the multi-valued map $BL_{T^{(1)}} : T^{(1)} \to \mathcal{H}(C^{\alpha_1})$ is an inverse of the quotient map $c^{(1)} : T^{(3)} \to T^{(1)}$. This shows (10.34) and (10.35).

(c) $s(\omega_0,\alpha_2)(t_1,0) = 0$ because of formula (7.52) in lemma 7.20 (a).

$$\begin{aligned}
\partial_{t_2} s(\omega_0,\alpha_2)(t_1,t_2) &= (-\partial_\tau)s(p_{>1}\omega_0,\alpha_2+1)(t_1,t_2)\\
&= (-\partial_\tau)s[p_{>1}\omega_0](t_1,0)\\
\text{thus } s(\omega_0,\alpha_2)(t_1,t_2) &= t_2\cdot(-\partial_\tau)s[p_{>1}\omega_0](t_1,0)\\
&\equiv t_2\cdot v_2 \mod \mathbb{C}\cdot s[y\omega_0](t_1,0)\\
\text{with a suitable } v_2 &\in \psi_{\alpha_2}^{-1}(\overline{\psi_{\alpha_1}(s[\omega_0](t_1,0))}) - \{0\}.
\end{aligned} \tag{10.44}$$

Here $v_2 \neq 0$ follows from (10.37) which is a consequence of the fact that $p_{>1}$ is not in the Jacobi ideal of $f_{(t_1,0)}$.

(d) This follows from (10.34) and part (c).

(e) The proof is similar to the calculations which prove part (a), but simpler.

$$\begin{aligned}
&\partial_{t_1} s[x\omega_0](t_1,0)\\
=\ & (-\partial_\tau)s[\partial_{t_1} f_{(t_1,0)}\cdot x\omega_0](t_1,0) = (-\partial_\tau)s[x^3y^2\omega_0](t_1,0)\\
\overset{(*)}{=}\ & \frac{2t_1-1}{6t_1(t_1-1)}(-\partial_\tau)s[f_{(t_1,0)}\cdot x\omega_0](t_1,0)\\
=\ & \frac{2t_1-1}{6t_1(t_1-1)}(-\partial_\tau\tau)s[x\omega_0](t_1,0) = \frac{2t_1-1}{6t_1(t_1-1)}(-\frac{6}{5})s[x\omega_0](t_1,0)\\
=\ & \frac{2t_1-1}{5t_1(1-t_1)}s[x\omega_0](t_1,0).
\end{aligned}$$

For $\overset{(*)}{=}$ one has to find 3 relations in $df_{(t_1,0)}\wedge d\Omega^1_{\mathbb{C}^3}$ between the monomial differential forms $x^3y^2\omega_0, xy^3z\omega_0, xyz^2\omega_0$ and $x^3z\omega_0$ in $f_{(t_1,0)}\cdot x\omega_0$ and $x^3y^2\omega_0$. □

The last step before the proof of theorem 10.1 is the following result on a transversal monodromy group. Its proof uses formula (6.8) in theorem 6.1.

Theorem 10.7. *Consider a bimodal family of quadrangle surface singularities in table (10.1). The pull back to $T^{(3)}$ with $c^{(1)}$ of the homology group $\bigcup_{t_1\in T^{(1)}} Ml(f_{(t_1,0)}) \to T^{(1)}$ comes equipped with a monodromy representation $\pi^{(3)} : \pi_1(T^{(3)},\tau^{(3)}) \to G_{\mathbb{Z}}$ (with $c^{(1)}(\tau^{(3)}) = i$) which is called transversal monodromy group.*

(a) The following table lists the local monodromies around elliptic fixed points in $(c^{(2)})^{-1}(0)$, $(c^{(2)})^{-1}(1)$ and $(c^{(2)})^{-1}(\infty)$.

	$W_{1,0}$	$S_{1,0}$	$U_{1,0}$	$E_{3,0}$	$Z_{1,0}$	$Q_{2,0}$
$(c^{(2)})^{-1}(\{0,1\})$	id	id	id	id	id	id
$(c^{(2)})^{-1}(\infty)$	id	M_h^5	id	id	id	M_h^6

(10.45)

Therefore $\mathrm{Im}(\pi^{(3)}) = \{\mathrm{id}\}$ *for* $W_{1,0}, U_{1,0}, E_{3,0}, Z_{1,0}$, *and* $\mathrm{Im}(\pi^{(3)}) = \{\mathrm{id}, M_h^{m_\infty}\}$ *for* $S_{1,0}$ *and* $Q_{2,0}$.

(b)

$$\begin{aligned} & \{g \in G_{\mathbb{Z}} \,|\, g \text{ acts trivially on } D_{BL}\} \\ = & \{g \in G_{\mathbb{Z}} \,|\, g = \pm\,\mathrm{id} \text{ on } Ml_\zeta\} \\ = & \{\pm\,\mathrm{id}, \pm M_h^{m_\infty}\} \\ = & \left\{ \begin{array}{ll} \{\pm\,\mathrm{id}\} & \text{for } U_{1,0}, E_{3,0}, Z_{1,0} \\ \{\pm\,\mathrm{id}, \pm M_h^{m_\infty}\} & \text{for } W_{1,0}, S_{1,0}, Q_{2,0}. \end{array} \right. \end{aligned} \tag{10.46}$$

(c) $G_{\mathcal{R}}^{smar,gen}$ *is here the group in* (8.13) *for the singularities of multiplicity* ≥ 3, *namely the curve singularities* $W_{1,0}, E_{3,0}, Z_{1,0}$ *and the surface singularities* $S_{1,0}, U_{1,0}, Q_{2,0}$.

$$G_{\mathcal{R}}^{smar,gen} = \left\{ \begin{array}{ll} \{\mathrm{id}\} & \text{for } U_{1,0}, E_{3,0}, Z_{1,0}, \\ \{\mathrm{id}, M_h^{m_\infty}\} & \text{for } W_{1,0}, S_{1,0}, Q_{2,0}. \end{array} \right. \tag{10.47}$$

Proof: We start with part (b). Suppose that $g \in G_{\mathbb{Z}}$ acts trivially on D_{BL}. Then it acts trivially on $\mathcal{H}(C^{\alpha_1})$, so $g = \lambda \cdot \mathrm{id}$ on Ml_ζ for some $\lambda \in \mathbb{C}^*$. And $\mathbb{C} \cdot (v_1 + v_2) = \mathbb{C} \cdot (\lambda v_1 + \overline{\lambda} v_2)$, so $\lambda = \overline{\lambda} \in \{\pm 1\}$. This together with formula (6.8) and the set of eigenvalues of M_h gives (10.46).

(a) The Papperitz-Riemann symbol

$$\left\{ \begin{array}{cccc} 0 & 1 & \infty & \\ 0 & 0 & a & z \\ 1-c & c-a-b & b & \end{array} \right\} \tag{10.48}$$

encodes the local behaviour near 0, 1 and ∞ of scalar solutions of the hypergeometric equation. Locally suitable solutions have the following form (h.o.t. = higher order terms):

$$\begin{array}{llll} \text{near } 0: & t_1^0 + \text{h.o.t.} & \text{and} & t_1^{1-c} + \text{h.o.t.}, \\ \text{near } 1: & (t_1 - 1)^0 + \text{h.o.t.} & \text{and} & (t_1 - 1)^{c-a-b} + \text{h.o.t.}, \\ \text{near } \infty: & t_1^{-a} + \text{h.o.t.} & \text{and} & t_1^{-b} + \text{h.o.t.} \end{array} \tag{10.49}$$

Especially, the local monodromy of the space of solutions has the eigenvalues

$$\begin{array}{llll} \text{around } 0: & 1 & \text{and} & e^{2\pi i(1-c)}, \\ \text{around } 1: & 1 & \text{and} & e^{2\pi i(c-a-b)}, \\ \text{around } \infty: & e^{-2\pi i a} & \text{and} & e^{-2\pi i b}. \end{array} \tag{10.50}$$

In our situation $(1-c, c-a-b, a-b) = (\frac{1}{m_0}, \frac{1}{m_1}, \frac{1}{m_\infty})$,

$$\begin{array}{ccccccc} & W_{1,0} & S_{1,0} & U_{1,0} & E_{3,0} & Z_{1,0} & Q_{2,0} \\ a & \frac{1}{2} & \frac{1}{2} & \frac{4}{9} & \frac{4}{9} & \frac{3}{7} & \frac{5}{12} \\ b & \frac{1}{3} & \frac{3}{10} & \frac{1}{3} & \frac{1}{3} & \frac{2}{7} & \frac{1}{4} \\ c & \frac{11}{12} & \frac{9}{10} & \frac{8}{9} & \frac{8}{9} & \frac{6}{7} & \frac{5}{6} \end{array} \tag{10.51}$$

The branched covering $c^{(2)} : T^{(4)} \to T^{(2)}$ has at elliptic fixed points the orders m_0, m_1, m_∞. Therefore the local monodromies of the pull back to $T^{(3)}$ of the solutions on

$$T^{(1)} = \mathbb{C} - \{0, 1\} \subset T^{(2)} = \mathbb{P}^1\mathbb{C}$$

become $+\,\mathrm{id}$ except around the elliptic fixed points in $(c^{(2)})^{-1}(\infty)$ in the cases $S_{1,0}$ and $Q_{2,0}$ where they become $-\,\mathrm{id}$.

The same holds for the restrictions to Ml_ζ of the local monodromies in $\pi^{(3)}$.

With (6.8) we obtain (10.45) for $U_{1,0}, E_{3,0}, Z_{1,0}$ and the following approximation of (10.45) for $W_{1,0}, S_{1,0}, Q_{2,0}$.

$$\begin{array}{llll} & W_{1,0} & S_{1,0} & Q_{2,0} \\ (c^{(2)})^{-1}(\{0,1\}) & \mathrm{id} \text{ or } -M_h^6 & \mathrm{id} \text{ or } -M_h^5 & \mathrm{id} \text{ or } -M_h^6 \\ (c^{(2)})^{-1}(\infty) & \mathrm{id} \text{ or } -M_h^6 & -\mathrm{id} \text{ or } M_h^5 & -\mathrm{id} \text{ or } M_h^6 \end{array} \tag{10.52}$$

The case $W_{1,0}$: The sublattice $Ml_{-1,\mathbb{Z}}$ has rank 1. Therefore the local transversal monodromies of the homology bundle on $T^{(1)}$ around 0, 1 and ∞ have on $Ml_{-1,\mathbb{Z}}$ eigenvalues in $\{\pm 1\}$. The branched covering $T^{(4)} \to T^{(2)}$ is at the elliptic fixed points of even order. Thus $\pi^{(3)}$ restricts to the trivial monodromy on $Ml_{-1,\mathbb{Z}}$. This excludes $-M_h^6$ in (10.52).

The case $S_{1,0}$: The local transversal monodromies of the homology bundle on $T^{(1)}$ around 0, 1 and ∞ have on $Ml_{e^{-2\pi i/5}}$ eigenvalues in $\mathrm{Eiw}(\zeta)$. The branched covering is at the elliptic fixed points in $(c^{(2)})^{-1}(\{0,1\})$ of order 10. Thus the local monodromies of $\pi^{(3)}$ around points in $(c^{(2)})^{-1}(\{0,1\})$ are trivial on $Ml_{e^{-2\pi i/5}}$. This excludes $-M_h^5$ in the first line of (10.52). The branched covering is at the elliptic fixed points in $(c^{(2)})^{-1}(\infty)$ of order 5. Formula (10.40) in theorem 10.6 gives near ∞

$$s[x\omega_0](t_1, 0) = (t_1^{-2/5} + \text{h.o.t.}) \cdot (\text{a flat multi-valued section}). \tag{10.53}$$

Therefore also the local monodromy of $\pi^{(3)}$ around points in $(c^{(2)})^{-1}(\infty)$ is trivial. This excludes $-\mathrm{id}$ in the second line of (10.52).

The case $Q_{2,0}$: The local transversal monodromies of the homology bundle on $T^{(1)}$ around 0, 1 and ∞ have on $Ml_{e^{-2\pi i/3}}$ eigenvalues in $\mathrm{Eiw}(e^{2\pi i/6})$. The branched covering $T^{(4)} \to T^{(2)}$ is at the elliptic fixed points of order 6. Thus $\pi^{(3)}$ restricts to the trivial monodromy on $Ml_{e^{-2\pi i/3}}$. This excludes $-M_h^6$ in the first line and $-\mathrm{id}$ in the second line of (10.52).

(c) $-\mathrm{id} \notin G_{\mathcal{R}}^{smar,gen}$ by theorem 8.8 (d). $G_{\mathcal{R}}^{smar,gen}$ fixes $BL(f, \pm\rho)$ for any $(f, \pm\rho) \in M_\mu^{mar}$. Because $T^{(7)} \to D_{BL}$ is an open embedding, $G_{\mathcal{R}}^{smar,gen}$ fixes D_{BL}. By part (b) $G_{\mathcal{R}}^{smar,gen} = \{\mathrm{id}\}$ for $U_{1,0}, E_{3,0}, Z_{1,0}$, and $G_{\mathcal{R}}^{smar,gen} = \{\mathrm{id}\}$ or $\{\mathrm{id}, M_h^{m_\infty}\}$ or $\{\mathrm{id}, -M_h^{m_\infty}\}$ for $W_{1,0}, S_{1,0}, Q_{2,0}$. The coordinate changes φ of the curve singularities $W_{1,0}$ and the surface singularities $S_{1,0}$ and $Q_{2,0}$ in the following table give a nontrivial element of $G_{\mathcal{R}}^{smar,gen}$.

$$\begin{array}{lll} W_{1,0} & S_{1,0} & Q_{2,0} \\ (x,y) \mapsto (-x,y) & (x,y,z) \mapsto (-x,y,z) & (x,y,z) \mapsto (x,y,-z) \end{array} \tag{10.54}$$

The coordinate change φ maps ω_0 to $-\omega_0$ and $s[\omega_0](t_1,0)$ to $-s[\omega_0](t_1,0)$. Therefore $(\varphi)_{hom}|_{Ml_\zeta} = -\mathrm{id}$ and $(\varphi)_{hom} = M_h^{m_\infty}$ (and not $-M_h^{m_\infty}$). This shows (10.46) for $W_{1,0}, S_{1,0}, Q_{2,0}$. □

Finally we come to the proof of theorem 10.1. Within this proof, we will also finish the proof of theorem 6.1. After it, we will finish the proof of theorem 10.3.

Proof of theorem 10.1: By theorem 10.7 (a)+(c), the transversal monodromy representation $\pi^{(7)}$ of the pull back to $T^{(7)}$ with $c^{(5)}$ of the homology bundle $\bigcup_{(t_1,t_2)\in T^{(5)}} Ml(f_{(t_1,t_2)}) \to T^{(5)}$ is trivial in the cases $W_{1,0}, U_{1,0}, E_{3,0}, Z_{1,0}$ and has image in $G_{\mathcal{R}}^{smar,gen} = \{\mathrm{id}, M_h^{m_\infty}\}$ in the cases $S_{1,0}$ and $Q_{2,0}$. Thus the strong marking $+\mathrm{id}$ on $f_{(i,0)}$ induces for each $f_{(t_1,t_2)}$ two strong markings in the same right equivalence class in the cases $S_{1,0}$ and $Q_{2,0}$ and one strong marking in the other cases. In any case, this gives a map $T^{(7)} \to (M_\mu^{smar})^0$.

The composition $T^{(7)} \to (M_\mu^{smar})^0 \to D_{BL}$ is an open embedding by theorem 10.6. Also recall that $(M_\mu^{smar})^0 \to D_{BL}$ is an immersion and that all three spaces are 2-dimensional manifolds.

Therefore $T^{(7)} \to (M_\mu^{smar})^0$ and $(M_\mu^{smar})^0 \to D_{BL}$ are open embeddings. We postpone the proof that the map $T^{(7)} \to (M_\mu^{smar})^0$ is an isomorphism.

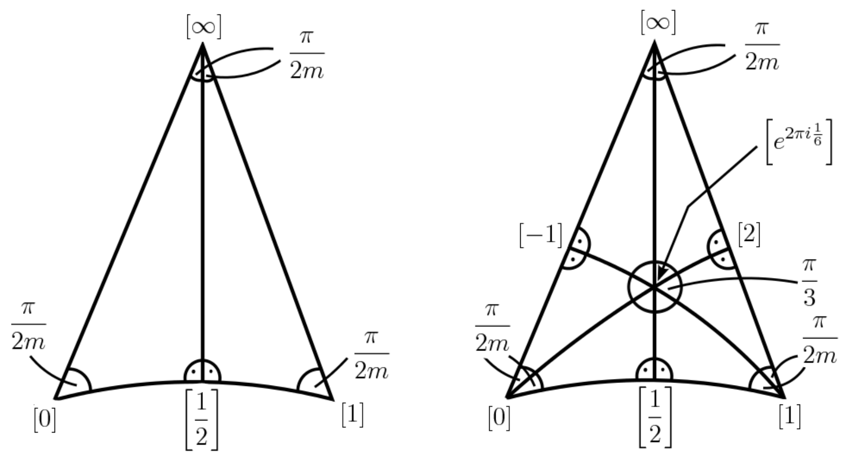
Part (b) follows now easily: Consider the case of singularities of multiplicity ≥ 3. $-\mathrm{id} \in G_\mathbb{Z}$ acts trivially on D_{BL}. It acts nontrivially on M_μ^{smar} by theorem 8.5 (c). The map

$$(M_\mu^{smar})^0 \to D_{BL}$$

is an embedding. Therefore $-\mathrm{id} \in G_\mathbb{Z}$ does not act on $(M_\mu^{smar})^0$. Therefore $-\mathrm{id} \notin G^{smar}$. This shows part (b). In this case $(M_\mu^{smar})^0 \cong (M_\mu^{mar})^0$ by theorem 8.5 (c).

In the case of singularities of multiplicity 2, $M_\mu^{smar} = M_\mu^{mar}$ and $(M_\mu^{smar})^0 = (M_\mu^{mar})^0$ hold anyway.

$c^{(2)} : T^{(4)} = \mathbb{H} \to T^{(2)} = \mathbb{P}^1\mathbb{C}$ is the branched covering from an action of a triangle group Γ of type $(\frac{1}{m_0}, \frac{1}{m_1}, \frac{1}{m_\infty})$ on $\mathbb{H}$. The group Γ is a normal subgroup of index 2 respectively 6 of a triangle group Γ^{qh} of type $(2, 2m, 2m)$ for $W_{1,0}$ and $S_{1,0}$ and of type $(2, 3, 2m)$ for $U_{1,0}, E_{3,0}, Z_{1,0}$ and $Q_{2,0}$ such that $\Gamma^{qh}/\Gamma = (G_2$ respectively $G_3)$. The following pictures show hyperbolic triangles associated to Γ and Γ^{qh}. The symbols $[0], [1], [\infty], [\frac{1}{2}], [2], [-1], [e^{2\pi i/6}]$ at special points indicate the images of these points under $c^{(2)}$.



The group Γ^{qh} maps the set of elliptic fixed points $(c^{(2)})^{-1}(\{0, 1, \infty\}) = T^{(4)} - T^{(3)}$ of Γ to itself, so it acts on $T^{(3)}$.

By the proved implication $\Leftarrow$ in (10.12) in theorem 10.3, the orbits of Γ^{qh} in $T^{(3)}$ are contained in the right equivalence classes of quasihomogeneous singularities. By the embedding

$$T^{(3)} \to \mathcal{H}(C^{\alpha_1})$$

in theorem 10.6, Γ^{qh} acts also on $\mathcal{H}(C^{\alpha_1})$, and the orbits are contained in the orbits of $\Psi(G^{mar})$, because the orbits of G^{mar} on $(M_\mu^{mar})^0$ are the right equivalence classes in $(M_\mu^{mar})^0$.

Now compare the actions of Γ^{qh} and $\Psi(G^{mar})$ on $\mathcal{H}(C^{\alpha_1})$. Γ^{qh} acts as a triangle group of type $(2, 2m, 2m)$ respectively $(2, 3, 2m)$, and $\Psi(G^{mar})$ acts by theorem 6.1 (b) as a subgroup of a triangle group of the same type. And the orbits of Γ^{qh} are contained in the orbits of $\Psi(G^{mar})$. Therefore the actions coincide, and $\Psi(G^{mar}) = \Psi(G_\mathbb{Z})$ is a triangle group of the claimed type in (6.7). This gives the surjectivity in theorem 6.1 and finishes the proof of theorem 6.1.

It also shows that G^{mar} acts on $T^{(3)}$. Because $T^{(3)}$ contains representatives of the right equivalence classes of all quasihomogeneous singularities in the given μ-homotopy family, the marked quasihomogeneous singularities in $(M_\mu^{mar})^0$ must all be in $T^{(3)}$. This proves that the open embedding $T^{(7)} \to (M_\mu^{mar})^0$ is an isomorphism.

Next we will prove $G_{\mathbb{Z}} = G^{mar}$. Consider an element $g_1 \in G_{\mathbb{Z}}$. Because of $\Psi(G^{mar}) = \Psi(G_{\mathbb{Z}})$, we can multiply it with an element $g_2 \in G^{mar}$ such that $g_3 = g_1 g_2$ satisfies $\Psi(g_3) = \mathrm{id}$. By formula (6.8) in theorem 6.1 $g_3 \in \{\pm M_h^k \,|\, k \in \mathbb{Z}\} \subset G^{mar}$. This proves $G_{\mathbb{Z}} = G^{mar}$.

Now $M_\mu^{mar} = (M_\mu^{mar})^0$ holds. Because $BL : (M_\mu^{mar})^0 \to D_{BL}$ is an embedding, $BL : M_\mu^{mar} \to D_{BL}$ is an embedding. This finishes the proof of theorem 10.1. $\square$

Proof of $\Rightarrow$ in (10.12) in theorem 10.3: $G_{\mathbb{Z}}$ acts as Γ^{qh} on $\mathcal{H}(C^{\alpha_1})$ and thus as G_2 respectively G_3 on $T^{(1)}$. This shows $\Rightarrow$ in (10.12) for the quasihomogeneous singularities.

An element $g \in G_{\mathbb{Z}}$ which acts trivially on $T^{(3)}$ is in $\{\pm M_h^k \,|\, k \in \mathbb{Z}\}$ and restricts to $\lambda \cdot \mathrm{id}$ on Ml_ζ for some $\lambda \in \mathrm{Eiw}(\zeta)$. Because of

$$g: \ \mathbb{C}\cdot(v_1+v_2) \mapsto \mathbb{C}(\lambda\cdot v_1 + \overline{\lambda}\cdot v_2) = \mathbb{C}\cdot(v_1 + \overline{\lambda}^2\cdot v_2)$$

it acts on the fibers of the projection $D_{BL} \to \mathcal{H}(C^{\alpha_1})$ by multiplication with $\overline{\lambda}^2$, and it acts in the same way on the fibers of the projection $T^{(7)} \to T^{(3)}$. But $(\overline{\lambda}^2)^{m_\infty} = 1$. This shows $\Rightarrow$ in (10.12) for all singularities. $\square$

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PICARD GROUPS FOR LINE BUNDLES WITH CONNECTIONS

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To the memory of Egbert Brieskorn

Abstract. We study analogues of the usual Picard group for complex manifolds or non-singular complex algebraic varieties but instead of line bundles we study line bundles with connections. We choose an approach which works for both cases. We identify obstructions for the existence of a connection, or a connection which is even integrable or regular (integrable), and point out where one should be careful when passing from the analytic to the algebraic case.

Introduction

It was Egbert Brieskorn who brought the authors together already in 1970 when he was professor at Göttingen. As a result of the first meeting a cooperation started which lasted over decades up to now, the main subject being theorems of Lefschetz type, we are therefore very grateful to him! In this context it was natural for us to turn to the Picard group. In the present paper we consider Picard groups of line bundles with a connection.

In order to be more precise, let X be a reduced complex analytic space. It is known that the isomorphism classes of line bundles on X define a group, called the analytic Picard group $Pic^{an}(X)$ of X.

Remember that one can pass from a line bundle to the invertible sheaf of its sections, after all we may work with invertible sheaves instead of line bundles because we have an equivalence of categories.

If X is a complex manifold, it is natural to consider line bundles on the space X with a connection or with an integrable connection. The isomorphism classes of these line bundles define groups that we shall denote by $Pic^{an}_c(X)$ for line bundles with a connection and $Pic^{an}_{ci}(X)$ for line bundles with an integrable connection.

We are going to compare these groups with the original Picard group $Pic^{an}(X)$ using certain exact sequences. In particular, these give obstructions for the existence of a connection resp. an integrable connection. As we will see these results are not really new (in the analytic case) but the important point is that we use an elementary approach which also goes over to the algebraic case without problems. It avoids hypercohomology (which is basic for Deligne cohomology) or the curvature of differentiable connections. But in order to make the results plausible we relate our approach to one which uses the well-known relation to Deligne cohomology.

An important special case is the one of compact Kähler manifolds. Here we show that we can avoid to go back to (p,q)-forms explicitly but we can argue with the abstract framework of Hodge theory. This has the advantage that we can easily pass afterwards to smooth complete algebraic varieties which might not be projective. We prove that in the compact Kähler (or complete algebraic) case every connection on a line bundle is automatically integrable - a fact which may be surprising before seeing the proof (which is easy).

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The essential point for us is to pass to the algebraic case. As already said our approach goes over easily. To work with hypercohomology, similar to Deligne cohomology, requires some care but we discuss how to argue then. Also, we deal with regular (integrable) connections and study different ways to describe the obstructions for their existence. After all we show that an algebraic line bundle admits a regular integrable connection if and only if its complex first Chern class vanishes - a result which does not follow from the Riemann-Hilbert correspondence!!

By the way, the theory of D-modules will not be considered here because it is only related to the integrable case

At the end we discuss some illustrative examples.

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1. Analytic Comparisons

1.1. $Pic^{an}(X)$ and $Pic_c^{an}(X)$

In this section let X be a complex manifold which is paracompact (e.g. Stein or compactifiable; the condition is not automatically fulfilled, see [4]). A connection on an invertible $\mathcal{O}_X$-module $\mathcal{L}$ is a $\mathbb{C}$-linear morphism $\nabla : \mathcal{L} \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{L}$ such that $\nabla(fs) = f\nabla(s) + df \otimes s$, see [5] I Déf. 2.4, p. 7.

If $\mathcal{L} = \mathcal{O}_X$, a connection is defined by a form $\omega \in H^0(X, \Omega^1_X)$: $\nabla(1) = \omega$, so $\nabla(f) = df + f\omega$. If $\mathcal{L}$ is trivial, s a nowhere vanishing section of $\mathcal{L}$ and $\omega \in H^0(X, \Omega^1_X)$, there is a uniquely defined connection ∇ on $\mathcal{L}$ such that $\nabla(s) = \omega \otimes s$: we say that it is defined by ω with respect to s. Two line bundles $(\mathcal{L}, \nabla)$, $(\mathcal{L}', \nabla')$ are called isomorphic if there is an isomorphism $\phi : \mathcal{L} \to \mathcal{L}'$ such that the diagram

$$\begin{array}{ccc} \mathcal{L} & \stackrel{\nabla}{\to} & \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{L} \\ \phi\downarrow & & id \otimes \phi \downarrow \\ \mathcal{L}' & \stackrel{\nabla'}{\to} & \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{L}' \end{array}$$

is commutative.

The isomorphism classes of invertible $\mathcal{O}_X$-modules with connection form a group $Pic_c^{an}(X)$.

We have an exact sequence of sheaves:

$$0 \to \mathbb{C}^*_X \to \mathcal{O}^*_X \to d\mathcal{O}_X \to 0$$

where $\mathbb{C}^*_X \to \mathcal{O}^*_X$ is given by the inclusion and $\mathcal{O}^*_X \to d\mathcal{O}_X$ is defined by $f \mapsto df/f$.

This latter morphism is surjective, because, if $\omega \in d\mathcal{O}_{X,x}$, there is $f \in \mathcal{O}_{X,x}$ such that $\omega = df$. Then $e^f \in \mathcal{O}^*_{X,x}$ has its image equal to ω. The rest of the sequence is exact because of Poincaré Lemma.

This exact sequence of sheaves gives an exact sequence of cohomology:

$$\ldots \to H^0(X, d\mathcal{O}_X) \to H^1(X, \mathbb{C}^*_X) \to H^1(X, \mathcal{O}^*_X) \to H^1(X, d\mathcal{O}_X) \to \ldots$$

Here we only use the mapping $H^1(X, \mathcal{O}^*_X) \to H^1(X, d\mathcal{O}_X)$ from the exact sequence (see also the proof of Theorem 2.2.22 of [3]).

Now we can prove the following exact sequence in an elementary way. We will see that it can also be obtained easily using hypercohomology (Deligne cohomology).

Theorem 1.1. *We have an exact sequence:*

$$H^0(X, \mathcal{O}^*_X) \to H^0(X, \Omega^1_X) \to Pic_c^{an}(X) \to Pic^{an}(X) \to H^1(X, \Omega^1_X)$$

Proof. The map $H^0(X, \mathcal{O}_X^*) \to H^0(X, \Omega_X^1)$ is defined by $g \mapsto \frac{dg}{g}$, $H^0(X, \Omega_X^1) \to Pic_c^{an}(X)$ by $\omega \mapsto (\mathcal{O}_X, \nabla(f) = df + f\omega)$.

The map $Pic^{an}(X) \to H^1(X, \Omega_X^1)$ is the composition of $H^1(X, \mathcal{O}_X^*) \to H^1(X, d\mathcal{O}_X)$ (see above) and the natural map from $H^1(X, d\mathcal{O}_X)$ to $H^1(X, \Omega_X^1)$, since $Pic^{an}(X) \simeq H^1(X, \mathcal{O}_X^*)$.

Now, notice that we have a group structure on $Pic_c^{an}(X)$. According to Deligne in [5] p. 8, consider the invertible sheaves (i.e. invertible $\mathcal{O}_X$-modules) $\mathcal{L}$ and $\mathcal{L}'$ defined by the (s_i) and (s'_i) on an open covering $\mathcal{U}$, with the connections ∇ and ∇' defined by (α_i) and (α'_i) on the open covering $\mathcal{U}$, then $\mathcal{L} \otimes \mathcal{L}'$ is invertible and defined by $(s_i \otimes s'_i)$, and the connection ∇_0 on this invertible sheaf is defined by $(\alpha_i + \alpha'_i)$.

(i) Now let us prove the exactness. First, the function $g \in H^0(X, \mathcal{O}_X^*)$ is mapped onto $\frac{dg}{g} \in H^0(X, \Omega_X^1)$, and this in turn to the element of $Pic_c^{an}(X)$ represented by $(\mathcal{O}_X, \nabla)$, where $\nabla(f) := df + f\frac{dg}{g}$. This is the inverse image of $(\mathcal{O}_X, d)$ under the isomorphism $\cdot g : \mathcal{O}_X \to \mathcal{O}_X$, so its class in $Pic_c^{an}(X)$ is trivial: we have a commutative diagram

$$\begin{array}{ccc} H^0(X, \mathcal{O}_X) & \overset{\nabla}{\to} & H^0(X, \Omega_X^1) = H^0(X, \Omega_X^1 \otimes \mathcal{O}_X) \\ \cdot g \downarrow & & \cdot g \downarrow \\ H^0(X, \mathcal{O}_X) & \overset{d}{\to} & H^0(X, \Omega_X^1) = H^0(X, \Omega_X^1 \otimes \mathcal{O}_X) \end{array}$$

Suppose now that $\omega \in H^0(X, \Omega_X^1)$ is mapped onto the trivial element of $Pic_c^{an}(X)$, which means that $(\mathcal{O}_X, d)$ is isomorphic to $(\mathcal{O}_X, f \mapsto df + \omega f)$. The isomorphism gives a mapping from $\mathcal{O}_X$ onto itself, which is of the form $\cdot g$ for some $g \in H^0(X, \mathcal{O}_X^*)$. Then, the image of $1 \in H^0(X, \mathcal{O}_X)$ is $\omega \in H^0(X, \Omega_X^1)$ and by the multiplication by g, it is dg. Therefore $\omega = \frac{dg}{g}$.

(ii) It is obvious that the composition of the two middle arrows gives the trivial mapping.

The kernel of the map $Pic_c^{an}(X) \to Pic^{an}(X)$ defined by $(\mathcal{L}, \nabla) \mapsto \mathcal{L}$ is given by the pairs $(\mathcal{O}_X, \nabla)$, so it coincides with the image of the morphism $H^0(X, \Omega_X^1) \to Pic_c^{an}(X)$ defined by $\omega \mapsto (\mathcal{O}_X, \nabla(f) = df + f\omega)$. So, the middle part of the sequence is exact.

(iii) Now let $\mathcal{L}$ be an invertible sheaf which is in the kernel of $Pic^{an}(X) \to H^1(X, \Omega_X^1)$. Let $\mathcal{U} = (U_i)_{i\in I}$ be a covering of X, such that $\mathcal{L}|U_i$ is isomorphic to $\mathcal{O}_X|U_i$ by a map $\mathcal{O}_X|U_i \to \mathcal{L}|U_i$ which corresponds to $1 \mapsto s_i$. Let g_{ij} be the complex analytic transition map defined on $U_{ij} = U_i \cap U_j$ from $\mathcal{L}|U_i$ to $\mathcal{L}|U_j$. We have $s_j = g_{ij}s_i$ on $U_i \cap U_j$.

Since $s_j = g_{ij}s_i = g_{ij}g_{ki}s_k = g_{kj}s_k$ on $U_i \cap U_j \cap U_k$, we have $g_{kj} = g_{ij}g_{ki}$ on $U_i \cap U_j \cap U_k$. The family (g_{ij}) defines a 2-cocycle of $H^1(X, \mathcal{O}_X^*)$, a fact which is well-known. Since

$$H^1(\mathcal{U}, \Omega_X^1) \subset H^1(X, \Omega_X^1)$$

cf. [9] Hilfssatz 12.4, p. 91, the image of $\mathcal{L}$ in $H^1(X, \Omega_X^1)$ being trivial, the 2-cocycle (dg_{ij}/g_{ij}) is trivial, i.e. a coboundary. Therefore there are differential forms ω_i and ω_j defined respectively on U_i and U_j, such that:

$$\frac{dg_{ij}}{g_{ij}} = \omega_j - \omega_i$$

on $U_i \cap U_j$.

Consider for each i the connection $\tilde{\nabla}_i$ on $\mathcal{O}_X|U_i$ defined by:

$$\tilde{\nabla}_i(f) = df + f\omega_i$$

This defines on $\mathcal{L}|U_i$ a connection:

$$\nabla_i(fs_i) = df \otimes s_i + f\omega_i \otimes s_i,$$

which gives for $f = 1$:

$$\nabla_i(s_i) = \omega_i \otimes s_i.$$

On $U_i \cap U_j$, we have $g_{ij}s_i = s_j$. Therefore, on $U_i \cap U_j$:

$$\nabla_i(fg_{ij}s_i) = d(fg_{ij}) \otimes s_i + fg_{ij}\omega_i \otimes s_i = g_{ij}df \otimes s_i + fdg_{ij} \otimes s_i + fg_{ij}\omega_i \otimes s_i,$$

which implies, with $f = 1$, on $U_i \cap U_j$:

$$\nabla_i(g_{ij}s_i) = dg_{ij} \otimes s_i + g_{ij}\omega_i \otimes s_i.$$

Therefore:

$$\nabla_i(s_j) = g_{ij}(\frac{dg_{ij}}{g_{ij}} + \omega_i) \otimes s_i = (\frac{dg_{ij}}{g_{ij}} + \omega_i) \otimes g_{ij}s_i = (\omega_j - \omega_i + \omega_i) \otimes s_j$$

which yields:

$$\nabla_i(s_j) = \nabla_j(s_j)$$

on $U_i \cap U_j$.

Therefore the $(\nabla_i)_{i\in I}$ define on $\mathcal{L}$ a connection ∇ and the class of the element $\mathcal{L}$ which lies in the kernel of the map $Pic^{an}(X) \to H^1(X, \Omega^1_X)$ is the image of the class of $(\mathcal{L}, \nabla)$.

It remains to prove that the image of $(\mathcal{L}, \nabla)$ in $H^1(X, \Omega^1_X)$ in the above sequence vanishes.

Let $(U_i)_{i\in I}$ be an open covering of X such that $\mathcal{L}|U_i$ is isomorphic to $\mathcal{O}_X|U_i$ by a map $s_i \mapsto 1$. We write $\nabla s_i = \omega_i \otimes s_i$. Let (g_{ij}) be the cocycle of transition functions such that $s_j = g_{ij}s_i$. Then (dg_{ij}/g_{ij}) is a cocycle which represents an element of $H^1(X, \Omega^1_X)$. Since:

$$\nabla(s_j) = \nabla(g_{ij} \otimes s_i) = dg_{ij} \otimes s_i + g_{ij}\omega_i \otimes s_i = \omega_j \otimes s_j = g_{ij}\omega_j \otimes s_i,$$

we obtain:

$$\frac{dg_{ij}}{g_{ij}} = \omega_j - \omega_i.$$

Therefore the class of the element given by the elements (dg_{ij}/g_{ij}) vanishes in $H^1(X, \Omega^1_X)$.

This shows that the above sequence is exact.

We shall give an interpretation of this exact sequence below.

Implicitly we have used:

Lemma 1.2. *Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module which is represented by a cocycle (g_{ij}) in $C^1(\mathcal{U}, \mathcal{O}^*_X)$. Then, a connection ∇ on $\mathcal{L}$ is represented by an element (ω_i) in $C^0(\mathcal{U}, \Omega^1_X)$ which is mapped by $\delta : C^0(\mathcal{U}, \Omega^1_X) \to C^1(\mathcal{U}, \Omega^1_X)$ onto $(\frac{dg_{ij}}{g_{ij}}) \in C^1(\mathcal{U}, \Omega^1_X)$.*

Note that $(d\omega_i) \in \mathcal{C}^0(\mathcal{U}, \Omega^2_X)$ is a cocycle, i.e. defines an element of $H^0(X, \Omega^2_X)$, which is the curvature of ∇, see below.

Particularly easy is the case of Stein manifolds. Then $H^1(X, \Omega^1_X) = 0$, because of Cartan's Theorem B, so from Theorem 1.1 we obtain:

Lemma 1.3. *Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module on a Stein manifold X. Then there is a complex analytic connection on $\mathcal{L}$.*

In the following subsection we shall show how our reasoning above is related to the literature ("Atiyah obstruction").

1.2. Atiyah obstruction.

Atiyah ([1] §2) has studied complex analytic connections on a holomorphic principal fibre bundle P. Whereas differentiable connections always exist there is an obstruction to the existence of a complex analytic one. In particular, there is an obstruction $b(E)$ to the existence of a complex analytic connection on the principal fibre bundle which corresponds to a holomorphic vector bundle E (see [1] p. 194). We call it the Atiyah obstruction. In the case of a line bundle L we have that $b(L) \in H^1(X, \Omega^1_X)$.

Here we use again invertible sheaves $\mathcal{L}$ instead of line bundles L. Then a complex analytic connection on L corresponds to a connection on the sheaf $\mathcal{L}$ of holomorphic sections of L.

Let us recall the definition of $b(\mathcal{L})$, see [1] p. 193. Let $D(\mathcal{L})$ be the locally free $\mathcal{O}_X$-module defined as follows:
as a $\mathbb{C}_X$-module, $D(\mathcal{L}) := \mathcal{L} \oplus (\Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{L})$, and the $\mathcal{O}_X$-module structure is given by:

$$f \cdot (s, \beta) := (fs, f\beta + df \otimes s),$$

if f is a section of $\mathcal{O}_X$, s a section of $\mathcal{L}$ and β is a section of $\Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{L}$.
Then we get an exact sequence of $\mathcal{O}_X$-modules

$$0 \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{L} \to D(\mathcal{L}) \to \mathcal{L} \to 0$$

where the second arrow is given by $\beta \mapsto (0, \beta)$ and the third one by $(s, \beta) \mapsto s$.
Applying $Hom(\mathcal{L}, \cdots)$ we obtain a long exact cohomology sequence

$$\ldots \to H^0(X, Hom(\mathcal{L}, D(\mathcal{L})) \to H^0(X, Hom(\mathcal{L}, \mathcal{L})) \to H^1(X, Hom(\mathcal{L}, \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{L})) \to \ldots$$

Now $b(\mathcal{L})$ is defined as the image of $1 \in H^0(X, \mathcal{O}_X)$ in $H^1(X, \Omega^1_X)$ under the mapping:

$$H^0(X, \mathcal{O}_X) \overset{\simeq}{\to} H^0(X, Hom(\mathcal{L}, \mathcal{L})) \to H^1(X, Hom(\mathcal{L}, \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{L})) \overset{\simeq}{\to} H^1(X, \Omega^1_X)$$

(so the mapping depends on $\mathcal{L}$!).

Lemma 1.4. $b(\mathcal{L}) = 0$ *if and only if* $\mathcal{L}$ *admits a connection.*

Proof: A splitting of the first exact sequence above is given by an $\mathcal{O}_X$-linear mapping of the form $s \mapsto (s, \nabla(s))$, such that ∇ is a connection on $\mathcal{L}$, and vice versa.

Look at the second exact sequence. The second arrow maps 1 onto $b(\mathcal{L})$, by definition of $b(\mathcal{L})$, with the identifications made in the definition. The inverse images of 1 with respect to the first arrow correspond to the splittings of the first exact sequence, i.e. to the connections on $\mathcal{L}$. This implies our statement.

Lemma 1.5. $b(\mathcal{L})$ *is the image of* $-[\mathcal{L}] \in H^1(X, \mathcal{O}^*_X)$ *in* $H^1(X, \Omega^1_X)$*, i.e.* $b(\mathcal{L})$ *is represented by the cocycle* $-(\frac{dg_{ij}}{g_{ij}})$*.*

Proof: Let $\mathcal{U} = (U_i)$ be an open Stein covering of X such that $\mathcal{L}|U_i$ is trivial. Let s_i be a nowhere vanishing section of $\mathcal{L}|U_i$. Then, $s_j = g_{ij}s_i$, where g_{ij} are the corresponding transition functions. Let ∇_i be the connection on $\mathcal{L}|U_i$ such that $\nabla_i(s_i) = 0$. Now, let us describe $H^0(\mathcal{U}, Hom(\mathcal{L}, \mathcal{L})) \to H^1(\mathcal{U}, Hom(\mathcal{L}, \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{L}))$ using the exact sequence of complexes:

$$0 \to C^{\cdot}(\mathcal{U}, Hom(\mathcal{L}, \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{L})) \to C^{\cdot}(\mathcal{U}, Hom(\mathcal{L}, D(\mathcal{L})) \to C^{\cdot}(\mathcal{U}, Hom(\mathcal{L}, \mathcal{L})) \to 0.$$

Consider $(\sigma_i) \in C^0(\mathcal{U}, Hom(\mathcal{L}, D(\mathcal{L}))$, where σ_i is the homomorphism $\mathcal{L}|U_i \to D(\mathcal{L})|U_i$ which maps s_i to $(s_i, 0)$ (note that $\nabla_i(s_i) = 0$), i.e. $s_j = g_{ij}s_i$ to $(s_j, \frac{dg_{ij}}{g_{ij}} \otimes s_j)$. Then (σ_i) is mapped to $(\tau_i) \in C^0(\mathcal{U}, Hom(\mathcal{L}, \mathcal{L}))$ with $\tau_i = id : \mathcal{L}|U_i \to \mathcal{L}|U_i$.

The coboundary of (σ_i) is given by $\sigma_j - \sigma_i : \mathcal{L}|U_i \cap U_j \to D(\mathcal{L})|U_i \cap U_j$:

$$(\sigma_j - \sigma_i)(s_j) = (0, -\frac{dg_{ij}}{g_{ij}} \otimes s_j),$$

so $\sigma_j - \sigma_i$ can be identified with $-\frac{dg_{ij}}{g_{ij}} \in H^0(U_i \cap U_j, \Omega^1_X)$.

Note that the relation established in the preceding lemma is taken up to sign as definition of the Atiyah class in [18] Def. 4.2.18.

Corollary 1.6. *An invertible sheaf $\mathcal{L}$ admits a connection if and only if its image in $H^1(X, \Omega^1_X)$ is* 0.

This corollary is consequence of Lemmas 1.4 and 1.5. This coincides with our result from Theorem 1.1.

1.3. $Pic^{an}_c(X)$ and $Pic^{an}_{ci}(X)$

Recall that a connection ∇ is integrable if its curvature vanishes.

When $\mathcal{L} = \mathcal{O}_X$ and $\nabla(f) = df + f\omega$, the value of the curvature R_∇ of the connection ∇ on $\mathcal{L}$ is $d\omega$ (see I 3.2.2 of [5], p. 23).

More generally, recall that a connection is given by a $\mathbb{C}$-linear morphism:

$$\nabla^1 : \mathcal{L} \to \Omega^1_X \otimes \mathcal{L} = \Omega^1_X(\mathcal{L})$$

It defines a $\mathbb{C}$-linear morphism:

$$\nabla^2 : \Omega^1_X(\mathcal{L}) \to \Omega^2_X(\mathcal{L})$$

by the formula: $\nabla^2(\omega \otimes s) = d\omega \otimes s - \omega \wedge \nabla(s)$ (see I (2.4) and (2.9) of [5]).

Definition 1.7. *The connection $\nabla = \nabla^1$ is said to be integrable if $\nabla^2 \circ \nabla^1 = 0$.*

In particular, if s is a global nowhere vanishing section of $\mathcal{L}$ and if ∇ is defined by ω with respect to s we have $R_\nabla(s') = d\omega \otimes s'$ for every section of $\mathcal{L}$. So ∇ is integrable if and only if $d\omega = 0$.

Obviously we have, similarly to Lemma 1.2:

Lemma 1.8. *Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module which is represented by a cocycle (g_{ij}) in $C^1(\mathcal{U}, \mathcal{O}^*_X)$. Then, an integrable connection ∇ on $\mathcal{L}$ is represented by an element (ω_i) in $C^0(\mathcal{U}, \Omega^1_X)$, ω_j closed, which is mapped by $\delta : C^0(\mathcal{U}, \Omega^1_X) \to C^1(\mathcal{U}, \Omega^1_X)$ onto $(\frac{dg_{ij}}{g_{ij}}) \in C^1(\mathcal{U}, \Omega^1_X)$.*

In particular the trivial connection d on $\mathcal{O}_X$ is integrable. As we did for the group $Pic^{an}_c(X)$, the isomorphism classes of analytic invertible sheaves with integrable connection form a group $Pic^{an}_{ci}(X)$ in which the neutral element is the class of $(\mathcal{O}_X, d)$ and the product of the classes of $(\mathcal{L}_1, \nabla_1)$ and of $(\mathcal{L}_2, \nabla_2)$ is the class of $(\mathcal{L}_1 \otimes \mathcal{L}_2, \nabla)$, where:

$$\nabla(s_1 \otimes s_2) = \nabla_1(s_1) \otimes s_2 + s_1 \otimes \nabla_2(s_2).$$

One can prove (see [5] using Théorème 2.17 Chap. I p. 12) that, if $(\mathcal{L}_1, \nabla_1)$ and $(\mathcal{L}_2, \nabla_2)$ are integrable connections, the connection:

$$(\mathcal{L}_1 \otimes \mathcal{L}_2, \nabla)$$

is also integrable. One can see this directly, too, using that the sum of closed forms is closed.

The curvature of a connection $(\mathcal{L}, \nabla)$ defines an $\mathcal{O}_X$-homomorphism:

$$\mathcal{L} \to \Omega^2_X \otimes \mathcal{L}$$

Now $\mathrm{Hom}(\mathcal{L}, \Omega_X^2 \otimes \mathcal{L}) \simeq H^0(X, Hom(\mathcal{L}, \Omega_X^2 \otimes \mathcal{L})) \simeq H^0(X, \Omega_X^2)$, so it is given by an element ω of $H^0(X, \Omega_X^2)$. If this cohomology group vanishes, we have $Pic_{ci}^{an}(X) \simeq Pic_c^{an}(X)$.

One can prove the following proposition also by Deligne cohomology, see below, but it is much easier to proceed directly.

Proposition 1.9. *Let X be a complex manifold. We have an exact sequence*

$$0 \to Pic_{ci}^{an}(X) \to Pic_c^{an}(X) \to H^0(X, \Omega_X^2).$$

Proof. Let $(\mathcal{L}, \nabla)$ be an integrable connection.

Assume this connection is isomorphic to the trivial connection $(\mathcal{O}_X, d)$, the class of the connection $(\mathcal{L}, \nabla)$ is therefore the class of the trivial connection. This means that the map $Pic_{ci}^{an}(X) \to Pic_c^{an}(X)$ is an injection.

The mapping $Pic_c^{an}(X) \to H^0(X, \Omega_X^2)$ associates the curvature of ∇ with the isomorphism class of $(\mathcal{L}, \nabla)$. It is well-defined: if $(\mathcal{L}, \nabla)$ and $(\mathcal{L}', \nabla')$ are isomorphic and if we take local sections of $\mathcal{L}$ and $\mathcal{L}'$ which correspond each other with respect to the isomorphism, the two connections are defined by the same differential forms with respect to these sections. The exactness at $Pic_c^{an}(X)$ is obvious.

In fact the following proposition shows that $Pic_{ci}^{an}(X)$ is of topological nature:

Proposition 1.10. *We have the isomorphism:*

$$Pic_{ci}^{an}(X) \simeq H^1(X, \mathbb{C}^*).$$

Proof. According to Théorème 2.17 in chapter I of [5] there is an equivalence of categories between the category of local systems of one-dimensional complex vector spaces on X with the category of line bundles with an integrable connection.

The resulting bijection is compatible with the group structure given by the tensor product.

We can observe that the group $H^1(X, \mathbb{C}^*)$ classifies the local systems of one dimensional complex vector spaces on X (see Theorem 3.3 of [23]), up to isomorphism, because the local transition functions are locally constant. The same is true for $Pic_{ci}^{an}(X)$ as mentioned at the beginning of this paragraph.

Corollary 1.11. *Let $f : X \to Y$ be a holomorphic map between two complex manifolds such that it induces an isomorphism $H_1(X, \mathbb{Z}) \to H_1(Y, \mathbb{Z})$, then:*

$$Pic_{ci}^{an}(X) \simeq Pic_{ci}^{an}(Y).$$

Proof: Note that

$$\mathrm{Ext}^1(H_0(X, \mathbb{Z}), \mathbb{C}^*) = 0,$$

because the abelian group $H_0(X, \mathbb{Z})$ is free, and the Universal coefficient formula implies

$$H^1(X, \mathbb{C}^*) \simeq Hom(H_1(X, \mathbb{Z}), \mathbb{C}^*).$$

So we get isomorphisms

$$Hom(H_1(X, \mathbb{Z}), \mathbb{C}^*) \simeq H^1(X, \mathbb{C}^*) \simeq Pic_{ci}^{an}(X)$$

1.4. **Relation to Deligne cohomology.** The preceding subsection is closely related to special cases of Deligne cohomology. We start by recalling the notion of Čech hypercohomology.

Let $\mathcal{S}^\cdot$ be a non-negative complex of sheaves of abelian groups on a topological space X. If $\mathcal{U}$ is an open covering of X we can define $\mathbb{H}^k(\mathcal{U}, \mathcal{S}^\cdot) := H^k(C^\cdot(\mathcal{U}, \mathcal{S}^\cdot)_{tot})$ where $(C^\cdot(\mathcal{U}, \mathcal{S}^\cdot))_{tot}$ is the total (or the simple) complex associated to the bi-graded complex $C^\cdot(\mathcal{U}, \mathcal{S}^\cdot)$ (see e.g. [3] p. 14, p. 28). Taking the direct limit with respect to open coverings $\mathcal{U}$, we get $\check{\mathbb{H}}^k(X, \mathcal{S}^\cdot) := \lim\limits_{\rightarrow} \mathbb{H}^k(\mathcal{U}, \mathcal{S}^\cdot)$, see [3] p. 32. We can proceed in a slightly different way, similarly to [11] II 5.8

p.223 in the case of sheaves : we consider only open coverings $\mathcal{U} = (U_x)_{x\in X}$ with $x \in U_x$, put $\check{C}^\cdot(X, \mathcal{S}^\cdot) := \lim\limits_{\rightarrow} C^\cdot(\mathcal{U}, \mathcal{S}^\cdot)$, then $\check{\mathbb{H}}^k(X, \mathcal{S}^\cdot) = H^k((\check{C}^\cdot(X, \mathcal{S}^\cdot))_{tot})$.

Now let X be as before and let $\mathcal{U} = (U_i)$ be an open covering of X. We assume that the U_i are Stein, which can be achieved by refinement. Let $Pic^{an}\,\mathcal{U}$ be the group of isomorphism classes of invertible $\mathcal{O}_X$-modules which are trivial on the U_i, and let $Pic_c^{an}\,\mathcal{U}$ be the group of isomorphism classes of such sheaves with connection. First, $Pic^{an}\,\mathcal{U} \simeq H^1(\mathcal{U}, \mathcal{O}_X^*)$.

Let $\mathcal{S}^\cdot$ be the non-negative complex:

$$\mathcal{O}_X^* \overset{g\mapsto \frac{dg}{g}}{\longrightarrow} \Omega_X^1 \to 0 \to \dots .$$

Then we have a description of $Pic_c^{an}\, X$ as a (Čech) hypercohomology group:

Lemma 1.12. *a)* $Pic_c^{an}\,\mathcal{U} \simeq \mathbb{H}^1(\mathcal{U}, \mathcal{S}^\cdot)$.
b) $Pic_c^{an}\, X \simeq \check{\mathbb{H}}^1(X, \mathcal{S}^\cdot) \simeq \mathbb{H}^1(X, \mathcal{S}^\cdot)$ (cf. [3] Theorem 2.2.20, p. 80).

Proof: a) Argue as in the proof of Lemma 1.2 (See 2.2).
b) Take the direct limit with respect to open Stein coverings $\mathcal{U}$. The second isomorphism holds because X is paracompact (see [3] Theorem 1.3.13, p. 32).

As a consequence, we obtain the exact sequence of Theorem 1.1 again:
We have an exact sequence of complexes:

$$0 \to C^{\cdot+1}(\mathcal{U}, \Omega_X^1) \to (C^\cdot(\mathcal{U}, \mathcal{S}^\cdot))_{tot} \to C^\cdot(\mathcal{U}, \mathcal{O}_X^*) \to 0$$

Note that $H^1(V, \Omega_X^1) = 0$ for $V = U_{i_0} \cap \ldots \cap U_{i_q}$ because V is Stein: recall that the intersection of two open Stein subets is Stein, see [19] Prop. 51.7, p. 225. So we have exactness on the right.

This exact sequence induces a long exact cohomology sequence

$$\ldots \to H^k(\mathcal{U}, \mathcal{O}_X^*) \to H^k(\mathcal{U}, \Omega_X^1) \to \mathbb{H}^{k+1}(\mathcal{U}, \mathcal{S}^\cdot) \to H^{k+1}(\mathcal{U}, \mathcal{O}_X^*) \to \ldots$$

After this take the direct limit and replace Čech (hyper)cohomology by the usual one.

In fact, using Proposition 2.2 below we have an easier proof.

Now let us turn to Deligne cohomology. Let us recall its definition (see [8] p. 45). Put $\mathbb{Z}(p) := (2\pi i)^p\mathbb{Z} \subset \mathbb{C}$. Let $\mathbb{Z}(p)_{\mathcal{D}}$ be the following non-negative complex:

$$\mathbb{Z}(p)_X \to \Omega_X^0 \to \ldots \to \Omega_X^{p-1} \to 0 \to \ldots$$

where the first arrow is the inclusion. Then the Deligne cohomology $H^*_{\mathcal{D}}(X, \mathbb{Z}(p))$ is defined as the hypercohomology $\mathbb{H}^*(X, \mathbb{Z}(p)_{\mathcal{D}})$.

Looking at the commutative diagram

$$\begin{array}{ccccccc}
\mathbb{Z}(p)_X & \to & \mathcal{O}_X & \to & \Omega_X^1 & \to \ldots \to & \Omega_X^{p-1} \\
\downarrow & & \downarrow & & \downarrow \cdot(2\pi i)^{-p+1} & & \downarrow \cdot(2\pi i)^{-p+1} \\
0 & \to & \mathcal{O}_X^* & \overset{f\mapsto \frac{df}{f}}{\to} & \Omega_X^1 & \to \ldots \to & \Omega_X^{p-1}
\end{array}$$

where the second verical arrow is given by $f \mapsto exp((2\pi i)^{-p+1} f)$ we see that the complex above is quasi-isomorphic to

$$0 \to \mathcal{O}_X^* \overset{f\mapsto \frac{df}{f}}{\to} \Omega_X^1 \to \ldots \to \Omega_X^{p-1} \to 0 \to \ldots$$

For $p = 1$, we obtain that $\mathbb{Z}(1)_{\mathcal{D}}$ is quasi-isomorphic to $\mathcal{O}_X^*(-1)$, cf. [2] p. 2038, so $H^1_{\mathcal{D}}(X, \mathbb{Z}(1)) \simeq H^0(X, \mathcal{O}_X^*)$ and $H^2_{\mathcal{D}}(X, \mathbb{Z}(1)) \simeq Pic^{an}(X)$.

For $p = 2$, we get that $\mathbb{Z}(2)_{\mathcal{D}}$ is quasi-isomorphic to $\mathcal{S}^\cdot(-1)$, cf. [8] p. 46, so $Pic_c^{an}(X) \simeq H^2_{\mathcal{D}}(X, \mathbb{Z}(2))$ because of Lemma 1.12 (see the remark of Deligne quoted in [2] at the bottom of p. 2039).

For $p \geq \dim X + 1$ the complex is quasi-isomorphic to $0 \to \mathcal{O}_X^* \to d\mathcal{O}_X \to 0 \to \ldots$, see beginning of subsection 1.1; by Poincaré Lemma, it is also quasi-isomorphic to

$$0 \to \mathbb{C}_X^* \to 0 \to \ldots.$$

So $H^2_{\mathcal{D}}(X, \mathbb{Z}(p)) \simeq H^1(X, \mathbb{C}_X^*) \simeq Pic^{an}_{ci}(X)$, using Proposition 1.10.

For $p > 2$, $H^2_{\mathcal{D}}(X, \mathbb{Z}(p))$ does not depend on p:

Let $\pi : \mathbb{Z}(p+1)_{\mathcal{D}} \to \mathbb{Z}(p)_{\mathcal{D}}$ be the projection, then $\mathbb{H}^k(X, ker\,\pi) \simeq H^{k-p-1}(X, \Omega_X^p) = 0$, $k \leq 3$.

We obtain altogether, cf. [10] p. 156:

Lemma 1.13. *a)* $H^2_{\mathcal{D}}(X, \mathbb{Z}(1)) \simeq Pic^{an}(X)$.
b) $H^2_{\mathcal{D}}(X, \mathbb{Z}(2)) \simeq Pic^{an}_c(X)$.
c) $H^2_{\mathcal{D}}(X, \mathbb{Z}(p)) \simeq Pic^{an}_{ci}(X)$ *for* $p > 2$.

1.5. $Pic^{an}(X)$ and $Pic^{an}_{ci}(X)$

The first exact sequence of §1.1 gives a long exact sequence which fits into a commutative diagram:

Theorem 1.14. *We have a commutative diagram with exact rows:*

$$\begin{array}{ccccccccccccc} 0 & \to & H^0(X, \mathbb{C}_X^*) & \to & H^0(X, \mathcal{O}_X^*) & \to & H^0(X, d\mathcal{O}_X) & \to & Pic^{an}_{ci}(X) & \to & Pic^{an}(X) & \to & H^1(X, d\mathcal{O}_X) \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \to & H^0(X, \mathbb{C}_X^*) & \to & H^0(X, \mathcal{O}_X^*) & \to & H^0(X, \Omega_X^1) & \to & Pic^{an}_c(X) & \to & Pic^{an}(X) & \to & H^1(X, \Omega_X^1) \end{array}$$

Proof. The exactness of the upper line is consequence of Proposition 1.10 and the exactness of the sequence $0 \to \mathbb{C}_X^* \to \mathcal{O}_X^* \to d\mathcal{O}_X \to 0$.

Since the vertical map $H^0(X, d\mathcal{O}_X) \to H^0(X, \Omega_X^1)$ is injective we conclude that

$$0 \to H^0(X, \mathbb{C}_X^*) \to H^0(X, \mathcal{O}_X^*) \to H^0(X, \Omega_X^1)$$

is exact. Because of Theorem 1.1 we conclude that the lower line is exact, too.

Remark: We may also argue using hypercohomology:

In the upper row compare $\mathcal{O}_X^* \to d\mathcal{O}_X \to 0$ with $\mathcal{O}_X^* \to 0$, in the lower row $\mathcal{O}_X^* \to \Omega_X^1 \to 0$ with $\mathcal{O}_X^* \to 0$.

In particular, we observe that:

Lemma 1.15. *If the complex manifold X is compact with an invertible $\mathcal{O}_X$-module $\mathcal{L}$ on X and if ∇_1 and ∇_2 are two connections on $\mathcal{L}$ such that $(\mathcal{L}, \nabla_1) \simeq (\mathcal{L}, \nabla_2)$, we must have $\nabla_1 = \nabla_2$.*

Proof. We have $(\nabla_1 - \nabla_2)(s) = \omega \otimes s$ where $\omega \in H^0(X, \Omega_X^1)$ is mapped to $0 \in Pic^{an}_c(X)$. So there is $g \in H^0(X, \mathcal{O}_X^*)$ such that $\omega = \frac{dg}{g}$. Since $H^0(X, \mathbb{C}^*) = H^0(X, \mathcal{O}_X^*)$ because global functions on X are locally constant on a compact space, we have that $\omega = 0$.

Now let us drop the compactness condition again.

Lemma 1.16. *a) An element $x \in H^2(X, \mathbb{Z})$ is sent onto 0 in $H^2(X, \mathbb{C})$ if and only if it is the first Chern class of an invertible $\mathcal{O}_X$-module which can be endowed with an integrable connection.*
b) If X is Stein, an invertible sheaf $\mathcal{L}$ admits an integrable complex analytic connection on X if and only if the complex first Chern class vanishes.

Proof. a) We have a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{Z} & \to & \mathbb{C} & \to & \mathbb{C}^* & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathbb{Z} & \to & \mathcal{O}_X & \to & \mathcal{O}_X^* & \to & 0 \end{array}$$

with exact rows. This leads to a commutative diagram:

$$\begin{array}{ccc} H^1(X,\mathbb{C}^*_X) & \to & H^2(X,\mathbb{Z}) \\ \downarrow & & \downarrow \\ H^1(X,\mathcal{O}^*_X) & \to & H^2(X,\mathbb{Z}) \end{array}$$

The lower arrow associates to each invertible sheaf its first Chern class, therefore the upper arrow associates to each invertible sheaf with an integrable connection the first Chern class of the invertible sheaf. Now consider the upper row of the first diagram. It leads to an exact sequence:

$$H^1(X,\mathbb{C}^*) \to H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{C}),$$

which gives our result.

b) Note that we have $H^1(X,\mathcal{O}^*_X) \simeq H^2(X,\mathbb{Z})$, too, because $H^k(X,\mathcal{O}_X) = 0$, $k = 1, 2$.

Remark: We can make Proposition 1.9 more precise: There is an exact sequence

$$0 \to Pic^{an}_{ci}(X) \to Pic^{an}_c(X) \to H^0(X, d\Omega^1_X) \to H^2(X,\mathbb{C}^*_X)$$

Compare the non-negative complexes $\mathcal{O}^*_X \to d\mathcal{O}_X \to 0$ and $\mathcal{O}^*_X \to \Omega^1_X \to 0$, see subsection 1.4. The cokernel is quasi-isomorphic to $0 \to \Omega^1_X/d\mathcal{O}_X \to 0$, i.e. to $0 \to d\Omega^1_X \to 0$.

1.6. **Compact Kähler manifolds.** In the case X is a compact Kähler manifold, we can apply Hodge Theory.

We prefer an approach which can be transferred later on to the case of smooth complete complex algebraic varieties which might not be Kähler:

We have $H^k(X;\mathbb{C}) \simeq \mathbb{H}^k(X,\Omega^\cdot_X)$, by Poincaré lemma.

Let us look at the Hodge filtration F on $\Omega^\cdot$: let $F^p\Omega^\cdot$ be the subcomplex

$$0 \to \ldots \to 0 \to \Omega^p_X \to \Omega^{p+1}_X \to \ldots$$

The corresponding spectral sequence degenerates at E_1, cf. [6] p. 28, so $\mathbb{H}^{k-p}(X,\Omega^{\geq p}) = \mathbb{H}^k(X, F^p\Omega^\cdot)$ can be considered as a subspace $F^pH^k(X;\mathbb{C})$ of $H^k(X;\mathbb{C})$.

Let $\overline{F^p}H^k(X;\mathbb{C})$ be the image of $F^pH^k(X;\mathbb{C})$ under conjugation in $H^k(X;\mathbb{C})$. Assume $p+q = k$. Then $H^{p,q}(X) := F^pH^k(X;\mathbb{C}) \cap \overline{F^q}H^k(X;\mathbb{C}) \simeq F^pH^k(X;\mathbb{C})/F^{p+1}H^k(X;\mathbb{C}) \simeq H^q(X,\Omega^p_X)$.

In particular, $H^{1,1}(X)$ is a subspace of $H^2(X;\mathbb{C})$ which is isomorphic to $H^1(X,\Omega^1_X)$.

Then the first part of the following Lemma is well-known:

Lemma 1.17. *Let X be a compact Kähler manifold, $\mathcal{L}$ an invertible sheaf on X.*

a) (see [12] Ch. 3.3, p. 417) *The complex first Chern class $c_1(\mathcal{L})_\mathbb{C}$ of $\mathcal{L}$ is in $H^{1,1}(X)$.*

b) (see [1] Prop. 12, p. 196) *With the identifications above, $b(\mathcal{L}) = -2\pi i c_1(\mathcal{L})_\mathbb{C}$.*

Proof. We have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{Z}_X & \to & \mathcal{O}_X & \overset{f\mapsto e^{2\pi i f}}{\to} & \mathcal{O}^*_X & \to & 0 \\ & & \downarrow \cdot 2\pi i & & \downarrow \cdot 2\pi i & & \downarrow & & \\ 0 & \to & \mathbb{C}_X & \to & \mathcal{O}_X & \to & d\mathcal{O}_X & \to & 0 \end{array}$$

We get a commutative diagram

$$\begin{array}{ccc} H^1(X,\mathcal{O}^*_X) & \to & H^2(X;\mathbb{Z}) \\ \downarrow & & \downarrow \cdot 2\pi i \\ H^1(X,d\mathcal{O}_X) & \to & H^2(X;\mathbb{C}) \\ \downarrow & & \\ H^1(X,\Omega^1_X) & & \end{array}$$

Note that $d\mathcal{O}_X$ is quasiisomorphic to $\Omega_X^{\geq 1}$, hence we may replace $H^1(X, d\mathcal{O}_X)$ by $F^1H^2(X;\mathbb{C})$. In particular, the middle horizontal arrow is injective.

a) Look at the images of (g_{ij}).

By [17] Theorem 4.3.1, p. 62, we have that the image in $H^2(X;\mathbb{C})$ is $2\pi i c_1(\mathcal{L})_{\mathbb{C}}$. The second commutative diagram shows that $2\pi i c_1(\mathcal{L})_{\mathbb{C}} \in F^1H^2(X;\mathbb{C})$. Since the first Chern class is real it is invariant under conjugation, so we obtain our statement.

b) By Lemma 1.5, the image of (g_{ij}) in $H^1(X,\Omega_X^1)$ is $-b(\mathcal{L})$. If we identify $H^{1,1}$ with $H^1(X,\Omega_X^1)$ we obtain our statement because of a).

Note that the proof of b) in [1] loc. cit. works only if dim $X = 1$ because it uses an exact sequence of the form

$$0 \to \mathbb{C}_X \to \mathcal{O}_X \to \Omega_X^1 \to 0$$

Now in the Kähler case we have a stronger result than Lemma 1.16:

Lemma 1.18. *Let X be a compact Kähler manifold, $\mathcal{L}$ an invertible sheaf on X. Then the following conditions are equivalent:*

a) $\mathcal{L}$ admits an integrable connection,

b) $\mathcal{L}$ admits a connection,

c) the first Chern class of $\mathcal{L}$ is a torsion element.

For b) $\Rightarrow$ c) cf. [3] Cor. 2.2.25.

Proof. That the first Chern class is a torsion element means that the complex first Chern class vanishes, because it is known that the cohomology group $H^2(X,\mathbb{Z})$ is finitely generated when X is compact, hence triangulable.

a) $\Leftrightarrow$ c): $\mathcal{L}$ admits an integrable connection if and only if the image of $\mathcal{L}$ in $H^1(X, d\mathcal{O}_X)$ vanishes, by Theorem 1.14.

The composition $Pic^{an}\, X \to H^1(X, d\mathcal{O}_X) \to H^2(X;\mathbb{C})$ is given by $[\mathcal{L}] \mapsto 2\pi i c_1(\mathcal{L})_{\mathbb{C}}$, see the proof of the preceding lemma.

To prove a) $\Leftrightarrow$ c) it is therefore sufficient to show that the mapping $H^1(X, d\mathcal{O}_X) \to H^2(X;\mathbb{C})$ is injective, which has been done in the preceding proof.

Now b) $\Leftrightarrow$ c), because we know that b) holds if and only if $b(\mathcal{L}) = 0$ by Lemma 1.4. The rest follows from the preceding lemma 1.17.

In the preceding Lemma 1.18, i.e. in the case of compact Kähler manifolds, we can sharpen the fact that a) $\Leftrightarrow$ b):

Theorem 1.19. *If X is a compact Kähler manifold, a connection on an invertible sheaf is integrable.*

Proof. We have another connection ∇' which is integrable, by Lemma 1.18. Then the difference of the connections is the multiplication by a form $\omega \in H^0(X,\Omega_X^1)$. By Hodge theory, the Hodge spectral sequence degenerates at E_1, so $d\omega = 0$. Hence the two connections have the same curvature, so the original connection must be integrable, too.

2. Algebraic case

2.1. Suppose now that X is a smooth complex algebraic variety. The underlying analytic space X^{an} is a paracompact complex manifold. One has an analogue of Theorem 1.14 but one has to be careful with the upper row because one has no longer a Poincaré lemma. In fact we have to replace the sheaf $d\mathcal{O}_X$ by the sheaf

$$^c\Omega^1_X := ker(d : \Omega^1_X \to \Omega^2_X)$$

of closed Pfaffian forms on X.

We will always use Zariski topology (even in the case of $H^0(X, \mathbb{C}^*_X)$ below) if we do not write X^{an}. However, $c_1(X) := c_1(X^{an})$.

We will see that the following theorem can be proved using an algebraic analogue of Deligne cohomology, too, i.e. using hypercohomology, but we can proceed in an elementary way:

Theorem 2.1. *Let X be a smooth complex algebraic variety. Then we have a commutative diagram with exact rows*

$$\begin{array}{ccccccccccc} 0 & \to & H^0(X,\mathbb{C}^*_X) & \to & H^0(X,\mathcal{O}^*_X) & \to & H^0(X,{}^c\Omega^1_X) & \to & Pic_{ci}(X) & \to & Pic(X) & \to & H^1(X,{}^c\Omega^1_X) \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \to & H^0(X,\mathbb{C}^*_X) & \to & H^0(X,\mathcal{O}^*_X) & \to & H^0(X,\Omega^1_X) & \to & Pic_c(X) & \to & Pic(X) & \to & H^1(X,\Omega^1_X) \end{array}$$

Proof. We can no longer use the exact sequence of the beginning of section 1.1. Therefore we must proceed in a different way.

Let us check first that the lower row is exact.

Note that the sequence of sheaves: $0 \to \mathbb{C}^*_X \to \mathcal{O}^*_X \to \Omega^1_X$ is exact, where $\mathcal{O}^*_X \to \Omega^1_X$ is defined by $h \mapsto \frac{dh}{h}$. In fact:

Suppose that $h \in \mathcal{O}^*_{X,x}$, where x is a closed point of X, $\frac{dh}{h} = 0$: Then $h^{an} \in \mathcal{O}^*_{X^{an},x}$ is mapped to $0 \in \Omega^1_{X^{an},x}$, so h^{an} is constant, which implies that h is constant.

Therefore the sequence:

$$0 \to H^0(X, \mathbb{C}^*_X) \to H^0(X, \mathcal{O}^*_X) \to H^0(X, \Omega^1_X)$$

is exact.

The rest goes as in the proof of Theorem 1.1.

The upper row is treated in an analogous way. Note that the connection ∇ on $\mathcal{O}_X$:

$$\nabla(f) = df + f\omega$$

is integrable if and only if ω is closed, because the curvature of ∇ is $d\omega$.

Note that $0 \to \mathbb{C}^*_X \to \mathcal{O}^*_X \to {}^c\Omega^1_X \to 0$ is in general not exact, in contrast to the analytic case: take $X = \mathbb{C}^*$, $\omega := \frac{dz}{z} \in {}^c\Omega^1_X$.

Proposition 1.9 has an algebraic counterpart:

Proposition 2.2. *Let X be a non-singular complex algebraic variety. We have an exact sequence*

$$0 \to Pic_{ci}(X) \to Pic_c(X) \to H^0(X, \Omega^2_X).$$

The proof is similar to the one of Proposition 1.9.

2.2. **Use of Čech hypercohomology.** Similarly as in the analytic case (see §1.4) we can observe that $Pic_c(X)$ is isomorphic to the first Čech hypercohomology group $\check{\mathbb{H}}^1(X, \mathcal{S}^\cdot)$ of the complex $\mathcal{S}^\cdot$:

$$\mathcal{O}^*_X \to \Omega^1_X \to 0 \to \dots$$

on X (but not, up to a shift, of the complex $\mathbb{Z}_X \to \mathcal{O}_X \to {}^c\Omega^1_X \to 0 \to \dots$).

For Čech hypercohomology, we refer to subsection (1.4).

More precisely:

Lemma 2.3. *If X is a non-singular complex variety, we have:*

$$Pic_c(X) \simeq \check{\mathbb{H}}^1(X, \mathcal{S}^\cdot) \simeq \mathbb{H}^1(X, \mathcal{S}^\cdot).$$

Proof: Let $\mathcal{U} = (U_i)$ be a covering of X by open Zariski subsets of X. An element of $\mathbb{H}^1(\mathcal{U}, \mathcal{S}^\cdot)$ is given by an element $((\omega_i), (g_{ij})) \in C^0(\mathcal{U}, \Omega^1_X) \oplus C^1(\mathcal{U}, \mathcal{O}^*_X)$ such that (g_{ij}) is a cocycle, i.e. $g_{ij} = g_{ik}g_{kj}$ on $U_i \cap U_j \cap U_k$, and $\omega_j - \omega_i = \frac{dg_{ij}}{g_{ij}}$ on $U_i \cap U_j$.

Assume now that $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module on X which is endowed with a connection ∇. There is a Zariski open covering $\mathcal{U}$ of X such that for each U_i we have a trivialization of $\mathcal{L}|U_i$. Then $\mathcal{L}$ is represented by some cocycle (g_{ij}) in $C^1(\mathcal{U}, \mathcal{O}^*_X)$, and $\nabla|U_i$ corresponds to a connection $g \mapsto dg + g\omega_i$ on $\mathcal{O}_{U_i}$. Then $\omega_j - \omega_i = \frac{dg_{ij}}{g_{ij}}$ on $U_i \cap U_j$, so we obtain an element of $\mathbb{H}^1(\mathcal{U}, \mathcal{S}^\cdot)$, hence of $\check{\mathbb{H}}^1(X, \mathcal{S}^\cdot)$.

On the other hand, an element of $\check{\mathbb{H}}^1(X, \mathcal{S}^\cdot)$ comes from an element of $\mathbb{H}^1(\mathcal{U}, \mathcal{S}^\cdot)$ which is represented by a cocycle (g_{ij}) and (ω_i) for a suitable open Zariski covering $\mathcal{U}$ of X. Then (g_{ij}) defines an invertible $\mathcal{O}_X$-module $\mathcal{L}$, and (ω_i) defines a connection on $\mathcal{L}$.

One verifies that one obtains well-defined mappings between $Pic_c(X)$ and $\check{\mathbb{H}}^1(X, \mathcal{S}^\cdot)$. We obtain $Pic_c(X) \simeq \check{\mathbb{H}}^1(X, \mathcal{S}^\cdot)$.

Now in the case of sheaves we have isomorphisms $\check{H}^k \to H^k$ for $k = 0, 1$, see [11] II 5.9 Corollaire, p. 227 (note that X is not paracompact and that we are not only dealing with coherent algebraic sheaves!). This result still holds in the case of hypercohomology, as shown in the following proposition. So our lemma is proved.

Proposition 2.4. *Let X be a topological space and $\mathcal{S}^\cdot$ a non-negative complex of sheaves of abelian groups on X. Then the homomorphism $\check{\mathbb{H}}^k(X, \mathcal{S}^\cdot) \to \mathbb{H}^k(X, \mathcal{S}^\cdot)$ is bijective for $k \leq 1$ and injective for $k = 2$.*

Proof: (i) First we may reduce to the case that $\mathcal{S}^\cdot$ is a bounded complex:
Choose $p > 0$. Let $\pi : \mathcal{S}^\cdot \to \mathcal{S}^{\leq p-1}$ be the canonical projection. Then the exact sequence $0 \to ker\, \pi \to \mathcal{S}^\cdot \to \mathcal{S}^{\leq p-1} \to 0$ of *presheaf*(!) complexes yields a short exact sequence of double complexes:

$$\check{C}^\cdot(X, ker\pi) \to \check{C}^\cdot(X), \mathcal{S}^\cdot) \to \check{C}^\cdot(X, \mathcal{S}^{\leq p-1}) \to 0$$

cf. [11] II Th. 5.8.1, p. 204, hence a long exact sequence

$$\check{\mathbb{H}}^{q-1}(X, \mathcal{S}^{\leq p-1}) \to \check{\mathbb{H}}^{q-p}(X, \mathcal{S}^{\geq p}) \to \check{\mathbb{H}}^q(X, \mathcal{S}^\cdot) \to \check{\mathbb{H}}^q(X, \mathcal{S}^{\leq p-1}) \to \check{\mathbb{H}}^{q-p+1}(X, \mathcal{S}^{\geq p})$$

Now put $p := 4$. Since $\check{\mathbb{H}}^q(X, \mathcal{S}^{\geq p}) = 0$ for $q < 0$ we obtain $\check{\mathbb{H}}^q(X, \mathcal{S}^\cdot) \simeq \check{\mathbb{H}}^q(X, \mathcal{S}^{\leq p-1})$ for $q \leq 2$. The same holds for $\mathbb{H}$ instead of $\check{\mathbb{H}}$.

(ii) So we may assume that $\mathcal{S}^\cdot$ is a bounded complex. Then we proceed by induction on the length of the complex, the case where the length is 0 being trivial.

Induction step: We may assume that $\mathcal{S}^0 \neq 0$. Putting $p = 1$ we obtain a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
\check{H}^{q-1}(X, \mathcal{S}^0) & \to & \check{\mathbb{H}}^{q-1}(X, \mathcal{S}^{\geq 1}) & \to & \check{\mathbb{H}}^q(X, \mathcal{S}^\cdot) & \to & \check{H}^q(X, \mathcal{S}^0) & \to & \check{\mathbb{H}}^q(X, \mathcal{S}^{\geq 1}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^{q-1}(X, \mathcal{S}^0) & \to & \mathbb{H}^{q-1}(X, \mathcal{S}^{\geq 1}) & \to & \mathbb{H}^q(X, \mathcal{S}^\cdot) & \to & H^q(X, \mathcal{S}^0) & \to & \mathbb{H}^q(X, \mathcal{S}^{\geq 1})
\end{array}$$

Using the fact that the case of a sheaf is established by [11] p. 227, see above, and the Five Lemma we obtain the induction step.

Remark: The proof of the preceding Theorem gives the following exact sequence:

$$\check{H}^0(X, \mathcal{O}^*_X) \to \check{H}^0(X, \Omega^1_X) \to \check{\mathbb{H}}^1(X, \mathcal{S}^\cdot) \to \check{H}^1(X, \mathcal{O}^*_X) \to \check{H}^1(X, \Omega^1_X)$$

This exact sequence can also be obtained as follows:

Look at the exact sequence of *presheaf* (!) complexes:

$$0 \to \Omega^1_X\{1\} \to \mathcal{S}^\cdot \to \mathcal{O}^*_X\{0\} \to 0$$

where, for any presheaf $\mathcal{T}$, the complex $\mathcal{T}\{k\}$ denotes the complex $\mathcal{T}^\cdot$ with $\mathcal{T}^l = \mathcal{T}$ for $l = k$ and $= 0$ otherwise.

This gives the long exact Čech cohomology sequence in question.

We can proceed in the same way to prove the exactness of the upper line of the diagram of Theorem 2.1 by replacing Ω^1_X by ${}^c\Omega^1_X$. See Remark after Theorem 1.14.

We have special cases:

Lemma 2.5. *Let X be complete, $\mathcal{L}$ an invertible $\mathcal{O}_X$-module on X.*

a) $Pic(X) \simeq Pic^{an}(X^{an})$, similarly for Pic_c, Pic_{ci}.

b) $\mathcal{L}$ admits an integrable connection if and only if $c_1(\mathcal{L})$ is a torsion element.

c) Every connection on $\mathcal{L}$ is integrable, so $Pic_c(X) \simeq Pic_{ci}(X)$.

Proof: a) This follows from GAGA (see [22] and also [20] p. 152/153) if X is projective. In general, use [13] Théorème 4.4 instead of [22].

Instead of [20] we can also compare 2.1 and 2.2 with the corresponding analytic statements. b), c): If X is projective we know that X^{an} is compact Kähler, so the results follows by GAGA and Lemma 1.18, Theorem 1.19.

In general we know by [7] §5 that we can still apply Hodge theory to X, so Lemma 1.18 and Theorem 1.19 still hold. In fact, the Hodge filtration is still defined via $\Omega^\cdot_X$.

For part b) of the lemma it will turn out that the hypothesis that X is complete is unnecessary, see Corollary 2.11 below. For c) we must in general restrict to regular connections, see below (Theorem 2.13).

Remember that compact Kähler manifolds are not automatically algebraic, cf. the case of complex tori, see [21] Cor. p. 35.

Lemma 2.6. *Let X be affine. Then every invertible $\mathcal{O}_X$-module on X admits a connection.*

Proof: Obvious from Theorem 2.1, because $H^1(X, \Omega^1_X) = 0$.

2.3. **Regularity.** It is useful to take the notion of regularity into account.

The regularity has been introduced by P.Deligne in [5] Chap II §4. For the sake of convenience we define here the regularity of integrable connections on an invertible sheaf:

Definition 2.7. *Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module and ∇ an integrable connection on $\mathcal{L}$. Then ∇ is called regular if there exists a smooth compactification $\bar{X}$ of X such that $D := \bar{X} \setminus X$ is a divisor with normal crossings and that, for all $x \in D$, there exists an open Zariski neighbourhood V of x and there exists $s \in H^0(V, j_*\mathcal{L})$, s nowhere vanishing on $V' := V \setminus D$, such that $\nabla(s|V') = (\alpha|V') \otimes (s|V')$ with $\alpha \in H^0(V, \Omega^1_{\bar{X}}(log\, D))$. Here $j : X \to \bar{X}$ is the inclusion.*

Note that we can replace:

"there exists $s \in H^0(V, j_*\mathcal{L}), s$ nowhere vanishing on $V' = V \setminus D$, such that $\nabla(s|V') = (\alpha|V') \otimes s|V'$"

by

"for any $s \in H^0(V, j_*\mathcal{L}), s$ nowhere vanishing on $V' = V \setminus D$, we have $\nabla(s|V') = (\alpha|V') \otimes s|V'$".

Here it is important that we deal with invertible sheaves!

In fact, let $s, s' \in H^0(V, j_*\mathcal{L})$, s, s' nowhere vanishing on $V' = V \setminus D$. Then $s' = hs$, where h is a rational function on V which has neither zeroes nor poles inside V'. If $\nabla(s|V') = (\alpha|V') \otimes (s|V')$ with $\alpha \in H^0(V, \Omega^1_{\bar{X}}(log\, D))$ we get $\nabla(s'|V') = (\alpha'|V') \otimes (s'|V')$ with $\alpha' = \frac{dh}{h} + \alpha \in H^0(V, \Omega^1_{\bar{X}}(log\, D))$.

As P. Deligne noticed, the notion of regularity does not depend on the compactification of X such that the divisor at ∞ is a normal crossing divisor (see [5] p. 90).

We can define the Picard group $Pic_{cir}X$ of regular integrable connections in an obvious way. Now let us fix a compactification $\bar{X}$ of X as in the preceding definition.

Lemma 2.8. *There is an exact sequence:*

$$H^0(X, \mathcal{O}_X^*) \to H^0(\bar{X}, {}^c\Omega^1_{\bar{X}}(log\, D)) \to Pic_{cir}(X) \to Pic(X) \to H^1(\bar{X}, {}^c\Omega^1_{\bar{X}}(log\, D))$$

Proof: The proof is analogous to the proof of Theorem 1.1.

But first observe that, for any invertible $\mathcal{O}_X$-module $\mathcal{L}$, there is a Zariski open covering $\mathcal{U} = (\bar{U}_i)$ of $\bar{X}$ such that the restriction of $\mathcal{L}$ to $U_i = \bar{U}_i \setminus D$ is trivial.

For this, we may assume that X is connected, hence irreducible. One considers a non-empty and therefore dense Zariski open subspace U of X on which $\mathcal{L}$ is trivial. On U, the restriction $\mathcal{L}|U$ has a nowhere vanishing section s. This section extends to $\bar{X}$ as a rational section s_1 of $\mathcal{L}$. Let D_1 be the divisor of this section - this makes sense because $\mathcal{L}$ is locally trivial. Now D_1 extends to a divisor $\bar{D}_1$ on $\bar{X}$. For any $x \in \bar{X}$ there is an open affine neighbourhood $\bar{V}$ such that $\bar{D}_1|\bar{V}$ is a principal divisor, i.e. divisor of some rational function ϕ_x. Then $\phi_x^{-1}s_1$ is a nowhere vanishing section of $\mathcal{L}|V$ with $V := \bar{V} \setminus D$; it gives a trivialization of $\mathcal{L}|V$.

The first arrow is induced by the homomorphism $j_*\mathcal{O}_X^* \to {}^c\Omega^1_{\bar{X}}(log\, D)$ which is defined as follows. Locally, a section g of $j_*\mathcal{O}_X^*$ is of the form $h^{-1}\tilde{g}$, where $h, \tilde{g}$ are regular functions which do not vanish inside X. Then the image is defined to be $\frac{dg}{g} = \frac{d\tilde{g}}{\tilde{g}} - \frac{dh}{h}$ which is indeed a closed logarithmic form.

Assume now that $g \in H^0(X, \mathcal{O}_X^*)$ is given. Then the image in $Pic_{cir}(X)$ is given by $\mathcal{O}_X$, together with the connection $f \mapsto df + f\frac{dg}{g}$. This is isomorphic to $\mathcal{O}_X$, together with the connection $f \mapsto df$, so we have the trivial element of $Pic_{cir}(X)$.

On the other hand, suppose that $\omega \in H^0(\bar{X}, {}^c\Omega^1_{\bar{X}}(log\, D))$ is mapped onto the trivial element of $Pic_{cir}(X)$. Then there is a $g \in H^0(X, \mathcal{O}_X^*)$ such that $\omega = \frac{dg}{g}$.

This shows the exactness at $H^0(\bar{X}, {}^c\Omega^1_{\bar{X}}(log\, D))$.

Then, an element of $Pic\, X$ is represented by a cocycle (g_{ij}) on a covering $\mathcal{U}$ as defined before. This covering comes from an affine covering $\bar{\mathcal{U}}$ of $\bar{X}$, where each g_{ij} extends as a rational function with poles inside D which is a regular and non-vanishing function on $\bar{U}_i \cap \bar{U}_j \setminus D$. Then $\frac{dg_{ij}}{g_{ij}}$ is a closed logarithmic form on $\bar{U}_i \cap \bar{U}_j$: After refining $\mathcal{U}$ if necessary we may assume that we can write $g_{ij} = h_{ij}^{-1}\tilde{g}_{ij}$ where h_{ij} and $\tilde{g}_{ij}$ are regular on $\bar{U}_i \cap \bar{U}_j$ and without zeroes in $U_i \cap U_j$. Then:

$$\frac{dg_{ij}}{g_{ij}} = \frac{d\tilde{g}_{ij}}{\tilde{g}_{ij}} - \frac{dh_{ij}}{h_{ij}}$$

is a closed logarithmic form. This defines the map:

$$Pic\, X \to H^1(\bar{X}, {}^c\Omega^1_{\bar{X}}(log\, D)).$$

On the other hand, a regular integrable connection on $\mathcal{O}_X$ is of the form $g \mapsto dg + g\omega$ with $\omega \in H^0(\bar{X}, {}^c\Omega^1_{\bar{X}}(log\, D))$, and the map from $H^0(\bar{X}, {}^c\Omega^1_{\bar{X}}(log\, D))$ into $Pic_{cir}(X)$ is given by:

$$\omega \mapsto (\mathcal{O}_X, \nabla)$$

where $\nabla(g) = dg + g\omega$. Then, the composition:

$$H^0(\bar{X}, {}^c\Omega^1_{\bar{X}}(log\, D)) \to Pic_{cir}(X) \to Pic\, X$$

is zero. Let $(\mathcal{L}, \nabla)$ a regular integrable connection on the invertible $\mathcal{O}_X$-module $\mathcal{L}$ where $\mathcal{L}$ is isomorphic to $\mathcal{O}_X$. The pair $(\mathcal{L}, \nabla)$ is isomorphic to $(\mathcal{O}_X, \nabla_0)$ for some connection ∇_0, and there is a closed logarithmic form $\omega \in H^0(\bar{X}, {}^c\Omega^1_{\bar{X}}(log\, D))$, such that $\nabla_0(g) = dg + g\omega$. This proves the exactness of the sequence at $Pic_{cir}(X)$.

Now fix an element of $Pic\, X$ whose image in $H^1(\bar{X}, {}^c\Omega^1_{\bar{X}}(log\, D))$ is trivial. Such an element is given by an affine covering $\mathcal{U}$ and a cocycle

$$(\frac{dg_{ij}}{g_{ij}})$$

such that:

$$\frac{dg_{ij}}{g_{ij}} = \omega_j - \omega_i$$

where ω_i is a closed form in ${}^c\Omega^1_{\bar{X}}(log\, D)$ over the Zariski open subset U_i of X.

As we did in the proof of Theorem 1.1, the element (ω_i) defines a regular integrable connection ∇ on an invertible $\mathcal{O}_X$-module $\mathcal{L}$ and the image of the isomorphism class of $\mathcal{L}$ is the element of $H^1(\bar{X}, {}^c\Omega^1_{\bar{X}}(log\, D))$ given by the cocycle $(\frac{dg_{ij}}{g_{ij}})$.

It remains to prove that the composition:

$$Pic_{cir}(X) \to Pic\, X \to H^1(\bar{X}, {}^c\Omega^1_{\bar{X}}(log\, D))$$

is zero. As in the proof of Theorem 1.1, an element of $Pic_{cir}(X)$ is given by $(\mathcal{L}|U_i, \nabla|U_i)_i$ such that $(\mathcal{L}|U_i, \nabla|U_i)$ is isomorphic over the Zariski open subspace U_i to $(\mathcal{O}_{U_i}, \tilde{\nabla}_i)$ where:

$$\tilde{\nabla}_i(f) = df + \omega_i f$$

for some $\omega_i \in H^0(U_i, {}^c\Omega^1_{\bar{X}}(log\, D))$, and, if the element (g_{ij}) is the cocycle which defines $\mathcal{L}$, we have:

$$\frac{dg_{ij}}{g_{ij}} = \omega_j - \omega_i.$$

Since the forms ω_i are closed, reasoning as in the proof of Theorem 1.1, we obtain our assertion.

Remarks. 1. In fact, at the beginning we have shown that $j_*\mathcal{L}$ is an invertible $j_*\mathcal{O}_X$-module, $j : X \to \bar{X}$ being the inclusion.
2. Again we can prove the lemma by showing that $Pic_{cir}X \simeq \check{\mathbb{H}}^1(\bar{X}, \mathcal{T}^\cdot) \simeq \mathbb{H}^1(\bar{X}, \mathcal{T}^\cdot)$, where $\mathcal{T}^\cdot$ is the non-negative complex

$$j_*\mathcal{O}^*_X \overset{g \mapsto \frac{dg}{g}}{\longrightarrow} {}^c\Omega^1_{\bar{X}}(log\, D) \longrightarrow 0 \longrightarrow \dots$$

with $j : X \to \bar{X}$ being the inclusion.

In this context it is useful to have:

Lemma 2.9. *$Pic\, X \simeq H^1(\bar{X}, j_*\mathcal{O}^*_X)$.*

Proof: It is sufficient to show that $R^1 j_*\mathcal{O}^*_X = 0$. An element of $(R^1 j_*\mathcal{O}^*_X)_x$ is represented by an element of $H^1(U \cap X, \mathcal{O}^*_X)$, where U is an open neighbourhood of x, so by a line bundle $\mathcal{L}$ on $U \cap X$. After shrinking U if necessary we know that $\mathcal{L}$ is trivial, by the proof of Lemma 2.8. This implies our assertion.

Theorem 2.10. *Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module on X. Then $\mathcal{L}$ admits a regular integrable connection if and only if its first Chern class is a torsion element.*

Proof: Since the integral cohomology of X is an abelian group of finite type, the implication $\Rightarrow$ is proved by Lemma 1.16.

Now, consider the implication $\Leftarrow$.

Suppose first that $c_1(\mathcal{L}) = 0$.

Let $\bar{X}$ be a smooth compactification of X such that $D := \bar{X} \setminus X$ is a normal crossing divisor. Suppose that D has r irreducible components. Then $\mathcal{L}$ extends to an algebraic invertible sheaf $\mathcal{L}'$ on $\bar{X}$ with first Chern class $c_1(\mathcal{L}') = 0$.

To prove this, we consider the diagram with exact rows:

$$\begin{array}{ccccccc} \mathbb{Z}^r & \to & Pic\,\bar{X} & \to & Pic\,X & \to & 0 \\ \downarrow\simeq & & \downarrow c_1 & & \downarrow c_1 & & \\ H^2(\bar{X}^{an}, X^{an}; \mathbb{Z}) & \to & H^2(\bar{X}^{an}; \mathbb{Z}) & \to & H^2(X^{an}; \mathbb{Z}) & & \end{array}$$

Let $[\mathcal{L}]$ be the class of $\mathcal{L}$. We have assumed that its first Chern class is $c_1(\mathcal{L}) = 0$. Let $\mathcal{L}_1$ be a invertible $\mathcal{O}_{\bar{X}}$-module whose class has its image equal to $[\mathcal{L}]$. The first Chern class of $\mathcal{L}_1$ comes from an element of $H^2(\bar{X}^{an}, X^{an}; \mathbb{Z})$ which corresponds to an element of $\mathbb{Z}^r$ whose image in $Pic\,\bar{X}$ is $\mathcal{L}_2$ which has the same first Chern class as $\mathcal{L}_1$. The invertible sheaf $\mathcal{L}' := \mathcal{L}_1 \otimes \mathcal{L}_2^{-1}$ has a first Chern class equal to 0 and it extends $\mathcal{L}$.

On the complete non-singular variety $\bar{X}$ we have obtained an invertible sheaf $\mathcal{L}'$ which extends $\mathcal{L}$ and has first Chern class $c_1(\mathcal{L}') = 0$. By Lemma 2.5 the invertible sheaf $\mathcal{L}'$ is endowed with a integral connection ∇'. The restriction of ∇' to $\mathcal{L}$ is a regular integral connection.

If $c_1(\mathcal{L}) = c$, c being a torsion element, by Lemma 1.16 there is an analytic invertible sheaf $\mathcal{L}'$ with integrable connection on X^{an} having c as first Chern class. By Deligne's existence theorem (Théorème 5.9 Chap. II of [5] p. 97) we can find an invertible sheaf $\mathcal{L}_1$ on X with an integrable connection such that $\mathcal{L}_1^{an} = \mathcal{L}'$, so $c_1(\mathcal{L}_1) = c$. Now $c_1(\mathcal{L} \otimes (\mathcal{L}_1)^{-1}) = 0$, so by the preceding result there is a regular integrable connection on $\mathcal{L} \otimes (\mathcal{L}_1)^{-1}$. So we get a regular integrable connection on $\mathcal{L} = \mathcal{L}_1 \otimes (\mathcal{L}' \otimes (\mathcal{L}_1)^{-1})$, too.

Therefore if the first Chern class of $\mathcal{L}$ is a torsion element, the invertible sheaf $\mathcal{L}$ admits a regular integrable connection.

Corollary 2.11. *Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Then the following conditions are equivalent:*

1. *$\mathcal{L}$ admits a regular integrable connection;*
2. *$\mathcal{L}$ admits an integrable connection;*
3. *$\mathcal{L}^{an}$ admits an analytic integrable connection;*
4. *the first Chern class $c_1(\mathcal{L})$ of $\mathcal{L}$ is a torsion element.*

2.4. **Remark on integrability and regularity.** One may define a notion of regularity for connections which does not suppose that the connection is integrable - at least in the case of invertible sheaves.

This may seem to be superfluous because we will see that such a connection is automatically integrable. The situation changes, however, if we generalize the notions of regularity by asking regularity with respect to a partial compactification only.

Definition 2.12. *Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module and ∇ a connection on $\mathcal{L}$. Then ∇ is called regular if there exists a smooth compactification $\bar{X}$ of X such that $D := \bar{X} \setminus X$ is a divisor with normal crossings and that, for all $x \in D$ there exists an affine neighbourhood V of x and there exists $s \in H^0(V, j_*\mathcal{L})$ which does not vanish on D, such that $\nabla(s|V) = (\alpha|V) \otimes s|V$ with $\alpha \in H^0(V, \Omega^1_{\bar{X}}(log\,D))$. Here $j : X \to \bar{X}$ is the inclusion.*

As in the definition of a regular integrable connection we may again replace "there exists s... such that..." by "for all s... we have...".

The independence of the compactification will follow from the next theorem.

We can define the group $Pic_{cr}X$ of isomorphism classes of invertible $\mathcal{O}_X$-modules with a regular connection in an obvious way.

In fact, such a regular connection is automatically integrable, because we have:

Theorem 2.13. *If $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module, every regular connection on $\mathcal{L}$ is integrable.*

Proof: We proceed as in the proof of Theorem 1.19.

First we show that the mapping:

$$Pic_{cir}X \to Pic_{cr}X$$

is surjective. In fact, we have the following Lemma:

Lemma 2.14. *There is a commutative diagram with exact rows*

$$\begin{array}{ccccccccc} H^0(X,\mathcal{O}_X^*) & \to & H^0(\bar{X},{}^c\Omega^1_{\bar{X}}(log\,D)) & \to & Pic_{cir}(X) & \to & Pic(X) & \to & H^1(\bar{X},{}^c\Omega^1_{\bar{X}}(log\,D)) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^0(X,\mathcal{O}_X^*) & \to & H^0(\bar{X},\Omega^1_{\bar{X}}(log\,D)) & \to & Pic_{cr}(X) & \to & Pic(X) & \to & H^1(\bar{X},\Omega^1_{\bar{X}}(log\,D)) \end{array}$$

Proof: As in Lemma 2.8, the proof is analogous to the proof of Theorem 1.1.

The upper line is exact, as we saw in Lemma 2.8. Concerning the lower row, we define the map $Pic\,X \to H^1(\bar{X},\Omega^1_{\bar{X}}(log\,D))$ as the composition $Pic\,X \to H^1(\bar{X},{}^c\Omega^1_{\bar{X}}(log\,D)) \to H^1(\bar{X},\Omega^1_{\bar{X}}(log\,D))$.

The map $H^0(\bar{X},\Omega^1_{\bar{X}}(log\,D)) \to Pic_{cr}(X)$ is given by:

$$\omega \mapsto (\mathcal{O}_X,\nabla)$$

where the connection ∇ is defined by $\nabla(f) = df + f\omega$. This defines a connection on $\mathcal{O}_X$ which is regular since $\omega \in H^0(\bar{X},\Omega^1_{\bar{X}}(log\,D))$. Therefore, the composition:

$$H^0(\bar{X},\Omega^1_{\bar{X}}(log\,D)) \to Pic_{cr}(X) \to Pic\,X$$

is zero.

Let $(\mathcal{L},\nabla)$ be an invertible sheaf with a regular connection whose image is zero in $Pic(X)$. Then $\mathcal{L}$ is isomorphic to the trivial invertible sheaf $\mathcal{O}_X$ and there is a connection ∇_0 on $\mathcal{O}_X$ such that $(\mathcal{L},\nabla)$ is isomorphic to $(\mathcal{O}_X,\nabla_0)$. So ∇_0 is a regular connection. On the other hand there is a global form ω on X, such that $\nabla_0(f) = df + f\omega$. If ∇_0 is regular, one can choose the form ω as a global rational form on $\bar{X}$ in $H^0(\bar{X},\Omega^1_{\bar{X}}(log\,D))$. Then the lower row is exact at $Pic_{cr}(X)$.

Now, let us check the exactness at $Pic(X)$. Let $\bar{\mathcal{U}} = \bar{U}_i$ be an affine covering of $\bar{X}$ as in the proof of Lemma 2.8, such that (U_i) is a covering of X and $(\mathcal{L}|U_i,\nabla|U_i = \nabla_i)$ is isomorphic to $(\mathcal{O}_X|U_i,\tilde{\nabla}_i)$, where:

$$\tilde{\nabla}_i(f) = df + f\omega_i$$

with a rational differential form ω_i defined on $\bar{U}_i$ with poles contained in D. On this covering (U_i) of X, the invertible sheaf $\mathcal{L}$ defines the cocycle (g_{ij}) and its image in $H^1(\bar{X},\Omega^1_{\bar{X}}(log\,D))$ is the cocycle $\frac{d\hat{g}_{ij}}{\hat{g}_{ij}}$ defined by the rational functions on the covering $(\bar{U}_i)$ which extend (g_{ij}) and, again:

$$\frac{d\hat{g}_{ij}}{\hat{g}_{ij}} = \omega_j - \omega_i.$$

If the image of the class of $\mathcal{L}$ in $H^1(\bar{X}, \Omega^1_{\bar{X}}(log\, D))$ is trivial, we have:

$$\frac{d\hat{g}_{ij}}{\hat{g}_{ij}} = \omega_j - \omega_i.$$

where $\hat{g}_{ij}$ is a rational function which extends g_{ij} to $\bar{X}$ and ω_i is a logarithmic differential form along D on $\bar{U}_i$. The invertible sheaf $\mathcal{L}$ is endowed with a regular connection ∇ locally defined on U_i by:

$$\tilde{\nabla}_i(f) = df + f\omega_i.$$

This ends the proof of Lemma 2.14.

Then, we have:

Lemma 2.15. $H^0(\bar{X}, {}^c\Omega^1_{\bar{X}}(log\, D)) = H^0(\bar{X}, \Omega^1_{\bar{X}}(log\, D))$

Proof. We know that the spectral sequence $E_1^{pq} = H^q(\bar{X}, \Omega^p_{\bar{X}}(log\, D)) \to H^{p+q}(X^{an}; \mathbb{C})$ degenerates at E_1 (see [6] Corollaire 3.2.13 page 38), so the mapping:

$$H^0(\bar{X}, \Omega^1_{\bar{X}}(log\, D)) \stackrel{d}{\to} H^0(\bar{X}, \Omega^2_{\bar{X}}(log\, D))$$

is the zero map which precisely means that the forms in $H^0(\bar{X}, \Omega^1_{\bar{X}}(log\, D))$ are closed as stated in the lemma.

This proves the Lemma.

Proof of Theorem 2.13:

The two preceding lemmas show that $Pic_{cir}(X) \to Pic_{cr}(X)$ is surjective.

Now let ∇ be a regular connection on $\mathcal{L}$. Because of the surjectivity just mentioned there is a line bundle $\mathcal{L}'$ on X and an integrable regular connection ∇' on $\mathcal{L}'$ such that $\mathcal{L}' \simeq \mathcal{L}$; we may assume moreover that $\mathcal{L}' = \mathcal{L}$. Then $\nabla(s) = \nabla'(s) + \omega \otimes s$ with $\omega \in H^0(\bar{X}, \Omega^1_{\bar{X}}(log\, D))$. Because of the last lemma: $d\omega = 0$, hence ∇ is integrable, too.

Remark: Since $Pic_{cir}\, X = Pic_{cr}\, X$ we have $Pic_{cir}\, X \simeq \mathbb{H}^1(X, \tilde{\mathcal{T}}^{\cdot})$, where $\tilde{\mathcal{T}}^{\cdot}$ is the complex $j_*\mathcal{O}^*_X \to \Omega^1_{\bar{X}}(log\, D) \to 0 \to \ldots$.

So it may seem that discussing regular connections without the hypothesis of integrability was useless.

What is useful, however, is the lower exact sequence of Lemma 2.14.

Furthermore let us look at the following situation: $X \subset \bar{X}$, $\bar{X}$ being a smooth complex algebraic variety which is not assumed to be complete, $D := \bar{X} \setminus X$ divisor with normal crossings. Let ∇ be connection on an invertible sheaf on X. Then we may define when ∇ is regular resp. regular integrable with respect to D in an obvious way. In the case $\bar{X} = X$ this means that no regularity condition is imposed at all, so we can no longer expect coincidence of the two notions.

3. Some examples

In the following examples we only consider complex algebraic varieties.

3.1. For the complex projective line, the invertible sheaf $\mathcal{O}(k)$ has no connection whenever $k \neq 0$. Consider $X = \mathbb{P}^1$. One knows that $Pic(X) = \mathbb{Z}$. We shall see that $Pic_{ci}(X) \simeq Pic_c(X) = \{0\}$.

In fact, as we have proved in the section 2, for any compact connected complex Kähler manifold X (in particular any complex projective variety without singularities) we have:

$$Pic^{an}_{ci}(X) \simeq Pic^{an}_c(X) \simeq H^1(X, \mathbb{C}^*).$$

For the complex line $\mathbb{P}^1$ the cohomology $H^1(X^{an}, \mathbb{C}^*) = 0$. By GAGA (see [22], [20]) we have $Pic_{ci}(X) \simeq Pic^{an}_{ci}(X^{an})$ and $Pic_c(X) \simeq Pic^{an}_c(X^{an})$ which yields our result.

3.2. We give an example of an invertible $\mathcal{O}_X$-module which has a connection but no integrable connection.

Let $\bar{X} := \{z_0z_1 - z_2z_3 = 0\} \subset \mathbb{P}^3$. Notice that $\bar{X}$ is a complex surface isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Let $D := \bar{X} \cap \{z_0 + z_1 + z_2 - z_3 = 0\}$. Let $X := \bar{X} \setminus D$.

One verifies that D is a smooth hypersurface of $\bar{X}$. Using the Lefschetz Theorem on hyperplane sections, one shows that D is connected. In fact, D is a non-singular projective plane curve of degree 2. So $D \simeq \mathbb{P}^1$. Then $H^1(D^{an}; \mathbb{Z}) = 0$.

By [14] (p. 75) we have a commutative diagram whose lines are exact:

$$\begin{array}{ccccccc} \mathbb{Z} & \to & Pic\,\bar{X} & \to & Pic\,X & \to & 0 \\ \downarrow\simeq & & \downarrow & & \downarrow & & \\ H^2(\bar{X}^{an}, X^{an}; \mathbb{Z}) & \to & H^2(\bar{X}^{an}; \mathbb{Z}) & \to & Im\,\phi & \to & 0 \end{array}$$

where $\phi : H^2(\bar{X}^{an}; \mathbb{Z}) \to H^2(X^{an}; \mathbb{Z})$.

We have (see [16] Chap. III Exercise 12.6, p. 292):

$$Pic\,\bar{X} \simeq Pic\,\mathbb{P}^1 \times Pic\,\mathbb{P}^1 \simeq \mathbb{Z} \times \mathbb{Z}.$$

According to Künneth formula, we have:

$$H^2(\bar{X}^{an}; \mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z}.$$

One verifies that the middle vertical arrow in the diagram above given by the first Chern class is an isomorphism: one has to compute $c_1(p_i^*(\mathcal{O}_{\mathbb{P}^1}(n)), i = 1, 2$, where p_1 and p_2 are the projections of $\bar{X}$ onto $\mathbb{P}^1$.

By the Five Lemma, the last vertical arrow is an isomorphism.

Moreover the lower line of the diagram gives an exact sequence:

$$\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to Im\,\phi$$

because $H^2(\bar{X}^{an}, X^{an}; \mathbb{Z}) \simeq H_2(D^{an}; \mathbb{Z})$ by Lefschetz duality and:

$$H_2(D^{an}; \mathbb{Z}) \simeq \mathbb{Z}$$

because $D \simeq \mathbb{P}^1$.

Therefore, there is an element $c \in Im\,\phi$ which is not a torsion element.

The surjectivity of the third vertical arrow gives that there is an invertible sheaf $\mathcal{L}$ on X such that $c_1(\mathcal{L}) = c$.

Since X is affine, we have:

$$H^1(X, \Omega^1_X) = 0.$$

According to Lemma 2.6 there is a connection on the sheaf $\mathcal{L}$. But according to Lemma 1.16, there is no integrable connection on $\mathcal{L}$.

3.3. Notice that it is easier to find an example where there are connections which are not integrable or regular. One may consider $X = \mathbb{C}^2$. In this case both $Pic^{an}(X^{an})$ and $Pic(X)$ are trivial.

A connection on $\mathcal{O}_X$ (resp. $\mathcal{O}_{X^{an}}$) is given by a global algebraic (resp. analytic) differential form ω:

$$\nabla(f) = df + f\omega.$$

If one considers $\omega = dz_1$, the corresponding connection is integrable but not regular.

If $\omega = z_1dz_2$, the corresponding connection is not integrable because the form is not closed.

We can compute $Pic_c(X)$ and $Pic_{ci}(X)$ by using the diagram of Theorem 2.1. Then:

$$Pic_c(X) \simeq H^0(X, \Omega^1_X)$$

because for $X = \mathbb{C}^2$, the map $H^0(X, \mathbb{C}^*_X) \to H^0(X, \mathcal{O}^*_X)$ is an isomorphism.

Similarly, we have:

$$Pic_{ci}(X) \simeq H^0(X, {}^c\Omega^1_X).$$

In the analytic case, we know that:

$$Pic^{an}_{ci}(X^{an}) \simeq H^1(X^{an}, \mathbb{C}^*),$$

so it is trivial.

For $Pic^{an}_c(X^{an})$ the exact sequence of 1.14 gives that $Pic^{an}_c(X^{an})$ is isomorphic to the group $H^0(X^{an}, d\Omega^1_{X^{an}})$. The elements of $Pic^{an}_c(X^{an})$ are given by their curvature. Note that $H^2(X^{an}; \mathbb{C}^*) = 0$.

3.4. Let X be a non-singular algebraic variety. It may happen that all invertible sheaves on X admit an integrable connection whereas this is not true for X^{an}, as shown by the following example:

Consider the algebraic variety $X = \mathbb{C}^* \times \mathbb{C}^*$.

Notice that for this variety $Pic(X) = 0$, because $X = \mathbb{C}^2 \setminus Z$ where Z is the closed algebraic subspace given by the union of the lines $\mathbb{C} \times \{0\}$ and $\{0\} \times \mathbb{C}$, then using the Proposition 6.5 in Chapter II of [16] p. 133, we have a surjection:

$$Pic(\mathbb{C}^2) \to Pic(X).$$

Then, $Pic(X) = 0$.

On the other hand $Pic^{an}(X^{an}) \simeq H^2(X^{an}; \mathbb{Z}) = \mathbb{Z}$ because X^{an} is a Stein space; use the exact exponential sequence.

Therefore, there are invertible $\mathcal{O}_{X^{an}}$-modules for which the complex first Chern class is $\neq 0$. According to Lemma 1.16 these sheaves do not have integrable connections. However, by Theorem 1.1 they have a connection because $H^1(X^{an}, \Omega^1_{X^{an}}) = 0$. But these do not come from an algebraic invertible sheaf, because the latter ones are trivial.

3.5. Let X be a non-singular complex algebraic variety and $\mathcal{L}$ an invertible sheaf on X. By Corollary 2.11, there is a connection on $\mathcal{L}$ (and even an integrable one) as soon as $c^1_{\mathbb{C}}(\mathcal{L}) = 0$. This is no longer true if we pass to the analytic situation as shown by the following example:

Put $X := \mathbb{C}^2 \setminus \{0\}$. Note that X^{an} is simply connected.

On the other hand, $H^1(X^{an}, \mathcal{O}_{X^{an}}) \neq 0$: Let $\mathcal{U}$ be the open Stein covering by $U_1 = \mathbb{C} \times \mathbb{C}^*$, $U_2 = \mathbb{C}^* \times \mathbb{C}$. Then $H^1(X^{an}, \mathcal{O}_{X^{an}})$ is the cokernel of :

$$H^0(U_1, \mathcal{O}_{U_1}) \oplus H^0(U_2, \mathcal{O}_{U_1}) \to H^0(U_1 \cap U_2, \mathcal{O}_{U_1 \cap U_2})$$

$$(a, b) \mapsto r_1(a) - r_2(b)$$

where r_1, r_2 are restrictions, so

$$H^1(X^{an}, \mathcal{O}_{X^{an}}) \simeq V$$

where V is the vector space of all globally convergent Laurent series in two variables with negative exponents, so $V \neq 0$.

As usual, let $Pic_0(X^{an})$ be the group of isomorphism of line bundles on X^{an} with trivial first Chern class. The exact sequence:

$$0 = H^1(X; \mathbb{Z}) \to H^1(X^{an}, \mathcal{O}_{X^{an}}) \to Pic_0(X^{an}) \to 0$$

shows that $Pic_0(X^{an}) \neq 0$. On the other hand, $Pic(X) = Pic(\mathbb{C}^2) = 0$. So there are invertible $\mathcal{O}_{X^{an}}$-modules with first Chern class 0 which are not algebraizable. These cannot admit a connection: The composition $H^1(X^{an}, \mathcal{O}_{X^{an}}) \overset{\cong}{\to} Pic(X^{an}) \to H^1(X^{an}, \Omega^1_{X^{an}})$ is given by $(f_{ij}) \mapsto (2\pi i df_{ij})$, so $b(\mathcal{L}) \neq 0$ if (f_{ij}) does not represent the trivial element: note that the mapping $H^1(X^{an}, \mathcal{O}_{X^{an}}) \to H^1(X^{an}, \Omega^1_{X^{an}})$ corresponds to the mapping $V \to V^2$: $h \mapsto (\frac{\partial h}{\partial z_1}, \frac{\partial h}{\partial z_2})$ which is injective.

This shows that there are invertible sheaves on X^{an} whose first Chern class vanishes and which do not admit a holomorphic connection. In particular, we cannot improve Lemma 1.16 in general. On the other hand, cf. Lemma 1.18.

3.6. Let X be a non-singular complex algebraic variety, $\mathcal{L}$ an invertible $\mathcal{O}_X$-module, ∇ a connection on $\mathcal{L}$.

Then we have:

$$\nabla \text{ regular integrable } \Rightarrow \nabla \text{ integrable}$$

This implication is not invertible, as shown by the example $X = \mathbb{C}^2, \mathcal{L} = \mathcal{O}_X$ (see above 3.3).

Note that ∇ is integrable if and only if ∇^{an} is integrable.

In fact, we can consider the existence of connections on $\mathcal{L}$ (resp. $\mathcal{L}^{an}$):

$$\begin{array}{ccccc} \exists\,\nabla \text{ regular integrable} & \Leftrightarrow & \exists\,\nabla \text{ integrable} & \Rightarrow & \exists\,\nabla \\ & & \Updownarrow & & \Downarrow \\ & & \exists\,\nabla \text{ analytic integrable} & \Rightarrow & \exists\,\nabla \text{ analytic} \end{array}$$

For the left upper and the middle vertical equivalence see Corollary 2.11.

Note that there may be no connection at all on $\mathcal{L}$ or $\mathcal{L}^{an}$, as shown by the example $X = \mathbb{P}_1, \mathcal{L} = \mathcal{O}(k), k \neq 0$.

The right horizontal arrows are not invertible, as shown by the complicated example 3.2.

The right vertical arrow is not invertible if the answer to the following question is positive:

Let X be the Serre example of a non-singular algebraic surface which is not affine but the corresponding complex analytic manifold is Stein (see [15] p. 232 Example 3.2). Is there an invertible $\mathcal{O}_X$-module $\mathcal{L}$ on X such that its image in $H^1(X, \Omega^1_X)$ does not vanish? (Note that X is not affine, so it is possible that $H^1(X, \Omega^1_X) \neq 0$). Then, $\mathcal{L}$ does not admit a connection.

On the other hand, X^{an} is Stein, so $H^1(X^{an}, \Omega^1_{X^{an}}) = 0$, which implies that there is a connection on $\mathcal{L}^{an}$.

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SOME PROPERTIES AND APPLICATIONS OF BRIESKORN LATTICES

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ABSTRACT. After reviewing the main properties of the Brieskorn lattice in the framework of tame regular functions on smooth affine complex varieties, we prove a conjecture of Katzarkov-Kontsevich-Pantev in the toric case.

1. Introduction

The *Brieskorn lattice*, introduced by Brieskorn in [Bri70] in order to provide an algebraic computation of the Milnor monodromy of a germ of complex hypersurface with an isolated singularity, has also proved central in the Hodge theory for vanishing cycles of such a singularity, as emphasized by Pham [Pha80, Pha83]. Hodge theory for vanishing cycles, as developed by Steenbrink [Ste76, Ste77, SS85] and Varchenko [Var82], makes it an analogue of the Hodge filtration in this context, and fundamental results have been obtained by M. Saito [Sai89] in order to characterize it among other lattices in the Gauss-Manin system of an isolated singularity of complex hypersurface. As such, it leads to the definition of a period mapping, as introduced and studied with much detail by K. Saito for some singularities [Sai83]. It is also a basic constituent of the period mapping restricted to the μ-constant stratum [Sai91], where a natural Torelli problem occurs (see [Sai91], [Her99]).

For a holomorphic germ $f:(\mathbb{C}^{n+1},0)\to(\mathbb{C},0)$ with an isolated singularity, denoting by t the coordinate on the target space $\mathbb{C}$, the space

$$\Omega^{n+1}_{\mathbb{C}^{n+1},0}/\mathrm{d}f\wedge\mathrm{d}\Omega^{n-1}_{\mathbb{C}^{n+1},0} \tag{1.1}$$

is naturally endowed with a $\mathbb{C}\{t\}$-module structure (where t acts as the multiplication by f), and the *Brieskorn lattice* is the $\mathbb{C}\{t\}$-module (see [Bri70, p. 125])

$$''H^n_{f,0}=\Big(\Omega^{n+1}_{\mathbb{C}^{n+1},0}/\mathrm{d}f\wedge\mathrm{d}\Omega^{n-1}_{\mathbb{C}^{n+1},0}\Big)\Big/\mathbb{C}\{t\}\text{-torsion}. \tag{1.2}$$

Brieskorn shows that (1.2) is free of finite rank equal to the Milnor number $\mu(f,0)$, and Sebastiani [Seb70] shows the torsion freeness of (1.1), which can thus also serve as an expression for $''H^n_{f,0}$. It is also endowed with a meromorphic connection ∇ having a pole of order at most two at $t=0$, and the $\mathbb{C}(\{t\})$-vector space with connection generated by $''H^n_{f,0}$ is isomorphic to the Gauss-Manin connection, which has a regular singularity there. $''H^n_{f,0}$ is thus a $\mathbb{C}\{t\}$-lattice of this $\mathbb{C}(\{t\})$-vector space. While the action of ∇_{∂_t}, simply written as ∂_t, introduces a pole, there is a well-defined action of its inverse ∂_t^{-1} that makes $''H^n_{f,0}$ a module over the ring of $\mathbb{C}\{\{\partial_t^{-1}\}\}$ of 1-Gevrey series (i.e., formal power series $\sum_{n\geqslant0}a_n\partial_t^{-n}$ such that the series $\sum_n a_nu^n/n!$ converges). It happens to be also free of rank μ over this ring ([Mal74, Mal75]). The relation between the rings $\mathbb{C}\{t\}$ and $\mathbb{C}\{\{\partial_t^{-1}\}\}$ is called *microlocalization*. In the global case below, we will use instead the Laplace transformation. The mathematical richness of this object leads to various generalizations.

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For non-isolated hypersurface singularities, the objects with definition as in (1.2) (but in various degrees) have been introduced by Hamm in his Habilitationsschrift (see [Ham75, §II.5]), who proved that they are $\mathbb{C}\{t\}$-free of finite rank, but do not coincide with (1.1) in general. A natural $\mathbb{C}\{\{\partial_t^{-1}\}\}$-structure still exists on (1.1), and Barlet and Saito [BS07] have shown that the $\mathbb{C}\{t\}$-torsion and the $\mathbb{C}\{\{\partial_t^{-1}\}\}$-torsion coincide, so that $''H^k_{f,0}$ remains $\mathbb{C}\{\{\partial_t^{-1}\}\}$-free of finite rank.

The Brieskorn lattice has also a global variant. On the one hand, the Brieskorn lattice for tame regular functions on smooth affine complex varieties (see Section 2) is a direct analogue of the case of an isolated singularity, but the double pole of the action of t with respect to the variable ∂_t^{-1} cannot in general be reduced to a simple one by a meromorphic (even formal) gauge transformation i.e., the Gauss-Manin system with respect to the variable ∂_t^{-1} has in general an irregular singularity. The properties of the Brieskorn module for regular functions on affine manifolds which are not tame have been considered by Dimca and M. Saito [DS01].

On the other hand, given a *projective* morphism $f: X \to \mathbb{A}^1$ on a smooth quasi-projective variety X, the Brieskorn modules, defined as the hypercohomology $\mathbb{C}[\partial_t^{-1}]$-modules of the twisted de Rham complex $(\Omega^\bullet_X[\partial_t^{-1}], \mathrm{d} - \partial_t^{-1}\mathrm{d}f)$, have been shown to be $\mathbb{C}[\partial_t^{-1}]$-free (Barannikov-Kontsevich, see [Sab99b]), and a similar result holds when one replaces $\Omega^\bullet_X$ with $\Omega^\bullet_X(\log D)$ for some divisor with normal crossings. More generally, one can adapt the definition of the Brieskorn modules for the twisted de Rham complex attached to a mixed Hodge module, and the $\mathbb{C}[\partial_t^{-1}]$-freeness still holds, so that they can be called Brieskorn lattices (see loc. cit.). This enables one to use the push-forward operation by the map f and reduce the study to that of Brieskorn lattices attached to mixed Hodge modules on the affine line, as for example the mixed Hodge modules that the Gauss-Manin systems of f underlie. In such a way, the Brieskorn lattice has a *purely Hodge-theoretic definition*, which does not refer to the underlying geometry, and can thus be attached, for example, to any polarizable variation of Hodge structure on a punctured affine line (see [Sab08, §1.d]).

The Brieskorn lattice of tame functions is of particular interest and has been considered in [Sab06] for example. The Brieskorn lattice for families of such functions, considered in [DS03], has been investigated with much care for families of Laurent polynomials in relation with mirror symmetry by Reichelt and Reichelt-Sevenheck [RS15, Rei14, Rei15, RS17].

Lastly, in the global setting as above, the pole of order two of the action of t with respect to the variable ∂_t^{-1} produces in general a truly irregular singularity, and the Brieskorn lattice is an essential tool to produce the *irregular Hodge filtration* attached to such a singularity (see [SY15, Sab17]).

The contents of this article is as follows. In Section 2, we review known results on the Brieskorn lattice for a tame function. We show in Section 3 how these results enables one to obtain a simple proof of a conjecture of Katzarkov-Kontsevich-Pantev in the toric case.

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2. The Brieskorn lattice of a tame function

In this section, we review the main properties of the Brieskorn lattice attached to a tame function on an affine manifold, following [Sab99a, Sab06, DS03].

Let U be a smooth complex affine variety of dimension n and let $f \in \mathscr{O}(U)$ be a regular function on U. There are various notions of tameness for such a function, which are not known to be equivalent, but for what follows they have the same consequences. One of the definitions, given by Katz in [Kat90, Th. 14.13.3], is that the cone of $f_!\mathbb{C}_U \to \boldsymbol{R}f_*\mathbb{C}_U$ should have constant cohomology on $\mathbb{A}^1$. We will use the notion of a weakly tame function, as defined in [NS99], that is, either cohomologically tame or M-tame.

We assume that f is weakly tame. Let θ be a new variable. The *Brieskorn lattice* attached to f is the $\mathbb{C}[\theta]$-module

$$G_0 := \Omega^n(U)[\theta]\big/(\theta\mathrm{d} - \mathrm{d}f)\Omega^{n-1}(U)[\theta].$$

An expression like (1.1) also exists if U is the affine space $\mathbb{A}^{n+1}$, but the above one is valid for any smooth affine variety U. The variable θ is for ∂_t^{-1}. We already notice that

$$G_0/\theta G_0 \simeq \Omega^n(U)/\mathrm{d}f \wedge \Omega^{n-1}(U) \tag{2.1}$$

has dimension equal to the sum $\mu = \mu(f)$ of the Milnor numbers of f at all its critical points in U. The following properties are known in this setting.

(1) The algebraic Gauss-Manin systems $\mathscr{H}^k f_+\mathscr{O}_U$ are isomorphic to powers of the $\mathbb{C}[t]\langle\partial_t\rangle$-module $(\mathbb{C}[t], \partial_t)$, except for $k = 0$, so their localized Laplace transforms vanish except that for $k = 0$. If we regard the Laplace transform of $\mathscr{H}^0 f_+\mathscr{O}_U$ as a $\mathbb{C}[\tau]\langle\partial_\tau\rangle$-module, we know that it has finite type as such, and its localized Laplace transform G, that is, the $\mathbb{C}[\tau, \tau^{-1}]$-module obtained by localization, is free of rank μ. We have

$$G = \Omega^n(U)[\tau, \tau^{-1}]\big/(\mathrm{d} - \tau\mathrm{d}f)\Omega^{n-1}(U)[\tau, \tau^{-1}].$$

(2) Setting $\theta = \tau^{-1}$, we write

$$G = \Omega^n(U)[\theta, \theta^{-1}]\big/(\theta\mathrm{d} - \mathrm{d}f)\Omega^{n-1}(U)[\theta, \theta^{-1}],$$

and there is therefore a natural morphism $G_0 \to G$. This morphism is *injective*, so that G_0 is a *free* $\mathbb{C}[\theta]$-module of rank μ such that $\mathbb{C}[\theta, \theta^{-1}] \otimes_{\mathbb{C}[\theta]} G_0 = G$, i.e., G_0 is a $\mathbb{C}[\theta]$-lattice of G, on which the restriction of the Gauss-Manin connection has a pole of order at most two. Moreover, the action of $\theta^2\partial_\theta$ on the class $[\omega]$ of $\omega \in \Omega^n(U)$ in G_0 is given by

$$\theta^2\partial_\theta[\omega] = [f\omega],$$

and the action of $\theta^2\partial_\theta$ on a polynomial $\sum_{k\geqslant 0}[\omega_k\theta^k]$ is obtained by the usual formulas.

(3) Let $V_\bullet G$ be the (increasing) V-filtration of G with respect to the function τ (recall that G has a regular singularity at $\tau = 0$, while that at infinity is usually irregular). It is a filtration by free $\mathbb{C}[\tau]$-modules of rank μ indexed by $\mathbb{Q}$. The jumping indices of the induced filtration $V_\bullet(G_0/\theta G_0)$, together with their multiplicities (the dimension of $\mathrm{gr}^V_\beta(G_0/\theta G_0)$) form the *spectrum of f at ∞*. The jumping indices are contained in the interval $[0, n] \cap \mathbb{Q}$ and the spectrum is symmetric with respect to $n/2$.

(4) On the other hand, for $\alpha \in [0, 1) \cap \mathbb{Q}$, the vector space $\mathrm{gr}^V_\alpha G$ is endowed with the nilpotent endomorphism N induced by the action of $-(\tau\partial_\tau + \alpha)$ and with the increasing filtration $G_\bullet \mathrm{gr}^V_\alpha G$ naturally induced by the filtration $G_p = \theta^{-p}G_0$, i.e.,

$$G_p \mathrm{gr}^V_\alpha G = (G_p \cap V_\alpha G)/(G_p \cap V_{<\alpha}G),$$

where the intersections are taken in G. As a consequence, we have isomorphisms

$$p \in \mathbb{Z},\ \alpha \in [0, 1), \qquad \mathrm{gr}^G_p \mathrm{gr}^V_\alpha G \xrightarrow[\sim]{\theta^p} \mathrm{gr}^V_{\alpha+p}(G_0/\theta G_0).$$

(5) The $\mathbb{C}$-vector space $H_{\neq 1} := \bigoplus_{\alpha\in(0,1)\cap\mathbb{Q}} \mathrm{gr}^V_\alpha G$, resp. $H_1 := \mathrm{gr}^V_0 G$, endowed with
- the filtration

$$F^p H_{\neq 1} := \bigoplus_{\alpha\in(0,1)\cap\mathbb{Q}} G_{n-1-p} \mathrm{gr}^V_\alpha G \quad \text{resp. } F^p H_1 = G_{n-p} \mathrm{gr}^V_0 G,$$

- and the weight filtration $W_\bullet = \mathrm{M}(\mathrm{N})[n-1]$ (resp. $\mathrm{M}(\mathrm{N})[n]$), i.e., the monodromy filtration of N centered at $n-1$ (resp. n), is part of a mixed Hodge structure. In particular, N strictly shifts by one the filtration $G_\bullet \operatorname{gr}_\alpha^V G$ and acts on the graded space $\operatorname{gr}_\bullet^G \operatorname{gr}_\alpha^V G$ as the degree-one morphism induced by $-\tau\partial_\tau$. We therefore have a commutative diagram, for any $\alpha \in [0,1)$ and $p \in \mathbb{Z}$, (see [Var81] and [SS85, §7] in the singularity case):

$$
\begin{tikzcd}
\operatorname{gr}_p^G \operatorname{gr}_\alpha^V G \arrow[r, "\theta^p", "\sim"'] \arrow[d, "{[\mathrm{N}]}"'] & \operatorname{gr}_{\alpha+p}^V(\Omega^n(U)/\mathrm{d}f\wedge\Omega^{n-1}(U)) \arrow[d, "{[f]}"] \\
\operatorname{gr}_{p+1}^G \operatorname{gr}_\alpha^V G \arrow[r, "\theta^{p+1}", "\sim"'] & \operatorname{gr}_{\alpha+p+1}^V(\Omega^n(U)/\mathrm{d}f\wedge\Omega^{n-1}(U)),
\end{tikzcd}
\tag{2.2}
$$

by using the relation $-\tau\partial_\tau = \theta\partial_\theta$.

To see this, write the commutative diagram

$$
\begin{tikzcd}
\operatorname{gr}_p^G \operatorname{gr}_\alpha^V G \arrow[r, "\theta^p", "\sim"'] \arrow[d, "\theta\partial_\theta - \alpha"'] & \operatorname{gr}_{\alpha+p}^V \operatorname{gr}_0^G G \arrow[d, "\theta\partial_\theta - (\alpha+p)"'] \arrow[dr] & \\
\operatorname{gr}_{p+1}^G \operatorname{gr}_\alpha^V G \arrow[r, "\theta^p", "\sim"'] & \operatorname{gr}_{\alpha+p}^V \operatorname{gr}_1^G G \arrow[r, "\theta"] & \operatorname{gr}_{\alpha+p+1}^V \operatorname{gr}_0^G G
\end{tikzcd}
$$

and use that in the vertical morphisms, the constant part α or $\alpha+p$ induces the morphism 0.

(6) Recall that a mixed Hodge structure $(H_\mathbb{Q}, F^\bullet H_\mathbb{C}, W_\bullet H_\mathbb{Q})$ is said to be *of Hodge-Tate type* if
 (a) the filtration $W_\bullet$ has only even jumping indices
 (b) and $W_{2\bullet}H_\mathbb{C}$ is opposite to $F^\bullet H_\mathbb{C}$.

The description of the mixed Hodge structure given in (5) implies the following criterion. We will set $\nu = n-1$ when considering $H_{\neq 1}$ and $\nu = n$ when considering H_1. We will then denote by H either $H_{\neq 1}$ or H_1.

Corollary 2.3. *The mixed Hodge structure that the triple $(H, F^\bullet H, W_\bullet H)$ underlies is of Hodge-Tate type if and only if, for any integer k such that $0 \leqslant k \leqslant [\nu/2]$, the $(\nu-2k)$th power of* N *induces an isomorphism*

$$[\mathrm{N}]^{\nu-2k} : \operatorname{gr}_k^G H \xrightarrow{\sim} \operatorname{gr}_{\nu-k}^G H.$$

Proof. We define the filtration $W'_\bullet H$ indexed by $2\mathbb{Z}$ by the formula $W'_{2k}H = G_{\nu-k}H$, so that $G_k H = W'_{2(\nu-k)}H$. If we set $\ell = \nu - 2k$ for $0 \leqslant k \leqslant \nu/2$, we have $0 \leqslant \ell \leqslant \nu$ and the isomorphism in the corollary is written

$$[\mathrm{N}]^\ell : \operatorname{gr}_{\nu+\ell}^{W'} H \xrightarrow{\sim} \operatorname{gr}_{\nu-\ell}^{W'} H.$$

We can conclude that $W'_\bullet H = W_\bullet H$ if we know that $\mathrm{N}^{\nu+1} = 0$, that is, $\operatorname{gr}_{\nu+1}^G H = 0$. This is a consequence of the positivity of the spectrum [Sab06, Cor. 13.2], which says that, if $\alpha \in [0,1)$, we have $\operatorname{gr}_k^G \operatorname{gr}_\alpha^V G = 0$ for $k \notin [0,\nu] \cap \mathbb{N}$. $\square$

The following lemma was pointed out to me by Claus Hertling.

Lemma 2.4. *A mixed Hodge structure $(H_\mathbb{Q}, F^\bullet H_\mathbb{C}, W_\bullet H_\mathbb{Q})$ is Hodge-Tate if and only if we have, for all $p \in \frac{1}{2}\mathbb{Z}$,*

$$\dim \operatorname{gr}_F^p H_\mathbb{C} = \dim \operatorname{gr}_{2p}^W H_\mathbb{Q}.$$

Proof. Indeed, one direction is clear. Conversely, if the equality of dimensions holds, then (6a) holds since $F^\bullet H$ has only integral jumps; moreover, up to a Tate twist, one can assume that $W_{<0}H = 0$, so $\mathrm{gr}_F^k H = 0$ for $k < 0$. It is enough to prove that $\mathrm{gr}_F^p \mathrm{gr}_{2\ell}^W H = 0$ for all $p \neq \ell$. We prove this by induction on ℓ. If $\ell = 0$, the result follows from the property that $F^pH = 0$ for $p < 0$ and Hodge symmetry. Assume the result up to ℓ. For $j \leqslant \ell$ we thus have $\dim \mathrm{gr}_F^j \mathrm{gr}_{2j}^W H = \dim \mathrm{gr}_{2j}^W H = \dim \mathrm{gr}_F^j H$ (the latter equality by the assumption), and therefore $\mathrm{gr}_{2i}^W \mathrm{gr}_F^j H = 0$ for $i \neq j$. In particular, taking $i = \ell + 1$, we have $\mathrm{gr}_F^p \mathrm{gr}_{2(\ell+1)}^W H = 0$ for all $p \leqslant \ell$. By Hodge symmetry, we obtain $\mathrm{gr}_F^p \mathrm{gr}_{2(\ell+1)}^W H = 0$ for all $p \neq \ell + 1$, as wanted. □

(7) We now consider the case where $U = (\mathbb{C}^*)^n$, endowed with coordinates $x = (x_1, \dots, x_n)$. Let $f \in \mathbb{C}[x, x^{-1}]$ be a Laurent polynomial in n variables, with Newton polyhedron $\Delta(f)$. We assume that f is *nondegenerate with respect to its Newton polyhedron* and *convenient* (see [Kou76]). In particular, 0 belongs to the interior of its Newton polyhedron. It is known that such a function is M-tame.

For any face σ of dimension $n-1$ of the boundary $\partial\Delta(f)$, we denote by L_σ the linear form with coefficients in $\mathbb{Q}$ such that $L_\sigma \equiv 1$ on σ. For $g \in \mathbb{C}[x, x^{-1}]$, we set $\deg_\sigma(g) = \max_m L_\sigma(m)$, where the max is taken on the exponents of monomials x^m appearing in g, and $\deg_{\Delta^*}(g) = \max_\sigma \deg_\sigma(g)$. We denote the volume form $\mathrm{d}x_1/x_1 \wedge \cdots \wedge \mathrm{d}x_n/x_n$ by ω, giving rise to an identification $\mathbb{C}[x, x^{-1}] \xrightarrow{\sim} \Omega^n(U)$ and $\mathbb{C}[x, x^{-1}]/(\partial f) \xrightarrow{\sim} G_0/\theta G_0$ (see (2.1)).

The Newton increasing filtration $\mathcal{N}_\bullet\Omega^n(U)$ indexed by $\mathbb{Q}$ is defined by

$$\mathcal{N}_\beta\Omega^n(U) := \{g\omega \in \Omega^n(U) \mid \deg_{\Delta^*}(g) \leqslant \beta\}.$$

We have $\mathcal{N}_\beta\Omega^n(U) = 0$ for $\beta < 0$ and $\mathcal{N}_0\Omega^n(U) = \mathbb{C}\cdot\omega$. We can extend this filtration to $\Omega^n(U)[\theta]$ by setting

$$\mathcal{N}_\beta\Omega^n(U)[\theta] := \mathcal{N}_\beta\Omega^n(U) + \theta\mathcal{N}_{\beta-1}\Omega^n(U) + \cdots + \theta^k\mathcal{N}_{\beta-k}\Omega^n(U) + \cdots$$

and then naturally induce this filtration on G_0, to obtain a filtration $\mathcal{N}_\bullet G_0$ and then on $G_0/\theta G_0$. We have

$$\mathcal{N}_\bullet G_0 = V_\bullet G \cap G_0 \quad \text{and} \quad \mathcal{N}_\bullet(G_0/\theta G_0) = V_\bullet(G_0/\theta G_0). \tag{2.5}$$

Corollary 2.3 now reads, according to (2.2) and by using the above identification through multiplication by ω:

Corollary 2.6. *The mixed Hodge structure that the triple $(H, F^\bullet H, W_\bullet H)$ underlies is of Hodge-Tate type if and only if, for any integer k such that $0 \leqslant k \leqslant [\nu/2]$ ($\nu = n-1$, resp. n), we have isomorphisms*

$$\mathrm{gr}^{\mathcal{N}}_{\alpha+k}\big(\mathbb{C}[x, x^{-1}]/(\partial f)\big) \xrightarrow[\sim]{[f]^{n-1-2k}} \mathrm{gr}^{\mathcal{N}}_{\alpha+n-1-k}\big(\mathbb{C}[x, x^{-1}]/(\partial f)\big) \quad \forall \alpha \in (0, 1),$$

resp.
$$\mathrm{gr}^{\mathcal{N}}_{k}\big(\mathbb{C}[x, x^{-1}]/(\partial f)\big) \xrightarrow[\sim]{[f]^{n-2k}} \mathrm{gr}^{\mathcal{N}}_{n-k}\big(\mathbb{C}[x, x^{-1}]/(\partial f)\big).$$

3. On a conjecture of Katzarkov-Kontsevich-Pantev

In this section we use the algebraic Brieskorn lattice of a convenient nondegenerate Laurent polynomial to solve the toric case of the part "$f^{p,q} = h^{p,q}$" of Conjecture 3.6 in [KKP17] (the other equality "$h^{p,q} = i^{p,q}$" is obviously not true by simply considering the case of the standard Laurent polynomial mirror to the projective space $\mathbb{P}^n$, see also another counter-example in

[LP18]). We refer to [LP18, Har17, Sha17] for further discussion and positive results on this conjecture.

3.a. **The Brieskorn lattice and the conjecture of Katzarkov-Kontsevich-Pantev.** Given a smooth quasi-projective variety U and a morphism $f : U \to \mathbb{A}^1$, every twisted de Rham cohomology $H^k_{\mathrm{DR}}(U, \mathrm{d} + \mathrm{d}f)$, i.e., the kth hypercohomology of the twisted de Rham complex $(\Omega^\bullet_U, \mathrm{d} + \mathrm{d}f)$, is endowed with a decreasing filtration $F^\bullet_{\mathrm{Yu}} H^k_{\mathrm{DR}}(U, \mathrm{d} + \mathrm{d}f)$ indexed by $\mathbb{Q}$ (see [Yu14]). For $\alpha \in [0, 1)$, the filtration indexed by $\mathbb{Z}$ defined by $F^p_{\mathrm{Yu},\alpha} = F^{p-\alpha}_{\mathrm{Yu}}$ can also be computed in terms of the Kontsevich complex $\Omega^\bullet_f(\alpha)$ together with its stupid filtration (see [ESY17, Cor. 1.4.5]). The irregular Hodge numbers $h^{p,q}_\alpha(f)$ are defined as

$$h^{p,q}_\alpha(f) := \dim \mathrm{gr}^{p-\alpha}_{F_{\mathrm{Yu}}} H^{p+q}_{\mathrm{DR}}(U, \mathrm{d} + \mathrm{d}f). \tag{3.1}$$

It is well-known that $\dim H^k_{\mathrm{DR}}(U, \mathrm{d} + \mathrm{d}f) = \dim H^k(U, f^{-1}(t))$ for $|t| \gg 0$. This space is endowed with a monodromy operator (around $t = \infty$), and we will consider the case where this monodromy operator is *unipotent*. In such a case, the filtration $F^\bullet_{\mathrm{Yu}} H^{p+q}_{\mathrm{DR}}(U, \mathrm{d} + \mathrm{d}f)$ is known to jump at integers only, and in (3.1) only $\alpha = 0$ occurs. We then simply denote this number by $h^{p,q}(f)$, so that, in such a case,

$$h^{p,q}(f) := \dim \mathrm{gr}^p_{F_{\mathrm{Yu}}} H^{p+q}_{\mathrm{DR}}(U, \mathrm{d} + \mathrm{d}f).$$

Let $W_\bullet$ be the monodromy filtration on $H^k(U, f^{-1}(t))$ centered at k. The conjecture of [KKP17] that we consider is the possible equality (see [LP18, Har17, Sha17])

$$h^{p,q}(f) = \dim \mathrm{gr}^W_{2p} H^{p+q}(U, f^{-1}(t)). \tag{3.2}$$

If moreover U is affine and f is weakly tame, so that $H^{p+q}_{\mathrm{DR}}(U, \mathrm{d} + \mathrm{d}f) = 0$ unless $p + q = n$, [SY15, Cor. 8.19] gives, using the notation of Section 2:[1]

$$h^{p,q}(f) = \begin{cases} \dim \mathrm{gr}^V_{n-p}(G_0(f)/\theta G_0(f)) = \dim \mathrm{gr}^p_F \mathrm{gr}^V_0 G & \text{if } p + q = n, \\ 0 & \text{if } p + q \neq n, \end{cases}$$

and this is the number denoted by $f^{p,q}$ in [KKP17]. In such a case, we have $H = H_1$ in the notation of Section 2(5).

The following criterion has been obtained, with a different approach of the irregular Hodge filtration, by Y. Shamoto.

Proposition 3.3 ([Sha17]). *Assume U affine and f weakly tame with unipotent monodromy operator at infinty. Then (3.2) holds true if and only if the mixed Hodge structure of Section 2(5) on $H = H_1$ is of Hodge-Tate type.*

Proof. According to Lemma 2.4, proving the result amounts to identifying the space $\mathrm{gr}^V_0 G$ endowed with its nilpotent operator N with the space $H^n(U, f^{-1}(t))$ endowed with the nilpotent part of the (unipotent) monodromy (up to a nonzero constant). Choosing an extension $F : \mathcal{X} \to \mathbb{P}^1$ of f as a projective morphism on a smooth variety $\mathcal{X}$ such that $\mathcal{X} \smallsetminus U$ is a divisor, and setting $\mathcal{F} = \boldsymbol{R}j_*\mathbb{C}_U$ ($j : U \hookrightarrow \mathcal{X}$), we identify the dimension of $H^k(U, f^{-1}(t))$ with that of the kth-hypercohomology on $\mathcal{X}$ of the Beilinson extension $\Xi_F\mathcal{F}$. Then the desired identification is given by [Sab97, Cor. 1.13]. ☐

[1] The definition of δ_γ in [SY15] should read $\dim \mathrm{gr}^V_\gamma(G_0(f)/uG_0(f))$.

3.b. **The toric case of the conjecture of Katzarkov-Kontsevich-Pantev, first part.** As usual in toric geometry, we denote by M the lattice $\mathbb{Z}^n$ in $\mathbb{C}^n$ and by N its dual lattice. We fix a reflexive simplicial polyhedron $\Delta \subset \mathbb{R} \otimes M$ with vertices in M and having 0 in its interior (it is then the unique integral point in its interior), see [Bat94, §4.1]. We denote by Δ^* the dual polyhedron with vertices in N, which is also simplicial reflexive and has 0 in its only interior point, and by $\Sigma \subset N$ the fan dual to Δ, which is also the cone on Δ^* with apex 0. We assume that Σ is the fan of nonsingular toric variety X of dimension n, that is, each set of vertices of the same $(n-1)$-dimensional face of $\partial\Delta^*$ is a $\mathbb{Z}$-basis of N. We know that

- X is Fano ([Bat94, Th. 4.1.9]),
- the Chow ring $A^*(X) \simeq H^{2*}(X, \mathbb{Z})$ is generated by the divisor classes D_v corresponding to vertices $v \in V(\Delta^*)$ of Δ^*, i.e., primitive elements on the rays of Σ (see [Ful93, p. 101]),
- we have $c_1(TX) = c_1(K_X^\vee) = \sum_{v \in V(\Delta^*)} D_v$ in $H^{2*}(X, \mathbb{Z})$ (see [Ful93, p. 109]), which satisfies Hard Lefschetz on $H^{2*}(X, \mathbb{Q})$, by ampleness of $K_X^\vee$.

Let us fix coordinates $x = (x_1, \dots, x_n)$ such that $\mathbb{Q}[N] = \mathbb{Q}[x, x^{-1}]$. We use the notation of Section 2(7). Due to the reflexivity of Δ^*, L_σ has coefficients in $\mathbb{Z}$ (it corresponds to a vertex of Δ). For $g \in \mathbb{C}[x, x^{-1}]$, the σ-degree $\deg_\sigma(g) = \max_m L_\sigma(m)$ and the Δ^*-degree $\deg_{\Delta^*}(g) = \max_\sigma \deg_\sigma(g)$ are thus nonnegative integers.

Proposition 3.4. *The case "$f^{p,q} = h^{p,q}$" of* [KKP17, Conj. 3.6] *holds true if f is the Laurent polynomial*

$$f(x) = \sum_{v \in V(\Delta^*)} x^v \in \mathbb{Q}[x, x^{-1}].$$

The idea of the proof is to notice that the property for the second morphism in Corollary 2.6 to be an isomorphism is exactly the property that $c_1(TX)$ satisfies the Hard Lefschetz property, and thus to identify its source and target as the cohomology of X in suitable degree.

Lemma 3.5. *For Δ as above, any Laurent polynomial*

$$f_{\boldsymbol{a}}(x) = \sum_{v \in V(\Delta^*)} a_v x^v \in \mathbb{C}[x, x^{-1}], \quad \boldsymbol{a} = (a_{v \in V}) \in (\mathbb{C}^*)^{V(\Delta^*)}.$$

is convenient and non-degenerate in the sense of Kouchnirenko.

Proof. The Newton polyhedron of $f_{\boldsymbol{a}}$ is equal to Δ^*, and 0 belongs to its interior. In order to prove the non-degeneracy, we note that the vertices of any $(n-1)$-dimensional face σ of $\partial\Delta^*$ form a $\mathbb{Z}$-basis. It follows that, in suitable toric coordinates $y_1, \dots, y_n$, the restriction $f_{\boldsymbol{a}|\sigma}$ can be written as $y_1 + \cdots + y_n$, and the non-degeneracy is then obvious. ◻

Proof of Proposition 3.4. Note that $\deg_{\Delta^*}(f) = 1$, as well as $\deg_{\Delta^*}(x_i \partial f / \partial x_i) = 1$. The Jacobian ring $\mathbb{Q}[x, x^{-1}]/(\partial f)$ is endowed with the Newton filtration $\mathcal{N}_\bullet$ induced by the Δ^*-degree $\deg_{\Delta^*}$, and corresponds to $\mathcal{N}_\bullet(G_0/\theta G_0)$ by multiplication by ω. In the present setting, [BCS05, Th. 1.1] identifies the graded ring $A^*(X)_{\mathbb{Q}}$ with the graded ring

$$\mathrm{gr}^{\mathcal{N}}_\bullet\big(\mathbb{Q}[x, x^{-1}]/(\partial f)\big).$$

By applying Hard Lefschetz to $c_1(TX)$, we deduce that, for every $k \in \mathbb{N}$ such that $0 \leqslant k \leqslant [n/2]$, multiplication by the $(n-2k)$th power of the $\mathcal{N}$-class $[f]$ of f induces an isomorphism

$$[f]^{n-2k} : \mathrm{gr}^{\mathcal{N}}_k\big(\mathbb{Q}[x, x^{-1}]/(\partial f)\big) \xrightarrow{\ \sim\ } \mathrm{gr}^{\mathcal{N}}_{n-k}\big(\mathbb{Q}[x, x^{-1}]/(\partial f)\big).$$

By Corollary 2.6 for $H = H_1$, we deduce the assertion of the proposition from Proposition 3.3. ◻

3.c. **The toric case of the conjecture of Katzarkov-Kontsevich-Pantev, second part.** We now prove the main result of this note.

Theorem 3.6. *The case "$f^{p,q} = h^{p,q}$" of* [KKP17, Conj. 3.6] *holds true for any Laurent polynomial*

$$f_{\boldsymbol{a}}(x) = \sum_{v \in V(\Delta^*)} a_v x^v \in \mathbb{C}[x, x^{-1}], \quad \boldsymbol{a} = (a_{v \in V}) \in (\mathbb{C}^*)^{V(\Delta^*)}.$$

Remark 3.7. The case where $n = 3$ was already proved differently by Y. Shamoto [Sha17, §4.2].

Proof. Let us set $H(f_{\boldsymbol{a}}) = H_1(f_{\boldsymbol{a}}) = \mathrm{gr}_0^V G(f_{\boldsymbol{a}})$, where $G(f_{\boldsymbol{a}})$ is the localized Laplace transform of the Gauss-Manin system for $f_{\boldsymbol{a}}$ as in Section 2(2). By Lemma 3.5, we can apply the results of Section 2 to $f_{\boldsymbol{a}}$ for any $\boldsymbol{a} \in (\mathbb{C}^*)^{V(\Delta^*)}$. We will prove that, for fixed p, both terms $\dim \mathrm{gr}^G_{n-p} H(f_{\boldsymbol{a}})$ and $\dim \mathrm{gr}^W_{2p} H(f_{\boldsymbol{a}})$ in Lemma 2.4 are independent of $\boldsymbol{a}$. Since they are equal if $\boldsymbol{a} = (1, \dots, 1)$, after Proposition 3.4, they are equal for any $\boldsymbol{a} \in (\mathbb{C}^*)^{V(\Delta^*)}$, as wanted.

(1) For the first term, we will use [NS99]. We have denoted there $\dim \mathrm{gr}^G_p H(f_{\boldsymbol{a}})$ by $\nu_p(f_{\boldsymbol{a}})$ and, since $\mathrm{gr}^V_\alpha G = 0$ for $\alpha \notin \mathbb{Z}$, it is also equal to the number denoted there by $\Sigma_{p-1}(f_{\boldsymbol{a}})$. By the theorem in [NS99] and Lemma 3.5, $\Sigma_{p-1}(f_{\boldsymbol{a}})$ depends semi-continuously on $\boldsymbol{a}$. On the other hand, according to [Kou76], $\dim H(f_{\boldsymbol{a}})$ is independent of $\boldsymbol{a}$ and is computed only in terms of Δ^*. Since $\dim H(f_{\boldsymbol{a}}) = \sum_p \Sigma_{p-1}(f_{\boldsymbol{a}})$, each term in this sum is also constant with respect to $\boldsymbol{a}$.

(2) We will prove the local constancy of $\dim \mathrm{gr}^W_{2p} H(f_{\boldsymbol{a}})$ near any $\boldsymbol{a}_o \in (\mathbb{C}^*)^{V(\Delta^*)}$. As noticed in [DS03, §4], we can apply the results of Section 2 of loc. cit. to $f_{\boldsymbol{a}_o}$. We fix a Stein open set $\mathcal{B}^o$ adapted to $f_{\boldsymbol{a}_o}$ as in [DS03, §2a], and fix a neighbourhood X of $\boldsymbol{a}_o$ so that it is also adapted to any $f_{\boldsymbol{a}}$ for $\boldsymbol{a}$ in this neighbourhood. By construction, all the critical points of $f_{\boldsymbol{a}_o}$ are contained in the interior of $\mathcal{B}^o$ if X is chosen small enough, and since $\mu(f_{\boldsymbol{a}})$ is constant, the same property holds for $\boldsymbol{a} \in X$. By using successively Theorem 2.9, Remark 2.11 and Proposition 1.20(1) in [DS03], we deduce that, when $\boldsymbol{a}$ varies in X, the localized partial Laplace transformed Gauss-Manin systems $G(f_{\boldsymbol{a}})$ form an $\mathscr{O}_X[\tau, \tau^{-1}]$-free module with integrable connection and regular singularity along $\tau = 0$, which is compatible with base change with respect to X. As a consequence, the monodromy of each $G(f_{\boldsymbol{a}})$ around $\tau = 0$ is constant, and the assertion follows. ☐

Remark 3.8 (suggested by the referee). If we relax the condition in Section 3.b that the toric Fano variety X is *nonsingular*, then we have to consider the orbifold Chow ring of X as in [BCS05], or the Chen-Ruan orbifold cohomology of X. For the cohomology of the untwisted sector (i.e., the usual cohomology), the Hard Lefschetz theorem is still valid (see [Ste77]) and Proposition 3.4 still holds, i.e., (3.2) holds for f. Moreover, Part (2) of the proof of Theorem 3.6 also extends to this setting. However, the semicontinuity result of [NS99] used in Part (1) of the proof is not enough to imply the constancy (with respect to $\boldsymbol{a}$) of $\nu_p(f_{\boldsymbol{a}})$.

On the other hand, one can also consider the various $h^{p,q}_\alpha(f)$ for $\alpha \in (0,1) \cap \mathbb{Q}$ and, correspondingly, the twisted sectors of the orbifold X. In such a case, Hard Lefschetz for f may already give trouble (see [Fer06]).

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ON THE STRUCTURE OF BRIESKORN LATTICES, II

MORIHIKO SAITO

To the memory of Egbert Brieskorn

Abstract. We give a simple proof of the uniqueness of extensions of good sections for formal Brieskorn lattices, which can be used in a paper of C. Li, S. Li, and K. Saito for the proof of convergence in the non-quasihomogeneous polynomial case. Our proof uses an exponential operator argument as in their paper, although we do not use polyvector fields nor smooth differential forms. We also present an apparently simpler algorithm for an inductive calculation of the coefficients of primitive forms in the Brieskorn-Pham polynomial case. In a previous paper on the structure of Brieskorn lattices, there were some points which were not yet very clear, and we give some explanations about these, e.g. on the existence and the uniqueness of primitive forms associated with good sections, where we present some rather interesting examples. In Appendix we prove the uniqueness up to a nonzero constant multiple of the higher residue pairings in some formal setting which is different from the one in the main theorem. This is questioned by D. Shklyarov.

Introduction

Let $f:(X,0)\to(\Delta,0)$ be a holomorphic function on a complex manifold, where Δ is an open disk with coordinate t. Assume $X_0:=f^{-1}(0)$ has an isolated singularity at 0. We have the associated Gauss-Manin system G_f and the Brieskorn lattice $H''_f\subset G_f$, where G_f is a regular holonomic $\mathcal{D}_{\Delta,0}$-module on which the action of ∂_t is bijective, and H''_f is a finite submodule over $\mathbb{C}\{t\}$ and also over $\mathbb{C}\{\{\partial_t^{-1}\}\}$ (the latter comes from the theory of microdifferential operators [SKK]), see [Br], [Ph], [ScSt], [Sa3], etc. There is a surjection

$$\mathrm{pr}_0: H''_f \twoheadrightarrow H''_f/\partial_t^{-1}H''_f \cong \Omega_f\ \big(:=\Omega^{n+1}_{X,0}/df\wedge\Omega^n_{X,0}\cong\mathbb{C}\{x\}/(\partial f)\big),$$

where $(\partial f)\subset\mathbb{C}\{x\}$ is the Jacobian ideal generated by the partial derivatives $\partial_{x_i}f$ with $x=(x_0,\dots,x_n)$ a local coordinate system of $(X,0)$, and $n:=\dim X_0=\dim X-1$.

For a $\mathbb{C}$-linear section σ_0 of pr_0, set $I_0:=\mathrm{Im}\,\sigma_0$. We say that σ_0 is *good* in this paper if

$$tI_0\subset I_0+\partial_t^{-1}I_0,\quad\text{i.e.}\quad t\sigma_0=\sigma_0A_0+\partial_t^{-1}\sigma_0A_1\quad\big(A_0,A_1\in\mathrm{End}_{\mathbb{C}}(\Omega_f)\big).\tag{0.1}$$

Let V be the filtration of Kashiwara [Ka] and Malgrange [Ma1] on G_f indexed decreasingly by $\mathbb{Q}$ so that the action of $\partial_t t-\alpha$ on $\mathrm{Gr}_V^\alpha G_f$ is nilpotent. It induces the filtration V on H''_f and Ω_f. A good section is called *very good* in this paper if it is strictly compatible with V. (It is called *good* in [Sa3].) In the weighted homogeneous polynomial case, every good section is very good (see Proposition 3.1 below) although it does not hold in general. The eigenvalues of A_1, which are called the exponents associated with a good section, do not necessarily coincide with the usual exponents defined as in [St] unless the section is very good (see Example 4.1 below). Note that A_1 is not necessarily semisimple in general (see [Sa3]). This causes a certain problem when we have to take an eigenvector of A_1 which generates the Jacobian ring over $\mathbb{C}\{x\}$. It is needed to construct a primitive form associated with a good section satisfying the orthogonality condition for the canonical pairing.

The existence of a very good section is proved in [Sa3] by using Deligne's canonical splitting of the mixed Hodge structure [De] (which is applied to the canonical mixed Hodge structure on the vanishing cohomology [St]) together with the relation with the Brieskorn lattice as in [ScSt]. Note that very good sections correspond to opposite filtrations to the Hodge filtration on the vanishing cohomology which are stable by the action of $N := \log T_u$ where T_u is the unipotent part of the monodromy (see [Sa3, Theorem 3.6]). In the weighted homogeneous polynomial case, N vanishes and the existence of very good sections is trivial so that we do not need to use the above arguments at all. The orthogonality condition for the higher residue pairings in [SK1], [SK2] follows from the orthogonality of the corresponding splitting of the Hodge filtration with respect to the canonical self-pairing of the vanishing cohomology, since the pairings can be identified with this self-pairing, see [Sa3]. Using the extension argument as below, we can get a unique primitive form associated with a very good section satisfying the orthogonality condition, see Remark 3.7 below. However, the existence and the uniqueness of the associated primitive form do not hold in general unless a good section is very good, see Examples 4.3 and 4.4 below.

Let $F : Y \to \Delta$ be a deformation of f with $Y = X \times S$, $S = \Delta^m$, and $F|_{X\times\{0\}} = f$. Here we assume that the singular locus C of $(F, pr) : Y \to \Delta \times S$ is *proper* over S. Then the calculation of the Gauss-Manin system and the Brieskorn lattice can be reduced to the case $C \cap (X \times \{0\}) = \{0\}$ by shrinking S and restricting to an open neighborhood of each connected component of C. We have the Gauss-Manin system $G_{F,S}$ and the Brieskorn lattice $H''_{F,S} \subset G_{F,S}$, where $G_{F,S}$ is a regular holonomic $\mathcal{D}_{\Delta\times S,0}$-module on which the action of ∂_t is bijective, and $H''_{F,S}$ is a finite submodule over $\mathbb{C}\{t, \mathbf{s}\}$ and also over $\mathbb{C}\{\mathbf{s}\}\{\{u\}\}$ (see (1.1.1) for the latter). Here $u := \partial_t^{-1}$, and $\mathbf{s} = (s_1, \dots, s_m)$ is the coordinate system of $\Delta^m \subset \mathbb{C}^m$. Let $\mathfrak{m}_0 \subset \mathbb{C}\{\mathbf{s}\} := \mathbb{C}\{s_1, \dots, s_m\}$ be the maximal ideal generated by the s_i. There is a surjection

$$\mathrm{pr}_S : H''_{F,S} \twoheadrightarrow H''_{F,S}/\partial_t^{-1} H''_{F,S} \cong \Omega_{F,S} \bigl(:= \Omega^{n+1}_{Y/S,0}/dF \wedge \Omega^n_{Y/S,0}\bigr),$$

together with the canonical isomorphisms

$$G_{F,S}|_0 = G_f, \quad H''_{F,S}|_0 = H''_f, \quad \Omega_{F,S}|_0 = \Omega_f,$$

where $G_{F,S}|_0 := G_{F,S}\big/\,\mathfrak{m}_0\, G_{F,S}$, etc. For a $\mathbb{C}\{\mathbf{s}\}$-linear section σ_S of pr_S, set $I_S := \operatorname{Im} \sigma_S$. We say that σ_S is *good* if

$$tI_S \subset I_S + \partial_t^{-1} I_S, \quad \partial_{s_i} I_S \subset I_S + \partial_t I_S. \tag{0.2}$$

It is shown by B. Malgrange (see [Ma2], [Ma3]) that any good section σ_0 of pr_0 can be uniquely extended to a good $\mathbb{C}\{\mathbf{s}\}$-linear section σ_S of pr_S by solving Birkhoff's Riemann-Hilbert problem in this case, see also [SK2], [He], [Sab], etc. (Here the orthogonality condition for the higher residue pairings can be reduced easily to the case $S = pt$.)

We can also consider the formal Gauss-Manin system $\widehat{G}_f$ and the formal Brieskorn lattice $\widehat{H}''_f$, which are free modules of rank μ over $\mathbb{C}((u))$ and $\mathbb{C}[[u]]$ respectively (where $u = \partial_t^{-1}$). They can be obtained by taking the u-adic completion of G_f and H''_f as in [Sa2]. There is a natural projection

$$\widehat{\mathrm{pr}}_0 : \widehat{H}''_f \twoheadrightarrow \widehat{H}''_f/\partial_t^{-1}\widehat{H}''_f \cong \Omega_f,$$

where the last isomorphism follows from the u-adic completion argument.

We also have the formal Gauss-Manin system $\widehat{G}_{F,\widehat{S}}$ and the formal Brieskorn lattice $\widehat{H}''_{F,\widehat{S}}$, which are free modules of rank μ over $\mathbb{C}((u))[[\mathbf{s}]]$ and $\mathbb{C}[[u, \mathbf{s}]] := \mathbb{C}[[u, s_1, \dots, s_m]]$ respectively. There is a natural projection

$$\widehat{\mathrm{pr}}_{\widehat{S}} : \widehat{H}''_{F,\widehat{S}} \twoheadrightarrow \widehat{H}''_{F,\widehat{S}}/\partial_t^{-1}\widehat{H}''_{F,\widehat{S}} \cong \Omega_{F,\widehat{S}},$$

where $\Omega_{F,\widehat{S}}$ is the $\mathfrak{m}_0$-adic completion of $\Omega_{F,S}$ so that $\Omega_{F,\widehat{S}} := \Omega_{F,S} \otimes_{\mathbb{C}\{\mathbf{s}\}} \mathbb{C}[[\mathbf{s}]]$. We can define the notion of good sections $\widehat{\sigma}_0$, $\widehat{\sigma}_{\widehat{S}}$ in the same way as in the convergent case by using the analogues of conditions (0.1) and (0.2) where I_0 is defined by $\operatorname{Im} \widehat{\sigma}_0$, and I_S is replaced by $I_{\widehat{S}} := \operatorname{Im} \widehat{\sigma}_{\widehat{S}}$. We have the following.

Theorem 1. *Any good $\mathbb{C}$-linear section $\widehat{\sigma}_0$ of $\widehat{\mathrm{pr}}_0$ satisfying* (0.1) *can be extended uniquely to a good $\mathbb{C}[[\mathbf{s}]]$-linear section $\widehat{\sigma}_{\widehat{S}}$ of $\widehat{\mathrm{pr}}_{\widehat{S}}$ satisfying* (0.2) *with I_S replaced by $I_{\widehat{S}} := \operatorname{Im} \widehat{\sigma}_{\widehat{S}}$.*

In fact, this easily follows from an assertion which is irrelevant to the action of t, see Theorem 1.4 below. Theorem 1 does not seem to be stated explicitly in [LLS], although it seems to be used there in an essential way for the proof of the coincidence with the Malgrange's construction [Ma2], [Ma3], which gives the convergence of their extensions of good sections. Here it seems rather difficult to prove directly the convergent version of Theorem 1 by using the exponential operator argument without using Malgrange's result in the convergent case. The advantage of this method seems to be that one can calculate step by step the coefficients of the Taylor expansion of primitive forms explicitly (see (2.3) below for a special case). However, it is not very clear how much it is useful for the original purpose of the primitive form, i.e. the associated period mapping, since the radius of convergence, for instance, does not seem to be calculated easily. It might be rather difficult to expect it theoretically since the partial Fourier transformation is used in an essential way.

It seems that Theorem 1 is proved in [LLS] provided that "uniquely" is replaced by "canonically" in the statement. In a more recent version of it, they seem to show the uniqueness statement in terms of primitive forms together with a rather complicated proof in the weighted homogeneous case. Actually Theorem 1 can be proved more easily as is shown in the proof of Theorem 1.4 below by using an exponential operator argument given in [LLS]. However, the latter argument is a rather amazing one for many complex geometers and their paper is not necessarily easy to read for non-specialists of mathematical physics. So we present in this paper a possibly simpler proof without using polyvector fields nor C^∞ differential forms and by using a hopefully more precise argument than [LLS].

As a corollary of the exponential operator argument, we also present an algorithm for an inductive calculation of the coefficients of primitive forms for Brieskorn-Pham polynomials, which seems simpler than the one in [LLS] in case of these polynomials. By using it, we can calculate the coefficients of the first few terms of the Taylor expansion of the primitive forms without computers in this case, see (2.3) below. (The argument in this paper cannot be applied to the situation of [DoSa] where the Brieskorn lattices are stable by ∂_t^{-1}, but the V-filtration is stable by ∂_t, instead of ∂_t^{-1}, in their case.)

In Appendix we prove the uniqueness up to a nonzero constant multiple of the higher residue pairings in some formal setting which is different from the one in Theorem 1 because of the difference between $\mathbb{C}((u))[[\mathbf{s}]]$ and $\mathbb{C}[[\mathbf{s}]]((u))$. It is written to answer a question of Dmytro Shklyarov. This uniqueness does not hold for the formal Gauss-Manin systems as in Theorem 1 because of the isomorphism in Proposition 1.3 below which is obtained by using the exponential operator argument. This shows a clear difference between the two kinds of formal Gauss-Manin systems.

We thank C. Hertling for useful comments about this paper, D. Shklyarov for a good question which became a source of Appendix, and C. Li for a good question that led us to a correct formulation of an algorithm for the inductive calculation of the coefficients of primitive forms. This work is partially supported by Kakenhi 24540039.

In Section 1 we review formal Gauss-Manin systems and Brieskorn lattices, and explain an exponential operator argument as in [LLS]. In Section 2 we present an algorithm for an inductive computation of the coefficients of the Taylor expansion of primitive forms in the Brieskorn-Pham polynomial case, which is apparently simpler in this case than the one in [LLS]. In Section 3 we give some remarks related to good sections and very good sections in the sense of this paper. In Section 4 we present some interesting examples. In Appendix we show the uniqueness up to a nonzero constant multiple of the higher residue pairings in some formal setting.

1. Formal Gauss-Manin systems and Brieskorn lattices

In this section we review formal Gauss-Manin systems and Brieskorn lattices, and explain an exponential operator argument as in [LLS] without using polyvector fields nor C^∞ differential forms, but using more precise arguments.

Notation 1.1. Let $f : X \to \Delta$, and $F : Y \to \Delta$ be as in the introduction, where $Y = X \times S$ with $S = \Delta^m$. We have the microlocal Gauss-Manin system defined by

$$G_{F,S} := H^{n+1}C^\bullet_{F,Y} \quad \text{with} \quad C^\bullet_{F,Y} := \big(\Omega^\bullet_{Y/S,0}\{\{u\}\}[u^{-1}],\, ud - dF\wedge\big),$$

where $u = \partial_t^{-1}$, and $n = \dim X - 1$. Here $\Omega^\bullet_{Y/S,0}\{\{u\}\}$ can be defined by using

$$\mathbb{C}\{y\}\{\{u\}\} := \big\{\textstyle\sum_{\nu,k} a_{\nu,k}\, y^\nu u^k \in \mathbb{C}[[y,u]] \,\big|\, \sum_{\nu,k} |a_{\nu,k}|\, r^{|\nu|+k}/k! < \infty \ \ (\exists\, r > 0)\big\}, \tag{1.1.1}$$

where $y = (y_0, \dots, y_{n+m})$ is a local coordinate system of Y with $y^\nu := \prod_i y_i^{\nu_i}$ and $|\nu| := \sum_i \nu_i$ for $\nu = (\nu_0, \dots, \nu_{n+m}) \in \mathbb{N}^{n+m+1}$.

The Brieskorn lattice is defined by

$$H''_{F,S} := H^{n+1}C^{(0),\bullet}_{F,Y} \quad \text{with} \quad C^{(0),\bullet}_{F,Y} := \big(\Omega^\bullet_{Y/S,0}\{\{u\}\},\, ud - dF\wedge\big).$$

These are obtained by the microlocalization of the usual Gauss-Manin systems and Brieskorn lattices, see [Ph], [Sa3], etc. (Note that $G_{F,S}$ and $H''_{F,S}$ are finite free modules of rank μ over $\mathbb{C}\{\mathbf{s}\}\{\{u\}\}[u^{-1}]$ and $\mathbb{C}\{\mathbf{s}\}\{\{u\}\}$ respectively although it is not used in this paper.)

The action of ∂_{x_j}, ∂_{s_i} can be defined by using the canonical generator $\delta(t - F)$ which is not explicitly written in $C^\bullet_{F,Y}$ to simplify the notation (see also [Sa3]). More precisely $\delta(t - F)$ is a generator of an $\mathcal{E}$-module $\mathcal{C}_F$ which is the microlocalization of a $\mathcal{D}$-module $\mathcal{B}_F$, and the latter is the direct image of $\mathcal{O}_Y$ by the graph embedding of F as a $\mathcal{D}$-module. Here $\mathcal{E}$ is the ring of microdifferential operators (see [SKK]). This generator satisfies the relations

$$\begin{aligned} t\,\delta(t-F) &= F\,\delta(t-F), \\ \partial_{x_j}\delta(t-F) &= -(\partial F/\partial x_j)\,\partial_t\,\delta(t-F), \\ \partial_{s_i}\delta(t-F) &= -(\partial F/\partial s_i)\,\partial_t\,\delta(t-F). \end{aligned} \tag{1.1.2}$$

Note that the second relation is compatible with the differential $ud - dF\wedge$ of the complex $C^\bullet_{F,Y}$ (up to the multiplication by u), and the latter can be identified with the relative de Rham complex $\mathrm{DR}_{Y/S}(\mathcal{C}_F)$ up to a shift of complex. These are compatible with the theory of Gauss-Manin connections on Brieskorn lattices as in [Gre].

We have the formal Gauss-Manin system defined by

$$\widehat{G}_{F,S} := H^{n+1}\widehat{C}^\bullet_{F,Y} \quad \text{with} \quad \widehat{C}^\bullet_{F,Y} := \big(\Omega^\bullet_{Y/S,0}((u)),\, ud - dF\wedge\big),$$

see also [SaSa], etc. for the case $S = pt$. It has the formal Brieskorn lattice defined by

$$\widehat{H}''_{F,S} := H^{n+1}\widehat{C}^{(0),\bullet}_{F,Y} \quad \text{with} \quad \widehat{C}^{(0),\bullet}_{F,Y} := \big(\Omega^\bullet_{Y/S,0}[[u]],\, ud - dF\wedge\big).$$

We also have the bi-formal Gauss-Manin system defined by

$$\widehat{G}_{F,\widehat{S}} := H^{n+1}\widehat{C}^{\bullet}_{F,\widehat{Y}} \quad \text{with} \quad \widehat{C}^{\bullet}_{F,\widehat{Y}} := \big(\,\Omega^{\bullet}_{X,0}((u))[[\mathbf{s}]],\, ud - dF\wedge\big),$$

with $[[\mathbf{s}]] := [[s_1,\dots,s_\mu]]$, and similarly for $\widehat{H}''_{F,\widehat{S}}$ and $\widehat{C}^{(0),\bullet}_{F,\widehat{Y}}$ with $((u))$ replaced by $[[u]]$.

We can define similarly

$$G_{f,S},\quad \widehat{G}_{f,S},\quad \widehat{G}_{f,\widehat{S}},\quad H''_{f,S},\quad \widehat{H}''_{f,S},\quad \widehat{H}''_{f,\widehat{S}},$$

by replacing F with f in the above definitions, where f is viewed as a trivial deformation.

We also have $\widehat{G}_f$, $\widehat{H}''_f$ by replacing $\Omega^{\bullet}_{Y/S,0}$ with $\Omega^{\bullet}_{X,0}$ in the definition of $\widehat{G}_{f,S}$, $\widehat{H}''_{f,S}$. There are canonical isomorphisms

$$\widehat{G}_{F,\widehat{S}}\big|_0 = \widehat{G}_f,\quad \widehat{H}''_{F,\widehat{S}}\big|_0 = \widehat{H}''_f, \tag{1.1.3}$$

and similar isomorphisms with F replaced by f. Here we set for any $\mathbb{C}[[\mathbf{s}]]$-module N

$$N|_0 := N/\mathfrak{m}_0 N = N\otimes_{\mathbb{C}[[\mathbf{s}]]}\mathbb{C}, \tag{1.1.4}$$

where $\mathfrak{m}_0$ is the maximal ideal of $\mathbb{C}[[\mathbf{s}]]$. We also have a canonical injection

$$\iota : \widehat{G}_f \hookrightarrow \widehat{G}_{f,\widehat{S}}. \tag{1.1.5}$$

There are natural isomorphisms

$$\Omega_{F,\widehat{S}} = \widehat{H}''_{F,\widehat{S}}/\partial_t^{-1}\widehat{H}''_{F,\widehat{S}},\quad \Omega_{f,\widehat{S}} = \widehat{H}''_{f,\widehat{S}}/\partial_t^{-1}\widehat{H}''_{f,\widehat{S}},\quad \Omega_f = \widehat{H}''_f/\partial_t^{-1}\widehat{H}''_f,$$

where $\Omega_{F,\widehat{S}}$, Ω_f are as in the introduction, and $\Omega_{f,\widehat{S}} = \Omega_f[[\mathbf{s}]]$. We have the canonical isomorphisms

$$\Omega_{F,\widehat{S}}\big|_0 = \Omega_f,\quad \Omega_{f,\widehat{S}}\big|_0 = \Omega_f. \tag{1.1.6}$$

Proposition 1.2. *With the above notation, $\widehat{G}_{F,\widehat{S}}$ and $\widehat{H}''_{F,\widehat{S}}$ are finite free modules of rank μ over $\mathbb{C}((u))[[\mathbf{s}]]$ and $\mathbb{C}[[u,\mathbf{s}]] = \mathbb{C}[[u,s_1,\dots,s_m]]$ respectively, where μ is the Milnor number of f. We have a similar assertion with F replaced by f.*

Proof. It is enough to show the assertion for F since the assertion for f is the special case of a trivial deformation.

Let $U^{\bullet}$ be the $\mathfrak{m}_0$-adic filtration on $\widehat{C}^{\bullet}_{F,Y}$, $\widehat{C}^{\bullet}_{F,\widehat{Y}}$, i.e.

$$U^k\,\widehat{C}^{\bullet}_{F,Y} = \mathfrak{m}_0^k\,\widehat{C}^{\bullet}_{F,Y},\ \text{ etc.}$$

Then $\widehat{C}^{\bullet}_{F,\widehat{Y}}$ is the $\mathfrak{m}_0$-adic completion of $\widehat{C}^{\bullet}_{F,Y}$ so that

$$\widehat{C}^{\bullet}_{F,\widehat{Y}} = \varprojlim_k \widehat{C}^{\bullet}_{F,\widehat{Y}}/\mathfrak{m}_0^k\,\widehat{C}^{\bullet}_{F,\widehat{Y}} = \varprojlim_k \widehat{C}^{\bullet}_{F,Y}/\mathfrak{m}_0^k\,\widehat{C}^{\bullet}_{F,Y}. \tag{1.2.1}$$

Moreover the filtration U induces a strict filtration on the complexes, and the induced filtration U on the cohomology groups coincides with the $\mathfrak{m}_0$-adic filtration on these $\mathbb{C}[[\mathbf{s}]]$-modules so that

$$\widehat{G}_{F,\widehat{S}} = \varprojlim_k \widehat{G}_{F,\widehat{S}}/\mathfrak{m}_0^k\,\widehat{G}_{F,\widehat{S}} = \varprojlim_k \widehat{G}_{F,S}/\mathfrak{m}_0^k\,\widehat{G}_{F,S}, \tag{1.2.2}$$

(and a similar assertion holds for the corresponding Brieskorn lattices). These are shown by an argument similar to [Sa1], [Sa2] using the acyclicity of the complexes $\mathrm{Gr}_U^k\,\widehat{C}^{\bullet}_{F,Y}$ except for the highest degree together with the Mittag-Leffler condition [Gro]. Here the acyclicity follows from the canonical isomorphisms

$$\mathrm{Gr}_U^0\,\widehat{C}^{\bullet}_{F,Y}\otimes_{\mathbb{C}}\mathrm{Gr}_U^k\,\mathbb{C}[[\mathbf{s}]] \xrightarrow{\sim} \mathrm{Gr}_U^k\,\widehat{C}^{\bullet}_{F,Y}. \tag{1.2.3}$$

Taking the cohomology of the last isomorphism and using the strictness of the filtration U, we then get the isomorphisms

$$\mathrm{Gr}_U^0 \widehat{G}_{F,S} \otimes_{\mathbb{C}} \mathrm{Gr}_U^k \mathbb{C}[[\mathbf{s}]] \xrightarrow{\sim} \mathrm{Gr}_U^k \widehat{G}_{F,S} \,(= \mathrm{Gr}_U^k \widehat{G}_{F,\widehat{S}}). \tag{1.2.4}$$

This implies that $\widehat{G}_{F,\widehat{S}}$ is free of rank μ over $\mathbb{C}((u))[[\mathbf{s}]]$ since $\mathrm{Gr}_U^0 \widehat{G}_{F,S} = \widehat{G}_f$ is free of rank μ over $\mathbb{C}((u))$. The argument is similar for $\widehat{H}''_{F,\widehat{S}}$. This finishes the proof of Proposition 1.2.

Proposition 1.3 (compare to [LLS]). *We have the exponential operator*

$$\Psi := e^{(F-f)/u} : \widehat{G}_{f,\widehat{S}} \to \widehat{G}_{F,\widehat{S}}, \tag{1.3.1}$$

which is an isomorphism of finite free $\mathbb{C}((u))[[\mathbf{s}]]$*-modules with inverse given by*

$$\Phi := e^{(f-F)/u} : \widehat{G}_{F,\widehat{S}} \to \widehat{G}_{f,\widehat{S}}. \tag{1.3.2}$$

Moreover, these are compatible with the actions of t and ∂_{s_i}.

Proof. Since $F - f \in \mathfrak{m}_0\mathcal{O}_{Y,0}$, we can verify that Ψ and Φ induce $\mathbb{C}((u))[[\mathbf{s}]]$-linear morphisms between the complexes $\widehat{C}^{\bullet}_{F,\widehat{Y}}$ and $\widehat{C}^{\bullet}_{f,\widehat{Y}}$, and these are inverse of each other. Moreover they are compatible with the actions of t and ∂_{s_i} which are defined by using (1.1.2). (For t, set $v := u^{-1} = \partial_t$, which gives the Fourier transform of t, i.e. t is identified with $-\partial_v$.) This finishes the proof of Proposition 1.3.

Theorem 1.4. *Let $\sigma_{\widehat{S}} : \Omega_{F,\widehat{S}} \to \widehat{H}''_{F,\widehat{S}}$ be a* $\mathbb{C}[[\mathbf{s}]]$*-linear section of the canonical projection $p_{F,\widehat{S}} : \widehat{H}''_{F,\widehat{S}} \to \Omega_{F,\widehat{S}}$ satisfying the condition*

$$\partial_{s_i} I_{\widehat{S}} \subset I_{\widehat{S}} + u^{-1} I_{\widehat{S}} \quad \textit{with} \quad I_{\widehat{S}} := \mathrm{Im}\,\sigma_{\widehat{S}}. \tag{1.4.1}$$

Such a section of $p_{F,\widehat{S}}$ is uniquely determined by $I_0 := I_{\widehat{S}}\big|_0 \subset \widehat{G}_f$ so that

$$I_{\widehat{S}} = \widehat{H}''_{F,\widehat{S}} \cap \Psi\big(\iota\big(I_0[u^{-1}]\big)[[\mathbf{s}]]\big). \tag{1.4.2}$$

Proof. By the isomorphism (1.3.1), the assertion is equivalent to

$$\Phi(I_{\widehat{S}}) = \Phi\big(\widehat{H}''_{F,\widehat{S}}\big) \cap \iota\big(I_0[u^{-1}]\big)[[\mathbf{s}]] \quad \text{in} \quad \widehat{G}_{f,\widehat{S}}. \tag{1.4.3}$$

We will show the inclusion $\subset$ together with the assertion that the right-hand side of (1.4.3) is isomorphic to $\Phi\big(\Omega_{F,\widehat{S}}\big)$ by the projection $\Phi(p_{F,\widehat{S}})$ so that it also gives a section of $\Phi(p_{F,\widehat{S}})$.

By Propositions 1.2 and 1.3, $\widehat{H}''_{F,\widehat{S}}$ and $\Phi(\widehat{H}''_{F,\widehat{S}})$ are free $\mathbb{C}[[u,\mathbf{s}]]$-submodules of $\widehat{G}_{F,\widehat{S}}$ and $\widehat{G}_{f,\widehat{S}}$ respectively with rank μ. We have moreover

$$\mathfrak{m}_0^k\, \Phi(\widehat{H}''_{F,\widehat{S}}) = \Phi(\widehat{H}''_{F,\widehat{S}}) \cap \mathfrak{m}_0^k\, \widehat{G}_{f,\widehat{S}}, \tag{1.4.4}$$

i.e. the inclusion $\Phi(\widehat{H}''_{F,\widehat{S}}) \hookrightarrow \widehat{G}_{f,\widehat{S}}$ is strictly compatible with the $\mathfrak{m}_0$-adic filtration. This follows from the injective morphism of short exact sequences

$$\begin{array}{ccccccccc}
0 & \to & \mathfrak{m}_0^k\, \Phi(\widehat{H}''_{F,\widehat{S}}) & \to & \Phi(\widehat{H}''_{F,\widehat{S}}) & \to & \Phi(\widehat{H}''_{F,\widehat{S}})/\mathfrak{m}_0^k\, \Phi(\widehat{H}''_{F,\widehat{S}}) & \to & 0 \\
 & & \cap & & \cap & & \cap & & \\
0 & \to & \mathfrak{m}_0^k\, \widehat{G}_{f,\widehat{S}} & \to & \widehat{G}_{f,\widehat{S}} & \to & \widehat{G}_{f,\widehat{S}}/\mathfrak{m}_0^k\, \widehat{G}_{f,\widehat{S}} & \to & 0
\end{array}$$

Here the injectivity of the last vertical morphism is reduced to the case $k = 1$ by using the graded quotients Gr_U^j of the $\mathfrak{m}_0$-adic filtration U together with isomorphisms similar to (1.2.4) (which hold also for $\Phi(\widehat{H}''_{F,\widehat{S}})$ since it is a finite free $\mathbb{C}[[u,s]]$-module).

Using again the graded quotients Gr_U^j together with (1.4.4) and isomorphism similar to (1.2.4), we then get

$$\widehat{G}_{f,\widehat{S}} = \Phi(\widehat{H}''_{F,\widehat{S}}) \oplus \iota\big(u^{-1}I_0[u^{-1}]\big)[[\mathbf{s}]], \tag{1.4.5}$$

since

$$\widehat{G}_f = \widehat{H}''_f \oplus u^{-1}I_0[u^{-1}] \quad \text{and} \quad \Phi(\widehat{H}''_{F,\widehat{S}})/\mathfrak{m}_0\Phi(\widehat{H}''_{F,\widehat{S}}) = \widehat{H}''_f.$$

By (1.4.5) we get the isomorphism between the right-hand side of (1.4.3) and $\Phi\big(\Omega_{F,\widehat{S}}\big)$.

It now remains to show

$$\Phi(I_{\widehat{S}}) \subset \iota\big(I_0[u^{-1}]\big)[[\mathbf{s}]]. \tag{1.4.6}$$

But this follows immediately from condition (1.4.1). In fact, $\widehat{G}_{f,\widehat{S}}$ is identified with $\widehat{G}_f[[\mathbf{s}]]$ so that any element of $\widehat{G}_{f,\widehat{S}}$ has a Taylor expansion in s, and moreover, the above identification and Φ are compatible with the iterated actions of the ∂_{s_i} and also with the restriction to $\mathbf{s}=0$. This finishes the proof of Theorem 1.4.

Remarks 1.5. (i) Formal Gauss-Manin systems and formal Brieskorn lattices are treated also in [LLS] where the use of polyvector fields does not seem to be quite essential for them.

(ii) The commutativity of the projective limit and the cohomology does not seem to be explained in [LLS]. Here the Mittag-Leffler condition as in [Gro] is usually needed. This point is not completely trivial even if we have the acyclicity of the complex except for the top degree. For instance, it is not quite clear whether any surjective morphism of projective systems induces a surjective morphism by passing to the projective limit, unless we know that the Mittag-Leffler condition is satisfied for the projective system defined by the kernel, see [Gro]. This might be applied to the surjection from the top term of the complex to the cohomology, where the strictness of the last differential is related.

(iii) The construction in [LLS] is slightly different from the one in earlier papers [SK1], [SK2], where the deformation F of f was defined over a space of dimension $\mu-1$, instead of μ, and the value of F together with the natural projection is used in order to define a morphism to a space S of dimension μ. Note also that one gets a formal Gauss-Manin system of $\mu+1$ variables in [LLS], where the relative critical locus C is finite and *flat* over S, although the image of C in S is the discriminant locus in [SK1], [SK2], since F is used for the morphism to S.

(iv) It seems to be quite difficult to prove the convergent version of Theorem 1. Even in case $f = x^a + y^b$ with $1/a + 1/b < 1/2$, for instance, the convergence of the image of a monomial in x, y by Ψ seems to be quite non-trivial. (Note that, even if we get a divergent power series by this, it does not contradict the result of Malgrange since the procedure of extending good sections is not so simple.) Here the calculation seems easier for Φ. It may be possible to show the convergence in $\mathbf{s}$ for each fixed degree part for the variable u provided that we take a standard representative of the versal deformation of f (i.e. $F = f + \sum_i g_i s_i$ with g_i monomial generators of the Jacobian ring).

(v) It does not seem to be very clear what kind of argument is used for the proof of the coincidence of the new construction of the higher residue pairings in [LLS] with the old one. It could be shown, for instance, by using the uniqueness (up to a constant multiple) of the pairing in the versal unfolding case by generalizing an argument in [Sa3, 2.7] about the duality of simple holonomic $\mathcal{E}$-modules to the $\widehat{\mathcal{E}}$-module case and using the compatibility with the base change by $\{0\} \hookrightarrow S$ for the one variable case. Here it does not seem easy to conclude it only by using the coincidence after taking the graded quotients of the Hodge filtration, since an automorphism of a filtered Gauss-Manin system of one variable is not necessarily the identity even if it induces the identity by taking the graded quotients. (Note that a non-degenerate pairing can be identified with an isomorphism with the dual up to a shift of filtration. If there are two non-degenerate pairings, then we can compose one isomorphism with the inverse of the other so that we get an automorphism.)

(vi) If polyvector fields are used in the theory of primitive forms as in [LLS], one may have to divide a representative of a primitive form by a holomorphic relative differential form of the highest degree $\Omega_{Z/S}$ in order to get a representative in the polyvector fields. In this case one might get a "primitive function" rather than a primitive form (and this may be more natural for the product structure). In the simple singularity case, it is a constant function, and this seems always possible provided that one can take the relative differential form $\Omega_{Z/S}$ to be the primitive form in the usual sense.

2. Some explicit calculations

In this section we present an algorithm for an inductive computation of the coefficients of the Taylor expansion of primitive forms in the Brieskorn-Pham polynomial case, which is apparently simpler in this case than the one in [LLS].

2.1. Primitive forms. In the notation of the introduction, assume F is a miniversal deformation of f as in [LLS] so that

$$\dim S=\mu\,(:=\dim\mathcal{O}_{X,0}/(\partial f)).$$

Let $\sigma_0:\Omega_f\hookrightarrow H''_f$ be a good section of $\mathrm{pr}_0:H''_f\to\Omega_f$ in (0.1) satisfying

$$S_K(\omega,\omega')\subset\mathbb{C}\,u^{n+1}\quad\text{for }\ \omega,\omega'\in\operatorname{Im}\sigma_0. \tag{2.1.1}$$

Here $u:=\partial_t^{-1}$, and we denote in this paper the higher residue pairings by

$$S_K:G_f\times G_f\to K:=\mathbb{C}\{\{u\}\}[u^{-1}]. \tag{2.1.2}$$

Note that

$$S_K(\omega,\omega')\subset\mathbb{C}\{\{u\}\}\,u^{n+1}\quad\text{for any }\ \omega,\omega'\in H''_f. \tag{2.1.3}$$

This implies a rather strong restriction on H''_f.

By Malgrange's theory on Birkhoff's Riemann-Hilbert problem (see [Ma2], [Ma3]), any good section σ_0 of $\mathrm{pr}_0:H''_f\twoheadrightarrow H''_f/\partial_t^{-1}H''_f\cong\Omega_f$ in (0.1) can be uniquely extended to a good $\mathbb{C}\{\mathbf{s}\}$-linear section

$$\sigma_S:\Omega_{F,S}\hookrightarrow H''_{F,S}$$

of

$$\mathrm{pr}_S:H''_{F,S}\twoheadrightarrow H''_{F,S}/\partial_t^{-1}H''_{F,S}\cong\Omega_{F,S},$$

as is explained in the introduction. Moreover the good section σ_0 is uniquely lifted to a $\mathbb{C}$-linear morphism

$$\sigma_S^{\nabla}:\Omega_f\hookrightarrow H''_{F,S},$$

so that

$$\operatorname{Im}\sigma_S^{\nabla}\subset\operatorname{Im}\sigma_S,\quad \partial_{s_j}(\operatorname{Im}\sigma_S^{\nabla})\subset\partial_t\,(\operatorname{Im}\sigma_S). \tag{2.1.4}$$

In fact, the second condition of (0.2) in the introduction implies an integrable connection on Ω_S (by considering the action of ∂_{s_j} on I_S mod ∂_tI_S), and σ_S^{∇} is defined by using the flat sections of this connection so that only the component of the second term ∂_tI_S in the second condition of (0.2) remains (see [SK1], [SK2]). Thus the second condition of (2.1.4) holds. Here (2.1.1) is also extended to the case of σ_S^{∇}. Note that, by the uniqueness of the extension in Theorem 1.4, these constructions are compatible with the formal completion and we have similarly $\sigma_{\widehat{S}}^{\nabla}$, etc.

Assume there is $\overline{\zeta}_0 \in \Omega_f$ which is an eigenvector of A_1 in (0.1), and generates Ω_f over $\mathbb{C}\{x\}$. Set

$$\zeta_0 := \sigma_0(\overline{\zeta}_0) \in H''_f.$$

In the weighted homogeneous polynomial case, we have up to a nonzero constant multiple

$$\zeta_0 = [dx_0 \wedge \cdots \wedge dx_n], \tag{2.1.5}$$

where $x_0, \dots, x_n$ are coordinates such that $\sum_i w_i x_i \partial_{x_i} f = f$ with $w_i \in \mathbb{Q}_{>0}$. (This follows from Proposition 3.1 below.)

The primitive form ζ_S associated with σ_0 and $\overline{\zeta}_0$ is then defined by

$$\zeta_S := \sigma_S^{\nabla}(\overline{\zeta}_0) \in H''_{F,S}.$$

Similarly the formal primitive form $\zeta_{\widehat{S}}$ associated with σ_0 and $\overline{\zeta}_0$ is defined by

$$\zeta_{\widehat{S}} := \sigma_{\widehat{S}}^{\nabla}(\overline{\zeta}_0) \in \widehat{H}''_{F,\widehat{S}}.$$

The latter coincides with the image of ζ_S in $\widehat{H}''_{F,\widehat{S}}$ by Theorem 1.4 together with a remark after (2.1.4).

2.2. Relation with the exponential operators Ψ and Φ. In the notation of (2.1) and Proposition 1.3, the formal primitive form $\zeta_{\widehat{S}}$ is the unique element of $\widehat{H}''_{F,\widehat{S}}$ satisfying

$$\Phi(\zeta_{\widehat{S}}) = \iota(\zeta_0) \mod \iota\big(u^{-1} I_0[u^{-1}]\big)[[\mathbf{s}]], \tag{2.2.1}$$

where $I_0 := \operatorname{Im} \sigma_0$, $u := \partial_t^{-1}$, and ι is as in (1.1.5). In fact, the uniqueness of $\zeta_{\widehat{S}}$ follows from the direct sum decomposition (1.4.5), and (2.2.1) holds since

$$\Phi(\zeta_{\widehat{S}})\big|_0 = \zeta_{\widehat{S}}\big|_0 = \zeta_0, \quad \partial_{s_j} \Phi(\zeta_{\widehat{S}}) = \Phi(\partial_{s_j} \zeta_{\widehat{S}}) \in \iota\big(u^{-1} I_0[u^{-1}]\big)[[\mathbf{s}]],$$

where the last assertion follows from the proof of Theorem 1.4 together with the second condition of (2.1.4).

This characterization of formal primitive forms is compatible with the construction in [LLS], since (2.2.1) is equivalent to

$$\zeta_{\widehat{S}} = \Psi\big(\iota(\zeta_0)\big) \mod \Psi\big(\iota\big(u^{-1} I_0[u^{-1}]\big)[[\mathbf{s}]]\big). \tag{2.2.2}$$

2.3. Case of Brieskorn-Pham polynomials. Assume

$$f := \textstyle\sum_{i=0}^n x_i^{m_i} \quad (m_i \geqslant 2),$$

i.e. f is a Brieskorn-Pham polynomial. In this case we can calculate the first few terms of the coefficients of the Taylor expansion of $\zeta_{\widehat{S}}$ without using a computer program as follows.

Set

$$\Gamma := \mathbb{N}^{n+1} \cap \textstyle\prod_{i=0}^n [0, m_i - 2],$$

so that

$$\#\Gamma = \textstyle\prod_{i=0}^n (m_i - 1) = \mu.$$

We have the natural coordinates s_ν of $S = \mathbb{C}^\mu$ for $\nu = (\nu_1, \dots, \nu_n) \in \Gamma$. We may assume

$$F = f + \textstyle\sum_{\nu \in \Gamma} g_\nu s_\nu \quad \text{with} \quad g_\nu = x^\nu := \prod_i x_i^{\nu_i} \ \ (\nu \in \Gamma). \tag{2.3.1}$$

Moreover we have the canonical good section σ_0 such that

$$I_0\,(:= \operatorname{Im} \sigma_0) = \textstyle\sum_{\nu \in \Gamma} \mathbb{C}\,[g_\nu \omega_0] \subset H''_f \quad \text{with} \quad \omega_0 := dx_0 \wedge \cdots \wedge dx_n.$$

In the Brieskorn-Pham polynomial case we have for any $\nu=(\nu_0,\dots,\nu_n)\in\mathbb{N}^{n+1}$

$$\partial_t\,[x^\nu\omega_0]=\frac{\nu_i-m_i+1}{m_i}[x^\nu x_i^{-m_i}\omega_0]\quad\text{if }\ \nu_i\geqslant m_i-1. \tag{2.3.2}$$

This implies

$$[x^\nu\omega_0]=0\ \text{ in }\ H_f''\quad\text{if }\ \nu_i+1\in m_i\mathbb{N}\ \text{ for some }\ i. \tag{2.3.3}$$

(These become more complicated in the general weighted homogeneous polynomial case.)

Let $\zeta_{S,k}$ be the image of ζ_S in $H''_{F,S}/\mathfrak{m}_0^{k+1}H''_{F,S}$, where $\mathfrak{m}_0$ is the maximal ideal of $\mathcal{O}_{S,0}$, and k is a positive integer (which may be determined by the computational ability). Set

$$A_k:=\bigl\{a=(a_\nu)\in\mathbb{N}^\Gamma\bigm| |a|\leqslant k\bigr\}\quad\text{with}\quad |a|:=\textstyle\sum_{\nu\in\Gamma}a_\nu.$$

For $a=(a_\nu)\in\mathbb{N}^\Gamma$, define

$$p(a)=\bigl(p(a)_0,\dots,p(a)_n\bigr)\in\mathbb{N}^{n+1}\quad\text{by}\quad p(a)_i:=\textstyle\sum_{\nu\in\Gamma}\nu_i a_\nu,$$

so that

$$g^a:=\textstyle\prod_\nu g_\nu^{a_\nu}=\prod_{i,\nu}x_i^{\nu_i a_\nu}=\prod_i x_i^{p(a)_i}=:x^{p(a)}.$$

Define further

$$q(a)=\bigl(q(a)_0,\dots,r(a)_n\bigr),\ \ r(a)=\bigl(r(a)_0,\dots,r(a)_n\bigr)\ \text{ in }\ \mathbb{N}^{n+1},$$

by the condition

$$p(a)_i=q(a)_i\,m_i+r(a)_i\quad\text{with}\quad 0\leqslant r(a)_i<m_i\quad(\forall\,i\in[0,n]).$$

In particular, we have

$$q(a)_i=\left\lfloor\frac{p(a)_i}{m_i}\right\rfloor. \tag{2.3.4}$$

(Note that $\lfloor\alpha\rfloor:=\max\{k\in\mathbb{Z}\mid k\leqslant\alpha\}$ for $\alpha\in\mathbb{R}$.) Set

$$e_a=\textstyle\sum_{i=0}^n q(a)_i\,-\,|a|,$$

and

$$A'_k:=\bigl\{a\in A_k\bigm| e_a\geqslant 0,\ r(a)\in\Gamma\bigr\}.$$

Note that the last condition $r(a)\in\Gamma$ is equivalent to that $r(a)_i\neq m_i-1$ $(\forall i)$.

Using the characterization of $\zeta_{S,k}$ in (2.2.1), we then get the following Taylor expansion in s by increasing induction on $|\nu|:=\sum_i\nu_i\leqslant k$:

$$\zeta_{S,k}=\sum_{a\in A'_k}c_a\,\partial_t^{-e_a}\bigl[\,g_{r(a)}\,s^a\,\omega_0\,\bigr]_F\in H''_{F,S}/\mathfrak{m}_0^{k+1}H''_{F,S}, \tag{2.3.5}$$

with $c_a\in\mathbb{C}$, $s^a:=\prod_{\nu\in\Gamma}s_\nu^{a_\nu}$, and $g_{r(a)}=x^{r(a)}$ by definition. Here $[\eta]_F$ for $\eta\in\Omega_{Y/S}^{n+1}$ denotes its class in $H''_{F,S}$ (mod $\mathfrak{m}_0^{k+1}$). For $\omega\in\Omega_{X,0}^{n+1}$, its class in H''_f is simply denoted by $[\omega]$. We have

$$[s^\nu\eta]_F=s^\nu[\eta]_F,$$

since the differential of the Gauss-Manin complex is $\mathcal{O}_S$-linear. Note, however, that

$$[s^\nu\omega]_F\neq s^\nu[\omega]\quad\bigl(\text{i.e.,}\quad[\omega]_F\neq[\omega]\bigr)\quad\text{for}\quad\omega\in\Omega_{X,0}^{n+1}.$$

In fact, they belong to different groups $H''_{F,S}$ and $H''_{f,S}$ or H''_f. (This is related with a question of C. Li. It is a source of an error in a previous version where the formula was too much simplified.)

By the characterization (2.2.1) the summation in (2.3.5) is actually taken over

$$A''_k:=\bigl\{a\in A_k\bigm|\partial_t^{|a|}[g^a\omega_0]\notin\partial_t I_0[\partial_t]\bigr\}.$$

In the Brieskorn-Pham polynomial case we have

$$\partial_t^{|a|}[g^a\omega_0] \notin \partial_t I_0[\partial_t] \iff \partial_t^{|a|}[g^a\omega_0] \in H''_f \setminus \{0\}, \tag{2.3.6}$$

by (2.3.2) and (2.3.3). Using the last two formulas again, we then get

$$A'_k = A''_k,$$

together with the Taylor expansion (2.3.5) inductively.

The coefficients c_a for $a \in A'_k$ are inductively determined by comparing the coefficients of both sides of (2.2.1). Since

$$e^{(f-F)\partial_t} = e^{-\sum_{\nu\in\Gamma} g_\nu s_\nu \partial_t} = \textstyle\prod_{\nu\in\Gamma} e^{-g_\nu s_\nu \partial_t}, \tag{2.3.7}$$

we get by using (2.3.2)

$$c_a = -\sum_{0\leqslant b<a}\left((-1)^{|a-b|}\,\frac{c_b}{(a-b)!}\prod_{i=0}^{n}\prod_{k_i=1}^{q(a,b)_i}\frac{r(b)_i+p(a-b)_i-k_i m_i+1}{m_i}\right), \tag{2.3.8}$$

with

$$q(a,b)_i := \left\lfloor \frac{r(b)_i+p(a-b)_i}{m_i}\right\rfloor.$$

Here $\lfloor * \rfloor$ is as in a remark after (2.3.4), $(a-b)! := \prod_{\nu\in\Gamma}(a_\nu-b_\nu)!$, and we have by definition

$$b \leqslant a \iff b_\nu \leqslant a_\nu \ (\forall\, \nu \in \Gamma), \quad \text{and} \quad b < a \iff b \leqslant a \ \text{and}\ b \neq a.$$

2.4. Example. Assume $f = x_1^7 + x_2^3$ and $k = 6$. Then the $s^a = \prod_\nu s_\nu^{a_\nu}$ for $a \in A'_k \setminus \{0\}$ are

$$s_{(5,1)}^3, \quad s_{(4,1)}s_{(5,1)}^2, \quad s_{(5,1)}^6, \quad s_{(4,1)}s_{(5,1)}^5, \quad s_{(4,1)}^2 s_{(5,1)}^4, \quad s_{(3,1)}s_{(5,1)}^5. \tag{2.4.1}$$

The corresponding $g_{r(a)} = x^{r(a)}$ in (2.3.5) are respectively

$$x_1, \quad 1, \quad x_1^2, \quad x_1, \quad 1, \quad 1, \tag{2.4.2}$$

and we have $e_a = 0$ for $a \in A'_k$ in this case. We denote the corresponding coefficients c_a by

$$c_{(1)}, \ \ldots\ , c_{(6)}. \tag{2.4.3}$$

Using (2.3.8), we first get

$$c_{(1)} = \tfrac{1}{3!}\cdot\tfrac{9\cdot 2}{7^2\cdot 3} = \tfrac{1}{7^2}\,,$$
$$c_{(2)} = \tfrac{1}{2!}\cdot\tfrac{8}{7^2\cdot 3} = \tfrac{2^2}{7^2\cdot 3}\,,$$

and then verify that $c_{(3)},\ldots,c_{(6)}$ are respectively equal to

$$-\tfrac{1}{6!}\cdot\tfrac{24\cdot 17\cdot 10\cdot 3\cdot 4}{7^4\cdot 3^2}+\tfrac{1}{3!}\cdot\tfrac{1}{7^2}\cdot\tfrac{10\cdot 3}{7^2\cdot 3} = -\tfrac{17\cdot 2^2}{7^4\cdot 3^2}+\tfrac{5}{7^4\cdot 3} = \tfrac{-68+15}{7^4\cdot 3^2} = -\tfrac{53}{7^4\cdot 3^2}\,,$$
$$-\tfrac{1}{5!}\cdot\tfrac{23\cdot 16\cdot 9\cdot 2\cdot 4}{7^4\cdot 3^2}+\tfrac{1}{2!}\cdot\tfrac{1}{7^2}\cdot\tfrac{9\cdot 2}{7^2\cdot 3}+\tfrac{1}{3!}\cdot\tfrac{2^2}{7^2\cdot 3}\cdot\tfrac{9\cdot 2}{7^2\cdot 3} = -\tfrac{23\cdot 2^4}{7^4\cdot 5\cdot 3}+\tfrac{3}{7^4}+\tfrac{2^2}{7^4\cdot 3} = \tfrac{-368+45+20}{7^4\cdot 5\cdot 3} = -\tfrac{101}{7^4\cdot 5}\,,$$
$$-\tfrac{1}{4!\cdot 2!}\cdot\tfrac{22\cdot 15\cdot 8\cdot 4}{7^4\cdot 3^2}+\tfrac{1}{2!}\cdot\tfrac{1}{7^2}\cdot\tfrac{8}{7^2\cdot 3}+\tfrac{1}{2!}\cdot\tfrac{2^2}{7^2\cdot 3}\cdot\tfrac{8}{7^2\cdot 3} = -\tfrac{11\cdot 5\cdot 2^2}{7^4\cdot 3^2}+\tfrac{2^2}{7^4\cdot 3}+\tfrac{2^4}{7^4\cdot 3^2} = \tfrac{(-55+3+4)2^2}{7^4\cdot 3^2} = -\tfrac{2^6}{7^4\cdot 3}\,,$$
$$-\tfrac{1}{5!}\cdot\tfrac{22\cdot 15\cdot 8\cdot 4}{7^4\cdot 3^2}+\tfrac{1}{2!}\cdot\tfrac{1}{7^2}\cdot\tfrac{8}{7^2\cdot 3} = -\tfrac{11\cdot 2^3}{7^4\cdot 3^2}+\tfrac{2^2}{7^4\cdot 3} = \tfrac{(-22+3)2^2}{7^4\cdot 3^2} = -\tfrac{19\cdot 2^2}{7^4\cdot 3^2}\,.$$

The conclusion agrees with a calculation in [LLS] using a different algorithm together with a computer program.

3. Good sections and very good sections

In this section we give some remarks related to good sections and very good sections in the sense of this paper.

Proposition 3.1. *In the notation of the introduction, any good section of* pr_0 *is very good, if* f *is a weighted homogeneous polynomial.*

Proof. By definition (see (1.1.2)), A_0 in (0.1) is identified with the action of f on the Jacobian ring $\mathbb{C}\{x\}/(\partial f)$, and it vanishes in the weighted homogeneous case. Hence the image of the section is stable by the action of $\partial_t t$ which is identified with A_1. So the assertion follows.

The following proposition implies a formula for the dimension of the parameter space of very good sections satisfying the orthogonality condition for the self-duality in the case $N = 0$ (including the weighted homogeneous polynomial case), see Corollary (3.3) below.

Proposition 3.2. *Let* H *be a finite dimensional* $\mathbb{C}$*-vector space with a finite filtration* F*. Let* S *be a self-pairing of* H *such that* $S(F^pH, F^qH) = 0$ *for* $p+q = m+1$*, and the induced pairing of* $\mathrm{Gr}_F^p H$ *and* $\mathrm{Gr}_F^q H$ *is non-degenerate for* $p+q = m$*, where* $m \in \mathbb{Z}$ *is a fixed number. Assume* S *is* $(-1)^m$*-symmetric, i.e.* $S(u,v) = (-1)^m S(v,u)$*. Set* $e_p := \dim \mathrm{Gr}_F^p H$*. Then splittings* $H = \bigoplus_k G^k$ *of the filtration* F *(i.e.* $F^P H = \bigoplus_{k\geqslant p} G^k$*) satisfying the condition* $S(G^p, G^q) = 0$ $(p+q \ne m)$ *are parametrized by* $\mathbb{C}^{\,d(H,F,S)}$ *with*

$$d(H,F,S) := \begin{cases} \sum_{p<q<m-p} e_p e_q + \sum_{p<m/2} \binom{e_p}{2} & \text{if } m \text{ is even,} \\ \sum_{p<q<m-p} e_p e_q + \sum_{p<m/2} \binom{e_p+1}{2} & \text{if } m \text{ is odd.} \end{cases} \tag{3.2.1}$$

Proof. Let $\overline{S}$ denote the induced pairing of $\mathrm{Gr}_F^p H \times \mathrm{Gr}_F^{m-p} H$. We have $e_p = e_{m-p}$ since $\overline{S}$ is non-degenerate. Take bases $(\overline{v}_{p,i})_{i\in[1,e_p]}$ of $\mathrm{Gr}_F^p H$ $(p \in \mathbb{Z})$ satisfying

$$\overline{S}(\overline{v}_{p,i}, \overline{v}_{m-p,j}) = \varepsilon_p\, \delta_{i,j} \quad \text{with} \quad \varepsilon_p = \pm 1,$$

where $\delta_{i,j} = 1$ if $i = j$, and 0 otherwise. Since $S(u,v)$ is $(-1)^m$-symmetric, we have

$$\varepsilon_p = (-1)^m \varepsilon_{m-p}. \tag{3.2.2}$$

We can lift $\overline{v}_{p,i}$ to $v_{p,i} \in F^pH \subset H$ so that

$$S(v_{p,i}, v_{q,j}) = \varepsilon_p\, \delta_{p,m-q}\, \delta_{i,j} \quad (\text{with } \varepsilon_p \text{ as above}). \tag{3.2.3}$$

This will be shown in Lemma 3.4 below. (In the case of polarized Hodge structures as in the case of Corollary (3.3) below, this easily follows from the Hodge decomposition.)

Set

$$I := \{(p,i) \in \mathbb{Z}^2 \mid i \in [1, e_p]\},$$

where $[1, e_p] = \emptyset$ if $e_p = 0$. Set

$$J := \{((p,i),(q,j)) \in I^2 \mid p < q\} \subset I^2.$$

Then any splitting of the filtration F is expressed by

$$(\theta_{(p,i),(q,j)}) \in \mathbb{C}^J,$$

since it defines a lift $w_{p,i} \in F^pH$ of $\overline{v}_{p,i} \in \mathrm{Gr}_F^p H$ for each (p,i) by

$$w_{p,i} := v_{p,i} + \sum_{(q,j)\in I,\, q>p} \theta_{(p,i),(q,j)}\, v_{q,j} \in F^pH,$$

which is the image of $\overline{v}_{p,i}$ by the splitting of the canonical surjection

$$F^pH \to \mathrm{Gr}_F^p H.$$

Note that the ambiguity of the splitting is given by the vector space

$$\mathrm{Hom}(\mathrm{Gr}_F^p H, F^{p+1}H), \tag{3.2.4}$$

and its dimension is $\sum_{q>p} e_p e_q$ for each p.

The orthogonality condition of the splitting for the pairing S is given by the relations

$$S(w_{p,i}, w_{q,j}) = 0 \quad \text{for } ((p,i),(q,j)) \in R,$$

with

$$R := \begin{cases} \big\{((p,i),(q,j)) \in I^2 \mid p+q<m,\ (p,i) \leqslant (q,j)\big\} & \text{if } m \text{ is even,} \\ \big\{((p,i),(q,j)) \in I^2 \mid p+q<m,\ (p,i) < (q,j)\big\} & \text{if } m \text{ is odd.} \end{cases}$$

Here we use the lexicographic order on I, i.e. $(p,i) < (q,j) \iff p<q$ or $p=q,\ i<j$.

By (3.2.3) we have

$$S(w_{p,i}, w_{q,j}) = \begin{cases} 0 & \text{if } p+q>m, \\ \varepsilon_p\, \delta_{i,j} & \text{if } p+q=m, \end{cases}$$

and $S(w_{p,i}, w_{q,j})$ for $p+q<m$ is given by

$$\begin{aligned} S(w_{p,i}, w_{q,j}) &= \varepsilon_{m-q}\, \theta_{(p,i),(m-q,j)} + \varepsilon_p\, \theta_{(q,j),(m-p,i)} \\ &\quad + \textstyle\sum_{(r,k)\in I,\, p<r<m-q} \varepsilon_r\, \theta_{(p,i),(r,k)}\, \theta_{(q,j),(m-r,k)}. \end{aligned} \tag{3.2.5}$$

Here note that we have by (3.2.2)

$$\varepsilon_{m-q} + \varepsilon_p \neq 0 \ \text{ in the case where } (p,i)=(q,j) \text{ and } m \text{ is even.} \tag{3.2.6}$$

Consider the map

$$\gamma : R \hookrightarrow J \quad ((p,i),(q,j)) \mapsto ((p,i),(m-q,j)).$$

We say that $\theta_{\gamma((p,i),(q,j))} = \theta_{(p,i),(m-q,j)}$ is the *depending parameter* of the relation

$$S(w_{p,i}, w_{q,j}) = 0 \quad \text{for} \quad ((p,i),(q,j)) \in R.$$

By (3.2.5), $\theta_{(p,i),(m-q,j)}$ appears in $S(w_{p,i}, w_{q,j})$ as a linear term with a nonzero coefficient, where (3.2.6) is used in the case $(p,i)=(q,j)$ and m is even. Moreover $\theta_{(p',i'),(m-q',j')}$ appearing in the relation $S(w_{p,i}, w_{q,j}) = 0$ must satisfy the inequality

$$p'+q' \geqslant p+q.$$

(In fact, (p',i') must coincide with (p,i) or (q,j), and the inequality follows from (3.2.5).) This implies that $\theta_{(p,i),(m-q,j)}$ does not appear in the relations

$$S(w_{p',i'}, w_{q',j'}) \quad \text{with} \quad p'+q' > p+q.$$

We can now prove by induction on $p+q$ and using (3.2.5) that the values of the depending parameters are given as polynomials of the remaining parameters

$$\theta_{(p,i),(q,j)} \quad \text{with} \quad ((p,i),(q,j)) \in J \setminus \gamma(R),$$

which are called independent parameters. Thus splittings of the filtration F, which are orthogonal to each other with respect to the pairing S, are parametrized by

$$\mathbb{C}^{J\setminus\gamma(R)}.$$

Moreover we have

$$d(H,F,S) = \#\big(J \setminus \gamma(R)\big).$$

So the assertion follows.

Corollary 3.3 *Let $f : (X,0) \to (\Delta,0)$ be as in the introduction. Let $n = \dim X_0$. Assume the Milnor monodromy is semisimple. Let n_α be the multiplicity of the exponents of f for*

$\alpha \in \mathbb{Q} \cap (0,n)$ as is defined in [St]. Then very good sections of pr_0 in the introduction are parametrized by $\mathbb{C}^{d_f}$ with $d_f = \sum_{|\lambda|=1,\,\mathrm{Im}\,\lambda \geqslant 0} d_{f,\lambda}$ and

$$d_{f,\lambda} := \begin{cases} \sum_{p<q<n+1-p} n_p n_q + \sum_{p<(n+1)/2} \binom{n_p}{2} & \text{if } \lambda = 1 \text{ and } n \text{ is odd,} \\ \sum_{p<q<n+1-p} n_p n_q + \sum_{p<(n+1)/2} \binom{n_p+1}{2} & \text{if } \lambda = 1 \text{ and } n \text{ is even.} \\ \sum_{p<q<n-p} n_{p+\alpha} n_{q+\alpha} + \sum_{p<n/2} \binom{n_{p+\alpha}}{2} & \text{if } \lambda = -1 \text{ and } n \text{ is even,} \\ \sum_{p<q<n-p} n_{p+\alpha} n_{q+\alpha} + \sum_{p<n/2} \binom{n_{p+\alpha}+1}{2} & \text{if } \lambda = -1 \text{ and } n \text{ is odd.} \\ \sum_{p<q} n_{p+\alpha} n_{q+\alpha} & \text{if } |\lambda| = 1 \text{ and } \mathrm{Im}\,\lambda > 0, \end{cases}$$

where $p,q \in \mathbb{Z}$, and $\lambda = e^{2\pi i\alpha}$ with $\alpha \in [0, \frac{1}{2}]$.

Proof. By [St] there is a canonical mixed Hodge structure on the vanishing cohomology $H^n(F_{f,0}, \mathbb{C})$, where $F_{f,0}$ is the Milnor fiber of f around $0 \in X$, and the Hodge filtration F is compatible with the direct sum decomposition by the eigenvalues of the monodromy T

$$H^n(F_{f,0}, \mathbb{C}) = \textstyle\bigoplus_{\lambda \in \mathbb{C}^*} H_\lambda.$$

Moreover there are canonical non-degenerate pairings of mixed Hodge structures

$$S : H_{\neq 1} \otimes H_{\neq 1} \to \mathbb{C}(-n), \quad S : H_1 \otimes H_1 \to \mathbb{C}(-n-1), \tag{3.3.1}$$

where $H_{\neq 1} := \bigoplus_{\lambda \neq 1} H_\lambda$, and these are compatible with the action of the monodromy T, i.e.

$$S(Tu, Tv) = S(u,v). \tag{3.3.2}$$

So the assumption on S in Proposition 3.2 is satisfied for $H_{\neq 1}$ and H_1 with $m = n$ and $n+1$ respectively. The multiplicities n_α of the Steenbrink exponents can be defined by

$$n_\alpha := \dim \mathrm{Gr}_F^p H_\lambda \quad \text{with} \quad p = [\alpha],\ \lambda = e^{2\pi i\alpha}, \tag{3.3.3}$$

where we use the symmetry of the exponents in [St] i.e.

$$n_\alpha = n_\beta \quad \text{if} \quad \alpha + \beta = n+1. \tag{3.3.4}$$

For $\lambda = \pm 1$, the assertion of Corollary (3.3) then follows from Proposition 3.2. If $\lambda \neq \pm 1$, we get the assertion by using the remark around (3.2.4) together with the duality isomorphism

$$\big(H_{\overline{\lambda}}, F[n]\big) = \mathbf{D}(H_\lambda, F) := \mathrm{Hom}_{\mathbb{C}}\big((H_\lambda, F), \mathbb{C}\big), \tag{3.3.5}$$

which follows from the first non-degenerate pairing in (3.3.1). (In fact, the latter implies that any splitting of F on H_λ determines uniquely its dual splitting of F on $H_{\overline{\lambda}}$ by using the orthogonality condition with respect to S.) This finishes the proof of Corollary (3.3).

Lemma 3.4 *With the notation in the proof of Proposition* 3.2*, the $\overline{v}_{p,i}$ can be lifted to $v_{p,i} \in F^pH$ so that* (3.2.3) *holds.*

Proof. We show the assertion by induction on

$$\max\{p \mid \mathrm{Gr}_F^p H \neq 0\} - \min\{p \mid \mathrm{Gr}_F^p H \neq 0\}.$$

Set $a := \min\{p \mid \mathrm{Gr}_F^p H \neq 0\}$, $b := \max\{p \mid \mathrm{Gr}_F^p H \neq 0\}$, and

$$H' = F^{a+1}H/F^bH.$$

Let S' be the induced pairing on H'. By inductive hypothesis, $\overline{v}_{p,i}$ for $p \in [a+1, b-1]$ can be lifted to $v'_{p,i} \in F^pH' \subset H'$ so that

$$S'(v'_{p,i}, v'_{q,j}) = \varepsilon_p\, \delta_{p,m-q}\, \delta_{i,j} \quad (p,q \in [a+1, b-1]).$$

We can lift $\overline{v}_{a,i}$ to $v_{a,i} \in H$ by induction on i so that

$$S(v_{a,i}, v_{a,j}) = 0 \quad (i,j \in [1, e_a]).$$

Note that $\overline{v}_{b,i}$ is identified with $v_{b,i} \in F^b H = \mathrm{Gr}_F^b H$, and we have

$$S(v_{a,i}, v_{b,j}) = \overline{S}(\overline{v}_{a,i}, \overline{v}_{b,j}) = \varepsilon_a\, \delta_{i,j}.$$

Then we can lift $v'_{p,i}$ to $v_{p,i} \in F^p H$ for $p \in [a+1, b-1]$ so that

$$S(v_{p,i}, v_{a,j}) = 0 \quad (p \in [a+1, b-1]).$$

Here we have

$$S(v_{p,i}, v_{q,j}) = S'(v'_{p,i}, v'_{q,j}) = \varepsilon_p\, \delta_{p,m-q}\, \delta_{i,j} \quad (p, q \in [a+1, b-1]).$$

So (3.2.3) follows (since $S(v_{p,i}, v_{b,j}) = 0$ for $p > a$). This finishes the proof of Lemma 3.4.

Remark 3.5. In the weighted homogeneous polynomial case, it seems that the formula in Corollary (3.3) is essentially equivalent to a formula for the parameter space of primitive forms in [LLS]. (Its verification is left to the reader.) Condition (3.2.3) does not seem to be absolutely necessary for the argument in the proof of Proposition 3.2, since it seems to be enough to assume (3.2.3) for $p+q \geqslant m$ (which trivially holds) although (3.2.5) becomes more complicated without assuming condition (3.2.3) for $p+q < m$, see also [LLS]. Note, however, that the parameter space does not necessarily coincide with the origin in the case it is 0-dimensional, since it would imply (3.2.3) also for $p+q < m$.

Remark 3.6. We have in general

$$V^{>\alpha_\mu - 1} H''_f = V^{>\alpha_\mu - 1} G_f, \tag{3.6.1}$$

where α_μ is the maximal exponent. In fact, setting $F^p H''_f := \partial_t^{-p} H''_f$, we have

$$\mathrm{Gr}_F^p \mathrm{Gr}_V^\alpha H''_f = 0 \quad \text{for} \quad \alpha > \alpha_\mu + p, \tag{3.6.2}$$

(in particular, for $\alpha > \alpha_\mu - 1$ and $p \leqslant -1$).

Remark 3.7. It is known that the minimal exponent α_1 in the usual sense (i.e. as is defined in (3.3.3)) has multiplicity 1, and moreover $V^{>\alpha_1}\Omega_f \subset \Omega_f$ is identified with the maximal ideal of the Jacobian ring $\mathbb{C}\{x\}/(\partial f)$, see [DiSa, 4.11] (and also [Sa4], Remark 3.11). Here the theories of mixed Hodge modules [Sa1] and microlocal b-functions [Sa5] are used. We need the commutativity of taking the graded quotients Gr_F^p, Gr_V^α and the cohomology functor H^{n+1} in an essential way, since there is no canonical $\mathcal{O}_X$-module structure if one takes the cohomology functor first. (In case $\alpha_1 < 1$, the assertion may also follow from [Va].)

The above assertion implies that there is a unique primitive form associated with any very good section (in the sense of this paper) satisfying the orthogonality condition for the higher residue pairings (which follows from the orthogonality condition as in [Sa3, Lemma 2.8]). However, A_1 in (0.1) is not necessarily semisimple as is seen in Example 4.2 below, and there is not always a primitive form associated with any good section satisfying the orthogonality condition unless the section is very good, see Example 4.3 below. We also have a problem about the uniqueness of the associated primitive form, see Example 4.4 below. If we assume that the eigenvalue of the Euler vector field is the minimal exponent, then this may make the existence of the associated primitive form more difficult in general.

4. Examples.

In this section we present some interesting examples.

Example 4.1. If f is not a weighted homogeneous polynomial, it may be possible that there is a good section of pr_0 which is not very good, see [Sa3]. For instance, consider the case

$$f = x^a + y^b + x^{a-2} y^{b-2} \quad (1/a + 1/b < 1/2),$$

where we have a good section such that the eigenvalues of A_1 in (0.1) are

$$\alpha'_1 := \alpha_1 + 1, \quad \alpha'_\mu := \alpha_\mu - 1, \quad \alpha'_k := \alpha_k \ (k \in [2, \mu - 1]). \tag{4.1.1}$$

Here $\alpha_1 \leqslant \cdots \leqslant \alpha_\mu$ are the exponents of f as is defined in [St] (see also (3.3.3) above), which can be expressed in this case by

$$\textstyle\sum_k t^{\alpha_k} = \sum_{0<i<a,\, 0<j<b} t^{i/a+j/b}, \tag{4.1.2}$$

with $\mu = (a-1)(b-1)$. (Note that $\alpha'_i \leqslant \alpha'_{i+1}$ does not hold for $i = 1$ and $\mu - 1$.)

To show (4.1.1), set

$$R := \mathbb{C}\{\{\partial_t^{-1}\}\}, \quad K := \mathbb{C}\{\{\partial_t^{-1}\}\}[\partial_t]. \tag{4.1.3}$$

Put

$$\omega^{(i,j)} = x^{i-1}y^{j-1}dx \wedge dy.$$

By using (1.1.2) restricted to $X \times \{0\}$, we get

$$\begin{gathered} t\,[\omega^{(i,j)}] - \alpha^{(i,j)}\,\partial_t^{-1}[\omega^{(i,j)}] = c^{(i,j)}\,[\omega^{(i+a-2,j+b-2)}] \quad \text{in} \quad H''_f, \\ \text{with} \qquad \alpha^{(i,j)} = \deg_{(a,b)} \omega^{(i,j)} := i/a + j/b, \quad c^{(i,j)} \in \mathbb{C}^*. \end{gathered} \tag{4.1.4}$$

These imply that we have free generators v_k $(k \in [1, \mu])$ of the Gauss-Manin system G_f over K satisfying

$$\partial_t t\, v_k = \alpha_k v_k \quad (k \in [1, \mu]), \tag{4.1.5}$$

and we have the following free generators of the Brieskorn lattice H''_f over R :

$$v_1 + e\,\partial_t v_\mu, \quad v_k \ (k \in [2, \mu]) \quad \text{with} \quad e \in \mathbb{C}^*. \tag{4.1.6}$$

More precisely the above calculation implies that

$$[\omega^{(i,j)}] = v_k \mod V^{\alpha_k + 2 - 2\alpha_1} G_f, \tag{4.1.7}$$

where k is determined by $(i,j) \in [1, a-1] \times [1, b-1]$ with condition $i/a + j/b = \alpha_k$ satisfied. Here V is the filtration of Kashiwara and Malgrange on the Gauss-Manin system G_f as in the introduction. This is closely related with the modified degree $\deg_{(a,b)} \omega^{(i,j)}$ defined above, and we have

$$\deg_{(a,b)} \omega^{(i,j)} \leqslant \max\{\alpha \in \mathbb{Q} \,|\, [\omega^{(i,j)}] \in V^\alpha H''_f\}, \tag{4.1.8}$$

where the equality holds if $(i,j) \in [1, a-1] \times [1, b-1]$. In fact, we have by [Sa2]

$$\mathrm{Gr}_V^{\alpha_k} \omega^{(i,j)} \neq 0 \quad \text{for} \ (i,j) \in [1, a-1] \times [1, b-1] \ \text{with} \ \alpha_k := i/a + j/b.$$

(Here we can also use the μ-constant deformation $f_s = x^a + y^b + s\,x^{a-2}y^{b-2}$ $(s \in \Delta^*)$ together with the graded quotients of the decreasing filtration defined by $\deg_{(a,b)} \omega \geqslant \alpha$ for $\omega \in \Omega_X^2$.)

Take a good section whose image is spanned by

$$v'_1 := \partial_t^{-1} v_1, \quad v'_\mu := \frac{1}{e}\, v_1 + \partial_t\, v_\mu, \quad v'_k := v_k \ (k \in [2, \mu - 1]), \tag{4.1.9}$$

where $e \in \mathbb{C}^*$ is as above. Then the eigenvalues of the associated A_1 are as in (4.1.1).

Note that the image of $v'_\mu = \frac{1}{e}\, v_1 + \partial_t\, v_\mu$ in the Jacobian ring modulo the maximal ideal does not vanish (i.e., it generates the Jacobian ring over it), and the other images vanish, where Ω_X^2 is trivialized by $dx \wedge dy$. So r in [SK1], [SK2] seems to be $\alpha'_\mu = \alpha_\mu - 1$ (instead of α_1) which may be bigger than α_2 in general. It will be shown in Examples 4.3 and 4.4 below that this can cause serious problems related with the existence and the uniqueness of the associated primitive form.

Example 4.2. It is not very difficult to construct an *abstract* example of a Brieskorn lattice H''_f with a good section such that A_1 in (0.1) is non-semi-simple. (The following argument seems to be easier than the one in [Sa3], Remark after 3.10, where it seems rather difficult to determine the structure of the Brieskorn lattice for *geometric* examples.)

Let (H', F) be the underlying filtered $\mathbb{C}$-vector space of a mixed $\mathbb{R}$-Hodge structure endowed with the self-duality pairing S, an automorphism T_s of finite order, and a nilpotent endomorphism N of type $(-1,-1)$, satisfying the usual conditions

$$S(T_s u, T_s v) = S(u,v), \quad S(Nu,v) + S(u,Nv) = 0, \quad T_s N = N T_s.$$

We have the eigenvalue decomposition $(H', F) = \bigoplus_\lambda (H'_\lambda, F)$ by the action of T_s. Assume for simplicity

$$(H', F) = (H'_\lambda, F) \oplus (H'_{\overline{\lambda}}, F),$$

for some $\lambda \neq 1, -1$. Then $(H'_{\overline{\lambda}}, F)$ is the dual of (H'_λ, F) up to a shift of filtration by S. Assume further

$$\dim \mathrm{Gr}_F^p H'_\lambda = \begin{cases} 1 & \text{if } p = 1 \\ 2 & \text{if } p = 2, \\ 0 & \text{otherwise,} \end{cases} \qquad \dim \mathrm{Gr}_F^p H'_{\overline{\lambda}} = \begin{cases} 2 & \text{if } p = 1, \\ 1 & \text{if } p = 2, \\ 0 & \text{otherwise,} \end{cases} \tag{4.2.1}$$

together with the non-vanishing (i.e. the surjectivity and the injectivity) of the morphisms

$$N : \mathrm{Gr}_F^2 H'_\lambda \twoheadrightarrow \mathrm{Gr}_F^1 H'_\lambda, \quad N : \mathrm{Gr}_F^2 H'_{\overline{\lambda}} \hookrightarrow \mathrm{Gr}_F^1 H'_{\overline{\lambda}}.$$

Then we have a splitting of the short exact sequence

$$0 \to \mathrm{Gr}_F^2 H'_\lambda \to H'_\lambda \to \mathrm{Gr}_F^1 H'_\lambda \to 0, \tag{4.2.2}$$

such that the image of $\mathrm{Gr}_F^1 H'_\lambda$ in H'_λ by the splitting is contained in $\mathrm{Ker}\, N$, but does not coincide with $\mathrm{Im}\, N$. For $H'_{\overline{\lambda}}$, we take the dual splitting by using S. We will show that this splitting leads to an example of a good section of an *abstract* Brieskorn lattice ${G'_f}^{(0)}$ such that A_1 is non-semisimple.

By the above decompositions of H', we have a decomposition of regular holonomic $\mathcal{D}_{S,0}$-modules

$$G' = G'_\lambda \oplus G'_{\overline{\lambda}}. \tag{4.2.3}$$

Here G' is actually defined by the above isomorphism, and G'_λ, $G'_{\overline{\lambda}}$ are unique regular holonomic $\mathcal{D}_{S,0}$-modules of rank 3 over K together with isomorphisms

$$\mathrm{Gr}_V^{\beta+k} G'_\lambda = H'_\lambda, \quad \mathrm{Gr}_V^{\beta'+k} G'_{\overline{\lambda}} = H'_{\overline{\lambda}}, \tag{4.2.4}$$

in a compatible way with the actions of $\partial_t t - \beta - k$, $\partial_t t - \beta' - k$, and $(2\pi i)^{-1} N$, where $\beta, \beta' \in \mathbb{Q} \cap (1,2)$ with $\lambda = e^{-2\pi i \beta}$, $\overline{\lambda} = e^{-2\pi i \beta'}$, and the action of ∂_t^{-1} is used for the above identification. Then there are unique R-submodules ${G'_\lambda}^{(0)}$, ${G'_{\overline{\lambda}}}^{(0)}$ of G'_λ, $G'_{\overline{\lambda}}$ satisfying

$$\mathrm{Gr}_V^{\beta+p} {G'_\lambda}^{(0)} = F^{2-p} H'_\lambda, \quad \mathrm{Gr}_V^{\beta'+p} {G'_{\overline{\lambda}}}^{(0)} = F^{2-p} H'_{\overline{\lambda}} \quad (\forall\, p \in \mathbb{Z}), \tag{4.2.5}$$

where R, K are as in (4.1.3). Moreover ${G'_\lambda}^{(0)}$ has free generators e_1, e_2, e_3 over R satisfying

$$(\partial_t t - \beta)\, e_1 = \partial_t\, e_3, \quad (\partial_t t - \beta)\, e_2 = 0, \quad (\partial_t t - \beta - 1)\, e_3 = 0. \tag{4.2.6}$$

(In fact, this follows from the vanishing of $\mathrm{Gr}_V^\alpha {G'_\lambda}^{(0)}$ for $\alpha \neq \beta, \beta+1$.)

The above choice of the splitting of (4.2.2) then gives free generators $\widetilde{e}_1$, $\widetilde{e}_2$, $\widetilde{e}_3$ of ${G'_\lambda}^{(0)}$ over R defined by

$$\widetilde{e}_1 := e_1, \quad \widetilde{e}_2 := e_2, \quad \widetilde{e}_3 := e_3 - c\,\partial_t^{-1} e_2, \tag{4.2.7}$$

where $c \in \mathbb{C}^*$. Then we have

$$(\partial_t t - \beta)\,\widetilde{e}_1 = \partial_t\,\widetilde{e}_3 + c\,\widetilde{e}_2, \quad (\partial_t t - \beta)\,\widetilde{e}_2 = 0, \quad (\partial_t t - \beta - 1)\,\widetilde{e}_3 = 0. \tag{4.2.8}$$

So the action of t on the generators $\widetilde{e}_1$, $\widetilde{e}_2$, $\widetilde{e}_3$ is expressed as in (0.1) by using the matrices

$$A_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad A_1 = \begin{pmatrix} \beta & 0 & 0 \\ c & \beta & 0 \\ 0 & 0 & \beta+1 \end{pmatrix} \tag{4.2.9}$$

and A_1 is non-semi-simple.

Example 4.3. It seems rather complicated to construct an example as in Example 4.2 above in a geometric way, and we need some more calculations as follows. Here the Thom-Sebastiani type theorem as in [ScSt] seems quite useful. For instance, set

$$f = g + h \quad \text{with} \quad g = x^{10} + y^3 + x^2y^2, \;\; h = z^6 + w^5 + z^4w^3.$$

Let G_f, H''_f denote the Gauss-Manin system and the Brieskorn lattice associated to f, and similarly with f replaced by g, h. Let $\alpha_{f,i}$ be the exponents of f, and similarly for $\alpha_{g,i}$, $\alpha_{h,i}$. Then H''_g has a basis u_i over R (with R as in (4.1.3)) satisfying

$$(\partial_t t - \alpha_{g,1})\,u_i = \begin{cases} \partial_t\,u_{14} & \text{if } i = 1, \\ 0 & \text{if } i \neq 1, \end{cases} \tag{4.3.1}$$

where $\mu_g = 14$, and we assume $\alpha_{g,i} \leqslant \alpha_{g,i+1}$. In this case the $\alpha_{g,i}$ are given by

$$\textstyle\sum_{i=1}^{14} t^{\alpha_{g,i}} = t^{1/2} + t + t^{3/2} + \sum_{k=1}^{9} t^{1/2+k/10} + \sum_{k=1}^{2} t^{1/2+k/3}.$$

In fact, this equality together with the non-triviality of the action of N on H_{-1} follows from a result in [St] for functions with non-degenerate Newton boundary. Then (4.3.1) follows from [ScSt] together with Remark 3.6, since

$$\alpha_{g,\mu_g} - \alpha_{g,1} = 1. \tag{4.3.2}$$

As for H''_h, we have a basis $(v_1, \dots, v_{20})$ of G_h over K and free generators $v'_1, \dots, v'_{20}$ of H''_h over R satisfying (4.1.5) and (4.1.9) as in Example 4.1, where $\mu_h = 20$, and R, K are as in (4.1.3). We will denote α_j, α'_j in (4.1.1) by $\alpha_{h,j}$, $\alpha'_{h,j}$ here.

We can actually take any h in Example 4.1 satisfying the following condition:

$$\alpha_{g,i} + \alpha_{h,j} = \alpha_{g,\mu_g} + \alpha_{h,\mu_h} - 2 \quad \text{for some } i, j \geqslant 2, \tag{4.3.3}$$

where g may be replaced by $x^{a'} + y^{b'} + x^2y^2$ with $1/a' + 1/b' < 1/2$. In the case of the above g and h, condition (4.3.3) holds for $(i,j) = (2,2)$ as is shown later.

By the Thom-Sebastiani type theorem as in [ScSt], there are canonical isomorphisms

$$G_f = G_g \otimes_K G_h, \quad H''_f = H''_g \otimes_R H''_h, \tag{4.3.4}$$

such that the action of t on the left-hand side is identified with $t \otimes id + id \otimes t$ on the right-hand side. Let $w_{i,j}$ and $w'_{i,j}$ be respectively the element of G_f corresponding to $u_i \otimes v_j$ and $u_i \otimes v'_j$ in $G_g \otimes_K G_h$ under the isomorphism (4.3.4). Set

$$G'_f := G'_{f,\lambda} \oplus G'_{f,\overline{\lambda}} \subset G_f,$$

with

$$G'_{f,\lambda} := K\,w_{1,20} \oplus K\,w_{2,2} \oplus K\,w_{14,20},$$
$$G'_{f,\bar\lambda} := K\,w_{1,1} \oplus K\,w_{13,19} \oplus K\,w_{14,1},$$

where $\lambda = \exp(-2\pi i(2/15))$, and $\beta = 17/15$ in the notation of Example 4.2. In fact, we have

$$\begin{aligned} &\alpha_{g,1} = 15/30, \quad \alpha_{g,2} = 18/30, \quad \alpha_{g,13} = 42/30, \quad \alpha_{g,14} = 45/30,\\ &\alpha_{h,1} = 11/30, \quad \alpha_{h,2} = 16/30, \quad \alpha_{h,19} = 44/30, \quad \alpha_{h,20} = 49/30,\end{aligned}$$

hence

$$\begin{aligned} \alpha_{1,20} &= 32/15, \quad & \alpha_{2,2} &= 17/15, \quad & \alpha_{14,20} &= 47/15,\\ \alpha_{1,1} &= 13/15, \quad & \alpha_{13,19} &= 43/15, \quad & \alpha_{14,1} &= 28/15,\end{aligned}$$

and

$$\begin{aligned} \alpha'_{1,20} &= 17/15, \quad & \alpha'_{2,2} &= 17/15, \quad & \alpha'_{14,20} &= 32/15,\\ \alpha'_{1,1} &= 28/15, \quad & \alpha'_{13,19} &= 43/15, \quad & \alpha'_{14,1} &= 43/15,\end{aligned}$$

where $\alpha_{i,j} := \alpha_{g,i} + \alpha_{h,j}$, $\alpha'_{i,j} := \alpha_{g,i} + \alpha'_{h,j}$. Note that

$$(\partial_t t - \alpha_{i,j})^k w_{i,j} = 0,$$

with $k = 2$ if $i = 1$, and $k = 1$ otherwise.

If we consider the image of

$$R\,w'_{1,20} \oplus R\,w'_{2,2} \oplus R\,w'_{14,20},$$

by the natural projection $G'_f \twoheadrightarrow G'_{f,\lambda}$, then it coincides with

$$R\,\partial_t w_{1,20} \oplus R\,w_{2,2} \oplus R\,\partial_t w_{14,20}.$$

So the situation is quite close to the one in Example 4.2.

Set

$$\widetilde{w}'_{i,j} := \begin{cases} w'_{14,20} - c\,\partial_t^{-1} w'_{2,2} & \text{if } (i,j) = (14,20)\\ w'_{13,19} + c'\partial_t^{-1} w'_{1,1} & \text{if } (i,j) = (13,19)\\ w'_{i,j} & \text{otherwise.}\end{cases}$$

Here $c, c' \in \mathbb{C}^*$ are chosen appropriately so that $\widetilde{w}'_{14.20}$ and $\widetilde{w}'_{13,19}$ are orthogonal to each other. Then $\widetilde{w}'_{i,j}$ and $\widetilde{w}'_{i',j'}$ are orthogonal to each other unless $(i,j) = (15-i', 21-j')$. Here we use the compatibility of the Thom-Sebastiani type isomorphism with the self-duality (i.e. with the higher residue pairings) up to a constant multiplication. (This can be shown by using the fact that the discriminant of a deformation of the form $F := f + \sum_i x_i s_i$ is reduced.)

Let G''_f be the orthogonal complement of $G'_f \subset G_f$ by the self-duality (i.e. the higher residue pairings). Then the decomposition $G_f = G'_f \oplus G''_f$ is compatible with the Brieskorn lattice, and induces the decomposition

$$H''_f = {G'_f}^{(0)} \oplus {G''_f}^{(0)}.$$

In fact, we have the direct sum decompositions

$$\begin{aligned} G_g &= G'_g \oplus G''_g \quad \text{with} \quad G'_g := K\,u_1 \oplus K\,u_{14}, \quad G''_g := \textstyle\bigoplus_{2\leqslant i\leqslant 13} K\,u_i,\\ G_h &= G'_h \oplus G''_h \quad \text{with} \quad G'_h := K\,v_1 \oplus K\,v_{20}, \quad G''_h := \textstyle\bigoplus_{2\leqslant i\leqslant 19} K\,v_i,\end{aligned}$$

which are compatible with the Brieskorn lattices. They induce the decomposition compatible with the Brieskorn lattice

$$G_g \otimes_K G_h = (G'_g \otimes_K G'_h) \oplus (G''_g \otimes_K G''_h) \oplus (G'_g \otimes_K G''_h) \oplus (G''_g \otimes_K G'_h).$$

Then ${G'_f}^{(0)}$ is identified with the direct sum of

$$G'_g \otimes_K G'_h \quad \text{and a direct factor of } G''_g \otimes_K G''_h,$$

via the isomorphism (4.3.4) in a compatible way with the Brieskorn lattice.

By a calculation similar to (4.2.8), the action of t on the free generators

$$\widetilde{w}'_{1,20},\ \widetilde{w}'_{2,2},\ \widetilde{w}'_{14,20},\ \widetilde{w}'_{1,1},\ \widetilde{w}'_{13,19},\ \widetilde{w}'_{14,1}$$

of ${G'_f}^{(0)}$ over R can be expressed as in (0.1) by using the matrices

$$(4.3.5)\qquad A_0=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\1&0&0&0&0&0\\\gamma&0&0&0&0&0\\0&0&0&0&0&0\\0&0&\gamma&1&0&0\end{pmatrix}\qquad A_1=\begin{pmatrix}\beta&0&0&0&0&0\\c&\beta&0&0&0&0\\0&0&\beta+1&0&0&0\\0&0&0&\beta'&0&0\\0&0&0&0&\beta'+1&0\\0&0&0&0&c'&\beta'+1\end{pmatrix}$$

where $\beta=17/15$, $\beta'=28/15$, and $\gamma\in\mathbb{C}^*$. In this case it is rather difficult to get an associated primitive form. In fact, $\widetilde{w}'_{1,20}$ is the unique member of the generators whose class in the Jacobian ring $\mathcal{O}_{X,0}/(\partial f)$ generates the ring over it, where Ω_X^2 is trivialized by $dx\wedge dy$. However, $\widetilde{w}'_{1,20}$ is annihilated only by $(A_1-\beta)^2$, and the kernel of $A_1-\beta$ in the Jacobian ring is generated over $\mathbb{C}$ by the class of $\widetilde{w}'_{2,2}=w_{2,2}$ which is contained in the maximal ideal. (The details are left to the reader.)

Example 4.4. We first consider an abstract example. Let G be a regular holonomic $\mathcal{D}_{S,0}$-module which is a free K-module of rank 4 with generators u_i ($i\in[1,4]$) satisfying

$$\partial_t t\,u_i=\gamma_i u_i,$$

with

$$(4.4.1)\qquad 0<\gamma_1<\gamma_k<\gamma_4<1\quad(k=2,3).$$

Assume u_i and u_j are orthogonal to each other by the self-duality pairing (i.e. the higher residue pairings) S_K in (2.1.2) unless $i+j=5$. More precisely, assume

$$S_K(u_i,u_j)=\varepsilon_i\,\delta_{i,5-j}\,\partial_t^{-1},$$

with $\varepsilon_i\in\mathbb{C}^*$ satisfying $\varepsilon_1=\varepsilon_2=-\varepsilon_3=-\varepsilon_4$. Note that the above condition implies

$$\gamma_i+\gamma_{5-i}=1.$$

Let $c,c'\in\mathbb{C}^*$. Put

$$u'_i:=\begin{cases}u_1+cu_3+c'u_4 & \text{if } i=1,\\ u_2+cu_4 & \text{if } i=2,\\ \partial_t^{-1}u_i & \text{if } i=3,4.\end{cases}$$

Then

$$S_K(u'_i,u'_j)=\varepsilon'_i\,\delta_{i,5-j}\,\partial_t^{-2}\quad(\varepsilon'_i\in\mathbb{C}^*).$$

Set $c'':=c'/c$. Define

$$\begin{aligned}w_1&:=\partial_t^{-1}u_1,\\ w_2&:=u'_1-c''u'_2=u_1-c''u_2+cu_3,\\ w_3&:=\partial_t^{-1}u_3,\\ w_4&:=u'_1=u_1+cu_3+c'u_4.\end{aligned}$$

Then we have

$$H''_f:=\textstyle\sum_{i=1}^4 R\,u'_i=\sum_{i=1}^4 R\,w_i,$$

and moreover

$$S_K(w_i, w_j) = \varepsilon_i''\,\delta_{i,5-j}\,\partial_t^{-2} \quad (\varepsilon_i'' \in \mathbb{C}^*).$$

In this case the action of t on the generators $w_1, \dots, w_4$ can be expressed as in (0.1) by using the matrices

$$(4.4.2) \qquad A_0 = \begin{pmatrix} 0 & * & 0 & * \\ 0 & 0 & 0 & 0 \\ 0 & * & 0 & * \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad A_1 = \begin{pmatrix} \gamma_1+1 & 0 & 0 & 0 \\ 0 & \gamma_2 & 0 & 0 \\ 0 & 0 & \gamma_3+1 & 0 \\ 0 & 0 & 0 & \gamma_4 \end{pmatrix}$$

This abstract example can be realized as a direct factor of the Brieskorn lattice associated with

$$f = x^a + y^b + x^{a-3}y^{b-2} + x^{a-2}y^{b-2},$$

if $a > b$ and $3/a + 2/b < 1$ (where the last condition corresponds to (4.4.1)). In fact, setting

$$g_1 := 1, \quad g_2 := x, \quad g_3 := x^{a-3}y^{b-2}, \quad g_4 := x^{a-2}y^{b-2},$$

we have

$$\begin{aligned} [g_i\,dx \wedge dy] &= u_i \ \ \mathrm{mod}\ \ V^{\gamma_i+2-\gamma_1-\gamma_2}G_f \quad (i = 1, 2), \\ \partial_t\,[g_i\,dx \wedge dy] &= u_i \ \ \mathrm{mod}\ \ V^{\gamma_i+1-\gamma_1-\gamma_2}G_f \quad (i = 3, 4), \end{aligned}$$

where

$$\gamma_1 = 1/a + 1/b, \quad \gamma_2 = 2/a + 1/b, \quad \gamma_3 = 1 - 2/a - 1/b, \quad \gamma_4 = 1 - 1/a - 1/b.$$

The argument is similar to the proof of (4.1.7). (The details are left to the reader.) In this case, both w_2 and w_4 can be a primitive form associated with the good section whose image is spanned by the w_i.

Appendix: Uniqueness of higher residue pairings in some formal setting

This Appendix is written to answer a question of Dmytro Shklyarov.

Let $R = \mathbb{C}[[\mathbf{s}]]$ with $\mathbf{s} = (s_1, \dots, s_m)$, and $u := \partial_t^{-1}$. Let $\widehat{G}_R$ and $\widehat{H}''_R$ respectively denote the 'formal' Gauss-Manin system and the 'formal' Brieskorn lattice associated with a deformation $F = f + \sum_{i=1}^m g_i s_i$ of $f \in \mathbb{C}\{x\}$ with an isolated singularity. Here 'formal' means that $\widehat{G}_R$ and $\widehat{H}''_R$ are finite free modules of rank r over $R((u))$ and $R[[u]]$ respectively. They are endowed with the actions of t and ∂_{s_i} or $u\partial_{s_i}$ satisfying the usual relations. (Note that the uniqueness of the higher residue pairings does not hold over $\mathbb{C}((u))[[\mathbf{s}]]$ because of the isomorphism in Proposition 1.3. In fact, $\mathbb{C}((u))[[\mathbf{s}]]$ is much bigger than $R((u))$, and has much larger flexibility as is shown by the proposition.)

The dual of $\widehat{G}_R$ can be defined by

$$\mathbb{D}(\widehat{G}_R) := \mathrm{Hom}_{R((u))}\big(\widehat{G}_R, R((u))\big),$$

where the actions of $R((u))$, t, and ∂_{s_i} are given appropriately as usual, see e.g. [Sa3]. Then the self-duality pairing (i.e. the higher residue pairings) can be identified with an isomorphism of $R((u))\langle\partial_{s_i}, t\rangle$-modules

$$\widehat{G}_R \simeq \mathbb{D}(\widehat{G}_R).$$

So the uniqueness up to a nonzero constant multiple of the higher residue pairings in this formal setting is equivalent to

$$(A.1) \qquad \mathrm{End}_{R((u))\langle\partial_{s_i}, t\rangle}(\widehat{G}_R) = \mathbb{C},$$

under the assumption that the discriminant is *reduced*, e.g. if F is a miniversal deformation of f. Here the discriminant D is a divisor on $(\mathbb{C}\times\mathbb{C}^m,0)$ having the coordinates $t,s_1,\dots,s_m$, and D is the image of the relative critical locus defined by the $\partial_{x_i}F$. We can also get D by using the graded quotients of the filtration on the usual Gauss-Manin system defined by the usual Brieskorn lattice shifted by the action of ∂_t^{-i}, where the latter is a coherent sheaf on $(\mathbb{C}\times\mathbb{C}^m,0)$. Passing to the completion by the maximal ideal of $\mathbb{C}\{\mathbf{s}\}$, we get the isomorphisms of $R[t]$-modules

$$\widehat{H}''_R/\partial_t^{-k}\widehat{H}''_R\cong R[t]/(h)^k, \tag{A.2}$$

where $h\in\mathbb{C}\{\mathbf{s}\}[t]$ is a defining function of the discriminant D, and Ω_X^{n+1} is trivialized by $dx_0\wedge\dots\wedge dx_n$.

There is a divisor Σ on $(\mathbb{C}^m,0)$ such that $D\subset\mathbb{C}\times\mathbb{C}^m$ is etale over the complement of Σ by the projection $\mathbb{C}\times\mathbb{C}^m\to\mathbb{C}^m$. By Hironaka's resolution of singularities using blowing-ups with smooth centers, the assertion can be reduced to the case where Σ is a divisor with normal crossings. In fact, the pull-back induces an injective morphism of local rings under smooth center blow-ups of $\mathbb{C}^m$, and we still have the injectivity after taking the formal completion for s_i and u. Then, changing the coordinates s_i appropriately, we may assume that the discriminant D is defined in $(\mathbb{C}\times\mathbb{C}^m,0)$ by the function

$$h:=t^r-s_1^{a_1}\cdots s_m^{a_m}. \tag{A.3}$$

Here we can forget the relation with f,F from now on.

We take the ramified covering

$$\rho:(\mathbb{C}^m,0)\ni(\widetilde{s}_i)\mapsto(s_i):=(\widetilde{s}_i^{\,b_i})\in(\mathbb{C}^m,0),$$

where $b_i:=r/\mathrm{GCD}(r,a_i)$. Set $c_i:=a_i/\mathrm{GCD}(r,a_i)$. Then $rc_i=a_ib_i$, and the pull-back of the equation (A.3) under ρ is given by

$$\widetilde{h}:=t^r-(\widetilde{s}_1^{\,c_1}\cdots\widetilde{s}_m^{\,c_m})^r.$$

We now pass to the localization $\widetilde{R}_{\widetilde{s}}:=\widetilde{R}[1/\widetilde{s}_1\cdots\widetilde{s}_m]$ of $\widetilde{R}:=\mathbb{C}[[\widetilde{s}_1,\dots,\widetilde{s}_m]]$. This is a finite etale Galois extension of $R_s:=R[1/s_1\cdots s_m]$ with Galois group $G=\prod_{i=1}^m\mu_{b_i}$, where μ_{b_i} is the group of roots of 1 of order b_i in $\mathbb{C}$. Let $\widehat{G}_{\widetilde{R}_{\widetilde{s}}}$ be the pull-back of $\widehat{G}_{R_s}:=R_s\otimes_R\widehat{G}_R$ by ρ. This can be defined by $\widetilde{R}_{\widetilde{s}}\otimes_{R_s}\widehat{G}_{R_s}$ since $\widetilde{R}_{\widetilde{s}}$ is finite over R_s. We have the canonical decomposition

$$\widehat{G}_{\widetilde{R}_{\widetilde{s}}}=\bigoplus\nolimits_{\lambda\in\mu_r}\widehat{G}_{\widetilde{R}_{\widetilde{s}},\lambda}, \tag{A.4}$$

where $\mu_r:=\{\lambda\in\mathbb{C}\mid\lambda^r=1\}$. In fact, let F be the decreasing filtration on $\widehat{G}_{\widetilde{R}_{\widetilde{s}}}$ defined by $u^j\widehat{H}''_{\widetilde{R}_{\widetilde{s}}}$ where $\widehat{H}''_{\widetilde{R}_{\widetilde{s}}}$ is the localization by $\widetilde{s}_1\cdots\widetilde{s}_n$ of the pull-back by ρ of the formal Brieskorn lattice. Then we can get the decomposition by taking the inductive limit by p of the projective limit by q of the canonical decompositions

$$(F^p/F^q)\widehat{G}_{\widetilde{R}_{\widetilde{s}}}=\bigoplus\nolimits_{\lambda\in\mu_r}(F^p/F^q)\widehat{G}_{\widetilde{R}_{\widetilde{s}},\lambda}, \tag{A.5}$$

which can be defined by setting

$$(F^p/F^q)\widehat{G}_{\widetilde{R}_{\widetilde{s}},\lambda}=\mathrm{Ker}\big((t-\lambda\,\widetilde{s}_1^{\,c_1}\cdots\widetilde{s}_m^{\,c_m})^{q-p}:(F^p/F^q)\widehat{G}_{\widetilde{R}_{\widetilde{s}}}\to(F^p/F^q)\widehat{G}_{\widetilde{R}_{\widetilde{s}}}\big), \tag{A.6}$$

since the discriminant is reduced. In fact, there is a canonical direct sum decomposition

$$\begin{aligned}&\mathbb{C}\big[t,\widetilde{s}_1,\dots,\widetilde{s}_m,\tfrac{1}{\widetilde{s}_1\cdots\widetilde{s}_m}\big]\big/\big(t^r-(\widetilde{s}_1^{\,c_1}\cdots\widetilde{s}_m^{\,c_m})^r\big)^{q-p}\\&\quad=\bigoplus\nolimits_{\lambda\in\mu_r}\mathbb{C}\big[t,\widetilde{s}_1,\dots,\widetilde{s}_m,\tfrac{1}{\widetilde{s}_1\cdots\widetilde{s}_m}\big]\big/\big(t-\lambda\,\widetilde{s}_1^{\,c_1}\cdots\widetilde{s}_m^{\,c_m}\big)^{q-p}.\end{aligned}$$

Taking its tensor product with $\widetilde{R} = \mathbb{C}[[\widetilde{s}_1, \dots, \widetilde{s}_m]]$ over $\mathbb{C}[\widetilde{s}_1, \dots, \widetilde{s}_m]$, we then get

$$(A.7) \qquad \begin{aligned}(F^p/F^q)\widehat{G}_{\widetilde{R}_{\widetilde{s}}} &\cong \widetilde{R}\big[t, \tfrac{1}{\widetilde{s}_1 \cdots \widetilde{s}_m}\big]/\big(t^r - (\widetilde{s}_1^{c_1} \cdots \widetilde{s}_m^{c_m})^r\big)^{q-p} \\ &= \textstyle\bigoplus_{\lambda \in \mu_r} \widetilde{R}\big[t, \tfrac{1}{\widetilde{s}_1 \cdots \widetilde{s}_m}\big]/\big(t - \lambda\, \widetilde{s}_1^{c_1} \cdots \widetilde{s}_m^{c_m}\big)^{q-p},\end{aligned}$$

where the first isomorphism follows from (A.2). (In fact, ρ is flat and the pull-back is an exact functor.) This implies that the decomposition (A.5) can be obtained by (A.6). (Note that F cannot be exhaustive if we use the formal Gauss-Manin system as in Theorem 1.)

For $\theta \in \mathrm{End}_{R((u))\langle \partial_{s_i}, t\rangle}(\widehat{G}_R)$, its pull-back $\widetilde{\theta} := \rho^*\theta$ is an endomorphism of $\widehat{G}_{\widetilde{R}_{\widetilde{s}}}$ preserving the decomposition (A.4). (In fact, $\widetilde{\theta}$ preserves the filtration F up to a shift by some integer k, i.e., $\widetilde{\theta}(F^p\widehat{G}_{\widetilde{R}_{\widetilde{s}}}) \subset F^{p-k}\widehat{G}_{\widetilde{R}_{\widetilde{s}}}$ for any p.) Moreover $\widetilde{\theta}$ is compatible with the action of G (since it is the pull-back of θ by ρ), and G acts on the direct factors of the decomposition (A.4) transitively. Thus the assertion is reduced to

$$(A.8) \qquad \mathrm{End}_{\widetilde{R}_{\widetilde{s}}((u))\langle \partial_{\widetilde{s}_i}, t\rangle}(\widehat{G}_{\widetilde{R}_{\widetilde{s}},\lambda}) = \mathbb{C}.$$

We can verify (A.8) easily since $\widehat{G}_{\widetilde{R}_{\widetilde{s}},\lambda}$ is a free $\widetilde{R}((u))\big[\frac{1}{\widetilde{s}_1 \cdots \widetilde{s}_m}\big]$-module of rank 1 by (A.7). So (A.1) follows.

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A RESIDUAL DUALITY OVER GORENSTEIN RINGS WITH APPLICATION TO LOGARITHMIC DIFFERENTIAL FORMS

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Dedicated with admiration to Egbert Brieskorn

Abstract. Kyoji Saito's notion of a free divisor was generalized by the first author to reduced Gorenstein spaces and by Delphine Pol to reduced Cohen–Macaulay spaces. Starting point is the Aleksandrov–Terao theorem: A hypersurface is free if and only if its Jacobian ideal is maximal Cohen–Macaulay. Pol obtains a generalized Jacobian ideal as a cokernel by dualizing Aleksandrov's multi-logarithmic residue sequence. Notably it is essentially a suitably chosen complete intersection ideal that is used for dualizing. Pol shows that this generalized Jacobian ideal is maximal Cohen–Macaulay if and only if the module of Aleksandrov's multi-logarithmic differential k-forms has (minimal) projective dimension $k-1$, where k is the codimension in a smooth ambient space. This equivalent characterization reduces to Saito's definition of freeness in case $k=1$. In this article we translate Pol's duality result in terms of general commutative algebra. It yields a more conceptual proof of Pol's result and a generalization involving higher multi-logarithmic forms and generalized Jacobian modules.

1. Introduction

Logarithmic differential forms along hypersurfaces and their residues were introduced by Kyoji Saito (see [22]). They are part of his theory of primitive forms and period mappings where the hypersurface is the discriminant of a universal unfolding of a function with isolated critical point (see [23, 24]). The special case of normal crossing divisors appeared earlier in Deligne's construction of mixed Hodge structures (see [8]). Here the logarithmic differential 1-forms form a locally free sheaf. In general a divisor with this property is called a free divisor. Further examples include plane curves (see [22, (1.7)]), unitary reflection arrangements and their discriminants (see [29, Thm. C]) and discriminants of versal deformations of isolated complete intersection singularities and space curves (see [17, (6.13)] and [30]). Free divisors also occur as discriminants in prehomogeneous vector spaces (see [10]). In case of hyperplane arrangements the study of freeness attracted a lot of attention (see [31]).

Let D be a germ of reduced hypersurface in $Y \cong (\mathbb{C}^n, 0)$ defined by $h \in \mathscr{O}_Y$. The $\mathscr{O}_Y$-modules $\Omega^q(\log D)$ of logarithmic differential q-forms along D and the $\mathscr{O}_D$-modules ω_D^p of regular meromorphic differential p-forms on D fit into a short exact logarithmic residue sequence (see [22, §2] and [2, §4])

$$0 \longrightarrow \Omega_Y^q \longrightarrow \Omega^q(\log D) \xrightarrow{\operatorname{res}_D^q} \omega_D^{q-1} \longrightarrow 0.$$

Denoting by $\nu_D \colon \widetilde{D} \to D$ the normalization of D, $(\nu_D)_* \mathscr{O}_{\widetilde{D}} \subseteq \omega_D^0$ (see [22, (2.8)]). For plane curves Saito showed that equality holds exactly for normal crossing curves (see [22, (2.11)]).

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Granger and the first author (see [11]) generalized this fact and thus extended the Lê–Saito Theorem (see [16]) by an equivalent algebraic property. They showed that $(\nu_D)_*\mathscr{O}_{\widetilde{D}} = \omega_D^0$ if and only if D is normal crossing in codimension one, that is, outside of an analytic subset of Y of codimension at least 3. The proof uses the short exact sequence

$$0 \longleftarrow \mathscr{J}_D \xleftarrow{\langle -,dh\rangle} \Theta_Y \longleftarrow \mathrm{Der}(-\log D) \longleftarrow 0$$

obtained as the $\mathscr{O}_Y$-dual of the logarithmic residue sequence. It involves the Jacobian ideal $\mathscr{J}_D$ of D, the $\mathscr{O}_Y$-module $\Theta_Y := \mathrm{Der}_{\mathbb{C}}(\mathscr{O}_Y) \cong (\Omega_Y^1)^*$ of vector fields on Y and its submodule $\mathrm{Der}(-\log D) \cong \Omega^1(\log D)^*$ of logarithmic vector fields along D. It is shown that $\omega_D^0 = \mathscr{J}_D^*$ and that $\mathscr{J}_D = (\omega_D^0)^*$ if D is a free divisor. In fact freeness of D is equivalent to $\mathscr{J}_D$ being a Cohen–Macaulay ideal by the Aleksandrov–Terao theorem (see [2, §2] and [28, §2]).

As observed by first author (see [27]) the inclusion $(\nu_D)_*\mathscr{O}_{\widetilde{D}} \subseteq \omega_D^0$ can be seen as

$$(\nu_D)_*\omega_{\widetilde{D}}^0 \hookrightarrow \omega_D^0.$$

He showed that $(\nu_X)_*\omega_{\widetilde{X}}^0 = \omega_X^0$ is equivalent to X being normal crossing in codimension one for reduced equidimensional spaces X which are free in codimension one. Here freeness means Gorenstein with Cohen–Macaulay ω-Jacobian ideal. As the latter coincides with the Jacobian ideal for complete intersections (see [19, Prop. 1]), this generalizes the classical freeness of divisors which holds true in codimension one.

Multi-logarithmic differential forms generalize Saito's logarithmic differential forms replacing hypersurfaces $D \subseteq Y$ by subspaces $X \subseteq Y$ of codimension $k \geq 2$. They were first introduced with meromorphic poles along reduced complete intersections by Aleksandrov and Tsikh (see [5, 6]), later with simple poles by Aleksandrov (see [3, §3]) and recently along reduced Cohen–Macaulay and reduced equidimensional spaces by Aleksandrov (see [4, §10]) and by Pol (see [21, §4.1]). The precise relation of the forms with simple and meromorphic poles was clarified by Pol (see [21, Prop. 3.1.33]). Here we consider only multi-logarithmic forms with simple poles.

The $\mathscr{O}_Y$-modules $\Omega^q(\log X/C)$ of multi-logarithmic q-forms on Y along X depend on the choice of divisors $D_1, \ldots, D_k$ defining a reduced complete intersection $C = D_1 \cap \cdots \cap D_k \subseteq Y$ such that $X \subseteq C$. Consider the divisor $D = D_1 \cup \cdots \cup D_k$ defined by $h = h_1 \cdots h_k \in \mathscr{O}_Y$. Due to Aleksandrov and Pol there is a multi-logarithmic residue sequence

$$0 \longrightarrow \Sigma\Omega_Y^q \longrightarrow \Omega^q(\log X/C) \xrightarrow{\mathrm{res}^q_{X/C}} \omega_X^{q-k} \longrightarrow 0 \tag{1.1}$$

where $\Sigma = \mathcal{I}_C(D)$ is obtained from the ideal $\mathcal{I}_C$ of $C \subseteq Y$ and ω_X^p is the $\mathscr{O}_X$-module of regular meromorphic p-forms on X (see [4, §10] and [21, §4.1.3]). Pol introduced an $\mathscr{O}_Y$-module $\mathrm{Der}^k(-\log X/C)$ of logarithmic k-vector fields on Y along X and a kind of Jacobian ideal $\mathscr{J}_{X/C}$ of X that fit into the short exact sequence dual to (1.1) for $q = k$

$$0 \longleftarrow \mathscr{J}_{X/C} \xleftarrow{\langle -,\alpha_X\rangle} \Theta_Y^k \longleftarrow \mathrm{Der}^k(-\log X/C) \longleftarrow 0 \tag{1.2}$$

where $\Theta_Y^q = \bigwedge_{\mathscr{O}_Y}^q \Theta_Y$ and $\begin{bmatrix} \alpha_X \\ h_1, \ldots, h_k \end{bmatrix} \in \omega_X^0$ is a fundamental form of X (see [21, §4.2.2-3]). Notably the duality applied here is $-^\Sigma = \mathrm{Hom}_{\mathscr{O}_Y}(-, \Sigma)$. Pol showed that Cohen–Macaulayness of $\mathscr{J}_{X/C}$ serves as a further generalization of freeness. In fact the property is independent of C (see [21, Prop. 4.2.21]) and $\mathscr{J}_{X/C}$ coincides with the ω-Jacobian ideal in case X is Gorenstein (see [21, §4.2.5]). By relating Σ- and $\mathscr{O}_Y$-duality Pol established the following major result (see [21, Thm. 4.2.22] or [20]). In particular it generalizes Saito's original definition of freeness to the case $k > 1$.

Theorem 1.1 (Pol)**.** *Let $X \subseteq C \subseteq Y \cong (\mathbb{C}^n, 0)$ where X is a reduced Cohen–Macaulay germ and C a complete intersection germ, both of codimension $k \geq 1$ in Y. Then*

$$\operatorname{pdim}(\Omega^k(\log X/C)) \geq k-1$$

with equality equivalent to freeness of X.

In §2 we pursue the main objective of this article: a translation of Theorem 1.1 in terms of general commutative algebra. The role of $\mathscr{O}_Y \twoheadrightarrow \mathscr{O}_C = \mathscr{O}_Y/\mathcal{I}_C$ is played by a map of Gorenstein rings $R \to \overline{R} = R/I$ of codimension $k \geq 2$. For dualizing we use

$$-^I = \operatorname{Hom}_R(-, I), \quad -^\vee = \operatorname{Hom}_R(-, \omega_R), \quad -^{\overline{\vee}} = \operatorname{Hom}_{\overline{R}}(-, \overline{\omega}_R)$$

where ω_R is a canonical module for R and $\overline{\omega}_R = \overline{R} \otimes_R \omega_R$, which is a canonical module for $\overline{R}$ due to the Gorenstein hypothesis (see Notation 2.1). Modelled after the multi-logarithmic residue sequence (1.1) along $X = C$ we define an I-free approximation of a finitely generated R-module M as a short exact sequence

$$0 \longrightarrow IF \xrightarrow{\iota} M \longrightarrow W \longrightarrow 0$$

where F is free and W is an $\overline{R}$-module. More precisely M plays the role of $\Omega^q(\log X/C)(-D)$ which, as opposed to $\Omega^q(\log X/C)$, is independent of the choice of D. The I-dual sequence

$$0 \longleftarrow V \xleftarrow{\alpha} F^\vee \xleftarrow{\lambda} M^I \longleftarrow 0$$

plays the role of the Σ-dual sequence (1.2) for $X = C$. In Proposition 2.13 we show that M is I-reflexive if and only if W is the $\overline{R}$-dual of V. Our main result is

Theorem 1.2. *Let R be a Gorenstein local ring and let I be an ideal of R of height $k \geq 2$ such that $\overline{R} = R/I$ is Gorenstein. Consider an I-free approximation*

$$0 \longrightarrow IF \xrightarrow{\iota} M \xrightarrow{\rho} W \longrightarrow 0$$

of an I-reflexive finitely generated R-module M with $W \neq 0$ and the corresponding I-dual

$$0 \longleftarrow V \xleftarrow{\alpha} F^\vee \xleftarrow{\lambda} M^I \longleftarrow 0.$$

Then $W = V^{\overline{\vee}}$ and V is a maximal Cohen–Macaulay $\overline{R}$-module if and only if $\operatorname{G-dim}(M) \leq k-1$. *In this latter case $V = W^{\overline{\vee}}$ is ($\overline{\omega}_R$-)reflexive. Unless $\overline{\alpha} := \overline{R} \otimes \alpha$ is injective,* $\operatorname{G-dim}(M) \geq k-1$.

Due to the Gorenstein hypothesis, Theorem 1.2 applies to the complete intersection ring $\overline{R} = \mathscr{O}_C$, but in general not to $\overline{R} = \mathscr{O}_X$. In §2.5 we describe a construction to restrict the support of an I-free approximation to the locus defined by an ideal $J \trianglelefteq R$ with $I \subseteq J$. Lemma 3.15 shows that it is made in a way such that the multi-logarithmic residue sequence along X is obtained from that along C by restricting with $J = \mathcal{I}_X$. Corollary 2.29 extends Theorem 1.2 to this generalized setup.

In §3 we apply our results to multi-logarithmic forms. We define $\mathscr{O}_Y$-submodules

$$\operatorname{Der}^q(-\log X) \subseteq \Theta_Y^q$$

of logarithmic q-vector fields on Y along X independent of C and show that

$$\operatorname{Der}^k(-\log X) = \operatorname{Der}^k(-\log X/C).$$

We further define Jacobian $\mathscr{O}_X$-modules $\mathcal{J}_X^{n-q} \subseteq \mathscr{O}_X \otimes_{\mathscr{O}_Y} \Theta_Y^{q-k}$ of X independent of C and Y such that $\mathcal{J}_X^{\dim X} = \mathcal{J}_{X/C}$. The Σ-dual of the multi-logarithmic residue sequence reads

$$0 \longleftarrow \mathcal{J}_X^{n-q} \xleftarrow{\alpha^X} \Theta_Y^q \longleftarrow \operatorname{Der}^q(-\log X) \longleftarrow 0$$

where α^X is contraction by α_X. As a consequence of Corollary 2.29 we obtain the following result which is due to Pol in case $q = k$ (see [21, Prop. 4.2.17, Thm. 4.2.22]).

Theorem 1.3. *Let $X \subseteq C \subseteq Y \cong (\mathbb{C}^n, 0)$ where X is a reduced Cohen–Macaulay germ and C a complete intersection germ, both of codimension $k \geq 2$ in Y. For $k \leq q < n$, $\omega_X^{q-k} = \operatorname{Hom}_{\mathscr{O}_X}(\mathscr{J}_X^{n-q}, \omega_X)$ where $\omega_X = \operatorname{Hom}_{\mathscr{O}_C}(\mathscr{O}_X, \mathscr{O}_C)(D)$ and $\operatorname{pdim}(\Omega^q(\log X/C)) \geq k-1$. Equality holds if and only if $\mathscr{J}_X^{n-q}$ is maximal Cohen–Macaulay. In this latter case $\mathscr{J}_X^{n-q} = \operatorname{Hom}_{\mathscr{O}_X}(\omega_X^{q-k}, \omega_X)$ is ω_X-reflexive.*

The analogy with the hypersurface case (see [22, (1.8)]) now raises the question whether $\mathscr{J}_X^{n-q}$ being maximal Cohen–Macaulay for $q = k$ implies the same for all $q > k$. An explicit description of the Jacobian modules is given in Remark 3.25.

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2. Residual duality over Gorenstein rings

For this section we fix a Cohen–Macaulay local ring R with $n := \dim(R)$ and an ideal $I \trianglelefteq R$ with $k := \operatorname{height}(I) \geq 2$ defining a Cohen–Macaulay factor ring $\overline{R} := R/I$. These fit into a short exact sequence

$$0 \longrightarrow I \longrightarrow R \xrightarrow{\pi} \overline{R} \longrightarrow 0. \tag{2.1}$$

Note that (see [7, Thm. 2.1.2.(b), Cor. 2.1.4])

$$n - \dim(\overline{R}) = \operatorname{grade}(I) = \operatorname{height}(I) = k \geq 2.$$

In particular I is a regular ideal of R and hence any $\overline{R}$-module is R-torsion.

We assume further that R admits a canonical module ω_R. Then also $\overline{R}$ admits a canonical module $\omega_{\overline{R}}$ (see [7, Thm. 3.3.7]).

Notation 2.1. Abbreviating $\overline{\omega}_R := \overline{R} \otimes_R \omega_R$ we deal with the following functors

$$\begin{aligned} -^* &:= \operatorname{Hom}_R(-, R), & -^\vee &:= \operatorname{Hom}_R(-, \omega_R), \\ -^I &:= \operatorname{Hom}_R(-, I\omega_R), & -^{\overline{\vee}} &:= \operatorname{Hom}_R(-, \overline{\omega}_R). \end{aligned}$$

In general $\overline{\omega}_R \not\cong \omega_{\overline{R}}$ and $-^{\overline{\vee}}$ is not the duality of $\overline{R}$-modules. For an $\overline{R}$-module N,

$$N^* = \operatorname{Hom}_{\overline{R}}(N, \overline{R})$$

but $N^\vee$ means either $\operatorname{Hom}_R(N, \omega_R)$ or $\operatorname{Hom}_{\overline{R}}(N, \omega_{\overline{R}})$, depending on the context. For R-modules M and N, we denote the canonical evaluation map by

$$\delta_{M,N} \colon M \to \operatorname{Hom}_R(\operatorname{Hom}_R(M, N), N), \quad m \mapsto (\varphi \mapsto \varphi(m)).$$

Whenever applicable we use an analogous notation for $\overline{R}$-modules. We denote canonical isomorphisms as equalities.

Lemma 2.2. *Let N be an $\overline{R}$-module. Then $\operatorname{Ext}_R^i(N, \omega_R) = 0$ for $i < k$ and $N^I = 0$.*

Proof. The first vanishing is due to Ischebeck's Lemma (see [12, Satz 1.9]), the second holds because ω_R and hence $I\omega_R$ is torsion free (see [7, Thm. 2.1.2.(c)]) whereas N is torsion. ☐

2.1. ***I*-duality and *I*-free approximation.**

Lemma 2.3. *There is a canonical identification $\omega_R = I^I$ and a canonical inclusion $I \hookrightarrow \omega_R^I$. They combine to the map $\delta_{I,I\omega_R}\colon I \to I^{II}$ which is an isomorphism if R is Gorenstein.*

Proof. Applying $-^\vee$ to (2.1) and $\operatorname{Hom}_R(I,-)$ to $I\omega_R \hookrightarrow \omega_R$ yields an exact sequence with a commutative triangle

$$\begin{tikzcd} \operatorname{Ext}^1_R(\overline{R},\omega_R) & I^\vee \arrow[l] & \omega_R \arrow[l] \arrow[dl, "\mu"] & \overline{R}^\vee \arrow[l] & 0 \arrow[l] \\ & I^I. \arrow[u, hook] & & & \end{tikzcd} \tag{2.2}$$

The diagonal map sends $\varepsilon \in \omega_R$ to the multiplication map $\mu(\varepsilon)\colon I \to I\omega_R$, $x \mapsto x \cdot \varepsilon$. With Lemma 2.2 it follows that $\omega_R = I^\vee = I^I$.

There is an isomorphism $R \cong \operatorname{End}_R(\omega_R)$ sending each element to the corresponding multiplication map (see [7, Thm. 3.3.4.(d))]). Applying $\operatorname{Hom}_R(\omega_R,-)$ to $I\omega_R \hookrightarrow \omega_R$ yields a commutative square

$$\begin{tikzcd} R \arrow[r, "\cong"'] & \operatorname{End}_R(\omega_R) \\ I \arrow[u, hook] \arrow[r, "\delta'"] & \omega_R^I. \arrow[u, hook] \end{tikzcd} \tag{2.3}$$

If R is Gorenstein, then $\omega_R^I = \operatorname{Hom}_R(R,I) = I$ and δ' is an isomorphism.

Combined with the above identification $\omega_R = I^I$, δ' defines a map $\delta\colon I \to I^{II}$. Since

$$\delta(x)(\mu(\varepsilon)) = \delta'(x)(\varepsilon) = x \cdot \varepsilon = \mu(\varepsilon)(x) = \delta_{I,I\omega_R}(x)(\mu(\varepsilon))$$

for all $x \in I$ and $\varepsilon \in \omega_R$, in fact $\delta = \delta_{I,I\omega_R}$. □

Definition 2.4. If F is a free R-module, then we call $IF = I \otimes_R F$ an *I-free module*. An R-module M is called *I-reflexive* if $\delta_{M,I\omega_R}\colon M \to M^{II}$ is an isomorphism.

Proposition 2.5. *Let F be a free R-module F. Then $F^\vee = (IF)^I$ by restriction. The adjunction map $IF \to F^{\vee I}$ is induced by the isomorphism δ_{F,ω_R} and identifies with $\delta_{IF,I\omega_R}$. In case R is Gorenstein, IF is I-reflexive.*

Proof. Applying $\operatorname{Hom}_R(F,-)$ to μ in (2.2) yields $F^\vee = (IF)^I$ by Hom-tensor adjunction.

Applying $F \otimes_R -$ to (2.3) yields a commutative square

$$\begin{tikzcd} F \arrow[r, "\delta_{F,\omega_R}", "\cong"'] & F^{\vee\vee} \\ IF \arrow[u, hook] \arrow[r] & F^{\vee I} \arrow[u, hook] \end{tikzcd}$$

where the bottom row is adjunction. In fact, using Lemma 2.3,

$$\begin{aligned} IF = I \otimes_R F \to F \otimes_R \omega_R^I &= F \otimes_R \operatorname{Hom}_R(\omega_R, I\omega_R) \\ &= \operatorname{Hom}_R(F \otimes_R \omega_R, I\omega_R) \\ &= \operatorname{Hom}_R(F \otimes_R \operatorname{Hom}_R(R, \omega_R), I\omega_R) \\ &= \operatorname{Hom}_R(\operatorname{Hom}_R(F \otimes_R R, \omega_R), I\omega_R) \\ &= \operatorname{Hom}_R(\operatorname{Hom}_R(F, \omega_R), I\omega_R) = F^{\vee I}, \\ x \cdot e &\mapsto (\psi \mapsto x \cdot \psi(e)). \end{aligned}$$

Identifying $F^\vee = (IF)^I$ using Lemma 2.3 yields with the map μ in diagram (2.2)

$$\varepsilon = \psi(e) \leftrightarrow \mu(\varepsilon) \implies x \cdot \psi(e) = x \cdot \varepsilon = \mu(\varepsilon)(x).$$

Adjunction thus becomes identified with $\delta_{IF, I\omega_R}$. The last claim is due to Lemma 2.3. $\square$

Definition 2.6. Let M be a finitely generated R-module. We call a short exact sequence

$$0 \longrightarrow IF \xrightarrow{\iota} M \xrightarrow{\rho} W \longrightarrow 0 \tag{2.4}$$

where F is free and $IW = 0$ an *I-free approximation of M* with *support* $\operatorname{Supp}(W)$. We consider W as an $\overline{R}$-module. The inclusion map $\iota\colon IF \hookrightarrow F = M$ defines the *trivial I-free approximation*

$$0 \longrightarrow IF \longrightarrow F \longrightarrow F/IF \longrightarrow 0.$$

A *morphism of I-free approximations* is a morphism of short exact sequences.

Lemma 2.7. *For any I-free approximation (2.4), ι fits into a unique commutative triangle*

$$\begin{tikzcd} & F \\ IF \arrow[ur, hook] \arrow[r, hook, "\iota"] & M. \arrow[u, "\kappa"'] \end{tikzcd} \tag{2.5}$$

If ι^{-1} denotes the choice of any preimage under ι, then $\kappa(m) = \iota^{-1}(xm)/x$ for any $x \in I \cap R^{\mathrm{reg}}$. If M is maximal Cohen–Macaulay, then κ is surjective. In particular, (2.4) becomes trivial if in addition κ injective.

Proof. Applying $\operatorname{Hom}_R(-, F)$ to (2.4) yields

$$\operatorname{Ext}^1_R(W, F) \longleftarrow \operatorname{Hom}_R(IF, F) \xleftarrow{\iota^*} \operatorname{Hom}_R(M, F) \longleftarrow \operatorname{Hom}_R(W, F) \longleftarrow 0.$$

By Ischebeck's Lemma (see [12, Satz 1.9]), $\operatorname{Ext}^1_R(W, F) = 0 = \operatorname{Hom}_R(W, F)$ making ι^* an isomorphism. Then κ is the preimage of the canonical inclusion $IF \hookrightarrow F$ under ι^*. The formula for κ follows immediately.

Since $\operatorname{coker}(\kappa)$ is a homomorphic image of F/IF, $\dim(\operatorname{coker}(\kappa)) \leq n - k \leq n - 2$. If M is maximal Cohen–Macaulay, then $\operatorname{depth}(\operatorname{coker}(\kappa)) \geq n - 1$ by the Depth Lemma (see [7, Prop. 1.2.9]). This forces $\operatorname{coker}(\kappa) = 0$ (see [7, Prop. 1.2.13]) and makes κ surjective. $\square$

By functoriality of the cokernel, any $\varphi \in F^\vee$ gives rise to a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & I\omega_R \arrow[r] & \omega_R \arrow[r, "\pi_\omega"] & \overline{\omega}_R \arrow[r] & 0 \\
& & F \arrow[u, "\varphi"'] & & \\
0 \arrow[r] & IF \arrow[r, "\iota"] \arrow[uu, "\varphi|_{IF}"] \arrow[ur, hook] & M \arrow[r, "\rho"] \arrow[u, "\kappa"'] & W \arrow[r] \arrow[uu, dashed, "\overline{\varphi}"'] & 0
\end{tikzcd} \tag{2.6}$$

with top exact row induced by (2.1) and bottom row (2.4). This defines a map

$$\begin{aligned} W^{\overline{\vee}} &\longleftarrow F^\vee \\ \overline{\varphi} &\longleftarrow\!\shortmid \varphi. \end{aligned} \tag{2.7}$$

Applying $\operatorname{Hom}_R(F, -)$ to the upper row of (2.6) yields a short exact sequence

$$0 \longrightarrow F^I \longrightarrow F^\vee \longrightarrow F^{\overline{\vee}} \longrightarrow 0. \tag{2.8}$$

By Lemma 2.2 applying $-^I$ to (2.4) and (2.5) yields the exact diagonal sequence and the triangle of inclusions with vertex F^I in the following commutative diagram.

$$\begin{tikzcd}
0 & V \arrow[l] \arrow[d, equal] & F^{\overline{\vee}} \arrow[l, "\overline{\alpha}"'] & M^I/F^I \arrow[l, "\overline{\lambda}"'] & 0 \arrow[l] \arrow[dl] \\
0 & V \arrow[l] \arrow[ddd, hook] & F^\vee \arrow[l, "\alpha"'] \arrow[u, two heads] \arrow[d, equal] & M^I \arrow[l, "\lambda"'] \arrow[u, two heads] \arrow[dl, "\iota^I"'] & 0 \arrow[l] \\
& & (IF)^I \arrow[ddl] & F^I \arrow[l, hook'] \arrow[u, hook, "\kappa^I"'] & \\
& & & & \\
& \operatorname{Ext}^1_R(W, I\omega_R) & & &
\end{tikzcd} \tag{2.9}$$

By Proposition 2.5, the identification $F^\vee = (IF)^I$ in diagram (2.9) is given by

$$\varphi \leftrightarrow \varphi|_{IF} = \varphi \circ \kappa \circ \iota$$

in diagram (2.6). It defines the map λ with cokernel α. For $\psi \in M^I$, $\lambda(\psi)$ is defined by

$$\lambda(\psi)|_{IF} = \psi \circ \iota.$$

With $\operatorname{Ext}^1_R(W, I\omega_R)$ also V is an $\overline{R}$-module. Using (2.8) the Snake Lemma yields the short exact upper row of (2.9). By Lemma 2.2 the commutative square $\operatorname{Hom}_R(IF \hookrightarrow M, I\omega_R \hookrightarrow \omega_R)$ reads

$$\begin{tikzcd}
(IF)^I \arrow[d, hook] & M^I \arrow[l, "\iota^I"'] \arrow[d, hook] \\
(IF)^\vee & M^\vee. \arrow[l, "\iota^\vee"', "\cong"]
\end{tikzcd}$$

This allows one to check equalities of maps $M \to \omega_R$ after precomposing with ι. It follows that

$$\varphi \circ \kappa \in M^I \iff \varphi \in \lambda(M^I) \implies \varphi = \lambda(\varphi \circ \kappa) \tag{2.10}$$

for any $\varphi \in F^\vee$.

Definition 2.8. We call the middle row

$$0 \longleftarrow V \xleftarrow{\alpha} F^\vee \xleftarrow{\lambda} M^I \longleftarrow 0 \tag{2.11}$$

of diagram (2.9) the I-*dual* of the I-free approximation (2.4). We set

$$W' := \operatorname{Ext}^1_R(V, I\omega_R). \tag{2.12}$$

Lemma 2.9. *For any I-free approximation (2.4) the map (2.7) factors through the map α in (2.9) defining an inclusion $\nu\colon V \to W^{\triangledown}$, that is,*

$$\begin{tikzcd} W^{\triangledown} & V \arrow[l, hook', "\nu"'] & F^\vee, \arrow[l, two heads, "\alpha"'] \\ \overline{\varphi} & & \varphi. \arrow[ll, maps to] \end{tikzcd}$$

Proof. By diagrams (2.6) and (2.9), equivalence (2.10) and exactness properties of Hom,

$$\overline{\varphi} = 0 \iff \overline{\varphi} \circ \rho = 0 \iff \varphi \circ \kappa \in M^I \iff \varphi \in \lambda(M^I) \iff \alpha(\varphi) = 0. \qquad \square$$

Remark 2.10. By Lemma 2.2 applying $\operatorname{Hom}_R(W, -)$ to the upper row of diagram (2.6) yields

$$W^{\triangledown} = \operatorname{coker} \operatorname{Hom}_R(W, \pi_\omega) \cong \operatorname{Ext}^1_R(W, I\omega_R).$$

The inclusion of V in the latter in diagram (2.9) uses $\operatorname{coker} \iota^I \hookrightarrow \operatorname{Ext}^1_R(W, I\omega_R)$. The relation with the inclusion ν in Lemma 2.9 is clarified by the double complex obtained by applying $\operatorname{Hom}_R(-,-)$ to (2.4) and the upper row of (2.6). By Lemma 2.2 it expands to a commutative diagram with exact rows and columns

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & & \\ \operatorname{Ext}^1_R(W, I\omega_R) \arrow[d] & (IF)^I \arrow[l] \arrow[d] & M^I \arrow[l, "\iota^I"'] \arrow[d] & 0 \arrow[l] \arrow[d] & \\ 0 & (IF)^\vee \arrow[l] \arrow[d] & M^\vee \arrow[l, "\iota^\vee"'] \arrow[d] & 0 \arrow[l] \arrow[d, "\operatorname{Hom}_R(W,\pi_\omega)"] & \\ & (IF)^{\triangledown} & M^{\triangledown} \arrow[l] & W^{\triangledown} \arrow[l] \arrow[d, "\cong"'] & 0 \arrow[l] \\ & & & \operatorname{Ext}^1_R(W, I\omega_R). & \end{tikzcd}$$

An element $\alpha(\varphi) \in V$ with $\varphi \in F^\vee$ maps to $\varphi|_{IF} \in (IF)^I$, to $\varphi \circ \kappa \in M^\vee$ and to $\overline{\varphi} \in W^{\triangledown}$.

2.2. ***I*-reflexivity over Gorenstein rings.** In this subsection we assume that R is Gorenstein and study I-reflexivity of modules M in terms of an I-free approximation (2.4). With the Gorenstein hypothesis $F^\vee$ is free and hence

$$\operatorname{Ext}^1_R(F^\vee, -) = 0. \tag{2.13}$$

Proposition 2.11. *Assume that R is Gorenstein. For any I-free approximation (2.4) and W' as in (2.12) there is a commutative square*

$$\begin{tikzcd}
M \arrow[r, two heads, "\rho"] \arrow[d, "\delta_{M,I\omega_R}"] & W \arrow[d, "\overline{\delta}"] \\
M^{II} \arrow[r, two heads, "\rho'"] & W'
\end{tikzcd}$$

and $\overline{\delta}$ is an isomorphism if and only if M is I-reflexive.

Proof. Consider the following commutative diagram whose rows are (2.4) and obtained by applying $-^I$ to the triangle with vertex $F^\vee$ in diagram (2.9).

(2.14)
$$\begin{tikzcd}
 & & F \arrow[rrr, bend left, "\delta_{F,\omega_R}", "\cong"'] & & & \\
0 \arrow[r] & IF \arrow[r, "\iota"] \arrow[ur, hook] \arrow[d, "\delta_{IF,I\omega_R}", "\cong"'] & M \arrow[r, "\rho"] \arrow[u, "\kappa"] \arrow[d, "\delta_{M,I\omega_R}"] & W \arrow[r] \arrow[d, dashed, "\overline{\delta}"] & 0 & \\
0 \arrow[r] & (IF)^{II} \arrow[r, "\iota^{II}"] \arrow[d, equal] & M^{II} \arrow[r, "\rho'"] & W' \arrow[r] & 0 & \\
 & F^{\vee I} \arrow[r, hook] \arrow[ur, "\lambda^I"] & F^{\vee\vee}. & & &
\end{tikzcd}$$

The latter is a short exact sequence by Lemma 2.2 and (2.13). The commutative squares in diagram (2.14) are due to functoriality of δ and the cokernel. The claimed equivalence then follows from the Snake Lemma. Proposition 2.5 yields the part of diagram (2.14) involving δ_{F,ω_R}. This part is just added for clarification but not needed for the proof. ☐

Lemma 2.12. *Assume that R is Gorenstein and consider an I-free approximation (2.4). Then the maps ν from Lemma 2.9 and $\overline{\delta}$ from Proposition 2.11 fit into a commutative square*

$$\begin{tikzcd}
W \arrow[r, "\delta_{W,\overline{\omega}_R}"] \arrow[d, "\overline{\delta}"] & W^{\triangledown\triangledown} \arrow[d, "\nu^{\triangledown}"] \\
W' & V^{\triangledown}. \arrow[l, "\xi", "\cong"']
\end{tikzcd}$$

Proof. Consider the double complex obtained by applying $\mathrm{Hom}_R(-,-)$ to the middle and top rows of diagrams (2.9) and (2.6). By Lemma 2.2 and (2.13) it expands to a commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & & \\
 & 0 \arrow[r] \arrow[d] & F^{\vee I} \arrow[r, "\lambda^I"] \arrow[d] & M^{II} \arrow[r, "\rho'"] \arrow[d] & W' \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] \arrow[d] & F^{\vee\vee} \arrow[r, "\lambda^\vee"] \arrow[d, "(\pi_\omega)_*"] & M^{I\vee} \arrow[r] \arrow[d] & 0 & \\
0 \arrow[r] & V^{\triangledown} \arrow[r, "\alpha^{\triangledown}"] & F^{\vee\triangledown} \arrow[r, "\lambda^{\triangledown}"] \arrow[d] & M^{I\triangledown} & & \\
 & & 0. & & &
\end{tikzcd}$$

The Snake Lemma yields an isomorphism $\xi\colon V^{\triangledown} \to W'$. Attaching the square of Proposition 2.11, the relation $\overline{\delta}(w) = \xi(\widetilde{\psi})$ is given by the diagram chase

$$\begin{tikzcd}
 & & & m \arrow[r, mapsto] \arrow[d, mapsto] & w \arrow[d, mapsto] \\
 & & & \delta_{M,I\omega_R}(m) \arrow[r, mapsto] \arrow[d, mapsto] & \overline{\delta}(w) \\
 & \psi \arrow[r, mapsto] \arrow[d, mapsto] & \psi \circ \lambda = \delta_{M,I\omega_R}(m) & & \\
\widetilde{\psi} \arrow[r, mapsto] & \widetilde{\psi} \circ \alpha = \pi_\omega \circ \psi. & & &
\end{tikzcd}$$

Using implication (2.10), diagram (2.6) and Lemma 2.9, one deduces that, with $x \in I \cap R^{\mathrm{reg}}$ and $v = \alpha(\varphi)$,

$$\begin{aligned}
x\varphi \circ \kappa \in M^I \implies x\varphi &= \lambda(x\varphi \circ \kappa) \\
\implies x\psi(\varphi) &= \psi(x\varphi) = (\psi \circ \lambda)(x\varphi \circ \kappa) \\
&= \delta_{M,I\omega_R}(m)(x\varphi \circ \kappa) = x(\varphi \circ \kappa)(m) \\
\implies \psi(\varphi) &= (\varphi \circ \kappa)(m) \\
\implies \widetilde{\psi}(v) &= (\widetilde{\psi} \circ \alpha)(\varphi) = (\pi_\omega \circ \psi)(\varphi) = (\pi_\omega \circ \varphi \circ \kappa)(m) = \overline{\varphi}(w) \\
&= (\nu \circ \alpha)(\varphi)(w) = \nu(\alpha(\varphi))(w) = \nu(v)(w) \\
&= \delta_{W,\overline{\omega}_R}(w)(\nu(v)) = \nu^{\triangledown}(\delta_{W,\overline{\omega}_R}(w))(v) = (\nu^{\triangledown} \circ \delta_{W,\overline{\omega}_R})(w)(v) \\
\implies \widetilde{\psi} &= (\nu^{\triangledown} \circ \delta_{W,\overline{\omega}_R})(w) \\
\implies \overline{\delta}(w) &= \xi(\widetilde{\psi}) = (\xi \circ \nu^{\triangledown} \circ \delta_{W,\overline{\omega}_R})(w) \\
\implies \overline{\delta} &= \xi \circ \nu^{\triangledown} \circ \delta_{W,\overline{\omega}_R}.
\end{aligned}$$

$\square$

Proposition 2.13. *Assume that R is Gorenstein and consider an I-free approximation* (2.4)*. Then M is I-reflexive if and only if the map $\nu^{\triangledown} \circ \delta_{W,\overline{\varpi}_R}$ with ν from Lemma 2.9 identifies $W = V^{\triangledown}$.*

Proof. The claim follows from Proposition 2.11 and Lemma 2.12. ☐

Lemma 2.14. *Assume that R is Gorenstein and consider an I-free approximation* (2.4)*. Then the map ν from Lemma 2.9 fits into a commutative diagram*

$$\begin{tikzcd}
W^{\triangledown} & & V \arrow[ll, "\nu"'] \arrow[d, "\delta_{V,\overline{\varpi}_R}"] \\
W^{\triangledown\triangledown\triangledown} \arrow[u, "\delta^{\triangledown}_{W,\overline{\varpi}_R}"] & & V^{\triangledown\triangledown}. \arrow[ll, "\nu^{\triangledown\triangledown}"] \arrow[ull, "(\nu^{\triangledown}\circ\delta_{W,\overline{\varpi}_R})^{\triangledown}"']
\end{tikzcd}$$

Proof. For any $v \in V$ and $w \in W$ we have

$$\begin{aligned}
(\delta^{\triangledown}_{W,\overline{\varpi}_R} \circ \nu^{\triangledown\triangledown} \circ \delta_{V,\overline{\varpi}_R})(v)(w) &= \delta^{\triangledown}_{W,\overline{\varpi}_R}(\nu^{\triangledown\triangledown}(\delta_{V,\overline{\varpi}_R}(v)))(w) \\
&= \delta^{\triangledown}_{W,\overline{\varpi}_R}(\delta_{V,\overline{\varpi}_R}(v) \circ \nu^{\triangledown})(w) \\
&= (\delta_{V,\overline{\varpi}_R}(v) \circ \nu^{\triangledown})(\delta_{W,\overline{\varpi}_R}(w)) \\
&= \delta_{V,\overline{\varpi}_R}(v)(\delta_{W,\overline{\varpi}_R}(w) \circ \nu) \\
&= \delta_{W,\overline{\varpi}_R}(w)(\nu(v)) \\
&= \nu(v)(w)
\end{aligned}$$

and hence $\nu = \delta^{\triangledown}_{W,\overline{\varpi}_R} \circ \nu^{\triangledown\triangledown} \circ \delta_{V,\overline{\varpi}_R}$ as claimed. ☐

Corollary 2.15. *Assume that R is Gorenstein and consider an I-free approximation* (2.4) *of an I-reflexive R-module M. Then V in diagram* (2.9) *is ($\overline{\varpi}_R$-)reflexive if and only if ν in Lemma 2.9 identifies $V = W^{\triangledown}$.*

Proof. The claim follows from Proposition 2.13 and Lemma 2.14. ☐

2.3. ***R*-dual *I*-free approximation.** In this subsection we consider the R-dual of an I-free approximation (2.4). The interesting part of the long exact Ext-sequence of $-^{\vee}$ applied to (2.4) turns out to be

$$0 \leftarrow \operatorname{Ext}^k_R(M,\omega_R) \leftarrow \operatorname{Ext}^k_R(W,\omega_R) \xleftarrow{\beta} \operatorname{Ext}^{k-1}_R(IF,\omega_R) \leftarrow \operatorname{Ext}^{k-1}_R(M,\omega_R) \leftarrow 0. \tag{2.15}$$

In fact, applying $-^{\vee}$ to (2.1) yields (see Lemma 2.17 and [7, Thm. 3.3.10.(c).(ii)])

$$\operatorname{Ext}^i_R(IF,\omega_R) = F^* \otimes_R \operatorname{Ext}^i_R(I,\omega_R) = F^* \otimes_R \operatorname{Ext}^{i+1}_R(\overline{R},\omega_R) = 0 \text{ for } i \neq 0, k-1.$$

In case both R and $\overline{R}$ are Gorenstein, we will identify the map β to its image with the map $\overline{\alpha}$ in (2.9) (see Corollary 2.21). In §2.4 this fact will serve to relate the Gorenstein dimension of M to the depth of V.

In order to describe the map β in (2.15) we fix a canonical module ω_R of R with an injective resolution $(E^{\bullet},\partial^{\bullet})$,

$$0 \longrightarrow \omega_R \longrightarrow E^0 \xrightarrow{\partial^0} E^1 \xrightarrow{\partial^1} E^2 \xrightarrow{\partial^2} \cdots.$$

We use it to fix representatives

$$\operatorname{Ext}^i_R(-,\omega_R) := H^i \operatorname{Hom}_R(-,E^{\bullet}).$$

Then (see [7, Thms. 3.3.7.(b), 3.3.10.(c).(ii)])

$$H^i \operatorname{Ann}_{E^\bullet}(I) = H^i \operatorname{Hom}(\overline{R}, E^\bullet) = \operatorname{Ext}^i_R(\overline{R}, \omega_R) = \delta_{i,k} \cdot \omega_{\overline{R}} \tag{2.16}$$

where

$$\omega_{\overline{R}} := H^k \operatorname{Ann}_{E^\bullet}(I)$$

is a canonical module of $\overline{R}$.

In the sequel we explicit the maps of the following commutative diagram

$$\begin{tikzcd}
\operatorname{Ext}^k_R(W, \omega_R) \arrow[dddd, "\gamma"', "\cong"] & & \operatorname{Ext}^{k-1}_R(IF, \omega_R) \arrow[ll, "\beta"'] \\
& & F^* \otimes_R \operatorname{Ext}^{k-1}_R(I, \omega_R) \arrow[u, "\cong"', "\chi"] \\
& & F^* \otimes_R H^{k-1}(E^\bullet / \operatorname{Ann}_{E^\bullet}(I)) \arrow[u, "\cong"', "F^* \otimes H^{k-1}(\tau^\bullet)"] \arrow[d, "\cong"', "F^* \otimes \zeta"] \\
& & F^* \otimes_R \omega_{\overline{R}} \arrow[ld, phantom] \\
\operatorname{Hom}_{\overline{R}}(W, \omega_{\overline{R}}) = W^\vee & V' \arrow[l, hook', "\nu'"'] & F^* \otimes_R \omega_{\overline{R}} \arrow[l, two heads, "\alpha'"']
\end{tikzcd} \tag{2.17}$$

which defines the map $\nu' \circ \alpha'$ and its image V'. The maps $\tau^\bullet$, χ, ζ, γ and α' are described in Lemmas 2.16, 2.17, 2.18, 2.19 and Proposition 2.20 respectively.

Lemma 2.16. *For any injective R-module E there is a canonical isomorphism*

$$\tau \colon E/\operatorname{Ann}_E(I) \to \operatorname{Hom}_R(I, E), \quad \overline{e} \mapsto - \cdot e = (x \mapsto x \cdot e).$$

In particular, there is a canonical isomorphism $\tau^\bullet \colon E^\bullet / \operatorname{Ann}_{E^\bullet}(I) \to \operatorname{Hom}_R(I, E^\bullet)$.

Proof. Applying the exact functor $\operatorname{Hom}_R(-, E)$ to (2.1) yields a short exact sequence

$$0 \leftarrow \operatorname{Hom}_R(I, E) \leftarrow \operatorname{Hom}_R(R, E) \leftarrow \operatorname{Hom}_R(\overline{R}, E) \leftarrow 0.$$

Identifying $E = \operatorname{Hom}_R(R, E)$, $e \mapsto - \cdot e$, and hence

$$\operatorname{Hom}_R(\overline{R}, E) = \operatorname{Ann}_E(I) \tag{2.18}$$

yields the claim. $\square$

Lemma 2.17. *For any $i \in \mathbb{N}$ there is a canonical isomorphism*

$$\begin{tikzcd}
F^* \otimes_R \operatorname{Ext}^i_R(I, \omega_R) \arrow[r, "\chi_i", "\cong"'] \arrow[d, equal] & \operatorname{Ext}^i_R(IF, \omega_R) \arrow[d, equal] \\
F^* \otimes_R H^i \operatorname{Hom}_R(I, E^\bullet) \arrow[r] & H^i \operatorname{Hom}_R(IF, E^\bullet) \\
\varphi \otimes [\psi] \arrow[r, mapsto] & {[\varphi|_{IF} \cdot \widetilde{\psi}(1)] = [(\kappa \circ \iota)^*(\varphi) \cdot \widetilde{\psi}(1)]}
\end{tikzcd}$$

where $\widetilde{\psi} \in \operatorname{Hom}_R(R, E^\bullet)$ extends $\psi \in \operatorname{Hom}_R(I, E^\bullet)$. We set $\chi := \chi_{k-1}$.

Proof. For any $i \in \mathbb{N}$ there is a sequence of canonical isomorphisms

$$\begin{aligned}
F^* \otimes_R H^i \operatorname{Hom}_R(I, E^\bullet) &= \operatorname{Hom}_R(F, H^i \operatorname{Hom}_R(I, E^\bullet)) \\
&= H^i \operatorname{Hom}_R(F, \operatorname{Hom}_R(I, E^\bullet)) \\
&= H^i \operatorname{Hom}_R(IF, E^\bullet),
\end{aligned}$$

the latter one being Hom-tensor adjunction, sending

$$\begin{aligned}\varphi\otimes[\psi] &\mapsto (f\mapsto\varphi(f)\cdot[\psi]=[\varphi(f)\cdot\psi])\\ &\mapsto [f\mapsto\varphi(f)\cdot\psi]\\ &\mapsto [x\cdot f\mapsto\varphi(f)\cdot\psi(x)=\varphi(x\cdot f)\cdot\widetilde{\psi}(1)]=[\varphi|_{IF}\cdot\widetilde{\psi}(1)]\end{aligned}$$

where $x\in I$ and $f\in F$. □

Lemma 2.18. *There is a connecting isomorphism*

$$\begin{aligned}\zeta\colon H^{k-1}(E^\bullet/\operatorname{Ann}_{E^\bullet}(I))&\to H^k\operatorname{Ann}_{E^\bullet}(I)=\omega_{\overline{R}},\\ [\overline{e}]&\mapsto[\partial^{k-1}(e)].\end{aligned}$$

Proof. The connecting homomorphism ζ in degree k of the short exact sequence

$$0\to\operatorname{Ann}_{E^\bullet}(I)\to E^\bullet\to E^\bullet/\operatorname{Ann}_{E^\bullet}(I)\to 0$$

is an isomorphism since $E^\bullet$ is a resolution and hence $H^i(E^\bullet)=0$ for $i\ge k-1\ge 1$. □

Lemma 2.19. *For any $\overline{R}$-module N there is a canonical isomorphism*

$$\begin{aligned}\gamma\colon H^k\operatorname{Hom}_R(N,E^\bullet)&\to\operatorname{Hom}_{\overline{R}}(N,H^k\operatorname{Ann}_{E^\bullet}(I))=N^\vee,\\ [\phi]&\mapsto(n\mapsto[\phi(n)]).\end{aligned}$$

Proof. Fix an $\overline{R}$-projective resolution $(P_\star,\delta_\star)$ of N and consider the double complex

$$A^{\star,\bullet}:=\operatorname{Hom}_R(P_\star,E^\bullet)=\operatorname{Hom}_{\overline{R}}(P_\star,\operatorname{Hom}_R(\overline{R},E^\bullet))=\operatorname{Hom}_{\overline{R}}(P_\star,\operatorname{Ann}_{E^\bullet}(I))$$

whose alternate representation is due to Hom-tensor adjunction and (2.18). It yields two spectral sequences with the same limit. By exactness of $\operatorname{Hom}_{\overline{R}}(P_\star,-)$ and (2.16) and using the alternate representation the E_2-page of the first spectral sequence identifies with

$${}'E_2^{p,q}=H^p(H^{\star,q}(A^{\star,\bullet}))=H^p\operatorname{Hom}_{\overline{R}}(P_\star,H^q\operatorname{Ann}_{E^\bullet}(I))=\delta_{k,q}\cdot H^p\operatorname{Hom}_{\overline{R}}(P_\star,\omega_{\overline{R}}).$$

By exactness of $\operatorname{Hom}_R(-,E^\bullet)$ the E_2-page of the second spectral sequence reads

$${}''E_2^{p,q}=H^q(H^{p,\bullet}(A^{\star,\bullet}))=H^q\operatorname{Hom}_R(H^pP_\star,E^\bullet)=\delta_{p,0}\cdot H^q\operatorname{Hom}_R(N,E^\bullet).$$

So both spectral sequences degenerate. The resulting isomorphism ${}''E_2^{0,k}\to{}'E_2^{0,k}$ is γ. □

Proposition 2.20. *Assume that R is Gorenstein and consider an I-free approximation (2.4). Then the map α' in diagram (2.17) is induced by*

$$\begin{aligned}\nu'\circ\alpha'\colon F^*\otimes_R\omega_{\overline{R}}=F^*\otimes_R H^k\operatorname{Ann}_{E^\bullet}(I)&\to\operatorname{Hom}_{\overline{R}}(W,H^k\operatorname{Ann}_{E^\bullet}(I))=W^\vee,\\ \varphi\otimes[a]&\mapsto\overline{\varphi}\cdot[a],\end{aligned}$$

where $\varphi\mapsto\overline{\varphi}$ is (2.7) with $\omega_R=R$. In particular, $\operatorname{Ext}_R^k(M,R)=0$ if ν' is surjective.

Proof. The proof is done by chasing diagram (2.17) and the diagram

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}_R(W,E^{k-1}) \arrow[r, "\rho^*"] \arrow[d, "(\partial^{k-1})_*"] & \operatorname{Hom}_R(M,E^{k-1}) \arrow[r, "\iota^*"] \arrow[d, "(\partial^{k-1})_*"] & \operatorname{Hom}_R(IF,E^{k-1}) \arrow[r] \arrow[d, "(\partial^{k-1})_*"] & 0\\
0 \arrow[r] & \operatorname{Hom}_R(W,E^k) \arrow[r, "\rho^*"] & \operatorname{Hom}_R(M,E^k) \arrow[r, "\iota^*"] & \operatorname{Hom}_R(IF,E^k) \arrow[r] & 0.
\end{tikzcd}$$

This latter defines the connecting homomorphism β in (2.15) on representatives as

$$(\rho^*)^{-1}\circ(\partial^{k-1})_*\circ(\iota^*)^{-1},$$

where $(\iota^*)^{-1}$ denotes the choice of any preimage under ι^*.

Let $\varphi \otimes [\overline{e}] \in F^* \otimes_R H^{k-1}(E^\bullet / \operatorname{Ann}_{E^\bullet}(I))$. Then by Lemmas 2.16, 2.17, 2.18 and 2.19, and diagram (2.6) with $\omega_R = R$

$$\begin{tikzcd}
 & {[\kappa^*(\varphi)\cdot e]} \arrow[r, mapsto, "H^{k-1}(\iota^*)"] \arrow[d, mapsto] & {[(\iota^*\circ\kappa^*)(\varphi)\cdot e]} \\
{[((\rho^{-1})^*\circ\kappa^*)(\varphi)\cdot\partial^{k-1}(e)]} \arrow[r, mapsto, "H^k(\rho^*)"] \arrow[ddd, mapsto, "\gamma"'] & {[\kappa^*(\varphi)\cdot\partial^{k-1}(e)]} & \varphi\otimes[-\cdot e] \arrow[u, mapsto, "\chi"'] \\
 & & \varphi\otimes[\overline{e}] \arrow[u, mapsto, "F^*\otimes H^{k-1}(\tau^\bullet)"'] \arrow[d, mapsto, "F^*\otimes\zeta"'] \\
 & & \\
(\pi\circ\varphi\circ\kappa\circ\rho^{-1})\cdot[\partial^{k-1}(e)] = \overline{\varphi}\cdot[\partial^{k-1}(e)] & & \varphi\otimes[\partial^{k-1}(e)] \arrow[ll, mapsto, "\nu'\circ\alpha'"']
\end{tikzcd}$$

where ρ^{-1} denotes the choice of any preimage under ρ. By diagram (2.6) and Lemma 2.18 the ambiguity of this choice is cancelled when multiplying $(\rho^{-1})^* \circ \kappa^*(\varphi) = \varphi \circ \kappa \circ \rho^{-1}$ with $\partial^{k-1}(e) \in \operatorname{Ann}_{E^\bullet}(I)$.

The particular claim follows from diagram (2.17) and the exact sequence (2.15). $\square$

Corollary 2.21. *Assume that both R and $\overline{R}$ are Gorenstein and consider an I-free approximation (2.4). Then identifying $\overline{\omega}_R = \omega_{\overline{R}}$ (see diagrams (2.9) and (2.17)) makes*

$$\alpha' = \overline{\alpha}, \quad V' = V, \quad \operatorname{Ext}_R^{k-1}(M,R) \cong \ker(\overline{\alpha}) = M^I/F^I.$$

In particular, if M is I-reflexive, then $\operatorname{Ext}_R^k(M,R) = 0$ if and only if V is ($\overline{\omega}_R$-)reflexive.

Proof. Let $\varphi \mapsto \overline{\varphi}$ be (2.7) with $\omega_R = R$. Pick free generators $\varepsilon \in \omega_R$ and $\widetilde{\varepsilon} \in \omega_{\overline{R}}$ inducing the identification $\overline{\omega}_R = \omega_{\overline{R}}$ by sending $\overline{\varepsilon} = \pi_\omega(\varepsilon) \mapsto \widetilde{\varepsilon}$. Then

$$\begin{aligned}
F^\vee \otimes_R \overline{R} = F^* \otimes_R \overline{\omega}_R = F^* \otimes_R \omega_{\overline{R}}, &\quad W^{\overline{\vee}} = W^\vee, \\
(\varphi\cdot\varepsilon)\otimes\overline{1} \leftrightarrow \varphi\otimes\overline{\varepsilon} \leftrightarrow \varphi\otimes\widetilde{\varepsilon}, &\quad \overline{\varphi}\cdot\overline{\varepsilon} \leftrightarrow \overline{\varphi}\cdot\widetilde{\varepsilon}.
\end{aligned}$$

By diagram (2.6) and Lemma 2.9 the map $F^\vee \otimes_R \overline{R} \to W^{\overline{\vee}}$ induced by $\nu \circ \alpha$ sends

$$(\varphi\cdot\varepsilon)\otimes\overline{1} \mapsto \overline{\varphi\cdot\varepsilon} = \pi_\omega \circ ((\varphi\circ\kappa\circ\rho^{-1})\cdot\varepsilon) = (\pi\circ\varphi\circ\kappa\circ\rho^{-1})\cdot\pi_\omega(\varepsilon) = \overline{\varphi}\cdot\overline{\varepsilon}.$$

By Proposition 2.20 this map coincides with $\nu' \circ \alpha'$ subject to the above identifications. This shows that $\alpha' = \overline{\alpha}$ and $V' = V$. By the exact sequence (2.15), the commutative diagram (2.17) and the exact upper row of diagram (2.9),

$$\begin{aligned}
\operatorname{Ext}_R^{k-1}(M,R) &= \ker(\beta) \cong \ker(\alpha') = \ker(\overline{\alpha}) = M^I/F^I, \\
\operatorname{Ext}_R^{k}(M,R) &= \operatorname{coker}(\beta) \cong \operatorname{coker}(\nu') = W^\vee/\nu'(V').
\end{aligned}$$

In particular $\operatorname{Ext}_R^k(M,R) = 0$ if and only if ν' identifies $V' = W^\vee$ or, equivalently, if ν identifies $V = W^{\overline{\vee}}$. The particular claim now follows with Corollary 2.15. $\square$

2.4. **Projective dimension and residual depth.** Assume that R is Gorenstein. Then every finitely generated R-module M has finite Gorenstein dimension $\operatorname{G-dim}(M) < \infty$ (see [18, Thm. 17]). Recall that if M has finite projective dimension $\operatorname{pdim}(M) < \infty$, then

$$\operatorname{G-dim}(M) = \operatorname{pdim}(M)$$

(see [18, Cor. 21]). Consider an I-free approximation (2.4) of an R-module M. In the following we relate the case of minimal Gorenstein dimension of M to Cohen–Macaulayness of V, proving our main result.

Lemma 2.22. *Assume that R is Gorenstein and consider an I-free approximation (2.4) with $W \neq 0$. Then W is a maximal Cohen–Macaulay $\overline{R}$-module if and only if $\operatorname{G-dim}(M) \leq k$. In this case $\operatorname{G-dim}(M) \leq k-1$ if and only if $\operatorname{Ext}^k_R(M,R) = 0$. If $\overline{R}$ is Gorenstein, then $\operatorname{G-dim}(M) \geq k-1$ unless $\overline{\alpha}$ in diagram (2.9) is injective.*

Proof. By hypothesis $M \neq 0$ is finitely generated over the Gorenstein ring R. It follows that (see [18, Thm. 17, Lem. 23.(c)])

$$\operatorname{G-dim}(M) = \max\left\{i \in \mathbb{N} \;\middle|\; \operatorname{Ext}^i_R(M,R) \neq 0\right\} < \infty. \tag{2.19}$$

The Auslander–Bridger Formula (see [18, Thm. 29]) then states that

$$\operatorname{depth}(M) = \operatorname{depth}(R) - \operatorname{G-dim}(M) = \dim(R) - \operatorname{G-dim}(M) = n - \operatorname{G-dim}(M). \tag{2.20}$$

By the Depth Lemma (see [7, Prop. 1.2.9]) applied to the short exact sequence (2.1)

$$\begin{aligned} n-k+1 = \operatorname{depth}(\overline{R}) + 1 &\geq \min\{\operatorname{depth}(R), \operatorname{depth}(I) - 1\} + 1 = \operatorname{depth}(I) \\ &\geq \min\left\{\operatorname{depth}(R), \operatorname{depth}(\overline{R}) + 1\right\} = n-k+1 \end{aligned}$$

and hence

$$\operatorname{depth}(IF) = \operatorname{depth}(I) = n-k+1. \tag{2.21}$$

($\Longrightarrow$) Using (2.21) and (2.20) the Depth Lemma applied to the short exact sequence (2.4) gives

$$\operatorname{G-dim}(M) = n - \operatorname{depth}(M) \leq n - \min\{\operatorname{depth}(IF), \operatorname{depth}(W)\} \leq n - (n-k) = k.$$

($\Longleftarrow$) Using (2.20) and (2.21) the Depth Lemma applied to the short exact sequence (2.4) gives

$$n-k = \dim(\overline{R}) \geq \dim(W) \geq \operatorname{depth}(W) \geq \min\{\operatorname{depth}(M), \operatorname{depth}(IF) - 1\} \geq n-k.$$

By (2.19) this latter inequality becomes $\operatorname{G-dim}(M) \leq k-1$ if and only if $\operatorname{Ext}^k_R(M,R) = 0$ (see [18, Lem. 23.(c)]).

If $\overline{R}$ is Gorenstein and $\overline{\alpha}$ is not injective, then $\operatorname{Ext}^{k-1}_R(M,R) \neq 0$ by Corollary 2.21 and hence $\operatorname{G-dim}(M) \geq k-1$ by (2.19). □

We can now conclude the proof of our main result.

Proof of Theorem 1.2. Since M is I-reflexive, $W = V^{\triangledown}$ by Proposition 2.13.

($\Longrightarrow$) Suppose that V is maximal Cohen–Macaulay. Then also W is maximal Cohen–Macaulay and V is ($\overline{\omega}_R$-)reflexive (see [7, Prop. 3.3.3.(b).(ii), Thm. 3.3.10.(d).(iii)]). By Corollary 2.21 $\operatorname{Ext}^k_R(M,R) = 0$ and by Lemma 2.22 $\operatorname{G-dim}(M) = k-1$.

($\Longleftarrow$) Suppose that $\operatorname{G-dim}(M) \leq k-1$. By Lemma 2.22 W is maximal Cohen–Macaulay and $\operatorname{Ext}^k(M,R) = 0$. By Corollary 2.21 $V = W^{\triangledown}$ is ($\overline{\omega}_R$-)reflexive and maximal Cohen–Macaulay (see [7, Prop. 3.3.3.(b).(ii)]).

The last claim is due to Lemma 2.22. □

2.5. **Restricted I-free approximation.** In this subsection we describe a construction that reduces the support of an I-free approximation (2.4) and preserves I-reflexivity of M under suitable hypotheses. In §3.2 this will be related to the definition of multi-logarithmic differential forms and residues along Cohen–Macaulay spaces (see [4, §10] and [21, Ch. 4]).

Fix an ideal $J \trianglelefteq R$ with $I \subseteq J$ and set $S := \overline{R}$ and $T := R/J$. By hypothesis S is Cohen–Macaulay and hence (see [7, Prop.1.2.13])

$$\operatorname{Ass}(S) = \operatorname{Min}\operatorname{Spec}(S). \tag{2.22}$$

Lemma 2.23. *There is an inclusion*

$$\operatorname{Supp}_S(T) \cap \operatorname{Ass}(S) \subseteq \operatorname{Ass}_S(T).$$

In particular, equality in $\operatorname{Hom}_S(N, S)$ *for any* T*-module* N*, or in* $\operatorname{Hom}_S(N, T)$ *for any* S*-module* N*, can be checked at* $\operatorname{Ass}_S(T)$*.*

Proof. The inclusion follows from (2.22) and $\operatorname{Min}\operatorname{Supp}_S(T) \subseteq \operatorname{Ass}_S(T)$. For any T-module N (see [7, Exe. 1.2.27])

$$\operatorname{Ass}_S(\operatorname{Hom}_S(N, S)) = \operatorname{Supp}_S(N) \cap \operatorname{Ass}(S) \subseteq \operatorname{Supp}_S(T) \cap \operatorname{Ass}(S) \subseteq \operatorname{Ass}_S(T)$$

and the first particular claim follows, the second holds for a similar reason. □

Definition 2.24. For any S-module N we consider the submodule supported on $V(J)$

$$N_T := \operatorname{Hom}_S(T, N) = \operatorname{Ann}_N(J) \subseteq N.$$

For an I-free approximation (2.4) its *J-restriction* is the I-free approximation

$$0 \longrightarrow IF \xrightarrow{\iota_J} M_J \xrightarrow{\rho_T} W_T \longrightarrow 0 \tag{2.23}$$

defined as its image under the map $\operatorname{Ext}^1_R(W, IF) \to \operatorname{Ext}^1_R(W_T, IF)$.

In explicit terms it is the source of a morphism of I-free approximations

$$\begin{tikzcd} 0 \arrow[r] & IF \arrow[r, "\iota"] & M \arrow[r, "\rho"] & W \arrow[r] & 0 \\ 0 \arrow[r] & IF \arrow[r, "\iota_J"] \arrow[u, equal] & M_J \arrow[r, "\rho_T"] \arrow[u, hook] & W_T \arrow[r] \arrow[u, hook] & 0. \end{tikzcd} \tag{2.24}$$

The right square is obtained as the pull-back of ρ and $W_T \hookrightarrow W$, whose universal property applied to ι and $0\colon IF \to W_T$ gives the left square. The analogue of κ in (2.5) for the J-restriction (2.23) is the composition

$$\kappa_J\colon M_J = IF :_M J \subseteq M \xrightarrow{\kappa} F. \tag{2.25}$$

By Lemma 2.2 and the Snake Lemma, applying $-^I$ to (2.24) yields (see Definition 2.8)

$$\begin{tikzcd} 0 & V \arrow[l] \arrow[d, two heads] & F^\vee \arrow[l, "\alpha"'] \arrow[d, equal] & M^I \arrow[l, "\lambda"'] \arrow[d, hook] & 0 \arrow[l] \\ 0 & V^T \arrow[l] & F^\vee \arrow[l, "\alpha^T"'] & M^I_J \arrow[l, "\lambda^J"'] & 0 \arrow[l] \end{tikzcd} \tag{2.26}$$

where the bottom row

$$0 \longleftarrow V^T \xleftarrow{\alpha^T} F^\vee \xleftarrow{\lambda^J} M^I_J \longleftarrow 0 \tag{2.27}$$

is the I-dual (2.11) of the J-restriction (2.23). In diagram (2.26), we denote

$$U := \ker(V \twoheadrightarrow V^T). \tag{2.28}$$

The J-restriction behaves well under the following hypothesis on T.

$$T_{\mathfrak{p}} = \begin{cases} S_{\mathfrak{p}} & \text{if } \mathfrak{p} \in \mathrm{Ass}_S(T), \\ 0 & \text{if } \mathfrak{p} \in \mathrm{Ass}(S) \setminus \mathrm{Ass}_S(T). \end{cases} \tag{2.29}$$

This is due to the following

Remark 2.25. Our constructions commute with localization. As special cases of the J-restriction and its I-dual we record

$$(\iota_J, \rho_T) = \begin{cases} (\iota, \rho) & \text{if } T = S, \\ (\mathrm{id}_{IF}, 0) & \text{if } T = 0, \end{cases} \quad (\lambda^J, \alpha^T) = \begin{cases} (\lambda, \alpha) & \text{if } T = S, \\ (\mathrm{id}_{F^\vee}, 0) & \text{if } T = 0. \end{cases}$$

Localizing (2.24) and (2.26) at the image of $\mathfrak{p} \in \mathrm{Ass}(S)$ under the map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ yields these special cases under hypothesis (2.29).

In the setup of our applications in §3 condition (2.29) holds true due to the following

Lemma 2.26. *If S is reduced and T is unmixed with $\dim(T) = \dim(S)$, then condition (2.29) holds and $\mathrm{Ass}_S(T) \subseteq \mathrm{Ass}(S)$.*

Proof. By hypothesis on T and (2.22)

$$\mathrm{Ass}_S(T) = \mathrm{Min}\,\mathrm{Supp}_S(T) \subseteq \mathrm{Min}\,\mathrm{Spec}(S) = \mathrm{Ass}(S). \tag{2.30}$$

By hypothesis on S, for any $\mathfrak{p} \in \mathrm{Ass}(S)$, $S_{\mathfrak{p}}$ is a field with factor ring $T_{\mathfrak{p}}$. If $\mathfrak{p} \in \mathrm{Ass}_S(T)$, then $T_{\mathfrak{p}} \neq 0$ and hence $T_{\mathfrak{p}} = S_{\mathfrak{p}}$. Otherwise, $\mathfrak{p} \notin \mathrm{Supp}_S(T)$ by (2.30) and hence $T_{\mathfrak{p}} = 0$. ☐

Lemma 2.27. *Assume that R is Gorenstein and consider the J-restriction (2.23) of an I-free approximation. If T satisfies condition (2.29), then for U as defined in (2.28)*

$$\alpha^{-1}(U) = \{\varphi \in F^\vee \mid \varphi \circ \kappa(M) \subseteq J\omega_R\}.$$

In particular, $JV \subseteq U$.

Proof. Let $\varphi \in F^\vee$ and denote by $\overline{\varphi}_T$ the map $\overline{\varphi}$ in diagram (2.6) for the J-restriction (2.23). Consider the map ψ defined by the commutative diagram

$$\begin{tikzcd} W \arrow[r, "\psi"] \arrow[dr, "\overline{\varphi}"] & T \otimes_R \omega_R \\ W_T \arrow[u, hook] \arrow[r, "\overline{\varphi}_T"] & S \otimes_R \omega_R. \arrow[u, two heads] \end{tikzcd} \tag{2.31}$$

By Lemma 2.23 and since $\omega_R \cong R$ both $\overline{\varphi}_T = 0$ and $\psi = 0$ can be checked at $\mathrm{Ass}_S(T)$. There the vertical maps in diagram (2.31) induce the identity by condition (2.29) and Remark 2.25. With diagram (2.26), Lemma 2.9 applied to (2.23) and diagram (2.6) it follows that

$$\alpha(\varphi) \in U \iff \alpha^T(\varphi) = 0 \iff \overline{\varphi}_T = 0 \iff \psi = 0 \iff \varphi \circ \kappa(M) \subseteq J\omega_R.$$

This proves the equality and the inclusion follows with $JV = J\alpha(F^\vee) = \alpha(JF^\vee)$. ☐

Proposition 2.28. *Assume that R is Gorenstein and consider the J-restriction (2.23) of an I-free approximation. If T satisfies condition (2.29), then with M also M_J is I-reflexive.*

Proof. By Lemma 2.27 there is a short exact sequence

$$0 \to U/JV \to V/JV \to V^T \to 0. \tag{2.32}$$

By condition (2.29) and Remark 2.25

$$JS_{\mathfrak{p}} = \begin{cases} 0 & \text{if } \mathfrak{p} \in \operatorname{Ass}_S(T), \\ S_{\mathfrak{p}} & \text{if } \mathfrak{p} \in \operatorname{Ass}(S) \setminus \operatorname{Ass}_S(T), \end{cases}$$

$$(V \twoheadrightarrow V^T)_{\mathfrak{p}} = \begin{cases} \mathrm{id}_{V_{\mathfrak{p}}} & \text{if } \mathfrak{p} \in \operatorname{Ass}_S(T), \\ 0 & \text{if } \mathfrak{p} \in \operatorname{Ass}(S) \setminus \operatorname{Ass}_S(T), \end{cases}$$

and hence

$$\begin{aligned} \forall \mathfrak{p} \in \operatorname{Ass}(S)\colon (JV)_{\mathfrak{p}} = JS_{\mathfrak{p}} V_{\mathfrak{p}} = U_{\mathfrak{p}} &\implies (U/JV)_{\mathfrak{p}} = 0 \\ &\implies \dim(U/JV) < \dim(S) = \operatorname{depth}(\overline{\omega}_R). \end{aligned}$$

Then $(U/JV)^{\triangledown} = 0$ by Ischebeck's Lemma (see [12, Satz 1.9]). Using sequence (2.32) and Hom-tensor adjunction it follows that

$$(V^T)^{\triangledown} = (V/JV)^{\triangledown} = (T \otimes_S V)^{\triangledown} = (V^{\triangledown})_T.$$

Denote by ν_T the map ν from Lemma 2.9 applied to the J-restriction (2.23). We obtain a diagram

$$\begin{tikzcd} W_T \arrow[rr, "(\nu^{\triangledown} \circ \delta_{W,\overline{\omega}_R})_T"] \arrow[d, equal] & & (V^{\triangledown})_T \arrow[d, equal] \\ W_T \arrow[r, "\delta_{W_T,\overline{\omega}_R}"] & (W_T)^{\triangledown\triangledown} \arrow[r, "(\nu_T)^{\triangledown}"] & (V^T)^{\triangledown}. \end{tikzcd} \tag{2.33}$$

By Lemma 2.23 and since $\overline{\omega}_R \cong S$, its commutativity can be checked at $\operatorname{Ass}_S(T)$. By condition (2.29) and Remark 2.25 top and bottom horizontal maps in diagram (2.33) identify at $\operatorname{Ass}_S(T)$. Diagram (2.33) thus commutes and Proposition 2.13 yields the claim. $\square$

The Cohen–Macaulay property is invariant under restriction of scalars $S \to T$ and by Hom-tensor adjunction $\operatorname{Hom}_S(-, \omega_S) = \operatorname{Hom}_T(-, \omega_T)$ on T-modules where (see [7, Thm. 3.3.7.(b)])

$$\omega_T = \operatorname{Hom}_S(T, \omega_S). \tag{2.34}$$

Combining Theorem 1.2 and Proposition 2.28 yields (see diagram (2.26))

Corollary 2.29. *In addition to the hypotheses of Theorem 1.2, let $J \trianglelefteq R$ with $J \subseteq I$ be such that $T = R/J$ satisfies condition (2.29) and $W_T \neq 0$. Consider the J-restriction (2.23) with I-dual (2.27). Then $W_T = \operatorname{Hom}_T(V^T, \omega_T)$ and V^T is a maximal Cohen–Macaulay T-module if and only if* $\operatorname{G-dim}(M_J) \leq k-1$. *In this latter case $V^T = \operatorname{Hom}_T(W_T, \omega_T)$ is ω_T-reflexive. Unless $T \otimes \alpha^T$ (and hence $\overline{\alpha}$) is injective* $\operatorname{G-dim}(M_J) \geq k-1$. $\square$

Finally we mention a construction analogous to Definition 2.24 not used in the sequel.

Remark 2.30. Assume that J satisfies the hypotheses on I and consider an I-free approximation (2.4) where W is already a T-module. Then $W_T = W$ and $M_J = M$ and the image of (2.4) under the map $\operatorname{Ext}^1_R(W, IF) \to \operatorname{Ext}^1_R(W, JF)$ is a J-free approximation that fits into a commutative diagram with cartesian left square

$$\begin{tikzcd}
0 \arrow[r] & JF \arrow[r] & M^J \arrow[r] & W \arrow[r] & 0 \\
0 \arrow[r] & IF \arrow[r] \arrow[u, hook] & M \arrow[r] \arrow[u, hook] & W \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

where $M^J/M_J \cong JF/IF$. In particular, $M^J = M_J$ if and only if $I = J$.

3. Application to logarithmic forms

In this section results from §2 are used to give a more conceptual approach to and to generalize a duality of multi-logarithmic forms found by Pol [21] as a generalization of result by Granger and the first author [11].

Let Y be a germ of a smooth complex analytic space of dimension n. Then $Y \cong (\mathbb{C}^n, 0)$ and $\mathscr{O}_Y \cong \mathbb{C}\{x_1, \dots, x_n\}$ by a choice of coordinates $x_1, \dots, x_n$ on Y. We denote by

$$\mathscr{Q}_- := Q(\mathscr{O}_-)$$

the total ring of fractions of $\mathscr{O}_-$. In this section we set $-^* := \operatorname{Hom}_{\mathscr{O}_Y}(-, \mathscr{O}_Y)$.

Let $\Omega^\bullet_Y$ denote the *De Rham algebra on* Y, that is,

$$\mathscr{O}_Y \to \Omega^1_Y, \quad f \mapsto df,$$

is the universally finite $\mathbb{C}$-linear derivation of $\mathscr{O}_Y$ (see [25, §2] and [15, §11]) and $\Omega^q_Y = \bigwedge^q_{\mathscr{O}_Y} \Omega^1_Y$ for all $q \geq 0$. In terms of coordinates $\Omega^1_Y \cong \bigoplus_{i=1}^n \mathscr{O}_Y dx_i$ and hence

$$\Omega^q_Y = \bigwedge^q_{\mathscr{O}_Y} \Omega^1_Y \cong \bigoplus_{i_1 < \cdots < i_q} \mathscr{O}_Y dx_{i_1} \wedge \cdots \wedge dx_{i_q}$$

is a free $\mathscr{O}_Y$-module. By definition the dual

$$(\Omega^1_Y)^* = \operatorname{Der}_{\mathbb{C}}(\mathscr{O}_Y) =: \Theta_Y \cong \bigoplus_{i=1}^n \mathscr{O}_Y \frac{\partial}{\partial x_i}$$

is the module of $\mathbb{C}$-*linear derivations on* $\mathscr{O}_Y$, or of vector fields on Y. The module of q-*vector fields on* Y is then the free $\mathscr{O}_Y$-module

$$(\Omega^q_Y)^* = \bigwedge^q_{\mathscr{O}_Y} \Theta_Y =: \Theta^q_Y \cong \bigoplus_{i_1 < \cdots < i_q} \mathscr{O}_Y \frac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_q}}.$$

Notation 3.1. We set $N := \{1, \dots, n\}$ and $N^q_< := \{\underline{j} \in N^q \mid j_1 < \cdots < j_q\}$. For $\underline{j} \in N^q$ and $\underline{f} = (f_1, \dots, f_\ell) \in \mathscr{O}_Y^\ell$ we abbreviate

$$dx_{\underline{j}} := dx_{j_1} \wedge \cdots \wedge dx_{j_q}, \quad \frac{\partial}{\partial x_{\underline{j}}} := \frac{\partial}{\partial x_{j_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{j_q}},$$
$$\underline{j}_{\hat{i}} := (j_1, \dots, \widehat{j_i}, \dots, j_q), \quad d\underline{f} = df_1 \wedge \cdots \wedge df_\ell.$$

The perfect pairing

$$\Theta^q_Y \times \Omega^q_Y \to \mathscr{O}_Y, \quad (\delta, \omega) \mapsto \langle \delta, \omega \rangle, \tag{3.1}$$

then satisfies

$$\left\langle \frac{\partial}{\partial x_{\underline{j}}}, dx_{\underline{k}} \right\rangle = \delta_{\underline{j},\underline{k}} := \delta_{j_1,k_1} \cdots \delta_{j_q,k_q}. \tag{3.2}$$

3.1. **Log forms along complete intersections.** Let $C \subseteq Y$ be a reduced complete intersection of codimension $k \geq 1$. Then $\mathscr{O}_C = \mathscr{O}_Y/\mathcal{I}_C$ where $\mathcal{I}_C = \mathcal{I}_{C/Y}$ is the ideal of $C \subseteq Y$. Let $\underline{h} = (h_1, \dots, h_k) \in \mathscr{O}_Y^k$ be any regular sequence such that $\mathcal{I}_C = \langle h_1, \dots, h_k \rangle$. Geometrically $C = D_1 \cap \dots \cap D_k$ where $D_i := \{h_i = 0\}$ for $i = 1, \dots, k$.

Notation 3.2. We denote $D := D_1 \cup \dots \cup D_k = \{h = 0\}$ where $h := h_1 \cdots h_k$,

$$-(D) := - \otimes_{\mathscr{O}_Y} \mathscr{O}_Y \frac{1}{h}, \qquad\qquad -(-D) := - \otimes_{\mathscr{O}_Y} \mathscr{O}_Y h,$$

$$\Sigma = \Sigma_{C/D/Y} := \mathcal{I}_C(D) = \sum_{i=1}^{k} \frac{h_i}{h} \mathscr{O}_Y \subseteq \mathscr{Q}_Y, \qquad -^{\Sigma} := \operatorname{Hom}_{\mathscr{O}_Y}(-, \Sigma).$$

Note that $\Sigma = \mathscr{O}_Y$ in case $k = 1$.

The following definition due to Aleksandrov (see [3, §3] and [21, Def. 3.1.4]) generalizes Saito's logarithmic differential forms (see [22]) from the hypersurface to the complete intersection case.

Definition 3.3. The module of *multi-logarithmic differential q-forms on Y along C* is defined by

$$\begin{aligned}\Omega^q(\log C) = \Omega^q_Y(\log C) &:= \left\{\omega \in \Omega^q_Y \;\middle|\; d\mathcal{I}_C \wedge \omega \subseteq \mathcal{I}_C \Omega^{q+1}_Y\right\}(D) \\ &= \left\{\omega \in \Omega^q_Y(D) \;\middle|\; \forall i = 1, \dots, k\colon dh_i \wedge \omega \in \Sigma\Omega^{q+1}_Y\right\}\end{aligned}$$

where the equality is due to the Leibniz rule. Observe that

$$\Sigma\Omega^q_Y \subseteq \Omega^q(\log C) \subseteq \mathscr{Q}_Y \otimes_{\mathscr{O}_Y} \Omega^q_Y$$

with $\Omega^q(\log C)(-D) \subseteq \mathscr{Q}_Y \otimes_{\mathscr{O}_Y} \Omega^q_Y$ independent of D (see [21, Prop. 3.1.10]).

Extending Saito's theory (see [22, §1-2]) Aleksandrov (see [3, §3-4,6]) gives an explicit description of multi-logarithmic differential forms and defines a multi-logarithmic residue map. We summarize his results.

Proposition 3.4. *An element $\omega \in \Omega^q_Y(D)$ lies in $\Omega^q(\log C)$ if and only if there exist $g \in \mathscr{O}_Y$ inducing a non zero-divisor in $\mathscr{O}_C$, $\xi \in \Omega^{q-k}_Y$ and $\eta \in \Sigma\Omega^q_Y$ such that*

$$g\omega = \frac{d\underline{h}}{h} \wedge \xi + \eta.$$

This representation defines a multi-logarithmic residue map

$$\operatorname{res}^q_C \colon \Omega^q(\log C) \to \mathscr{Q}_C \otimes_{\mathscr{O}_C} \Omega^{q-k}_C, \quad \omega \mapsto \frac{\xi}{g},$$

that fits into a short exact multi-logarithmic residue sequence

$$0 \longrightarrow \Sigma\Omega^q_Y \longrightarrow \Omega^q(\log C) \xrightarrow{\operatorname{res}^q_C} \omega^{q-k}_C \longrightarrow 0 \tag{3.3}$$

where ω^p_C is the module of regular meromorphic p-forms on C. □

Corollary 3.5. *For $q < k$, $\Omega^q(\log C) = \Sigma\Omega^q_Y$ and $\Omega^n(\log C) = \Omega^n_Y(D)$.* □

Remark 3.6. The multi-logarithmic residue map can be written in terms of residue symbols as $\operatorname{res}^q_C(\omega) = \begin{bmatrix} h\omega \\ \underline{h} \end{bmatrix}$ (see [27, §1.2][1]). In particular $\operatorname{res}^k_C(\frac{d\underline{h}}{h}) = \begin{bmatrix} d\underline{h} \\ \underline{h} \end{bmatrix} \in \omega^k_C$ is the fundamental form of C (see [13, §5]). □

[1]This remark was made in the first author's talk "Normal crossings in codimension one" at the 2012 Oberwolfach conference "Singularities" (see [26]).

Higher logarithmic derivation modules play a prominent role in arrangement theory (see for instance [1]). Here we extend the definitions of Granger and the first author (see [9, §5]) and by Pol (see [21, Def. 3.2.1]) as follows.

Definition 3.7. We define the module of *multi-logarithmic q-vector fields on Y along C* by

$$\begin{aligned}\mathrm{Der}^q(-\log C) = \mathrm{Der}^q_Y(-\log C) &:= \left\{\delta \in \Theta^q_Y \;\middle|\; \left\langle \delta, \wedge^k d\mathcal{I}_C \wedge \Omega^{q-k}_Y \right\rangle \subseteq \mathcal{I}_C \right\}\\ &= \left\{\delta \in \Theta^q_Y \;\middle|\; \left\langle \delta, d\underline{h} \wedge \Omega^{q-k}_Y \right\rangle \subseteq \mathcal{I}_C \right\}\end{aligned}$$

where the equality is due to the Leibniz rule. Observe that

$$\mathcal{I}_C \Theta^q_Y \subseteq \mathrm{Der}^q(-\log C).$$

Lemma 3.8. *We can identify the functors on $\mathscr{O}_Y$-modules (see Notation 2.1)*

$$-^\Sigma = -(-D)^{\mathcal{I}_C}, \quad (\Sigma \otimes_{\mathscr{O}_Y} -)^\Sigma = -^*,$$

and hence $-^{\Sigma\Sigma} = -^{\mathcal{I}_C\mathcal{I}_C}$.

Proof. Since $\mathscr{O}_Y(D)$ is invertible and by Hom-tensor adjunction

$$-^\Sigma = \mathrm{Hom}_{\mathscr{O}_Y}(-, \mathcal{I}_C(D)) = \mathrm{Hom}_{\mathscr{O}_Y}(-, \mathrm{Hom}_{\mathscr{O}_Y}(\mathscr{O}_Y(-D), \mathcal{I}_C)) = -(-D)^{\mathcal{I}_C}$$

By Lemma 2.3 in case $k \geq 2$, $\mathscr{O}_Y = \mathcal{I}_C^{\mathcal{I}_C} = \Sigma^\Sigma$ and again by Hom-tensor adjunction

$$(\Sigma \otimes_{\mathscr{O}_Y} -)^\Sigma = \mathrm{Hom}_{\mathscr{O}_Y}(\Sigma \otimes_{\mathscr{O}_Y} -, \Sigma) = \mathrm{Hom}_{\mathscr{O}_Y}(-, \Sigma^\Sigma) = -^*. \qquad \square$$

Lemma 3.9. *Any elements $\delta \in \mathrm{Der}^q(-\log C)$ and $\omega \in \Omega^q(\log C)$ pair to $\langle \delta, \omega \rangle \in \Sigma$.*

Proof. Let g, ξ and η be as in Proposition 3.4. Then by definition

$$g\langle \delta, h\omega\rangle = \langle \delta, hg\omega \rangle = \langle \delta, d\underline{h} \wedge \xi + h\eta \rangle = \langle \delta, d\underline{h} \wedge \xi \rangle + h\langle \delta, \eta \rangle \in \mathcal{I}_C.$$

Since g induces a non zero-divisor in $\mathscr{O}_C = \mathscr{O}_Y/\mathcal{I}_C$ this implies that $\langle \delta, h\omega \rangle \in \mathcal{I}_C$ and hence $\langle \delta, \omega \rangle \in \frac{1}{h}\mathcal{I}_C = \Sigma$. $\square$

The following proofs for $q \geq k \geq 1$ proceed along the lines of Saito's base case $q = k = 1$ (see [22, (1.6)]) or Pol's generalization to $q = k \geq 1$ (see [21, Prop. 3.2.13]).

Lemma 3.10. *If $\omega \in \Omega^q_Y(D)$ with $\langle \mathrm{Der}^q(-\log C), \omega \rangle \subseteq \Sigma$, then $\omega \in \Omega^q(\log C)$.*

Proof. For every $\ell \in \{1, \ldots, k\}$ and $\underline{j} \in N^{q+1}_<$ consider

$$\delta^\ell_{\underline{j}} := \sum_{i=1}^{q+1} (-1)^{i+1} \frac{\partial h_\ell}{\partial x_{j_i}} \frac{\partial}{\partial x_{\underline{j}_{\hat{i}}}} \in \Theta^q_Y.$$

For every $\underline{i} \in N^{q-k}$

$$d\underline{h} \wedge dx_{\underline{i}} = \sum_{\underline{k} \in N^q_<} \frac{\partial(\underline{h}, x_{\underline{i}})}{\partial x_{\underline{k}}} dx_{\underline{k}},$$

where $\frac{\partial(\underline{h}, x_{\underline{i}})}{\partial x_{\underline{k}}}$ is the $q \times q$-minor of the Jacobian matrix of $(\underline{h}, x_{\underline{i}})$ with column indices $\underline{k}$, and hence using (3.2)

$$\begin{aligned}\left\langle \delta^\ell_{\underline{j}}, d\underline{h} \wedge dx_{\underline{i}} \right\rangle &= \sum_{i=1}^{q+1} (-1)^{i+1} \frac{\partial h_\ell}{\partial x_{j_i}} \sum_{\underline{k} \in N^q_<} \frac{\partial(\underline{h}, x_{\underline{i}})}{\partial x_{\underline{k}}} \left\langle \frac{\partial}{\partial x_{\underline{j}_{\hat{i}}}}, dx_{\underline{k}} \right\rangle\\ &= \sum_{i=1}^{q+1} (-1)^{i+1} \frac{\partial h_\ell}{\partial x_{j_i}} \frac{\partial(\underline{h}, x_{\underline{i}})}{\partial x_{\underline{j}_{\hat{i}}}} = \frac{\partial(h_\ell, \underline{h}, x_{\underline{i}})}{\partial x_{\underline{j}}} = 0.\end{aligned}$$

It follows that $\delta^\ell_{\underline{j}} \in \mathrm{Der}^q(-\log C)$ for all $\ell = 1, \ldots, k$ and $\underline{j} \in N^{q+1}_<$.

Now let $\omega = \sum_{\underline{k}\in N^q_<} \frac{a_{\underline{k}}}{h} dx_{\underline{k}} \in \Omega^q_Y(D)$ where $a_{\underline{k}} \in \mathscr{O}_Y$. For all $\ell = 1, \ldots, k$ and $\underline{j} \in N^{q+1}_<$

$$\left\langle \delta^\ell_{\underline{j}}, \omega \right\rangle = \sum_{i=1}^{q+1} (-1)^{i+1} \frac{\partial h_\ell}{\partial x_{j_i}} \sum_{\underline{k}\in N^q_<} \frac{a_{\underline{k}}}{h} \left\langle \frac{\partial}{\partial x_{\underline{j}_{\hat{i}}}}, dx_{\underline{k}} \right\rangle = \sum_{i=1}^{q+1} (-1)^{i+1} \frac{\partial h_\ell}{\partial x_{j_i}} \frac{a_{\underline{j}_{\hat{i}}}}{h}$$

by (3.2) and hence

$$\begin{aligned} dh_\ell \wedge \omega &= \sum_{j=1}^{n} \frac{\partial h_\ell}{\partial x_j} dx_j \wedge \sum_{\underline{k}\in N^q_<} \frac{a_{\underline{k}}}{h} dx_{\underline{k}} = \sum_{\underline{j}\in N^{q+1}_<} \sum_{i=1}^{q+1} \frac{\partial h_\ell}{\partial x_{j_i}} \frac{a_{\underline{j}_{\hat{i}}}}{h} dx_{j_i} \wedge dx_{\underline{j}_{\hat{i}}} \\ &= \sum_{\underline{j}\in N^{q+1}_<} \sum_{i=1}^{q+1} (-1)^{i+1} \frac{\partial h_\ell}{\partial x_{j_i}} \frac{a_{\underline{j}_{\hat{i}}}}{h} dx_{\underline{j}} = \sum_{\underline{j}\in N^{q+1}_<} \left\langle \delta^\ell_{\underline{j}}, \omega \right\rangle dx_{\underline{j}}. \end{aligned}$$

If $\langle \mathrm{Der}^q(-\log C), \omega\rangle \subseteq \Sigma$, then $dh_\ell \wedge \omega \in \Sigma\Omega^q_Y$ for all $\ell = 1, \ldots, k$ and hence $\omega \in \Omega^q(\log C)$. $\square$

Proposition 3.11. *There are chains of $\mathscr{O}_Y$-submodules of $\mathscr{Q}_Y \otimes_{\mathscr{O}_Y} \Omega^q_Y$ and $\mathscr{Q}_Y \otimes_{\mathscr{O}_Y} \Theta^q_Y$*

$$\Omega^q_Y \subseteq \Sigma\Omega^q_Y \subseteq \Omega^q(\log C) \subseteq \Omega^q_Y(D) \subseteq \Sigma\Omega^q_Y(D), \tag{3.4}$$

$$\Sigma\Theta^q_Y \supseteq \Theta^q_Y \supseteq \mathrm{Der}^q(-\log C) \supseteq \mathcal{I}_C\Theta^q_Y \supseteq \Theta^q_Y(-D) \tag{3.5}$$

that are Σ-duals of each other.

Proof. Tensoring with $\mathscr{Q}_Y$ makes both chains collapse. The cokernels of all inclusions are therefore torsion whereas Σ is torsion free. Applying $-^\Sigma$ thus results in a chain of $\mathscr{O}_Y$-modules again. In case of (3.4) this yields

$$(\Omega^q_Y)^\Sigma \supseteq (\Sigma\Omega^q_Y)^\Sigma \supseteq \Omega^q_Y(\log C)^\Sigma \supseteq \Omega^q_Y(D)^\Sigma \supseteq (\Sigma\Omega^q_Y(D))^\Sigma$$

and, with Lemma 3.8 and freeness of Ω^q_Y and Θ^q_Y, the chain of $\mathscr{O}_Y$-submodules of $\mathscr{Q}_Y \otimes_{\mathscr{O}_Y} \Theta^q_Y$

$$\Sigma\Theta^q_Y \supseteq \Theta^q_Y \supseteq \Omega^q_Y(\log C)^\Sigma \supseteq \mathcal{I}_C\Theta^q_Y \supseteq \Theta^q_Y(-D).$$

For every $\delta \in \Omega^q(\log C)^\Sigma$ and $\xi \in \Omega^{q-k}$, $\frac{d\underline{h}}{h} \wedge \xi \in \Omega^q(\log C)$ by Proposition 3.4, hence

$$\langle \delta, d\underline{h} \wedge \xi\rangle = h\left\langle \delta, \frac{d\underline{h}}{h} \wedge \xi \right\rangle \in h\Sigma = \mathcal{I}_C$$

and $\delta \in \mathrm{Der}^q(-\log C)$. With Lemma 3.9, it follows that $\Omega^q_Y(\log C)^\Sigma = \mathrm{Der}^q(-\log C)$.

By the same reasoning $-^\Sigma$ applied to (3.5) yields a chain of $\mathscr{O}_Y$-modules

$$(\Sigma\Theta^q_Y)^\Sigma \subseteq (\Theta^q_Y)^\Sigma \subseteq \mathrm{Der}^q(-\log C)^\Sigma \subseteq (\Sigma\Theta^q_Y)(-D)^\Sigma \subseteq \Theta^q_Y(-D)^\Sigma$$

that can be rewritten as the chain of $\mathscr{O}_Y$-submodules of $\mathscr{Q}_Y \otimes_{\mathscr{O}_Y} \Omega^q_Y$

$$\Omega^q_Y \subseteq \Sigma\Omega^q_Y \subseteq \mathrm{Der}^q(-\log C)^\Sigma \subseteq \Omega^q_Y(D) \subseteq \Sigma\Omega^q_Y(D).$$

The missing equality $\mathrm{Der}^q(-\log C)^\Sigma = \Omega^q(\log C)$ follows from Lemmas 3.9 and 3.10. $\square$

3.2. **Log forms along Cohen–Macaulay spaces.** Let $X \subseteq Y$ be a reduced Cohen-Macaulay germ of codimension $k \geq 2$. Then $\mathscr{O}_X = \mathscr{O}_Y/\mathcal{I}_X$ where $\mathcal{I}_X := \mathcal{I}_{X/Y}$ denotes the ideal $X \subseteq Y$. There is a reduced complete intersection $C \subseteq Y$ of codimension k such that $X \subseteq C$ and hence $\mathcal{I}_X \supseteq \mathcal{I}_C$ (see [21, Prop. 4.2.1]). Set $X' := \overline{C \setminus X}$ such that $C = X \cup X'$. The link with §2.5 is made by setting

$$S := \mathscr{O}_C, \quad T := \mathscr{O}_X.$$

By Lemma 2.26 condition (2.29) holds and

$$\mathscr{Q}_C = \prod_{\mathfrak{p} \in \operatorname{Ass}_{\mathscr{O}_C}(\mathscr{O}_X)} \mathscr{O}_{X,\mathfrak{p}} \times \prod_{\mathfrak{p} \in \operatorname{Ass}_{\mathscr{O}_C}(\mathscr{O}_{X'})} \mathscr{O}_{X',\mathfrak{p}} = \mathscr{Q}_X \times \mathscr{Q}_{X'}. \tag{3.6}$$

This decomposition extends to differential forms as follows.

Lemma 3.12. *We have $\mathscr{Q}_X d\mathcal{I}_C = \mathscr{Q}_X d\mathcal{I}_X \subseteq \mathscr{Q}_X \otimes_{\mathscr{O}_Y} \Omega^1_Y$ and hence*

$$\mathscr{Q}_C \otimes_{\mathscr{O}_C} \Omega^p_C = \mathscr{Q}_X \otimes_{\mathscr{O}_X} \Omega^p_X \oplus \mathscr{Q}_{X'} \otimes_{\mathscr{O}_{X'}} \Omega^p_{X'}.$$

Proof. By (3.6) we may localize at $\mathfrak{p} \in \operatorname{Ass}_{\mathscr{O}_C}(\mathscr{O}_X)$. We may further assume $p = 1$ since exterior product commutes with extension of scalars. Let $\mathfrak{p} \mapsto \mathfrak{q}$ under $\operatorname{Spec}(\mathscr{O}_C) \to \operatorname{Spec}(\mathscr{O}_Y)$. Then $\mathcal{I}_{C,\mathfrak{q}} = \mathcal{I}_{X,\mathfrak{q}}$ by (3.6) and hence $u\mathcal{I}_X \subseteq \mathcal{I}_C$ for some $u \in \mathscr{O}_Y \setminus \mathfrak{q}$. By the Leibniz rule $ud\mathcal{I}_X \subseteq d\mathcal{I}_C + \mathcal{I}_X du$ and hence the first claim. Since $\Omega^1_C = \Omega^1_Y/(\mathscr{O}_Y d\mathcal{I}_C + \mathcal{I}_C\Omega^1_Y)$ this yields $\Omega^1_{C,\mathfrak{p}} = \Omega^1_{X,\mathfrak{p}}$ and the second claim follows. □

The following fact is well-known (see [27, (2.14)]); we only sketch a proof.

Lemma 3.13. *The modules of regular differential p-forms on X and C are related by*

$$\omega^p_X = \operatorname{Hom}_{\mathscr{O}_C}(\mathscr{O}_X, \omega^p_C) \subseteq \omega^p_C.$$

Proof. Kersken explicitly describes (see [14, (1.2)])

$$\omega^p_X = \left\{ \begin{bmatrix} \xi \\ \underline{h} \end{bmatrix} \;\middle|\; \xi \in \Omega^{p+k}_Y,\ \mathcal{I}_X\xi \subseteq \mathcal{I}_C\Omega^{p+k}_Y,\ d\mathcal{I}_X \wedge \xi \subseteq \mathcal{I}_C\Omega^{p+k+1}_Y \right\} \tag{3.7}$$

where $\begin{bmatrix} \xi \\ \underline{h} \end{bmatrix} = 0$ if and only if $\xi \in \mathcal{I}_C\Omega^{p+k}_Y$. In particular, $\omega^p_X \subseteq \operatorname{Hom}_{\mathscr{O}_C}(\mathscr{O}_X, \omega^p_C) \subseteq \omega^p_C$ and equality in ω^p_C can be checked at $\operatorname{Ass}(\mathscr{O}_C)$. Lemma 3.12 yields the claim. □

The following modules of differential forms on Y due to Aleksandrov (see [4, Def. 10.1] and [21, Def. 4.1.3]) are defined by the relations in (3.7).

Definition 3.14. The module of *multi-logarithmic differential q-forms on Y along X relative to C* is defined by

$$\begin{aligned}\Omega^q(\log X/C) = \Omega^q_Y(\log X/C) &:= \left\{\omega \in \Omega^q_Y \;\middle|\; \mathcal{I}_X\omega \subseteq \mathcal{I}_C\Omega^q_Y,\ d\mathcal{I}_X \wedge \omega \subseteq \mathcal{I}_C\Omega^{q+1}_Y\right\}(D)\\ &= \left\{\omega \in \Omega^q_Y(D) \;\middle|\; \mathcal{I}_X\omega \subseteq \Sigma\Omega^q_Y,\ d\mathcal{I}_X \wedge \omega \subseteq \Sigma\Omega^{q+1}_Y\right\}.\end{aligned}$$

Observe that

$$\Sigma\Omega^q_Y \subseteq \Omega^q(\log X/C) \subseteq \Omega^q(\log C)$$

with $\Omega^q(\log X/C)(-D) \subseteq \mathscr{Q}_Y \otimes_{\mathscr{O}_Y} \Omega^q_Y$ independent of D (see [21, Prop. 4.1.5]).

Lemma 3.15. *There is an equality $\Omega^q(\log X/C) = \Sigma\Omega^q_Y :_{\Omega^q(\log C)} \mathcal{I}_X$. In other words,*

$$\Omega^q(\log X/C)(-D) = \mathcal{I}_X\Omega^q_Y :_{\Omega^q(\log C)} \mathcal{I}_X.$$

Proof. There are obvious inclusions

$$\Sigma\Omega^q_Y \subseteq \Omega^q(\log X/C) \subseteq \Sigma\Omega^q_Y :_{\Omega^q(\log C)} \mathcal{I}_X \subseteq \Omega^q(\log C).$$

By Proposition 3.4 and Lemma 3.12

$$\begin{aligned} \omega \in \Sigma\Omega^q_Y :_{\Omega^q(\log C)} \mathcal{I}_X &\implies \mathcal{I}_X \operatorname{res}^q_C(\omega) \subseteq \operatorname{res}^q_C(\Sigma\Omega^q_Y) = 0 \\ &\implies \operatorname{res}^q_C(\omega) \in \mathscr{Q}_X \otimes_{\mathscr{O}_X} \Omega^{q-k}_X \\ &\implies 0 = d\mathcal{I}_X \wedge \operatorname{res}^q_C(\omega) = \operatorname{res}^{q+1}_C(d\mathcal{I}_X \wedge \omega) \\ &\implies d\mathcal{I}_X \wedge \omega \subseteq \Sigma\Omega^{q+1}_Y \\ &\implies \omega \in \Omega^q(\log X/C). \end{aligned}$$

□

The idea of Remark 3.6 is used by Aleksandrov (see [4, §10]) to define multi-logarithmic residues along X as the restriction of those along C. The bottom sequence of the diagram in the following Proposition 3.16 appears in his work (see [4, Thm. 10.2]); Pol proved exactness on the right (see [21, Prop. 4.1.21]). An alternative argument is suggested by §2.5. The following data

$$R := \mathscr{O}_Y, \quad I := \mathcal{I}_C, \quad J := \mathcal{I}_X, \quad F := \Omega^q_Y, \quad M := \Omega^q(\log C)(-D), \quad \rho := \frac{1}{h}\operatorname{res}^q_C \tag{3.8}$$

give rise to an I-free approximation (2.4) with J-restriction (2.23). By Corollary 3.5 $W = 0$ if $q < k$ and (2.4) is trivial for $q = n$. We are therefore concerned with the case $k \le q < n$. By Lemmas 3.13 and 3.15 (see Definition 2.24 and (2.25))

$$W_T = \omega^{q-k}_X, \quad M_J = \Omega^q(\log X/C)(-D). \tag{3.9}$$

Now twisting diagram (2.24) by D yields the following result.

Proposition 3.16. *Applying* $\operatorname{Ext}^1_{\mathscr{O}_Y}(\omega^{q-k}_X \hookrightarrow \omega^{q-k}_C, \Sigma\Omega^q_Y)$ *to the multi-logarithmic residue sequence* (3.3) *yields a commutative diagram with exact rows and cartesian right square*

$$\begin{tikzcd} 0 \arrow[r] & \Sigma\Omega^q_Y \arrow[r] \arrow[d, equal] & \Omega^q(\log C) \arrow[r, "\operatorname{res}^q_C"] & \omega^{q-k}_C \arrow[r] & 0 \\ 0 \arrow[r] & \Sigma\Omega^q_Y \arrow[r] & \Omega^q(\log X/C) \arrow[r, "\operatorname{res}^q_{X/C}"] \arrow[u, hook] & \omega^{q-k}_X \arrow[r] \arrow[u, hook] & 0 \end{tikzcd} \tag{3.10}$$

where ω^p_X *is the module of regular meromorphic* p*-forms on* X*.* □

3.3. **Higher log vector fields and Jacobian modules.** Pol gives a description of $\operatorname{res}^q_{X/C}$ preserving the analogy with the definition of res^q_C in Proposition 3.4 (see [21, §4.2.1]). As suggested by Remark 3.6 the role of $\frac{dh}{h} \in \Omega^k(\log C)$ is played by a preimage $\frac{\alpha_X}{h} \in \Omega^k(\log X/C)$ of the fundamental form $\begin{bmatrix} \alpha_X \\ \underline{h} \end{bmatrix} \in \omega^0_X$ of X (see [13, §5]).

Definition 3.17. Let $\mathbf{1}_X := (1, 0) \in \mathscr{Q}_X \times \mathscr{Q}_{X'} = \mathscr{Q}_C$ (see Lemma 3.12). A *fundamental form of* X *in* Y is an $\alpha_X = \alpha_{X/C/Y} \in \Omega^k_Y$ such that $\overline{\alpha_X} = \overline{\mathbf{1}_X d\underline{h}} \in \mathscr{Q}_C \otimes_{\mathscr{O}_Y} \Omega^k_Y$.

Such a fundamental form exists and the explicit description of multi-logarithmic differential forms in Proposition 3.4 generalizes verbatim (see [21, Prop. 4.2.6]).

Proposition 3.18. *An element* $\omega \in \Omega^q_Y(D)$ *lies in* $\Omega^q(\log X/C)$ *if and only if there exist* $g \in \mathscr{O}_Y$ *inducing a non zero-divisor in* $\mathscr{O}_C$, $\xi \in \Omega^{q-k}_Y$ *and* $\eta \in \Sigma\Omega^q_Y$ *such that*

$$g\omega = \frac{\alpha_X}{h} \wedge \xi + \eta$$

and the map $\operatorname{res}^q_{X/C}$ *in* (3.10) *is defined by* $\operatorname{res}^q_{X/C}(\omega) = \frac{\xi}{g}$. ☐

In the same spirit we extend Definition 3.7. We start with the first option as definition.

Definition 3.19. We define the module of *multi-logarithmic q-vector fields on Y along X* by

$$\operatorname{Der}^q(-\log X) = \operatorname{Der}^q_Y(-\log X) := \left\{\delta \in \Theta^q_Y \;\middle|\; \left\langle \delta, \wedge^k d\mathcal{I}_X \wedge \Omega^{q-k}_Y \right\rangle \subseteq \mathcal{I}_X \right\}.$$

The following result completes the analogy with Definition 3.7. In particular $\operatorname{Der}^k(-\log X)$ is Pol's module $\operatorname{Der}^k(-\log X/C)$ (see [21, Def. 4.2.8]) which is thus independent of C.

Lemma 3.20. *We have*

$$\begin{aligned}\operatorname{Der}^q(-\log C) \subseteq \left\{\delta \in \Theta^q_Y \;\middle|\; \left\langle \delta, \alpha_X \wedge \Omega^{q-k}_Y \right\rangle \subseteq \mathcal{I}_X \right\} &= \operatorname{Der}^q(-\log X)\\ = \left\{\delta \in \Theta^q_Y \;\middle|\; \left\langle \delta, \alpha_X \wedge \Omega^{q-k}_Y \right\rangle \subseteq \mathcal{I}_C \right\}&.\end{aligned}$$

Proof. By Definition 3.17 $\overline{\alpha_X} = \overline{\mathbf{1}_X d\underline{h}} = \overline{d\underline{h}} \in \mathscr{Q}_X \otimes_{\mathscr{O}_Y} \Omega^k_Y$. For $\delta \in \Theta^q_Y$ and $\xi \in \Omega^{q-k}_Y$

$$\begin{aligned}\langle \delta, \alpha_X \wedge \xi\rangle \in \mathcal{I}_X \iff 0 = \overline{\langle \delta, \alpha_X \wedge \xi\rangle} &= \left\langle \overline{\delta}, \overline{\alpha_X} \wedge \overline{\xi}\right\rangle\\ &= \left\langle \overline{\delta}, \overline{d\underline{h}} \wedge \overline{\xi}\right\rangle = \overline{\langle \delta, d\underline{h} \wedge \xi\rangle} \in \mathscr{Q}_X\end{aligned}$$

where $\overline{\delta} \in \mathscr{Q}_X \otimes_{\mathscr{O}_Y} \Theta^q_Y$ and $\overline{\xi} \in \mathscr{Q}_X \otimes_{\mathscr{O}_Y} \Omega^{q-k}_Y$. The claimed inclusion follows. Using the Leibniz rule and that $\mathscr{Q}_X d\mathcal{I}_C = \mathscr{Q}_X d\mathcal{I}_X \subseteq \mathscr{Q}_X \otimes_{\mathscr{O}_Y} \Omega^1_Y$ by Lemma 3.12

$$\begin{aligned}0 = \left\langle \overline{\delta}, \overline{d\underline{h}} \wedge \overline{\xi}\right\rangle \in \mathscr{Q}_X \iff 0 = \left\langle \overline{\delta}, \wedge^k \overline{d\mathcal{I}_C} \wedge \overline{\xi}\right\rangle &= \left\langle \overline{\delta}, \wedge^k \overline{d\mathcal{I}_X} \wedge \overline{\xi}\right\rangle\\ &= \overline{\langle \delta, \wedge^k d\mathcal{I}_X \wedge \xi\rangle} \subseteq \mathscr{Q}_X \iff \left\langle \delta, \wedge^k d\mathcal{I}_X \wedge \xi\right\rangle \subseteq \mathcal{I}_X.\end{aligned}$$

This proves the first equality. With $\mathcal{I}_C = \mathcal{I}_X \cap \mathcal{I}_{X'}$ the second equality follows from $\alpha_X \in \mathcal{I}_{X'}\Omega^k_Y$ (see [21, Prop. 4.2.5]). ☐

Using Proposition 3.18 and Lemma 3.20 we obtain the following analogue of Lemma 3.9 and of the equality $\operatorname{Der}^q(-\log C) = \Omega^q(\log C)^\Sigma$ from Proposition 3.11.

Lemma 3.21. *For* $\delta \in \operatorname{Der}^q(-\log X)$ *and* $\omega \in \Omega^q(\log X/C)$ *we have* $\langle \delta, \omega\rangle \in \Sigma$. ☐

Lemma 3.22. *There is an equality* $\operatorname{Der}^q(-\log X) = \Omega^q(\log X/C)^\Sigma$. ☐

The following proposition extends Proposition 3.11 and includes the counterpart of Lemma 3.10.

Proposition 3.23. *There are chains of* $\mathscr{O}_Y$*-submodules of* $\mathscr{Q}_Y \otimes_{\mathscr{O}_Y} \Omega^q_Y$ *and* $\mathscr{Q}_Y \otimes_{\mathscr{O}_Y} \Theta^q_Y$

$$\begin{gathered}\Omega^q_Y \subseteq \Sigma\Omega^q_Y \subseteq \Omega^q(\log X/C) \subseteq \Omega^q(\log C) \subseteq \Omega^q_Y(D) \subseteq \Sigma\Omega^q_Y(D),\\ \Sigma\Theta^q_Y \supseteq \Theta^q_Y \supseteq \operatorname{Der}^q(-\log X) \supseteq \operatorname{Der}^q(-\log C) \supseteq \mathcal{I}_C\Theta^q_Y \supseteq \Theta^q_Y(-D)\end{gathered}$$

that are Σ*-duals of each other.*

Proof. By Lemma 3.8 and Proposition 3.11 M in (3.8) is I-reflexive. By Proposition 2.28 and (3.9) $\Omega^q(\log X/C)(-D)$ is therefore $\mathcal{I}_C$-reflexive and, again by Lemma 3.8, $\Omega^q(\log X/C)$ Σ-reflexive. The claim follows with Proposition 3.11 and Lemmas 3.20 and 3.22. ☐

Definition 3.24. Contraction with α_X defines a map

$$\alpha^X\colon \Theta^q_Y \to \mathscr{O}_X \otimes_{\mathscr{O}_Y} \Theta^{q-k}_Y = \operatorname{Hom}_{\mathscr{O}_Y}(\Omega^{q-k}_Y, \mathscr{O}_X), \quad \delta \mapsto (\omega \mapsto \overline{\langle \delta, \alpha_X \wedge \omega\rangle}).$$

Taking $p + q = n$ we define the *pth Jacobian module of* X as the $\mathscr{O}_X$-module

$$\mathcal{J}^p_X := \alpha^X(\Theta^q_Y).$$

The Jacobian module $\mathscr{J}_X^{\dim X}$ agrees with Pol's Jacobian ideal $\mathscr{J}_{X/C}$ (see [21, Not. 4.2.14]) which coincides with the ω-Jacobian ideal if X is Gorenstein (see [21, Prop. 4.2.34]).

Remark 3.25. In explicit terms

$$\alpha^X\colon \Theta_Y^q \to \bigoplus_{\underline{i}\in N_<^{q-k}} \mathscr{O}_X dx_{\underline{i}}, \quad \delta \mapsto \sum_{\underline{i}\in N_<^{q-k}} \langle \delta, \alpha_X \wedge dx_{\underline{i}}\rangle dx_{\underline{i}}.$$

In case $X = C$, $\alpha_C = d\underline{h}$ and

$$\left\langle \delta, d\underline{h} \wedge dx_{\underline{i}} \right\rangle = \sum_{\underline{j}\in N_<^q} \frac{\partial(\underline{h}, x_{\underline{i}})}{\partial x_{\underline{j}}} \left\langle \delta, dx_{\underline{j}} \right\rangle.$$

In particular, $\mathscr{J}_C^{\dim C}$ is the Jacobian ideal of C.

Lemma 3.26. *If $k \le q \le n$, then $\omega_X^{q-k} \neq 0$ and, unless $q = n$, $\mathscr{O}_X \otimes \alpha^X$ is not injective.*

Proof. This can be checked at smooth points of $X = C$ where $\underline{h} = (x_1, \dots, x_k)$ and $\alpha_X = d\underline{h}$. Here $\omega_X^{q-k} = \Omega_X^{q-k} \neq 0$ and $0 \neq \frac{\partial}{\partial x_{\underline{j}}} \in \ker(\mathscr{O}_X \otimes \alpha^X)$ if $\{1, \dots, k\} \not\subseteq \{j_1, \dots, j_q\}$. $\square$

By Lemma 3.20 there is a short exact sequence (see [21, Prop. 4.2.16] for $q = k$)

$$0 \longleftarrow \mathscr{J}_X^{n-q} \xleftarrow{\alpha^X} \Theta_Y^q \longleftarrow \operatorname{Der}_Y^q(-\log X) \longleftarrow 0. \tag{3.11}$$

Lemma 3.27. *There is a pairing*

$$\mathscr{J}_X^{n-q} \otimes \omega_X^{q-k} \to \operatorname{Hom}_{\mathscr{O}_C}(\mathscr{O}_X, \mathscr{O}_C)(D) = \omega_X, \quad \left(\alpha^X(\delta), \operatorname{res}_{X/C}^q(\omega)\right) \mapsto \langle \delta, \omega\rangle.$$

Proof. By Lemma 3.21 the pairing $\Omega_Y^q(D) \times \Theta_Y^q \to \mathscr{O}_Y(D)$ obtained from (3.1) maps both $\Omega_Y^q(\log X/C) \times \operatorname{Der}_Y^q(-\log X)$ and $\Sigma\Omega_Y^q \otimes \Theta_Y^q$ to Σ. Using the bottom row of (3.10) and (3.11) this yields a pairing $\mathscr{J}_X^{n-q} \otimes \omega_X^{q-k} \to \mathscr{O}_Y(D)/\Sigma = \mathscr{O}_C(D) = \omega_C$. Both $\mathscr{J}_X^{n-q}$ and ω_X^{q-k} are supported on X and applying $\operatorname{Hom}_{\mathscr{O}_C}(\mathscr{O}_X, -)$ yields the claim (see (2.34)). $\square$

We can now prove our main application.

Proof of the Theorem 1.3. By Lemmas 3.8 and 3.22 sequence (3.11) in terms of (3.8) is the I-dual J restriction (2.27) twisted by D, that is, $V^T = \mathscr{J}_X^{n-q}$ and $\alpha^T = \alpha^X$ up to a twist by D. With (3.9) and Lemma 3.26 the claim now reduces to Corollary 2.29. The identifications are induced by the pairing in Lemma 3.27. $\square$

Proposition 3.28. *The $\mathscr{O}_X$-modules $\mathscr{J}_X^{n-q}$ depend only on X.*

Proof. We identify $\mathscr{J}_X^{n-q} = \Theta_Y^q / \operatorname{Der}_Y^q(-\log X)$ by the exact sequence (3.11). Any isomorphism $Y' \cong Y$ of minimal embeddings of X induces an isomorphism $\varphi\colon \mathscr{O}_Y \cong \mathscr{O}_{Y'}$ over $\mathscr{O}_X$ identifying $\mathcal{I}_{X/Y} \cong \mathcal{I}_{X/Y'}$. There are induced compatible isomorphisms $\Theta_Y^q \cong \Theta_{Y'}^q$ and $\Omega_Y^p \cong \Omega_{Y'}^p$ over φ resulting in an isomorphism over φ

$$\operatorname{Der}_Y^q(-\log X) \cong \operatorname{Der}_{Y'}^q(-\log X).$$

Any general embedding $X \subseteq Y'$ arises from a minimal embedding $X \subseteq Y$ up to isomorphism of the latter as $Y' = Y \times Z$ where $Z \cong (\mathbb{C}^m, 0)$ and hence

$$\mathcal{I}_{X/Y'} = \mathscr{O}_Y \hat{\otimes} \mathfrak{m}_Z + \mathcal{I}_{X/Y} \hat{\otimes} \mathscr{O}_Z.$$

Pick coordinates $z_1, \dots, z_m$ on Z and abbreviate $d\underline{z} := dz_1 \wedge \cdots \wedge dz_m$ and $\frac{\partial}{\partial \underline{z}} := \frac{\partial}{\partial z_1} \wedge \cdots \wedge \frac{\partial}{\partial z_m}$. Then there are decompositions

$$\Omega^{q+m}_{Y'} = \mathscr{O}_Z \hat{\otimes} \Omega^q_Y \wedge d\underline{z} \oplus \widetilde{\Omega}^{q+m}_{Y'}, \quad \Theta^{q+m}_{Y'} = \mathscr{O}_Z \hat{\otimes} \Theta^q_Y \wedge \frac{\partial}{\partial \underline{z}} \oplus \widetilde{\Theta}^{q+m}_{Y'}$$

where the modules with tilde are generated by basis elements not involving $d\underline{z}$ and $\frac{\partial}{\partial \underline{z}}$ respectively. Fundamental forms of X in Y' and Y can be chosen compatibly as

$$\alpha_{X/C/Y'} = \alpha_{X/C/Y} \wedge d\underline{z} \in \Omega^{k+m}_{Y'}.$$

With Lemma 3.20 this yields inclusions

$$\operatorname{Der}^q_Y(-\log X) \wedge \frac{\partial}{\partial \underline{z}} + \widetilde{\Theta}^{q+m}_{Y'} \subseteq \operatorname{Der}^{q+m}_{Y'}(-\log X) \supseteq \mathcal{I}_{X/Y'} \Theta^{q+m}_{Y'} \supseteq \mathfrak{m}_Z \hat{\otimes} \Theta^q_Y \wedge \frac{\partial}{\partial \underline{z}}$$

and a cartesian square

$$\begin{tikzcd} \mathscr{O}_Z \hat{\otimes} \Theta^q_Y \arrow[r, hook, "- \wedge \frac{\partial}{\partial \underline{z}}"] & \Theta^{q+m}_{Y'} \\ \operatorname{Der}^q_Y(-\log X) + \mathfrak{m}_Z \hat{\otimes} \Theta^q_Y \arrow[u, hook] \arrow[r, hook] & \operatorname{Der}^{q+m}_{Y'}(-\log X). \arrow[u, hook] \end{tikzcd}$$

It gives rise to an isomorphism of $\mathscr{O}_X$-modules

$$\Theta^{q+m}_{Y'} / \operatorname{Der}^{q+m}_{Y'}(-\log X) \;\cong\; \mathscr{O}_Z \hat{\otimes} \Theta^q_Y / (\operatorname{Der}^q_Y(-\log X) + \mathfrak{m}_Z \hat{\otimes} \Theta^q_Y \;\cong\; \Theta^q_Y / \operatorname{Der}^q_Y(-\log X). \quad \square$$

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A ZOO OF GEOMETRIC HOMOLOGY THEORIES

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1. Introduction

A homology theory is on the one hand given by a spectrum - and from this point of view homology theories are almost as general as spaces. Originally they occurred in a completely different form by geometric constructions like simplicial or singular homology theories or later bordism theories, K-theory (a cohomology theory) and others. In this note we introduce a zoo of homology theories which both generalize singular homology and bordism theory in a natural way. More precisely for each subset A of the natural numbers $\mathbb{N}$ we construct a homology theory h_*^A which for $A = \mathbb{N} - \{1\}$ is ordinary mod 2 singular homology and for $A = \{0\}$ is singular bordism.

The theories in our zoo are all bordism groups, which generalize the case of smooth manifolds by allowing singularities. There are many concepts of manifolds with singularities one could use here. For our pupose the objects the author introduced some years ago, which are called **stratifolds**, work particularly well [4]. The theory of stratifolds was further elaborated in [3] in the thesis of the author's PhD student Anna Grinberg. The zoo comes from forcing certain strata indexed by the subset A to be empty.

Despite their simple construction computations of these groups seem to be very complicated. We give a few simple examples. However there are no interesting applications so far and the zoo looks a bit like a curiosity. But one never knows for what these theories might be good in the future. We mention a concrete question which might be useful in connection with the Griffiths group consisting of algebraic cycles in a smooth algebraic variety over the complex numbers which vanish in singular homology.

I dedicate these notes to my friend Egbert Brieskorn. Egbert is (in a very different way like our common teacher Hirzebruch) a person who had a great influence on me. When I had to make a complicated decision I often had him in front of my eyes and asked myself: What would Egbert suggest? Conversations with him were always intense and fruitful. I miss him very much.

When I thought about a subject for this note I also asked myself, what would Egbert say about this or that mathematics. I have no idea what he would say about this zoo. But I hope he would at least like the occurrence of manifolds with singularities. And it would probably find his interest that if Y is a compact complex singular variety in a non-singular complex algebraic variety X it admits a natural structure of a stratifold with all odd-dimensional strata empty and so represents a homology class in the special case where A consists of all odd numbers.

I would like to thank Peter Lendweber for careful reading of a first version of these notes leading to several clarifications and improvements.

2. Generalized homology theories and singular bordism

To motivate the construction let me recall the definition of singular bordism groups. Let X be a topological space. Then a cycle is a pair $f : M \to X$, where M is a closed smooth n-dimensional manifold and f a continuous map. Two cycles (M, f) and (M', f') represent the same bordism class if and only if there is a compact manifold W with $\partial W = M + M'$, and an extension $F : W \to X$ of the maps f and f'. This is an equivalence relation and the equivalence classes form a group under disjoint union denoted by $\mathcal{N}_n(X)$, the n-th singular bordism group. If $g : X \to Y$ is a continuous map it induces a homomorphism

$$g_* : \mathcal{N}_n(X) \to \mathcal{N}_n(Y)$$

given by post-composition and this way we obtain for each n a functor from the category of topological spaces to the category of abelian groups. By construction (using the cylinder as a bordism) this is a **homotopy functor**, meaning that if g and g' are homotopic, then $g_* = g'_*$.

This functor is a homology theory, which normally is expressed as an extension to the category of topological pairs fulfilling the Eilenberg-Steenrod axioms apart from the dimension axiom. But an equivalent simple characterization is the following. As in the case of relative homology groups one has to add data to a functor h_*, namely a boundary operator, which in our case is the boundary operator for a Mayer-Vietoris sequence: for open subsets U and V of a topological space a natural operator

$$d : h_k(U \cup V) \to h_{k-1}(U \cap V).$$

Then a **homology theory** is a homotopy functor h_* together with a natural boundary operator as above, such that the Mayer-Vietoris sequence

$$... \to h_{k+1}(U \cup V) \to h_k(U \cap V) \to h_k(U) \oplus h_k(V) \to h_k(U \cup V) \to ....$$

is exact. Here the maps are given by the boundary operator, the induced maps of the inclusions and the difference of the induced maps of the inclusions.

Examples of homology theories are singular homology and the npn-oriented bordism groups $\mathcal{N}_*(X)$. In the latter case the boundary operator is given a follows. If $f : M \to U \cup V$ is a continuous map, then consider A, the complement of $f^{-1}(U)$ in M, and B, the complement of $f^{-1}(V)$ in M. These are disjoint closed subsets. Thus there is a smooth function $\rho : M \to \mathbb{R}$, which on A is 0 and on B is 1. Let $t \in (0, 1)$ be a regular value of ρ. Then $d[(M, f)]$ is represented by $f|_{\rho^{-1}(t)} : \rho^{-1}(t) \to U \cap V$. The construction of singular bordism was carried out in [2] on the category of pairs of spaces. The proof that our absolute bordism theory is a homology theory uses the same ideas, see [1], Chapter II. For manifolds with singularities there is a problem with this proof since then there are no bicollars in general. But it was shown in [4] that there is a bicollar up to bordism. The same arguments apply to the generalized bordism theories constructed below.

3. Stratifolds

There are plenty of definitions of stratified spaces, starting from Whitney stratified spaces and Mather's abstract stratified spaces [5], which are both differential topological concepts, as well as purely topological concepts. All of them have in common that it is a topological space together with a decomposition into manifolds, which are called strata. Since we want to generalize bordism of smooth manifolds we restrict ourselves to differential topological stratifolds.

Our approach to stratifolds is motivated by a definition of smooth manifolds in the spirit of algebraic geometry as topological spaces together with a sheaf of functions, which in the traditional definition corresponds to the smooth functions. Then a manifold is a Hausdorff space M with countable basis together with a sheaf $\mathcal{C}$ of continuous functions, which is locally diffeomorphic to $\mathbb{R}^n$ equipped with the sheaf of all smooth functions. Here a morphism between spaces X and X' equipped with subsheaves of the sheaf of smooth functions is a continuous map f such that if ρ' is in the sheaf over X', then $\rho' f$ is in the sheaf over X. An isomorphism or here called **diffeomorphism** is a bijective map f such that f and f^{-1} are morphisms.

Having this in mind it is natural to generalize this by considering locally compact Hausdorff spaces $\mathcal{S}$ with countable basis together with a sheaf $\mathcal{C}$ of continuous functions, such that for

$f_1, ..., f_k$ in $\mathcal{C}$ and f a smooth function on $\mathbb{R}^k$, the composition $f(f_1, .., f_k)$ is in $\mathcal{C}$. A stratifold is defined as a pair $(\mathcal{S}, \mathcal{C})$ such that the following properties are fulfilled. Given $\mathcal{C}$ one can define the tangent space $T_x\mathcal{S}$ at a point $x \in \mathcal{S}$ as the vector space of all derivations of the germs $\Gamma_x(\mathcal{C})$ of smooth functions at x. This gives a decomposition of $\mathcal{S}$ into subspaces

$$\mathcal{S}^k := \{x \in \mathcal{S} \mid \dim T_x\mathcal{S} = k\}.$$

These subspaces are called the **k-strata** of $\mathcal{S}$. The union of all strata of dimension $\leq k$ is called the **k-skeleton** Σ^k.

Definition 1. *An n-dimensional stratifold is a pair $(\mathcal{S}, \mathcal{C})$ as above such that*
(1) For each k the stratum $\mathcal{S}^k$ together with the restriction $\mathcal{S}_{S^k}$ of the sheaf to it is a smooth k-dimensional manifold, i.e. is locally diffeomorphic to $\mathbb{R}^k$.
(2) All skeleta are closed subsets of $\mathcal{S}$.
(3) All strata of dimension $> n$ are empty.
(4) For each $x \in \mathcal{S}$ and open neighborhood U there is a so-called bump function $\rho : \mathcal{S} \to \mathbb{R}_{\geq 0}$ in $\mathcal{C}$, such that supp $\rho \subset U$ and $\rho(x) > 0$.
(5) For each $x \in S^k$ the restriction gives an isomorphism $\Gamma_x(\mathcal{C}) \to \Gamma_x(\mathcal{C}|_{S^k})$.
A continuous map $f : \mathcal{S} \to \mathcal{S}'$ is called a morphism or **smooth**, *if $f\rho \in \mathcal{C}$ for each $\rho \in \mathcal{C}'$. If f is a homeomorphism and f and f' are smooth, then f is called a* **diffeomorphism**.

A smooth map f induces, as for smooth manifolds, a linear map between the tangent spaces, the differential. It is given by pre-composition with the map f mapping a derivation at $x \in \mathcal{S}$ to a derivation of $\mathcal{S}'$ at $f(x)$. This induced map is called the **differential** of f at x.

Whereas the other conditions are natural, one might wonder where the last condition comes from. Looking at Mather's abstract stratified spaces, he gives a decomposition of the space into the strata plus additional data. Among them there are neighborhoods of the strata together with retractions π to the strata. Then Mather defines smooth (also called controlled) functions f as continuous functions such that the restriction of f to each stratum is smooth and there is a smaller neighborhood such that π restricted to the smaller neighborhood commutes with f. This implies our condition (5) and actually one can reconstruct the retraction π from our data if (5) is fulfilled ([4], p. 18ff).

All smooth manifolds are stratifolds. In this note we will only use the following comparatively simple class of stratifolds, which is similar to the construction of CW-complexes, which we call **polarizable stratifolds**, abbreviated as **p-stratifolds**. A 0-dimensional p-stratifold is a 0-dimensional smooth manifold. Let $(\mathcal{S}, \mathcal{C})$ be a $(k-1)$-dimensional p-stratifold and W be a k-dimensional manifold with boundary and $f : \partial W \to \mathcal{S}$ a proper smooth map. Then we define a k-dimensional p-stratifold $(\mathcal{S}' := W \cup_f \mathcal{S}, \mathcal{C}')$, where $\mathcal{C}'$ is constructed as follows. Choose a collar $\varphi : \partial W \times [0, 1) \to U \subset W$. Then f is in $\mathcal{C}'$ if and only if $f|_{\mathcal{S}}$ and $f|_W$ are smooth and there is an open subset $U' \subset U$ such that f commutes with the retraction to ∂W given by the collar. The last condition guarantees condition (5) above. It is easy to check that this is a k-dimensional stratifold.

This way one obtains plenty of explicit stratifolds. For example let W be a compact manifold with boundary and f the constant map from the boundary to a point. Then if we choose a collar of the boundary and attach W to the point (equivalently collapse the boundary to a point) and define the sheaf as above, we obtain a stratifold with 0-stratum a point and top-stratum the interior of W. A special case of this is the cone over a smooth manifold.

If $\mathcal{S}$ is an n-dimensional p-stratifold and M is a m-dimensional smooth manifold then the product $\mathcal{S} \times M$ is naturally an $(n+m)$-dimensional p-stratifold. In the construction above one replaces W by $W \times M$ and each attaching map f by $f \times \mathrm{id}$.

We define **an n-dimensional p-stratifold $\mathcal{T}$ with boundary** as a pair of topological spaces $(T, \partial T)$ together with the structure of an n-dimensional stratifold on $T - \partial T$, the structure of an $(n-1)$-dimensional stratifold on ∂T such that there is a homeomorphism $\varphi : \partial T \times [0,1)$ onto an open neigbourhood $U \subset T$ of ∂T, which on ∂T is the identity, such that $T - U$ is a closed subset of T (implying that ∂T is an end) and its restriction to $\partial T \times (0,1)$ is a diffeomorphism of stratifolds onto $U - \partial T$. Such a homeomorphism φ is called a **collar**.

Using a collar one can glue p-stratifolds with boundary the same way one glues manifolds over common boundary components. Thus one can define bordism groups and, if one adds a continuous map to a topological space X, singular bordism groups.

The following observation is central for our construction of the zoo of bordism groups. If $\mathcal{T}$ and $\mathcal{T}'$ are p-stratifolds with boundary whose stratum of dimension r is in both cases empty, then the same holds for the glued stratifold. Similarly if $\mathcal{S}$ and $\mathcal{S}'$ are stratifolds with empty k-stratum, then the same holds for the disjoint union. Let $A \subseteq \mathbb{N}$ be a set. Here $\mathbb{N}$ contains 0. An **n-dimensional A-stratifold** is a p-stratifold $\mathcal{S}$ such that for $a \in \mathbb{N} - A$ the stratum of dimension $n - a$ is empty. For example, if $A = \{0\}$, then an A-stratifold is a smooth manifold, all strata except the top stratum being empty. Or, if $A = \mathbb{N} - \{1\}$, then $\mathcal{S}$ is an A-stratifold if the stratum of dimension $n-1$ is empty. Or, if A consists of the even numbers, then $\mathcal{S}$ is an A-stratifold if and only if the strata of odd codimension are empty.

4. The zoo and the main theorem

With this, it is possible to define the zoo of bordism theories.

Definition 2. *Let X be a topological space and n a natural number and $A \subseteq \mathbb{N}$. An n-dimensional* **singular A-stratifold in** X *is a closed (compact without boundary) n-dimensional A-stratifold $\mathcal{S}$ together with a continuous map $f : \mathcal{S} \to X$.*

A **singular A-bordism between** *two n-dimensional singular A-stratifolds $(\mathcal{S}, f)$ and $(\mathcal{S}, f')$ is a compact singular A-stratifold $\mathcal{T}$ with boundary $\mathcal{S} + \mathcal{S}'$ together with a continuous map $F : \mathcal{T} \to X$ extending f and f'.*

Since one can glue n-dimensional singular A-stratifolds over common boundary components, singular A-bordism is an equivalence relation. Thus one can consider the equivalence classes, which form a group under disjoint union denoted by $\mathcal{N}_n^A(X)$. The proof is the same as in the case of smooth manifolds.

If $g : X \to Y$ is a continuous map, the post-composition induces a homomorphism

$$g_* : \mathcal{N}_n^A(X) \to \mathcal{N}_n^A(Y),$$

which makes $\mathcal{N}_n^A(X)$ a **functor** from the category of topological spaces and continuous maps to the category of graded abelian groups and homomorphisms.

To formulate our main theorem, namely that for each A we obtain a homology theory, we have to construct boundary operators. We have described above how this is done for bordism groups of smooth manifolds. To generalize this to stratifolds one has to consider **regular values** of smooth maps ρ from a p-stratifold $\mathcal{S}$ to $\mathbb{R}$. A value $t \in \mathbb{R}$ is a regular value if the restriction to all strata is a regular value. We note that by definition of the sheaf $\mathcal{C}$, if $\mathcal{S}$ is constructed inductively by attaching smooth manifolds W via a smooth map to the lower skeleta, t is also a regular value of the restriction of ρ to the boundary of W. The reason is that ρ commutes with the retractions given by the collar. This implies that the preimage of ρ restricted to W is a smooth manifold with boundary and the restriction of the collar chosen on W is a collar on this preimage. This implies that the preimage $\rho^{-1}(t)$ is in a natural way a p-stratifold of codimension 1. Furthermore, if $f : \mathcal{S} \to X$ is a continuous map, we can consider its restriction to $\rho^{-1}(t)$.

Finally, if $\mathcal{S}$ is an A-stratifold, then the codimensions of the strata of $\rho^{-1}(t)$ are unchanged and so $\rho^{-1}(t)$ is again an A-stratifold.

Thus one can define the boundary operator in the Mayer-Vietoris sequence as for smooth manifolds as follows. Let U and V be open subsets of a topological space X and $f : \mathcal{S} \to U \cup V$ a singular A-stratifold. Then we consider the complements C of $f^{-1}(U)$ and D of $f^{-1}(V)$ in M. These are closed disjoint subsets of $\mathcal{S}$. In a stratifold one has partitions of unity [4], Proposition 2.3, and so there is a smooth function ρ, which on C is zero and on D is 1. In a stratifold one can apply Sard's Theorem ([4], Proposition 2.6), and so there is a regular value $t \in (0, 1)$. By the considerations above $\rho^{-1}(t)$ is a codimension 1 stratifold and the restriction of f to it gives a singular A-stratifold in $U \cap V$. We will next show that this is well-defined and gives a natural boundary operator.

Our main Theorem is the following:

Theorem 3. *Let A be a subset of $\mathbb{N}$. For open subsets U and V in a topological space X the boundary operator*

$$d : \mathcal{N}_n^A(U \cup V) \to \mathcal{N}_{n-1}^A(U \cap V)$$

is well-defined and natural.

The functor $\mathcal{N}_n^A(X)$ together with the boundary operator d is a homology theory.

Proof. We first note that since a homotopy is a special bordism, the functor is a homotopy functor. Thus one only has to prove that there is an exact Mayer-Vietoris sequence. This amounts to showing that for all open subsets U and V of X the boundary operator

$$d : \mathcal{N}_n^A(U \cup V) \to \mathcal{N}_{n-1}^A(U \cap V)$$

is well-defined and natural and that the sequence is exact.

We begin with the proof that d is well-defined. In the case of bordism of smooth manifolds this is easy using that $\rho^{-1}(t)$ has a bicollar. In the case of stratifolds this is not the case. But it was shown in [4], Lemma B.1, page 197 that up to bordism one has a bicollar. This was proved there for so-called regular stratifolds. The regularity was used only at one place, namely to guarantee that the set of regular values is an open subset if $\mathcal{S}$ is compact [4], Proposition 4.3, page 44. Once this is the case, then the proof of [4], Lemma B.1 goes through without any change for p-stratifolds.

Next we show that the set of regular values of ρ is an open set if $\mathcal{S}$ is a compact p-stratifold. For this we consider the regular points, the points in $\mathcal{S}$ where the differential of ρ is non-trivial. But $x \in \mathcal{S}$ is a regular point if and only its restriction to the interior of the attached manifold W is regular. This restriction to W extends to ∂W and commutes with the retraction given by a collar. This implies that the regular points form an open subset. The singular points are the complement of the regular points (and so they are a closed subset) and the image of the singular points are the singular values. The complement of the singular values are the regular values. If $\mathcal{S}$ is compact, the image of a closed set is a closed and so the image of the singular points is closed implying that the regular values form an open set. Thus the proof of [4], Lemma B.1 goes through for p-stratifolds.

With this the proof that the boundary operator is well-defined is the same as in [4] for regular stratifolds. The naturality follows more or less from the construction of the differential. Let $g : X \to X'$ be a continuous map and U, V be open subsets of X, and U' and V' be open subsets of X', such that $g(U) \subseteq U'$ and $g(V) \subseteq V'$. Then for a singular A-stratifold $f : \mathcal{S} \to X$ we denote the complements of the preimages of U and V by C and D, similarly we denote the complements of the preimage of U' and V' by C' and D'. We have chosen a smooth function ρ, which on C is 0 and on D is 1. Now we consider gf and notice that $C' \subseteq C$ and $D' \subseteq D$.

Thus we can take the same separating function for the definition of the boundary operator $d' : \mathcal{N}_n^A(U' \cup V') \to \mathcal{N}_{n-1}^A(U' \cap V')$.

Lemma B.1 in [4] is also the key to the proof of the special case considered in [4], that the Mayer-Vietoris sequence is exact. The case considered there is the case, where $A = \mathbb{N} - \{1\}$. That A is of that special form is nowhere used in this proof. The only thing that matters is that all constructions used in the proof stay within the world of A-stratifolds. These constructions are: gluing of stratifolds via parts of boundary components and taking the preimage of a regular value. The definition of A-stratifolds using conditions on the existence of non-empty strata of a certain codimension are compatible with these constructions. Thus the proof in [4] goes through. □

One can enlarge this zoo even more by adding additional structure to the strata of a stratifold, for example an orientation or a stable almost complex structure or a spin-structure or a framing. In all these cases one obtains again a homology theory.

Now we mention a few special cases which show that the A-homology theories give a unified picture of some of the most important homology theories which originally had rather different constructions. To formulate the result let me remind the reader of the Postnikov tower of a homology theory. As mentioned above one has a unified homotopy theoretic picture of homology theories in terms of spectra S. Given a spectrum S and a topological space X one can consider the stable homotopy groups $\pi_n(S \wedge X)$, which form a homology theory. As with spaces one can consider Postnikov towers of spectra. This is given by spectra S_k together with a map $S \to S_k$, where one requires that all stable homotopy groups of S_k vanish above degree k and the map induces an isomorphism up to degree k.

If we consider for example the Thom spectrum MO which represents singular bordism, then the 0-th stage of the Postnikov tower is a homology theory, which has coefficients $\mathbb{Z}/2$ in degree 0 and 0 in degree > 0. Thus this homology theory represents $H_*(X; \mathbb{Z}/2)$.

Returning to our zoo, we consider some special cases. For a positive integer k we consider the set $A_k := \mathbb{N} - \{1, ..., k\}$. For $k = \infty$ we define $A_\infty = \{0\}$. Then for $n \leq k$ an n-dimensional A_k-stratifold is the same as a smooth manifold and so for $n < k$ the bordism group $\mathcal{N}_n^{A_k}$ is equal to the bordism group of manifolds $\mathcal{N}_k$. In particular for $k = \infty$ the bordism groups $\mathcal{N}_*^{A_\infty}(X)$ are equal to the bordism group of smooth manifolds $N_*(X)$. On the other hand for $n \geq k$ the group is zero, since the cone over such a stratifold is a null bordism (the cone point is a stratum of codimension $n + 1 > k$). This implies that for $k = 1$ the coefficients $\mathcal{N}_*^{A_1}$ are $\mathbb{Z}/2$ in degree 0 and 0 else. Thus by the characterization of ordinary homology by the Eilenberg-Steerond Axioms (which include the dimension axiom) $\mathcal{N}_*^{A_1}(X)$ is equivalent to $H_*(X; \mathbb{Z}/2)$ for X a CW-complex. But since all p-stratifolds are homotopy equivalent to CW-complexes (all smooth manifolds with boundary are relative CW-complexes and a p-stratifold is inductively obtained by attaching smooth manifolds) this implies that the same holds for arbitrary topological spaces (exercise) and we have shown:

Theorem 4. *The homology theory $\mathcal{N}_*^{A_k}$ is equivalent to the homology theory given by the k-stage of the Postnikov tower of the Thom spectrum MO. In particular*

$$\mathcal{N}_*^{A_1}(X) \quad \text{is equivalent to} \quad H_*(X; \mathbb{Z}/2), \quad \text{and} \quad \mathcal{N}_*^{A_\infty}(X) = \mathcal{N}_*(X).$$

This is a good place to remark that the same result is not true if we use regular stratifolds instead of p-stratifolds. Then one also obtains homology theories. But although for A_1 both theories have the same coefficients, the theory based on regular stratifolds is only for CW-complexes equivalent to ordinary singular homology with $\mathbb{Z}/2$ coefficients. For more general spaces this is not true, for example for 1-point compactifications of non-compact manifolds the theory is in general different (see [4], page 187; the argument there for integral homology works also for $\mathbb{Z}/2$-homology).

We finish this note with a potential application of our theories to the Griffiths group. As mentioned before, one can add more structure to the strata of an A-stratifold. If we distinguish a stable almost complex structure on all strata (there is no compatibility between the structures on the different strata) we call the corresponding homology theory $\mathcal{U}_*^A(X)$.

In the discussion above, one obtains similar statements if one replaces non-oriented bordism by unitary bordism $U_*(X)$, MO by MU and $H_k(X;\mathbb{Z}/2)$ by $H_k(X;\mathbb{Z})$.

Now, we consider the special case of an A-homology theory for A_{even}-stratifolds with stable almost complex strata, where A_{even} consists of the set of even natural numbers. We have a forgetful transformation (replace A_{even} by $\mathbb{N}-\{1\}$ and use the orientation given by the almost complex structure to obtain an element in integral homology)

$$\varphi : \mathcal{U}_{2r}^{A_{even}}(X) \to H_{2r}(X;\mathbb{Z}).$$

Question: *What are the image and kernel of φ?*

This might be useful in connection with the Griffiths group consisting of the kernel of the natural transformation $\mathrm{H} : Z^*_{alg}X \to H^*(X;\mathbb{Z})$ (the letter H stands for Hodge), where X is a non-singular complex algebraic variety and $Z^*_{alg}X$ is the ring of cycles modulo algebraic equivalence on X. For simplicity we assume that X is compact, so that Poincaré duality holds and we can consider the corresponding map in homology $Z_*^{alg}X \to H_*(X,;\mathbb{Z})$. Totaro [6] has constructed a canonical lift of this transformation over $U_*(X) \otimes_{U_*} \mathbb{Z}$. We will construct another lift.

Since a complex algebraic variety is in a natural way an A_{even} p-stratifold [3], we obtain a transformation

$$Z_*^{alg}X \to \mathcal{U}_*^{A_{even}}(X).$$

If we compose this with the transformation given by the forgetful map φ above, the composition of these two transformations is the Poincaré dual of the transformation $\mathrm{H} : Z^*_{alg}X \to H^*(X;\mathbb{Z})$. Thus one might try to do the same as Totaro did, to find elements in the kernel of

$$\mathcal{U}_*^{A_{even}}(X) \to H_*(X;\mathbb{Z})$$

which are in the image of $Z_*^{alg}X \to \mathcal{U}_*^{A_{even}}(X)$. Unfortunately we have nothing to say about this at the moment. The reason why our lift might be interesting is that in contrast to $U_*(X) \otimes_{U_*} \mathbb{Z}$ our theory $\mathcal{U}_*^{A_{even}}(X)$ is a homology theory, which might be a useful fact. On the other hand a computation of $\mathcal{U}_*^{A_{even}}(X)$ is probably very hard.

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MORSIFICATIONS OF REAL PLANE CURVE SINGULARITIES

PETER LEVIANT AND EUGENII SHUSTIN

Dedicated to the memory of a great mathematician Egbert Brieskorn

Abstract. A real morsification of a real plane curve singularity is a real deformation given by a family of real analytic functions having only real Morse critical points with all saddles on the zero level. We prove the existence of real morsifications for real plane curve singularities having arbitrary real local branches and pairs of complex conjugate branches satisfying some conditions. This was known before only in the case of all local branches being real (A'Campo, Gusein-Zade). We also discuss a relation between real morsifications and the topology of singularities, extending to arbitrary real morsifications the Balke-Kaenders theorem, which states that the A'Campo–Gusein-Zade diagram associated to a morsification uniquely determines the topological type of a singularity.

Introduction

By a **singularity** we always mean a germ $(C, z) \subset \mathbb{C}^2$ of a plane reduced analytic curve at its singular point z. Irreducible components of the germ (C, z) are called **branches of** (C, z). Let $f(x, y) = 0$ be an (analytic) equation of (C, z), where f is defined in the closed ball $B(z, \varepsilon) \subset \mathbb{C}^2$ of radius $\varepsilon > 0$ centered at z. The ball $B(z, \varepsilon)$ is called the **Milnor ball** of (C, z) (and is denoted in the sequel $B_{C,z}$) if z is the only singular point of C in $B(z, \varepsilon)$, and $\partial B(z, \eta)$ intersects C transversally for all $0 < \eta \le \varepsilon$. A **nodal deformation** of a singularity (C, z) is a family of analytic curves $C_t = \{f_t(x, y) = 0\}$, where $f_t(x, y)$ is analytic in x, y, t for $(x, y) \in B(C, z)$ and t varying in an open disc $\mathbb{D}_\zeta \subset \mathbb{C}$ of some radius $\zeta > 0$ centered at zero, and where $C_0 = C$, C_t is smooth along $\partial B_{C,z}$, intersects $\partial B_{C,z}$ trasversally for all $t \in \mathbb{D}_\zeta$, for any $t \neq 0$, the curve C_t has only ordinary nodes in $B_{C,z}$, and the number of nodes does not depend on t. The maximal number of nodes in a nodal deformation of (C, z) in B equals $\delta(C, z)$, the δ-invariant (see, for instance, [17, §10]).

Let (C, z) be a real singularity, i.e., invariant with respect to the complex conjugation, $z \in C$ its real singular point. Denote by $\mathrm{ReBr}(C, z)$, $\mathrm{ImBr}(C, z)$ the numbers of real branches and the pairs of complex conjugate branches centered at z, respectively. Let $C_t = \{f_t(x, y) = 0\}$, $t \in \mathbb{D}_\zeta$, be an equivariant[1] nodal deformation of a real singularity (C, z). Its restriction to $t \in [0, \zeta)$ is called a **real nodal deformation**. A real nodal deformation is called a **real morsification** of (C, z) if each function f_t, $0 < t < \zeta$, has only real critical points in $B(C, z)$, all critical points are Morse, and all the saddle points have the zero critical level. Clearly, then all maxima have positive critical values, and all minima negative ones.

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[1] Here and further on, *equivariant* means commuting with the complex conjugation.

N. A'Campo [1, 2, 4] and S. Gusein-Zade [15, 16] performed a foundational research on this subject. In particular, they showed that real morsifications carry a lot of information on singularities and allow one to compute such invariants as the monodromy and intersection form in vanishing homology in a simple and efficient way. However, some questions have remained open, in particular:

Question: *Does any real plane curve singularity admit a real morsification?*

Our main result is a partial answer to this question. Before precise formulation, we should mention that an affirmative answer was given before in the case of all branches of (C,z) being real (below referred to as a **totally real** singularity), see [1, Theorem 1][2] and [14, Theorem 4] (see also [6, Section 4.3]). Notice that any topological type of a curve singularity is presented by a totally real singularity, see [14, Theorem 3].

Now we give necessary definitions. A singularity is called **Newton non-degenerate**, if in some local coordinates, it is **strictly Newton non-degenerate**, that is given by an equation $f(x,y)=0$ with a convenient Newton diagram at $z=(0,0)$ and such that the truncation of $f(x,y)$ to any edge of the Newton diagram is a quasihomogeneous polynomial without critical points in $(\mathbb{C}^*)^2$ (i.e., it has no multiple binomial factors). We say that a singularity (C,z) is **admissible along its tangent line** L if the singularity (C_L,z) formed by the union of the branches of (C,z) tangent to L is as follows: (C_L,z) is the union of a Newton non-degenerate singularity with a singularity, whose all branches are smooth.

Theorem 1. *Let (C,z) be a real singularity, $\mathcal{T}(C,z)=\{z_0=z,z_1,...\}$ the vertices of its minimal resolution tree. For any $z_i\in\mathcal{T}(C,z)$ denote by (C_i,z_i) the germ at z_i of the corresponding strict transform of (C,z). If, for any real point $z_i\in\mathcal{T}(C,z)$, the singularity (C_i,z_i) is admissible along each of its non-real tangent lines, then the real singularity (C,z) admits a real morsification.*

Note that the case of totally real singularities is included, since then the restrictions asserted in Theorem are empty. We illustrate the range of singularities covered by Theorem 1 with a few examples.

Example 1. *(1) Any quasihomogeneous (in real coordinates) singularity satisfies the hypotheses of Theorem 1, and their morsifications can be constructed in the same manner as for the totally real singularities even if the singularity contains complex conjugate branches, see Section 2.1.2.*

(2) The simplest singularity satisfying the hypotheses of Theorem 1 and whose morsification is constructed by a new method suggested in the present paper is a pair of transversal ordinary cuspidal branches, given, for instance, by an equation $(x^2+y^2)^2+x^5=0$. The real part of its morsification looks as shown in Figure 1. One can show that all possible morsifications are isotopic to this one.

(3) The simplest singularity beyond the range of Theorem 1 is a pair of two transversal complex conjugate branches of order 4 with two Puiseux pairs $(2,3)$ and $(2,7)$ (equivalently, with the Puiseux characteristic exponents $(4,6,7)$), given, for instance, by an equation

$$((w_+^2-x^3)^2-x^5w_+)((w_-^2-x^3)^2-x^5w_-)=0,\quad w_\pm=y\pm x\sqrt{-1}\ .$$

On the other hand, a singularity consisting of a pair of complex conjugate branches with the same Puiseux pairs $(2,3)$, $(2,7)$ as above, but having a common real tangent does satisfy the hypotheses of Theorem 1, since after one blow up it turns into a singularity with two complex conjugate branches having only one Puiseux pair.

We believe that the following holds:

[2]As pointed to us by S. Gusein-Zade, there is a gap in the proof of [1, Theorem 1]: namely, the function in [1, Formula (1) in page 12] does not possess the claimed properties.

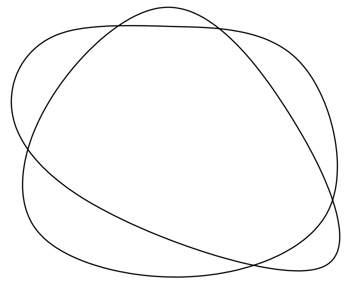


FIGURE 1. Morsification of a pair of complex conjugate cuspidal branches

Conjecture 1. *Any real plane curve singularity possesses a real morsification.*

In the proof of Theorem 1 presented in Section 2, we combine a relatively elementary inductive blow-up construction in the spirit of [1] with the patchworking construction as appears in [20, 21] and some explicit formulas for real morsifications of pairs of complex conjugate smooth branches and pairs of branches of topological type $x^p+y^q=0$, $(p,q)=1$. We expect that suitable formulas for real morsifications of pairs of complex conjugate branches with several Puiseux pairs would lead to a complete solution of the existence problem of real morsifications.

A real morsification of a totally real singularity yields a so-called A'Campo-Gusein-Zade diagram, which uniquely determines the topological type of the singular point, as shown by L. Balke and R. Kaenders [7, Theorem 2.5 and Corollary 2.6]. In Section 4, we extend this result to morsifications of arbitrary real singularities.

1. Elementary geometry of real morsifications

For the reader's convenience, we present here few simple and in fact known claims on morsifications. In what follows we consider only real singularities.

Recall that a real node of a real curve can be either hyperbolic or elliptic, that is, analytically equivalent over $\mathbb{R}$ either to $x^2-y^2=0$, or $x^2+y^2=0$, respectively. For a real nodal deformation $C_t=\{f_t(x,y)=0\}$, $0\le t<\zeta$, the saddle critical points of f_t on the zero level correspond to real hyperbolic nodes of C_t and vice versa.

Lemma 2. *The number of hyperbolic nodes in any real nodal deformation C_t, $0\le t<\zeta$, of (C,z) does not exceed $\delta(C,z)-\mathrm{ImBr}(C,z)$.*

Proof. As we noticed in Introduction, the maximal number of nodes in a nodal deformation of a singularity (C,z) is the δ-invariant $\delta(C,z)$. In a real nodal deformation, a pair $Q,\overline{Q}$ of complex conjugate branches either glues up into one surface immersed into $B(C,z)$ thus reducing the total number of nodes by at least one, or Q and $\overline{Q}$ do not glue up to each other and to other branches and then their intersection points are either complex conjugate nodes or real elliptic nodes, and, at last, if Q and $\overline{Q}$ do not glue up to each other, but glue up to some other branches of (C,z), we loose at least two nodes. So, the bound follows. ☐

The following lemma is a version of [1, Lemma 4 and Theorem 3]. Let C_t, $0\le t<\zeta$, be a a real morsification of a real singularity (C,z). The sets $\mathbb{R}C_t$, $0<t<\zeta$, are isotopic in the disc $\mathbb{R}B_{C,z}$. Each of them is called a **divide** of the given morsification (more information on divides see in Section 4.1). Given a divide $D\subset\mathbb{R}B_{C,z}$ of a real morsification of the real singularity (C,z), the connected components of $\mathbb{R}B_{C,z}\setminus D$ disjoint from $\partial\mathbb{R}B_{C,z}$ are called inner components. Denote by $I(D)$ the union of the closures of the inner components of $\mathbb{R}B_{C,z} \setminus D$ (called **body of the divide** in [3]).

Lemma 3. *Let $D = \mathbb{R}C_t$ be a divide of a real morsification of a real singularity (C, z). Then*

(i) *if (C, z) is not a hyperbolic node then $I(D)$ is non-empty, connected, and simply connected;*
(ii) *D has $\delta(C, z) - \mathrm{ImBr}(C, z)$ singularities, which are hyperbolic nodes of C_t;*
(iii) *each inner component of $\mathbb{R}B_{C,z} \setminus D$ is homeomorphic to an open disc;*
(iv) *the number $h(C, z)$ of the inner components of $\mathbb{R}B_{C,z} \setminus D$ does not depend on the morsification and satisfies the relation*

$$h(C, z) + \delta(C, z) - \mathrm{ImBr}(C, z) = \mu(C, z) \ ,$$

$\mu(C, z)$ being the Milnor number.

Proof. In claim (i) suppose that $I(D)$ is not connected. Then the associated Coxeter-Dynkin diagram of the singularity (C, z) constructed in [15] (see also [16, §3]) appears to be disconnected contrary to the fact that it is always connected [12, 14]. Furthermore, $I(D)$ is simply connected since is has no holes by construction.

Statements (ii)-(iv) follow from claim (i), from the bound

$$\#\mathrm{Sing}(D) \le \delta(C, z) - \mathrm{ImBr}(C, z)$$

of Lemma 2, from the Milnor formula [17, Theorem 10.5]

$$\mu(C, z) = 2\delta(C, z) - \mathrm{ReBr}(C, z) - 2\mathrm{ImBr}(C, z) + 1 \ ,$$

from the fact that each inner component of $\mathbb{R}B_{C,z} \setminus D$ contains a critical point of the function $f_t(x, y)$, and hence

$$h(C, z) + \delta(C, z) - \mathrm{ImBr}(C, z) \le \mu(C, z) \ ,$$

and from the calculation of the Euler characteristic of $I(D)$

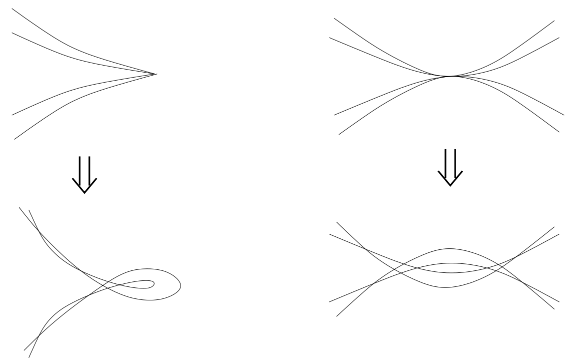
$$h(C, z) - (2 \cdot \#\mathrm{Sing}(D) - \mathrm{ReBr}(C, z)) + \#\mathrm{Sing}(D) \ge 1.$$

□

Remark 4. *In fact, one could equivalently define real morsifications as real nodal deformations having precisely $\delta(C, z) - \mathrm{ImBr}(C, z)$ hyperbolic nodes as their only singularities.*

Lemma 5. *Given a real morsification C_t, $0 \le t < \zeta$, of a real singularity (C, z),*

- *any real branch P of (C, z) does not glue up with other branches and deforms into a family of immersed discs P_t, $t > 0$, whose real point sets $\mathbb{R}P_t \subset \mathbb{R}B_{C,z}$ are immersed segments with $\delta(P)$ selfintersetions and endpoints on $\partial\mathbb{R}B_{C,z}$;*
- *any pair of complex conjugate branches $Q, \overline{Q}$ of (C, z) do not glue up to other branches, but glue up to each other so that they deform into a family of immersed cylinders Q_t, $t > 0$, with the real point set $\mathbb{R}Q_t \subset \mathbb{R}B_{C,z}$ being an immersed circle disjoint from $\partial B(C, z)$ and having $\delta(Q \cup \overline{Q}) - 1 = 2\delta(Q) + (Q \cdot \overline{Q}) - 1$ selfintersections (here $(Q \cdot \overline{Q})$ denotes the intersection number);*
- *for any two real branches P', P'', the intersection $\mathbb{R}P'_t \cap \mathbb{R}P''_t$, $t > 0$, consists of $(P' \cdot P'')$ points;*
- *for any real branch P and a pair of complex conjugate branches $Q, \overline{Q}$, the intersection $\mathbb{R}P_t \cap \mathbb{R}Q_t$, $t > 0$, consists of $2(P \cdot Q)$ points;*
- *for any two pairs of complex conjugate branches $Q', \overline{Q}'$ and $Q'', \overline{Q}''$, the intersection $\mathbb{R}Q'_t \cap \mathbb{R}Q''_t$, $t > 0$, consists of $2(Q' \cdot Q'') + 2(Q' \cdot \overline{Q}'')$ points.*



(a) (b)

FIGURE 2. Non-partitions

Proof. Straightforward from Lemmas 2 and 3. □

Lemma 6. *Let (C_1, z), (C_2, z) be two real singularities without branches in common. If the real singularity $(C_1 \cup C_2, z)$ possesses a real morsification, then each of the real singularities (C_1, z), (C_2, z) possesses a real morsification too.*

Proof. Straightforward from Lemma 5. □

Given a divide D of a real morsification of a real singularity (C, z), it follows from Lemma 3 that $I(D)$ possesses a cellular decomposition into $\mathrm{Sing}(D)$ as vertices, the components of $D \setminus \mathrm{Sing}(D)$, disjoint from $\partial\mathbb{R}B_{C,z}$, as the 1-cells, and the inner components of $\mathbb{R}B_{C,z} \setminus D$ as the 2-cells. Following [1, §1], we say that the given real morsification defines a **partition**, if, in the above cellular decomposition of $I(D)$, the intersection of the closures of any two 2-cells is either empty, or a vertex, or the closure of a 1-cell.

This property was assumed in the Balke-Kaenders theorem [7, Theorem 2.5 and Corollary 2.6] about the recovery of the topological type of a singularity out of the A'Campo-Gusein-Zade diagram. In fact, this assumption is not needed (see Section 4). Here we just notice the following:

Lemma 7. *There are real morsifications that do not define a partition.*

Proof. For the proof, we present two simple examples: Figure 2(a) shows a real morsification of the singularity $(y^2 + x^3)(y^2 + 2x^3) = 0$ (two cooriented real cuspidal branches with a common tangent), while Figure 2(b) shows a real morsification of the real singularity

$$(y^2 - x^4)(y^2 - 2x^4) = 0$$

(four real smooth branches quadratically tangent to each other). A construction is elementary. For example, the morsification shown in Figure 2(a) can be defined by

$$(y^2 + x^2(x - \varepsilon_1(t)))(y^2 + 2(x - \varepsilon_2(t))^2(x - \varepsilon_3(t))) = 0\ ,$$

where $0 < \varepsilon_2(t) < \varepsilon_3(t) \ll \varepsilon_1(t) \ll 1$. □

2. Existence of real morsifications

2.1. **Blow-up construction.** Let us recall that the multiplicity of a singularity (C,z), resp. of a branch P, is the intersection numbers $\mathrm{mt}(C,z)=(C\cdot L)_z$, resp. $(P\cdot L)_z$ with a generic line L through z. Recall that the proper transform of (C,z) under the blowing up of z consists of several germs (C_i^*,z_i) with z_i being distinct points on the exceptional divisor E associated with distinct tangents to (C,z). It is know that (see, for instance, [13, Page 185 and Proposition 3.34])

$$\delta(C,z)=\sum_i\delta(C_i^*,z_i)+\frac{\mathrm{mt}(C,z)(\mathrm{mt}(C,z)-1)}{2},\quad \mathrm{mt}(C,z)=\sum_i(C_i^*\cdot E)_{z_i}\ . \tag{1}$$

2.1.1. *The totally real singularities.* The existence of real morsifications for totally real singularities was proved in [1, Theorem 1]. We present here a proof (similar to the A'Campo's one) in order to be self-contained and to use elements of that proof in the general case.

(1) Consider, first, the case of a totally real singularity (C,z) whose all branches are smooth. We proceed by induction on the maximal δ-invariant $\Delta_1(C,z)$ of the union of any subset of branches tangent to each other.

The base of induction, $\Delta_1(C,z)=0$, corresponds to the union of $d\ge 2$ smooth branches with distinct tangents. Here $\delta(C,z)=d(d-1)/2$, and we construct a real morsification by shifting the branches to a general position.

Assuming that $\Delta_1(C,z)>0$ in the induction step, we blow up the point z into an exceptional divisor E. The strict transform of (C,z) splits into components (C_i^*,z_i), $z_i\in\mathbb{R}E$, corresponding to different tangents of (C,z). Notice that E is transversal to all branches of (C_i^*,z_i), and hence $\Delta_1(C_i^*\cup E,z_i)<\Delta_1(C,z)$ for all i (cf. (1)). Then we construct real morsifications of each real singularity $(C_i^*\cup E,z_i)$ in which the germs (E,z_i) stay fixed (in view of Lemma 5 these germs do not glue up with other branches, and hence can be kept fixed by suitable local equivariant diffeomorphisms). Thus, we get the union of real curves $(C_i^*)^+$ in neighborhoods of z_i, having

$$\sum_i\delta(C_i^*,z_i)=\delta(C,z)-\frac{\mathrm{mt}(C,z)\cdot(\mathrm{mt}(C,z)-1)}{2}$$

real hyperbolic nodes and $\mathrm{mt}(C,z)$ real intersetion points with E. Then we blow down E and obtain a deformation whose elements have $\delta(C,z)-\frac{\mathrm{mt}(C,z)\cdot(\mathrm{mt}(C,z)-1)}{2}$ real hyperbolic nodes and a point of transversal intersection of $\mathrm{mt}(C,z)$ smooth branches. Deforming the latter real singularity, we complete the construction of a real morsification.

(2) Now we prove the existence of real morsifications for arbitrary totally real singularities, using induction on $\Delta_2(C,z)$, the δ-invariant of the union of all singular branches of (C,z). The preceding consideration serves as the base of induction. The induction step is very similar: we blow up the point z and notice that $\sum_i\Delta_2(C_i^*\cup E,z_i)<\Delta_2(C,z)$; then proceed as in the preceding paragraph.

2.1.2. *Semiquasihomogeneous singularities.* The same blow-up construction of real morsifications works well in the important particular case of semiquasihomogeneous singularities. Let

$$F(x,y)=\sum_{pi+qj=pq}a_{ij}x^iy^j$$

be a real square-free quasihomogeneous polynomial, where $1\le p\le q$. Then

$$(C,z)=\Big\{F(x,y)+\sum_{pi+qj>pq}a_{ij}x^iy^j=0\Big\}$$

is called a real semiquasihomogeneous singularity of type (p, q). This real singularity has $d = \gcd(p, q)$ branches, among which we allow complex conjugate pairs.

(1) A semiquasihomogeneous singularity of type (p, p) is just the union of smooth transversal branches. If they all are real the existence of a real morsification is proved in Section 2.1.1. Thus, suppose that $F(x, y)$ splits into the product $F_1(x, y)$ of real linear forms and the product $F_2(x, y)$ of positive definite quadratic forms $q_i(x, y)$, $1 \le i \le k$, $k \ge 1$. The forms q_i are not proportional to each other, and there are $b_i > 0$, $i = 1, ..., k$, such that any two quadrics $q_i - b_i = 0$ and $q_j - b_j = 0$, $1 \le i < j \le k$, intersect in four real points, and all their intersection points are distinct. So, we obtain a real morsification by deforming (C, z) in the family

$$F(x, y, t) = F_1(x, y) \prod_{i=1}^{k} (q_i(x, y) - b_i t), \quad 0 \le t \ll 1 ,$$

and then by shifting each of the lines defined by $F_1 = 0$ to a general position.

(2) Let (C, z) be a real semiquasihomogeneous singularity of type (p, q), $2 \le p < q$. We simultaneously prove the existence of real morsifications of (C, z) and of the following additional singularities:

- (f1) $(C \cup L, z)$, where L is a real line intersecting (C, z) at z with multiplicity p (i.e. transversally) or q (tangent);
- (f2) $(C \cup L_1 \cup L_2, z)$, where a real line L_1 intersects (C, z) with multiplicity p and a real line $L_2 \ne L_1$ intersects (C, z) at z with multiplicity p or q.

We proceed by induction on $\delta(C, z)$. The base of induction, $\delta(C, z) = 1$, corresponds to $p = 2$, $q = 3$, that is, an ordinary cusp. Here (C, z), $(C \cup L, z)$, and $(C \cup L_1 \cup L_2, z)$ are totally real, hence possess a real morsification. Suppose that $\delta(C, z) > 1$, blow up the point z, and consider the union of the strict transform of the studied singularity with the exceptional divisor E. Notice that the strict transform of a real semiquasihomogeneous singularity of type (p, q) is also a real semiquasihomogeneous singularity either of type $(p, q-p)$ if $2p \le q$, or of type $(q-p, p)$ if $2p > q$, and in both cases it intersects E with multiplicity p. It is easy to see that the strict transform of singularities of the form (f1) and (f2) with added E is again a real singularity of one of these forms with parameters $(p, q-p)$ or $(q-p, p)$ and, may be, an extra real node. We then complete the proof as in Section 2.1.1.

2.2. **Singularities without real tangents.** The constructions of morsifications presented in this section is the mein novelty of the present paper. In the case of singularities with only smooth branches, Lemma 8 presents a rather simple direct formula for the morsification. In the case of non-smooth branches with one Puiseux pair (Lemma 9 below), we apply an ad hoc deformation argument (a kind of the pathchworking construction). The geometric background for this argument is as follows. We extend the pair $(\mathbb{C}^2, (C, z))$ to a trivial family $(\mathbb{C}^2, (C, z)) \times (\mathbb{C}, 0)$, then blow up the point $z \in \mathbb{C}^2 \times \{0\}$. The central fiber of the new family is the union of the blown-up plane $\mathbb{C}^2_1$ and the exceptional divisor $E \simeq \mathbb{P}^2$. The germ (C, z) yields in $\mathbb{P}^2$ a real conic C_2 with multiplicity $p \ge 2$ that intersects the line $\mathbb{C}^2_1 \cap E$ in two imaginary points. Our deformation gives an inscribed equivariant family of curve germs, whose real part appears to be a deformation of the above p-multiple conic C_2.

2.2.1. *The case of one pair of complex conjugate tangents.* Let a real singularity (C, z) have exactly two tangent lines, and they are complex conjugate. In suitable local equivariant coordinates x, y in $B_{C,z}$, we have $z = (0, 0)$, and the tangent lines are

$$L = \{x + (\alpha + \beta\sqrt{-1})y = 0\}, \ \overline{L} = \{x + (\alpha - \beta\sqrt{-1})y = 0\} ,$$

where $\alpha, \beta \in \mathbb{R}$, $\beta \ne 0$.

Denote by (C_i,z), $i=1,...,s$, the branches of (C,z) tangent to L; respectively $(\overline{C}_i,z)$, $i=1,...,s$, are the branches of (C,z) tangent to $\overline{L}$. Introduce the new coordinates

$$w=x+(\alpha+\beta\sqrt{-1})y,\quad \widehat{w}=x+(\alpha-\beta\sqrt{-1})y\ .$$

Notice that $\widehat{w}=\overline{w}$ if $x,y\in\mathbb{R}$. We also will use for $\mathbb{R}^2\setminus\{0\}$ the coordinates $\rho>0$, $\theta\in\mathbb{R}/2\pi\mathbb{Z}$ such that

$$x+\alpha y=\rho\cos\theta,\quad \beta y=\rho\sin\theta,\quad \rho=\sqrt{w\widehat{w}}\ . \tag{2}$$

Lemma 8. *Let (C,z) have only smooth branches. Then (C,z) possesses a real morsification.*

Proof. A branch (C_i,z), $1\le i\le s$, has an analytic equation

$$\widehat{w}=\sum_{n\in I_i}a_{in}w^n,\quad I_i\subset\{n\in\mathbb{Z}\ :\ n>1\},\ a_{in}\in\mathbb{C}^*\text{ as }n\in I_i\ .$$

Correspondingly, $(\overline{C}_i,z)$ is given by $w=\sum_{n\in I_i}\overline{a}_{in}\widehat{w}^n$. We claim that the equation

$$F_t(w,\widehat{w}):=\prod_{i=1}^{s}(\Phi_i(w,\widehat{w})-t^2)=0,\quad 0\le t<\zeta\ , \tag{3}$$

defines a real morsification of (C,z), where

$$\Phi_i(w,\widehat{w})=\left(\widehat{w}-\sum_{n\in I_i}a_{in}w^n\right)\left(w-\sum_{n\in I_i}\overline{a}_{in}\widehat{w}^n\right)$$

and $\zeta>0$ is sufficiently small. First, $F_t(w,\widehat{w})$ (the left-hand side of (3)) is an analytic function in $w,\widehat{w}$ and t. A separate factor in $F_t(w,\widehat{w})$ is

$$\Phi_i(w,\widehat{w})-t^2=w\widehat{w}-t^2+\sum_{n\in I_i}|a_{in}|^2(w\widehat{w})^n-\sum_{n\in I_1}(\overline{a}_{in}w^{n+1}+a_{in}\widehat{w}^{n+1})$$
$$+2\sum_{\substack{n_1<n_2\\ n_1,n_2\in I_i}}(w\widehat{w})^{n_1}(a_{in_1}\overline{a}_{in_2}w^{n_2-n_1}+\overline{a}_{in_1}a_{in_2}\widehat{w}^{n_2-n_1})\ .$$

Restricting the equation $\Phi_i(w,\widehat{w})$ to $\mathbb{R}B_{C,z}$ (in coordinates x,y), passing in $\mathbb{R}^2\setminus\{0\}$ to coordinates $\rho>0$, $\theta\in\mathbb{R}/2\pi\mathbb{Z}$ defined via (2, and rescaling by substitution of $t\rho$ for ρ, we obtain a family of curves depending on the parameter $0\le t<\zeta$

$$\Psi_{i,t}:=\rho^2-1+\sum_{n\in I_i}t^{2n-2}|a_{in}|^2\rho^{2n}-2\sum_{n\in I_i}t^{n-1}|a_{in}|\rho^{n+1}\cos((n+1)\theta-\theta_{in})$$
$$+\ 2\sum_{\substack{n_1<n_2\\ n_1,n_2\in I_i}}t^{n_1+n_2-2}|a_{in_1}a_{in_2}|\rho^{n_1+n_2}\cos((n_2-n_1)\theta+\theta_{in_1}-\theta_{in_2})=0\ ,$$

where $a_{in}=|a_{in}|\exp(\sqrt{-1}\theta_{in})$, $n\in I_i$. It is easy to see that each of them a circle embedded into an annulus $\{|\rho-1|<Kt\}\subset\mathbb{R}^2$ with $K>0$ a constant determined by the given singularity (C,z), and, furthermore, the normal projection of each curve to the circle $\rho=1$ is a diffeomorphism. Let $1\le i<j\le s$. Set

$$n_{ij}=\min\{n\in I_i\cup I_j\ :\ a_{in_{ij}}\ne a_{jn_{ij}}\}\ .$$

Note that $n_{ij}=(C_i\cdot C_j)$, the intersection number of branches C_i,C_j. On the other hand,

$$\Psi_{i,t}(\rho,\theta)-\Psi_{j,t}(\rho,\theta)=2t^{n_{ij}-1}|a_{in_{ij}}-a_{jn_{ij}}|\rho^{n_{ij}+1}\cos((n_{ij}+1)\theta-\theta_{ij,n_{ij}})+O(t^{n_{ij}})\ ,$$

where $\theta_{ij,n_{ij}} \in \mathbb{R}/2\pi\mathbb{Z}$, and hence, for a sufficiently small $t > 0$, the curves $\Psi_{i,t} = 0$ and $\Psi_{j,t} = 0$ intersect transversally in $2n_{ij} + 2$ points. In total, we obtain

$$2 \sum_{1 \le i < j \le s} (n_{ij} + 1) = 2 \sum_{1 \le i < j \le s} (C_i \cdot C_j) + s^2 - s = \delta(C, z) - \mathrm{ImBr}(C, z)$$

hyperbolic nodes as required for a real morsification. □

Lemma 9. *Let the singularity (C_L, z) be formed by a pair of branches of topological type*

$$x^p + y^q = 0, \quad 2 \le p < q, \quad (p, q) = 1,$$

that are tangent to L and $\overline{L}$ respectively. Then (C, z) possesses a real morsification.

Proof. **(1)** We start with the very special case of (C, z) given by

$$F(w, \widehat{w}) = w^p \widehat{w}^p - a\widehat{w}^{p+q} - \overline{a}w^{p+q} = 0, \quad a \in \mathbb{C}^* \ . \tag{4}$$

Denote by $P(\lambda) = \lambda^p + b^{(0)}_{p-2}\lambda^{p-2} + ... + b^{(0)}_0 \in \mathbb{R}[\lambda]$ the monic polynomial of degree p having $\left[\frac{p}{2}\right]$ critical points on the level $-2|a|$ and $\left[\frac{p-1}{2}\right]$ critical points on the level $2|a|$, whose roots sum up to zero (a kind of the p-th Chebyshev polynomial). We claim that there exist real functions $b_0(t), ..., b_{p-2}(t)$, analytic in $t^{\frac{1}{p}}$ such that $b_i(0) = b^{(0)}_i$, $0 \le i \le p - 2$, and the family

$$F_t(w, \widehat{w}) = (w\widehat{w} - t^2)^p + \sum_{i=p-2}^{0} t^{\frac{(p-i)(p+q)}{p}} b_i(t)(w\widehat{w} - t^2)^i - a\widehat{w}^{p+q} - \overline{a}w^{p+q} = 0 \ , \tag{5}$$

$$0 \le t < \zeta \ ,$$

is a real morsification of (C, z). To prove this, we rescale the latter equation by substituting $(tw, t\widehat{w})$ for $(w, \widehat{w})$ and restrict our attention to $\mathbb{R}B_{C,z}$ passing to the coordinates ρ, θ in (2):

$$(\rho^2 - 1)^p + \sum_{i=p-2}^{0} t^{\frac{(p-i)(q-p)}{p}} b_i(t)(\rho^2 - 1)^i - 2|a|\rho^{p+q} \cos((p + q)\theta - \theta_a) = 0 \ ,$$

where $a = |a| \exp(\sqrt{-1}\theta_a)$. Next, we substitute $\rho^2 = 1 + t^{\frac{q-p}{p}}\sigma$ and come to

$$(1 + t^{\frac{q-p}{p}}\sigma)^{-(p+q)/2} \left(\sigma^p + \sum_{i=p-2}^{0} b_i(t)\sigma^i\right) = 2|a| \cos((p + q)\theta - \theta_a) \ . \tag{6}$$

Finally, we recover the unknown functions $b_{p-2}(t), ..., b_0(t)$ from the following conditions.

Let $P(\lambda) > 3|a|$ as $|\lambda| > \lambda_0$. Suppose that $|\sigma| \le \lambda_0$ and that t is small so that the function of σ

$$P_t(\sigma) := (1 + t^{\frac{q-p}{p}}\sigma)^{-(p+q)/2} \left(\sigma^p + \sum_{i=p-2}^{0} b_i(t)\sigma^i\right)$$

has simple critical points $\mu_1(t), ..., \mu_{p-1}(t)$ arranged in the growing order and respectively close to the critical points $\mu^{(0)}_1, ..., \mu^{(0)}_{p-1}$ of $P(\lambda)$. So, we require

$$P_t(\mu_i(t)) = (-1)^i \cdot 2|a|, \quad i = 1, ..., p - 1 \ . \tag{7}$$

These conditions hold true for $t = 0$ by construction, and we only need to verify that the Jacobian with respect to $\mu_1, ..., \mu_{p-1}$ does not vanish. To this end, we observe that there exists a diffeomorphism of a neighborhood of the point $(\mu^{(0)}_1, ..., \mu^{(0)}_{p-1}) \in \mathbb{R}^{p-1}$ onto a neighborhood of the point $(b^{(0)}_{p-2}, ..., b^{(0)}_0) \in \mathbb{R}^{p-1}$ sending the critical points of a polynomial $\lambda^p + \widetilde{b}_{p-2}\lambda^{p-2} + ... + \widetilde{b}_0$

to its coefficients. Then the Jacobian of the left-hand side of the system (7) with respect to $\mu_1,...,\mu_{p-1}$ at $t=0$ turns to be

$$\det\left((\mu_i^{(0)})^j\frac{\partial b_j}{\partial\mu_i}\Big|_{t=0}\right)_{i=1,...,p-1}^{j=0,...,p-2}=\pm\prod_{1\le i<j\le p-1}(\mu_i^{(0)}-\mu_j^{(0)})\cdot\det\frac{D(\widetilde{b}_{p-2},...,\widetilde{b}_0)}{D(\mu_1,...,\mu_{p-1})}\Big|_{t=0}\ne 0\ .$$

It follows from (7) that, for any $\theta\in\mathbb{R}/2\pi\mathbb{Z}$, the equation (6) on σ has p real solutions (counting multiplicities) in the interval $|\sigma|<\lambda_0$, and we have exactly $(p-1)(p+q)=\delta(C,z)-\mathrm{ImBr}(C,z)$ double roots as

$$\sigma=\mu_{2i-1}(t),\quad \cos((p+q)\theta-\theta_a)=-1,\quad 1\le i\le\frac{p}{2}\ ,$$

or

$$\sigma=\mu_{2i}(t),\quad \cos((p+q)\theta-\theta_a)=1,\quad 1\le i\le\frac{p-1}{2}\ .$$

That is, family (5) indeed describes a real morsification of (C,z).

Note, that the real curve $\{F_t=0\}\subset\mathbb{R}B_{C,z}$ is an immersed circle lying in the $\lambda_0 t^{\frac{p+q}{p}}$-neighborhood of the ellipse $\rho=t$ and transversally intersecting in $2p$ points (counting multiplicities) with each real line through the origin.

(2) Consider the general case. By a coordinate change

$$(w,\widehat{w})\mapsto\left(w+\sum_{i\ge2}\alpha_i\widehat{w}^i,\ \widehat{w}+\sum_{i\ge2}\overline{\alpha}_i w^i\right)$$

one can bring (C,z) to a strictly Newton non-degenerate form with the Newton diagram

$$\Gamma(F)=[(p+q,0),p,p)]\cup[(p,p),(0,p+q)]$$

in the coordinates $w,\widehat{w}$ (see Figure 3(a)), and with an equation

$$F(w,\widehat{w})=(w\widehat{w})^p-a\widehat{w}^{p+q}-\overline{a}w^{p+q}+\sum_{\substack{pi+qj>p(p+q)\\ qi+pj>p(p+q)}}a_{ij}w^i\widehat{w}^j=0\ ,$$

where $a\in\mathbb{C}^*$ and $a_{ij}=\overline{a}_{ji}$ for all i,j (cf. (5)). We construct a real morsification of (C,z) combining the result of the preceding step with the patchworking construction as developed in [21, Section 2].

Denote by $\Delta(F)$ the Newton polygon of $F(w,\widetilde{w})$ and divide the domain under $\Gamma(F)$ by the segment $[(0,0),(p,p)]$ into two triangles T_1,T_2 (see Figure 3(b)). So, $\Delta(F)$, T_1, and T_2 form a convex subdivision of the convex polygon $\widetilde{\Delta}(F)=\mathrm{Conv}(\Delta(F)\cup\{(0,0)\})$, i.e., there exists a convex piecewise linear function $\nu:\widetilde{\Delta}(F)\to\mathbb{R}$ taking integral values at integral points and whose linearity domains are $\Delta(F)$, T_1, and T_2. The overgraph $\mathrm{Graph}^+(\nu)$ of ν is a three-dimensional convex lattice polytope, and we have a natural morphism $\mathrm{Tor}(\mathrm{Graph}^+(\nu))\to\mathbb{C}$ whose fibers for $t\in\mathbb{C}^*$ are isomorphic to $\mathrm{Tor}(\widetilde{(F)})$, and the central fiber is the union

$$\mathrm{Tor}(\Delta(F))\cup\mathrm{Tor}(T_1)\cup\mathrm{Tor}(T_2).$$

In the toric surface $\mathrm{Tor}(\Delta(F))$, we have a curve $C=\{F(w,\widehat{w})=0\}$, in the toric surfaces $\mathrm{Tor}(T_1)$ and $\mathrm{Tor}(T_2)$, we define curves

$$R_1=\{(w\widehat{w}-1)^p-\overline{a}w^{p+q}=0\}\quad\text{and}\quad R_2=\{(w\widehat{w}-1)^p-a\widehat{w}=0\}\ ,$$

respectively. The complex conjugation interchanges the pairs $(\mathrm{Tor}(T_1),R_1)$ and $(\mathrm{Tor}(T_2),R_2)$. Note that R_1,R_2 transversally intersect the toric divisors

$$\mathrm{Tor}([(p,p),(p+q,0)])\quad\text{and}\quad\mathrm{Tor}([(p,p),(0,p+q)])$$

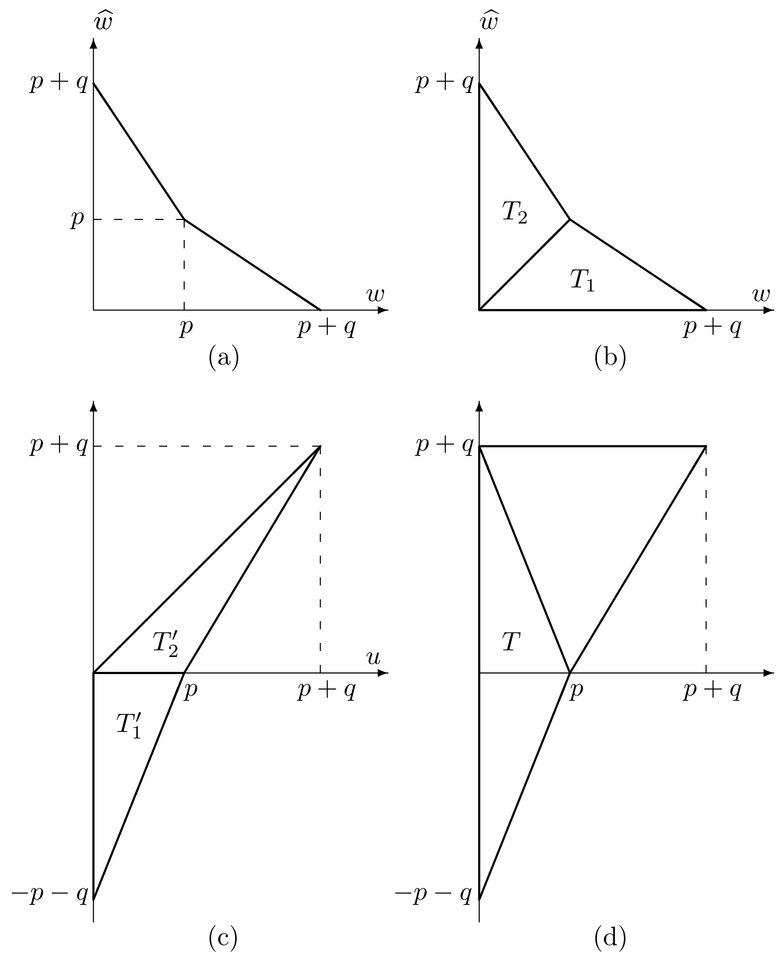


FIGURE 3. Patchworking a real morsification

in the same points as C. Furthermore, R_1, R_2 are rational curves intersecting the toric divisor $\mathrm{Tor}([(0,0),(p,p)] = \mathrm{Tor}(T_1)\cap\mathrm{Tor}(T_2)$ in the same point z_1, where each of them has a singular point of topological type $x^p+y^{p+q}=0$. To apply the patchworking statement of [21, Theorem 2.8], we perform the weighted blow up $\mathfrak{X}\to\mathrm{Tor}(\mathrm{Graph}^+(\nu))$ of the point z_1 with the exceptional divisor $E=\mathrm{Tor}(T)$, $T=\mathrm{Conv}\{(p,0),(0,p+q),(0,-p-q)\}$ (see [21, Figure 1]) being a part of the central fiber of $\mathfrak{X}\to\mathbb{C}$.

One can view this blow up via the refinement procedure developed in [20, Section 3.5]. Namely, we perform the toric coordinate change $u=w\widehat{w}$, $v=w^{-1}$, transforming the triangles T_1,T_2 to T'_1,T'_2 as shown in Figure 3(c), and respectively transforming the curves R_1,R_2 and the function ν. Note that this coordinate change defines an automorphism of the punctures real plane $\mathbb{R}^2\setminus\{0\}$. Next we perform another coordinate change $u=u_1+1$, $v=v_1$, bringing the singular points of R_1,R_2 to the origin and transforming their Newton triangles T'_1,T'_2 into the edge $T''_1=[(p,0),(0,-p-q)]$ and the triangle $T''_2=\mathrm{Conv}\{(0,p+q),(p,0),(p+q,p+q)\}$, respectively (see Figure 3(d)). The triangle $T=\mathrm{Conv}\{(0,-p-q),(0,p+q),(p,0)\}$ corresponds to the exceptional surface, in which we have to define a real curve by an equation with Newton triangle T, having the coefficients at the vertices determined by the equations of R_1 and R_2 and having $(p-1)(p+q) = \delta(C,z) - \mathrm{ImBr}(C,z)$ real hyperbolic nodes. We just borrow the required curve from the special example studied in the first step. Namely, we do the above transformations with the data given by (4), and arrive at the curve given by a polynomial having coefficient a at $(0, p+q)$, coefficient $\overline{a}$ at $(0, -p-q)$, coefficient 1 at $(p, 0)$, and coefficients $b_i^{(0)}$ at $(i, 0)$, $i = 0, ..., p-2$.

To apply [21, Theorem 2.8], we have to verify the following transversality conditions:

- for $i = 1, 2$, the germ at R_i of the family of curves on the surface $\mathrm{Tor}(T_i)$ in the tautological linear system that have a singularity of the topological type $x^p + y^{p+q} = 0$ in a fixed position, is smooth of expected dimension;
- the germ at R of the family of curves on the surface $\mathrm{Tor}(T)$ in the tautological linear system that intersect the toric divisors $\mathrm{Tor}([(0, -p-q), (p, 0)])$ and $\mathrm{Tor}([(p, 0), (0, p+q)])$ in fixed points and have $(p-1)(p+q)$ nodes, is smooth of expected dimension.

Both conditions are particular cases of the S-transvesality property, and they follow from the criterion in [19, Theorem 4.1(1)]. In the former case, one needs the inequality $-R_iK_i > b$, where K_i is the canonical divisor of the surface $\mathrm{Tor}(T_i)$, and b a topological invariant of the singularity defined by

$$b(x^p + y^{p+q} = 0) = \begin{cases} p + (p+q) - 1, & \text{if } q \not\equiv 1 \mod p, \\ p + (p+q) - 2, & \text{if } q \equiv 1 \mod p \end{cases}$$

and the inequality holds, since $-R_iK_i = p + (p+q) + 1$. In the latter case, one needs the inequality

$$R \cdot \mathrm{Tor}([(0, p+q), (0, -p-q)]) > 0$$

(nodes do not count in the criterion), which evidently holds.

Thus, [21, Theorem 2.8] yields the existence of an analytic equivariant deformation of $F(w, \widehat{w})$ defining in $\mathbb{R}B_{C,z}$ curves with $(p-1)(p+q) = \delta(C,z) - \mathrm{ImBr}(C,z)$ hyperbolic nodes. $\square$

Lemma 10. *Let a real singularity (C, z) with exactly two tangent lines $L, \overline{L}$ be admissible along its tangent lines. Then (C, z) possesses a real morsification.*

Proof. We apply construction presented in the proof of Lemmas 8 and 9 for the bunch of smooth branches a nd for pairs of singular complex conjugate branches separately, and we shall show that, for any two pairs $(C_1, \overline{C}_1)$, $(C_2, \overline{C}_2)$ of complex conjugate branches of (C, z), their divides intersect in $2(C_1 \cdot C_2) + 2\mathrm{mt}C_1 \cdot \mathrm{mt}C_2$ (real) points.

For C_1, C_2 smooth this follows from Lemma 8. In other situations, we can assume that $C_1 \cup C_2$ (and $\overline{C}_1 \cup \overline{C}_2$) form a strictly Newton non-degenerate singularity so that C_1 os of topological type $x^p + y^q = 0$ with $2 \le p < q$, $(p, q) = 1$, and C_2 is of topological type $x^{p'} + y^{q'} = 0$ with $1 \le p' < q'$, $(p', q') = 1$.

If $q/p = q'/p'$, then $p = p'$, $q = q'$, and hence $C_1 \cup \overline{C}_1$ and $C_2 \cup C_2$ are given by

$$F(w, \widehat{w}) = (w\widehat{w})^p - a\widehat{w}^{p+q} - \overline{a}w^{p+q} + \sum_{\substack{pi+qj>p(p+q) \\ qi+pj>p(p+q)}} a_{ij} w^i \widehat{w}^j = 0 \ ,$$

and

$$F'(w, \widehat{w}) = (w\widehat{w})^p - a'\widehat{w}^{p+q} - \overline{a}'w^{p+q} + \sum_{\substack{pi+qj>p(p+q) \\ qi+pj>p(p+q)}} a'_{ij} w^i \widehat{w}^j = 0 \ ,$$

respectively, where $a, a', a - a' \in \mathbb{C}^*$. The patchworking construction in the second step of the proof of Lemma 9 can be applied to both the pairs of the branches simultaneously, and the considered question on the intersection of the divides reduces then to the intersection of the curves R, R' in the toric surface $\mathrm{Tor}(T)$, $T = \mathrm{Conv}\{(0,-p-q),(p,0),(0,p+q)\}$. The real parts $\mathbb{R}R, \mathbb{R}R'$ of these curves, in suitable coordinates $\sigma > 0$, $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ are given by

$$\sigma^p + \sum_{i=p-2}^{0} b_i^{(0)}\sigma^i = 2|a|\cos((p+q)\theta - \theta_a), \quad \sigma^p + \sum_{i=p-2}^{0} b_i^{(0)}\sigma^i = 2|a'|\cos((p+q)\theta - \theta_{a'}) \ ,$$

respectively. The number of their (real) intersection points is p times the number of solutions of the equation

$$|a|\cos((p+q)\theta - \theta_a) = |a'|\cos((p+q)\theta - \theta_{a'}), \quad \theta \in \mathbb{R}/2\pi\mathbb{Z} \ .$$

The latter number is $2(p+q)$, and hence the total number of intersection points is

$$2p(p+q) = 2pq + 2p^2 = 2(C_1 \cdot C_2) + 2\mathrm{mt}C_1 \cdot \mathrm{mt}C_2$$

as required.

Suppose that $\tau = \frac{q'}{p'} - \frac{q}{p} > 0$. Then $C_1 \cup \overline{C}_1$ and $C_2 \cup C_2$ are given by

$$F(w,\widehat{w}) = (w\widehat{w})^p - a\widehat{w}^{p+q} - \overline{a}w^{p+q} + \sum_{\substack{pi+qj>p(p+q)\\ qi+pj>p(p+q)}} a_{ij}w^i\widehat{w}^j = 0 \ ,$$

and

$$F'(w,\widehat{w}) = (w\widehat{w})^{p'} - a'\widehat{w}^{p'+q'} - \overline{a}'w^{p'+q'} + \sum_{\substack{p'i+q'j>p'(p'+q')\\ q'i+p'j>p'(p'+q')}} a'_{ij}w^i\widehat{w}^j = 0 \ ,$$

respectively. Along the construction of Lemmas 8 and 9, we substitute in the above equations $(w\widehat{w} - t^2)^p$ for $(w\widehat{w})^p$ and $(w\widehat{w} - t^2)^{p'}$ for $w\widehat{w})^{p'}$, respectively, then make the same rescaling $(w,\widehat{w}) \mapsto (tw, t\widehat{w})$. Next, we pass to the real coordinates σ, θ via

$$\rho^2 = w\widehat{w} = 1 + t^{\frac{q-p}{p}}\sigma, \quad w = \rho\exp(\sqrt{-1}\theta), \ \widehat{w} = \rho\exp(-\sqrt{-1}\theta) \ ,$$

(adapted to the pair p, q, not p', q' !). Then the real morsification of $C_1 \cup \overline{C}_1$ is given by

$$\sigma^p + \sum_{i=p-2}^{0} b_i^{(0)}\sigma^i = 2|a|\cos((p+q)\theta - \theta_a) + O(t^{\frac{1}{p}}) \ ,$$

while the real morsification of $C_2 \cup \overline{C}_2$ is given by $\sigma^{p'} = O(t^{tau})$. The divide of the real morsification of $C_2 \cup \overline{C}_2$ is the circle immersed into the $O(t^{\frac{1}{p'}})$-neighborhood of the level line $\sigma = 0$ in the annulus $\{(\sigma,\theta) \in (-\lambda_0,\lambda_0) \times (\mathbb{R}/2\pi\mathbb{Z})\}$ so that the normal projection onto the circle $\sigma = 0$ is a p'-fold covering. Hence, this divide intersects with the divide of the real morsification of $C_1 \cup \overline{C}_1$ in $2p'(p+q) = 2p'q + 2p'p = 2(C_1 \cdot C_2) + 2\mathrm{mt}C_1 \cdot \mathrm{mt}C_2$ real points.

The case of $\tau = \frac{q}{p} - \frac{q'}{p'} < 0$ can be considered in the same manner. $\square$

2.2.2. *The case of several pairs of complex conjugate tangents.* Suppose now that (C,z) has $r \geq 2$ pairs of complex conjugate tangent lines

$$L_i = \{x + (\alpha_i + \beta_i\sqrt{-1})y = 0\}, \quad \overline{L}_i = \{x + (\alpha_i - \beta_i\sqrt{-1})y = 0\}, \quad i = 1,...,r \ ,$$

where $\alpha_i, \beta_i \in \mathbb{R}$, $\beta_i \neq 0$ for all $i = 1,...,r$. Set

$$w_i = x + (\alpha_i + \beta_i\sqrt{-1})y, \quad \widehat{w}_i = x + (\alpha_i - \beta_i\sqrt{-1})y, \quad i = 1,,,.,r \ .$$

Equations $\rho_i^2 := w_i\widehat{w}_i = \mathrm{const} > 0$, $i = 1,...,r$, define distinct ellipses in $\mathbb{R}^2$, and there are $\gamma_1,...,\gamma_r > 0$ such that each two ellipses $\rho_i^2 = \gamma_i$, $\rho_j^2 = \gamma_j$, $1 \leq i < j \leq r$, intersect in four (real) points, and all $2r(r-1)$ intersection points are distinct.

For any $i = 1, ..., r$, we introduce a real singularity $(C^{(i)}, z)$ formed by the union of all the branches of (C, z) tangent either to L_i, or to $\overline{L}_i$, and then construct a real morsification of $(C^{(i)}, z)$ following the procedure of Section 2.2.1, in which t should be replaced by $t\sqrt{\gamma_i}$. For a given $t > 0$, the divide of this morsification lies in $O(t^{>2})$-neighborhood of the ellipse $\rho_i^2 = \gamma_i t^2$, and it is the union of several immersed circles so that the normal projection onto the ellipse is a covering of multiplicity $\frac{1}{2}\mathrm{mt}(C^{(i)}, z)$. Hence, the divides of the morsifications of $(C^{(i)}, z)$ and $C^{(j)}, z)$, $1 \le i < j \le r$, intersect in $\mathrm{mt}C^{(i)} \cdot \mathrm{mt}C^{(j)}$ real points. So, in total the union of all r divides contains

$$\sum_{i=1}^{r} \left(\delta(C^{(i)}, z) - \mathrm{ImBr}(C^{(i)}, z)\right) + \sum_{1 \le i < j \le r} (C^{(i)} \cdot C^{(j)})_z = \delta(C, z) - \mathrm{ImBr}(C, z)$$

real hyperbolic nodes.

2.3. **Proof of Theorem 1: general case.** Suppose now that (C, z) is a real singularity satisfying hypotheses of Theorem 1. Denote by (C^{re}, z), resp. (C^{im}, z), the union of the branches of (C, z) that have real, resp. complex conjugate tangents.

If $C^{re} = \emptyset$, the existence of a real morsification follows from the results of Sections 2.2.1 and 2.2.2. Assume that $C^{re} \neq \emptyset$, and it contains only smooth branches. We settle this case by induction on $\Delta_3(C, z)$, the maximal δ-invariant of a subgerm of (C^{re}, z) having a unique tangent line. If $\Delta_3(C, z) = 0$, then all branches of (C^{re}, z) are smooth real and transversal to each other. Then we first construct a real morsification of (C^{im}, z) as in Sections 2.2.1 and 2.2.2 with $t > 0$ chosen so small that each branch of (C^{re}, z) intersects the divide of the morsification of (C^{im}, z) in $\mathrm{mt}(C^{im}, z)$ real points. Then we slightly shift the branches of (C^{re}, z) in general position keeping the above real intersection points and obtaining additional $\delta(C^{re}, z)$ hyperbolic nodes as required. In the case of $\Delta_3(C, z) > 0$, we blow up the point z and consider the strict transform of (C^{re}, z), which consists of germ (C_i, z_i) with real centers z_i on the exceptional divisor E. Clearly, for each germ $(C_i \cup E, z_i)$, its branches with real tangents are smooth and transversal to E, and, furthermore, $\Delta_3(C_i \cup E, z_i) < \Delta_3(C, z)$ for all i. Hence, there are real morsifications of the germs $(C_i \cup E, z_i)$, in which we cam assume the germs (E, z_i) to be fixed. Then we blow down E and obtain a deformation of (C^{re}, z) with $\mathrm{mt}(C^{re}, z)$ real smooth transversal branches at z and additional $\delta(C^{re}, z) - \mathrm{ImBr}(C^{re}, z) - \frac{1}{2}\mathrm{mt}(C^{re}, z)(\mathrm{mt}(C^{re}, z) - 1)$ real hyperbolic nodes (cf. computations in Section 2.1.1(1)). Returning back the subgerm (C^{im}, z), we obtain a real singularity at z with $\Delta_3 = 0$, and thus, complete the construction of a real morsification of (C, z) as in the beginning of this paragraph.

Now we get rid of all extra restrictions on (C^{re}, z) and prove the existence of a real morsification of (C, z) by induction on $\Delta_4(C, z)$, which is the δ-invariant of the union of singular branches of (C^{re}, z). The preceding consideration serves as the base of induction. The induction step is precisely the same, and we only notice that (in the above notations)

$$\max \Delta_4(C_i \cup E, z_i) < \Delta_4(C, z).$$

The proof of Theorem 1 is completed.

3. Real morsifications and Milnor fibers

3.1. **A'Campo surface and Milnor fiber.** In [2, Section 3], A'Campo constructs the link of a divide of a real morsification of a singularity (which we call **A'Campo link**). This link is embedded into the 3-sphere, the boundary of the Milnor ball, and the fundamental result by A'Campo [2, Theorem 2] states that it is isotopic to the link of the given singularity in the 3-sphere. In this section, we discuss a somewhat stronger isotopy. Namely, in [2, Section 3], A'Campo associates with a real morsification a surface (which we call **A'Campo surface**), whose boundary is the A'Campo link. It is natural to ask whether the pair (A'Campo surface, A'Campo link) is isotopic to the pair (Milnor fiber, its boundary).

In [2, Page 22], A'Campo conjectures a certain transversality condition for the known morsifications that ensure the discussed transversality. Here we prove this transversality condition for all morsifications constructed in Section 2. We also show that the spoken transversality condition may fail even for morsifications of simple singularities. Hence, the question on the isotopy between the A'Campo surface and the Milnor fiber remains open in a general case.

Let $(C,0)\subset\mathbb{C}^2$ be a real singularity given by an equivariant analytic equation $f(x,y)=0$. Following [2, Section 3], we replace the standard Milnor ball $B(C,0)$ by the bi-disc

$$B(0,\rho_0):=\{u+v\sqrt{-1}\in\mathbb{C}^2\ :\ u,v\in D(0,\rho_0)\subset\mathbb{R}^2\},$$

where $\rho_0>0$ and $\mathbb{C}^2=\mathbb{R}^2\oplus\mathbb{R}^2\sqrt{-1}$. It is easy to verify that $\partial B(0,\rho)$ transversally intersects with C for each $0<\rho\le\rho_0$ if ρ_0 is small enough, and we assume this further on. For $\xi\in\mathbb{C}$ with $0<|\xi|\ll 1$ all curves $M_\xi=\{f(x,y)=\xi\}\subset B(0,\rho_0)$ are smooth and transversally intersect $\partial B(0,\rho_0)$. They are called **Milnor fibers** of the given singularity $(C,0)$. Respectively, the links $LM_\xi=M_\xi\cap\partial B(0,\rho_0)$ are isotopic in the sphere $\partial B(0,\rho_0)$ to the link $L(C,z)=C\cap\partial B(0,\rho_0)$ of the singularity (C,z), and the pairs (M_ξ,LM_ξ), $0<|\xi|\ll 1$, are isotopic in $(B(0,\rho_0),\partial B(0,\rho_0))$.

Introduce the family of bi-discs

$$B'_\rho(0,\rho_0)=\{u+v\sqrt{-1}\in\mathbb{C}^2\ :\ u\in D(0,\rho_0),\ v\in D(0,\rho)\},\quad 0<\rho\le\rho_0\ .$$

By definition, $B'_{\rho_0}(0,\rho_0)=B(0,\rho_0)$. Let $C_t=\{f_t(x,y)=0\}$, $0\le t\le t_0$, $f_0=f$, be a real morsification of $(C,0)$ defined in $B(0,\rho_0)$. Without loss of generality, we can assume that C_t intersects with $\partial B(0,\rho_0)$ transversally for all $0\le t\le t_0$.

We have two families of singular surfaces in $B(0,\rho_0)$:

- $F(\rho)=C_{t_0}\cap B'_\rho(0,\rho_0)$, $0\le\rho\le\rho_0$,
- $R(\rho)=\{u+v\sqrt{-1}\in B'_\rho(0,\rho_0)\ :\ u\in\mathbb{R}C_{t_0},\ v\in T_u\mathbb{R}C_{t_0},\ v\in D(0,\rho)\}$, $0\le\rho\le\rho_0$ (here $\mathbb{R}C_{t_0}\subset D(0,\rho_0)$ is an immersed real analytic curve with nodes, and at each node $u\in\mathbb{R}C_{t_0}$ we understand $T_u\mathbb{R}C_{t_0}$ as the union of the tangent lines to the branches centered at u).

Denote $LF(\rho)=F(\rho)\cap\partial B'_\rho(0,\rho_0)$ and $LR(\rho)=R(\rho)\cap\partial B'_\rho(0,\rho_0)$ for all $0<\rho\le\rho_0$.

Lemma 11. *[cf. [2], Theorem 2] (1) The set $LR(\rho)$ is a link in the sphere $\partial B'(\rho)$ for any $0<\rho\le\rho_0$. The set $LF(\rho)$ is a link in the sphere $\partial B'_\rho(0,\rho_0)$ for all but finitely many values $\rho\in(0,\rho_0]$. Furthermore, $LF(\rho_0)$ is a link equivariantly isotopic in $\partial B(0,\rho_0)$ to the singularity link $L(C,z)$.*

(2) There exists $\rho'=\rho'(t_0)$ such that the links $LF(\rho')$ and $LR(\rho')$ are equivariantly isotopic in $\partial B'_{\rho'}(0,\rho_0)$, and the pairs $(F(\rho'),LF(\rho'))$ and $(R(\rho'),LR(\rho'))$ are equivariantly isotopic in $(B'_{\rho'}(0,\rho_0),\partial B'_{\rho'}(0,\rho_0))$.

Proof. The first statement is straightforward. The second one immediately follows from the fact that $F(\rho)$ and $R(\rho)$ are immersed surfaces having the same real point set with the same tangent planes along it. $\square$

For $\eta>0$ small enough, the algebraic curves

$$F^{sm}(\rho)=\{f_{t_0}(x,y)=\eta\}\cap B'_\rho(0,\rho_0)$$

are smooth for all $\rho'(t_0)\le\rho\le\rho_0$, and each of them is obtained from $F(\rho)$ by a small deformation in a neighborhood U_u of each node $u\in\mathbb{R}C_{t_0}$ that replaces two trasversally intersecting discs with a cylinder. Respectively, for all $\rho'(t_0)\le\rho\le\rho_0$, we define C^∞-smooth equivariant **A'Campo**

surfaces $R^{sm}(\rho) \subset B'_\rho(0,\rho_0)$, obtained from $R(\rho)$ by replacing $R(\rho)\cap U_u$ with the cylinder $F^{sm}(\rho)\cap U_u$ smoothly attached to $R(\rho)\setminus U_u$ for each node $u \in \mathbb{R}C_{t_0}$.

If $\xi \in \mathbb{C}\setminus\{0\}$ with $|\xi|$ small enough, then the intersections $M_\xi \cap \partial B'(\rho)$ are transversal for all $\rho'(t_0) \le \rho \le \rho_0$. We would like to address

Question. *Is the pair $(R^{sm}(\rho_0), LR(\rho_0))$ isotopic to (M_ξ, LM_ξ) in $(B(0,\rho_0),\partial B(0,\rho_0))$, or, equivalently, is the pair $(R^{sm}(\rho'(t_0)), LR(\rho'(t_0)))$ isotopic to $(M_\xi \cap B'(\rho'(t_0)), M_\xi \cap \partial B'_{\rho'(t_0)}(0,\rho_0)$ in $(B'_{\rho'(t_0)}(0,\rho_0), \partial B'_{\rho'(t_0)}(0,\rho_0))$?*

This seems to be stronger that Lemma 11. We would like to comment on this question more. Since $(F^{sm}(\rho_0), F^{sm}(\rho_0)\cap\partial B(0,\rho_0))$ is isotopic to (M_ξ, LM_ξ) in $(B(0,\rho_0),\partial B(0,\rho_0))$, and, by Lemma 11, $(F^{sm}(\rho'(t_0)), F^{sm}(\rho'(t_0))\cap \partial B'_{\rho'(t_0)}(0,\rho_0))$ is (equivariantly) isotopic to $(R^{sm}(\rho'(t_0)), LR(\rho'(t_0)))$ in $B'_{\rho'(t_0)}(0,\rho_0), \partial B'_{\rho'(t_0)}(0,\rho_0))$, the answer to the above Question would be **yes**, if we could prove one of the following claims. Observe that the closure of $R^{sm}(\rho_0)\setminus R^{sm}(\rho'(t_0))$ as well as the closure of $F^{sm}(\rho_0)\setminus F^{sm}(\rho'(t_0))$ is the disjoint union of pairs of discs (corresponding to real branches of (C,z)) and cylinders (corresponding to pairs of complex conjugate branches of (C,z)), and the former surface defines a cobordism of $LR(\rho_0)$ and $LR(\rho'(t_0))$ trivially fibred over $[\rho'(t_0),\rho_0]$. So the requested claims are

(A) The surface $Closure(F^{sm}(\rho_0)\setminus F^{sm}(\rho'(t_0)))$ defines a trivial cobordism of

$$F^{sm}(\rho_0)\cap\partial B(0,\rho_0) \quad \text{and} \quad F^{sm}(\rho'(t_0))\cap\partial B'_{\rho'(t_0)}(0,\rho_0).$$

(B) The intersections $C_t \cap \partial B'_{\rho'(t_0)}(0,\rho_0)$ are trasversal for all $0\le t\le t_0$.

Claim (A) seems to be open in general so far, and it is proved in [18] for morsifications of totally real singularities obtained by the blowing up construction as in [1] (see also [11, Theorem 5.2]). Claim (B) is formulated in [2, Page 22] as a conjecture again for the morsifications of totally real singularities constructed in [1]. However, in general, it does not hold:

Proposition 12. *The totally real singularity (C,z) given by $y^2 - x^{2n} = 0$, $n \ge 4$, possesses a real morsification C_t, $0\le t\le t_0$ such that for arbitrary $0<\rho<\rho_0$ and $0<t<t_0$, there exist $0<\rho'<\rho$ and $0<t'<t$ for which the intersection of $C_{t'}$ and $\partial B'_{\rho'}(0,\rho_0)$ is not transversal.*

Proof. We have $\partial B'_\rho(0,\rho_0) = (\partial D(0,\rho_0)\times D(0,\rho)) \cup (D(0,\rho_0)\times\partial D(0,\rho))$. The intersection of C_t with $\partial D(0,\rho_0)\times D(0,\rho)$ is transversal for any real morsification of (C,z). On the other hand, the intersection of C_t with $D(0,\rho_0)\times\partial D(0,\rho)$ is not transversal at some point

$$p = u + v\sqrt{-1} \in D(0,\rho_0)\times\partial D(0,\rho)$$

if and only if the tangent line to C_t at this point has a real slope. Indeed, if C_t is given in a neighborhood of p by $y=\varphi(x)$, then the lack of transversality of the intersection of C_t and $D(0,\rho_0)\times\partial D(0,\rho)$ at p can be expressed as

$$\mathrm{Im}\frac{d\varphi}{dx}\Big|_p \cdot v_2 = v_1 - \mathrm{Re}\frac{d\varphi}{dx}\Big|_p \cdot v_2 = 0, \quad \text{where } v=(v_1,v_2)\ne 0\ ,$$

and hence $\mathrm{Im}\frac{d\varphi}{dx}\big|_p = 0$. In other words, the lack of transversality means the existence of a real slope tangent line to C_t at a non-real point.

Now we define

$$C_t = \left\{(y-tx^2)^2 - \prod_{k=1}^{n}(x-kt)^2 = 0\right\}, \quad 0\le t\le t_0, \quad 0<t_0\ll 1\ .$$

The real point set of C_t consist of two branches $y = tx^2 \pm \prod_{k=1}^{n}(x-kt)$ transversally intersecting in n points, and hence it is a real morsification. It is easy to compute that the branch

$$y = tx^2 + \prod_{k=1}^{n}(x-kt)$$

has $n-2$ tangent lines with the zero slope at the points

$$x_i(t) = \lambda^i \left(\frac{2}{n}\right)^{1/(n-2)} t^{1/(n-2)}(1+O(t^{>0})), \quad i = 0, ..., n-3\ ,$$

where $\lambda^{n-2} = -1$ is a primitive root of unity. Thus, we obtain at least $n-3$ zero slope tangents at imaginary points. Since $x_i(t) \to 0$ as $t \to 0$, the statement of Proposition follows. $\square$

3.2. **Real Milnor morsifications.** We say that a real morsification of a real singularity (C, z) is a **real Milnor morsification** if in the notation of Section 3.1, the pair $(R^{sm}(\rho_0), LR(\rho_0))$ is isotopic to (M_ξ, LM_ξ) in $(B'_\rho(z,\rho_0), \partial B'_\rho(z,\rho_0))$ for some $0 < \rho \le \rho_0$.

Theorem 2. *Any isolated real plane curve singularity satisfying the hypotheses of Theorem 1 admits a real Milnor morsification.*

Proof. We prove the theorem by establishing Claim (B) formulated in the preceding section.

Let (C, z) be a real singularity as in Theorem 1. Applying a suitable local diffeomorphism, we can assume that (C, z) does not contain (segments of) straight lines, and hence $(L \cdot C)_z < \infty$ for any line L through z. Denote by Λ the union of all real tangent lines to (C, z) at z. Under the assumption made, we apply the construction used in the proof of Theorem 1 and obtain a real morsification of $(C \cup \Lambda, z)$, in which Λ remains fixed. Then we get rid of Λ and obtain a real morsification C_t, $0 \le t \le t_0$, of (C, z). We shall show that it is a real Milnor morsification (possibly replacing t_0 with a smaller positive number).

As noticed in the proof of Proposition 12, the required property is equivalent to the absence of non-real lines with real slopes tangent to C_t, $0 \le t \le t_0$.

Our first observation is

Lemma 13. *Let (C, z) be a real singularity, L a real line passing through z and intersecting (C, z) only at z (in the Milnor ball), with a finite multiplicity $(L \cdot C)_0$. Denote by $\mathcal{P}_L$ the germ of the pencil of the lines parallel to L and by $\mathbb{R}\mathcal{P}_L$ its real point set. Let C_t, $0 \le t < \varepsilon$, be a real morsification of (C, z) as above, and let C_t and L_t intersect in $(L \cdot C)_z$ real points for any $t \in (0, \varepsilon)$. Then each line $L' \in \mathcal{P}_L \setminus \mathbb{R}\mathcal{P}_L$ intersects each element C_t, $0 < t < \varepsilon$, transversally.*

Proof. Let C' be a Milnor fiber. Then the lines of $\mathcal{P}_L$ in total are tangent to C' in

$$\kappa(C, z) + (L \cdot C)_z - \mathrm{mt}(C, z)$$

points, where $\kappa(C, z)$ is the class of the singularity (C, z) (see, for example, [13, Section I.3.4] for details). Since, for a node, $\kappa = 2$, and in general

$$\kappa(C, z) = 2\delta(C, z) + \mathrm{mt}(C, z) - \mathrm{Br}(C, z),$$

we get that the lines of $\mathcal{P}_L$ in total are tangent to C_t in

$$\kappa(C, z) + (L \cdot C)_z - \mathrm{mt}(C, z) - 2(\delta(C, z) - \mathrm{ImBr}(C, z)) = (L \cdot C)_z - \mathrm{ReBr}(C, z)$$

points. It follows that

- L intersects the morsification $C_{i,t}$ of any real branch (C_i, z) of (C, z) in $(L \cdot C_i)_z$ real points, while the real point set $\mathbb{R}C_{i,t}$ of $C_{i,t}$ is an immersed segment; that is, L cuts $\mathbb{R}C_{i,t}$ into $(L \cdot C_i) + 1$ immersed segments, among all but two have both endpoints on $\mathbb{R}L$; hence, varying L in $\mathbb{R}\mathcal{P}_L$, we encounter at least $(L \cdot C_i)_z - 1$ real tangency points;

- L intersects the morsification $C_{j,t}$ of a pair of complex conjugate branches (C_j, z), $(\overline{C}_j, z)$ of (C, z) in $2(L \cdot C_i)_z$ real points, and hence it cuts $\mathbb{R}C_{j,t}$ (which is an immersed circle) into $2(L \cdot C_i)_z$ immersed segments, whose all endpoints lie on $\mathbb{R}L$, and hence, varying L in $\mathbb{R}\mathcal{P}_L$, we encounter at least $2(L \cdot C_i)_z$ real tangency points.

The claim of Lemma follows. □

Remark that, under conditions of Lemma 13, there is an open neighborhood U_L of L in the dual plane $\mathbb{P}^{2,\vee}$ such that all non-real curves with real slopes intersect each curve C_t, $0 < t < \varepsilon$, transversally. Thus, Theorem 2 follows from

Lemma 14. *For any real line L through z, there exist $0 < \rho \le \rho_0$ satisfying the following conditions*

- $L \cap C \cap B'_\rho(z, \rho_0) = \{z\}$;
- *for some $\varepsilon > 0$, L intersects with any curve C_t, $0 < t < \varepsilon$, in $(L \cdot C)_z$ real points (counting multiplicities).*

Proof. Let $L_1, ..., L_k$ be all real tangent lines to (C, z) at z. Write $(C, z) = \bigcup_i (C_i, z)$, where (C_i, z) either has a unique (real) tangent line, or a pair of complex conjugate tangent lines, and (C_i, z), (C_j, z) have no tangent in common as $i \ne j$. We can consider morsifications of (C_i, z) separately.

Suppose that (C_i, z) has a pair of complex conjugate tangent lines. The morsification of (C_i, z) constructed in Section 2.2.1 is such that the real point set of C_t, $0 < t < \varepsilon$, consists of one or several immersed circles going in total $\frac{1}{2}\mathrm{mt}(C_i, z)$ times around z, and hence L (which is transversal to (C_i, z), i.e. $(L \cdot C_i)_z = \mathrm{mt}(C_i, z)$) intersects any curve C_t in $\mathrm{mt}(C_i, z)$ real points (counting multiplicities).

Suppose that (C_i, z) has a unique (real) tangent line L_z, and $L \ne L_z$. Then

$$(L \cdot C_i)_z = \mathrm{mt}(C_i, z).$$

The smooth real branches of (C_i, z) are deformed in any morsification so that they remain transversal to L and intersect L at one real point. For (C'_i, z), the union of the other branches of (C_i, z), the construction of a morsification presented in Section 2.3 goes inductively. Namely, we blow up z, construct a morsification of the strict transform of (C_i, z) united with the exceptional divisor and then blow down the exceptional divisor. Elements of this intermediate deformation have $\mathrm{mt}(C'_i, z)$ smooth real branches centered at z, all transversal to L, and in any further deformation they intersect with L in $\mathrm{mt}(C'_i, z)$ real points.

If (C_i, z) has a unique (real) tangent line L_z, and $L = L_z$, the statement follows from the construction. □

Therefore, we have proved Theorem 2. □

4. A'Campo-Gusein-Zade diagrams and topology of singularities

4.1. **AΓ-diagrams of real morsifications.** L. Balke and R. Kaenders proved [7, Theorem 2.5 and Corollary 2.6] that the A'Campo-Gusein-Zade diagram (briefly, AΓ-diagram) associated with a morsification of a totally real singularity determines the complex topological type of the given singularity. Here we extend this result to real morsifications of arbitrary real singularities. We get rid of the requirement for morsifications to define a partition (see Section 1 and [7, Definition 1.2]) and prove that an AΓ-diagram determines the topological type of the singularity as well as some additional information on its real structure.

Let us recall definitions from [5] and [7].

A subset D of a closed disc $\boldsymbol{D} \subset \mathbb{R}^2$ is called a **connected divide** if it is the image of an immersion of a disjoint union $\Sigma \neq \emptyset$ of a finite number of segments $I = [0, 1]$ and circles S^1 satisfying the following conditions:

- the set of the endpoints of all the segments in Σ is injectively mapped to $\partial\boldsymbol{D}$, whereas the other points of Σ are mapped to the interior of $\boldsymbol{D}$;
- each point of the complement $D \setminus \mathrm{Sing}(D)$ to a finite set $\mathrm{Sing}(D)$ has a unique preimage in Σ, each point of $\mathrm{Sing}(D)$ is a transversal intersection of two smooth local branches;
- the images of any two connected components of Σ intersect each other.

Note that Σ is uniquely determined by D. The image of any connected component of Σ is a divide, which is called a **branch** of the divide D.

The divide of a real morsification of a real singularity placed in the real Milnor disc (see Section 1) is a connected divide in the above sense.

Connected components of $\overline{D} \setminus D$ and of $D \setminus \mathrm{Sing}(D)$, disjoint from $\partial\boldsymbol{D}$, are called inner components. Clearly, each inner component of $\boldsymbol{D} \setminus D$ is homeomorphic to an open disc, and each inner component of $D \setminus \mathrm{Sing}(D)$ is homeomorphic either to an open interval, or to S^1 if $D \simeq S^1$.

It is straightforward that the set $\pi_0(\boldsymbol{D} \setminus D)$ of the connected components of $\boldsymbol{D} \setminus D$ can be 2-colored, i.e., there exists a function $\pi_0(\boldsymbol{D} \setminus D) \to \{\pm 1\}$ such that the components, whose boundaries intersect along one-dimensional pieces of D, have different signs, and there are precisely two functions like that (cf. [7, Proposition 1.4]). Fix a 2-coloring $s : \pi_0(\boldsymbol{D} \setminus D) \to \{\pm 1\}$. The **A'Campo-Gusein-Zade diagram** (AΓ**-diagram**) of a connected divide D is a 3-colored graph $\mathrm{A}\Gamma(D) = (V, E, c)$ such that

- the set V of its vertices is in one-to-one correspondence with the disjoint union of $\mathrm{Sing}(D)$ (the set of $\bullet$-vertices in the notation of [7]) and the set $\pi_0^{inn}(\boldsymbol{D} \setminus D)$ of the inner components of $\boldsymbol{D} \setminus D$ (the $\oplus$-vertices and $\ominus$-vertices in the notation of [7] in accordance with the chosen coloring);
- two distinct vertices $K_1, K_2 \in \pi_0^{inn}(\boldsymbol{D} \setminus D)$ such that $\partial K_1 \cap \partial K_2 \setminus \mathrm{Sing}(D) \neq \emptyset$ are joined by k edges, where k is the number of inner components of $D \setminus \mathrm{Sing}(D)$ inside $\partial K_1 \cap \partial K_2$;
- two vertices $K \in \pi_0^{inn}(\boldsymbol{D} \setminus D)$ and $p \in \mathrm{Sing}(D)$ such that $p \in \partial K$ are joined by k edges, where k is the number of components of the intersection of K with a small disc centered at p (clearly, here $k = 1$ or 2);
- the 3-coloring $c : V \to \{\pm 1, 0\}$ is defined by $c(K) = s(K)$, $K \in \pi_0^{int}(\boldsymbol{D} \setminus D)$, and $c(p) = 0$, $p \in \mathrm{Sing}(D)$.

Comparing with [7, Definition 1.5], we admit multi-graphs, i.e., vertices can be joined by several edges, while this is excluded in [7, Definition 1.5] by the partition requirement. On the other hand, there are no loops. By construction, the AΓ-diagram can be embedded into D (cf. [7, Remark in page 43]).

The AΓ-diagram associated with the divide of a real morsification of a real singularity is simply called an AΓ-diagram of that singularity.

4.2. AΓ**-diagram determines the weak real topological type of a singularity.** The topological type of a real singularity (C, z) is its equivalence class up to a homomorphism of the Milnor ball, and it is known [8, 23] (see also [9, Section 8.4]) that the topological type of a given singularity is determined by the collections of Puiseux pairs of its branches and by pairwise intersection numbers of the branches. We introduce the **weak real topological type** of (C, z) to be the topological type enriched with the following information:

- indication of real branches and pairs of complex conjugate branches;
- the cyclic order of real branches, that is, if (C, z) has $k \geq 1$ real branches, we number them somehow and introduce the cyclic order on the multiset $\{1, 1, 2, 2, ..., k, k\}$ induced

by the position of the $2k$ intersection points of the real branches with the circle $\partial\mathbb{R}B_{C,z}$ and defined up to reversing the orientation of $\partial\mathbb{R}B_{C,z}$ and renumbering the topological types of the real branches, their mutual intersection multiplicities and their intersection multiplicities with non-real branches.

Theorem 3. *An* $\mathrm{A}\Gamma$*-diagram of an arbitrary real singularity determines its weak real topological type.*

Proof. Balke and Kaenders [7] proved that the $\mathrm{A}\Gamma$-diagram determines the topological type of a totally real singularity, and we closely follow the lines of their proof referring for details to [7, Section 2] and presenting necessary modifications for the general case.

First, we remark that the partition requirement (see Section 1) was not, in fact, used in [7]. In particular, it is not needed in the construction of the Coxeter-Dynkin diagram from the given divide as presented in [15].

(1) The main step in the proof of [7, Theorem 2.5 and Corollary 2.6] is to show that an $\mathrm{A}\Gamma$-diagram of a totally real singularity determines the branch structure of the divide, pairwise intersection numbers of the branches, and an $\mathrm{A}\Gamma$-diagram of each branch. Their argument literally applies in the general case. We notice in addition that one can easily distinguish between $\mathrm{A}\Gamma$-diagrams of non-closed and closed branches of the divide, i.e., between an $\mathrm{A}\Gamma$-diagram of a real branch of (C, z) and an $\mathrm{A}\Gamma$-diagram of a pair of complex conjugate branches. Namely, in the former case, the $\mathrm{A}\Gamma$-diagram contains either a univalent $\bullet$-vertex, or a bivalent $\bullet$-vertex joined with a $\oplus$-vertex and $\ominus$-vertex, while in the latter case, the $\mathrm{A}\Gamma$-diagram has no such $\bullet$-vertices.

We only comment on the persistence of the cyclic order of real branches of the singularity (aka, non-closed branches of the divide). An embedding of the $\mathrm{A}\Gamma$-diagram into $\mathbb{R}B_{C,z}$ defines the divide up to isotopy (see [7, Page 46]). The ambiguity in the construction of an embedding is related to the existence of the so-called **chains** in the $\mathrm{A}\Gamma$-diagram, i.e., connected subgraphs consisting of bivalent or univalent $\bullet$-vertices and bivalent $\oplus$-vertices (or bivalent $\ominus$-vertices) joined by arcs as shown in Figure 4(a) (cf. [7, Figure 6]). Figure 4(b) shows the corresponding fragment of the divide (cf. [7, Figure 7]). By [7, Lemma 2.8], the given $\mathrm{A}\Gamma$-diagram can be transformed by inserting new chains and extending the existing ones in a controlled way into a **chain separating** $\mathrm{A}\Gamma$-diagram, whose maximal (with respect to inclusion) chains have pairwise distinct lengths, and no new chain can be added.

Each chain of a divide shares the boundary with two non-inner components of the complement to the divide, and the disc $\mathbb{R}B_{C,z}$ can be cut into three parts as shown in Figure 4(b) by dashed lines (cf. [7, Figure 7]), and similarly one can cut $\mathbb{R}B_{C,z}$ with respect to the embedded chain of the $\mathrm{A}\Gamma$-diagram, Figure 4(a). Then a given embedding of a chain separating $\mathrm{A}\Gamma$-diagram can be changed in part A or in part B by a reflection with respect to the axis of the chain (and so for any other maximal chain). Note that the branches of the divide, which are disjoint from the chain of the divide, must all lie either in part A, or in part B, since any two of them must intersect each other. In the presence of such branches, located, say, in part A, and under the assumption that the chain is formed by two branches of the divide, all possible self-intersections of the latter branches must lie in part A too due to Lemma 3(i) applied to the divide with one of these two branches removed. All these observations yield that the cyclic order of non-closed branches of the divide is preserved under the changes of the embedding of the chain separating $\mathrm{A}\Gamma$-diagram described above. Finally, we note that the same cycling order of the divide is induced by the corresponding embedding of the original $\mathrm{A}\Gamma$-diagram.

(2) The topological type a real branch of the given singularity can be recovered from its $\mathrm{A}\Gamma$-diagram, see [7, Theorem 1.9]. In a similar way, we show that an $\mathrm{A}\Gamma$-diagram of a closed branch of the divide determines the topological type of a real singularity formed by a pair of

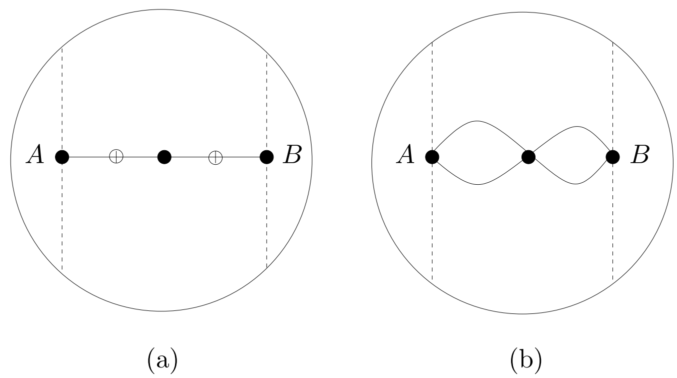


FIGURE 4. Chains of an AΓ-diagram and of a divide

complex conjugate branches. Namely, an AΓ-diagram defines the monodromy operator of such a singularity, see [4] and [16, Page 39], and hence its characteristic polynomial, which is the reduced Alexander polynomial of the link of the singularity [17, §8] (see also [22, Theorem 3.3]). Thus, we complete the proof with the following statement which is a particular case of [10, Proposition 3.2].

Lemma 15. *The reduced Alexander polynomial of a singularity formed by two topologically equivalent branches determines the topological type of the branches and their intersection multiplicity.*

(3) To complete the recovery of the topological type of the given singularity (C, z), we have to find pairwise intersection multiplicities of the branches of (C, z). By [7, Lemma 2.2], the intersection number of two non-closed branches of the divide equals the intersection multiplicity of the corresponding real branches of (C, z). Similarly, the intersection number of a non-closed and a closed branches of the divide equals twice the intersection multiplicity of the corresponding real branch of (C, z) with each of the two complex conjugate branches of (C, z) corresponding to the closed branch of the divide. At last, consider the intersection of two closed branches of the divide and suppose without loss of generality that these are the only branches of the divide. From Lemma 15 we know the topological type and the intersection multiplicity of complex conjugate branches of (C, z) associated with each of the branches of the divide. We claim that this information together with the intersection number of the branches of the divide determines the pairwise intersection multiplicities of all four branches of (C, z). Indeed, this can easily be proved by induction on the number of real infinitely near points in the resolution tree of (C, z). □

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SMOOTH MIXED PROJECTIVE CURVES AND A CONJECTURE

MUTSUO OKA

Dedicated to Professor Egbert Brieskorn

Abstract. Let $f(\mathbf{z},\bar{\mathbf{z}})$ be a strongly mixed homogeneous polynomial of three variables $\mathbf{z}=(z_1,z_2,z_3)$ of polar degree q with an isolated singularity at the origin. It defines a smooth Riemann surface C in the complex projective space $\mathbb{P}^2$. The fundamental group of the complement $\pi_1(\mathbb{P}^2\setminus C)$ is a cyclic group of order q if f is a homogeneous polynomial without $\bar{\mathbf{z}}$. We propose a conjecture that this may be even true for mixed homogeneous polynomials by giving several supporting examples.

1. Introduction

Let $f(\mathbf{z},\bar{\mathbf{z}})=\sum_{\nu,\mu}c_{\nu,\mu}\mathbf{z}^\nu\bar{\mathbf{z}}^\mu$ be a mixed polynomial of n-variables $\mathbf{z}=(z_1,\dots,z_n)\in\mathbb{C}^n$. A mixed polynomial $f(\mathbf{z},\bar{\mathbf{z}})$ is called *mixed weighted homogeneous* if there exist integers $q_1,\dots,q_n$ and $p_1,\dots,p_n$ and non-zero integers m_r, m_p such that

$$\gcd(q_1,\dots,q_n)=1,\quad \gcd(p_1,\dots,p_n)=1,$$
$$\textstyle\sum_{j=1}^n q_j(\nu_j+\mu_j)=m_r,\quad \sum_{j=1}^n p_j(\nu_j-\mu_j)=m_p,\quad \text{if } c_{\nu,\mu}\neq 0$$

We say $f(\mathbf{z},\bar{\mathbf{z}})$ is *mixed weighted homogeneous of radial weight type* $(q_1,\dots,q_n;m_r)$ *and of polar weight type* $(p_1,\dots,p_n;m_p)$.

Using polar coordinates r, η of $\mathbb{C}^*$ where $r>0$ and $\eta\in S^1$ with $S^1=\{\eta\in\mathbb{C}\,|\,|\eta|=1\}$, we define *a polar* $\mathbb{C}^*$*-action* on $\mathbb{C}^n$ by

$$(r,\eta)\circ\mathbf{z}=(r^{q_1}\eta^{p_1}z_1,\dots,r^{q_n}\eta^{p_n}z_n),\quad (r,\eta)\in\mathbb{R}^+\times S^1$$
$$(r,\eta)\circ\bar{\mathbf{z}}=\overline{(r,\eta)\circ\mathbf{z}}=(r^{q_1}\eta^{-p_1}\bar{z}_1,\dots,r^{q_n}\eta^{-p_n}\bar{z}_n).$$

Then f satisfies the Euler equality

$$f((r,\eta)\circ(\mathbf{z},\bar{\mathbf{z}}))=r^{m_r}\eta^{m_p}f(\mathbf{z},\bar{\mathbf{z}}).\tag{E1}$$

It is easy to see that such a polynomial defines a global fibration

$$f:\mathbb{C}^n-f^{-1}(0)\to\mathbb{C}^*\tag{GM}$$

without further assumption. A mixed polynomial $f(\mathbf{z},\bar{\mathbf{z}})$ is called a *strongly mixed weighted homogeneous polynomial* (respectively *strongly mixed homogeneous polynomial*) of n-variables $\mathbf{z}=(z_1,\dots,z_n)\in\mathbb{C}^n$ with polar degree q and radial degree d if $p_i=q_i$ for $i=1,\dots,n$ and $\sum_{i=1}p_i(\nu_i\pm\mu_i)=d$ and q (resp. $p_i=q_i=1$, $i=1,\dots,n$ and $|\nu|+|\mu|=d$ and $|\nu|-|\mu|=q$) for any ν,μ with $c_{\nu,\mu}\neq 0$. Here q is assumed to be a positive integer. For such a strongly mixed weighted homogenous polynomial, the associated $\mathbb{C}^*$-action on $\mathbb{C}^n$ is the holomorphic action defined by

$$(\zeta,(z_1,\dots,z_n))\mapsto\zeta\circ\mathbf{z}=(\zeta^{p_1}z_1\dots,\zeta^{p_n}z_n)$$

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and f satisfies the equality

$$f(\lambda\circ\mathbf{z},\overline{\lambda\circ\mathbf{z}})=r^d\exp{(iq\theta)}f(\mathbf{z},\bar{\mathbf{z}}),\quad \text{where } \lambda=r\exp(i\theta)\in\mathbb{R}^+\times S^1.$$

Assume that f is strongly mixed homogeneous. Then the action is reduced to

$$\lambda\circ\mathbf{z}\,=\,(\lambda z_1,\dots,\lambda z_n).$$

By the above equality, it defines canonically a real analytic projective variety V in $\mathbb{P}^{n-1}$:

$$V=\{[\mathbf{z}]\in\mathbb{P}^{n-1}\mid f(\mathbf{z},\bar{\mathbf{z}})=0\}.$$

Let $\widetilde{V}$ be the mixed affine hypersurface

$$\widetilde{V}=f^{-1}(0)=\{\mathbf{z}\in\mathbb{C}^n\mid f(\mathbf{z},\bar{\mathbf{z}})=0\}.$$

Let $f:\mathbb{C}^n\setminus\widetilde{V}\to\mathbb{C}^*$ be the global Milnor fibration defined by f and let F be the Milnor fiber, namely $F=f^{-1}(1)\subset\mathbb{C}^n$. The monodromy map $h:F\to F$ is defined by

$$h(\mathbf{z})=(\omega_q z_1,\dots,\omega_q z_n),\quad \omega_q=\exp(\frac{2\pi\, i}{q}).$$

and the restriction of the Hopf fibration to the Milnor fiber $\pi:\ F\to\mathbb{P}^{n-1}\setminus V$ is nothing but the quotient map by the cyclic action induced by h.

Remark 1. *We may also consider the case $q=0$ in the above strongly mixed homogeneous polynomial and consider the corresponding projective variety $V=\{[\mathbf{z}]\,|\,f(\mathbf{z},\bar{\mathbf{z}})=0\}$ but V need not be a real codimension 2 hypersurface. For example, $n=3$ and take*

$$f(\mathbf{z},\bar{\mathbf{z}}):=z_1\bar{z}_1-z_2\bar{z}_2+z_3\bar{z}_3.$$

Then $\dim_{\mathbb{R}}V=3$. Note that f does not have a Milnor fibration if $q=0$. Another extreme case is $g(\mathbf{z},\bar{\mathbf{z}}):=z_1\bar{z}_1+z_2\bar{z}_2+z_3\bar{z}_3$. Then $\tilde{V}=\{\mathbf{0}\}$ and V is empty. Such a polynomial is called a fake *strongly mixed homogeneous polynomial.*

A strongly mixed homogeneous polynomial is called a *true* strongly mixed homogeneous polynomial if f does not have any fake strongly mixed homogeneous factor in the polynomial ring $\mathbb{C}[z_1,\bar{z}_1,\dots,z_n,\bar{z}_n]$ which defines a non-empty projective variety.

In [10, 14], we have shown that

Theorem 2 (Theorem 11, [14])**.** *Assume that $f(\mathbf{z},\bar{\mathbf{z}})$ is a non-degenerate, strongly mixed homogeneous polynomial of n variables such that V is irreducible and mixed non-singular in an open dense subset. Then the embedding degree of V is equal to the polar degree q. In particular, $H_1(\mathbb{P}^{n-1}\setminus V)=\mathbb{Z}/q\mathbb{Z}$.*

Here "irreducible" means as a real algebraic variety.

Proposition 3. *Assume that $f(\mathbf{z},\bar{\mathbf{z}})$ is a non-degenerate, strongly mixed homogeneous polynomial of n variables such that V is irreducible and mixed non-singular in an open dense subset. Then the Euler characteristics satisfy the following equalities.*

(1) *$\chi(F)=q\,\chi(\mathbb{P}^{n-1}\setminus V)$ and $\chi(\mathbb{P}^{n-1}\setminus V)=n-\chi(V)$. In particular, if $n=3$ and V is smooth curve with the genus g, then $\chi(F)=q(1+2g)$.*

(2) *The following sequence is exact.*

$$1\to\pi_1(F)\xrightarrow{\pi_\sharp}\pi_1(\mathbb{P}^{n-1}\setminus V)\to\mathbb{Z}/q\mathbb{Z}\to 1.$$

In particular, F is simply-connected if and only if $\pi_1(\mathbb{P}^{n-1}\setminus V)\cong\mathbb{Z}/q\mathbb{Z}$.

(3) *If $q=1$, the projection $\pi:F\to\mathbb{P}^{n-1}\setminus V$ is a diffeomorphism.*

Remark 4. *The assumption that V is irreducible as a real algebraic variety is different from the irreduciblity of f in $\mathbb{C}[z_1, \bar{z}_1, \dots, z_n, \bar{z}_n]$.*

Using the periodic monodromy argument in [7], we have

Proposition 5. *Assume that f is a strongly mixed homogeneous polynomial of polar degree $q > 0$. The zeta function of the monodromy $h : F \to F$ is given by*

$$\zeta(t) = (1 - t^q)^{-\chi(F)/q}.$$

In particular, if $q = 1$, $h = \mathrm{id}_{\mathrm{F}}$ and $\zeta(t) = (1 - t)^{-\chi(F)}$.

If f is a holomorphic function with an isolated singularity at the origin, F is $(n-2)$-connected and it is homotopic to a bouquet of μ spheres of dimension $n - 1$ ([7]). For mixed polynomials, we do not have any connectivity theorem. But we do not have any examples of mixed weighted homogeneous polynomials which break the connectivity which holds in the holomorphic case. Thus we propose the following conjecture as a first working problem.

simply-connectedness Conjecture 6. *Assume $n \geq 3$ and that $f : \mathbb{C}^n \to \mathbb{C}$ is a non-degenerate strongly mixed homogeneous polynomial of polar degree q which has an isolated singularity at the origin. In other words, V is an irreducible mixed non-singular hypersurface of real codimension 2. Then*

(a) The Milnor fiber F is simply-connected.

By (2) of Proposition 3, this conjecture is equivalent to the following.

(b) The fundamental group of the complement $\pi_1(\mathbb{P}^{n-1} - V)$ is a cyclic group of order q.

The purpose of this paper is to give several non-trivial examples for the case $n = 3$ which support this conjecture.

Remark 7. *The condition "strongly non-degenerate" (with respect to the Newton boundary), introduced in [15], is necessary to have a Milnor fibration for a non-mixed weighted homogeneous polynomial. However for a mixed weighted homogeneous polynomial, the Milnor fibration (GM) always exists.*

For a mixed weighted homogneous polynomial, the notion 'non-degenerate' implies 'strongly non-degenerate'. We explain this assertion for a strongly mixed weighted homogenous polynomial f for simplicity. Take any face function f_Δ of f. f_Δ is also strongly mixed homogeneous and satisfies the Euler equality

$$f_\Delta(\lambda \circ \mathbf{z}, \overline{\lambda \circ \mathbf{z}}) = r^d \exp(iq\theta) f_\Delta(\mathbf{z}, \bar{\mathbf{z}}), \quad \text{where } \lambda = r\exp(i\theta) \in \mathbb{R}^+ \times S^1.$$

Take any point $\mathbf{w} \in \mathbb{C}^{*n} \setminus V(f_\Delta)$. Consider two tangent vectors $\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta} \in T_{\mathbf{w}}\mathbb{C}^{*n}$. By the above equality, it is easy to see that their images by $df_\Delta : T_{\mathbf{w}}\mathbb{C}^{*n} \to T_{f_\Delta(\mathbf{w})}\mathbb{C}^*$ are linearly independent. That is, non-zero numbers are regular values for f_Δ. (For a mixed weighted homogeneous polynomial, we use the equality (E1) and do the same argument.)

In Theorem 2 and Proposition 3, to make $\mathbb{P}^{n-1} \setminus V$ connected, we have to assume that V has no real codimension 1 component. This does not happen if f is non-degenerate or 'true'. To make the Milnor fiber F connected, we have to assume that $\dim_{\mathbb{R}} V = 2n - 4$ and the existence of a mixed smooth point (see (1) of Theorem 9). Thus in Theorem 2 and Proposition 3, we can replace the assumption on f by the assumption that f is a true strongly mixed homogeneous polynomial such that V is irreducible and mixed non-singular in an open dense subset.

2. Easy mixed polynomials

Unlike the holomorphic case, we do not know in general the connectivity of the Milnor fiber even under the assumption that $\tilde{V}$ has an isolated singularity at the origin. In this section, we study easy examples. Suppose that f is either a simplicial mixed polynomial or a join type or twisted join type polynomial of three variables. Then the connectivity behaves just as the holomorphic case. We will first explain these polynomials below.

2.1. **Simplicial polynomial.** Assume that $n = 3$ and $\mathbf{z} = (z_1, z_2, z_3)$. A mixed polynomial $f(\mathbf{z}, \bar{\mathbf{z}})$ is called simplicial if it is a linear sum of three mixed monomials

$$f(\mathbf{z}, \bar{\mathbf{z}}) = \sum_{i=1}^{3} c_i \mathbf{z}^{\nu_i} \bar{\mathbf{z}}^{\mu_i}$$

and if the two matrices

$$(\nu_i \pm \mu_i)_{i=1}^3 = \begin{pmatrix} \nu_{11} \pm \mu_{11} & \nu_{12} \pm \mu_{12} & \nu_{13} \pm \mu_{13} \\ \nu_{21} \pm \mu_{21} & \nu_{22} \pm \mu_{22} & \nu_{23} \pm \mu_{23} \\ \nu_{31} \pm \mu_{31} & \nu_{32} \pm \mu_{32} & \nu_{33} \pm \mu_{33} \end{pmatrix}$$

are non-degenerate where $\nu_i = (\nu_{i1}, \nu_{i2}, \nu_{i3})$, $\mu_i = (\mu_{i1}, \mu_{i2}, \mu_{i3})$. In this case, we may assume that $c_i = 1$ for $i = 1, 2, 3$. Among them, the following polynomials are strongly mixed homogeneous and have an isolated singularity at the origin.

$$\begin{aligned}
f_B &:= z_1^{q+r}\bar{z}_1^r + z_2^{q+r}\bar{z}_2^r + z_3^{q+r}\bar{z}_3^r, \text{ (Brieskorn Type)} \\
f_I &:= z_1^{q+r-1}\bar{z}_1^r z_2 + z_2^{q+r-1}\bar{z}_2^r z_3 + z_3^{q+r}\bar{z}_3^r, \text{ (Tree type a)} \\
f_{II} &:= z_1^{q+r-1}\bar{z}_1^r z_2 + z_2^{q+r-1}\bar{z}_2^r z_3 + z_3^{q+r-1}\bar{z}_3^r z_1, \text{ (Cyclic type a)} \\
f_{III} &:= z_1^{q+r-1}\bar{z}_1^r z_2 + z_2^{q+r-1}\bar{z}_2^r z_1 + z_3^{q+r}\bar{z}_3^r, \text{ (Simplicial+Join a)} \\
f_I' &:= z_1^{q+r}\bar{z}_1^{r-1}\bar{z}_2 + z_2^{q+r}\bar{z}_2^{r-1}\bar{z}_3 + z_3^{q+r}\bar{z}_3^r, \text{ (Tree type b)} \\
f_{II}' &:= z_1^{q+r}\bar{z}_1^{r-1}\bar{z}_2 + z_2^{q+r}\bar{z}_2^{r-1}\bar{z}_3 + z_3^{q+r}\bar{z}_3^{r-1}\bar{z}_1, \text{ (Cyclic type b)} \\
f_{III}' &:= z_1^{q+r}\bar{z}_1^{r-1}\bar{z}_2 + z_2^{q+r}\bar{z}_2^{r-1}\bar{z}_1 + z_3^{q+r}\bar{z}_3^r, \text{ (Simplicial+Join b).}
\end{aligned}$$

Here $q \geq 1$ and $r \geq 1$ are positive integers. All above polynomials have simply-connected Milnor fibers ([9]). For f_B, f_I, f_{II}, f_{III}, their Milnor fiberings and links are in fact isotopic to the holomorphic ones by the contraction $z_i^r \bar{z}_i^r \mapsto 1$ ([11, 5]):

$$\begin{aligned}
f_B &:= z_1^q + z_2^q + z_3^q, \text{ (Brieskorn Type)} \\
f_I &:= z_1^{q-1} z_2 + z_2^{q-1} z_3 + z_3^q, \text{ (Tree type a)} \\
f_{II} &:= z_1^{q-1} z_2 + z_2^{q-1} z_3 + z_3^{q-1} z_1, \text{ (Cyclic type a)} \\
f_{III} &:= z_1^{q-1} z_2 + z_2^{q-1} z_1 + z_3^q, \text{ (Simplicial+Join a).}
\end{aligned}$$

Remark 8. *The above list does not cover all possibilities. For example, we can combine f_I and f_I':*

$$f_I'' := z_1^{q+r}\bar{z}_1^{r-1}\bar{z}_2 + z_2^{q+r-1}\bar{z}_2^r z_3 + z_3^{q+r}\bar{z}_3^r.$$

2.2. **Join type mixed polynomials.** Let $f(\mathbf{z}, \bar{\mathbf{z}})$ be a true strongly mixed homogeneous convenient polynomial of n-variables $\mathbf{z} = (z_1, \ldots, z_n)$ of polar degree q and radial degree $q + 2r$ with an isolated singularity at the origin. Consider the join polynomial $g := f(\mathbf{z}, \bar{\mathbf{z}}) - w^{q+r}\bar{w}^r$ of $(n+1)$-variables. Let F_f, F_g be the respective Milnor fibers of f and g. Consider the projective Mixed hypersurfaces V_f and V_g defined by $f = 0$ and $g = 0$ respectively in $\mathbb{P}^{n-1}$ or $\mathbb{P}^n$.

Theorem 9. *Assume that $f(\mathbf{z}, \bar{\mathbf{z}})$ is a true strongly mixed homogeneous convenient polynomial of n-variables and the corresponding projective variety V_f has a mixed smooth point in V_f and $n \geq 2$. Then*

(1) *F_f is connected and*

(2) $\pi_1(\mathbb{P}^n \setminus V_g) = \mathbb{Z}/q\mathbb{Z}$ *and* F_g *is simply-connected.*

Proof. In this theorem, we do not assume that f is strongly non-degenerate. Note that

$$\begin{aligned} F_f =& \{\mathbf{z} \in \mathbb{C}^n \mid f(\mathbf{z}, \bar{\mathbf{z}}) - 1 = 0\} \\ V_f =& \{[\mathbf{z}] \in \mathbb{P}^{n-1} \mid f(\mathbf{z}, \bar{\mathbf{z}}) = 0\} \\ V_g =& \{[\mathbf{z} : w] \in \mathbb{P}^n \mid f(\mathbf{z}, \bar{\mathbf{z}}) - w^{q+r}\bar{w}^r = 0\}. \end{aligned}$$

Consider the affine chart $U_w := \{w \neq 0\}$ in $\mathbb{P}^n$. In this coordinate space, using affine coordinates $u_j = z_j/w$, $j = 1, \dots, n$, we see that

$$V_g \cap U_w = \{\mathbf{u} \in \mathbb{C}^n \mid f(\mathbf{u}, \bar{\mathbf{u}}) - 1 = 0\}.$$

This expression says that $F_f \cong V_g \cap U_w$. Note that $V_g \cap \{w = 0\} \cong V_f$ has a smooth point p. Consider the projection $\pi : \mathbb{P}^n \to \mathbb{P}^{n-1}$ which is defined by $[\mathbf{z} : w] \mapsto [\mathbf{z}] \in \mathbb{P}^{n-1}$. Then the restriction $\pi : V_g \to \mathbb{P}^{n-1}$ is a q-fold covering branched over $V_f \subset \mathbb{P}^{n-1}$. Take a non-singular point p of $V_f \subset \mathbb{P}^{n-1}$ and consider a small normal disk D centered at p. For simplicity, we assume that $p \in \{z_1 \neq 0\}$ and we choose the affine coordinate chart $\{z_1 \neq 0\}$ with affine coordinates $v_j = z_j/z_1$, $j = 2, \dots, n$ and $x = w/z_1$. In this chart, V_g is defined by $f(\mathbf{v}, \bar{\mathbf{v}}) - x^{q+r}\bar{x}^r = 0$ with $\mathbf{v} = (1, v_2, \dots, v_n)$. Then the covering $(\mathbf{v}, x) \mapsto \mathbf{v}$ is topologically equivalent to the holomorphic cyclic covering defined by $x^q - f = 0$ in a small disk D with center p. (In D, we can take the function $f : D \to \mathbb{C}$ as a real analytic complex-valued coordinate function and we may assume that the image $f(D)$ is a small unit disk Δ_ρ with radius ρ.) Thus the fiber of a boundary point p' , $f(p') = \rho e^{i\theta_0} \in \partial\Delta$, decomposes by $\{Re^{i(\theta_0+2j\pi)/q} \mid j = 0, \dots, q-1\}$ in x-coordinate with $R = |\rho|^{1/(q+2r)}$ and under the local monodromy along ∂D, those q points are cyclically rotated as $Re^{i(\theta_0+2j\pi)/q} \mapsto Re^{i(\theta_0+2(j+1)\pi)/q}$, $j = 0, \dots, q-1$. Thus $\pi^{-1}(D^*)$ is connected, where $D^* := D \setminus \{p\}$. As $\mathbb{P}^{n-1} \setminus V_f$ is connected, any point $y \in V_g \setminus V_f$ can be connected using the covering structure to one of the points $\pi^{-1}(p')$. Here we identify V_f with $V_g \cap \{w = 0\}$. As $V_g - V_f = V_g \cap U_\omega$, $V_g \cap U_\omega$ is connected.

Now we consider the fundamental group, assuming $n = 2$ for simplicity. V_g is defined by $z_3^{q+r}\bar{z}_3^r - f(\mathbf{z}, \bar{\mathbf{z}})$ where $\mathbf{z} = (z_1, z_2)$. Consider the pencil lines $L_\eta = \{z_2 = \eta z_1\}$ and let $b = (0 : 0 : 1)$ be the base point of the pencil. Let $\widetilde{\mathbb{P}}^2$ be the blow-up space at b. Then $\widetilde{\pi} : \widetilde{\mathbb{P}}^2 \to \mathbb{P}^1$ is well defined and $\pi_1(\widetilde{\mathbb{P}}^2 \setminus \widetilde{V}_g) \equiv \pi_1(\mathbb{P}^2 \setminus V_g)$ with $\widetilde{V}_g = \widetilde{\pi}^{-1}(V_g) \cong V_g$. The zero points $f(\mathbf{z}, \bar{\mathbf{z}}) = 0$ are the locus of singular pencil lines. Take a simple zero $p \in V_f$ and take p' nearby as a base line and put $L = \pi^{-1}(p')$. Take generators $\xi_1, \dots, \xi_q$ of $\pi_1(L \setminus V_g \cap L)$ as in Figure 1. They satisfy the vanishing relation at infinity: $\xi_q \dots \xi_1 = e$. The centers of the small circles are the points of $L \cap V_g$. We always orient the small circles counterclockwise. Then the monodromy relations at p are given by

$$\xi_1 = \xi_2 = \cdots = \xi_q, \quad \xi_q \dots \xi_1 = e$$

See [8]. The argument is exactly the same as for a complex algebraic curve with a maximal flex point in Zariski [18]. Thus we get $\xi_1^q = e$ and $\pi_1(\mathbb{P}^2 \setminus V_g) \cong \mathbb{Z}/q\mathbb{Z}$.

The assertion (2) of Theorem 9 is true for any $n \geq 2$. For $n > 2$, we take a generic hyperplane H of type $a_1 z_1 + \cdots + a_n z_n = 0$ which contains $[0 : \cdots : 0 : 1]$ and use the surjectivity $\pi_1(H \setminus V \cap H) \to \pi_1(\mathbb{P}^n \setminus V_g)$. The defining polynomial of $V_g \cap H$ is also of join type and use an induction argument. Here we do not use the Zariski Hyperplane section theorem [4] (we do not know if the same assertion holds for mixed hypersurfaces or not) but we only use the surjectivity for a non-singular mixed hypersurface of join type which is easy to be shown. We leave this assertion to reader. $\square$

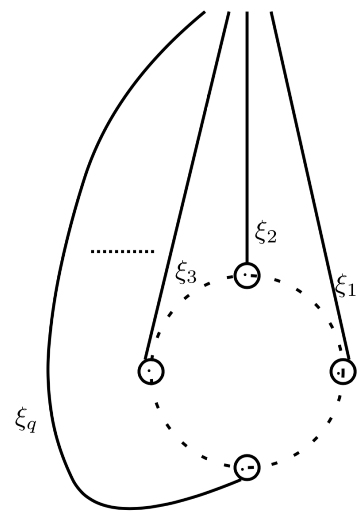

FIGURE 1. Generators of $\pi_1(L - L \cap V)$

Example 10. *Consider Rhie's Lens equation*

$$\varphi_n(z) := \bar{z} - \frac{z^{n-2}}{z^{n-1} - a^{n-1}} - \frac{\varepsilon}{z} = \frac{g(z,\bar{z})}{(z^{n-1} - a^{n-1})z} = 0,\ n \geq 2.$$

We can choose suitable positive numbers a, ε so that $0 < \varepsilon \ll a \ll 1$ and φ_n has $5(n-1)$ simple zeros (see [16] and also [13]). Let $g(z,\bar{z})$ be the numerator of φ_n and take the homogenization of $g(z,\bar{z})$

$$\begin{aligned} G(\mathbf{z}', \bar{\mathbf{z}'}) &:= g(z_1/z_2, \bar{z}_1/\bar{z}_2) z_2^n \bar{z}_2 \\ &= \bar{z}_1(z_1^n - a^{n-1} z_1 z_2^{n-1}) - \bar{z}_2 \left(z_1^{n-1} z_2 + \varepsilon (z_1^{n-1} z_2 - a^{n-1} z_2^{n-1} z_1) \right) \end{aligned}$$

where $\mathbf{z}' = (z_1, z_2)$. Consider the join type polynomial and the associated projective curve C:

$$C:\ f(\mathbf{z}, \bar{\mathbf{z}}) := z_3^n \bar{z}_3 + G(\mathbf{z}', \bar{\mathbf{z}'}) = 0, \quad \mathbf{z} = (z_1, z_2, z_3).$$

Observe that f is strongly mixed homogeneous of polar degree $q = n-1$ and radial degree $n+1$. Consider the affine chart $\{z_2 \neq 0\}$ and consider the affine coordinates $w_3 = z_3/z_2, w_1 = z_1/z_2$. Then the affine equation takes the form $w_3^n \bar{w}_3 - g(w_1, \bar{w}_1) = 0$. Consider the pencil of lines $L_\eta = \{z_1 - \eta z_2 = 0\}$ or in the affine equation, $w_1 = \eta$. There are exactly $5(n-1)$ singular pencil lines corresponding to the zeros of $g(w_1, \bar{w}_1) = 0$. These roots are all simple by the construction. In a small neighborhood of any such zero, the projection $\pi : C \to \mathbb{C}$ is locally equivalent to $w_3^n \bar{w}_3 - w_1 = 0$ or $w_3^n \bar{w}_3 - \bar{w}_1 = 0$ depending on the sign of the zero. Take a point η_0 near some zero of g and take generators $\xi_1, \dots, \xi_q$ of $\pi_1(L_{\eta_0} \setminus C)$ on the line L_{η_0} as in Figure 1, then we get that $\xi_1 = \cdots = \xi_q$ as the monodromy relation. Thus we get $\pi_1(\mathbb{P}^2 \setminus C) = \mathbb{Z}/q\mathbb{Z}$. Note that $L_\infty \cap C$ consists of q simple points. Thus the Euler number and the genus of C are calculated easily as

$$\begin{aligned} \chi(C) &= (n-1)(2 - (5n-5) - 1) + 5n - 5 + n - 1 = 17n - 5n^2 - 12, \\ g(C) &= \frac{(5n-7)(n-2)}{2}. \end{aligned}$$

In the moduli space of mixed polynomials of polar degree $n-1$ and radial degree $n+1$, the lowest genus is taken by

$$z_1^n\bar{z}_1 + z_2^n\bar{z}_2 + z_3^n\bar{z}_3$$

which is isotopic to the holomorphic curve

$$z_1^{n-1} + z_2^{n-1} + z_3^{n-1} = 0$$

of degree $n-1$ and therefore the genus is $(n-2)(n-3)/2$ by Plücker's formula.

Remark 11. *The genus of a non-singular mixed curve of polar degree q is greater or equal to $\frac{(q-1)(q-2)}{2}$ by the Thom inequality ([6]). In [10], it is shown that for any $g \geq 0$, there exists a mixed non-singular curve of polar degree 1 with genus g.*

2.3. **Twisted join type polynomials.** Let $f(\mathbf{z})$ be a strongly mixed homogeneous polynomial of polar degree q and radial degree $q+2r$ and consider the mixed homogeneous polynomial of $(n+1)$-variables:

$$g(\mathbf{z},\bar{\mathbf{z}},w,\bar{w}) \;=\; f(\mathbf{z},\bar{\mathbf{z}}) + \bar{z}_n w^{q+r}\bar{w}^{r-1}.$$

g is also strongly mixed homogeneous polynomial. Recall that $f(\mathbf{z},\bar{\mathbf{z}})$ is called to be *1-convenient* if the restriction of f to each coordinate subspace $f_i := f|_{\{z_i=0\}}$ is non-trivial for $i=1,\dots,n$ ([9])

Theorem 12. *([10]) Assume that $n \geq 2$ and f is 1-convenient with a connected Milnor fiber F_f and let $g(\mathbf{z},\bar{\mathbf{z}},w,\bar{w})$ be the twisted join polynomial as above.*

1. *The Milnor fiber $F_g = g^{-1}(1)$ of g is simply-connected.*
2. *The Euler characteristic of F_g is given by the formula:*

$$\chi(F_g) = -(q+r)\chi(F_f) + (q+r+1)\chi(F_{f_n})$$

where $f_n := f|\{z_n = 0\}$ and $F_{f_n} = f_n^{-1}(1)$.

Assume that $n=2$ and $f(\mathbf{z},\bar{\mathbf{z}})$ has an isolated singularity at the origin. Then we have

Corollary 13. $V = \{g = 0\} \subset \mathbb{P}^2$ *is a non-singular mixed curve and* $\pi_1(\mathbb{P}^2 - V) \cong \mathbb{Z}/q\mathbb{Z}$.

Example 14.

Consider the mixed curve defined by

$$f_I' = z_1^{q+r}\bar{z}_1^{r-1}\bar{z}_2 + z_2^{q+r}\bar{z}_2^{r-1}\bar{z}_3 + z_3^{q+r}\bar{z}_3^{r}, \ \text{(Tree type b)}$$

As f_I' is simplicial and also of twisted join type as

$$f_I' = z_1^{q+r}\bar{z}_1^{r-1}\bar{z}_2 + (z_2^{q+r}\bar{z}_2^{r-1}\bar{z}_3 + z_3^{q+r}\bar{z}_3^{r}),$$

we show that the Milnor fiber is simply-connected and

$$\pi_1(\mathbb{P}^2 \setminus C) \cong \mathbb{Z}/q\mathbb{Z}.$$

Here as $(z_2^{q+r}\bar{z}_2^{r-1}\bar{z}_3 + z_3^{q+r}\bar{z}_3^{r})$ is not 1- convenient, Theorem 12 can not be applied directly. Let us see this assertion directly. We take the coordinate chart $U_2 := \{z_2 \neq 0\}$ and put $w_1 = z_1/z_2, w_3 = z_3/z_2$. Then the affine equation of C in U_2 is

$$f(w_1,w_3) = w_1^{q+r}\bar{w}_1^{r-1} + \bar{w}_3 + w_3^{q+r}\bar{w}_3^{r}.$$

We consider the pencil $L_\eta := \{w_3 - \eta = 0\}$, $\eta \in \mathbb{C}$. It is easy to see that the branching locus is the set of the $q+2$ points given by

$$\Sigma := \{w_3 \,|\, \bar{w}_3(w_3^{q+r}\bar{w}_3^{r-1} + 1) = 0\}.$$

The base point of the pencil is $b = [1:0:0]$ and note that $b \in C$. $L_\eta \cap C$ has $q+1$ points over $\mathbb{C} \setminus \Sigma$ and 1 point over Σ. Taking a generic pencil L_{η_0} near a branching point $w \in \Sigma$ and taking generators $\xi_1, \dots, \xi_{q+1}$ of $\pi_1(L_{\eta_0} \setminus C)$ similarly as those in Figure 1, we get cyclic monodromy relations at each point of Σ:

$$\xi_1 = \xi_2 = \cdots = \xi_{q+1}.$$

This is enough to conclude that $\pi_1(\mathbb{P}^2 \setminus C)$ is abelian and therefore isomorphic to

$$H_1(\mathbb{P}^2 \setminus C) \cong \mathbb{Z}/q\mathbb{Z}.$$

As for the Euler characteristic, we get $\chi(C) = -(q+1)(q-1) + q + 3 = -q^2 + q + 4$. Thus the genus of C is $(q+1)(q-2)/2$.

3. Non-trivial examples

Let $F(\mathbf{z}, \bar{\mathbf{z}})$ be a true strongly non-degenerate mixed homogeneous polynomial of three variables $\mathbf{z} = (z_1, z_2, z_3)$ of polar degree q and radial degree $q + 2r$ and we consider the projective mixed curve

$$C := \{[\mathbf{z}] \in \mathbb{P}^2 \mid F(\mathbf{z}, \bar{\mathbf{z}}) = 0\}.$$

We study the geometric structure of C and the fundamental group $\pi_1(\mathbb{P}^2 - C)$ using the pencil $L_\eta := \{z_2 = \eta z_3\}$, $\eta \in \mathbb{C}$, or equivalently the projection

$$p : (\mathbb{P}^2, C) \to \mathbb{P}^1, \quad [\mathbf{z}] \mapsto [z_2, z_3].$$

Take the affine chart $U_3 := \{z_3 \neq 0\}$ with coordinate functions (z, w) with $z = z_1/z_3$, $w = z_2/z_3$. Then $C \cap U_3$ is defined by $f(z, w, \bar{z}, \bar{w}) = F(\mathbf{z}, \bar{\mathbf{z}})/z_3^{q+r}\bar{z}_3^r = 0$. Let $\Sigma \subset \mathbb{P}^1$ be the branching locus of p.

3.0.1. *Holomorphic case.* If F is homogeneous polynomial without complex conjugate variables, Σ is described by the discriminant locus of f as a polynomial in z. Put $\sigma(w) := \mathrm{discrim_z f(z, w)}$. Thus Σ is a finite set of points $\Sigma = \{\rho_1, \dots, \rho_\ell\}$ given by $\sigma(w) = 0$. For any $\rho_j \in \Sigma$ and $\rho_{j,k} \in p^{-1}(\rho_j)$, C is locally a cyclic covering of order $s_{j,k}$ at $\rho_{j,k}$ where $s_{j,k}$ is the multiplicity of $\rho_{j,k}$ in $p^{-1}(\rho_j)$ as the root of $f(z, \rho_j) = 0$ which is equal to the intersection multiplicity of L_{ρ_j} and C at $\rho_{j,k}$.

3.0.2. *Mixed polynomial case.* Let F be a mixed homogeneous polynomial. Usually it is not easy to compute Σ. Instead of computing Σ, we proceed as follows. Let $z = x + yi$ and $w = u + vi$ and write f as $f(x, y, u, v) := g(x, y, u, v) + ih(x, y, u, v)$ where g and h are real polynomials which are the real and imaginary part of f respectively. Consider the complex algebraic variety

$$C(\mathbb{C}) := \{(x, y, u, v) \in \mathbb{C}^4 \,|g(x, y, u, v) = h(x, y, u, v) = 0\}$$

which is the complexification of our curve. Note that $C(\mathbb{C}) \cap \mathbb{R}^4 = C$. The branching locus of $p_{\mathbb{C}} : C(\mathbb{C}) \to \mathbb{C}^2$ is obtained by a Groebner basis calculation from the ideal $[g, h, J]$ where $J = \frac{\partial g}{\partial x}\frac{\partial h}{\partial y} - \frac{\partial g}{\partial y}\frac{\partial h}{\partial x}$ and $[g, h, J]$ is the ideal generated by g, h, J. The defining ideal is generated by the polynomials $\mathbb{C}[u, v] \cap [g, h, J]$. It is usually a principal ideal and the generating polynomial $R(u, v)$ of this ideal describes the discriminant locus of the complexified variety. We define the branching locus $\Sigma_{\mathbb{R}}$ by the intersection $\Sigma_{\mathbb{C}} \cap \mathbb{R}^2$. Take a point $w \in \Sigma_{\mathbb{R}}$. It is not always true that a point $w \in \Sigma_{\mathbb{R}}$ is a branching point of $p : C \to \mathbb{R}^2$. It might come from the branching on the complex point of $C(\mathbb{C})$ outside of C. That is $\Sigma \subset \Sigma_{\mathbb{R}}$ but the equality does not hold in general. See Example 2 below. Also it might have some point η_0 such that $L_{\eta_0} \cap C$ contains a 1-dimensional intersection. See Remark 16.

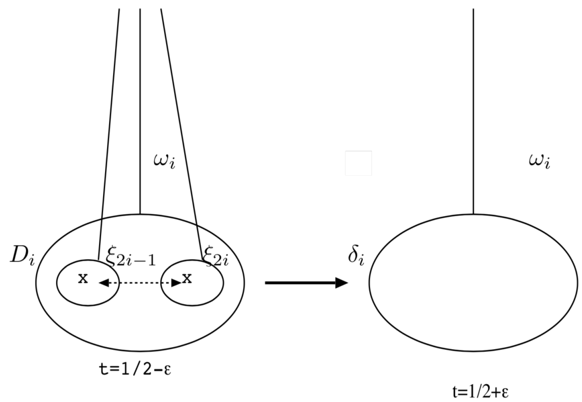

FIGURE 2. Vanishing loops

There are some cases for which these branching loci are comparatively simple. Suppose that f is a join type polynomial of $z_1^{q+r}\bar{z}_1^r$ and a strongly mixed homogeneous convenient polynomial $K(z_2, z_3, \bar{z}_2, \bar{z}_3)$ of two variables z_2 and z_3. Then the affine equation takes the form

$$f(\mathbf{z}, \bar{\mathbf{z}}) = z^{q+r}\bar{z}^r + k(w, \bar{w}) = 0$$

with respect to the affine coordinates $z = z_1/z_3$ and $w = z_2/z_3$. By the non-degeneracy assumption, the roots of $k(w, \bar{w}) = 0$ are all simple. Then the branching locus Σ is nothing but the set of those roots and over any of these roots, the projection is locally equivalent to the q-cyclic coverings $z^{q+r}\bar{z}^r - w = 0$ or $z^{q+r}\bar{z}^r - \bar{w} = 0$ respectively depending on the sign of the root.

However for a generic mixed polynomial, $\Sigma_{\mathbb{R}}$ and Σ are much more complicated. Usually they have real dimension 1 components and also they can have isolated points. We assume that for each η, $L_\eta \cap C$ is a finite point. We define $\gamma(\eta)$ to be the cardinality of $L_\eta \cap C$. We subdivide $\{\mathbb{C} \setminus \Sigma, \Sigma - S(\Sigma), S(\Sigma)\}$ by γ-values where $S(\Sigma)$ is the singular locus of Σ and let $\mathcal{D}$ be the corresponding subdivision. We call $\mathcal{D}$ *the γ-subdivision* of the parameter space. A 2-dimensional connected component $V \in \mathcal{D}$ (respectively 1-dimensional L, 0-dimensional P) is called a *region* (resp. an *edge*, a *vertex*). A region V is called *regular* if the inclusion map $V \subset \bar{V}$ is a homotopy equivalence. An edge L is called *regular* if there exist exactly two regions, say V_1, V_2 whose boundaries contain L and $\gamma(L) = (\gamma(V_1) + \gamma(V_2))/2$. A vertex P is called *regular* if there exist at most two regions which contain P in their boundary.

For a regular edge $M \in \mathcal{D}$, suppose that two regions S_1, S_2 are touching each other along M and suppose that $\gamma(S_1) > \gamma(S_2)$. Take a point $a \in M$ and a small transversal path

$$\sigma : [0, 1] \to \mathbb{R}^2 = \mathbb{C}$$

so that $\sigma(t) \in S_1$ for $t < 1/2$, $\sigma(1/2) = a$ and $\sigma(t) \in S_2$ for $t > 1/2$. Let $\gamma := (\gamma(S_1) - \gamma(S_2))/2$. Then for a sufficiently small $\varepsilon > 0$ and $1/2 - \varepsilon \leq \forall t < 1/2$, $p^{-1}(\sigma(t))$ consists of $\gamma(S_1)$ points, say $\xi_1(t), \dots, \xi_{\gamma(S_1)}(t)$ and among them there exist γ pairs of points $\{\xi_{2i-1}(t), \xi_{2i}(t)\}$, $i = 1, \dots, \gamma$ and we can choose a continuous family of disjoint γ disks $D_i(t), i = 1, \dots, \gamma$ in the pencil line $p^{-1}(\sigma(t)) = \mathbb{R}^2$ which contain only the corresponding pair of roots so that when t goes to $1/2$, two roots approach each other in the disk $D_i(t)$ and collapse to $\delta_i \in L_a \cap M$, a double point and then they disappear for $t > 1/2$. These pairs of roots $\{\xi_{2i-1}(t), \xi_{2i}(t)\}$ as roots of a polynomial equation $f(z, \sigma(t)) = 0$ have opposite signs (one positive and one negative). Take a base point

$b = [1 : 0 : 0]$ of the fundamental group at the base point of the pencil. Consider a loop $\omega_i \in \pi_1(L_{\sigma(1/2-\varepsilon)} - C, b)$ represented by the boundary loop of $D_i(1/2-\varepsilon)$, connected to the base point by a path outside of $L_{\sigma(1/2-\varepsilon)} \setminus \cup_{i=1}^k D_i(\sigma(1/2-\varepsilon))$. Then we get the following relation for $t : 1/2 - \epsilon \to 1/2 + \epsilon$

$$\omega_i = e, \quad i = 1, \dots, \gamma.$$

Take elements ξ_{2i-1}, ξ_{2i} as in Figure 2. Then this implies that

$$\omega_i = \xi_{2i-1}\xi_{2i} = e \text{ or equivalently } \xi_{2i-1} = \xi_{2i}^{-1},\ i = 1, \dots, \gamma.$$

We call these relations **vanishing monodromy relations**.

3.1. **Example 1.** Now we present several examples which are neither simplicial nor of join type but the complement has an abelian fundamental group.

3.1.1. *Example 1-1.* Consider the following mixed curve of polar degree 1

$$C_t : \quad F(\mathbf{z}, \bar{\mathbf{z}}) := {z_1}^2\bar{z}_1 + {z_2}^2\bar{z}_2 + z_3^2\bar{z}_3 + t\,{z_1}^2 z_2\bar{z}_3 = 0$$

with $t \in \mathbb{C}$ and let C_t be the corresponding projective curve. Let $M_t = F^{-1}(1)$ be the corresponding Milnor fiber. Then C_0 is of mixed Brieskorn type and isotopic to the standard line $z_1 + z_2 + z_3 = 0$, namely a sphere S^2 (see [11]) and M_0 is diffeomorphic to the plane $\mathbf{C}$. This is true for any small t. Observe that $\{z_3 = 0\} \cap C_t = \{[1 : -1 : 0]\}$.

We are interested in $C := C(-4) : z_1^2\bar{z}_1 + z_2^2\bar{z}_2 + z_3^2\bar{z}_3 - 4\,z_1 z_2\bar{z}_3$. We use the notation $M_{-4} = M$ for simplicity. Take the affine coordinate $z = z_1/z_3$, $w = z_2/z_3$. Then the affine equation is given as

$$C : \quad z^2\bar{z} + w^2\bar{w} + 1 - 4\,zw = 0.$$

To compute the Euler characteristic $\chi(C)$ and the fundamental group $\pi_1(\mathbf{P}^2 \setminus C)$, we consider the pencil $L_\eta := \{w = \eta\}$, $\eta = u + vi \in \mathbb{C}$. The branching locus $\Sigma_{\mathbb{R}}$ is given by $R(u, v) = 0$ where

$$\begin{aligned} R(u,v) = 27 &+ 11642\,v^2u^4 - 2640\,u^7v^2 + 405\,u^4v^8 + 162\,u^{10}v^2 + 16438\,v^6 \\ &- 6736\,v^6u^3 - 350\,u^6 + 405\,v^4u^8 + 162\,u^2v^{10} + 27\,v^{12} + 540\,v^6u^6 \\ &+ 28430\,u^2v^4 - 148\,u^3 - 2196\,uv^2 - 7032\,u^5v^4 + 27\,u^{12} - 2196\,uv^8 - 148\,u^9. \end{aligned}$$

See Appendix 1 (§3.3.1) for the practical computation of $R(u, v)$. Its diagram of the zero locus set $R = 0$ is given as Figure 3. Let A be the bounded region of $\mathbb{C} \setminus \Sigma_{\mathbb{R}}$ and let U be the complement $\mathbb{P}^1 \setminus \bar{A}$. There are four singular points $V_i, i = 1, \dots, 4$ of the boundary of $\bar{A}$. Actually -1 is an isolated point of $\Sigma_{\mathbb{R}}$ but $L_\eta \cap C$ consists of one simple point and it does not give any branching of the projection $p : C \to \mathbb{C}$. Thus $-1 \notin \Sigma$.

As the polar degree is 1, the number of intersection points of $L_\eta \cap C$ counted with sign is always 1. Observe that $L_\eta \cap C$ consist of 3 simple roots of $f(z, \eta) = 0$ for any $\eta \in A$. Observe further that over any point η of the complement U of $\bar{A}$, $L_\eta \cap C$ has a unique simple root, i.e. $\gamma(U) = 1$ and $\gamma(A) = 3$. For any smooth boundary point η of $\partial\bar{A}$, $L_\eta \cap C$ has two points, one simple and one double point. (Strictly speaking, there does not exist the notion of multiplicity in the mixed roots. See [12]. Here we use the terminology "double root" in the sense that it is a limit of two simple roots). As for four singular points, we have $\gamma(V_i) = 1$, $i = 1, \dots, 4$. Let $\overline{a_1a_2}$ be the line segment cut by $A \cap \{v = 0\}$ where $a_1 \approx 0.51, a_2 \approx 1.94$. For any $a_1 < \eta < a_2$, $L_\eta \cap C$ has three simple points which are all real. This can be observed by the diagram of $f = 0$ restricted on the real plane section $(w, z) \in \mathbb{R}^2$ (Figure 4). Consider the limit of $L_\eta \cap C$ when η goes to a_1 or a_2 along the real line segment $\overline{a_1a_2}$. There are two real positive roots and one real negative root and at both ends, two positive roots collapse in the double point, which is clear from Figure 4.

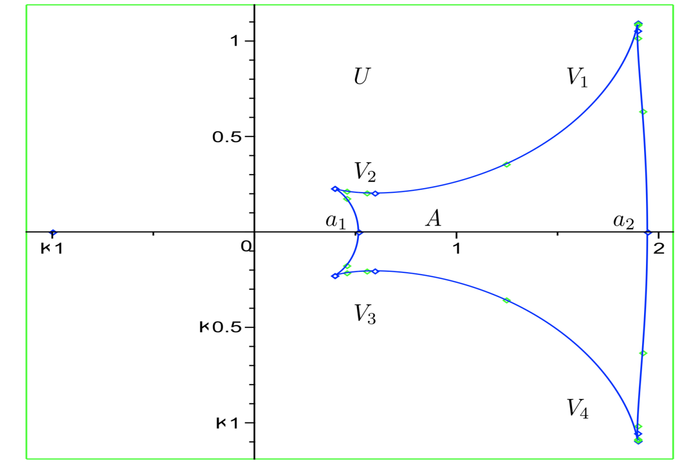

FIGURE 3. Diagram of $R = 0$, Example 1-1

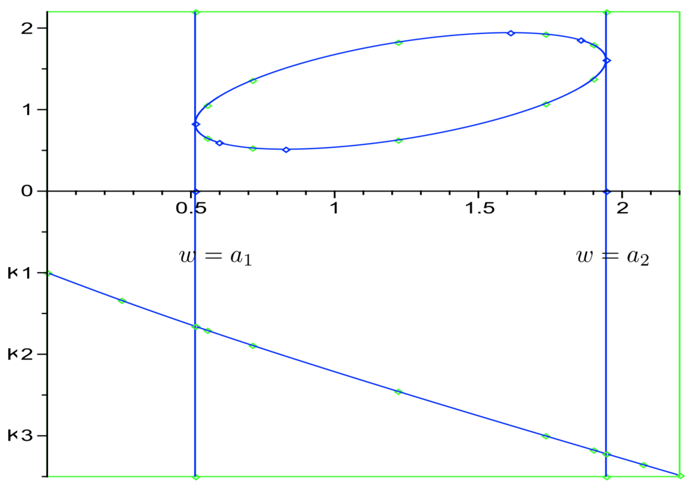

FIGURE 4. Diagram of $f = 0$, Example 1-1

Using these data, we can compute the Euler characteristic as

$$\chi(C) = \chi(p^{-1}(\bar{A})) + \chi(p^{-1}(U)) = -1 + 1 = 0.$$

This implies C is a torus and $\chi(\mathbb{P}^2 - C) = 3 - 0 = 3$. We claim

Proposition 15.

(1) $\pi_1(\mathbb{P}^2 - C) = \{e\}, \quad \pi_1(M) = \{e\}$.
(2) $\chi(M) = 3$, $H_1(M) = 0$, $H_2(M) = 2$.

Proof. We first compute the fundamental group. Put $b_0 = 1$ and we take L_{b_0} as a fixed regular pencil line. Then $L_{b_0} \cap C = \{x_1, x_2, x_3\}$ where

$$x_1 < 0 < x_2 < x_3.$$

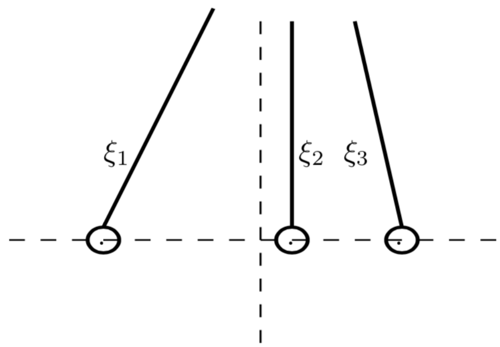

FIGURE 5. Generators of $\pi_1(L_{b_0} - C \cap L_{b_0})$

See Figure 4. It is not hard to see that $\pi_1(L_{b_0} \setminus C \cap L_{b_0}) \to \pi_1(\mathbb{P}^2 \setminus C)$ is surjective. See §3.3 for an explanation in detail. Take generators ξ_1, ξ_2, ξ_3 of $\pi_1(L_{b_0} \setminus C \cap L_{b_0})$ as in Figure 5. They are oriented counterclockwise. First, as a vanishing relation at infinity, they satisfy the relation

$$\xi_1\xi_2\xi_3 = e. \tag{1}$$

When η moves on the interval $[a_1, a_2]$ from $\eta = b_0$ to a_1 or a_2, we see that two positive roots collapse in the point for $\eta = a_1$ or $\eta = a_2$ and disappear for $\eta < a_1$ or $\eta > a_2$. Thus as a vanishing relation, we get

$$\xi_2 = \xi_3^{-1}.$$

Now we consider the movement from $\eta = 1$ along the vertical line to $\eta = 1 + v_0 i$ where $(1, v_0) \in \partial A$ and $v_0 \approx 0.26$. The generators are deformed as in Figure 6. Thus as a vanishing monodromy relation, we get $(\xi_2^{-1}\xi_1\xi_2)\xi_3 = e$. Thus combining the above relations, we get

$$\xi_1 = \xi_3,\ \xi_2 = \xi_1, \text{ and } \xi_1 = e.$$

We conclude that $\pi_1(\mathbb{P}^2 - C) = \{e\}$. □

Remark 16. *It can be observed that the set* $\Gamma := \{t = t_1 + t_2 i \in \mathbb{C} \,|\, C_t : \textit{singular}\}$ *is a real one-dimensional semi-algebraic set and the complement* $\mathbb{C} \setminus \Gamma$ *has two connected components in this case. The bounded region contains* 0 *and for any* t *in this region,* C_t *is isotopic to* C_0 *and it is a rational sphere.* Γ *is calculated by Groebner basis calculation. In our case, we found that* Γ *is defined by*

$$t_1^4 - 6t_1^2 + 8t_1 - 3 + 2t_2^2 t_1^2 - 6t_2^2 + t_2^4 = 0.$$

Certainly C_{-4} *is in the outside unbounded region. We may choose another one* $C_{\sqrt[3]{3}}$ *which must be isotopic to* C_{-4} *but the branching locus is very different and defined by* $R = 0$ *and its diagram is given by Figure 7.*

In this example, $\gamma(A) = \gamma(B) = 3$ *but the point* $(u, v) = (-1, 0)$ *is special as* $L_{-1} \cap C$ *has one simple point and one 1-dimensional component which is defined by* $|z| = \sqrt[6]{3}$. *Thus the geometry of the pencil is more complicated and it takes more careful consideration to compute*

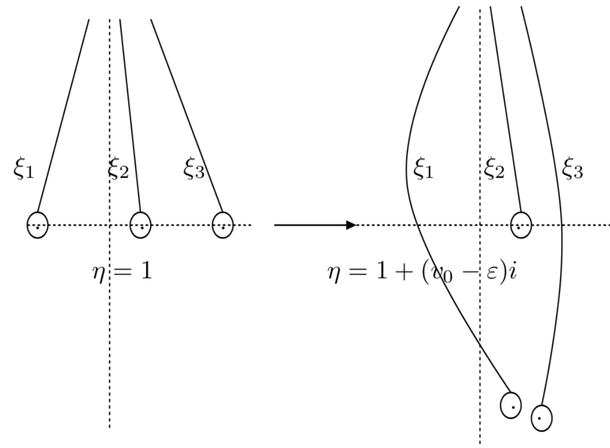

FIGURE 6. movement on $\eta = 1 + si$, Example 1-1

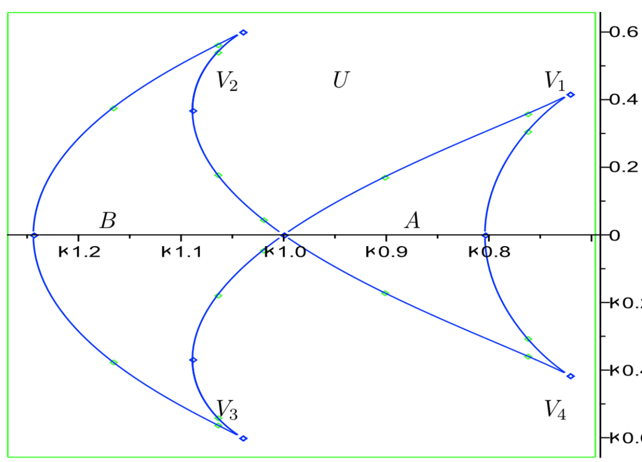

FIGURE 7. Diagram of $R = 0$, Remark16

the fundamental group.

$$R(u,v) := 27+540\,v^6u^6+405\,v^4u^8+162\,u^2v^{10}+120\,u^9+162\,u^{10}v^2+654\,u^4v^2+120\,u^3+216\,uv^2 +1008\,u^5v^4+216\,uv^8+27\,u^{12}+405\,v^8u^4+768\,v^6u^3+576\,u^7v^2+90\,v^6+558\,u^2v^4+186\,u^6+27\,v^{12}.$$

3.1.2. *Example 1-2.* We consider another example with polar degree 1 and radial degree 3. Let $F(\mathbf{z},\bar{\mathbf{z}}) := z_1^2\bar{z}_1 + z_2^2\bar{z}_2 + z_3^2\bar{z}_3 - 4z_2z_3\bar{z}_3 - 2z_3^2\bar{z}_1$. Taking the affine chart $\{z_3 \neq 0\}$ and coordinates $z = z_1/z_3, w = z_2/z_3$, the affine equation is $f(z,w) = z^2\bar{z} + w^2\bar{w} + 1 - 4w - 2\bar{z}$. Consider the pencil $L_\eta := \{w = \eta\}, \eta \in \mathbb{C}$. Putting $w = u + vi$, the branching locus is described by $R = 0$ where the explicit form is given in Appendix 2(§3.3.2) to show that the equation of R grows exponentially by the number of monomials and degree. However the diagram of $R = 0$ is not so

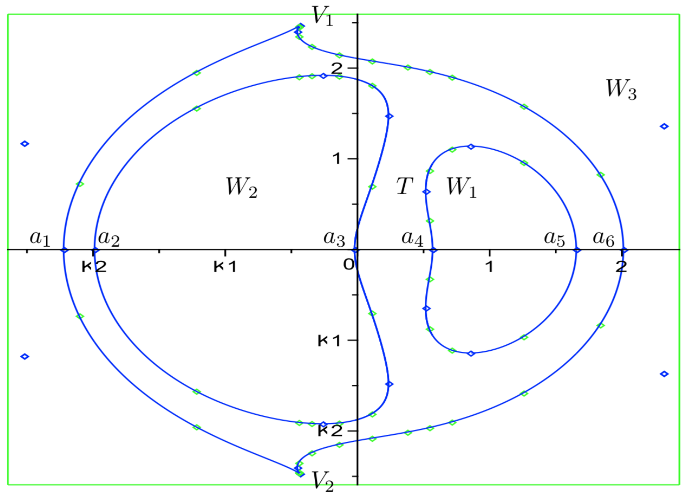

FIGURE 8. Diagram of $R = 0$, Example 1-2

complicated and it is given in Figure 8. We observe that $\gamma(W_i) = 1$, $i = 1, 2, 3$ and $\gamma(T) = 3$ where T is the complement of $\overline{W}_1 \cup \overline{W}_2 \cup \overline{W}_3$. There are two singular points of the boundary of T, V_1, V_2 and $\gamma(V_i) = 1$ and the other boundary points have 2 roots. Let $a_1, \dots, a_6$ be real roots of $R(u, 0) = 0$ and we assume that $a_1 < a_2 < \cdots < a_6$. Note that $a_1 \approx -2.22$ and $a_2 \approx -1.98$. See figure 8. In the Figure, the horizontal line is the w-coordinate axis. Take a base line L_{b_0} with $b_0 = a_2 - \varepsilon$, $0 < \varepsilon \ll 1$. See the diagram of $f = 0$ on $\mathbb{R}^2$ (Figure 9). Two vertical lines are $w = a_1$ and $w = a_2$. We take generators ξ_1, ξ_2, ξ_3 of $\pi_1(L_{b_0} \setminus C)$ as the left side of Figure 5. Considering a movement of $\eta = b_0$ to $\eta = a_1$ and from $\eta = b_0$ to $\eta = a_2$, we get the vanishing monodromy relations

$$\xi_1\xi_2 = e, \quad \xi_2\xi_3 = e. \tag{2}$$

This is also clear from Figure 9. Combining the vanishing relation $\xi_1\xi_2\xi_3 = e$, we get

$$\xi_2 = \xi_1^{-1} = \xi_3^{-1} = e.$$

Thus $\pi_1(\mathbb{P}^2 \setminus C)$ is abelian and we conclude that $\pi_1(\mathbb{P}^2 \setminus C) = H_1(\mathbb{P}^2 - C)$ is trivial. The Euler number is computed as

$$\chi(C) = \chi(W_1) + \chi(W_2) + \chi(W_3) + \chi(T) - \chi(\partial T) = 1 + 1 + 1 - 3 - 2 = -2.$$

Thus the genus of C is 2.

3.2. **Example 2.** Consider the next mixed curve of polar degree 2 and radial degree 4.

$$C_t \quad F(\mathbf{z}, \bar{\mathbf{z}}) := \bar{z}_1 z_1^3 + z_2^3 \bar{z}_2 + z_3^3 \bar{z}_3 + t\, z_1^2 \bar{z}_2 z_3$$

For t small, C_t is isotopic to the conic $z_1^2 + z_2^2 + z_3^2 = 0$ in $\mathbb{P}^2$ and a rational sphere ([11]). We take $t = -4$ and put $C = C_{-4}$ and $M = M_{-4}$ the Milnor fiber. The branching locus is defined by $R = R_1 R_2 = 0$ where

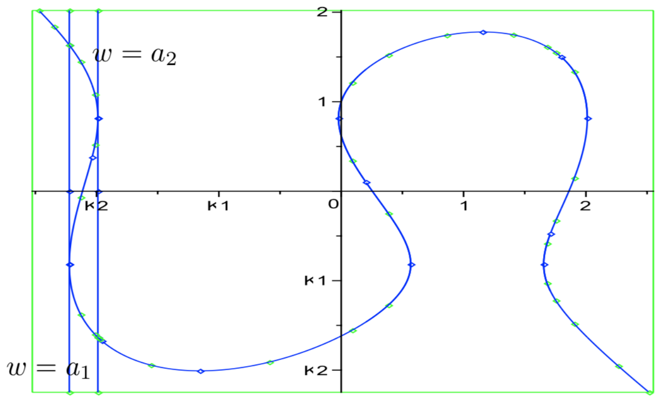

FIGURE 9. Diagram of $f = 0$ with $z, w \in \mathbb{R}$, Example 1-2

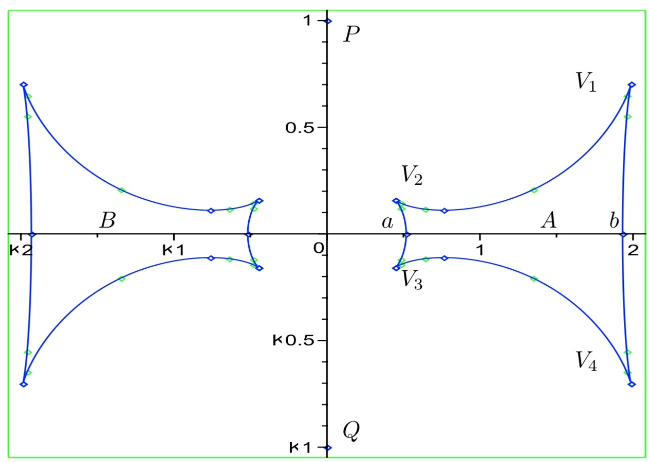

FIGURE 10. Diagram of $R = 0$, Example 2

$$R_1 := 1 + u^8 + 6\,v^4u^4 + 2\,u^4 - 2\,v^4 + 4\,v^6u^2 + 4\,u^6v^2 + v^8,$$
$$\begin{aligned} R_2 := 1 &- 12\,u^4 + 124\,v^4 - 320\,u^2v^2 + 8\,u^{14}v^2 + 12580\,v^4u^4 + 12936\,v^6u^2 + 3464\,u^6v^2 + 70\,v^8u^8 \\ &- 1228\,u^8v^4 + 56\,v^{10}u^6 - 1472\,v^6u^6 + 56\,v^6u^{10} - 548\,v^8u^4 - 26\,u^8 + 3846\,v^8 - 12\,u^{12} \\ &+ 124\,v^{12} + u^{16} + 28\,v^{12}u^4 + 8\,v^{14}u^2 + 28\,u^{12}v^4 - 368\,u^{10}v^2 + 176\,v^{10}u^2 + v^{16}. \end{aligned}$$

We claim that

Proposition 17. (1) *$\pi_1(\mathbb{P}^2 - C) \cong \mathbb{Z}/2\mathbb{Z}$.*
(2) *$\chi(C) = -2$. The genus of C is 2.*

Proof. The locus $R_1 = 0$ gives two isolated points $P = (0,1)$, $Q = (0,-1)$. Note that the affine equation of C is defined by $f(z,w) = z^3\bar{z} + w^3\bar{w} + 1 - 4z^2\bar{w} = 0$. Recall that the highest degree

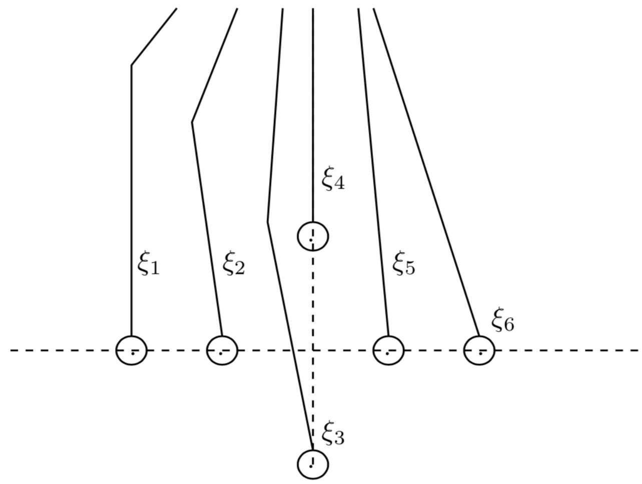

FIGURE 11. Generators of $\pi_1(L_1 - C \cap L_1)\,(\eta = 1)$

part of f as a polynomial of z is $z^3\bar{z}$, and therefore the number of roots counted with sign is two by [12]. We also observe that$f(z,w) = 0 \iff f(-z,w) = 0$. Thus the roots are symmetric with respect to the origin in z-coordinates. R is symmetric with respect to the v-axis but the region B does not give any branching. It comes from the complex part of the curve. Thus $\gamma(B) = 2$. Also we observe that $\gamma(A) = 6$ and $\gamma(\partial A) = 4$ except 4 singular points $V_1, \dots, V_4$ where $L_\eta \cap C$ has 2 multiple points. The complement region $E := \mathbb{P}^1 \setminus (\bar{A} \cup \{P, Q\})$ has 2 simple points for any fiber $L_\eta \cap C$ with $\eta \in E$. We have $\gamma(P) = \gamma(Q) = 1$. Take generators of $\pi_1(L_1 - C)$, $\xi_i, i = 1, \dots, 6$ as in Figure 11. Observe that $f(z,w) = 0$ implies $f(-z,w) = 0$. Thus the roots are always paired by $z, -z$ for a fixed w. Put $\bar{A} \cap \{v = 0\} = \{a, b\}$ with $a \approx 0.51$ and $b \approx 1.93$. First we consider the movement $\eta = 1 \to a$. Consider the diagram of $f_r := z^4 + w^4 + 1 - 4z^2 w$ (Figure12) where f_r is the restriction of f to $\mathbb{R}^2$. This says that on $[a, b]$, $L_\eta \cap C$ has exactly four real roots, which are symmetric with respect the origin and at $\eta = a$, they collapse to two double roots. Let f_i be the restriction of $f(iz, w)$ to $(z, w) \in \mathbb{R}^2$ and look at its diagram. See Figure 13. Using the real diagram of $f_i := -z^4 + w^4 + 1 + 4z^2 w$, we see also that there are exactly two purely imaginary roots of $f(z, \eta) = 0$ for any $w \in \mathbb{R}$. The above observation says that

$$\xi_1 \xi_2 = e, \quad \xi_5 \xi_6 = e. \tag{3}$$

(The Figure 12 shows that we get the same degeneration for $\eta \to b$.) Then we consider the movement of the line L_η further to the left until $\eta = 0$. Then we move L_η along the imaginary axis to $\eta = i$ which is a root of multiplicity 2 (P in the diagram). Note that the monodoromy along $|w - i| = \varepsilon$ is topologically the half turn of two roots. Thus we get

$$\xi_3 = \xi_4. \tag{4}$$

Now we will see the vanishing relation along the vertical line for $\eta = 1 \to 1 + c_0 i \dots$ where c_0 is the positive root of $R_2(1, v) = 0$. The root of $f(z, w) = 0$ with $w = 1 + ci$ is given as

$$\pm P_1, \pm P_2, \quad P_1 \approx 1.57 - 1.21i, P_2 \approx 0.76 + 0i$$

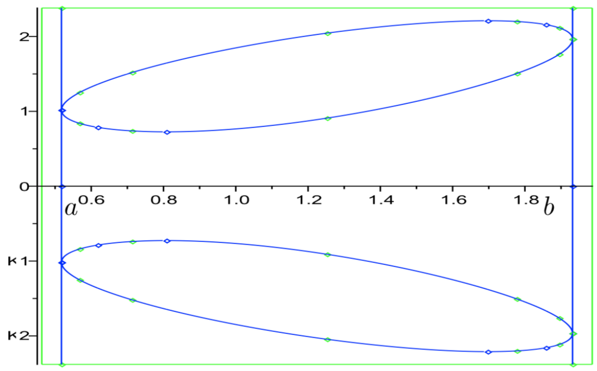

FIGURE 12. Diagram of $f = 0$, Example 2

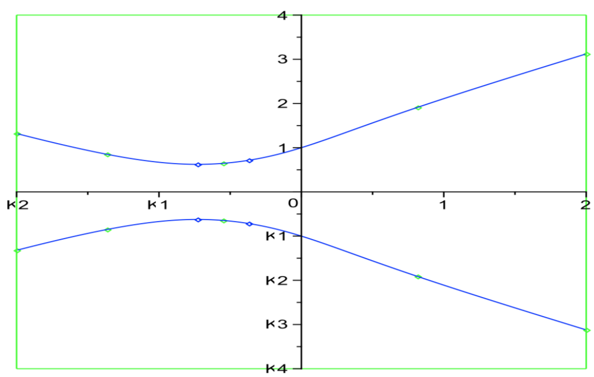

FIGURE 13. Diagram of $f_i(z, w) = 0$, Example 2

where $\pm P_1$ are double roots. Recall that $f(z, 1) = 0$ has roots

$$\pm Q_1, \pm Q_2, \pm Q_3, \quad Q_1 \approx 2.11i,\, Q_2 \approx 0.76, Q_3 \approx 1.84.$$

The movement of generators during the above movement is described in Figure 14. The dotted loops show the situation in $\eta = 1+(c_0-\varepsilon)i$ wit $0 < \varepsilon \ll 1$. They are denoted as $\xi_1', \dots, \xi_6'$. In this movement, ξ_2, ξ_5 do not move much. Other generators are deformed as indicated with arrows. At $\eta = 1 + c_0 i$, ξ_1', ξ_4' and ξ_3', ξ_6' collapse respectively. Thus $\xi_1 = \xi_1', \xi_4 = \xi_4'$ and $\xi_3 = \xi_3', \xi_6 = \xi_6'$ and we get vanishing relations which are written as

$$\xi_1\xi_4 = e, \quad (\xi_4\xi_5)^{-1}\xi_3(\xi_4\xi_5)\xi_6 = e. \tag{5}$$

Using (3), (4)and (5), we conclude that

$$\xi_2 = \xi_1^{-1},\, \xi_3 = \xi_1,\, \xi_4 = \xi_1^{-1}, \xi_5 = \xi_1,\, \xi_1^2 = e.$$

That is, $\pi_1(\mathbb{P}^2 - C) \cong \mathbb{Z}/2\mathbb{Z} \cong H_1(\mathbb{P}^2 - C)$. □

3.3. **Surjectivity.** Assume that $f(z, \bar{z}, w, \bar{w}) = 0$ is the affine equation of a non-singular mixed curve C of polar degree q and radial degree $q + 2r$. We assume that f is monic in the sense that

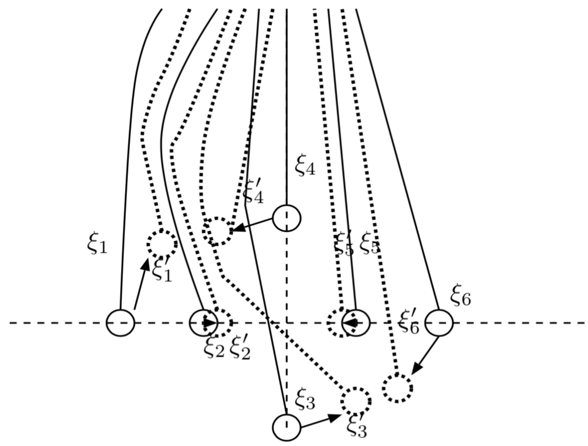

FIGURE 14. Movement of generators, Example 2

it has the monomial $z^{q+r}\bar{z}^r$ with a non-zero coefficient. Consider the pencil line

$$L_\eta = \{w = \eta\}, \eta \in \mathbb{C}$$

and we consider the γ-subdivision $\mathcal{D}$ of $\mathbb{C}$ (the parameter space) by the value of $\gamma(\eta)$ using the diagram of R. We assume that all regions, edges and vertices are regular. We assume also that the base point b of the pencil is not on C. Let $G = \{\gamma(\eta) \,|\, \eta \in \mathbb{C}\} \subset \mathbb{N}$ the possible number of roots of $f(z, \bar{z}, \eta, \bar{\eta}) = 0$ and γ_{max} be the maximum of G. We assume the following two conditions.
(1) The set $U_{max} := \{\eta \,|\, \gamma(\eta) = \gamma_{max}\}$ is connected and it is a region.
(2) Take a region U of $\mathcal{D}$ with $\gamma(U) < \gamma_{max}$. Put $\partial_+ U = \{q \in \partial U \,|\, \gamma(q) \geq \gamma(U)\}$. Then $\partial_+ U$ is connected.

Note that the above condition is satisfied in Example 1-1, Example 1-2, Example 2. Let B be the complement of the union of regions of $\mathcal{D}$, i.e. B is the union of the edges and vertices. We fix a generic line L_{η_0} with $\eta_0 \in U_{max}$ and a base point $b \in L_{\eta_0} \setminus C$. Let $\sigma : (I, \{0,1\}) \to (\mathbb{P}^2 \setminus C, b)$ be a loop. We may assume that $\{t \,|\, \sigma(t) \in B\}$ is finite. Let $\alpha := \min\{\gamma(\sigma(t)) \,|\, t \in I\}$ and we may assume that α is taken in a region V of $\mathcal{D}$. Put $\mathcal{D}_\beta$ be the union of $\overline{U}$ with $\gamma(U) \geq \beta$. Then we assert:

Assertion 18. *σ is homotopic in $\mathbb{P}^2 \setminus C$ to a loop $\hat{\sigma}$ in the pencil line $L_{\eta_0} \setminus C$.*

Proof. We may assume that $\pi \circ \sigma$ intersects B transversely at smooth points of B if it intersects.

Step 1. Suppose that $\alpha \neq \gamma_{max}$. Then the image of $\pi \circ \sigma$ intersects more than two regions. Take a path segment L of $\pi(\sigma(I)) \cap V$. Let P, Q be the end points of L and assume that $P = \pi(\sigma(t_1))$ and $Q = \pi(\sigma(t_2))$ with $t_1 < t_2$. By the assumption (2), P, Q belongs to the unique boundary component $\partial_+ V$ and there is a path L'' in the boundary $\partial_+ V$ connecting P, Q and $\gamma(\eta) \geq \alpha$ for any $\eta \in L''$. We want replace L by some path $L' \subset V$ which is homotopic to L'' relatively to the end points. See Figure 15. Consider the closed path at Q, $\omega := L^{-1} \cdot L'$. The composition of paths is to be read from the left. Take a lift $\tilde{\omega}$ which is a loop starting at $\sigma(t_2)$, passes through $\sigma(t_1)$ and comes back to $\sigma(t_2)$ which is null homotopic in $\mathbb{P}^2 \setminus C$. We can simply take $\tilde{\omega}$ near the infinity. Then replace σ by $\sigma_{[0,t_2]} \cdot \tilde{\omega} \cdot \sigma_{[t_2,1]}$ which is homotopic to σ. Now σ is clearly homotopic to σ' where $\sigma' := \sigma_{[0,t_2]} \cdot \tilde{\omega} \cdot \sigma_{[t_2,1]}$. Note that the image $\pi(\sigma'(I))$ replaces

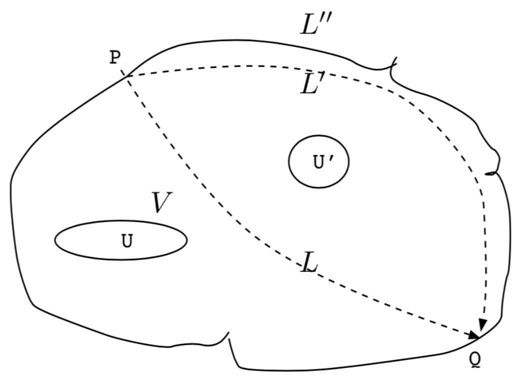

FIGURE 15. Segment L

the segment L by L'. Now we can deform L' to L'' and further to the other side of the region of L'', keeping the homotopy class. Doing this operation for any path segment cut by V, we get a loop σ'' whose image by π is in $\mathcal{D}_\beta$ where $\beta := \min\{G \setminus \{\alpha\}\}$. By induction, we can deform σ keeping the homotopy class to a loop σ_1 in $\pi^{-1}(U_{max})$.

Step 2. Now we assume that σ_1 is a loop in $\pi^{-1}(U_{max})$. We deform σ_1 further to a loop $\hat{\sigma}$ which is a loop in the line L_{η_0}.

If U_{max} is contractible, this is easy to deform using the fibration structure of π over U_{max}. This is the case for Example 1-1 and Example 2. In Example 1-2, $U_{max} = T$ and $\pi_1(U_{max}, \eta_0)$ is a free group of rank 2.

Assume that $\pi_1(U_{max})$ is non-trivial. Put $\tau := \pi \circ \sigma_1$, a loop in U_{max}. Take a lift $\tilde{\tau}$ starting at b which is a contractible closed curve in $\pi^{-1}U_{max} \setminus C$. Consider the loop $\sigma_1 \cdot \tilde{\tau}^{-1}$. This is homotopic to σ_1. The image of this modified loop by π is clearly homotopic to a constant loop at η_0. Using the fibration structure over U_{max}, we can deform this loop to a loop $\hat{\sigma}$ in $L_{\eta_0} \setminus C$. For the detail of lifting argument, see for example Spanier [17]. □

The surjectivity assertion is not true if η_0 does not belong to U_{max}. Also a loop $\tau \in (L_{\eta_0} \setminus C)$ cannot be expressed by a loop in $(L_\eta \setminus C)$ if $\gamma(\eta) < \gamma_{max}$ without using the monodromy relations. An example is given by ξ_{2i-1}, ξ_{2i} in Figure 2 can not deformed on the line $L_{\sigma(1/2+\varepsilon)}$. We close this paper by a question.

Question. Do the conditions (1) and (2) hold for any mixed function?

3.3.1. *Appendix 1.* Let f be a mixed strongly homogeneous polynomial. To compute the defining polynomial of the branching locus R in Example 1-1, Example 1-2 and Example 2, we proceed as follows. Let $z = x + yi$ and $w = u + vi$ and write f as $g + ih$ where g, h are polynomials of x, y, u, v with real coefficients. Let $J = \frac{\partial g}{\partial x}\frac{\partial h}{\partial y} - \frac{\partial g}{\partial y}\frac{\partial h}{\partial x}$ and let $A = [g, h, J]$, the ideal generated by g, h, J. Then we use the MAPLE command: Groebner[Basis](A,plex(x,y,u,v)). For further explanation for Groebner calculation, we refer [3] for example.

Acknowledgement. For the numerical calculation of roots of $f(z, w) = 0$ with fixed various complex numbers w's, we have used the following program on MAPLE which is kindly written by Pho Duc Tai, Hanoi University of Science. I am grateful to him for his help.

Pho's program to compute roots of mixed polynomial on MAPLE:

```
fsol3 := proc (f, z)
local aa, a, b, ff, f1, f2, h, i, j, k, s, temp; print(Factorization_of_Input = factor(f)); ff := factors(f)[2];
temp := {};
for k to nops(ff) do
if 1 < ff[k][2] then RETURN(printf("Input is not squarefree. Please solve each factor.")) end if;
assume(a, real); assume(b, real); h := expand(subs(z = a+I*b, ff[k][1]));
f1 := Re(h); f2 := Im(h); aa := RootFinding[Isolate]([f1, f2], [a, b]);
temp := `union`(temp, seq([[op(aa[i][1])][2], [op(aa[i][2])][2]], i = 1 .. nops(aa))) end do;
RETURN([op(temp)])
end proc
```

3.3.2. *Appendix 2: Equation of R for Example 1.2.*

The equation of the branching locus is the following.

$$
\begin{aligned}
R(u,v) = &-179685 + 129384576\,u^4v^4 + 2160\,u^{19}v^2 + 27\,u^{24} - 864\,v^{22} + 13590816\,u^7 - 102858240\,v^6u^5 \\
&- 47520\,u^{18}v^4 - 7631712\,u + 174564288\,v^6u^4 - 288581376\,v^6u^2 + 193050720\,v^8u^2 + 580608\,v^{14}u^3 \\
&+ 2032128\,u^{13}v^4 + 5080320\,u^9v^8 + 4064256\,u^7v^{10} + 72576\,uv^{16} - 142560\,u^6v^{16} - 9504\,u^2v^{20} - 142560\,u^{16}v^6 \\
&- 399168\,u^{12}v^{10} - 285120\,u^{14}v^8 - 285120\,u^8v^{14} - 47520\,u^4v^{18} - 399168\,u^{10}v^{12} + 441460992\,v^2u^6 + 27\,v^{24} \\
&+ 542688\,v^{16} + 55344648\,v^6u^6 + 216\,u^{21} + 12096\,v^{20} - 6048\,u^{19} + 20877120\,u^2 - 466968\,u^{15} + 4064256\,u^{11}v^6 \\
&+ 2032128\,u^5v^{12} + 580608\,u^{15}v^2 + 1550016\,v^2 - 21912872\,u^3 - 103122216\,v^2u + 15611136\,uv^{10} - 2415360\,v^8u^3 \\
&- 212093856\,u^3v^4 + 165770304\,u^7v^2 - 93452\,u^{18} + 33480480\,v^8u^5 + 27856256\,v^6u^3 + 138815904\,v^6u \\
&- 137971200\,u^7v^4 - 46557076\,v^6 + 134691072\,u^2v^2 + 20322912\,u^4 + 35835552\,v^4 - 15874316\,u^6 \\
&+ 588672\,u^5 - 425273772\,v^2u^4 + 404256\,u^{16} + 352376064\,v^2u^3 + 44453280\,v^4u^9 + 57198720\,v^6u^7 \\
&+ 6479040\,v^{10}u^3 - 596640\,v^{12}u + 6378146\,v^{12} + 271049040\,u^5v^4 - 55084800\,u^9v^2 - 242721696\,u^8v^2 \\
&- 283722816\,u^6v^4 + 85432284\,u^{10}v^2 + 15257280\,u^{11}v^2 - 9504\,u^{20}v^2 - 73189752\,v^8u + 324\,u^{22}v^2 \\
&+ 1597920\,u^{13} + 2630208\,v^{14}u^2 - 3738096\,u^{14}v^4 - 12425640\,u^{10}v^8 - 11851032\,u^8v^{10} - 8601296\,u^{12}v^6 \\
&- 7583152\,u^6v^{12} - 912372\,u^{16}v^2 - 752448\,u^{14} + 190612542\,u^8v^4 - 52716324\,u^2v^{10} - 107902338\,v^8u^4 \\
&+ 1782\,u^{20}v^4 - 50865792\,v^4u - 4232728\,u^9 + 9720\,u^{17}v^4 + 2160\,u^3v^{18} + 9720\,u^5v^{16} + 25920\,u^{15}v^6 \\
&+ 54432\,u^{11}v^{10} + 45360\,u^9v^{12} + 45360\,u^{13}v^8 + 25920\,u^7v^{14} + 216\,uv^{20} + 1451520\,u^6v^{14} + 120960\,u^2v^{18} \\
&+ 2155584\,v^{12}u^2 - 821676\,u^2v^{16} - 18081480\,u^7v^8 - 11114040\,u^{11}v^4 - 18630120\,u^9v^6 - 10126488\,u^5v^{10} \\
&- 3003624\,u^3v^{12} - 3552264\,u^{13}v^2 - 3198096\,u^4v^{14} + 72576\,u^{17} - 337318944\,u^5v^2 - 99220\,v^{18} - 2136768\,v^{14} \\
&+ 544320\,u^{16}v^4 + 2540160\,u^{12}v^8 + 3048192\,u^{10}v^{10} + 1451520\,u^{14}v^6 + 2540160\,u^8v^{12} + 120960\,u^{18}v^2 \\
&+ 544320\,u^4v^{16} + 12096\,u^{20} + 80914380\,v^4u^2 + 7271904\,u^{10} - 864\,u^{22} + 10619712\,u^4v^{10} + 18579648\,u^{12}v^4 \\
&- 22564800\,u^6v^8 + 33143040\,u^8v^8 + 19619136\,u^6v^{10} - 78557760\,u^8v^6 - 71293248\,u^{10}v^4 + 33409728\,u^{10}v^6 \\
&+ 7934784\,u^4v^{12} + 4945344\,u^{14}v^2 + 5940\,u^{18}v^6 + 5940\,u^6v^{18} + 324\,u^2v^{22} + 13365\,u^{16}v^8 + 24948\,u^{12}v^{12} \\
&+ 21384\,u^{14}v^{10} + 13365\,u^8v^{16} + 1782\,u^4v^{20} + 21384\,u^{10}v^{14} - 217728\,u^{15}v^4 - 762048\,u^{11}v^8 \\
&- 762048\,u^9v^{10} - 508032\,u^{13}v^6 - 508032\,u^7v^{12} - 54432\,u^{17}v^2 - 54432\,u^3v^{16} - 217728\,u^5v^{14} - 6048\,uv^{18} \\
&- 7068480\,u^8 - 357240\,uv^{14} + 30563520\,v^8 - 15242784\,v^{10} - 1027742\,u^{12} - 1945344\,u^{11} - 22380096\,u^{12}v^2.
\end{aligned}
$$

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LACUNAS AND LOCAL ALGEBRAICITY OF VOLUME FUNCTIONS

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To the memory of Egbert Brieskorn

Abstract. The volume cut off by a hyperplane from a bounded body with smooth boundary in $\mathbb{R}^{2k}$ never is an algebraic function on the space of hyperplanes: for k=1 it is the famous lemma XXVIII from Newton's Principia. Following an analogy of these volume functions with the solutions of hyperbolic PDE's, we study the local version of the same problem: can such a volume function coincide with an algebraic one at least in some domains of the space of hyperplanes, intersecting the body? We prove some homological and geometric obstructions to this integrability property. Based on these restrictions, we find a family of examples of such "locally integrable" bodies in Euclidean spaces.

1. Introduction

According to an Archimedes' theorem, the volume cut by a plane from a ball in $\mathbb{R}^3$ depends algebraically on the coordinates of the plane. The same is true also for all balls and ellipsoids in all odd-dimensional Euclidean spaces, but no additional examples are known by now.

On the contrary, Newton proved that for no bounded convex domain with smooth boundary in $\mathbb{R}^2$ the areas cut from it by the affine lines depend algebraically on the coordinates of these lines, see [12], [7], [2], [6]. V.I. Arnold [3] conjectured that similar statements hold also in higher dimensions. The even-dimensional part of this problem was completed in [16]: there is no bounded domain (convex or not) with smooth boundary in $\mathbb{R}^{2k}$, for which the volume cut off by a hyperplane is algebraic. The odd-dimensional part of Arnold's conjecture (stating that the ellipsoids in $\mathbb{R}^{2k+1}$ are unique bodies with this property) has only partial solutions: several geometric obstructions to the algebraicity of volumes are presented in [15]; however it is not clear whether they are sufficient for the proof of the general problem.

We study a local version of the same problem: given a body $W \subset \mathbb{R}^N$, can the corresponding volume function coincide with an algebraic one at least in some open subset of the space of all affine subspaces in $\mathbb{R}^N$ intersecting W? We prove some topological and geometric obstructions to this *local integrability* property, and find a series of new bodies satisfying it.

There is a deep analogy between this problem and the *lacuna problem* in the theory of hyperbolic PDE's developed in [13], [11], [4], [5]; for a list of parallel notions see page 138 in [15]. Many of our objects and terminology are borrowed from the theory of lacunas.

1.1. **Notation and definitions.** Denote by $\mathcal{P}$ the space of all affine hyperplanes in $\mathbb{R}^N$. It almost coincides with $\mathbb{R}P^N$: the homogeneous coordinates $(a_1 : \cdots : a_N : b)$ define the hyperplane with the equation

$$a_1x_1 + \cdots + a_Nx_N + b = 0, \tag{1}$$

and $(0 : \cdots : 0 : 1)$ is the only point in $\mathbb{R}P^N$ but not in $\mathcal{P}$.

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Let $W \subset \mathbb{R}^N$ be a *smooth body*, that is, a bounded (not necessarily connected) domain with smooth boundary. It defines a two-valued function V_W on $\mathcal{P}$: its values $V_W(X)$ on a hyperplane X are equal to the volumes of intersections of the body W with two halfspaces in $\mathbb{R}^N$ separated by X.

The space $\mathcal{P}$ consists of open domains whose points are the hyperplanes transversal to ∂W, and the walls between these domains formed by the hyperplanes tangent to it: these walls form the *projective dual* hypersurface of ∂W. Such an open domain in $\mathcal{P}$ is called a *lacuna* if the restriction of the volume functions to this domain coincides with an algebraic function on $\mathcal{P}$, that is, there exists a non-trivial polynomial $F(a_1, \dots, a_N, b, V)$ vanishing in any point $(a_1, \dots, a_N, b, V)$ such that V equals either of the two volumes cut off from the body W by the hyperplane with the equation (1) from our domain. The body W is called *algebraically integrable* if all domains of $\mathcal{P}$ are lacunas.

There is a trivial example of a lacuna: it is the domain consisting of hyperplanes not intersecting the body W, so that the corresponding volume function is equal identically to a pair of constants in it, 0 and the volume of entire W. Given a body, does it define nontrivial lacunas in $\mathcal{P}$ (so that the corresponding volume functions are not constant)?

In the case of convex $W \subset \mathbb{R}^{2k}$ and infinitely differentiable ∂W the answer is negative (there is only one non-trivial domain in $\mathcal{P}$, and it is not a lacuna); for $k = 1$ it is the Newton's lemma XXVIII. The main result of [16] says that for an arbitrary bounded body with C^∞-boundary in $\mathbb{R}^{2k}$ *all* regular domains in $\mathcal{P}$ cannot be lacunas simultaneously.

2. Obstructions to the integrability

In this section we assume that the boundary ∂W of the body $W \subset \mathbb{R}^N$ is a smooth component (or a collection of components) of the zero set of an irreducible polynomial with real coefficients.

For any generic real hyperplane X, we define an $(N-2)$-dimensional complex manifold, and some collection of elements of its $(N-2)$-dimensional homology group, one of which is given by the manifold $X \cup \partial W$, and the others are called *vanishing cycles*. Our main result (Theorem 1 below) says that if the intersection index of the first cycle with either of these vanishing cycles is not equal to 0, then the component of $\mathcal{P}$ containing X is not a lacuna. Let us introduce all these objects.

Let A be the zero set in $\mathbb{C}^N$ of the polynomial distinguishing ∂W. This set A can have singular points in the imaginary domain. Let us fix a Whitney stratification of the algebraic subvariety $A \cup \mathbb{C}P^{N-1}_\infty \subset \mathbb{C}P^N$, where $\mathbb{C}P^N$ is the standard compactification of $\mathbb{C}^N$, and $\mathbb{C}P^{N-1}_\infty$ is the "infinitely distant" hyperplane in it. An affine hyperplane $X \subset \mathbb{C}^N$ is called *generic* if its closure in $\mathbb{C}P^N$ is transversal to this chosen stratification of $A \cup \mathbb{C}P^{N-1}_\infty$. The set of generic hyperplanes contains a Zariski open subset in the space $\mathcal{P}_\mathbb{C}$ of all complex hyperplanes in $\mathbb{C}^N$. In particular, the real planes in $\mathbb{R}^N$, whose complexifications are generic, are dense in $\mathcal{P}$. Using the complexifications of real planes, we will consider $\mathcal{P}$ as a subset of $\mathcal{P}_\mathbb{C}$.

Denote by Reg the space of all generic hyperplanes in $\mathbb{C}^N$, and denote by $\mathrm{Reg}_\mathbb{R}$ the set of hyperplanes with real coefficients that are transversal to ∂W; in particular $\mathrm{Reg}_\mathbb{R} \supset \mathrm{Reg} \cap \mathcal{P}$. All elements of the difference $\mathrm{Reg}_\mathbb{R} \setminus (\mathrm{Reg} \cap \mathcal{P})$ correspond to real planes whose complexifications are not transversal to the stratified variety $A \cup \mathbb{C}P^{N-1}_\infty$ at some pairs of its complex conjugate imaginary points. The codimension of this difference in $\mathcal{P}$ is at least 2, in particular it does not separate different connected components of $\mathrm{Reg} \cap \mathcal{P}$.

The volume function is analytic inside any component of $\mathrm{Reg}_\mathbb{R}$.

Given a complex hyperplane X in $\mathbb{C}^N$, denote by $\breve{\mathbb{C}}^N, \breve{X}$ and $\breve{A}$ the sets $\mathbb{C}^N, X$ and A from which all singular points of the hypersurface A are removed.

Consider the chain of homomorphisms

$$H_N(\check{\mathbb{C}}^N, \check{X} \cup \check{A}) \to H_{N-1}(\check{X} \cup \check{A}) \to H_{N-2}(\check{X} \cap \check{A}), \tag{2}$$

where the first arrow is the usual boundary operator, and the second one is the Mayer-Vietoris differential. (All homology groups here and below are with integer coefficients only).

By the Thom isotopy lemma (see e.g. [10]), for all $X \in \mathrm{Reg}$ the groups of any of three kinds indicated in (2) are isomorphic to each other; moreover, any path in Reg identifies such groups for the endpoints of the path via the *Gauss–Manin connection* (that is, the homological realization of the covering homotopy property over this path).

Let $X_0 \in \mathrm{Reg} \cap \mathcal{P}$ be a generic plane. The group $H_N(\check{\mathbb{C}}^N, \check{X}_0 \cup \check{A})$ contains two important elements $\Lambda_\pm(X_0)$: the parts of the body $W \subset \mathbb{R}^N$ cut off by the real part of the hyperplane X_0 and taken with the canonical (once fixed) orientation of $\mathbb{R}^N$. Let $\Delta_\pm(X_0)$ be the images of these elements in the group $H_{N-2}(\check{X}_0 \cap \check{A})$ under the composite homomorphism (2). They are represented by the manifold $X_0 \cap \partial W$ taken with some (opposite) orientations, in particular $\Delta_-(X_0) + \Delta_+(X_0) = 0$.

For any $X \in \mathrm{Reg}$ the first and the last groups in (2) contain also some distinguished sets of elements, called *vanishing contours* and *vanishing cycles* respectively and defined in the following way.

Let u be a generic point of the hypersurface $\check{A}$, that is, a non-singular point of A such that the second fundamental form of A at this point is non-degenerate. Such points are dense in A since A is irreducible and bounds a body in $\mathbb{R}^N$. The set of all hyperplanes tangent to A at points close to u is then a smooth hypersurface in $\mathcal{P}_{\mathbb{C}}$.

Let B be a small ball in $\mathbb{C}^N$ centered at our generic point $u \in \check{A}$, and $X(u) \subset \mathbb{C}^N$ be the tangent hyperplane of A at u. For any hyperplane $X'(u)$ sufficiently close to $X(u)$ but lying in Reg, consider the sequence

$$H_N(B, X'(u) \cup A) \to H_{N-1}((X'(u) \cup A) \cap B) \to H_{N-2}(X'(u) \cap A \cap B), \tag{3}$$

whose maps are defined as in (2). All three groups in this sequence are then isomorphic to $\mathbb{Z}$, and both maps in it are the isomorphisms. Denote by $\Lambda(u)$ and $\Delta(u)$ some generators of the first and the last groups in (3) obtained one from another by this composite homomorphism. Denote by the same letters $\Lambda(u)$ and $\Delta(u)$ the images of these elements in the groups $H_N(\check{\mathbb{C}}^N, \check{X}'(u) \cup \check{A})$ and $H_{N-2}(\check{X}'(u) \cap \check{A})$ under the identical embedding.

An arbitrary path in Reg connecting the points $X'(u)$ and X_0 identifies the groups of any of three types (2) for these hyperplanes, in particular moves the elements $\Lambda(u)$ and $\Delta(u)$ into some two elements of the groups $H_N(\check{\mathbb{C}}^N, \check{X}_0 \cup \check{A})$ and $H_{N-2}(\check{X}_0 \cap \check{A})$ respectively. All elements of the latter two groups which can be obtained in this way from any choice of a generic point u, a path connecting X and $X'(u)$ in Reg, and a generator of the group $H_N(B, X'(u) \cup A)$, are called the *vanishing contours* and *vanishing cycles* respectively.

Theorem 1. *If the domain of* $\mathrm{Reg}_{\mathbb{R}} \subset \mathcal{P}$ *containing* X_0 *is a lacuna then the intersection indices* $\langle \Delta_+(X_0), \Delta \rangle \equiv -\langle \Delta_-(X_0), \Delta \rangle$ *of* $(n-2)$*-dimensional cycles in the complex* $(n-2)$*-dimensional manifold* $\check{X}_0 \cap \check{A}$ *are equal to 0 for all vanishing cycles* $\Delta \in H_{N-2}(\check{X}_0 \cap \check{A})$.

Proof. The integrals of the holomorphic volume form

$$dx_1 \wedge \cdots \wedge x_N \tag{4}$$

along the relative cycles define a linear function on the group $H_N(\mathbb{C}^N, X \cup A)$, and also on the group $H_N(\check{\mathbb{C}}^N, \check{X} \cup \check{A})$ for any $X \in \mathcal{P}$.

Every element Λ of the group

$$H_N(\check{\mathbb{C}}^N, \check{X}_0 \cup \check{A}) \tag{5}$$

defines a function germ $\operatorname{Int}(\Lambda)$ in a neighborhood of our point X_0 in Reg: its value at any point $X \approx X_0$ is equal to the integral of the form (4) along the relative cycle $\Lambda(X) \in H_N(\check{\mathbb{C}}^N, \check{X} \cup \check{A})$, obtained from Λ by the Gauss-Manin connection over the paths connecting X_0 and X in our neighborhood. By the construction, this function is complex analytic. If Λ is one of cycles Λ_+ or Λ_-, then the restriction of this function to $\mathrm{Reg}_{\mathbb{R}}$ coincides with the volume function, which also is analytic; therefore the analytic continuations of both functions to entire Reg coincide. If this analytic continuation is infinite-valued then the domain of $\mathrm{Reg}_{\mathbb{R}}$ containing X_0 is not a lacuna.

So we get a linear map Int from the group (5) to the space of all analytic function germs at the point $X_0 \in \mathcal{P}$. Denote by $\mathfrak{H}$ the image of the group (5) under this map (or, equivalently, the group (5) itself factored through the subgroup consisting of all elements defining zero germs). By the construction, $\mathfrak{H}$ is an integer lattice. The group $\pi_1(\mathrm{Reg}, X_0)$ acts on the group (5) by monodromy operators, and on $\mathfrak{H}$ by analytic continuations; these actions commute with our epimorphism $\operatorname{Int} : H_N(\check{\mathbb{C}}^N, \check{X}_0 \cup \check{A}) \to \mathfrak{H}$.

Now suppose that $\langle \Delta_+(X_0), \Delta \rangle \neq 0$ for some cycle Δ vanishing along a path connecting the points X_0 and $X'(u)$. Consider the loop in $\pi_1(\mathrm{Reg}, X_0)$ going along this path from X_0 to $X'(u)$, rotating around the set of planes tangent to A at points close to u, and coming back to X_0 along the same path. By the Picard–Lefschetz formula (and the functoriality of the maps (2)) this loop adds to the cycle $\Lambda_+(X_0)$ the class of the contour Λ vanishing along our path and taken with a non-zero coefficient c (equal to $\pm\langle \Delta_+(X_0), \Delta \rangle$).

If N is odd then we will pass this loop again and again. In this case the intersection index of $(N-2)$-dimensional cycles in $\check{X} \cap \check{A}$ is skew-symmetric, therefore any new travel along this loop adds to our integration chain a new copy of the cycle $c \cdot \Lambda$. The function germ defined by any vanishing cycle is not equal to zero, hence we get immediately an infinite number of leaves of the analytic continuation.

Lemma 1. *Let N be even, then the orbit of the germ defined by any vanishing contour Λ under our $\pi_1(\mathit{Reg}, X_0)$-action in $\mathfrak{H}$ is infinite.*

Proof of this lemma is based on considerations of §3 in [16]. The main tool there is a reflection group associated with any body like W. It acts on a lattice $\mathfrak{F}$ generated by finitely many elements corresponding to the vanishing contours, and the orbits of all these generators are not greater than the orbit of an arbitrary germ $\operatorname{Int}(\Lambda)$ defined by our vanishing contour under the action of the entire group $\pi_1(\mathrm{Reg}, X_0)$. (The action by reflections in $\mathfrak{F}$ is defined by the loops in Reg, all whose points are the planes parallel to X_0). Therefore if our $\pi_1(\mathrm{Reg}, X_0)$-orbit in $\mathfrak{H}$ of a germ defined by a vanishing contour is finite, then this reflection group also should be finite. However, it was proved in [16] that this reflection group always is infinite. □

Therefore the orbit of our contour $c \cdot \Lambda$ also is infinite. However, this orbit is a subset of the set of differences between the elements of the orbit ot the class $\operatorname{Int}(\Lambda_+(X)) \in \mathfrak{H}$. The latter orbit is thus also infinite, that is, the analytic continuation of the volume function has infinitely many leaves at the point X_0, and cannot be algebraic. □

Theorem 2. *If N is even then two* neighboring *domains of the set* $\mathrm{Reg}_{\mathbb{R}}$ *of generic hyperplanes in $\mathcal{P}$ (that is, two domains separated by only one piece of the variety projective dual to ∂W) cannot be lacunas simultaneously.*

Proof. Let X_1, X_2 be two points of $\mathrm{Reg} \cap \mathcal{P}$ separated by such a piece consisting of hyperplanes tangent to the surface ∂W close to some its generic point u; suppose that the planes X_1 and

X_2 are parallel and very close to the plane $X(u)$ tangent to A at this point. Then we have three important elements of the group $H_N(\check{\mathbb{C}}^N, \check{X}_1 \cup \check{A})$. The first one is our real contour $\Lambda_+(X_1)$ defined by the points of W cut off by the plane X_1. The second cycle, $M(\Lambda_+(X_2))$, is obtained from the similar element $\Lambda_+(X_2)$ of the group $H_N(\check{\mathbb{C}}^N, \check{X}_2 \cup \check{A})$ by the Gauss–Manin continuation over a small arc connecting the points X_2 and X_1 in the space Reg of generic complex hyperplanes. The third element is the vanishing cycle $\Lambda(u)$ generating the group $H_N(B, X_1 \cup A)$ where B is a small ball centered at the point u, see (3). By Lemma 3.3 of §III.3 in [15], these three cycles are related by the equality

$$\Lambda_+(X_1) - M(\Lambda_+(X_2)) = \pm\Lambda(u), \tag{6}$$

where the sign $\pm$ depends on the choice of the orientation of the last cycle. By Lemma 1, the orbit of the class $\mathrm{Int}(\Lambda(u)) \in \mathfrak{H}$ of the vanishing contour $\Lambda(u)$ under the monodromy action in $\mathfrak{H}$ is infinite in the case of even N, therefore the orbits of the classes of elements $\Lambda_+(X_1)$ and $\Lambda_+(X_2)$ cannot be finite simultaneously. □

Remark 1. It follows by induction from the identity (6) that either of the relative homology classes $\Lambda_+(X_0)$ and $\Lambda_-(X_0)$ is equal to the sum of several vanishing contours corresponding to the tangency points of ∂W with the hyperplanes parallel to X_0 and lying to the corresponding side from it.

3. Local geometry of the boundaries of lacunas and Davydova condition

Let X_1 and u be the same as in the previous proof. Let $\Delta_+(X_1)$ and $\Delta(u)$ be two elements of the group $H_{N-2}(\check{X}_1 \cap \check{A})$ obtained by the homomorphism (2) from the elements $\Lambda_+(X_1)$ and $\Lambda(u)$ used in this proof. If their intersection index in $\check{X}_1 \cap \check{A}$ is not equal to zero, then by Theorem 1 the domain of $\mathrm{Reg}_{\mathbb{R}}$ containing X_1 is not a lacuna. This property $\langle \Delta_+(X_1), \Delta(u)\rangle \neq 0$ can be checked directly in the terms of the local geometry of ∂W at the point u: more precisely, in the terms of its second fundamental form, cf. [8], [5].

Let us choose affine coordinates $y_1, \dots, y_N$ in $\mathbb{R}^N$ with the origin at the point u in such a way that $y_1 = 0$ on the tangent hyperplane $X(u)$, and $y_1 > 0$ on the examined hyperplane X_1 in our neighborhood B of the point u. The hypersurface ∂W is then defined by an equation of the form $y_1 = \chi(y_2, \dots, y_N)$ in a vicinity of the point u. The function χ is smooth and has a critical point at the origin: $d\chi(0) = 0$. This critical point is Morse since u is generic.

Proposition 1 (see e.g. [11] or Theorem 3.1 in page 183 of [15])**.** $\langle \Delta_+(X_1), \Delta(u)\rangle = 0$ *if and only if the* positive *inertia index of the quadratic part of the Taylor expansion of the function* χ *at the critical point is even.*

The trivial example occurs when this inertia index is equal to 0: in this case the cycle $\Delta_+(X_1)$ (consisting of all real points of $X_1 \cap A$) is empty close to u and certainly cannot intersect the vanishing cycle $\Delta(u)$ concentrated in the neighborhood of u.

Remark 2. This geometric condition is completely analogous to the *Davydova condition* in the theory of hyperbolic PDE's, see [8], although the integration cycles and forms in this theory are different. In both theories, the homology classes of the varieties like $X \cap A$ play the crucial role. However, in our case these cycles are related with the N-dimensional integration contours by the maps (2), while in the hyperbolic science the integration contours lie in some groups similar to our $H_N(\mathbb{C}^N \setminus (X \cup A))$, which in the case of generic X are related to the group $H_{N-2}(X \cap A)$ by the double Leray tube operation.

Now let U be a connected component of the space $\mathrm{Reg}_{\mathbb{R}} \subset \mathcal{P}$, and $Y \in \partial U$ a hyperplane tangent to ∂W.

Definition 1 (cf. [5])**.** The domain U is a *local lacuna* at the point Y if the volume function V_W coincides with a pair of regular analytic single-valued functions in the intersection of the domain U with some neighborhood of the point Y in $\mathcal{P}$.

Proposition 2 (cf. [5])**.** *1. Let $Y \in \mathcal{P}$ be a hyperplane having a generic tangency with ∂W at some point u. A domain of* $\mathrm{Reg}_{\mathbb{R}}$ *is a local lacuna close to this point Y if and only if the condition $\langle \Delta_+(X_1), \Delta(u)\rangle = 0$ from Proposition 1 is satisfied for some* (*and then for any*) *neighboring point X_1 of this domain.*

2. If a domain is not a local lacuna at some generic point of its boundary, then it also is not a lacuna.

The proof of statement 1 essentially repeats that of a similar statement in [5]: it follows from the removable singularity theorem. The proof of statement 2 uses additionally Theorem 1. $\square$

So, in the case of even N exactly one of neighboring domains of $\mathrm{Reg}_{\mathbb{R}}$ at a generic point $Y \in \partial W$ is a local lacuna, and the other is not.

In the case of odd N, either both neighboring domains are local lacunas or both are not. In particular, if N is odd and the hypersurface ∂W contains the points at which the inertia indices of its second fundamental quadratic form are odd, then the body W definitely is not algebraically integrable.

The study of geometric restrictions preventing a domain to be a local lacuna at more complicated points of its boundary also is parallel to that for hyperbolic PDE's, see [9], [14], [15].

4. Examples of lacunas

Let $m = N - 3$, so that $\mathbb{R}^N$ is decomposed into the sum $\mathbb{R}^3_x \oplus \mathbb{R}^m_y$.

Our easiest example is the tubular ε-neighborhood in $\mathbb{R}^N$ of the unit 2-sphere in $\mathbb{R}^3_x$, that is, the body defined by the inequality

$$\left(\sqrt{x_1^2 + x_2^2 + x_3^2} - 1\right)^2 + (y_1^2 + \cdots + y_m^2) \le \varepsilon^2 \ , \tag{7}$$

where $0 < \varepsilon < 1$. (This equation of its boundary is not polynomial, but is obviously equivalent to a polynomial one of degree 4).

There is a much more general class of examples. Instead of $y_1^2 + \cdots + y_m^2$, consider an arbitrary smooth function $\psi : \mathbb{R}^m_y \to \mathbb{R}_+$, invariant under the central symmetries in $\mathbb{R}^m_y$, whose unique critical point is a minimum point at the origin, $\psi(0) = 0$, and the entire set $\psi^{-1}([0, \varepsilon^2])$ is contained in some compact neighborhood of the origin in $\mathbb{R}^m_y$. Define the body W in $\mathbb{R}^3_x \oplus \mathbb{R}^m_y$ by the condition

$$\left(\sqrt{x_1^2 + x_2^2 + x_3^2} - 1\right)^2 + \psi(y_1, \ldots, y_m) \le \varepsilon^2. \tag{8}$$

Denote by C the volume of this body (8), and by Ω the $(N-1)$-dimensional Euclidean volume of its section by an arbitrary hyperplane in $\mathbb{R}^{m+3}$ containing the plane $\mathbb{R}^m_y$.

Theorem 3. *If a hyperplane $X \subset \mathbb{R}^{3+m}$ defined by some equation*

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \sum_{j=1}^m \beta_j y_j = \gamma$$

is sufficiently close to one containing the subspace $\mathbb{R}^m_y$ (*that is, X is nearly orthogonal to $\mathbb{R}^3_x$ and contains a point of $\mathbb{R}^3_x$ sufficiently close to the origin*), *then the volumes of two parts cut by X*

from the body (8) are equal to

$$\frac{C}{2} \ \pm \ \Omega \frac{\gamma}{\sqrt{\alpha_1^2+\alpha_2^2+\alpha_3^2}}. \tag{9}$$

In particular, the domain in $\mathcal{P}$ containing X is a lacuna.

Remark 3. The right-hand fraction in (9) is the distance from the plane $X \cap \mathbb{R}^3_x$ to the origin. The values (9) do not depend on the coefficients β_j in the equation of X.

Lemma 2. *In the conditions of Theorem 3, the $(m+2)$-dimensional volume of the intersection $X \cap W$ is equal to $\frac{\Omega}{\cos\alpha(X)}$ where $\alpha(X)$ is the angle between $\mathbb{R}^3_x$ and the normal vector of X.*

Proof of lemma. For any $y \in \mathbb{R}^m_y$, the preimage of y under the canonical projection $W \to \mathbb{R}^m_y$ is empty if $\psi(y) > \varepsilon^2$; if $\psi(y) < \varepsilon^2$ then it is a spherical layer in $\mathbb{R}^3_x$ between the spheres of radii $R = 1+\sqrt{\varepsilon^2-\psi(y)}$ and $r = 1-\sqrt{\varepsilon^2-\psi(y)}$. Let $\tilde{X}$ be the hyperplane in $\mathbb{R}^{3+m}$ containing the subspace $\mathbb{R}^m_y$ and such that the 2-planes $X \cap \mathbb{R}^3_x$ and $\tilde{X} \cap \mathbb{R}^3_x$ are parallel to one another. The orthogonal projection of $X \cap W$ to $\tilde{X}$ consists of points (x, y) such that $\psi(y) \leq \varepsilon^2$, and x belongs to a section of the above-described spherical layer (depending on y) by a 2-plane (depending also on X). If X is indeed sufficiently close to a vertical hyperplane containing $\mathbb{R}^m_y$, then for any y with $\psi(y) < \varepsilon^2$ this plane section of the layer is an annulus. The area of this annulus does not depend on the choice of this cutting 2-plane: if the distance of this plane from the origin in $\mathbb{R}^3_x$ is equal to $h < r$, then this area is equal to $\pi\left(\sqrt{R^2-h^2}^2 - \sqrt{r^2-h^2}^2\right) = \pi(R^2-r^2) = 4\pi\sqrt{\varepsilon^2-\psi(y)}$. So, the $(m+2)$-dimensional volume of the projection of $X \cap W$ to $\tilde{X}$ is equal to

$$4\pi \int_{\psi(y)\leq\varepsilon^2} \sqrt{\varepsilon^2-\psi(y)}dy,$$

which does not depend on X and hence is equal to the constant Ω. Further, the orthogonal projection of planes multiplies the volumes by the cosine of the angle between the normals of these planes. $\square$

Proof of Theorem 3. Let X_0 be the plane parallel to X and passing through the origin in $\mathbb{R}^{3+m}$. Both values of the volume function at the point X_0 are obviously equal to one another and hence to $\frac{C}{2}$. For any $\lambda \in [0, \mathrm{dist}(X_0, X)]$ denote by $X(\lambda)$ the plane obtained from X_0 by the parallel shift towards X by the distance λ. The derivatives of the volume functions $V_W(X(\lambda))$ over the parameter λ are then equal to $\pm$ the volume from Lemma 2. So, when we come to X, these volumes grow/decrease by

$$\frac{\Omega}{\cos\alpha(X)} \times \mathrm{dist}(X_0, X).$$

Consider the right triangle in $\mathbb{R}^{3+m}$ whose vertices are the origin and its projections to the planes X and $X \cap \mathbb{R}^3_x$. Its angle at the origin is equal to $\alpha(X)$, the leg at this vertex is equal to $\mathrm{dist}(X_0, X)$, and the hypotenuse is exactly the fraction in (9). $\square$

Remark 4. We see that a *locally* algebraically integrable body in $\mathbb{R}^N$ (that is, a body having nontrivial lacunas) does not need to be algebraic itself: in fact, only finite smoothness is demanded on the function $\psi(y_1, \dots, y_m)$ participating in the construction of our examples.

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SCHUBERT DECOMPOSITION FOR MILNOR FIBERS OF THE VARIETIES OF SINGULAR MATRICES

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Abstract. We consider the varieties of singular $m \times m$ complex matrices which may be either general, symmetric or skew-symmetric (with m even). For these varieties we have shown in another paper that they had compact "model submanifolds" for the homotopy types of the Milnor fibers which are classical symmetric spaces in the sense of Cartan. In this paper we use these models, combined with results due to a number of authors concerning the Schubert decomposition of Lie groups and symmetric spaces via the Cartan model, together with Iwasawa decomposition, to give cell decompositions of the global Milnor fibers.

The Schubert decomposition is in terms of "unique ordered factorizations" of matrices in the Milnor fibers as products of "pseudo-rotations". In the case of symmetric or skew-symmetric matrices, this factorization has the form of iterated "Cartan conjugacies" by pseudo-rotations. The decomposition respects the towers of Milnor fibers and symmetric spaces ordered by inclusions. Furthermore, the "Schubert cycles", which are the closures of the Schubert cells, are images of products of suspensions of projective spaces (complex, real, or quaternionic as appropriate). In the cases of general or skew-symmetric matrices the Schubert cycles have fundamental classes, and for symmetric matrices mod 2 classes, which give a basis for the homology. They are also shown to correspond to the cohomology generators for the symmetric spaces. For general matrices the duals of the Schubert cycles are represented as explicit monomials in the generators of the cohomology exterior algebra; and for symmetric matrices they are related to Stiefel-Whitney classes of an associated real vector bundle.

Furthermore, for a matrix singularity of any of these types. the pull-backs of these cohomology classes generate a characteristic subalgebra of the cohomology of its Milnor fiber.

We also indicate how these results extend to exceptional orbit hypersurfaces, complements and links, including a characteristic subalgebra of the cohomology of the complement of a matrix singularity.

Preamble: Motivation from the Work of Brieskorn

After Milnor developed the basic theory of the Milnor fibration and the properties of Milnor fibers and links for isolated hypersurface singularities, Brieskorn was involved in fundamental ways in developing a more complete theory of isolated hypersurface singularities. Furthermore through the work of his many students the theory was extended to isolated complete intersection singularities.

For isolated hypersurface singularities Brieskorn developed the importance of the intersection pairing on the Milnor fiber [Br]. This includes the computation of the intersection index for Pham-Brieskorn singularities, leading to the discovery that for a number of these singularities the link is an exotic topological sphere. He also demonstrated in a variety of ways that group theory in various forms plays an essential role in understanding the structure of singularities. This

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includes the relation between the monodromy and the Milnor fiber cohomology by the Gauss-Manin connection, and including the intersection pairing [Br2]. This includes the relation with Lie groups, especially for the ADE classification for simple hypersurface singularities, where he identified the intersection pairing with the Dynkin diagrams for the corresponding Lie groups. He also gave the structure of the discriminant for the versal unfoldings using the Weyl quotient map on the subregular elements of the Lie group [Br3]. In combined work with Arnold [Br4], he further showed that for the simple ADE singularities the complement of the discriminant is a $K(\pi, 1)$. He continued on beyond the simple singularities to understand the corresponding structures for unimodal singularities [Br5], setting the stage for further work in multiple directions.

The approaches which he initiated provide models for approaching questions for highly non-isolated hypersurface singularities which are used in this paper. For matrix singularities, the high-dimensional singular set means that the Milnor fiber, complement and link have low connectivity and hence can have (co)homology in many degrees [KMs]. To handle this complexity for matrix singularities of the various types, Lie group methods are employed to answer these questions. Partial answers were already given in [D3], including determining the (co)homology of the Milnor fibers using representations as symmetric spaces. This continues here by obtaining geometric models for the homology classes, understanding the analogue of the intersection pairing on the Milnor fiber via a Schubert decomposition, determining the structure of the link and complement, and their relations with the cohomology structure. We see that there is the analogue of the ADE classification which is given for the matrix singularities by the ABCD classification for the infinite families of simple Lie groups. We also indicate how these geometric methods extend to complements and links, including more general exceptional orbit hypersurfaces for prehomogeneous spaces.

Introduction

In this paper we derive the Schubert cell decomposition of the Milnor fibers of the varieties of singular matrices for $m \times m$ complex matrices which may be either general, symmetric, or skew-symmetric (with m even). We show that there is a homology basis obtained from "Schubert cycles", which are the closures of these cells. We further identify these homology classes with the cohomology. For general matrices we identify the correspondence with monomials of the generators for the exterior cohomology algebra and for symmetric matrices we identify the Schubert classes with monomials in the Stiefel-Whitney classes of an associated vector bundle. We also indicate how these results extend to more general exceptional orbit varieties and for the complements and links for all of these cases. Furthermore, for general matrix singularities defined from these matrix types, we define characteristic subalgebras of the cohomology of the Milnor fibers and complements representing them as modules over these subalgebras.

In [D3] we computed the topology of the exceptional orbit hypersurfaces for classes of prehomogeneous spaces which include these varieties of singular matrices. This included the topology of the Milnor fiber, link, and complement. This used the representation of the complements and the global Milnor fibers as homogeneous spaces which are homotopy equivalent to compact models which are classical symmetric spaces studied by Cartan. These symmetric spaces have representations as "Cartan models", which can be identified as compact submanifolds of the global Milnor fibers.

We use the Schubert decomposition for these symmetric spaces developed by Kadzisa-Mimura [KM] building on the earlier results for Lie groups and Stiefel manifolds by J. H. C. Whitehead [W], C.E. Miller, [Mi], I. Yokota [Y]. This allows us to give a Schubert decomposition for the compact models of the Milnor fibers, which together with Iwasawa decomposition provides a cell decomposition for the global Milnor fibers in terms of the Schubert decomposition for these symmetric spaces.

The Schubert decompositions are in terms of cells defined by the unique "ordered factorizations" of matrices in the Milnor fibers into "pseudo-rotations" of types depending on the matrix type, and their relation to a flag of subspaces. For symmetric or skew-symmetric matrices, this factorization has the form of iterated "Cartan conjugacies" by the pseudo-rotations. These are given by a modified form of conjugacy which acts on the Cartan models.

The Schubert decomposition is then further related to the co(homology) of the global Milnor fibers. We do so by showing the Schubert cycles for the symmetric spaces are images of products of suspensions of projective spaces of various types (complex, real, and quaternionic as appropriate). This allows us to relate the duals of the fundamental classes of the Schubert cycles (mod 2 classes for symmetric matrices) to the cohomology classes given for Milnor fibers in [D1]. These are given for the different matrix types and various coefficients as exterior algebras. In the symmetric matrix case the cohomology with $\mathbb{Z}/2\mathbb{Z}$ coefficients is given as an exterior algebra on the Stiefel-Whitney classes of an associated real vector bundle. For coefficient fields of characteristic zero the generators are classes which transgress to characteristic classes of appropriate types.

We further indicate how these methods also apply to exceptional orbit hypersurfaces in [D3] and how they further extend to the complements of the varieties and their links.

Lastly, we show that for matrix singularities of these matrix types, we can pull-back the cohomology algebras of the global Milnor fibers to identify characteristic subalgebras of the Milnor fibers for these matrix singularities. This represents the cohomology of the Milnor fiber of a matrix singularity of any of these types as a module over the corresponding characteristic subalgebra. We also indicate how this also holds for the cohomology of the complement.

1. Cell Decomposition for Global Milnor Fibers in Terms of their Compact Models

We consider the varieties of singular $m \times m$ complex matrices which may be either general, symmetric, or skew-symmetric (with m even). In [D1] we investigated the topology of these singularities, including the topology of the Milnor fiber, link and complement. This was done by viewing them as the exceptional orbit varieties obtained by the representation of a complex linear algebraic group G on a complex vector space V with open orbit. For example this includes the cases where $V = M$ is one of the spaces of complex matrices $M = Sym_m$ or $M = Sk_m$ (for $m = 2k$) acted on by $\mathrm{GL}_m(\mathbb{C})$ by $B \cdot A = BAB^T$, or , $M = M_{m,m}$ and $\mathrm{GL}_m(\mathbb{C})$ acts by left multiplication. Each of these representations have open orbits and the resulting prehomogeneous space has an exceptional orbit variety $\mathcal{E}$ which is a hypersurface of singular matrices.

Definition 1.1. The *determinantal hypersurface* for the space of $m \times m$ symmetric or general matrices, denoted by $M = Sym_m$ or $M = M_{m,m}$ is the hypersurface of singular matrices defined by det : $M \to \mathbb{C}$ and denoted by $\mathcal{D}_m^{(sy)}$ for $M = Sym_m$, or $\mathcal{D}_m$ for $M = M_{m,m}$. For the space of $m \times m$ skew-symmetric matrices $M = Sk_m$ (for $m = 2k$) the determinantal hypersurface of singular matrices is defined by the Pfaffian Pf : $Sk_m \to \mathbb{C}$, and is denoted by $\mathcal{D}_m^{(sk)}$. In the following we uniformly denote any of these functions as f.

Then, we showed in [D3] that the Milnor fibers for each of these singularities at 0 are diffeomorphic to their global Milnor fibers $f^{-1}(1)$ which are denoted by: F_m for general case, $F_m^{(sy)}$ for the symmetric case, and $F_m^{(sk)}$ for the skew-symmetric case. Then, we show in Theorem 3.1 in [D3, §3] that each global Milnor fiber is acted on transitively by a linear algebraic group and so is a homogeneous space. In particular, $F_m = SL_m(\mathbb{C})$, $F_m^{(sy)} \simeq SL_m(\mathbb{C})/SO_m(\mathbb{C})$, and

$F_{2m}^{(sk)} \simeq SL_{2m}(\mathbb{C})/Sp_m(\mathbb{C})$. Moreover, these spaces have as deformation retracts spaces which are symmetric spaces of classical type studied by Cartan: $SL_m(\mathbb{C})$ has as deformation retract SU_m; $SL_m(\mathbb{C})/SO_m(\mathbb{C})$ has as deformation retract SU_m/SO_m; and $SL_{2m}(\mathbb{C})/Sp_m(\mathbb{C})$ has as deformation retract SU_{2m}/Sp_m. These are compact models for the Milnor fibers and we denote them as F_m^c, $F_m^{(sy)\,c}$, and $F_{2m}^{(sk)\,c}$ respectively.

This allowed us to obtain the rational (co)homology (and integer cohomology for the general and skew-symmetric cases and the $\mathbb{Z}/2\mathbb{Z}$ cohomology for the symmetric cases), as well as using the Bott periodicity theorem to compute the homotopy groups in the stable range.

We will now further use the cell decompositions of the symmetric spaces together with Iwasawa decomposition to give the cell decompositions for the global Milnor fibers. We recall the Iwasawa decomposition for $SL_m(\mathbb{C})$ has the form KAN where $K = SU_m$, A_m consists of diagonal matrices with real positive entries of $\det = 1$, and N_m is the nilpotent group of upper triangular complex matrices with 1's on the diagonal. In particular, this means that the map

$$SU_m \times A_m \times N_m \to SL_m(\mathbb{C})$$

sending $(U, B, C) \mapsto U \cdot B \cdot C$ is a real algebraic diffeomorphism. Alternatively $A_m \cdot N_m$ consists of the upper triangular matrices of $\det = 1$ with complex entries except having real positive entries on the diagonal. As a manifold it is diffeomorphic to a Euclidean space of real dimension $2\binom{m}{2} + m - 1$. We denote this subgroup of $SL_m(\mathbb{C})$ as Sol_m, which is a real solvable subgroup of $SL_m(\mathbb{C})$.

For any of the preceding cases, let F denote the Minor fiber and Y the compact symmetric space associated to it. Suppose that Y has a cell decomposition with open cells $\{e_i : I = 1, \dots, r\}$. Then, we have the following simple proposition.

Proposition 1.2. *With the preceding notation, the cell decomposition of F is given by*

$$\{e_i \cdot \mathrm{Sol}_m : I = 1, \dots, r\}.$$

Moreover, if the closure $\bar{e}_i$ has a fundamental homology class (for Borel-Moore homology), then $\overline{e_i \cdot \mathrm{Sol}_m} = \bar{e}_i \cdot \mathrm{Sol}_m$ has a fundamental homology class with the same Poincaré dual.

Proof. By the Iwasawa decomposition $Y \times \mathrm{Sol}_m \simeq F$ via $(U, B) \mapsto U \cdot B$. Hence, if for $i \neq j$, $e_i \cap e_j = \emptyset$, then $(e_i \times \mathrm{Sol}_m) \cap (e_j \times \mathrm{Sol}_m) = \emptyset$ and $(e_i \cdot \mathrm{Sol}_m) \cap (e_j \cdot \mathrm{Sol}_m) = \emptyset$. Also, as $Y = \cup_i e_i$ is a disjoint union, so also is $F = \cup_i e_i \cdot \mathrm{Sol}_m$. Third, each $e_i \times \mathrm{Sol}_m$ is homeomorphic to a cell of dimension $\dim_{\mathbb{R}}(e_i) + 2\binom{m}{2} + m - 1$. Thus, F is a disjoint union of the cells $e_i \cdot \mathrm{Sol}_m$. Lastly, $\bar{e}_i = e_i \cup_{j_i} e_{j_i}$ where the last union is over cells of dimension less than $\dim e_i$. Hence, $e_i \cdot \bar{\mathrm{S}}\mathrm{ol}_m = \bar{e}_i \cdot \mathrm{Sol}_m = (e_i \cdot \mathrm{Sol}_m) \cup_{j_i} (e_{j_i} \cdot \mathrm{Sol}_m)$. Hence this is a cell decomposition.

Then, $\bar{e}_i$ is a singular manifold with open smooth $\underline{\text{manifold } e_i.}$ If it has a Borel-Moore fundamental class, which restricts to that of e_i, then so does $\overline{e_i \cdot \mathrm{Sol}_m}$ have a fundamental class that restricts to that for $e_i \cdot \mathrm{Sol}_m \simeq e_i \times \mathrm{Sol}_m$. Then, as $\bar{e}_i$ is the pull-back of $\overline{e_i \cdot \mathrm{Sol}_m}$ under the map $i : Y \to Y \times \mathrm{Sol}_m \simeq F$ which is transverse to $\bar{e}_i \times \mathrm{Sol}_m \simeq \overline{e_i \cdot \mathrm{Sol}_m}$, by a fiber-square argument for Borel-Moore homology, the Poincaré dual of $\overline{e_i \cdot \mathrm{Sol}_m}$ pulls-back via i^* to the Poincaré dual of $\bar{e}_i$. As i is a homotopy equivalence, via the isomorphism i^* the Poincaré duals agree. $\square$

2. Cartan Models for the Symmetric Spaces

The General Cartan Model.

By Cartan, a symmetric space is defined by a Lie group G with an involution $\sigma : G \to G$ so that the symmetric space is given by the quotient space G/G^σ, where G^σ denotes the subgroup of G invariant under σ. Furthermore this space can be embedded into the Lie group G. The embedding is called the *Cartan model*. It is defined as follows, where we follow the approach of

Kadzisa-Mimura [KM] and the references therein. They introduce two subsets M and N of G defined by:

$$M \;=\; \{g\sigma(g^{-1}) : g \in G\} \qquad \text{and} \qquad N \;=\; \{g \in G : \sigma(g^{-1}) = g\}.$$

Then, we have $G/G^\sigma \simeq M \subset N$. The inclusion is the obvious one, and the homeomorphism is given by $g \mapsto g\sigma(g^{-1})$. Via this homeomorphism, we may identify the symmetric space G/G^σ with the subset $M \subset G$. The subspace N is closed in G, and it can be shown that M is the connected component of N containing the identity element. In the three cases we consider, it will be the case that $M = N$.

We also note that while M and N are subspaces of G, they are not preserved under products nor conjugacy; however they do have the following properties.

Further Properties of the Cartan Model:

i) there is an action of G on both M and N defined by $g \cdot h = gh\sigma(g^{-1})$ and on M it is transitive;
ii) the homeomorphism $G/G^\sigma \simeq M$ is G-equivariant under left multiplication on G/G^σ and the preceding action on M;
iii) both M and N are invariant under taking inverses; and
iv) if $g, h \in N$ commute then $gh \in N$.

For U_n, $g^* = g^{-1}$ so an alternative way to write the action in i) is given by $g \mapsto h \cdot g \cdot \sigma(h^*)$. We will refer to this action as *Cartan conjugacy*.

Then, Kadzisa-Mimura use the cell decompositions for various G to give the cell decompositions for M and hence the symmetric space G/G^σ. There is one key difference with what we will do versus what Kadzisa-Mimura do. They give the cell decomposition; however we also want to represent the closed cells where possible as the images of specific singular manifolds, specifically products of suspensions of projective spaces of various types and to relate the fundamental homology classes to corresponding classes in cohomology. Together with the reasoning in §1 and the identification of the global Milnor fibers with the Cartan models, we will then be able to give the Schubert decomposition for the global Milnor fibers and identify the Schubert homology classes with dual cohomology classes.

The Cartan Models for SU_m, SU_m/SO_m, and SU_{2m}/Sp_m.

For the three cases we consider: SU_m , SU_m/SO_m, SU_{2m}/Sp_m, we first observe that the exact sequence of groups (2.1) does not split

$$1 \longrightarrow SU_m \longrightarrow U_m \xrightarrow{\det} S^1 \longrightarrow 1\,. \tag{2.1}$$

However, it does split as manifolds $U_m \simeq S^1 \times SU_m$ sending

$$C \mapsto (\det(C), I_{1,m-1}(\det(C)) \cdot C),$$

where $I_{1,m-1}(\det(C)^{-1})$ is the $m \times m$ diagonal matrix with 1's on the diagonal except in the first position where it is $\det(C)^{-1}$. Thus, topological statements about U_m have corresponding statements about SU_m and conversely.

We first give the representation for the symmetric spaces. For SU_m we just use itself as a compact Lie group.

Next, for $SU(m)/SO(m)$ we let the involution σ on $SU(m)$ be defined by $C \mapsto \overline{C}$. We see that $\sigma(C) = C$ is equivalent to $C = \overline{C}$. Thus C is a real matrix which is unitary; and hence C is real orthogonal. As $\det(C) = 1$, we see that $SU_m^\sigma = SO_m$.

The third case is SU_{2n}/Sp_n for $m = 2n$. In this case, the involution σ on SU_{2n} sends $C \mapsto J_n\overline{C}J_n^*$ where J_n is the $2n \times 2n$ block diagonal matrix with 2×2 diagonal blocks $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

As $J_n^* = J_n^T = -J_n = J_n^{-1}$, then $\sigma(C) = C$ is equivalent to $J_n\overline{C}J_n = -C$, or as $C^{-1} = \overline{C}^T$ we can rearrange to obtain $C^T J_n C = J_n$ (or alternatively $C J_n C^T = J_n$), which implies that C leaves invariant the bilinear form $(v, w) = v^T J_n w$ (for column vectors v and w) and so is an element of $Sp_n(\mathbb{C})$, and so an element of $Sp_n = SU_{2n} \cap Sp_n(\mathbb{C})$.

The corresponding Cartan models are then given as follows. We denote the Cartan models by respectively: $\mathcal{C}_m$, $\mathcal{C}_m^{(sy)}$, and $\mathcal{C}_m^{(sk)}$.

First, for $G = SU_m$, which is itself a symmetric space, and we let $\mathcal{C}_m = SU_m$. In this case, Cartan conjugacy is replaced by left multiplication.

Second, for SU_m/SO_m we claim

$$\mathcal{C}_m^{(sy)} \stackrel{def}{=} \{C \cdot C^T : C \in SU_m\} = \{B \in SU_m : B = B^T\}. \tag{2.2}$$

The inclusion of the LHS in the RHS is immediate. For the converse, we note that if $B \in SU_m$ and $B = B^T$, then by the following Lemma given in [KM] there is an orthonormal basis of eigenvectors which are real vectors so we may write $B = ACA^{-1}$ with A an orthogonal matrix and C a diagonal matrix with diagonal entries λ_j so that $|\lambda_j| = 1$. Thus, $A^{-1} = A^T$, and so $B = ADA^T \cdot ADA^T$ with D a diagonal matrix with entries $\sqrt{\lambda_j}$.

Lemma 2.1. *If $B \in SU_m$ and $B = B^T$ then there is a real orthonormal basis of eigenvectors for B.*

This is a simple consequence of the eigenspaces being invariant under conjugation, which is easily seen to follow from the conditions. In this case, Cartan conjugacy by A on B is checked to be given by $B \mapsto A \cdot B \cdot A^T$.

Third, for SU_{2n}/Sp_n with $m = 2n$, we may directly verify

$$\mathcal{C}_m^{(sk)} \stackrel{def}{=} \{C \cdot J_n \cdot C^T \cdot J_n^* : C \in SU_{2n}\} = \{B \in SU_{2n} : (B \cdot J_n)^T = -B \cdot J_n\}. \tag{2.3}$$

Then, Cartan conjugacy by A on B is given by $B \mapsto A \cdot (B \cdot J_n) \cdot A^T \cdot J_n^{-1}$, with $B \cdot J_n$ skew-symmetric for $B \in \mathcal{C}_m^{(sk)}$.

Hence, from (2.2), we have the compact model for $F_m^{(sy)}$ as a subspace is given by

$$F_m^{(sy)\,c} = SU_m \cap Sym_m(\mathbb{C})$$

and the Cartan model for the symmetric space SU_m/SO_m is given by $F_m^{(sy)\,c}$ itself. Similarly, from (2.3), we have the compact model for $F_m^{(sk)}$ with $m = 2n$ as a subspace is given by $F_m^{(sk)\,c} = SU_m \cap Sk_m(\mathbb{C})$ and the Cartan model for the symmetric space SU_{2n}/Sp_n is given by $F_m^{(sk)\,c} \cdot J_n^{-1}$.

Remark 2.2. Frequently for all three cases, we will want to apply a Cartan conjugate for an element of U_n instead of SU_n. The formula for the Cartan conjugate remains the same and the corresponding symmetric spaces are U_n, U_n/O_n, and U_{2n}/Sp_n. By the properties of Cartan conjugacy, an iteration of Cartan conjugacy by elements $A_i \in U_n$ whose product belongs to SU_n will be a Cartan conjugate by an element of SU_n and preserve the Cartan models of interest to us.

Tower Structures of Global Milnor fibers and Symmetric Spaces by Inclusion.

Lastly, these global Milnor fibers, symmetric spaces and compact models form towers via inclusions: i) sending $A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$ for $SU_m \subset SU_{m+1}$, $F_m \subset F_{m+1}$, or $F_m^{(sy)} \subset F_{m+1}^{(sy)}$ which induce inclusions of the symmetric spaces SU_m and SU_m/SO_m and corresponding global Milnor fibers, or ii) sending $A \mapsto \begin{pmatrix} A & 0 \\ 0 & I_2 \end{pmatrix}$ for the 2×2 identity matrix I_2 for $SU_m \subset SU_{m+2}$ for $m = 2n$ and the corresponding symmetric spaces SU_{2n}/Sp_n and Milnor fibers $F_m^{(sk)} \subset F_{m+2}^{(sk)}$. The Schubert decompositions will satisfy the additional property that they respect the inclusions.

We summarize these results by the following.

Proposition 2.3. *For the varieties of singular $m \times m$ complex matrices which are either general, symmetric or skew-symmetric, their global Milnor fibers, representations as homogeneous spaces, compact models given as symmetric spaces and Cartan models are summarized in Table 1.*

Milnor Fiber $F_m^{(*)}$	Quotient Space	Symmetric Space	Compact Model $F_m^{(*)\,c}$	Cartan Model
F_m	$SL_m(\mathbb{C})$	SU_m	SU_m	F_m^c
$F_m^{(sy)}$	$SL_m(\mathbb{C})/SO_m(\mathbb{C})$	SU_m/SO_m	$SU_m \cap Sym_m(\mathbb{C})$	$F_m^{(sy)\,c}$
$F_m^{(sk)}, m = 2n$	$SL_{2n}(\mathbb{C})/Sp_n(\mathbb{C})$	SU_{2n}/Sp_n	$SU_m \cap Sk_m(\mathbb{C})$	$F_m^{(sk)\,c} \cdot J_n^{-1}$

Table 1. Global Milnor fiber, its representation as a homogenenous space, compact model as a symmetric space, compact model as subspace and Cartan model.

3. Schubert Decomposition for Compact Lie Groups

We recall the "Schubert decomposition"for compact Lie groups, concentrating on SU_n. The cell decompositions of certain compact Lie groups, especially SO_n and U_n and SU_n were carried out by C. E. Miller [Mi] and I. Yokota [Y], building on the work of J. H. C. Whitehead [W] for the cell decomposition of Stiefel varieties. In the case of Grassmannians, the Schubert decomposition is in terms of the dimensions of the intersections of the subspaces with a given fixed flag of subspaces. For these Lie groups, elements are expressed as ordered products of (complex) "pseudo rotations"about complex hyperplanes (or reflections about real hyperplanes in the case of SO_n). The cell decomposition is based on the subspaces of a fixed flag that contain the orthogonal lines to the hyperplane axes of rotation (or reflection). We will concentrate on the complex case which is relevant to our situation.

(Complex) Pseudo-Rotations.

We note that given a complex 1–dimensional subspace $L \subset \mathbb{C}^n$, we can define a "(complex) pseudo-rotation"about the orthogonal hyperplane $L^{\perp}$ as follows. Let $x \in L$ be a unit vector. As L is complex we have a positive sense of rotation through an angle θ given by $x \mapsto e^{i\theta}x$. We extend this to be the identity on $L^{\perp}$. This is given by the following formula for any $x' \in \mathbb{C}^n$:

$$A_{(\theta,x)}(x') \;=\; x' - ((1 - e^{i\theta}) < x', x >)\, x\,.$$

This is not a true rotation as a complex linear transformation so we refer to this as a "pseudo-rotation". Then, $A_{(\theta,x)}$ can be written in matrix form as $A_{(\theta,x)} = (I_n - (1 - e^{i\theta})\, x \cdot \bar{x}^T)$ for x an n-dimensional column vector.

Remark 3.1. In the special case that $A_{(\theta,x)}$ has finite order as an element of the group U_n, it is called a "complex reflection".

We observe a few simple properties of pseudo-rotations:

i) $A_{(\theta,x)}$ only depends on $L =< x >$, so we will also feel free to use the alternate notation $A_{(\theta,L)}$;
ii) $A_{(\theta,x)}$ is a unitary transformation with $\det(A_{(\theta,x)}) = e^{i\theta}$;
iii) if $B \in U_n$, then $B \cdot A_{(\theta,x)} \cdot B^{-1} = A_{(\theta,Bx)}$ is again a pseudo-rotation; and
iv) $\overline{A_{(\theta,x)}} = A_{(-\theta,\bar{x})}$; $A^{-1}_{(\theta,x)} = A_{(-\theta,x)}$; and $A^T_{(\theta,x)} = A_{(\theta,\bar{x})}$.

Ordered Factorizations in SU_m and Schubert Symbols.

Then, given any $B \in SU_n$, we may diagonalize B using an orthonormal basis $\{v_1, \dots, v_n\}$ so if C denotes the unitary matrix with the v_i as columns, then we may write $B = CDC^{-1}$ where D is a diagonal matrix with diagonal entries λ_i of unit length so that $\prod_{i=1}^n \lambda_i = 1$. This can be restated as saying that B is a product of pseudo-rotations about the hyperplanes $< v_j >^{\perp}$ with angles θ_j where $\lambda_j = e^{i\theta_j}$. Thus, $B = \prod_{j=1}^n A_{(\theta_j,v_j)}$. However, we note that as certain eigenspaces may have dimension > 1, the terms and their order in the product are not unique.

There is a method introduced by Whitehead and used by Miller and Yokota for obtaining a unique factorization leading to the Schubert decomposition in SU_n. The product is rewritten as a product of different pseudo-rotations whose lines satisfy certain inclusion relations for a fixed flag leading to an ordering of the pseudo-rotations. We let $0 \subset \mathbb{C} \subset \mathbb{C}^2 \subset \cdots \subset \mathbb{C}^n$ denote the standard flag. Then, if $L =< x > \subset \mathbb{C}^k$ and $L =< x > \not\subset \mathbb{C}^{k-1}$, we will say that x *and* L *minimally belong to* $\mathbb{C}^k$ and introduce the notation $x \in_{\min} \mathbb{C}^k$ or $L \subset_{\min} \mathbb{C}^k$. If $x = (x_1, x_2, \dots, x_n)$ then $x \in_{\min} \mathbb{C}^k$ iff $x_{k+1} = \cdots = x_n = 0$ and $x_k \neq 0$. We observe two simple properties: if $x \in_{\min} \mathbb{C}^k$ then $\bar{x} \in_{\min} \mathbb{C}^k$; and if $x' \in_{\min} \mathbb{C}^{k'}$ with $k' < k$, then $A_{(\theta,x')}(x) \in_{\min} \mathbb{C}^k$.

Then to rewrite the product in a different form, we proceed, as in the other papers, to follow Whitehead with the following lemma.

Lemma 3.2. *Suppose that we have two pseudo-rotations $A_{(\theta,x)}$ and $A_{(\theta',x')}$ with $x \in_{\min} \mathbb{C}^m$ and $x' \in_{\min} \mathbb{C}^{m'}$.*

1) *If $m > m'$, then*

$$A_{(\theta,x)} \cdot A_{(\theta',x')} \;=\; A_{(\theta',x')} \cdot A_{(\theta,\tilde{x})}, \tag{3.1}$$

where $\tilde{x} = A^{-1}_{(\theta',x')}(x)$.

2) *If $m = m'$, and $< x > \neq < x' >$ let $W =< x, x' >$, which has dimension 2, and let $L =< \tilde{x} >= W \cap \mathbb{C}^{m-1}$, with $\tilde{x} \in_{\min} \mathbb{C}^k$ for $k \leq m-1$. Then, there exist pseudo-rotations $A_{(\tilde{\theta},\tilde{x})}$ and $A_{(\tilde{\theta}',\tilde{x}')}$ with $\tilde{x} \in_{\min} \mathbb{C}^k$ and $\tilde{x}' \in_{\min} \mathbb{C}^m$ such that*

$$A_{(\theta,x)} \cdot A_{(\theta',x')} \;=\; A_{(\tilde{\theta},\tilde{x})} \cdot A_{(\tilde{\theta}',\tilde{x}')} \,. \tag{3.2}$$

Moreover, for generic $x, x' \in_{\min} \mathbb{C}^m$, $\tilde{x} \in_{\min} \mathbb{C}^{m-1}$.

Proof. For 1), by property iii) of pseudo-rotations, $A^{-1}_{(\theta',x')} \cdot A_{(\theta,x)} \cdot A_{(\theta',x')}$ is a pseudo-rotation of the form $A_{(\theta,\tilde{x})}$ with $\tilde{x} = A^{-1}_{(\theta',x')}(x)$. Also, both $A_{(\theta,x)}$ and $A_{(\theta',x')}$ are the identity on $\mathbb{C}^{m\perp}$; hence $\tilde{x} \in_{\min} \mathbb{C}^m$.

For 2), if $< x >=< x' >$, then the pseudo-rotations commute. Next, suppose these lines differ so the complex subspace W spanned by x and x' is 2-dimensional. Then, $\dim_{\mathbb{C}} W \cap \mathbb{C}^{m-1} = 1$. We denote it by L and let it be spanned by a unit vector $\tilde{x}$ with say $\tilde{x} \in_{\min} \mathbb{C}^k$ for $k \leq m-1$ (and generically $k = m-1$). We note that both pseudo-rotations are the identity on $W^{\perp}$. Also, $W \subset \mathbb{C}^m$. It is sufficient to consider the pseudo-rotations restricted to $W \simeq \mathbb{C}^2$ with $\tilde{x}$ denoted by e_2 and orthogonal unit vector e_1. Then, let $(A_{(\theta,x)} \cdot A_{(\theta',x')})^{-1}(e_1) = v$. Then, we want a pseudo-rotation on W that sends $e_1 \mapsto v$. If $v \neq -e_1$, then reflection about the complex line spanned by $e_1 + v$, is a pseudo-rotation by π and sends e_1 to v. If $v = -e_1$, then reflection about the complex line spanned by e_2 works instead. If we denote this reflection by $A_{(\pi,\tilde{x}')}$, then

$A_{(\theta,x)} \cdot A_{(\theta',x')} \cdot A_{(\pi,\tilde{x}')}$ is a unitary transformation which fixes e_1 and is hence a pseudo-rotation about the line $< e_1 >$ and so sends $e_2 = \tilde{x}$ to $e^{i\tilde{\theta}}\tilde{x}$ for some angle $\tilde{\theta}$. Thus,

$$A_{(\theta,x)} \cdot A_{(\theta',x')} \; = \; A_{(\tilde{\theta},\tilde{x})} \cdot A_{(\tilde{\theta}',\tilde{x}')}$$

giving the result. □

This allows us to rewrite a product of pseudo-rotations as a product where the lines are minimally contained in successively larger subspaces of the flag.

Whitehead Algorithm for ordered factorization of Unitary matrices. Given $B \in SU_n$, we may write $B = \prod_{j=1}^{k} A_{(\theta_j,x_j)}$, with the $\{x_j\}$ an orthonormal set of vectors with say $x_j \in_{\min} \mathbb{C}^{m_j}$. Note that k may be less than n as we may exclude the eigenvectors x'_j with eigenvalue 1, which give $A_{(0,x'_j)} = I_n$. Then, we may use Lemma 3.2 to reduce the product into a standard form as follows. For the sequence $(m_1, m_2, \dots, m_k)$, we find the largest j so that $m_j \geq m_{j+1}$. If $m_j > m_{j+1}$, then by 1) of Lemma 3.2, we may replace $A_{(\theta_j,x_j)} \cdot A_{(\theta_{j+1},x_{j+1})}$ by $A_{(\theta_{j+1},x_{j+1})} \cdot A_{(\theta_j,\tilde{x}_j)}$, with $\tilde{x}_j \in_{\min} \mathbb{C}^{m_j}$. If instead $m_j = m_{j+1}$, then by 2) of Lemma 3.2, we may instead replace the product by $A_{(\theta'_j,x'_j)} \cdot A_{(\theta'_{j+1},x'_{j+1})}$, where $x'_{j+1} \in_{\min} \mathbb{C}^{m_j}$ and $x'_j \in_{\min} \mathbb{C}^{\ell}$, where $\ell < m_j$ satisfies $(< x_j, x_{j+1} > \cap \mathbb{C}^{m_j}) \subset_{\min} \mathbb{C}^{\ell}$.

Then, we relabel the angles and vectors to be (θ_j, x_j), where now $m_j < m_{j+1} < \cdots < m_k$. Then, we may repeat the procedure until we obtain $m_1 < m_2 < \cdots < m_k$. We summarize the final result of this process.

Lemma 3.3. *Given $B \in SU_n$, it may be written as a product*

$$B \; = \; A_{(\theta_1,x_1)} \cdot A_{(\theta_2,x_2)} \cdots A_{(\theta_k,x_k)}\,, \tag{3.3}$$

with $x_j \in_{\min} \mathbb{C}^{m_j}$ and $1 \leq m_1 < m_2 < \cdots < m_k \leq n$, and each $\theta_i \not\equiv 0 \bmod 2\pi$.

If B has the form given in Lemma 3.3 with $m_1 > 1$, then we will say that B has Schubert type $\mathbf{m} = (m_1, m_2, \cdots, m_k)$ and write $\mathbf{m}(B) = \mathbf{m}$. If instead $m_1 = 1$, then as $\det(B) = 1$

$$B = A_{(-\tilde{\theta},e_1)} \cdot A_{(\theta_2,x_2)} \cdot A_{(\theta_2,x_2)} \cdots A_{(\theta_k,x_k)},$$

where $\tilde{\theta} \equiv \sum_{j=2}^{k} \theta_j \mathrm{mod}\ 2\pi$ and we instead denote $\mathbf{m}(B) = (m_2, \cdots, m_k)$. For the case of an empty sequence with $k = 0$, we associate the unique identity element I. We refer to the tuple $\mathbf{m} = (m_1, m_2, \cdots, m_k)$ as the Schubert symbol of B. It will follow from Theorem 3.7 that this representation is unique.

There is also an alternative way to obtain a factorization (3.3) where instead $x_j \in_{\min} \mathbb{C}^{m'_j}$ with a decreasing sequence $m'_1 > m'_2 > \cdots > m'_k$. In fact, if we give a representation for B^{-1} as in (3.3) with the m_i increasing, then taking inverses gives a product of $A^{-1}_{(\theta_i,x_i)} = A_{(-\theta_i,x_i)}$ in decreasing order. There is a question for a given $B \in SU_n$ about the relation between the increasing and decreasing symbols. The relation between these is a consequence of the following lemma which is basically that given in [KM, Prop. 4.5] and is a consequence of the uniqueness of the Schubert symbol for one direction of ordering.

Lemma 3.4. *Suppose $x_i \in_{\min} \mathbb{C}^{m_i}$, for $1 \leq i \leq k$ and $m_1 < m_2 < \cdots < m_k$; and $y_j \in_{\min} \mathbb{C}^{m'_j}$, for $1 \leq j \leq k'$ and $m'_1 < m'_2 < \cdots < m'_k$. Also, suppose $\theta_i, \theta'_i \not\equiv 0 \bmod 2\pi$ for each i. Let $A_i = A_{(\theta_i,x_i)}$ and $B_j = A_{(\theta'_j,y_j)}$. If*

$$A_1 \cdot A_2 \cdots A_k \; = \; B_{k'} \cdot B_{k'-1} \cdots B_1,$$

then the following hold:

a) *$k = k'$ and $(m_1, m_2, \dots, m_k) = (m'_1, \dots, m'_{k'})$;*

b) $A_i \; = \; B_1^{-1} \cdot B_2^{-1} \cdots B_{i-1}^{-1} \cdot B_i \cdot B_{i-1} \cdots B_1$ *for* $1 \leq i \leq k$*; and*

c) $B_i \; = \; A_1 \cdot A_2 \cdots A_{i-1} \cdot A_i \cdot A_{i-1}^{-1} \cdots A_1^{-1}$ *for* $1 \leq i \leq k$.

In the cases of $k = 1$ *in b) and c), we let* $A_0 = B_0 = I_m$ *so they are understood to be* $A_1 = B_1$.

Proof. We let C_i denote the RHS of the equation in b) but for $1 \leq i \leq k'$. Since $B_{i-1} \cdot B_{i-2} \cdots B_1$ leaves pointwise invariant $(\mathbb{C}^{m'_i})^{\perp}$, we conclude $B_{i-1} \cdot B_{i-2} \cdots B_1(y_i) = y'_i \in_{min} \mathbb{C}^{m'_i}$; hence by property iii) for pseudo rotations, $C_i = A_{(\theta'_i, y'_i)}$. Thus, we have that A has two different Schubert factorizations with increasing Schubert symbols $(m_1, m_2, \dots, m_k)$ and $(m'_1, \dots, m'_{k'})$. By the uniqueness of the Schubert symbols, we obtain a).

Furthermore, by the uniqueness of the Schubert decomposition stated in Theorem 3.7 (for increasing Schubert decomposition) and Remark 3.8, it then furthermore follows that $A_i = C_i$ for all i so b) holds. Lastly, the uniqueness of the increasing order Schubert decomposition implies by taking inverses that we also have uniqueness of decreasing order Schubert decomposition. Then, the corresponding analogue of the argument for b) yields c). □

We then have the following corollary

Corollary 3.5. *If* $B \in SU_n$*, then*

$$\mathbf{m}(B) \; = \; \mathbf{m}(B^{-1}) \; = \; \mathbf{m}(\overline{B}) \; = \; \mathbf{m}(B^T)\,.$$

Proof. Given an increasing Schubert factorization $B = A_1 \cdot A_2 \cdots A_k$ for $A_i = A_{(\theta_i, x_i)}$ with Schubert symbol $\mathbf{m} = (m_1, m_2, \dots, m_k)$, then $B^{-1} = A_k \cdot A_{k-1} \cdots A_1$ is a Schubert factorization for decreasing order. This has the decreasing Schubert symbol $(m_k, m_{k-1}, \dots, m_1)$, and hence B^{-1} has the same increasing Schubert symbol $\mathbf{m}$.

Next, $\overline{B} = \overline{A_1} \cdot \overline{A_2} \cdots \overline{A_k}$, and by property iv) of pseudo-rotations $\overline{A_i} = A_{(-\theta_i, \bar{x_i})}$ so the Schubert Symbol is the same.

Lastly, as $B \in SU_n$, $B^T = \overline{B^{-1}}$, which combined with the two other properties implies that it has the same Schubert symbol. □

Remark 3.6. We will use the increasing order for the Schubert symbol to be in agreement with that used for the Schubert decomposition as in Milnor-Stasheff [MS]. In fact, if $A = A_1 \cdot A_2 \cdots A_k$ for $A_i = A_{(\theta_i, x_i)}$ with Schubert symbol $\mathbf{m} = (m_1, m_2, \dots, m_k)$, and we let $V = \mathbb{C} < x_1, \dots, x_k >$, then $\dim_{\mathbb{C}} V \cap \mathbb{C}^{m_i} = i$ so V as an element of the Grassmannian $G_k(\mathbb{C}^n)$ would also have Schubert symbol $\mathbf{m}$. In [KM], the decreasing order Schubert symbol is used; however, we easily change between the two.

We next state the form of the Schubert decomposition given in terms of the Schubert factorization giving the Schubert types for elements of SU_n.

Schubert Decomposition for SU_n.

In describing the Schubert decomposition for SU_n, we are giving a version of that contained in [W], [Mi], [Y] and summarized in [KM] (but using instead an increasing order).

Given an increasing sequence $m_1 < m_2 < \cdots < m_k$ with $1 < m_1$ and $m_k \leq n$, which we denote by $\mathbf{m} = (m_1, m_2, \dots, m_k)$, we define a map

$$\psi_{\mathbf{m}} : S\mathbb{C}P^{m_1 - 1} \times S\mathbb{C}P^{m_2 - 1} \times \cdots \times S\mathbb{C}P^{m_k - 1} \longrightarrow SU_n\,,$$

where SX denotes the suspension of X. This is given as follows:

First, we define a simpler map for $m \leq n$, $I = [0, 1]$ and a complex line $L \subset \mathbb{C}^m$,

$$\tilde{\psi}_m : I \times \mathbb{C}P^{m-1} \to SU_n$$

defined by $\tilde{\psi}_m(t, L) = A_{(2\pi t, L)}$. Since $A_{(0,L)} = A_{(2\pi,L)} = I_n$ independent of L, this descends to a map $\psi_m : S\mathbb{C}P^{m-1} \to SU_n$. Then, we define

$$\begin{aligned} \psi_{\mathbf{m}}((t_1, L_1), \ldots, (t_k, L_k)) &= A_{(-2\pi\tilde{t}, e_1)} \cdot \psi_{m_1}(t_1, L_1) \cdot \psi_{m_2}(t_2, L_2) \cdots \psi_{m_k}(t_k, L_k) \\ &= A_{(-2\pi\tilde{t}, e_1)} \cdot A_{(2\pi t_1, L_1)} \cdot A_{(2\pi t_2, L_2)} \cdots A_{(2\pi t_k, L_k)}\,, \end{aligned} \tag{3.4}$$

where $\tilde{t} = \sum_{j=1}^{k} t_j$. We note that the first factor $A_{(-2\pi\tilde{t}, e_1)}$ ensures the product is in SU_n as in the splitting for (2.1).

We observe that each $I \times \mathbb{C}P^{m-1}$ has an open dense cell

$$E_m = (0, 1) \times \{x = (x_1, \ldots, x_m, 0, \ldots 0) : (x_1, \ldots, x_m) \in S^{2m-1} \text{ and } x_m > 0\},$$

which is of dimension $2m - 1$ (as $x_m = \sqrt{1 - \sum_{j=1}^{m-1} |x_j|^2}$). Also, if $x = (x_1, \ldots, x_m, 0, \ldots 0)$ with $x_m > 0$, then $x \in_{\min} \mathbb{C}^m$.

We now introduce some notation and denote

$$\tilde{S}_{\mathbf{m}} = S\mathbb{C}P^{m_1-1} \times S\mathbb{C}P^{m_2-1} \times \cdots \times S\mathbb{C}P^{m_k-1}\,;$$

also, we consider the corresponding cell

$$E_{\mathbf{m}} = E_{m_1} \times E_{m_2} \times \cdots \times E_{m_k}\,,$$

and the image $S_{\mathbf{m}} = \psi_{\mathbf{m}}(E_{\mathbf{m}})$ in SU_n. Then, $E_{\mathbf{m}}$ is an open dense cell in $\tilde{S}_{\mathbf{m}}$ with

$$\dim_{\mathbb{R}} E_{\mathbf{m}} = \sum_{j=1}^{k} (2m_j - 1) = 2|\mathbf{m}| - \ell(\mathbf{m})$$

for $|\mathbf{m}| = \sum_{j=1}^{k} m_j$ and $\ell(\mathbf{m}) = k$, which we refer to as the length of $\mathbf{m}$. Also, the image $S_{\mathbf{m}} = \psi_{\mathbf{m}}(E_{\mathbf{m}})$ consists of elements of SU_n of Schubert type $\mathbf{m}$. Furthermore, $\overline{S_{\mathbf{m}}} = \psi_{\mathbf{m}}(\tilde{S}_{\mathbf{m}})$. Then the results of Whitehead, Miller and Yokota together give the following Schubert decomposition of SU_n.

Theorem 3.7. *The Schubert decomposition of SU_n has the following properties:*

a) *SU_n is the disjoint union of the $S_{\mathbf{m}}$ as $\mathbf{m} = (m_1, \ldots, m_k)$ varies over all increasing sequences with $1 < m_1$, $m_k \leq n$, and $0 \leq k \leq n - 1$.*
b) *The map $\psi_{\mathbf{m}} : E_{\mathbf{m}} \to S_{\mathbf{m}}$ is a homeomorphism.*
c) *$(\overline{S_{\mathbf{m}}} \backslash S_{\mathbf{m}}) \subset \cup_{\mathbf{m}'} S_{\mathbf{m}'}$, where the union is over all $S_{\mathbf{m}'}$ with $\dim S_{\mathbf{m}'} < \dim S_{\mathbf{m}}$.*
d) *the Schubert cells $S_{\mathbf{m}}$ are preserved under taking inverses, conjugates, and transposes.*

We note that d) follows from Corollary 3.5.

Hence, the Schubert decomposition by the cells $S_{\mathbf{m}}$ is a cell decomposition of SU_n. The cells $S_{\mathbf{m}}$ are referred to as the *Schubert cells of SU_n*. We note that as $\overline{S_{\mathbf{m}}}$ is the image of the "singular manifold" $\tilde{S}_{\mathbf{m}}$ which has a Borel-Moore fundamental class, we can describe in §5 the homology of SU_n in terms of the images of these fundamental classes.

Remark 3.8. There is an analogous Schubert decomposition for U_n where the Schubert symbols can include $m_1 = 1$.

4. Schubert Decomposition for Symmetric Spaces

For the Milnor fibers for the varieties of singular matrices, we have compact models which are symmetric spaces. To give the Schubert decomposition of these, we use the results of Kadzisa and Mimura [KM] which modifies the Schubert decomposition given for SU_n to apply to the Cartan models for the symmetric spaces. We have given the Schubert decomposition for SU_n in the previous section so we will consider the form it takes for both SU_n/SO_n and SU_{2n}/Sp_n.

We again use the standard flag $0 \subset \mathbb{C} \subset \mathbb{C}^2 \subset \cdots \subset \mathbb{C}^n$ and the same notation for pseudo-rotations as in §3.

Schubert Decomposition for SU_n/SO_n.

We consider an element of the Cartan model $\mathcal{C}_n^{(sy)}$ for SU_n/SO_n. If $B \in \mathcal{C}_n^{(sy)}$ we have that $B \in SU_n$ and $B = B^T$. By Lemma 2.1, there is an orthonormal basis of real eigenvectors x_i for B. Hence, each $< x_i > \in \mathbb{R}P^{n-1}$. Then B can be written as a product of pseudo-rotations about complexifications of real hyperplanes $\mathbb{C} < x_i >^{\perp}$. We will refer to such a pseudo-rotation $A_{(\theta,x)}$ for a real vector x as an $\mathbb{R}$-pseudo-rotation. There are two problems in trying to duplicate the reasoning used for the Schubert decomposition for SU_n. First, there is no analogue of Lemma 3.2 for products of $\mathbb{R}$-pseudo-rotations. Second, it need not be true that the ordered product of $\mathbb{R}$-pseudo-rotations $A_{(\theta,x_i)}$ is an element of $\mathcal{C}_n^{(sy)}$ if the vectors x_i are not mutually orthogonal.

The solution obtained by Kadzisa-Mimura is to use instead "ordered symmetric factorizations" by $\mathbb{R}$-pseudo-rotations. Specifically it will be a product resulting from the successive application of Cartan conjugates by $\mathbb{R}$-pseudo rotations, which always yields elements of $\mathcal{C}_n^{(sy)}$.

Then, in describing the Schubert decomposition for SU_n/SO_n, we are giving a version of that contained in [KM], except we again define maps from products of cones on real projective spaces whose open cells give the cell decomposition.

Given an increasing sequence $m_1 < m_2 < \cdots < m_k$ with $1 < m_1$ and $m_k \leq n$, which we denote by $\mathbf{m} = (m_1, m_2, \ldots, m_k)$ we define a map

$$\psi_{\mathbf{m}}^{(sy)} : (C\mathbb{R}P^{m_1-1}) \times (C\mathbb{R}P^{m_2-1}) \times \cdots \times (C\mathbb{R}P^{m_k-1}) \longrightarrow SU_n \,,$$

with $CX = (I \times X)/(\{0\} \times X)$ for $I = [0,1]$, denoting the cone on X. This is given as follows:

First, we define a simpler map for $m \leq n$, $I = [0,1]$ and a real line $L \subset \mathbb{R}^m$,

$$\tilde{\psi}_m^{(sy)} : C\mathbb{R}P^{m-1} \to SU_n$$

defined by $\tilde{\psi}_m^{(sy)}(t, L) = A_{(\pi t, L_{\mathbb{C}})}$, with $L_{\mathbb{C}}$ denoting the complexification of the real line L. Note this factors through the cone as $A_{(0,L_{\mathbb{C}})} = Id$, independent of L. We will henceforth abbreviate this to $A_{(\pi t, L)}$. Then, we extend this to a map

$$\tilde{\psi}_{\mathbf{m}}^{(sy)} : \prod_{i=1}^{k} (C\mathbb{R}P^{m_i-1}) \;\longrightarrow\; SU_n$$

defined by

$$\begin{aligned} \tilde{\psi}_{\mathbf{m}}^{(sy)}((t_1, L_1), \ldots, (t_k, L_k)) \;&=\; A_{(-\pi\tilde{t}, e_1)} \cdot \psi_{m_1}(t_1, L_1) \cdot \psi_{m_2}(t_2, L_2) \cdots \psi_{m_k}(t_k, L_k) \\ &=\; A_{(-\pi\tilde{t}, e_1)} \cdot A_{(\pi t_1, L_1)} \cdot A_{(\pi t_2, L_2)} \cdots A_{(\pi t_k, L_k)} \,. \end{aligned} \tag{4.1}$$

where $\tilde{t} = \sum_{j=1}^{k} t_j$. We note that the first factor $A_{(-\pi\tilde{t}, e_1)}$ ensures the product is in SU_n as in the splitting for (2.1). Then we define

$$\psi_{\mathbf{m}}^{(sy)}((t_1, L_1), \ldots, (t_k, L_k)) \;=\; \tilde{\psi}_{\mathbf{m}}^{(sy)}((t_1, L_1), \ldots, (t_k, L_k)) \cdot \left(\tilde{\psi}_{\mathbf{m}}^{(sy)}((t_1, L_1), \ldots, (t_k, L_k))\right)^T . \tag{4.2}$$

We note that the RHS is the Cartan conjugate of I by $\tilde{\psi}_{\mathbf{m}}((t_1, L_1), \ldots, (t_k, L_k)) \in SU_n$ and thus is in the Cartan model $\mathcal{C}_n^{(sy)}$. It can also be obtained by successively applying to I the Cartan conjugates by the $A_{(\pi t_j, L_j)}$, for $j = k, k-1, \ldots, 1, 0$, where we let $A_{(\pi t_0, L_0)}$ denote $A_{(-\pi\tilde{t}, e_1)}$ (each of these are, strictly speaking, Cartan conjugates for U_n but their product is in SU_n).

We observe that each $C\mathbb{R}P^{m-1}$ has an open dense cell

$$E_m^{(sy)} = (0,1) \times \{x = (x_1, \dots, x_m, 0, \dots 0) : (x_1, \dots, x_m) \in S^{m-1} \text{ and } x_m > 0\}$$

which is of dimension m. Also, if $x = (x_1, \dots, x_m, 0, \dots 0)$ with $x_m > 0$, then $x \in_{\min} \mathbb{C}^m$.

We now introduce some notation and denote

$$\tilde{S}_{\mathbf{m}}^{(sy)} = (C\mathbb{R}P^{m_1-1}) \times (C\mathbb{R}P^{m_2-1}) \times \cdots \times (C\mathbb{R}P^{m_k-1}),$$

the cell

$$E_{\mathbf{m}}^{(sy)} \;=\; E_{m_1}^{(sy)} \times E_{m_2}^{(sy)} \times \cdots \times E_{m_k}^{(sy)} \,,$$

and $S_{\mathbf{m}}^{(sy)} = \psi_{\mathbf{m}}(E_{\mathbf{m}}^{(sy)})$. Then, $E_{\mathbf{m}}^{(sy)}$ is an open dense cell in $\tilde{S}_{\mathbf{m}}^{(sy)}$ with

$$\dim_{\mathbb{R}} E_{\mathbf{m}}^{(sy)} = |\mathbf{m}| \overset{def}{=} \sum_{j=1}^{k} m_j .$$

Also, the image $S_{\mathbf{m}}^{(sy)} = \psi_{\mathbf{m}}(E_{\mathbf{m}}^{(sy)})$ consists of elements of SU_n of *real Schubert type* $\mathbf{m}$. Furthermore, $\overline{S_{\mathbf{m}}^{(sy)}} = \psi_{\mathbf{m}}^{(sy)}(\tilde{S}_{\mathbf{m}}^{(sy)})$. Then the results of Kadzisa-Mimura [KM, Thm 6.7] give the following Schubert decomposition of SU_n/SO_n.

Theorem 4.1. *The Schubert decomposition of SU_n/SO_n has the following properties:*

a) *$SU_n/SO_n$is the disjoint union of the $S_{\mathbf{m}}^{(sy)}$ as $\mathbf{m} = (m_1, \dots, m_k)$ varies over all increasing sequences with $1 < m_1$, $m_k \le n$, and $0 \le k \le n-1$.*
b) *The map $\psi_{\mathbf{m}}^{(sy)} : E_{\mathbf{m}}^{(sy)} \to S_{\mathbf{m}}^{(sy)}$ is a homeomorphism.*
c) *$(\overline{S_{\mathbf{m}}^{(sy)}} \backslash S_{\mathbf{m}}^{(sy)}) \subset \cup_{\mathbf{m}'} S_{\mathbf{m}'}^{(sy)}$, where the union is over all $S_{\mathbf{m}'}^{(sy)}$ with $\dim S_{\mathbf{m}'}^{(sy)} < \dim S_{\mathbf{m}}^{(sy)}$.*

Hence, the Schubert decomposition by the cells $S_{\mathbf{m}}^{(sy)}$ is a cell decomposition of SU_n/SO_n. We refer to the cells $S_{\mathbf{m}}^{(sy)}$ as the *symmetric Schubert cells of SU_n/SO_n*. We also refer to the factorization given by (4.2) for elements B of $S_{\mathbf{m}}^{(sy)}$ as the *ordered symmetric factorization* and the corresponding Schubert symbol is denoted by $\mathbf{m}^{(sy)}(B)$.

Remark 4.2. Unlike the case of SU_n, in general the $\tilde{S}_{\mathbf{m}}^{(sy)}$ do not carry a top-dimensional fundamental class. In the case of a simple Schubert symbol (m_1), since L is real, $A_{(\pi,L)}$ is the complexification of a real reflection about the real hyperplane $L_{\mathbb{C}}^{\perp}$ and hence it is its own inverse and transpose. This is independent of L. Then,

$$\begin{aligned} \psi_{(m_1)}^{(sy)}(\pi, L_1) \;&=\; A_{(-\pi,e_1)} \cdot A_{(\pi,L_1)} \cdot A_{(\pi,L_1)}^T \cdot A_{(-\pi,e_1)}^T \\ &=\; A_{(-\pi,e_1)} \cdot A_{(\pi,L_1)} \cdot A_{(\pi,L_1)}^{-1} \cdot A_{(-\pi,e_1)}^{-1} \;=\; Id \end{aligned} \tag{4.3}$$

Thus, $\psi_{(m_1)}^{(sy)}(\{1\} \times \mathbb{R}P^{m_1-1}) = Id$ and so factors to give a map $\psi_{(m_1)}^{(sy)} : S\mathbb{R}P^{m_1-1} \to \mathcal{C}_n^{(sy)}$. Hence, for the simple Schubert symbol (m_1), $\overline{E_{(m_1)}^{(sy)}} = \psi_{(m_1)}^{(sy)}(S\mathbb{R}P^{m_1-1})$ has a fundamental class which is the image of the fundamental class of $S\mathbb{R}P^{m_1-1}$.

For a general symmetric Schubert symbol $\mathbf{m} = \mathbf{m}^{(sy)} = (m_1, m_2, \dots, m_k)$, if $(SU_n/SO_n)^{(\ell)}$ denotes the ℓ-skeleton of SU_n/SO_n, then $\psi_{\mathbf{m}}^{(sy)}$ composed with the projection does factor through to give a map

$$\tilde{\psi}_{\mathbf{m}}^{(sy)\,\prime} : \prod_{i=1}^{k} S\mathbb{R}P^{m_i-1} \to (SU_n/SO_n)/(SU_n/SO_n)^{(|\mathbf{m}|-1)} .$$

The product again carries a fundamental class and in §5 we see how these images in homology correspond to generators.

Schubert Decomposition for SU_{2n}/Sp_n.

For the Schubert decomposition for SU_{2n}/Sp_n we will largely follow [KM, §7]; except that for the geometric properties of Milnor fibers we will emphasize the use of the quaternionic structure on $\mathbb{C}^{2n}$. We already have the complex structure giving multiplication by $\mathbf{i}$. We extend it to $\mathbb{H}$ by defining multiplication by $\mathbf{j}$ by $\mathbf{j}x = J_n\bar{x}$ for $x \in \mathbb{C}^{2n}$ with $\bar{x}$ complex conjugation (so $\mathbf{k}x = \mathbf{ij}x$). Then, it is a standard check (see e.g. [GW, §1.4.4]) that this defines a quaternionic action so $\mathbb{C}^{2n} \simeq \mathbb{H}^n$. For this quaternionic structure, each subspace $\mathbb{C}^{2m}$ spanned by $\{e_1, \dots, e_{2m}\}$ is a quaternionic subspace.

Let $< x, y >= x^T \cdot \bar{y}$ (for column vectors x and y) denote the Hermitian inner product on $\mathbb{C}^{2n}$. It has the following directly verifiable properties:

i) multiplication by J_n is $\mathbb{H}$-linear;
ii) $< \mathbf{j}x, \mathbf{j}y >= \overline{< x, y >}$; and
iii) (by ii)) both $< x, \mathbf{j}x >= 0$ and $< \mathbf{j}x, y >= -\overline{< x, \mathbf{j}y >}$.

An element B of the Cartan model for SU_{2n}/Sp_n is characterized from (2.3) by

$$(BJ_n)^T = -BJ_n.$$

so that BJ_n is an element of SU_{2n} and is skew-symmetric. This has the following consequence, which is basically equivalent to [KM, Thm 3.4].

Lemma 4.3. *If $B \in \mathcal{C}_{2n}^{(sk)}$, the Cartan model for SU_{2n}/Sp_n, then*
a) *$B\mathbf{j}x = \mathbf{j}B^*x$; and*
b) *if B satisfies the condition in a), then the eigenspaces of B are $\mathbb{H}$-subspaces.*

Proof. For a), this is a simple calculation.

$$B\mathbf{j}x \;=\; BJ_n\bar{x} \;=\; -(BJ_n)^T\bar{x} \;=\; -J_n^T B^T \bar{x} \;=\; J_n\overline{\bar{B}^T x} \;=\; J_n\overline{B^* x} \;=\; \mathbf{j}B^*x\,.$$

For b), we observe that if $Bx = \lambda x$, then as $B \in SU_{2n}$, $B^* = B^{-1}$ and $|\lambda| = 1$ so

$$B\mathbf{j}x \;=\; \mathbf{j}B^*x \;=\; \mathbf{j}B^{-1}x \;=\; \mathbf{j}\lambda^{-1}x \;=\; J_n\overline{\lambda^{-1}x} \;=\; \lambda J_n\bar{x} \;=\; \lambda\mathbf{j}x\,.$$

Thus, the λ-eigenspace of B is invariant under multiplication by $\mathbf{j}$. □

We will refer to a $B \in U_{2n}$ which satisfies the condition in a) of Lemma 4.3 as being $\mathbb{H}$*-linear. To factor such a matrix, we use a version of pseudo-rotation for $\mathbb{H}^n$. Given a quaternionic line $L \subset \mathbb{C}^{2n}$, let $L^\perp$ be the quaternionic hyperplane orthogonal to L. We define an $\mathbb{H}$-pseudo-rotation by an angle θ, $\tilde{A}_{(\theta,L)}$ which is the identity on $L^\perp$ and is multiplication by $e^{i\theta}$ on L. It is $\mathbb{C}$-linear and can be checked to be $\mathbb{H}$*-linear. If $x \in L$ is a unit vector, then by property iii), $\{x, \mathbf{j}x\}$ is an orthonormal basis for L. Then, $\tilde{A}_{(\theta,L)}$ can be written as a product of pseudo-rotations $A_{(\theta,x)}A_{(\theta,\mathbf{j}x)}$, which commute. By the properties of pseudo-rotations, we have the following properties of $\mathbb{H}$-pseudo-rotations.

i) $\tilde{A}^*_{(\theta,L)} = \tilde{A}^{-1}_{(\theta,L)} = \tilde{A}_{(-\theta,L)}$;
ii) $\overline{\tilde{A}_{(\theta,L)}} = \tilde{A}_{(-\theta,\bar{L})}$, where $\bar{L}$ is the $\mathbb{H}$-line generated by $\bar{x}$; and
iii) $\tilde{A}^T_{(\theta,L)} = \tilde{A}_{(\theta,\bar{L})}$;
iv) $\det(\tilde{A}_{(\theta,L)}) = e^{2i\theta}$;
v) If $L \perp L'$ then $\tilde{A}_{(\theta,L)}$ and $\tilde{A}_{(\theta,L')}$ commute;
vi) $\tilde{A}_{(\theta,L)}$ is $\mathbb{H}$*-linear.

Proof. All of i) - v) follow directly from the properties of pseudo-rotations. For vi) we observe that $\tilde{A}_{(\theta,L)}$ is characterized as a unitary matrix which has L for the eigenspace for $e^{i\theta}$ and $L^\perp$

as the eigenspace for the eigenvalue 1. Thus, for vi), as both L and $L^{\perp}$ are $\mathbb{H}$-subspaces we see $\tilde{A}_{(\theta,L)} \equiv Id$ on $L^{\perp}$ and for $x \in L$,

$$\tilde{A}_{(\theta,L)}(\mathbf{j}x) \;=\; e^{i\theta}\mathbf{j}x \;=\; \mathbf{j}e^{-i\theta}x \;=\; \mathbf{j}\tilde{A}^{-1}_{(\theta,L)}(x)\,.$$

As $\tilde{A}^{*}_{(\theta,L)} = \tilde{A}^{-1}_{(\theta,L)}$, we see that $\tilde{A}_{(\theta,L)}(\mathbf{j}x) = \mathbf{j}\tilde{A}^{*}_{(\theta,L)}(x)$ on each summand L and $L^{\perp}$; hence they are equal. $\square$

In addition, we can give a unique representation of $\tilde{A}_{(\theta,L)}$ as an ordered product of pseudo-rotations.

Lemma 4.4. *Given an $\mathbb{H}$-line $L \subset_{\min} \mathbb{C}^{2m}$, there is a unique unit vector $x \in L \cap \mathbb{C}^{2m-1}$ of the form $x = (x_1, \dots, x_{2m-1}, 0)$ with $x_{2m-1} > 0$ so that $\mathbf{j}x = (\bar{x_2}, -\bar{x_1}, \bar{x_4}, -\bar{x_3}, \dots, 0, -x_{2m-1})$. Hence, $\tilde{A}_{(\theta,L)}$ can be uniquely written $A_{(\theta,x)} \cdot A_{(\theta,\mathbf{j}x)}$.*

Proof. As $\dim_{\mathbb{C}} L = 2$. $\dim_{\mathbb{C}}(L \cap \mathbb{C}^{2m-1}) = 1$. It is ≥ 1, and otherwise it would be 2, i.e. $L \subset \mathbb{C}^{2m-1}$. Then, under the $\mathbb{H}$-linear projection $p : \mathbb{C}^{2m} \to \mathbb{C}^{2m}/\mathbb{C}^{2m-2}$ the image of L, which is an $\mathbb{H}$-subspace would have $\mathbb{C}$-dimension 1, a contradiction.

As $\dim_{\mathbb{C}}(L \cap \mathbb{C}^{2m-1}) = 1$, and $L \not\subset \mathbb{C}^{2m-2}$, we may find a unit vector $x \in L$ of the form $x' = (x'_1, \dots, x'_{2m-1}, 0)$ with $x'_{2m-1} \neq 0$. Multiplying x' by an appropriate unit complex number we obtain x with $x_{2m-1} > 0$. Then, $\mathbf{j}x$ is as stated and so is $\tilde{A}_{(\theta,L)}$. $\square$

Whitehead-Type Ordered Factorization.

For an $\mathbb{H}^*$-linear $B \in U_{2n}$, we may initially factor it as a product of $\mathbb{H}$-pseudo-rotations in a manner similar to the symmetric case as follows. Each eigenspace V_λ of B with $\lambda = e^{i\theta} \neq 1$ is an $\mathbb{H}$-subspace. We choose the smallest m'_1 so that $V_\lambda \cap \mathbb{C}^{2m'_1} \neq 0$, and hence is an $\mathbb{H}$-line $L_1^{(\lambda)}$. We successively repeat this for $(L_1^{(\lambda)})^{\perp} \cap V_\lambda$ and obtain an orthogonal decomposition $V_\lambda = L_1^{(\lambda)} \oplus L_2^{(\lambda)} \cdots L_{k'}^{(\lambda)}$ with $L_j^{(\lambda)} \subset_{\min} \mathbb{C}^{2m'_j}$ and $m'_1 < m'_2 < \cdots < m'_{k'}$. Each $L_j^{(\lambda)}$ gives an $\mathbb{H}$-pseudo-rotation $\tilde{A}_{(\theta, L_j^{(\lambda)})}$. We may do this for each eigenvalue $\lambda \neq 1$. Because different L_j are orthogonal, the corresponding $\mathbb{H}$-pseudo-rotations commute. Thus, we may factor B as a product of $\mathbb{H}$-pseudo-rotations

$$B \;=\; \tilde{A}_{(\theta_1,L_1)} \cdot \tilde{A}_{(\theta_2,L_2)} \cdots \tilde{A}_{(\theta_k,L_k)} \tag{4.4}$$

where $L_j \subset_{min} \mathbb{C}^{2m_j}$, $1 \leq m_1 \leq m_2 \leq \cdots \leq m_k$, and several θ_j may be equal. However, this is not an ordered factorization as some of the m_j may be equal.

We would like to apply an analogue of the Whitehead Lemma 3.2 to products of $\mathbb{H}$-pseudo-rotations. However, it is not possible to do so remaining in the category of $\mathbb{H}$-pseudo-rotations. For example, if $B \in U_{2n}$ then $B \cdot \tilde{A}_{(\theta,L)} \cdot B^{-1}$ is a unitary transformation with $B(L)$ as the eigenspace for $e^{i\theta}$ and $B(L^{\perp}) = (B(L))^{\perp}$ as the eigenspace for the eigenvalue 1. While $B(L)$ is a 2-dimensional complex space, it need not be an $\mathbb{H}$-subspace.

However, there is an alternate way to proceed which uses Lemma 4.4. We may uniquely decompose each $\mathbb{H}$-pseudo-rotation in (4.4) into a product of pseudo-rotations about orthogonal planes which thus all commute so that (4.4) may be rewritten

$$B \;=\; A_{(\theta_1,x_1)} \cdot A_{(\theta_2,x_2)} \cdots A_{(\theta_k,x_k)} \cdot A_{(\theta_k,\mathbf{j}x_k)} \cdots A_{(\theta_2,\mathbf{j}x_2)} \cdot A_{(\theta_1,\mathbf{j}x_1)} \tag{4.5}$$

Then, we can progressively apply Whitehead's Lemma to the factors $A_{(\theta_j,x_j)}$ beginning with the highest j and proceeding left to the lowest to obtain an ordered factorization for the product involving the $A_{(\theta_j,x_j)}$. Then for each application of Whitehead's Lemma for these, there is a corresponding application of it for the $A_{(\theta_j,\mathbf{j}x_j)}$ from the left proceeding to the right using the following lemma.

Lemma 4.5. *Given a relation between pseudo-rotations*

$$A_{(\theta,x)} \cdot A_{(\theta',x')} \;=\; A_{(\theta_1,x_1)} \cdot A_{(\theta_2,x_2)}\,, \tag{4.6}$$

there is a corresponding relation

$$A_{(\theta',\mathbf{j}x')} \cdot A_{(\theta,\mathbf{j}x)} \;=\; A_{(\theta_2,\mathbf{j}x_2)} \cdot A_{(\theta_1,\mathbf{j}x_1)}\,. \tag{4.7}$$

Proof. First, apply the transpose to each side of (4.6) and then conjugate with J_n to obtain

$$(J_n \cdot A^T_{(\theta',x')} \cdot J_n^{-1}) \cdot (J_n \cdot A^T_{(\theta,x)} \cdot J_n^{-1}) \;=\; (J_n \cdot A^T_{(\theta_2,x_2)} \cdot J_n^{-1}) \cdot (J_n \cdot A^T_{(\theta_1,x_1)} \cdot J_n^{-1}). \tag{4.8}$$

Then, for any pseudo-rotation $A_{(\theta,x)}$,

$$J_n \cdot A^T_{(\theta,x)} \cdot J_n^{-1} = J_n \cdot A_{(\theta,\bar{x})} \cdot J_n^{-1} = A_{(\theta,J_n\bar{x})} = A_{(\theta,\mathbf{j}x)}\,. \tag{4.9}$$

Thus, applying (4.9) to each product in (4.8) yields (4.7). □

Then, by applying Whitehead's Lemma successively to appropriate adjacent pairs $A_{(\theta_j,x_j)} \cdot A_{(\theta_{j'},x_{j'})}$ and Lemma 4.5 to the corresponding pairs $A_{(\theta_{j'},\mathbf{j}x_{j'})} \cdot A_{(\theta_j,\mathbf{j}x_j)}$ we may rewrite

$$B \;=\; A_{(\theta'_1,x'_1)} \cdot A_{(\theta'_2,x'_2)} \cdots A_{(\theta'_k,x'_k)} \cdot A_{(\theta'_k,\mathbf{j}x'_k)} \cdots A_{(\theta'_2,\mathbf{j}x'_2)} \cdot A_{(\theta'_1,\mathbf{j}x'_1)} \tag{4.10}$$

with the $A_{(\theta'_j,x'_j)}$ in increasing order and the $A_{(\theta'_j,\mathbf{j}x'_j)}$ in decreasing order.

Kadzisa-Mimura Ordered Skew-Symmetric Factorization.

In fact, this is the skew-symmetric factorization of $B \in \mathcal{C}_m^{(sk)}$ given by Kadzisa-Mimura. We further rewrite (4.10) using the properties of pseudo-rotations $\sigma(A_i^{-1}) = A_{(\theta_i,\mathbf{j}x_i)}$. Hence, B in (4.10) can be rewritten either as

$$B \;=\; \Big(A_{(\theta_1,x_1)} \cdot A_{(\theta_2,x_2)} \cdots A_{(\theta_k,x_k)} \cdot J_n \cdot A^T_{(\theta_k,x_k)} \cdots A^T_{(\theta_1,x_1)}\Big) \cdot J_n^{-1} \tag{4.11}$$

or alternatively for each $A_j = A_{(\theta_j,x_j)}$ as

$$B \;=\; A_1 \cdot A_2 \cdots A_k \cdot \sigma(A_k^{-1}) \cdots \sigma(A_1^{-1})\,, \tag{4.12}$$

which is a Cartan conjugate of I and hence belongs to $F_m^{(sk)\,c}$.

What we have not yet considered is the skew-symmetric Schubert symbol associated to this factorization. We shall do so in giving in the next section the Kadzisa-Mimura algorithm for obtaining the ordered skew-symmetric factorization from the full Whitehead ordered factorization.

We next define the maps for the cell decomposition of SU_{2n}/Sp_n via the Cartan Model $\mathcal{C}_{2n}^{(sk)}$. In describing the Schubert decomposition for SU_{2n}/Sp_n, we are giving a version that modifies that contained in [KM] to associate to the Borel-Moore fundamental classes of products of suspensions of quaternionic projective spaces the Borel-Moore fundamental classes of the "Schubert cycles" obtained as the closures of the Schubert cells. However, unlike the general and symmetric cases, we cannot directly do this by expressing the closures of Schubert cells as the images of the products of suspensions of quaternionic projective spaces. Instead we proceed through intermediate spaces which are products of suspensions of complex projective spaces.

For any $m > 0$, we define via the quaternionic structure on $\mathbb{C}^{2m} \simeq \mathbb{H}^m$ a map

$$\chi_m : \mathbb{C}P^{2m-2} \to \mathbb{H}P^{m-1}$$

by $\chi_m(L) = L + \mathbf{j}L$ for complex lines $L \subset \mathbb{C}^{2m-1}$. For a quaternionic line $Q \subset_{\min} \mathbb{H}^m$, Q has a unique element $x = (x_1, \dots, x_{4(m-1)}, x_{4m-3}, 0) \in S^{4m-3} \subset \mathbb{C}^{2m-1}$ with $x_{4m-3} > 0$. Then,

$$\mathbf{j}\,x = (\bar{x}_2, -\bar{x}_1, \bar{x}_4, -\bar{x}_3, \dots, \bar{x}_{4(m-1)}, -\bar{x}_{4m-5}, 0, -x_{4m-3})\,.$$

Hence, the set of such Q are parametrized by the cell E^{4m-4} in S^{4m-3} with $x_{4m-3} > 0$ (since $x_{4m-3} = \sqrt{1 - \sum_{j=1}^{4(m-1)} |x_j|^2}$). However, this cell also parametrizes the open dense subset of

$L \in \mathbb{C}P^{2m-2}$ with $L \subset_{\min} \mathbb{C}^{2m-1}$. The map χ_m acts as the identity on these parametrized cells of dimension $4m-4$, and the complements have lower dimensions. We may then take the suspension $S\chi_m : S\mathbb{C}P^{2m-2} \to S\mathbb{H}P^{m-1}$, which now is a homeomorphism on the cell $(0,1) \times E^{4m-4}$ of dimension $4m-3$. Thus, $S\chi_{m\,*}$ sends the Borel-Moore fundamental class of $S\mathbb{C}P^{2m-2}$ to that of $S\mathbb{H}P^{m-1}$.

Then, given an increasing sequence $1 < m_1 < m_2 < \cdots < m_k \leq n$, which we denote by $\mathbf{m}^{(sk)} = (m_1, m_2, \ldots, m_k)$, we may form the product map

$$\tilde{\chi}_{\mathbf{m}}^{(sk)} \;=\; S\chi_{m_1} \times S\chi_{m_2} \times \cdots \times S\chi_{m_k}\,,$$

which again sends the Borel-Moore fundamental class of the product $S\mathbb{C}P^{2m_1-2}\times\cdots\times S\mathbb{C}P^{2m_k-2}$ to that of $S\mathbb{H}P^{m_1-1} \times \cdots \times S\mathbb{H}P^{m_k-1}$.

Then, the correspondence we give between the fundamental homology classes of

$$S\mathbb{H}P^{m_1-1} \times \cdots \times S\mathbb{H}P^{m_k-1}$$

and the Schubert cycles will be via the fundamental homology classes of

$$S\mathbb{C}P^{2m_1-2} \times \cdots \times S\mathbb{C}P^{2m_k-2}.$$

We do so by defining a map

$$\psi_{\mathbf{m}}^{(sk)} : S\mathbb{C}P^{2m_1-2} \times S\mathbb{C}P^{2m_2-2} \times \cdots \times S\mathbb{C}P^{2m_k-2} \longrightarrow \mathcal{C}_m^{(sk)}.$$

This is given as follows:

$$\tilde{\psi}_m^{(sk)} : (I \times \mathbb{C}P^{2m_1-2}) \times (I \times \mathbb{C}P^{2m_2-2}) \times \cdots \times (I \times \mathbb{C}P^{2m_k-2}) \longrightarrow SU_n$$

is defined by

$$\begin{aligned} \tilde{\psi}_{\mathbf{m}}^{(sk)}((t_1, L_1), \ldots, (t_k, L_k)) \;&=\; A_{(-2\pi\tilde{t}, e_1)} \cdot A_{(2\pi t_1, L_1)} \cdot A_{(2\pi t_2, L_2)} \cdots A_{(2\pi t_k, L_k)} \\ &\quad \cdot A_{(2\pi t_k, \mathbf{j}L_k)} \cdots A_{(2\pi t_2, \mathbf{j}L_2)} \cdot A_{(2\pi t_1, \mathbf{j}L_1)} \cdot A_{(-2\pi\tilde{t}, -e_3)}\,, \end{aligned} \tag{4.13}$$

where $\tilde{t} = \sum_{j=1}^{k} t_j$. We note that the product is of the form (4.10) and hence (4.12). Also, the first and last factors $A_{(-2\pi\tilde{t}, e_1)}$ and $A_{(-2\pi\tilde{t}, -e_3)}$ ensure the product is in SU_n as in the splitting for (2.1).

Since $A_{(0,L)} = A_{(2\pi,L)} = I_n$ independent of a complex line $L \subset \mathbb{C}^{2m-1}$, (4.13) descends to a map

$$\psi_m^{(sk)} : S\mathbb{C}P^{2m_1-2} \times S\mathbb{C}P^{2m_2-2} \times \cdots \times S\mathbb{C}P^{2m_k-2} \longrightarrow \mathcal{C}_m^{(sk)}\,.$$

As remarked above, each $S\mathbb{C}P^{2m_j-2}$ has an open dense cell of dimension $4m_j - 3$ which we denote by

$$\begin{aligned} E_{m_j}^{(sk)} \;&=\; (0,1) \times \{x = (x_1, \ldots, x_{4(m_j-1)}, x_{4m_j-3}, 0, \ldots 0) \\ &\qquad : (x_1, \ldots, x_{4(m_j-1)}, x_{4m_j-3}), 0) \in S^{4m_j-3} \text{ and } x_{4m_j-3} > 0\} \end{aligned}$$

and we conclude $\mathbb{H} < x > \subset_{\min} \mathbb{C}^{2m_j}$.

We now introduce some notation and denote

$$\tilde{S}_{\mathbf{m}}^{(sk)} = S\mathbb{C}P^{2m_1-2} \times S\mathbb{C}P^{2m_2-2} \times \cdots \times S\mathbb{C}P^{2m_k-2}\,.$$

Also, we consider the corresponding cell $E_{\mathbf{m}}^{(sk)} = E_{m_1}^{(sk)} \times E_{m_2}^{(sk)} \times \cdots \times E_{m_k}^{(sk)}$, and the image $S_{\mathbf{m}}^{(sk)} = \psi_{\mathbf{m}}^{(sk)}(E_{\mathbf{m}}^{(sk)})$ in $\mathcal{C}_{2n}^{(sk)}$. Then, $E_{\mathbf{m}}^{(sk)}$ is an open dense cell in $\tilde{S}_{\mathbf{m}}^{(sk)}$ with

$$\dim_{\mathbb{R}} E_{\mathbf{m}}^{(sk)} = \sum_{j=1}^{k}(4m_j - 3) = 4|\mathbf{m}^{(sk)}| - 3k = 4|\mathbf{m}^{(sk)}| - 3\ell(\mathbf{m}^{(sk)})$$

for $|\mathbf{m}^{(sk)}| = \sum_{j=1}^{k} m_j$ (and $\ell(\mathbf{m}^{(sk)}) = k$). Also, the image $S_{\mathbf{m}}^{(sk)} = \psi_{\mathbf{m}}^{(sk)}(E_{\mathbf{m}}^{(sk)})$ consists of elements of $\mathcal{C}_{2n}^{(sk)}$ of skew Schubert type $\mathbf{m}$. Furthermore, $\overline{S_{\mathbf{m}}^{(sk)}} = \psi_{\mathbf{m}}^{(sk)}(\tilde{S}_{\mathbf{m}}^{(sk)})$. Then the results of Kadzisa-Mimura [KM, Thm 8.7] give the following Schubert decomposition of SU_{2n}/Sp_n.

Theorem 4.6. *The Schubert decomposition of SU_{2n}/Sp_n has the following properties via the diffeomorphism $SU_{2n}/Sp_n \simeq \mathcal{C}_{2n}^{(sk)}$:*

a) *SU_{2n}/Sp_n is the disjoint union of the $S_{\mathbf{m}}^{(sk)}$ as $\mathbf{m} = \mathbf{m}^{(sk)} = (m_1, \dots, m_k)$ varies over all increasing sequences with $1 < m_1 < \cdots < m_k \leq n$, and $0 \leq k \leq n-1$.*
b) *The map $\psi_{\mathbf{m}}^{(sk)} : E_{\mathbf{m}}^{(sk)} \to S_{\mathbf{m}}^{(sk)}$ is a homeomorphism.*
c) *$(\overline{S_{\mathbf{m}}^{(sk)}} \backslash S_{\mathbf{m}}^{(sk)}) \subset \cup_{\mathbf{m}'} S_{\mathbf{m}'}^{(sk)}$, where the union is over all $S_{\mathbf{m}'}^{(sk)}$ with $\dim S_{\mathbf{m}'}^{(sk)} < \dim S_{\mathbf{m}}^{(sk)}$.*

Hence, the Schubert decomposition by the cells $S_{\mathbf{m}}^{(sk)}$ gives a corresponding cell decomposition of SU_{2n}/Sp_n. The cells $S_{\mathbf{m}}^{(sk)}$ will be referred to as the *skew-symmetric Schubert cells of SU_{2n}/Sp_n or $\mathcal{C}_{2n}^{(sk)}$*. We note that $\overline{S_{\mathbf{m}}^{(sk)}}$ has a Borel-Moore fundamental class which we refer to as a *skew-symmetric Schubert cycle*. It is the image of the Borel-Moore fundamental class of the “singular manifold”$\tilde{S}_{\mathbf{m}}^{(sk)}$. It corresponds to the Borel-Moore fundamental class of the associated product of suspensions of quaternionic projective spaces. We describe in §5 the homology of SU_{2n}/Sp_n and $\mathcal{C}_{2n}^{(sk)}$ in terms of these skew-symmetric Schubert cycles. Furthermore, for $m = 2n$ the relation of $\mathcal{C}_m^{(sk)}$ with $F_m^{(sk)\,c}$ allows us to give a Schubert decomposition for the Milnor fiber.

Remark 4.7. If in the initial factorization of $B \in \mathcal{C}_{2n}^{(sk)}$ given in (4.4) into a product of $\mathbb{H}$-pseudo-rotations, the orders for all of the $L_j^{(\lambda_\ell)}$ are all distinct then $1 < m_1 < m_2 < \cdots < m_k$. By the commutativity of the $\mathbb{H}$-pseudo-rotations, we may arrange them in increasing order and obtain (4.10) without using Whitehead's Lemma. Hence, the skew-symmetric Schubert symbol is given by $\mathbf{m}^{(sk)} = (m_1, m_2, \cdots, m_k)$, which would be the corresponding Schubert symbol in the quaternionic Grassmannian. In general, the use of Whitehead's Lemma has the effect of twisting the $\mathbb{H}$-lines which then again reappear from the form of the skew-symmetric factorization.

5. Schubert Decomposition for Milnor Fibers

In this section we apply the results giving the Schubert decomposition for the associated symmetric spaces providing compact models for the global Milnor fibers. We first give the form that the Schubert decomposition gives for the specific Cartan models, and extending these to the Milnor fibers themselves. Second, in doing this we give an algorithm due to Whitehead and Kadzisa-Mimura for identifying for a given matrix in the global Milnor fiber the Schubert cell to which it belongs. Third, we will see the form that the Schubert decomposition takes for the global Milnor fibers using Iwasawa decomposition.

Whitehead-Kadzisa-Mimura Algorithm for Identifying Schubert Cells.
The algorithm given by Kadzisa-Mimura [KM] for the ordered factorizations of matrices in the various Cartan models uses the ordered factorization for SU_m based on the work of Whitehead [W] as developed by Miller [Mi] and Yokota [Y]. They cleverly combine the uniqueness of the factorization for U_m (and SU_m) and the Cartan conjugacy for the Cartan models to give the symmetric, respectively skew-symmetric, factorizations for the cases of SU_m/SO_m and for $m = 2n$, SU_{2n}/Sp_n. We explain this algorithm as it will apply to the compact models for global Milnor fibers and then for the global Milnor fibers themselves.

An element of any of the Cartan models is a matrix $B \in SU_m$ for appropriate m. Thus, by Lemma 3.3 we may obtain an ordered factorization by pseudo-rotations except with decreasing order for B.

$$B \;=\; A_k \cdot A_{k-1} \cdots A_1 \,, \tag{5.1}$$

where $A_j = A_{(\theta_j, x_j)}$ with the $\{x_j\}$ a set of unit vectors with $x_j \in_{\min} \mathbb{C}^{m_j}$ and

$$1 \leq m_1 < m_2 < \cdots < m_k \leq m,$$

and $\theta_i \not\equiv 0 \bmod 2\pi$ for each i. In addition, if $m_1 = 1$ then the Schubert symbol is $\mathbf{m} = (m_2, \ldots, m_k)$. Now from (5.1) we describe how to obtain either the symmetric or skew-symmetric ordered factorizations as obtained by Kadzisa-Mimura.

Ordered Symmetric Factorizations for $\mathcal{C}^{(sy)}$. As $B \in \mathcal{C}^{(sy)}$, $\sigma(B^{-1}) = B$. Hence, as

$$\sigma(B^{-1}) = \overline{B^{-1}} = B^T,$$

we obtain from (5.1)

$$A_k \cdot A_{k-1} \cdots A_1 \;=\; A_1^T \cdot A_2^T \cdots A_k^T \,.$$

As each $A_j = A_{(\theta_j, x_j)}$, $A_j^T = A_{(\theta_j, \bar{x}_j)}$ is a pseudo-rotation with $\bar{x}_j \in_{\min} \mathbb{C}^{m_j}$. Thus, it follows by Lemma 3.4 that $A_1 = A_1^T$ and x_1 is real. Let $C_1 = A_{(\frac{\theta_1}{2}, x_1)}$. We can write $A_1 = C_1 \cdot C_1$, and as $A_{(\theta_1, x_1)}$ is a pseudo-rotation about a real hyperplane, so is C_1. Hence, $C_1 = C_1^T$ and $\sigma(C_1) = C_1^*$. Then, from (5.1) since

$$B \;=\; A_k \cdot A_{k-1} \cdots A_1 \,, \tag{5.2}$$

we have

$$\begin{aligned} C_1^* \cdot B \cdot \sigma(C_1) \;&=\; (C_1^* \cdot A_k \cdot A_{k-1} \cdots A_2 \cdot C_1) \cdot C_1 \cdot \sigma(C_1) \\ &=\; (C_1^* \cdot A_k \cdot C_1) \cdot (C_1^* \cdot A_{k-1} \cdot C_1) \cdots (C_1^* \cdot A_2 \cdot C_1) \\ &=\; A_k^{(2)} \cdot A_{k-1}^{(2)} \cdots A_2^{(2)}, \end{aligned} \tag{5.3}$$

where each $A_j^{(2)} = C_1^* \cdot A_j \cdot C_1$ is again a pseudo-rotation $A_{(\theta_j, x_j^{(2)})}$, with $x_j^{(2)} = C_1^{-1}(x_j)$ satisfying $x_j^{(2)} \in_{\min} \mathbb{C}^{m_j}$ as $C_1 \equiv Id$ on $(\mathbb{C}^{m_1})^{\perp}$.

Also, the LHS of (5.3) is the Cartan conjugate of the symmetric matrix B and so is still symmetric (and in SU_n), except now it is a product of $k-1$ pseudo-rotations with Schubert symbol $(m_k, \ldots, m_2)$. Thus we can inductively repeat the argument to write.

$$C_j^* \cdots C_2^* \cdot C_1^* \cdot B \cdot \sigma(C_1) \cdot \sigma(C_2) \cdots \sigma(C_j) \;=\; A_k^{(j+1)} \cdot A_{k-1}^{(j+1)} \cdots A_{j+1}^{(j+1)}$$

which has Schubert symbol $(m_{j+1}, \ldots, m_k)$. After $k-1$ steps we obtain

$$C_{k-1}^* \cdots C_2^* \cdot C_1^* \cdot B\sigma(C_1) \cdot \sigma(C_2) \cdots \sigma(C_{k-1}) \;=\; A_k^{(k)} \,, \tag{5.4}$$

with $A_k^{(k)} = A_{(\theta_k, x_k^{(k)})}$ for $x_k^{(k)} \in_{\min} \mathbb{C}^{m_k}$. The last step then allows us to rewrite (5.4) as

$$B \;=\; C_1 \cdots C_{k-1} \cdot C_k \cdot \sigma(C_k^*) \cdot \sigma(C_{k-1}^*) \cdots \sigma(C_1^*) \,, \tag{5.5}$$

which gives the ordered symmetric factorization.

We obtain as a corollary of the algorithm

Corollary 5.1. *If $B \in F_m^{(sy)\,c} = \mathcal{C}_m^{(sy)}$, and has increasing Schubert symbol $\mathbf{m} = (m_1, \ldots, m_k)$, then the symmetric factorization has the same Schubert symbol $\mathbf{m}^{(sy)} = \mathbf{m}$.*

Ordered Skew-symmetric Factorizations for $\mathcal{C}_m^{(sk)}$. The algorithm for $\mathcal{C}_m^{(sk)}$, with $m = 2n$, is very similar and depends on the following lemma, see [KM, Lemma 7.2].

Lemma 5.2. *If* $B \in (U_{2n} \cap Sk_m(\mathbb{C})) \cdot J_n^{-1}$*, with* $m = 2n$*, has a factorization as in* (5.1)*, then:* k *is even,* m_1 *is odd,* $m_2 = m_1 + 1$*, and* $A_2 = \sigma(A_1^*)$.

Here $\sigma(A) = J_n \cdot \overline{A} \cdot J_n^{-1}$ and $A_1 = A_{(\theta_1, x_1)}$ with $x_1 \in_{\min} \mathbb{C}^{m_1}$, for which we may arrange $x_1 = (x_{1,1}, \ldots, x_{1,m_1})$ with $x_{1,m_1} > 0$. Then, by properties of pseudo-rotations

$$A_2 = \sigma(A_1^*) = A_{(\theta_1, \mathbf{j}x_1)}$$

(hence, $A_2 \cdot A_1$ is an $\mathbb{H}$-pseudo-rotation and A_1 and A_2 commute). We may then rewrite (5.1) as

$$\begin{aligned} A_1^* \cdot B \cdot \sigma(A_1) &= A_1^* \cdot A_k \cdot A_{k-1} \cdots A_3 \cdot A_1 \cdot \sigma(A_1^*) \cdot \sigma(A_1) \\ &= (A_1^* \cdot A_k \cdot A_1) \cdot (A_1^* \cdot A_{k-1} \cdot A_1) \cdots (A_1^* \cdot A_3 \cdot A_1) \\ &= A_k^{(2)} \cdot A_{k-1}^{(2)} \cdots A_3^{(2)}, \end{aligned} \tag{5.6}$$

where each $A_j^{(2)} = A_1^* \cdot A_j \cdot A_1$ is again a pseudo-rotation $A_{(\theta_j, x_j^{(2)})}$, with $x_j^{(2)} = A_1^{-1}(x_j)$ satisfying $x_j^{(2)} \in_{\min} \mathbb{C}^{m_j}$ as $A_1 \equiv Id$ on $(\mathbb{C}^{m_1})^{\perp}$.

Also, the LHS of (5.6) is the Cartan conjugate of B for which $B \cdot J_n$ is skew-symmetric (and in U_{2n}); and so it also has these properties, except now it is a product of $k-2$ pseudo-rotations with Schubert symbol $(m_k, \ldots, m_3)$. Thus we can inductively repeat the argument. After $\frac{k}{2}$ steps we obtain a factorization in the form

$$\begin{aligned} B &= A_{(\theta_1, x_1')} \cdots A_{(\theta_r, x_r')} \cdot \sigma(A^*_{(\theta_r, x_r')}) \cdots \sigma(A^*_{(\theta_1, x_1')}), \\ &= A_{(\theta_1, x_1')} \cdots A_{(\theta_r, x_r')} \cdot A_{(\theta_r, \mathbf{j}x_r')} \cdots A_{(\theta_1, \mathbf{j}x_1')} . \end{aligned} \tag{5.7}$$

Here $k = 2r$, and each $\mathbb{H} < x_r' > \subset_{\min} \mathbb{C}^{2m_j}$. This gives the ordered skew-symmetric factorization. By (4.9) we may write each $A_{(\theta_j, \mathbf{j}x_j')} = J_n \cdot A^T_{(\theta_j, x_j')} \cdot J_n^{-1}$, and then by (4.11) we may alternately write (5.7) in the form

$$B = A_{(\theta_1, x_1')} \cdots A_{(\theta_r, x_r')} \cdot J_n \cdot A^T_{(\theta_r, x_r')} \cdots A^T_{(\theta_1, x_1')} \cdot J_n^{-1} . \tag{5.8}$$

We obtain as a corollary of the algorithm.

Corollary 5.3. *If* $B \in \mathcal{C}_m^{(sk)} = F_m^{(sk)\,c} \cdot J_n^{-1}$ *(with* $m = 2n$*), then it has an increasing Schubert symbol of the form*

$$\mathbf{m} = (2m_1 - 1, 2m_1, 2m_2 - 1, 2m_2, \ldots, 2m_r - 1, 2m_r)$$

with $1 < m_1 < m_2, \cdots < m_r \leq n$*. Then the ordered skew-symmetric factorization has the skew-symmetric Schubert symbol* $\mathbf{m}^{(sk)} = (m_1, m_2, \ldots, m_r)$.

To use the preceding results for the global Milnor fibers, we use in each case the Iwasawa decomposition, which is given for SL_n by the Gram-Schmidt process, to determine the Schubert cell decomposition.

Global Milnor Fibers for the Variety of Singular $m \times m$-Matrices.

This is the simplest case and was essentially covered in Proposition 1.2. Given $B \in F_m$, the global Milnor fiber, we have $F_m = SL_m(\mathbb{C})$. To obtain its representation in the Iwasawa decomposition $SL_m(\mathbb{C}) = SU_m \cdot A_m \cdot N_m$ where A_m denotes the group of diagonal matrices with positive entries, and N_m is the nilpotent group of upper triangular complex matrices with 1'on the diagonal. We may apply the Gram-Schmidt process to the columns of B to obtain $B = A \cdot C$, where A is unitary and C is upper triangular with positive entries on the diagonal. As $\det(B) = 1$,

$\det(A)$ is a unit complex number, and $\det(C) > 0$; it follows that both $\det(A) = \det(C) = 1$; thus, C belongs to $\mathrm{Sol}_m = A_m \cdot N_m$. Then by applying the method of §3 for giving an ordered factorization for A gives the Schubert symbol for A, which we shall also use for B. Thus, we may describe the Schubert decomposition for the global Milnor fiber F_m as follows.

Theorem 5.4. *The Schubert decomposition of the global Milnor fiber F_m for the variety of $m \times m$ general complex matrices is given, via the diffeomorphism with $SL_m(\mathbb{C})$, by the disjoint union of the* Schubert cells *$S_{\mathbf{m}} \cdot Sol_m$ where the $S_{\mathbf{m}}$ are the Schubert cells of SU_m for all Schubert symbols $\mathbf{m} = (m_1, \dots, m_k)$ with $1 < m_1 < \cdots < m_k \leq m$.*

Global Milnor Fibers for the Variety of Singular $m \times m$-Symmetric Matrices.

If $B \in F_m^{(sy)}$, then we want to relate B to a matrix $C \in F_m^{(sy)\, c} = SU_m \cap Sym_m(\mathbb{C}) = \mathcal{C}_m^{(sy)}$. As B is symmetric and $\det(B) = 1$, as in [D3, Table 1] we may diagonalize the quadratic form $X^T \cdot B \cdot X$, for column vectors X so there is a $C \in SL_m(\mathbb{C})$ so that $(CX)^T \cdot B \cdot CX = X^T \cdot X$. Thus, $C^T \cdot B \cdot C = I_m$ or $B = (C^{-1})^T \cdot C^{-1}$. Then, by Iwasawa decomposition $C^{-1} = A \cdot E$, with $A \in SU_m$ and $E \in Sol_m$. Then, $B = E^T \cdot (A^T \cdot A) \cdot E$, and $A^T \cdot A \in \mathcal{C}_m^{(sy)}$. If

$$\mathbf{m} = (m_1, m_2, \dots, m_k)$$

is the Schubert symbol for $\tilde{A} = A^T \cdot A$, it is also the symmetric Schubert symbol and so

$$\tilde{A} = A^T \cdot A \in S_{\mathbf{m}}^{(sy)}$$

and conversely.

We let Sol_m^T denote the group of lower triangular complex matrices E with positive entries on the diagonal and $\det(E) = 1$. Then, there is the action of Sol_m^T on $\mathcal{C}_m^{(sy)}$ as follows:

$$Sol_m^T \times \mathcal{C}_m^{(sy)} \to \mathcal{C}_m^{(sy)} \qquad \text{sending} \qquad (E, \tilde{A}) \mapsto E \cdot \tilde{A} \cdot E^T\,.$$

Then, the action applied to each Schubert cell $S_{\mathbf{m}}^{(sy)}$ gives by Proposition 1.2 the Schubert cell for $F_m^{(sy)}$ which we denote by $Sol_m^T \cdot (S_{\mathbf{m}}^{(sy)})$. Combining this with Theorem 4.1 we obtain

Theorem 5.5. *The Schubert decomposition of the global Milnor fiber $F_m^{(sy)}$ for the variety of $m \times m$ symmetric complex matrices is given by the disjoint union of the* symmetric Schubert cells *$Sol_m^T \cdot (S_{\mathbf{m}}^{(sy)})$ for $S_{\mathbf{m}}^{(sy)}$ the symmetric Schubert cells of SU_m/SO_m for all symmetric Schubert symbols $\mathbf{m}^{(sy)} = (m_1, \dots, m_k)$ with $1 < m_1 < \cdots < m_k \leq m$.*

Furthermore, the preceding algorithm using ordered factorization gives the symmetric Schubert symbol for a given matrix in $F_m^{(sy)}$.

Global Milnor Fibers for the Variety of Singular $m \times m$ Skew-Symmetric Matrices.

For the case of $B \in F_m^{(sk)}$ with $m = 2n$, we follow an analogous argument to the preceding. We first want to relate B to a matrix $C \in F_m^{(sk)\, c} = SU_m \cap Sk_m(\mathbb{C})$, and then use the relation $F_m^{(sk)\, c} \cdot J_n^{-1} = \mathcal{C}_m^{(sk)}$ to determine the skew-symmetric factorization for $C \cdot J_n^{-1}$ to determine its skew-symmetric Schubert type.

As B is skew-symmetric with $\mathrm{Pf}(B) = 1$, as in [D3, Table 1] we may block diagonalize the quadratic form $X^T \cdot B \cdot X$, for column vectors X so there is a $C \in SL_m(\mathbb{C})$ so that

$$(CX)^T \cdot B \cdot CX = X^T \cdot J_n \cdot X.$$

Thus, $C^T \cdot B \cdot C = J_n$ or $B = (C^{-1})^T \cdot J_n \cdot C^{-1}$. Then, we again apply Iwasawa decomposition $C^{-1} = A \cdot E$, with $A \in SU_m$ and $E \in Sol_m$. Then,

$$B = E^T \cdot (A^T \cdot J_n \cdot A) \cdot E,$$

and

$$\tilde{A} = A^T \cdot J_n \cdot A \in SU_m \cap Sk_m(\mathbb{C}).$$

It follows $\tilde{A} \cdot J_n^{-1} \in \mathcal{C}_m^{(sk)}$. The Schubert symbol

$$\mathbf{m} = (2m_1 - 1, 2m_1, 2m_2 - 1, 2m_2, \ldots, 2m_k - 1, 2m_k)$$

for $\tilde{A} \cdot J_n^{-1}$ is obtained from the ordered factorization of $\tilde{A} \cdot J_n^{-1}$. By (5.8), this may be alternatively written as a skew-symmetric factorization of $\tilde{A}$

$$\tilde{A} \; = \; A_{(\theta_1, x_1')} \cdots A_{(\theta_k, x_k')} \cdot J_n \cdot A^T_{(\theta_k, x_k')} \cdots A^T_{(\theta_1, x_1')} \, . \tag{5.9}$$

By Corollary 5.3, $\mathbf{m}^{(sk)} = (m_1, m_2, \ldots, m_k)$ is the skew-symmetric Schubert symbol. Then, under the map $\mathcal{C}_m^{(sk)} \to F_m^{(sk)}$ given by right multiplication by J_n, i.e.

$$\tilde{A} \cdot J_n^{-1} \mapsto \tilde{A} \in SU_m \cap Sk_m(\mathbb{C}) = F_m^{(sk)},$$

we have $S_{\mathbf{m}}^{(sk)}$ mapping diffeomorphically to $S_{\mathbf{m}}^{(sk)} \cdot J_n \subset F_m^{(sk)}$. Hence, we again use the action of Sol_m^T but on $F_m^{(sk)}$ given by :

$$Sol_m^T \times F_m^{(sk)} \to F_m^{(sk)} \qquad \text{sending} \qquad (E, \tilde{A}) \mapsto E \cdot \tilde{A} \cdot E^T \, .$$

Then, from the action applied to each Schubert cell $S_{\mathbf{m}}^{(sk)}$ after right multiplication by J_n gives by Proposition 1.2 the Schubert cell for $F_m^{(sk)}$ which we denote by $Sol_m^T \cdot (S_{\mathbf{m}}^{(sk)} \cdot J_n)$. Combining this with Theorem 4.1 we obtain

Theorem 5.6. *The Schubert decomposition of the global Milnor fiber $F_m^{(sk)}$ for the variety of $m \times m$ skew-symmetric complex matrices (with $m = 2n$) is given by the disjoint union of the* skew-symmetric Schubert cells $Sol_m^T \cdot (S_{\mathbf{m}}^{(sk)} \cdot J_n)$ *corresponding to the skew-symmetric Schubert cells $S_{\mathbf{m}}^{(sk)}$ of $\mathcal{C}_m^{(sk)}$, for all skew-symmetric Schubert symbols $\mathbf{m}^{(sk)} = (m_1, \ldots, m_k)$ with $1 < m_1 < \cdots < m_k \leq n$.*

Furthermore, the preceding algorithm using ordered factorization gives the associated skew-symmetric Schubert symbol for a given matrix in $F_m^{(sk)}$.

6. Representation of the Dual Classes in Cohomology

Having given the Schubert decomposition for the global Milnor fibers in terms of the corresponding Cartan models, we now consider how the Schubert decomposition corresponds to the (co)homology of the global Milnor fibers as given in [D3], which was deduced from that of the corresponding symmetric spaces. We will refer to the closures of the Schubert cells in each case as *Schubert cycles* of the appropriate type. We shall see that for both the general and skew-symmetric cases the Schubert cycles are cycles whose fundamental classes define $\mathbb{Z}$-homology classes. For the symmetric case, the symmetric Schubert cycles are only mod 2-cycles which define unique $\mathbb{Z}/2\mathbb{Z}$-homology classes. The situation is somewhat similar to that for real Grassmannians where the $\mathbb{Z}/2\mathbb{Z}$-cohomology classes correspond to real Schubert cycles, while the rational classes are more difficult to identify in terms of the Schubert decomposition.

This identification is made using the standard method (see e.g. [Ma, Chap. IX, §4]) for computing the (co)homology of a finite CW-complex X with skeleta $\{X^{(k)}\}$ with coefficient ring R using the finite algebraic complex $C_k(\{X^{(k)}\}) = H_k(X^{(k)}, X^{(k-1)}; R)$, with boundary map given by the boundary map for the exact sequence of a triple. Then, $\mathrm{rk}_R(C_k(\{X^{(k)}\}))$ equals the number of cells q_k of dimension k. Thus, $\mathrm{rk}_R H_k(X; R) \leq q_k$ with equality iff the closures of the cells of dimension k give a free set of generators for $H_k(X; R)$. Likewise the cohomology is computed from the complex $C^k(\{X^{(k)}\}) = H^k(X^{(k)}, X^{(k-1)}; R)$ using the coboundary map for the exact sequence of a triple in cohomology.

Milnor Fiber for the Variety of Singular $m \times m$-Matrices.
We consider the Schubert decomposition for F_m obtained from that for the compact model $F^c_m = SU_m$ as a result of Theorem 5.4. Then, the homology of SU_m can be computed from the algebraic complex with basis formed from the Schubert cells $S_{\mathbf{m}}$. By a result of Hopf, the homology of SU_m (which is isomorphic as a graded $\mathbb{Z}$-module to its cohomology) is given as a graded $\mathbb{Z}$-module by

$$H_*(SU_m; \mathbb{Z}) \;\simeq\; \Lambda^*\mathbb{Z}\langle s_3, s_5, \ldots, s_{2m-1}\rangle\,,$$

where s_{2j-1} has degree $2j-1$. Then, a count shows that $H_q(SU_n; \mathbb{Z})$ is spanned by $s_{2m_1-1} \cdot s_{2m_2-1} \cdots s_{2m_k-1}$ where $1 < m_1 < m_2 < \cdots < m_k \leq m$ and $q = \sum_{j=1}^{k}(2m_j - 1)$. This equals the number of Schubert cells $S_{\mathbf{m}}$ of real dimension q. Thus, each $\overline{S_{\mathbf{m}}}$ defines a $\mathbb{Z}$-homology class of dimension $\dim_{\mathbb{R}} S_{\mathbf{m}}$. Together they form a basis for $H_q(SU_m; \mathbb{Z})$. Also, $\psi_{\mathbf{m}}(\tilde{S}_{\mathbf{m}}) = \overline{S_{\mathbf{m}}}$ and $\tilde{S}_{\mathbf{m}}$ has a top homology class in $H_q(\tilde{S}_{\mathbf{m}}; \mathbb{Z})$ for $q = \dim_{\mathbb{R}}(\tilde{S}_{\mathbf{m}})$, which we can view as a fundamental class for $\tilde{S}_{\mathbf{m}}$ for Borel-Moore homology. We have a similar dimension count in cohomology, so that the duals of the classes $\overline{S_{\mathbf{m}}}$ via the Kronecker pairing give a $\mathbb{Z}$-basis for cohomology.

Then, as $F^c_m = SU_m$ and the inclusion $i_m : F^c_m \hookrightarrow F_m$ is a homotopy equivalence, we obtain the following

Theorem 6.1. *The homology $H_*(F_m; \mathbb{Z})$ has for a free $\mathbb{Z}$-basis the fundamental classes of the Schubert cycles, given as images $i_{m\,*} \circ \psi_{\mathbf{m}\,*}([\tilde{S}_{\mathbf{m}}]) = \psi_{\mathbf{m}\,*}(\tilde{S}_{\mathbf{m}}) = \overline{S_{\mathbf{m}}}$ as we vary over the Schubert decomposition of SU_m. The Kronecker duals of these classes give the $\mathbb{Z}$-basis for the cohomology*

$$H^*(SU_m; \mathbb{Z}) \;\simeq\; \Lambda^*\mathbb{Z}\langle e_3, e_5, \ldots, e_{2m-1}\rangle\,.$$

Moreover, the Kronecker duals of the simple Schubert classes $S_{(m_1)}$ *are homogeneous generators of the exterior algebra cohomology.*

Proof. The preceding discussion establishes all of the theorem except for the last statement about the generators of the cohomology algebra. We prove this by induction on m. It is trivially true for $m = 1, 2$. Suppose it is true for $m < n$ and let $i_{n-1} : SU_{n-1} \hookrightarrow SU_n$ denote the natural inclusion $A \mapsto \left(\begin{smallmatrix} A & 0 \\ 0 & 1 \end{smallmatrix}\right)$. The Schubert decomposition preserves the inclusion so that any $S_{\mathbf{m}}$ for $\mathbf{m} = (m_1, m_2, \cdots, m_k)$ with $m_k < n$ is contained in the image of i_{n-1} and so is also a Schubert cell for SU_{n-1}; while if $m_k = n$, then $S_{\mathbf{m}}$ is in the complement of the image of SU_{n-1}. Thus, if the result is true for SU_{n-1}, the Kronecker duals to the simple $S_{(m_1)}$ with $m_1 < n$ restrict via i^*_{n-1} to the Kronecker duals of the $S_{(m_1)}$ with $m_1 < n$ viewed as Schubert cells of SU_{n-1}. Thus, they map to the generators of the exterior algebra $\Lambda^*\mathbb{Z} < e_3, e_5, \cdots e_{2n-3} >$. Also, the Kronecker dual to any $S_{\mathbf{m}}$ with $m_k = n$ is zero on any Schubert cell of SU_{n-1} so by a counting argument the kernel of i^*_{n-1}, which is the ideal generated by e_{2n-1}, is spanned by the Kronecker duals of the Schubert cells with $m_k = n$.

Now there is a unique Schubert class of this type of degree $2n - 1$, and hence its Kronecker dual is the added generator which together with the others for $S_{(m_1)}$ with $m_1 < n$ generate $H^*(SU_n; \mathbb{Z})$. □

There is also the question of identifying the Kronecker dual of the Schubert cycle $[\overline{S}_{\mathbf{m}}]$ for $\mathbf{m} = (m_1, m_2, \cdots, m_k)$, which we denote by $e_{\mathbf{m}}$. We claim it is given up to sign by the cohomology class $e_{2m_1-1} \cdot e_{2m_2-1} \cdots e_{2m_k-1}$ (where the products denote cup-products). We show this using the product structure of the group SU_m to give a product representation for the closures of Schubert cells together with the Hopf algebra structure of $H^*(SU_m)$.

We let $\overline{S_{\mathbf{m}}} \cdot \overline{S_{\mathbf{m}'}}$ denote the group product in SU_m of the closures of Schubert cells $\overline{S_{\mathbf{m}}}$ and $\overline{S_{\mathbf{m}'}}$. We also use the simpler notation S_{m_1} to denote the Schubert cell $S_{\mathbf{m}}$ when $\mathbf{m} = (m_1)$. In particular, we emphasize that

$$S_{m_1} \;=\; \{A_{(-\theta,e_1)} \cdot A_{(\theta,x_1)} : \theta \in (0,2\pi), x_1 \in_{\min} \mathbb{C}^{m_1}\}\,.$$

First, as result of Lemma 3.2, we obtain the following version of a lemma due to Whitehead (see e.g. [KM, Lemma 4.2] or [Mi, Lemma 2.2]).

Lemma 6.2. *For Schubert cells in $\mathcal{C}_m$ for SU_m,*

1) *If $1 < m_1 < m_2 \leq m$, then*
$$\overline{S_{m_2}} \cdot \overline{S_{m_1}} \;=\; \overline{S_{m_1}} \cdot \overline{S_{m_2}} \;=\; \overline{S_{(m_1,m_2)}}\,.$$
2) *If $1 < m' \leq m$, then*
$$\overline{S_{m'}} \cdot \overline{S_{m'}} \;\subseteq\; \overline{S_{(m'-1,m')}}\,.$$

We note that this differs slightly from the above referred to lemmas as each element in S_{m_1} is a product of two pseudo-rotations, one of which is $A_{(-\theta,e_1)}$. However, by the lemma, this pseudo-rotation can also be interchanged with other $A_{(\theta,x_j)}$, and combined via multiplication with other $A_{(-\theta',e_1)}$. We also note in the lemma that $\dim_{\mathbb{R}} S_{(m'-1,m')} \leq 2 \cdot \dim_{\mathbb{R}} S_{m'} - 2$.

We can inductively repeat this to obtain

Lemma 6.3. *For Schubert cells S_{m_j} in $\mathcal{C}_m$ (for SU_m):*

1) *If $\mathbf{m} = (m_1, m_2, \ldots, m_r)$ then*
$$\overline{S_{\mathbf{m}}} \;=\; \overline{S_{m_1}} \cdot \overline{S_{m_2}} \cdots \overline{S_{m_r}}\,.$$
2) *If $\mathbf{m} = (m_1, m_2, \ldots, m_r)$ and $\mathbf{m'} = (m'_1, m'_2, \ldots, m'_{r'})$ with*
$$\{m_1, m_2, \ldots, m_r\} \cap \{m'_1, m'_2, \ldots, m'_{r'}\} = \emptyset,$$
then
$$\overline{S_{\mathbf{m}}} \cdot \overline{S_{\mathbf{m'}}} \;=\; \overline{S_{\mathbf{m''}}}\,,$$
where $\mathbf{m''}$ is the union of $\mathbf{m}$ and $\mathbf{m'}$ in increasing order.
3) *If $\mathbf{m} = (m_1, m_2, \ldots, m_r)$ and $\mathbf{m'} = (m'_1, m'_2, \ldots, m'_{r'})$ with*
$$\{m_1, m_2, \ldots, m_r\} \cap \{m'_1, m'_2, \ldots, m'_{r'}\} \neq \emptyset,$$
then
$$\overline{S_{\mathbf{m}}} \cdot \overline{S_{\mathbf{m'}}} \;\subset\; \mathcal{C}_m^{(q)}\,,$$
where $q \leq \dim_{\mathbb{R}} S_{\mathbf{m}} + \dim_{\mathbb{R}} S_{\mathbf{m'}} - 2$.

Proof. For 1) we consider a product in $S_{m_1} \cdot S_{m_2} \cdots S_{m_r}$ which has the form

$$B \;=\; (A_{(-\theta_1,e_1)} \cdot A_{(\theta_1,x_1)}) \cdot (A_{(-\theta_2,e_1)} \cdot A_{(\theta_2,x_2)} \cdots (A_{(-\theta_r,e_1)} \cdot A_{(\theta_r,x_r)}), \tag{6.1}$$

where each $x_j \in_{\min} \mathbb{C}^{m_j}$. Then, we may repeatedly apply the Whitehead Lemma to move each $A_{(-\theta_j,e_1)}$ to the left and obtain a factorization in the form

$$B \;=\; A_{(-\tilde{\theta},e_1)} \cdot A_{(\theta_1,x'_1)} \cdot A_{(\theta_2,x'_2)} \cdots A_{(\theta_r,x'_r)}), \tag{6.2}$$

where $\tilde{\theta} = \sum_{j=1}^{r} \theta_j$ and each $x'_j \in_{\min} \mathbb{C}^{m_j}$. Hence, $B \in S_{\mathbf{m}}$. Conversely we can reverse the process beginning with B in (6.2) and obtain a factorization as in (6.1). This gives the equality for the Schubert cells. Since the closures are compact, we obtain the equality of 1) by taking closures of the Schubert cells.

Given 1) we may write

$$S_{\mathbf{m}} \cdot S_{\mathbf{m'}} \;=\; (S_{m_1} \cdot S_{m_2} \cdots S_{m_r}) \cdot (S_{m'_1} \cdot S_{m'_2} \cdots S_{m'_{r'}}). \tag{6.3}$$

If $\{m_1, m_2, \ldots, m_r\} \cap \{m'_1, m'_2, \ldots, m'_{r'}\} = \emptyset$, then we can repeatedly apply a) of the Whitehead Lemma to move an element of $S_{m'_j}$ across an element of S_{m_i} when $m_i > m'_j$ while preserving the order of the m_i's and m'_j's. We arrive at an ordered factorization with increasing order $\mathbf{m}''$, which is the union of $\mathbf{m}$ and $\mathbf{m}'$ in increasing order. Taking closures of the Schubert cells then gives 2).

Finally, for 3), we may begin with (6.3). There are smallest $m_\ell = m'_k$. Then, if $m'_j < m'_k$ then it differs from all m_i. Hence, we can first move the elements in $S_{m'_j}$ across all of those in S_{m_i} as in the previous case by 2) of Lemma 6.3. Next, we can move elements in $S_{m_{k'}}$ across those in S_{m_j} as long as $m_j > m_\ell$. Then, we arrive at a factorization where we have successive terms in S_{m_ℓ} and $S_{m_{k'}}$ with $m_\ell = m'_k$. Then, we may apply b) of the Whitehead lemma (or 2) of Lemma 6.2) and obtain a new pair in $S_{\tilde{m}}$ and S_{m_ℓ} with $\tilde{m} \leq m_\ell - 1$. This has the effect of reducing the sum of the Schubert symbol values in the product by at least 1. Also, further application of the Whitehead Lemma will not increase the sum. Hence, by further application of the Whitehead Lemma we obtain a product in the union of Schubert cells of dimension $q \leq \dim_{\mathbb{R}} S_{\mathbf{m}} + \dim_{\mathbb{R}} S_{\mathbf{m}'} - 2$. Thus, it lies in the q-skeleton of $\mathcal{C}_m$. This gives 3) when we take closures. $\square$

Now we will use the Hopf structure of $H^*(SU_n)$ to relate the fundamental classes from the Schubert decomposition with the cohomology classes via the Kronecker pairing. Let

$$\mu : SU_n \times SU_n \to SU_n$$

denote the multiplication map. Then, we can use Lemma 6.3 to determine the effect of μ_* for homology using the complex $C_k(\{X^{(k)}\})$ and then the coproduct map μ^* for the Hopf algebra. We obtain as a corollary of Lemma 6.3.

Corollary 6.4. *We let $s_{\mathbf{m}}$ denote the homology class obtained from $\psi_{\mathbf{m}\,*}([\tilde{S}_{\mathbf{m}}])$ with restriction to positive orientation for $E_{\mathbf{m}}$. For $\mathbf{m} = (m_1, m_2, \dots, m_r)$ and $\mathbf{m}' = (m'_1, m'_2, \dots, m'_{r'})$ we let $m = \{m_1, m_2, \dots, m_r\} \cap \{m'_1, m'_2, \dots, m'_{r'}\}$ and let $\mathbf{m}'' = (m''_1, m''_2, \dots, m''_{r''})$ denote the union of the elements of $\mathbf{m}$ and $\mathbf{m}'$ written in increasing order. Then,*

$$\mu_*(s_{\mathbf{m}} \otimes s_{\mathbf{m}'}) \;=\; \begin{cases} \varepsilon_{\mathbf{m},\mathbf{m}'} \cdot s_{\mathbf{m}''} & \text{if } m = \emptyset, \\ 0 & \text{if } m \neq \emptyset\,, \end{cases} \tag{6.4}$$

where $\varepsilon_{\mathbf{m},\mathbf{m}'}$ is the sign of the permutation which moves $(\mathbf{m}, \mathbf{m}')$ to increasing order.

The reason for the factor $\varepsilon_{\mathbf{m},\mathbf{m}'}$ is that each interchange of two factors $S_{(m_1)}$ and $S_{(m_2)}$ will change the orientation by a factor $(-1)^{(2m_1-1)(2m_2-1)} = -1$.

From the corollary we obtain a formula for the coproduct μ^* in terms of the (Kronecker) dual basis $\{e_{\mathbf{m}}\}$ in cohomology to Schubert basis for homology $\{s_{\mathbf{m}}\}$.

$$\mu^*(e_{\mathbf{m}}) \;=\; \sum (-1)^{\deg(e_{\mathbf{m}'})\deg(e_{\mathbf{m}''})} \varepsilon_{\mathbf{m}',\mathbf{m}''} \cdot e_{\mathbf{m}'} \otimes e_{\mathbf{m}''}\,, \tag{6.5}$$

where the sum is over all disjoint $\mathbf{m}'$ and $\mathbf{m}''$ whose union in increasing order gives $\mathbf{m}$ (and the terms $(-1)^{\deg(e_{\mathbf{m}'})\deg(e_{\mathbf{m}''})}$ arise from the property $(\varphi \otimes \psi)(\sigma \otimes \nu) = (-1)^{\deg(\varphi)\deg(\psi)}\varphi(\sigma)\psi(\nu)$). Since $S_{\mathbf{m}}$ is a product of odd dimensional cells, $\deg(e_{\mathbf{m}'})(= \dim_{\mathbb{R}} S_{\mathbf{m}}) \equiv \ell(\mathbf{m}) \bmod 2$ and the sign in (6.5) equals $(-1)^{\ell(\mathbf{m}')\ell(\mathbf{m}'')}$. Also, note the sum includes the empty symbol which denotes the Schubert cell consisting of just I_n. In the case of the simple Schubert symbol (m_1) we obtain

$$\mu^*(e_{(m_1)}) \;=\; e_{(m_1)} \otimes 1 \,+\, 1 \otimes e_{(m_1)}\,.$$

Hence, all of the $e_{(m_1)}$ are independent primitive classes. Then there is the following relation between the generators of $H^*(SU_n)$ and the Schubert classes.

Theorem 6.5. *$H^*(SU_n)$ is a free exterior algebra with generators $e_{(m)}$ of degrees $2m-1$, for $m = 2, \dots, n$. Moreover the Kronecker dual to $s_{\mathbf{m}}$ for $\mathbf{m} = (m_1, m_2, \dots, m_r)$ is*

$$e_{\mathbf{m}} = (-1)^{\beta(\mathbf{m})} e_{(m_1)} e_{(m_2)} \dots e_{(m_r)},$$

where $\beta(\mathbf{m}) = \binom{\ell(\mathbf{m})}{2}$ *(where we denote* $\binom{1}{2} = 0$*).*

Proof. We already have established the first statement about the algebra generators in Theorem 6.1. We note that it also follows from the Hopf algebra structure. Since the $e_{(m)}$, for $m = 2, \dots, n$ are primitive generators of degree $2m-1$, and $H^*(SU_n)$ is a Hopf algebra which is a free exterior algebra on generators of degrees $2m-1$ for $m = 2, \dots, n$, it follows by a theorem of Hopf-Samuelson that $H^*(SU_n)$ is the free exterior algebra generated by the primitive elements $e_{(m)}$, for $m = 2, \dots, n$.

We furthermore claim that the Kronecker dual to the Schubert class $s_{\mathbf{m}}$ for

$$\mathbf{m} = (m_1, m_2, \dots, m_r)$$

is given by $(-1)^{\beta(\mathbf{m})} e_{(m_1)} e_{(m_2)} \dots e_{(m_r)}$, which will follow from $e_{\mathbf{m}} = (-1)^{\ell(\mathbf{m}')} e_{(m_1)} e_{\mathbf{m}'}$ for

$$\mathbf{m}' = (m_2, m_3, \dots m_r).$$

We prove this by induction on r. It is already true for $r = 1$. Next, consider the case of $\mathbf{m} = (m_1, m_2)$; then $\varepsilon_{(m_1),(m_2)} = 1$, $\varepsilon_{(m_2),(m_1)} = -1$ and $(-1)^{\ell(m_1)\ell(m_2)} = -1$. Then, from (6.5)

$$\mu^*(e_{(m_1,m_2)}) \;=\; e_{(m_1,m_2)} \otimes 1 - e_{(m_1)} \otimes e_{(m_2)} + e_{(m_2)} \otimes e_{(m_1)} + 1 \otimes e_{(m_1,m_2)} \,. \tag{6.6}$$

Also, as μ^* is an algebra homomorphism,

$$\begin{aligned} \mu^*(e_{(m_1)} \cdot e_{(m_2)}) \;&=\; \mu^*(e_{(m_1)}) \cdot \mu^*(e_{(m_2)}) \\ &=\; \big(e_{(m_1)} \otimes 1 + 1 \otimes e_{(m_1)}\big) \cdot \big(e_{(m_2)} \otimes 1 + 1 \otimes e_{(m_2)}\big) \\ &=\; e_{(m_1)} \cdot e_{(m_2)} \otimes 1 + e_{(m_1)} \otimes e_{(m_2)} - e_{(m_2)} \otimes e_{(m_1)} + 1 \otimes e_{(m_1)} \cdot e_{(m_2)} \,, \end{aligned} \tag{6.7}$$

where the signs on the RHS result from both $e_{(m_1)}$ and $e_{(m_2)}$ having odd degree. Adding (6.7) and (6.6), we obtain

$$\mu^*(e_{(m_1,m_2)} + e_{(m_1)} \cdot e_{(m_2)}) = \big(e_{(m_1,m_2)} + e_{(m_1)} \cdot e_{(m_2)}\big) \otimes 1 + 1 \otimes \big(e_{(m_1,m_2)} + e_{(m_1)} \cdot e_{(m_2)}\big) \,. \tag{6.8}$$

This implies that if $e_{(m_1,m_2)} + e_{(m_1)} \cdot e_{(m_2)} \neq 0$, then it is a primitive element independent from the other primitive elements $e_{(m)}$. This contradicts the Hopf-Samuelson theorem. Thus, $e_{(m_1,m_2)} = -e_{(m_1)} \cdot e_{(m_2)}$.

Suppose by induction the result holds for $k < r$. Then, for $\mathbf{m} = (m_1, \dots, m_r)$, let $\mathbf{m}' = (m_2, \dots, m_r)$. First, by (6.5) we have

$$\mu^*(e_{\mathbf{m}}) \;=\; e_{\mathbf{m}} \otimes 1 + 1 \otimes e_{\mathbf{m}} + \sum (-1)^{\ell(\mathbf{m}')\ell(\mathbf{m}'')} \varepsilon_{\mathbf{m}',\mathbf{m}''} \cdot e_{\mathbf{m}'} \otimes e_{\mathbf{m}''} \,, \tag{6.9}$$

where the sum is over all $\mathbf{m}' = (m'_1, m'_2, \dots, m'_k)$ and $\mathbf{m}'' = (m''_1, m''_2, \dots, m''_{k'})$ which are both nonempty, disjoint, and whose union in increasing order is $\mathbf{m}$. Then, by induction we obtain

$$\begin{aligned} \mu^*(e_{(m_1)} \cdot e_{\mathbf{m}'}) \;&=\; \mu^*(e_{(m_1)}) \cdot \mu^*(e_{\mathbf{m}'}) \\ &=\; \big(e_{(m_1)} \otimes 1 + 1 \otimes e_{(m_1)}\big) \cdot \big(e_{\mathbf{m}'} \otimes 1 + 1 \otimes e_{\mathbf{m}'} + \\ &\qquad \sum (-1)^{\ell(\mathbf{m}'')\ell(\mathbf{m}''')} \varepsilon_{\mathbf{m}'',\mathbf{m}'''} \cdot e_{\mathbf{m}''} \otimes e_{\mathbf{m}'''}\big) \,, \end{aligned} \tag{6.10}$$

where the sum is over $\mathbf{m}''$ and $\mathbf{m}'''$ which are nonempty, disjoint and whose union in increasing order is $\mathbf{m}'$. In the sum on the RHS of (6.9), we have in addition to the terms $e_{\mathbf{m}} \otimes 1$ and $1 \otimes e_{\mathbf{m}}$ the four following types of terms :

Four Types of Terms in (6.9):

i) $(-1)^{\ell(\mathbf{m}')} \varepsilon_{(m_1),\mathbf{m}'} \cdot e_{(m_1)} \otimes e_{\mathbf{m}'} \;=\; (-1)^{\ell(\mathbf{m}')} e_{(m_1)} \otimes e_{\mathbf{m}'}$

ii) $(-1)^{\ell(\mathbf{m}')} \varepsilon_{\mathbf{m}',(m_1)} \cdot e_{\mathbf{m}'} \otimes e_{(m_1)} \;=\; e_{\mathbf{m}'} \otimes e_{(m_1)}$

iii) $(-1)^{\ell(\mathbf{m}'')\ell(\mathbf{m}''')} \varepsilon_{\mathbf{m}'',\mathbf{m}'''} \cdot e_{\mathbf{m}''} \otimes e_{\mathbf{m}'''}$ with m_1 in $\mathbf{m}''$

iv) $(-1)^{\ell(\mathbf{m}'')\ell(\mathbf{m}''')} \varepsilon_{\mathbf{m}'',\mathbf{m}'''} \cdot e_{\mathbf{m}''} \otimes e_{\mathbf{m}'''}$ with m_1 in $\mathbf{m}'''$

For comparison, we have in addition to the terms $(e_{(m_1)}e_{\mathbf{m}'}) \otimes 1$ and $1 \otimes (e_{(m_1)}e_{\mathbf{m}'})$ the corresponding terms from (6.10) which have the following types:

Corresponding Four Types of Terms in (6.10):

i) $e_{(m_1)} \otimes e_{\mathbf{m}'}$
ii) $(-1)^{\ell(\mathbf{m}')} e_{\mathbf{m}'} \otimes e_{(m_1)}$
iii) $(-1)^{\ell(\mathbf{m}'')\ell(\mathbf{m}''')} \varepsilon_{\mathbf{m}'',\mathbf{m}'''} \cdot (e_{(m_1)}e_{\mathbf{m}''}) \otimes e_{\mathbf{m}'''}$
iv) $(-1)^{\ell(\mathbf{m}'')}(-1)^{\ell(\mathbf{m}'')\ell(\mathbf{m}''')} \varepsilon_{\mathbf{m}'',\mathbf{m}'''} \cdot e_{\mathbf{m}''} \otimes (e_{(m_1)}e_{\mathbf{m}'''})$

In the first two cases for (6.10), we can view them as a decomposition of $\mathbf{m}$ either as $(\{m_1\}, \mathbf{m}')$ or $(\mathbf{m}', \{m_1\})$. We see that the corresponding coefficients for i) and ii) for (6.10) and (6.9) differ by a factor $(-1)^{\ell(\mathbf{m}')}$. The corresponding terms in iii) and iv) for (6.10) can be viewed as a decomposition either as $(\{m_1\} \cup \mathbf{m}'', \mathbf{m}''')$ or $(\mathbf{m}'', \{m_1\} \cup \mathbf{m}''')$. The corresponding coefficients will also be shown to differ by the same factor $(-1)^{\ell(\mathbf{m}')}$.

For example, for iv) let $\tilde{\mathbf{m}}''' = \{m_1\} \cup \mathbf{m}'''$. Then,

$$\varepsilon_{\mathbf{m}'',\tilde{\mathbf{m}}'''} = (-1)^{\ell(\mathbf{m}'')} \varepsilon_{\mathbf{m}'',\mathbf{m}'''}, \quad \ell(\tilde{\mathbf{m}}''') = \ell(\mathbf{m}''') + 1;$$

and by the induction hypothesis $e_{\tilde{\mathbf{m}}'''} = (-1)^{\ell(\mathbf{m}''')} e_{(m_1)} \cdot e_{\mathbf{m}'''}$. Then, substituting these values in iv) for (6.10) yields

$$\begin{aligned}
&(-1)^{\ell(\mathbf{m}'')}(-1)^{\ell(\mathbf{m}'')\ell(\mathbf{m}''')} \varepsilon_{\mathbf{m}'',\mathbf{m}'''} \cdot e_{\mathbf{m}''} \otimes (e_{(m_1)}e_{\mathbf{m}'''}) \;= \\
&(-1)^{\ell(\mathbf{m}'')}(-1)^{\ell(\mathbf{m}'')\ell(\tilde{\mathbf{m}}''')}(-1)^{\ell(\mathbf{m}'')}(-1)^{\ell(\mathbf{m}'')}(-1)^{\ell(\mathbf{m}''')} \varepsilon_{\mathbf{m}'',\tilde{\mathbf{m}}'''} \cdot e_{\mathbf{m}''} \otimes e_{\tilde{\mathbf{m}}'''} \\
&= (-1)^{\ell(\mathbf{m}'')\ell(\tilde{\mathbf{m}}''')}(-1)^{\ell(\mathbf{m}'')}(-1)^{\ell(\mathbf{m}''')} \varepsilon_{\mathbf{m}'',\tilde{\mathbf{m}}'''} \cdot e_{\mathbf{m}''} \otimes e_{\tilde{\mathbf{m}}'''} \\
&= (-1)^{\ell(\mathbf{m}')} \left((-1)^{\ell(\mathbf{m}'')\ell(\tilde{\mathbf{m}}''')} \varepsilon_{\mathbf{m}'',\tilde{\mathbf{m}}'''} \cdot e_{\mathbf{m}''} \otimes e_{\tilde{\mathbf{m}}'''} \right)
\end{aligned} \tag{6.11}$$

A similar, but somewhat simpler, argument works for the terms iii).

Then, we proceed as in the previous case. We compute $\mu^*(e_{\mathbf{m}} - (-1)^{\ell(\mathbf{m}')} e_{(m_1)}e_{\mathbf{m}'})$ from (6.10) and (6.9) and by the above all terms of types i) - iv) cancel so we obtain

$$\mu^*(e_{\mathbf{m}} - (-1)^{\ell(\mathbf{m}')} e_{(m_1)}e_{\mathbf{m}'}) \;=\; (e_{\mathbf{m}} - (-1)^{\ell(\mathbf{m}')} e_{(m_1)}e_{\mathbf{m}'}) \otimes 1 \;+\; 1 \otimes (e_{\mathbf{m}} - (-1)^{\ell(\mathbf{m}')} e_{(m_1)}e_{\mathbf{m}'}). \tag{6.12}$$

This again implies that $e_{\mathbf{m}} - (-1)^{\ell(\mathbf{m}')} e_{(m_1)}e_{\mathbf{m}'}$ is a primitive element if it is nonzero. Hence, it is zero and so $e_{\mathbf{m}} = (-1)^{\ell(\mathbf{m}')} e_{(m_1)}e_{\mathbf{m}'}$. Repeated inductive application of this implies that for $\mathbf{m} = (m_1, m_2, \dots, m_r)$

$$e_{\mathbf{m}} \;=\; (-1)^{\beta(\mathbf{m})} e_{(m_1)} \cdot e_{(m_2)} \cdots e_{(m_r)} \,.$$

with $\beta(\mathbf{m}) = 1 + 2 + \cdots + (r-1) = \binom{\ell(\mathbf{m})}{2}$. □

As a consequence we have determined the Poincaré duals to the Schubert classes.

Corollary 6.6. *For each Schubert symbol* $\mathbf{m} = (m_1, m_2, \dots, m_r)$ *let the ordered complement in* $\{2, 3, \dots, n\}$ *be denoted by* $\mathbf{m}' = (m'_1, m'_2, \dots, m'_{n-1-r})$.

i) *The Poincaré dual to the Schubert class* $\left[\overline{S_{\mathbf{m}}}\right]$ *in* F_n^c *and to the Schubert class* $\left[\overline{S_{\mathbf{m}}} \cdot Sol_n\right]$ *in* F_n *is given by*

$$(-1)^{(\beta(\mathbf{n}) + \beta(\mathbf{m}))} \varepsilon_{\mathbf{m},\mathbf{m}'} \, e_{(m'_1)} \cdot e_{(m'_2)} \cdots e_{(m'_{n-1-r})}$$

for $\mathbf{n} = (2, 3, \dots, n)$.

ii) *For Schubert symbols* $\mathbf{m}$ *and* $\mathbf{m}'$ *such that* $\ell(\mathbf{m}) + \ell(\mathbf{m}') = n-1$, *the intersection pairing satisfies*

$$\langle[\overline{S_{\mathbf{m}}}], [\overline{S_{\mathbf{m}'}}]\rangle = \begin{cases} (-1)^{(\beta(\mathbf{n})+\beta(\mathbf{m})+\beta(\mathbf{m}'))}\varepsilon_{\mathbf{m},\mathbf{m}'} & \text{if } \mathbf{m}' \text{ is the ordered complement to } \mathbf{m}, \\ 0 & \text{otherwise}. \end{cases} \tag{6.13}$$

Proof. By Theorem 6.5, the Kronecker dual to $\left[\overline{S_{\mathbf{m}}}\right]$ is given by

$$e_{\mathbf{m}} = (-1)^{\beta(\mathbf{m})} e_{(m_1)} \cdot e_{(m_2)} \cdots e_{(m_r)}.$$

Also, the fundamental class for $[SU_n]$ with orientation given by $\left[\overline{S_{\mathbf{n}}}\right]$ has Kronecker dual

$$(-1)^{\beta(\mathbf{n})} e_{(2)} \cdot e_{(3)} \cdots e_{(n)}.$$

Then, the Poincaré dual to $\left[\overline{S_{\mathbf{m}}}\right]$ is given by a cohomology class ν such that

$$e_{\mathbf{m}} \cup \nu = (-1)^{\beta(\mathbf{n})} e_{(2)} \cdot e_{(3)} \cdots e_{(n)}.$$

This is satisfied by

$$\nu = (-1)^{(\beta(\mathbf{n})+\beta(\mathbf{m}))}\varepsilon_{\mathbf{m},\mathbf{m}'}\, e_{(m'_1)} \cdot e_{(m'_2)} \cdots e_{(m'_{n-1-r})}\,.$$

In the case of the Schubert class $\left[\overline{S_{\mathbf{m}} \cdot Sol_n}\right]$ in F_n, we note that $\overline{S_{\mathbf{m}}}$ is the transverse intersection of $F_n^c = SU_n$ with $\overline{S_{\mathbf{m}}} \cdot Sol_n$ in F_n and that the inclusion $i_n : F_n^c \hookrightarrow F_n$ is a homotopy equivalence. Hence, by a fiber square argument, the Poincaré dual in $H^*(F_n;\mathbb{Z})$ to the fundamental class of $\overline{S_{\mathbf{m}}} \cdot Sol_n$ for Borel-Moore homology, agrees via i_n^* with that for the fundamental class of $\overline{S_{\mathbf{m}}}$ in $H^*(F_n^c;\mathbb{Z})$.

The consequence for the intersection pairing follows from the above and

$$\langle[\overline{S_{\mathbf{m}}}], [\overline{S_{\mathbf{m}'}}]\rangle = \langle e_{\mathbf{m}} \cup e_{\mathbf{m}'}, \left[\overline{S_{\mathbf{n}}}\right]\rangle. \tag{6.14}$$

□

Milnor Fiber for the Variety of Singular $m \times m$-Skew-Symmetric Matrices.

We second consider the case of the global Milnor fiber $F_m^{(sk)}$ for skew-symmetric matrices with $m = 2n$. Then, the homology of SU_{2n}/Sp_n can be computed from the algebraic complex with basis formed from the Schubert cells $S_{\mathbf{m}}^{(sk)}$. By a result of Cartan (see e.g. Mimura-Toda [MT, Theorem 6.7]) the homology of SU_{2n}/Sp_n (which is isomorphic as a graded $\mathbb{Z}$-module to its cohomology) is given as a graded $\mathbb{Z}$-module by

$$H_*(SU_{2n}/Sp_n;\mathbb{Z}) \simeq \Lambda^*\mathbb{Z}\langle s_5, s_9, \ldots, s_{4n-3}\rangle\,. \tag{6.15}$$

where s_{4j-3} has degree $4j-3$. By the universal coefficient theorem this holds as well as a vector space over a field $\mathbf{k}$ of characteristic zero.

Theorem 6.7. *The homology* $H_*(F_m^{(sk)\,c};\mathbb{Z})$ *for* $m = 2n$ *has for a free* $\mathbb{Z}$*-basis the fundamental classes of the skew-symmetric Schubert cycles,* $i_{m\,*} \circ \psi_{\mathbf{m}\,*}^{(sk)}([\tilde{S}_{\mathbf{m}}^{(sk)}]) = \psi_{\mathbf{m}\,*}^{(sk)}(\tilde{S}_{\mathbf{m}}^{(sk)}) = \overline{S_{\mathbf{m}}^{(sk)}}$ *as we vary over the Schubert decomposition of* $\mathcal{C}_m^{(sk)} \simeq SU_{2n}/Sp_n$*. Moreover, the Kronecker duals of the simple skew-symmetric Schubert cycles* $\overline{S_{(m_1)}^{(sk)}}$ *give homogeneous exterior algebra generators for the cohomology.*

This likewise extends to $H_*(F_m^{(sk)};\mathbb{Z})$ *(*$m = 2n$*) for Borel-Moore homology with basis given by the fundamental classes of the global skew-symmetric Schubert cycles* $Sol_m^T \cdot (\overline{S_{\mathbf{m}}^{(sk)}} \cdot J_n)$ *for* $F_m^{(sk)}$*. The Poincaré duals of these classes form a* $\mathbb{Z}$*-basis for the cohomology*

$$H^*(F_m^{(sk)};\mathbb{Z}) \simeq \Lambda^*\mathbb{Z}\langle e_5, e_9, \ldots, e_{4n-3}\rangle\,.$$

Proof. The proof follows the same lines as that of Theorem 6.1. Then, a count from (6.15) shows that $H_q(SU_{2n}/Sp_n;\mathbb{Z})$ is spanned by $s_{4m_1-3}\cdot s_{4m_2-3}\cdots s_{4m_k-3}$, where

$$1 < m_1 < m_2 < \cdots < m_k \leq n$$

and $q = \sum_{j=1}^{k}(4m_j - 3)$. By Theorem 5.6 this equals the number of skew-symmetric Schubert cells $S_{\mathbf{m}}^{(sk)}$ of real dimension q. Thus, each $\psi_{\mathbf{m}}^{(sk)}(\tilde{S}_{\mathbf{m}}^{(sk)}) = \overline{S_{\mathbf{m}}^{(sk)}}$ defines a $\mathbb{Z}$-homology class of dimension $\dim_{\mathbb{R}} S_{\mathbf{m}}^{(sk)}$. Together they form a basis for $H_q(SU_{2n}/Sp_n;\mathbb{Z})$. That the Kronecker duals of the simple Schubert cycles $S_{(m_1)}^{(sk)}$ give algebra generators for the cohomology follows by the same argument used in Theorem 6.1.

As $\tilde{S}_{\mathbf{m}}^{(sk)}$ has a top homology class in $H_q(\tilde{S}_{\mathbf{m}}^{(sk)};\mathbb{Z})$ for $q = \dim_{\mathbb{R}}(\tilde{S}_{\mathbf{m}}^{(sk)})$, we can view it as a fundamental class for $\tilde{S}_{\mathbf{m}}^{(sk)}$ for Borel-Moore homology. As $F_m^{(sk)\,c} \simeq \mathcal{C}_m^{(sk)} \simeq SU_{2n}/Sp_n$ by multiplication by J_n and the inclusion $i_m : F_m^{(sk)\,c} \hookrightarrow F_m^{(sk)}$ is a homotopy equivalence, we conclude that these classes form a $\mathbb{Z}$-basis for the cohomology via $H^*(F_m^{(sk)};\mathbb{Z}) \simeq H^*(F_m^{(sk)\,c};\mathbb{Z})$. Their Poincaré duals then form a $\mathbb{Z}$-basis for the Borel-Moore homology. □

Again there is the question of explicitly identifying the Kronecker dual of the fundamental class $\psi_{\mathbf{m}\,*}^{(sk)}([\tilde{S}_{\mathbf{m}}^{(sk)}])$ with a cohomology class as a polynomial in the cohomology algebra generators e_{4j-3}, $j = 2, \ldots, n$, and as a consequence explicitly identifying the generators for the cohomology algebra. We shall comment on this after next considering the symmetric case.

Milnor Fiber for the Variety of Singular $m \times m$-Symmetric Matrices.

We next consider the case of $F_m^{(sy)}$. Again the line of reasoning will be similar to the two preceding cases with the crucial difference that the (co)homology has two different forms for coefficients $\mathbb{Z}/2\mathbb{Z}$ or a field of characteristic zero. There is the compact model

$$F_n^{(sy)\,c} \simeq \mathcal{C}_n^{(sy)} \simeq SU_n/SO_n$$

for $F_n^{(sy)}$. Then, the homology of SU_n/SO_n can be computed from the algebraic complex with basis formed from the Schubert cells $S_{\mathbf{m}}^{(sy)}$. By a result of Borel and Hopf, see e.g. [Bo] and see [KM], the homology of SU_n/SO_n with $\mathbb{Z}/2\mathbb{Z}$-coefficients (which is isomorphic as a graded $\mathbb{Z}/2\mathbb{Z}$-vector space to its cohomology) is given as a graded vector space over the field $\mathbb{Z}/2\mathbb{Z}$

$$H_*(SU_n/SO_n;\mathbb{Z}/2\mathbb{Z}) \;\simeq\; \Lambda^*\mathbb{Z}/2\mathbb{Z}\langle s_2, s_3, \ldots, s_n\rangle\,,$$

where s_j has degree j. A count shows that

$$\dim_{\mathbb{Z}/2\mathbb{Z}} H_*(SU_n/SO_n;\mathbb{Z}/2\mathbb{Z}) \;=\; 2^{n-1}\,.$$

This is the same as the number of Schubert cells $S_{\mathbf{m}}^{(sy)}$, for

$$1 < m_1 < \cdots < m_k \leq n$$

in the cell decomposition of SU_n/SO_n. Thus, the Schubert cycles $\overline{S_{\mathbf{m}}^{(sy)}}$, which are mod 2-homology cycles, give a $\mathbb{Z}/2\mathbb{Z}$-basis for the homology $H_*(SU_n/SO_n;\mathbb{Z}/2\mathbb{Z})$. In particular the mod 2-homology cycles $\overline{S_{\mathbf{m}}^{(sy)}}$ for which $|\mathbf{m}| = q$ give a $\mathbb{Z}/2\mathbb{Z}$-basis for $H_q(SU_n/SO_n;\mathbb{Z}/2\mathbb{Z})$ for each $q \geq 0$.

Thus, we conclude by an analogous argument to that used in the preceding two cases

Theorem 6.8. *The homology $H_*(F_n^{(sy)\,c};\mathbb{Z}/2\mathbb{Z})$ has for a $\mathbb{Z}/2\mathbb{Z}$-basis the $\mathbb{Z}/2\mathbb{Z}$ fundamental classes of the symmetric Schubert cycles $\left[\overline{S_{\mathbf{m}}^{(sy)}}\right]$ as we vary over the Schubert decomposition of*

$\mathcal{C}_n^{(sy)} \simeq SU_n/SO_n$ for all symmetric Schubert symbols $\mathbf{m}^{(sy)} = (m_1, \dots, m_k)$ with

$$1 < m_1 < \cdots < m_k \leq n.$$

Moreover, the Kronecker duals of the simple symmetric Schubert cycles $\overline{S_{(m_1)}^{(sy)}}$ are algebra generators for the exterior cohomology algebra with $\mathbb{Z}/2\mathbb{Z}$-coefficients.

This extends to $H_*(F_n^{(sy)}; \mathbb{Z}/2\mathbb{Z})$ with $\mathbb{Z}/2\mathbb{Z}$-basis given by the Borel-Moore mod 2*-cycles given by the global symmetric Schubert cycles $\left[Sol_m^T \cdot (S_{\mathbf{m}}^{(sy)})\right]$ for $S_{\mathbf{m}}^{(sy)}$ over the symmetric Schubert symbols $\mathbf{m}^{(sy)}$. The Poincaré duals of these classes form a $\mathbb{Z}/2\mathbb{Z}$-basis for the cohomology.*

$$H^*(F_m^{(sy)}; \mathbb{Z}/2\mathbb{Z}) \;\simeq\; \Lambda^*\mathbb{Z}/2\mathbb{Z}\langle e_2, e_3, \dots, e_n\rangle \,.$$

There are several points to be made regarding this result and that for skew-symmetric matrices.

First, unlike the cases of SU_n and SU_{2n}/Sp_n, the closure of the Schubert cells are not the images of Borel-Moore homology classes of singular manifolds. As mentioned earlier, if we consider instead the quotient space $F_m^{(sy)\,c}/(F_m^{(sy)\,c})^{(q-1)}$, and $|\mathbf{m}| = q$, then the composition of the map

$$\tilde{\psi}_{\mathbf{m}}^{(sy)} : \prod_{i=1}^{k}(C\mathbb{R}P^{m_i-1}) \;\longrightarrow\; SU_n/SO_n \simeq F_m^{(sy)\,c}$$

with the quotient map $pr_q : F_m^{(sy)\,c} \to F_m^{(sy)\,c}/(F_m^{(sy)\,c})^{(q-1)}$ factors through to give a map

$$pr_q \circ \tilde{\psi}_{\mathbf{m}}^{(sy)} : \prod_{i=1}^{k} S\mathbb{R}P^{m_i-1} \;\longrightarrow\; F_m^{(sy)\,c}/(F_m^{(sy)\,c})^{(q-1)} \,.$$

As

$$pr_q : (F_m^{(sy)\,c}, (F_m^{(sy)\,c})^{(q-1)}) \to (F_m^{(sy)\,c}/(F_m^{(sy)\,c})^{(q-1)}, *),$$

for $*$ the point representing $(F_m^{(sy)\,c})^{(q-1)}$ in the quotient, is a relative homeomorphism,

$$pr_{q\,*} : H_q(F_m^{(sy)\,c}, (F_m^{(sy)\,c})^{(q-1)}; \mathbb{Z}/2\mathbb{Z}) \;\simeq\; H_q(F_m^{(sy)\,c}/(F_m^{(sy)\,c})^{(q-1)}, *; \mathbb{Z}/2\mathbb{Z}) \,.$$

Then, the closure $\overline{S_{\mathbf{m}}^{(sy)}}$ corresponds via the isomorphism to the image of the fundamental class of $\prod_{i=1}^{k}(S\mathbb{R}P^{m_i-1})$ under $pr_{q\,*} \circ \tilde{\psi}_{\mathbf{m}\,*}^{(sy)}$.

Moreover, as noted earlier for the simple Schubert symbol (m_1), there is a factored map $\tilde{\psi}_{(m_1)}^{(sy)} : S\mathbb{R}P^{m_i-1} \to SU_n/SO_n \simeq F_m^{(sy)\,c}$ with image $\overline{S_{(m_1)}^{(sy)}}$, giving it a Borel-Moore fundamental homology class for $\mathbb{Z}/2\mathbb{Z}$-coefficients.

However, for cohomology with rational coefficients, see e.g. [MT, Chap. 3, Thm 6.7 (2)] or Table 1 in [D3], many of these Schubert cells do not contribute homology classes. This is similar to the situation for oriented Grassmannians for $\mathbb{Z}/2\mathbb{Z}$ versus rational coefficients. This relation extends further. Over SU_n/SO_n is a natural n-dimensional real oriented vector bundle $E_n = (SU_n \times_{SO_n} \mathbb{R}^n)$ where $\mathbb{R}^n$ has the natural representation of SO_n. This bundle can be viewed geometrically as the set of oriented real subspaces $V \subset \mathbb{C}^n$ with $\dim_{\mathbb{R}} V = n$ such that $\mathbb{C}\langle V\rangle = \mathbb{C}^n$. Then, by e.g. [MT, Chap. 3, Thm 6.7 (3)] the cohomology of SU_n/SO_n, already quoted in Theorem 6.8 has $e_j = w_j(E_n)$, the j-th Stiefel-Whitney class. This bundle pulls-back by the homotopy equivalence $SU_n/SO_n \simeq F_n^{(sy)\,c} \simeq F_n^{(sy)}$ to give an n-dimensional real oriented vector bundle, which we denote by $\tilde{E}_n$ and then

$$H^*(F_n^{(sy)}; \mathbb{Z}/2\mathbb{Z}) \;\simeq \Lambda^*\mathbb{Z}/2\mathbb{Z} < w_2, w_3, \dots, w_n >$$

where $w_j = w_j(\tilde{E}_n)$ for each $j = 2, 3, \dots, n$. We will see in the next section that this algebra naturally pulls back to a characteristic subalgebra of Milnor fibers for general symmetric matrix singularities generated by the Stiefel-Whitney classes of the pull-back of $\tilde{E}_n$ to the Milnor fiber.

Although both

$$H^*(F_n^{(sy)}; \mathbb{Z}/2\mathbb{Z}) \simeq H^*(SU_n/SO_n; \mathbb{Z}/2\mathbb{Z}) \quad \text{and} \quad H^*(F_{2n}^{(sk)}; \mathbb{Z}) \simeq H^*(SU_{2n}/Sp_n; \mathbb{Z})$$

are exterior algebras, neither is a Hopf algebra. Hence, the full argument given for $H^*(F_n; \mathbb{Z})$ for the relation between the cohomology and the Schubert decomposition cannot be given using Hopf algebra methods. However, it does suggest the following conjecture is true and constitutes work in progress.

Conjecture: For both $F_n^{(sk)\,c}$ and $F_n^{(sy)\,c}$, the Kronecker duals to the Schubert classes $S_{(\mathbf{m})}^{(sk)}$, resp. $S_{(\mathbf{m})}^{(sy)}$ for Schubert symbols $\mathbf{m}^{(sk)}$, or $\mathbf{m}^{(sy)} = (m_1, m_2, \dots, m_r)$ are given up to sign by $e_{(m_1)} \cdot e_{(m_2)} \cdots e_{(m_r)}$ in the corresponding cohomology algebra.

7. Characteristic Subalgebra in the Cohomology of General Matrix Singularities

In the preceding section we have identified for the Milnor fibers F_m, $F_m^{(sy)}$, and $F_m^{(sk)}$ (for m = 2n), their cohomology and the decomposition of their homology using the Schubert decomposition. We see how this applies to the structure of Milnor fibers of general matrix singularities of each of these types.

Let M denoting any one of the three spaces of complex $m \times m$ matrices which are general $M_{m,m}(\mathbb{C})$, symmetric $Sym_m(\mathbb{C})$, or skew-symmetric $Sk_m(\mathbb{C})$ with $m = 2n$. Also, let $\mathcal{D}_m$, resp. $\mathcal{D}_m^{(sy)}$, or $\mathcal{D}_m^{(sk)}$ denote the variety of singular matrices of the corresponding type. We suppose that each type is defined by $H : M \to \mathbb{C}$, which denotes either the determinant det for $\mathcal{D}_m$ or $\mathcal{D}_m^{(sy)}$, or the Pfaffian Pf for $\mathcal{D}_m^{(sk)}$.

Matrix Singularities of a Given Type.

A matrix singularity of any of the given types is defined by a holomorphic germ

$$f_0 : \mathbb{C}^s, 0 \to M, 0,$$

and the singularity is defined by $X_0 = f_0^{-1}(\mathcal{V}), 0$ where $\mathcal{V}$ denotes the appropriate variety of singular matrices. We impose an additional condition on f which can take several forms based on forms of $\mathcal{K}$-equivalence preserving $\mathcal{V}$. There is the equivalence defined using the parametrized action by points in $\mathbb{C}^s$ of the group $G = GL_m(\mathbb{C})$ acting by $C \mapsto A \cdot C \cdot A^T$ in the symmetric or skew-symmetric cases. For the general $m \times m$ matrix case, the action of $G = GL_m(\mathbb{C})$ acting by left multiplication suffices for studying the Milnor fiber. However, for the general equivalence studying the pull-back of $\mathcal{D}_m$ the action is given by $G = GL_m(\mathbb{C}) \times GL_m(\mathbb{C})$ acting by $C \mapsto A \cdot C \cdot B^{-1}$. We denote the equivalence for any of the general, symmetric, or skew-symmetric cases as $\mathcal{K}_M$-equivalence. The second equivalence allows the action of germs of diffeomorphisms of $\mathbb{C}^s \times M, (0, 0)$ of the form $\varphi(x, y) = (\varphi_1(x), \varphi_2(x, y))$ which preserve $\mathbb{C}^s \times \mathcal{V}$, and is denoted $\mathcal{K}_{\mathcal{V}}$ equivalence. The third is a subgroup of $\mathcal{K}_{\mathcal{V}}$ which preserves the defining equation of $\mathbb{C}^s \times \mathcal{V}$, $H \circ pr_M$, with pr_M denoting projection onto M. It is denoted $\mathcal{K}_H$. See for example [DP2], [D2], or [D1] for more details about the groups of equivalence and their relations and the properties of germs which have finite codimension for one of these equivalences. In particular, for the three classes of varieties of singular matrices, $\mathcal{K}_{\mathcal{V}}$ and $\mathcal{K}_M$ equivalences agree.

If f_0 has finite $\mathcal{K}_{\mathcal{V}}$-codimension, then it may be deformed to f_t which is transverse to $\mathcal{V}$ in a neighborhood $B_\varepsilon(0)$ of $0 \in \mathbb{C}^s$ for $t \neq 0$. Then it is shown in [DM] that one measure of the vanishing topology of X_0 is by the"singular Milnor fiber" $\tilde{X}_t = f_t^{-1}(\mathcal{V}) \cap B_\varepsilon(0)$. It is homotopy equivalent to a bouquet of real spheres of dimension $s-1$. If $s < \mathrm{codim}_M(\mathrm{sing}(\mathcal{V}))$, then this is the usual Milnor fiber of $\mathcal{V}_0$. This condition requires $s < 4$, resp. 3, resp. 6, for the three types of matrices.

In the special case that $\mathcal{V}$ is a free divisor and holonomic in the sense of Saito [Sa] and satisfies a local weighted homogeneity condition [DM] or is a free divisor and H-holonomic [D1], then the singular Milnor number is given by the length of the normal space $N\mathcal{K}_{H\,e}f_0$, which is a determinantal module.

For the three classes of varieties of singular matrices, the varieties are not free divisors. Nonetheless, when $s \leq \mathrm{codim}_M(\mathrm{sing}(\mathcal{V}))$, Goryunov and Mond [GM] give a formula for the Milnor number which adds a correction term for the lack of freeness given by an Euler characteristic of a Tor complex. Instead, Damon-Pike [DP3] give a formula valid for all s but which is presently restricted to a limited range of matrices. It is given by a sum of terms which are lengths of determinantal modules, based on placing the varieties in a tower of free divisors [DP2].

Cohomology Structure of Milnor Fibers of General Matrix Singularities.

We explain how the results in earlier sections provide information about the cohomology of the Milnor fiber for a matrix singularity X_0 for all s.

We consider the defining equation $H : \mathbb{C}^N, 0 \to \mathbb{C}, 0$ for $\mathcal{V}$, where $M \simeq \mathbb{C}^N$ for each case. For $\mathcal{V}$ there exists $0 < \delta << \eta$ such that for balls $B_\delta \subset \mathbb{C}$ and$B_\eta \subset \mathbb{C}^N$ (with all balls centered 0), we let $\mathcal{F}_\delta = H^{-1}(B_\delta) \cap B_\eta$ so $H : \mathcal{F}_\delta \to B_\delta$ is the Milnor fibration of H, with Milnor fiber $\mathcal{V}_w = H^{-1}(w) \cap B_\eta$ for each $w \in B_\delta$. By continuity, there is an $\varepsilon > 0$ so that $f_0(B_\varepsilon) \subset \mathcal{F}_\delta$. By possibly shrinking all three values, $H \circ f_0 : f_0^{-1}(\mathcal{F}_\delta) \cap B_\varepsilon \to B_\delta$ is the Milnor fibration of $H \circ f_0$. Also, by the parametrized transversality theorem, for almost all $w \in B_\delta$, f_0 is transverse to $\mathcal{V}_w$ and so the Milnor fiber of $H \circ f_0$ is given by

$$X_w \;=\; (H \circ f_0)^{-1}(w) \cap B_\varepsilon \;=\; f_0^{-1}(\mathcal{V}_w) \cap B_\varepsilon\,.$$

Thus, if we denote $f_0|X_w = f_{0,w}$, then in cohomology with coefficient ring R,

$$f_{0,w}^* : H^*(\mathcal{V}_w; R) \to H^*(X_w; R).$$

For any of the three types of matrices with $(*)$ denoting $()$ for general matrices, (sy) for symmetric matrices, or (sk) for skew-symmetric matrices, we let

$$\mathcal{A}^{(*)}(f_0; R) \;\stackrel{def}{=}\; f_{0,w}^*(H^*(\mathcal{V}_w; R))\,,$$

which we refer to as the *characteristic subalgebra* of the cohomology of the Milnor fiber $H^*(X_w; R)$ of X_0. This is an algebra over R, and the cohomology of the Milnor fiber of the matrix singularity X_0 is a graded module over $\mathcal{A}^{(*)}(f_0; R)$ (both with coefficients R).

By Theorems 6.1 and 6.7 for the $m \times m$ general case or skew-symmetric case (with $m = 2n$), for $R = \mathbb{Z}$-coefficients (and hence for any coefficient ring R) $\mathcal{A}^{(*)}(f_0; R)$ is the quotient ring of a free exterior R-algebra on generators e_{2j-1}, for $j = 2, 3, \ldots, m$, resp. e_{4j-3} for $j = 2, 3, \ldots, n$. For the $m \times m$ symmetric case there are two important cases where either $R = \mathbb{Z}/2\mathbb{Z}$ or is a field of characteristic zero. In the first case, by Theorem 6.8, $\mathcal{A}^{(*)}(f_0; \mathbb{Z}/2\mathbb{Z})$ is the quotient ring of a a free exterior algebra on generators $e_j = w_j(\tilde{E}_m)$, for $j = 2, 3, \ldots, m$, for $w_j(\tilde{E}_m)$ the Stiefel-Whitney classes of the real oriented m-dimensional vector bundle $\tilde{E}_m$ on the Milnor fiber of $\mathcal{D}_m^{(sy)}$. Hence, $\mathcal{A}^{(*)}(f_0; \mathbb{Z}/2\mathbb{Z})$ is a subalgebra generated by the Stiefel-Whitney classes of the pull-back vector bundle $f_{0,w}^*(\tilde{E}_m)$ on X_w.

For the coefficient ring $R = \mathbf{k}$ a field of characteristic zero, the symmetric case breaks-up into two cases depending on whether m is even or odd (see [MT, (2),Thm. 6.7, Chap. 3] or Table 1 of [D3]).

(7.1) $$H^*(F_m^{(sy)}; \mathbf{k}) \;\simeq\; \begin{cases} \Lambda^*\mathbf{k}\langle e_5, e_9, \dots, e_{2m-1}\rangle & \text{if } m = 2k+1 \\ \Lambda^*\mathbf{k}\langle e_5, e_9, \dots, e_{2m-3}\rangle\{1, e_m\} & \text{if } m = 2k \end{cases},$$

where e_m is the Euler class of $\tilde{E}_m$. Hence, in both cases they are graded modules over an exterior algebra. Hence, the Milnor fiber of X_0 has cohomology over a field of characteristic zero which, via the characteristic subalgebra is a graded module over the exterior algebra in either case of (7.1).

We summarize these cases with the following.

Theorem 7.1. *Let $f_0 : \mathbb{C}^s, 0 \to M, 0$ be a matrix singularity of finite $\mathcal{K}_M$-codimension for M the space of $m \times m$ matrices which are either general, symmetric, or skew-symmetric (with $m = 2n$). Let $\mathcal{V}$ denote the variety of singular matrices. Then,*

i) *The cohomology (with coefficients in a ring R) of the Milnor fiber of $X_0 = f_0^{-1}(\mathcal{V})$ has a graded module structure over the characteristic subalgebra $\mathcal{A}^{(*)}(f_0; R)$ of f_0.*
ii) *In the general and skew-symmetric cases, $\mathcal{A}^{(*)}(f_0; R)$ is a quotient of the free R-exterior algebra with generators given in Theorems 6.1 and 6.7 .*
iii) *In the symmetric case with $R = \mathbb{Z}/2\mathbb{Z}$, $\mathcal{A}^{(sy)}(f_0; \mathbb{Z}/2\mathbb{Z})$ is the quotient of the free exterior algebra over $\mathbb{Z}/2\mathbb{Z}$ on the Stiefel-Whitney classes of the real oriented vector bundle $\tilde{E}_m$ on the Milnor fiber of $\mathcal{V}$.*
iv) *In the symmetric case with $R = \mathbf{k}$, a field of characteristic zero, $\mathcal{A}^{(sy)}(f_0; \mathbf{k})$ is a quotient of the $\mathbf{k}$-algebras in each of the cases in (7.1).*

In light of this theorem there are several problems to be solved for determining the cohomology of the Milnor fiber of the matrix singularity X_0 for coefficients R.

Questions for the Cohomology of the Milnor Fibers of Matrix Singularities

1) Determine the characteristic subalgebras as the images of the exterior algebras by determining which monomials map to nonzero elements in $H^*(X_w; R)$.
2) Find the non-zero monomials in the image by geometrically identifying the pull-backs of the Schubert classes.
3) For the symmetric case with $\mathbb{Z}/2\mathbb{Z}$-coefficients, compute the Stiefel-Whitney classes of the pull-back of the vector bundle $\tilde{E}_m$.
4) Determine a set of module generators for the cohomology of the Milnor fibers as modules over the characteristic subalgebras.

Transversality to Schubert Cycles.

We can give a first step for these using transversality. We let M denote one of the spaces of $m \times m$ matrices with variety of singular matrices denoted by $\mathcal{V}$. There is a transitive action on $SL_m(\mathbb{C})$ on the global Milnor fibers of the varieties of singular matrices in all three cases. We let $S_{\mathbf{m}}^{(*)}$ denote the Schubert cell in the global Milnor fiber of the corresponding type. For each Schubert class $S_{\mathbf{m}}^{(*)}$ and $A \in SL_m(\mathbb{C})$, we let $A \cdot S_{\mathbf{m}}^{(*)}$ denote the image under the action of A. Also, we let the germ $f_1 = A^{-1} \cdot f_0$ denote the germ obtained by applying the constant matrix A^{-1} to $f_0(x)$ independent of x. This action preserves the global Milnor fibers of $\mathcal{V}$. Then, deforming either the Schubert cells or f_0 by multiplication by A yields the following.

Lemma 7.2. *Given $f_0 : \mathbb{C}^s, 0 \to M, 0$ of finite $\mathcal{K}_M$-codimension, for almost all $A \in SL_m(\mathbb{C})$ the germ f_0 is transverse to $A \cdot S_{\mathbf{m}}^{(*)}$ for all Schubert cells $S_{\mathbf{m}}^{(*)}$ in a Milnor fiber $\mathcal{V}_w$ of $\mathcal{V}$. Then, for $f_1 = A^{-1} \cdot f_0$ and $e'_{\mathbf{m}}$ the Poincaré dual to $[\overline{S^*_{\mathbf{m}}}]$, $f_1^*(e'_{\mathbf{m}})$ is the Poincaré dual of $[f_1^{-1}(\overline{S_{\mathbf{m}}^{(*)}})]$.*

Then, f_1 is $\mathcal{K}_M$-equivalent to f_0, and $f_{0w}^* = f_{1w}^*$.

Proof. As $SL_m(\mathbb{C})$ is path-connected, the action of A is homotopic to the identity. Let A_t be such a path from I_m to A. Hence, $[A_t \cdot \overline{S_{\mathbf{m}}^{(*)}}] = [\overline{S_{\mathbf{m}}^{(*)}}]$ for all t.

Next, by the parametrized transversality theorem and the transitive acton of $SL_m(\mathbb{C})$ on the global Mlnor fiber, it follows that f_0 is transverse to $A \cdot \overline{S_{\mathbf{m}}^{(*)}}$ for almost all $A \in SL_m(\mathbb{C})$. As there are only a finite number of Schubert cells, then for almost all A this simultaneously holds for all of the Schubert cells $S_{\mathbf{m}}^{(*)}$. For such an A with $f_1 = A^{-1} \cdot f_0$, it follows that $f_1 = A \cdot f_0$ is transverse to all of the Schubert cells. If $e'_{\mathbf{m}}$ denotes the Poincaré dual to $[\overline{S_{\mathbf{m}}^{(*)}}]$, it is also the Poincaré dual to $[A \cdot \overline{S_{\mathbf{m}}^{(*)}}]$. Thus, by a fiber square argument $f_{1w}^*(e'_{\mathbf{m}})$ is the Poincaré dual to $[f_{1w}^{-1}(A \cdot \overline{S_{\mathbf{m}}^{(*)}})]$.

Lastly, the family $f_t = A_t^{-1} \cdot f_0$ is a $\mathcal{K}_M$-constant family so that $f_1 = A^{-1} \cdot f_0$ is $\mathcal{K}_M$-equivalent to f_0 and $f_{0w}^* = f_{1w}^*$. ☐

Remark 7.3. As a simple consequence of this lemma, we may replace f_0 by the $\mathcal{K}_M$-equivalent $f_1 = A^{-1} \cdot f_0$ transverse to $\overline{S_{\mathbf{m}}^{(*)}}$. If $s < \frac{1}{2}\mathrm{codim}_{\mathbb{R}}(S_{\mathbf{m}}^{(*)}) + 1$, then $f_{1w}^{-1}(A \cdot \overline{S_{\mathbf{m}}^{(*)}})$ is empty. Hence $f_{0w}^*(e'_{\mathbf{m}}) = 0$.

Module Structure for the Milnor Fibers.

We make several remarks regarding these questions concerning the module structure. These involve two cases at opposite extremes, namely $s < \mathrm{codim}_M(\mathrm{sing}(X_0))$ or f_0 is the germ of a submersion. In the first case when $s < \mathrm{codim}_M(\mathrm{sing}(\mathcal{V}))$, X_0 has an isolated singularity, and the singular Milnor fiber for f_0 is the Milnor fiber for X_0, so the Milnor number and singular Milnor number agree. Also, $f_{0w}^*(e'_{\mathbf{m}}) = 0$ for all $e'_{\mathbf{m}}$ of positive degree; thus

$$\mathcal{A}^{(*)}(f_0, R) = H^0(X_w; R) \simeq R.$$

As the Minor fiber is homotopy equivalent to a CW-complex of real dimension $s-1$, the corresponding classes which occur for the Milnor fiber will have a trivial module structure over $\mathcal{A}^{(*)}(f_0, R)$.

Second, if f_0 is the germ of a submersion, then the Milnor fiber has the form $\mathcal{V}_w \times \mathbb{C}^k$, where $k = s - \dim_{\mathbb{C}} M$ and so has the same cohomology, so we conclude that

$$f_0^* : H^*(\mathcal{V}_w; R) \simeq H^*(X_w; R)$$

so $\mathcal{A}^{(*)}(f_0, R) = H^*(X_w; R)$. Also, there are no singular vanishing cycles. Thus, for these two cases there is the following expression for the cohomology of the Milnor fiber, where the second summand has trivial module structure shifted by degree $s-1$.

$$H^*(X_w; R) \simeq \mathcal{A}^{(*)}(f_0, R) \oplus R^{\mu}[s-1], \tag{7.2}$$

where $\mu = \mu_{\mathcal{V}}(f_0)$ for $\mathcal{V} = \mathcal{D}_m^{(*)}$ the corresponding variety of singular matrices.

We ask whether this holds in general or at least for a large class of matrix singularities.

Question: How generally valid is (7.2) *for matrix singularities of the three types?*

For this question, we note that for the case of 2×3 complex matrices with $\mathcal{V}$ denoting the variety of singular matrices and $s = 5$, the matrix singularities define Cohen-Macaulay 3-fold singularities. A stabilization of these singularities gives a smoothing and Milnor fiber. In [DP3, Thm. 8.4] is given an algebraic formula for the vanishing Euler characteristic, which becomes the difference of the Betti numbers $b_3 - b_2$ of the Milnor fiber. While specific calculations in the Appendix of [DP3] show that the vanishing Euler characteristic typically increases in families with the $\mathcal{K}_V$-codimension, it is initially not clear how this increase is distributed as changes of b_3 and b_2. Surprisingly, Frühbis-Krüger and Zach [FZ], [Z] show that for a large class of such singularities that $b_2 = 1$. This suggests it may be possible to identify certain classes of $m \times m$ matrix singularities for which there are contributions from $\mathcal{A}^{(*)}(f_0, R)$ for the topology of the Milnor fiber. This is a fundamental question whose answer along with the preceding ones will clarify our understanding of the full cohomology of the Milnor fibers of matrix singularities.

8. Extensions to Exceptional Orbit Varieties, Complements, and Links

We indicate in this section how the methods of the previous sections can be extended to exceptional orbit hypersurfaces for prehomogeneous vector spaces in the sense of Sato, see [So] and [SK]. This includes equidimensional prehomogeneous spaces, see [D3], in the cases of both block representations of solvable linear algebraic groups [DP2] and the discriminants for quivers of finite type in the sense of Gabriel, see [G], [G2], represented as linear free divisors by Buchweitz-Mond [BM].

Second, we can also apply the preceding methods to the complements of exceptional orbit hypersurfaces arising as the varieties of singular $m \times m$ matrices just considered and the equidimensional prehomogeneous spaces just described. Third, in [D3], the cohomology of the link of one of these singularities is computed as a shift of the (co)homology of the complement. Thus, the Schubert classes for the complement correspond to cohomology classes in the link. However, we explain how the multiplicative cohomology structure of the complement contains more information than the cohomology of the link.

Exceptional Orbit Hypersurfaces for the Equidimensional Cases.

Block Representations of Linear Solvable Algebraic Groups.

First, for the case of block representations of solvable linear algebraic groups, in [DP, Thm 3.1] the complement was shown to be a $K(\pi, 1)$-space where π is a finite extension of $\mathbb{Z}^n$ (for n the rank of the solvable group) by the finite isotropy group of the action on the open orbit. The solvable group is an extension of an algebraic torus by a unipotent group which is contractible. The resulting cell decomposition follows from that for the torus times the unipotent group. Thus, the decomposition is that modulo the finite group. In important cases of (modified) Cholesky-type factorization for the three types of matrices and also $m \times (m+1)$ matrices the finite group is either the identity or $(\mathbb{Z}/2\mathbb{Z})^n$ and the resulting quotient is shown, see [DP, Thm 3.4], to still be the extension of a torus by a (contractible) unipotent group.

Thus, for these cases the cell decomposition follows from the product decomposition for the complex torus times the unipotent group, which has as a compact model a compact torus of the same rank. Moreover, by [DP, Thm 4.1], the cohomology with complex coefficients is an exterior algebra which has as generators 1-forms defined from the defining equation of the exceptional orbit hypersurface.

Also, by [DP, Thm 3.2] the Milnor fiber is again a $K(\pi', 1)$-space with π' a subgroup of π (for the complement) with quotient $\mathbb{Z}$. Again, by [DP, Thm 3.4] for the cases of (modified) Cholesky-type factorization of matrices, it is also true that the Milnor fiber for these cases is the extension of a torus, except of one lower rank, by the unipotent group. Likewise the cohomology with complex coefficients of the Milnor fiber is again an exterior algebra which has one fewer generator, as the result of a quotient by a single specified relation.

Discriminants of Quivers of Finite Type.

The quivers are defined by a finite ordered graph Γ having for each vertex v_i a space $\mathbb{C}^{n_i}$ and for each directed edge from v_i to v_j a linear map $\varphi_{ij} : \mathbb{C}^{n_i} \to \mathbb{C}^{n_j}$. Those quivers of finite type were classified by Gabriel [G], [G2]. The discriminants for the quiver spaces of finite type were shown by Buchweitz-Mond [BM] to be linear free divisors. As such these discriminants are exceptional orbit hypersurfaces for the action of the group $G = (\prod_{i=1}^{k} GL_{n_i}(\mathbb{C}))/\mathbb{C}^*$, where $k =$

$|\Gamma|$. Since each $GL_{n_i}(\mathbb{C})$ topologically factors as $SL_{n_i}(\mathbb{C}) \times \mathbb{C}^*$, the complement is diffeomorphic to $(\prod_{i=1}^{k} SL_{n_i}(\mathbb{C})) \times (\mathbb{C}^*)^{k-1}$. The earlier results for the Schubert decomposition for each $SL_n(\mathbb{C})$ via its maximal compact subgroup SU_n and the product cell decomposition for $(\mathbb{C}^*)^{k-1}$ gives a *product Schubert cell decomposition* for the complement.

The Milnor fiber has an analogous form $(\prod_{i=1}^{k} SL_{n_i}(\mathbb{C})) \times (\mathbb{C}^*)^{k-2}$, and a *product Schubert cell decomposition* for the Milnor fiber.

The cohomology of the complement is given by [D3, (5.11)] as an exterior algebra on a specific set of generators. The cohomology of the Milnor fiber is also an exterior algebra except with one fewer degree 1 generator, see [D3, (Thm 5.4)]. Furthermore, by Theorem 6.1 relating the Schubert decomposition for $SL_n(\mathbb{C})$ via its maximal compact subgroup SU_n with the cohomology classes, we conclude that for both the complement and the Milnor fiber of the discriminant of the space of quivers, the closures of the product Schubert cells provide a set of generators for the homology.

Complements of the Varieties of Singular Matrices.

We can likewise give a Schubert decomposition for the complements of the varieties of $m \times m$ matrices which are general, symmetric or skew-symmetric. We note that in [D3] the complements were given as $GL_m(\mathbb{C})$ for the general matrices, $GL_m(\mathbb{C})/O_m(\mathbb{C})$ for the symmetric matrices, and $GL_{2n}(\mathbb{C})/Sp_n(\mathbb{C})$ for the skew-symmetric case with $m = 2n$. These have as compact models the symmetric spaces U_m, resp. U_m/O_m, resp. U_{2n}/Sp_n. Each of these has a Schubert decomposition given in [KM]. As remarked in §3, U_m has a Schubert decomposition by cells $S_{\mathbf{m}}$ for $\mathbf{m} = (m_1, m_2, \ldots, m_r)$, where m_1 may equal 1 and it is not required that $\sum_{i=1}^{r} \theta_i \equiv 0 \bmod 2\pi$.

Second, in [KM, §5] is given a Schubert decomposition for U_m/O_m using for the symmetric Schubert cell $S_{\mathbf{m}}^{(sy)}$ the symmetric factorization into pseudo-rotations except again

$$\mathbf{m}^{(sy)} = (m_1, m_2, \ldots, m_r),$$

where m_1 may equal 1 and it is not required that $\sum_{i=1}^{r} \theta_i \equiv 0 \bmod \pi$.

Third, in [KM, §7] is given a Schubert decomposition for U_{2n}/Sp_n using for the skew-symmetric Schubert cell $S_{\mathbf{m}}^{(sk)}$ the skew-symmetric factorization into pseudo-rotations except again $\mathbf{m}^{(sk)} = (m_1, m_2, \ldots, m_r)$, where m_1 may equal 1 and it is not required that

$$\sum_{i=1}^{r} \theta_i \equiv 0 \bmod 2\pi.$$

In the case of U_m and U_{2n}/Sp_n the cohomology with integer coefficients is an exterior algebra with an added generator of degree 1; and for U_m/O_m the cohomology with $\mathbb{Z}/2\mathbb{Z}$ coefficients is an exterior algebra with an added generator of degree 1. Hence, a counting argument analogous to that for the Milnor fibers show that the closure of each Schubert class gives a homology generator for the complement.

Complements of the Varieties of Singular $m \times n$ Matrices.

The varieties of singular $m \times n$ complex matrices, $\mathcal{V}_{m,n}$, with $m \neq n$ were not considered earlier because they do not have Milnor fibers. However, the methods do apply to the complement and link as a result of work of J. H. C. Whitehead [W]. Let $M = M_{m,n}(\mathbb{C})$ denote the space of $m \times n$ complex matrices. We consider the case where $m > n$. The other case $m < n$ is equivalent by taking transposes. The left action of $GL_m(\mathbb{C})$ acts on M with an open orbit consisting of the matrices of rank n. This is the complement to the variety $\mathcal{V}_{m,n}$ of singular matrices and can be described as the ordered set of n independent vectors in $\mathbb{C}^m$. Then, the Gram-Schmidt procedure replaces them by an orthonormal set of n vectors in $\mathbb{C}^m$. This is the Stiefel variety $V_n(\mathbb{C}^m)$ and the Gram-Schmidt procedure provides a strong deformation retract of the complement $M\backslash\mathcal{V}_{m,n}$ onto the Stiefel variety $V_n(\mathbb{C}^m)$. Thus, the Stiefel variety is a compact model for the complement. Whitehead [W] computes both the (co)homology of the Stiefel variety using a Schubert decomposition which he gives. The cohomology for integer coefficients of the complement of the variety $\mathcal{V}_{m,n}$ is given by:

$$H^*(M_{m,n}\backslash\mathcal{V}_{m,n};\mathbb{Z}) \;\simeq\; \Lambda^*\mathbb{Z}\langle e_{2(m-n)+1}, e_{2(m-n)+3}, \dots, e_{2m-1}\rangle \tag{8.1}$$

with degree of e_j equal to j. Again the Schubert decomposition gives for the closure of each Schubert cell a homology generator.

Cohomology of the Links and Schubert Decomposition of the Complement.

Consider an exceptional orbit variety $\mathcal{E}$ of a prehomogeneous vector space V of $\dim_{\mathbb{C}} V = N$. Suppose there is a compact manifold $K \subset V\backslash\mathcal{E}$ oriented for a coefficients field $\mathbf{k}$, which is a compact model for the complement $V\backslash\mathcal{E}$. Then the cohomology of the link $L(\mathcal{E})$ is given, see [D3, Prop. 1.9], by the following formula

Cohomology of the Link $L(\mathcal{E})$:

$$\widetilde{H}^*(L(\mathcal{E});\mathbf{k}) \;\simeq \widetilde{H^*(K;\mathbf{k})}\,[2N-2-\dim_{\mathbb{R}} K]\ , \tag{8.2}$$

where the graded vector space $\widetilde{H^*(X;\mathbf{k})}\,[r]$ will denote the vector space $H^*(X;\mathbf{k})$, truncated at the top degree and shifted upward by degree r. Furthermore, to a basis of vector space generators of $H_q(K;\mathbf{k})$, $q < \dim_{\mathbb{R}} K$, there corresponds by Alexander duality a basis of vector space generators of $H^{2N-2-q}(K;\mathbf{k})$.

As a consequence of this and the preceding established relations between the Schubert decomposition (or product Schubert decomposition) of the complement and the homology, we obtain the following conclusions.

Theorem 8.1. *For the following exceptional orbit varieties $\mathcal{E}$ there are the following relations between the Schubert (or product Schubert) decomposition for a compact model of the complement and the cohomology of the link obtained by shifting the cohomology of the compact model (for coefficients a field of characteristic zero $\mathbf{k}$ unless otherwise stated).*

1) *For the equidimensional solvable case for (modified) Cholesky-type factorizations of $m\times m$ matrices of all three types or $(m+1)\times m$ matrices, the cohomology of the link is given by the shifted cohomology of the compact model torus, see* [D3, Thm 4.5]. *The closures of the cells of the product cell decomposition of nonmaximal dimension give a homology basis which correspond to the cohomology basis of the link after the shift.*
2) *For the discriminant of the quiver space for a quiver of finite type, the cohomology of the link is the shifted cohomology of the compact model described above with shift given by* [D3, Thm. 5.4]. *The closures of cells of the product Schubert decomposition of nonmaximal dimension for the complement give a homology basis which correspond after the shift to the cohomology basis for the link.*
3) *For the varieties of singular $m\times m$ complex matrices, in the general case or the skew-symmetric case with m even, the cohomology of the link is the shifted cohomology of the compact symmetric spaces U_m, resp. U_{2n}/Sp_n ($m = 2n$) given above with shift given in* [D3, Table 2]. *The closures of the Schubert cells of nonmaximal dimension in each case give a homology basis which corresponds to the cohomology basis of the link after the shift.*
4) *For the varieties of singular $m\times m$ complex symmetric matrices, the shifted cohomology of $H^*(U_m/O_m;\mathbb{Z}/2\mathbb{Z})$, described above, gives the cohomology of the link for $\mathbb{Z}/2\mathbb{Z}$-coefficients, where the shift is $\binom{m+1}{2}-2$. The closures of the Schubert cells of nonmaximal*

dimension in the Schubert decomposition give a basis of $\mathbb{Z}/2\mathbb{Z}$-homology classes corresponding to the cohomology basis of the link after the shift. For coefficients in a field $\mathbf{k}$ *of characteristic zero, the cohomology of U_m/O_m, is an exterior algebra which depends on whether m is even or odd and the shifts are given in* [D3, Table 2]*, without a direct relation with the Schubert decomposition.*

5) *For the variety of singular $m \times n$ complex matrices, $\mathcal{V}_{m,n}$ (with $m > n$), the cohomology of the compact model, the Stiefel variety $V_n(\mathbb{C}^m)$, for the complement is given by* (8.1). *The cohomology of the link is given in* (8.2) *as the upper truncated and cohomology $H^*(M_{m,n}\backslash\mathcal{V}_{m,n}, \mathbf{k})$ shifted by $n^2 - 2$ (as a graded vector space). The closures of the Schubert cells of nonmaximal dimension give a homology basis for the cohomology of the link after the shift.*

Complements of the Varieties of Matrix Singularities.

Given a matrix singularity $f_0 : \mathbb{C}^s, 0 \to M, 0$ with $\mathcal{V} \subset M$ the variety of singular matrices and $X_0 = f_0^{-1}(\mathcal{V})$. Here M can denote any of the spaces of matrices and of any sizes. In the preceding, we indicated how the cohomology of the link $L(\mathcal{V})$ is expressed as an upper truncated and shifted cohomology of the complement $M\backslash\mathcal{V}$. Because of the shift, we showed in [D3] that the cohomology product structure is essentially trivial. Thus, the link is a stratified real analytic set whose structure depends upon much more than just the group structure of the (co)homology. On the other hand, we showed in [D3] that the cohomology structure of the complement is an exterior algebra, and hence contributes considerably more that just the vector space structure of the cohomology of the link. This extra cohomology structure captures part of the additional structure.

Consequently, for the matrix singularity, using the earlier notation, we note that there is a map of complements $f_0 : (B_\varepsilon\backslash X_0) \to (B_\delta\backslash\mathcal{V})$. Also, $B_\delta\backslash\mathcal{V} \simeq M\backslash\mathcal{V}$, which has a compact model given by either a symmetric space or a Stiefel manifold. Thus, the cohomology of the complement $H^*(B_\varepsilon\backslash X_0; R)$ is a module over the characteristic subalgebra which is the image of $H^*(B_\delta\backslash\mathcal{V}; R)$ under f_0^*. In turn, this is an exterior algebra. Hence, the multiplicative structure considerably adds to the group structure that would result from the link. This is just as for the Milnor fiber described earlier.

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A MCKAY CORRESPONDENCE FOR THE POINCARÉ SERIES OF SOME FINITE SUBGROUPS OF $\mathrm{SL}_3(\mathbb{C})$

WOLFGANG EBELING

Dedicated to the memory of Egbert Brieskorn with great admiration

Abstract. A finite subgroup of $\mathrm{SL}_2(\mathbb{C})$ defines a (Kleinian) rational surface singularity. The McKay correspondence yields a relation between the Poincaré series of the algebra of invariants of such a group and the characteristic polynomials of certain Coxeter elements determined by the corresponding singularity. Here we consider some non-abelian finite subgroups G of $\mathrm{SL}_3(\mathbb{C})$. They define non-isolated three-dimensional Gorenstein quotient singularities. We consider suitable hyperplane sections of such singularities which are Kleinian or Fuchsian surface singularities. We show that we obtain a similar relation between the group G and the corresponding surface singularity.

Introduction

In [E4] we showed that the Poincaré series of the coordinate algebra of a two-dimensional quasihomogeneous singularity is the quotient of two polynomials one of which is related to the characteristic polynomial of the monodromy of the singularity. There are two special cases of this result. One is the case of a Kleinian singularity not of type A_{2n}. The Kleinian singularities are defined by quotients of $\mathbb{C}^2$ by finite subgroups of $\mathrm{SL}_2(\mathbb{C})$. In this case, the relation means that the Poincaré series is the quotient of the characteristic polynomials of the Coxeter element and the affine Coxeter element of the corresponding root system of type ADE. We derived this relation from the McKay correspondence. The other case is the case of a Fuchsian singularity. A Fuchsian singularity is defined by the action of a Fuchsian group (of the first kind) $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ on the tangent bundle $T_{\mathbb{H}}$ of the upper half plane $\mathbb{H}$. For a Fuchsian hypersurface singularity (or more generally for a Fuchsian singularity of genus 0 [EP]), we showed that the Poincaré series is the quotient of two characteristic polynomials of Coxeter elements [E5].

Here we consider a similar relation for the Poincaré series of some non-abelian finite subgroups of $\mathrm{SL}_3(\mathbb{C})$. The non-abelian finite subgroups of $\mathrm{SL}_3(\mathbb{C})$ define non-isolated three-dimensional Gorenstein quotient singularities. We consider those groups where the natural three-dimensional representation is irreducible and the corresponding quotient singularity has a certain hyperplane section which is a Kleinian or Fuchsian singularity. We show, that in this way, we again obtain relations between the Poincaré series of the algebra of invariants of the group and the characteristic polynomials of certain Coxeter elements determined by the corresponding Kleinian or Fuchsian singularity.

The famous paper [Br] of E. Brieskorn is fundamental for the study of Kleinian singularities. The Kleinian singularities were a central theme in Brieskorn's research and we owe Brieskorn many beautiful and important results about these singularities. Therefore I would like to express my great admiration for him in dedicating this paper to his memory.

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G	$\lvert G\rvert$	x, y, z	c_G	$R(x, y, z)$	Sing.	$\alpha_1, \ldots, \alpha_m$
$\mathcal{C}_{2n+1}$	$2n+1$	$2, 2n+1, 2n+1$	1	$x^{2n+1} + y^2 + z^2$	A_{2n}	$2n$
$\mathcal{C}_{2n}$	$2n$	$2, 2n, 2n$	2	$x^{2n} + y^2 + z^2$	A_{2n-1}	$2n-1$
$\mathcal{D}_n$	$4n$	$4, 2n, 2n+2$	2	$x^{n+1} + xy^2 + z^2$	D_{n+2}	$2, 2, n$
$\mathcal{T}$	24	$6, 8, 12$	2	$x^4 + y^3 + z^2$	E_6	$2, 3, 3$
$\mathcal{O}$	48	$8, 12, 18$	2	$x^3y + y^3 + z^2$	E_7	$2, 3, 4$
$\mathcal{I}$	120	$12, 20, 30$	2	$x^5 + y^3 + z^2$	E_8	$2, 3, 5$

TABLE 1. Subgroups of $\mathrm{SL}_2(\mathbb{C})$ and surface singularities

1. Finite subgroups of $\mathrm{SL}_2(\mathbb{C})$ and $\mathrm{SL}_3(\mathbb{C})$ and normal surface singularities

Let G be a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$. The classification of finite subgroups of $\mathrm{SL}_2(\mathbb{C})$ up to linear equivalence is well-known, see e.g. [Kl]. There are up to conjugacy five classes of such groups: the cyclic groups $\mathcal{C}_\ell$, the binary dihedral groups $\mathcal{D}_n$, the binary tetrahedral group $\mathcal{T}$, the binary octahedral group $\mathcal{O}$, and the binary icosahedral group $\mathcal{I}$. The quotients of $\mathbb{C}^2$ by these groups were studied by E. Brieskorn [Br]. Equations for these singularities can be obtained from generators and relations of the algebra of invariant polynomials with respect to G. This algebra has three generators x, y, z in each case which satisfy an equation $R(x, y, z) = 0$. The degrees of the generators and the equation $R(x, y, z) = 0$ are indicated in Table 1. (They can be found, e.g., in [Sp].) The equations define isolated hypersurface singularities in $\mathbb{C}^3$, the so called *Kleinian singularities*.

The finite subgroups of $\mathrm{SL}_3(\mathbb{C})$ were classified up to linear equivalence by H. F. Blichfeldt, G. A. Miller, and L. E. Dickson [Bl, MBD] with two missing cases (see [YY]). There are 12 types of finite subgroups of $\mathrm{SL}_3(\mathbb{C})$: (A)–(L). There are four infinite series (A)–(D). The groups of type (A) are the diagonal abelian groups and the groups of type (B) are isomorphic to transitive finite subgroups of $\mathrm{GL}_2(\mathbb{C})$. Here the natural 3-dimensional representation is not irreducible. Type (C) is the infinite series $\Delta(3n^2)$ of groups and type (D) the series $\Delta(6n^2)$ (for the notation see [HH, LNR, EL]). Moreover, we have 8 exceptional subgroups (E)–(L).

We consider those subgroups of type (C)–(L) which admit a certain hyperplane section which defines a Kleinian or Fuchsian singularity. Generators and relations for the algebra of invariant polynomials with respect to G have been computed in [YY] (see also [We] for some cases). They correspond to non-isolated Gorenstein quotient singularities $\mathbb{C}^3/G$. These singularities are either hypersurface singularities in $\mathbb{C}^4$ or complete intersection singularities in $\mathbb{C}^5$. We denote the coordinates of these spaces by w, x, y, z and w, x, y, z, u respectively. We consider hyperplane sections of these singularities, namely we consider the restrictions of the equations to the hyperplane $w = 0$. For the series (C) and (D) the hyperplane sections of the corresponding singularities for the first few elements of these series are listed in Table 2. It turns out that the singularities corresponding to the series (C) ($\Delta(3n^2)$) belong to Arnold's E-series whereas those of type (D) ($\Delta(6n^2)$) belong to Arnold's Z-series (n even) or are complete intersection singularities (n odd) (for the definition of these series see [Arn]). The subgroups which correspond to Kleinian singularities are the tetrahedral group $T = \Delta(3 \cdot 2^2)$ and the octahedral group $O = \Delta(6 \cdot 2^2)$ which correspond to the Kleinian singularities E_6 and E_7 respectively. Those which correspond to Fuchsian singularities are $\Delta(3 \cdot 4^2)$ (E_{14}), $\Delta(6 \cdot 4^2)$ (Z_{11}), $\Delta(6 \cdot 6^2)$ ($Z_{1,0}$), and $\Delta(6 \cdot 3^2)$ which corresponds to the elliptic complete intersection singularity $\delta 1$ in C. T. C. Wall's notation [Wa2]. (For a list of Fuchsian hypersurface and complete intersection singularities see [E5].) These are 6 cases. The remaining 8 exceptional subgroups of types (E)–(L) all correspond to Fuchsian singularities except in the case (H) which is the icosahedral group I corresponding

G	$\lvert G\rvert$	$w,x,y,z(,u)$	c_G	$R(0,x,y,z(,u))$	Sing.
$\Delta(3\cdot 2^2)$	12	$2,3,4,6$	1	$z^2+4y^3+27x^4$	E_6
$\Delta(3\cdot 3^2)$	27	$3,3,6,9$	3	$z^2+4y^3+27x^6$	$\widetilde{E}_8$
$\Delta(3\cdot 4^2)$	48	$4,3,8,12$	1	$z^2+4y^3+27x^8$	E_{14}
$\Delta(3\cdot 5^2)$	75	$5,3,10,15;30$	1	$z^2+4y^3+27x^{10}$	E_{18}
$\Delta(6\cdot 2^2)$	24	$2,4,6,9$	1	$z^2+4xy^3+27x^3$	E_7
$\Delta(6\cdot 3^2)$	54	$6,6,6,6,9$	3	$\left\{\begin{array}{c} z^2-xy \\ u^2+4xyz+27x^3 \end{array}\right\}$	$\delta 1$
$\Delta(6\cdot 4^2)$	96	$4,6,8,15$	1	$z^2+4xy^3+27x^5$	Z_{11}
$\Delta(6\cdot 5^2)$	150	$10,6,8,10,15$	1	$\left\{\begin{array}{c} z^2-xy \\ u^2+4x^2yz+27x^5 \end{array}\right\}$	no name
$\Delta(6\cdot 6^2)$	216	$6,6,12,21$	3	$z^2+4xy^3+27x^7$	$Z_{1,0}$
$\Delta(6\cdot 7^2)$	294	$14,6,10,14,21$	1	$\left\{\begin{array}{c} z^2-xy \\ u^2+4x^3yz+27x^7 \end{array}\right\}$	no name
$\Delta(6\cdot 8^2)$	384	$8,6,16,27;54$	1	$z^2+4xy^3+27x^9$	Z_{19}

TABLE 2. The first subgroups of types (C) and (D) and surface singularities

G	$\lvert G\rvert$	$w,x,y,z(,u)$	c_G	$R(0,x,y,z(,u))$	Sing.
(C): T	12	$2,3,4,6$	1	$z^2+4y^3+27x^4$	E_6
$\Delta(3\cdot 4^2)$	48	$4,3,8,12$	1	$z^2+4y^3+27x^8$	E_{14}
(D): O	24	$2,4,6,9$	1	$z^2+4xy^3+27x^3$	E_7
$\Delta(6\cdot 3^2)$	54	$6,6,6,6,9$	3	$\left\{\begin{array}{c} z^2-xy \\ u^2+4xyz+27x^3 \end{array}\right\}$	$\delta 1$
$\Delta(6\cdot 4^2)$	96	$4,6,8,15$	1	$z^2+4xy^3+27x^5$	Z_{11}
$\Delta(6\cdot 6^2)$	216	$6,6,12,21$	3	$z^2+4xy^3+27x^7$	$Z_{1,0}$
(E)	108	$6,6,9,12,12$	3	$\left\{\begin{array}{c} 9u^2-12z^2 \\ 432y^2-x^3-36xz \end{array}\right\}$	$K'_{1,0}$
(F)	216	$6,9,12,12$	3	$4z^3-144yz^2$ $+1728y^2z-186624x^4$	U_{12}
(G)	648	$9,12,18,18$	6	$4z^3-9yz^2+6y^2z-y^3+6912x^3y$	$U_{1,0}$
(H)$=I$	60	$2,6,10,15$	1	$z^2-y^3+1728x^5$	E_8
(I)	168	$4,6,14,21$	1	$z^2-y^3-1728x^7$	E_{12}
(J)	180	$6,6,12,15$	3	$y^3-xz^2+64x^2y^2$	$Q_{2,0}$
(K)	504	$6,12,18,21$	3	$y^3-xz^2-256x^3y$	Q_{11}
(L)	1080	$6,12,30,45$	3	$459165024z^2-25509168y^3$ $-(7558272-2519424\sqrt{15}i)x^5y$	E_{13}

TABLE 3. Subgroups of $\mathrm{SL}_3(\mathbb{C})$ and surface singularities

to the Kleinian singularity E_8. Altogether we have 14 cases which we will consider in this paper. They are listed in Table 3. These singularities are surface singularities and they are isolated except in the three cases $\Delta(6\cdot 3^2)$, (E) and (J). They correspond to Kleinian singularities in the cases T, O and (H) (the icosahedral group I) and to Fuchsian singularities in the other cases. They correspond to simple (T, O, I), unimodal ($\Delta(3\cdot 4^2)$, $\Delta(6\cdot 4^2)$, (F), (I), (K), (L)) and bimodal ($\Delta(6\cdot 6^2)$, (G), (J)) hypersurface singularities, to the unimodal complete intersection singularity of type $K'_{1,0}$ (type (E)) in Wall's notation [Wa1], and to the elliptic complete intersection singularity $\delta 1$. The names of the hypersurface singularities according to V. I. Arnold's classification [Arn] are indicated in the last column of Table 3.

2. Poincaré series and Coxeter elements

We now consider the isolated singularities corresponding to these singularities. They are quasihomogeneous. This means the following. A complex polynomial $f(x_1,\ldots,x_n)$ is called quasihomogeneous, if there are positive integers $w_1,\ldots,w_n$ (called *weights*) and d (called *degree*) such that $f(\lambda^{w_1}x_1,\ldots,\lambda^{w_n}x_n)=\lambda^d f(x_1,\ldots,x_n)$ for $\lambda\in\mathbb{C}^*$. A complete intersection singularity given as the zero set of polynomials $f_1(x_1,\ldots,x_n),\ldots,f_k(x_1,\ldots,x_n)$ is called quasihomogeneous, if $f_1,\ldots,f_k$ are quasihomogeneous with respect to the same weights $w_1,\ldots,w_n$ but degrees $d_1,\ldots,d_k$ respectively. We call the system $W:=(w_1,\ldots,w_n;d_1,\ldots,d_k)$ the *weight system* corresponding to the set of polynomials. Let c_W be the greatest common divisor of $w_1,\ldots,w_n,d_1,\ldots,d_k$. The weight system is called *reduced* if $c_W=1$.

We assume that $f_1(0)=\cdots=f_k(0)=0$ and the system of equations $f_1=\cdots=f_k=0$ has an isolated singularity at the origin. The coordinate algebra $A_f:=\mathbb{C}[x_1,\ldots,x_n]/(f_1,\ldots,f_k)$ is a $\mathbb{Z}$-graded algebra with respect to the system of weights $(w_1,\ldots,w_n;d_1,\ldots,d_k)$. Therefore we can consider the decomposition of A_f as a $\mathbb{Z}$-graded $\mathbb{C}$-vector space:

$$A_f:=\bigoplus_{k=0}^{\infty}A_{f,k},\quad A_{f,k}:=\left\{g\in A_f \;\middle|\; g(\lambda^{w_1}x_1,\ldots,\lambda^{w_n}x_n)=\lambda^k g(x_1,\ldots,x_n)\right\}.$$

The formal power series $p_f(t):=\sum_{k=0}^{\infty}(\dim_{\mathbb{C}}A_{f.k})t^k$ is called the *Poincaré series* of A_f. It is given by

$$p_f(t)=\frac{\prod_{i=1}^{k}(1-t^{d_i})}{\prod_{j=1}^{n}(1-t^{w_j})}.$$

Let (X,x) be a Kleinian singularity. Then the minimal resolution of the singularity x has an exceptional divisor with the dual graph depicted in Fig. 1 with $m=1$ in the case of the A_n-singularities and $m=3$ in the other cases (see, e.g., [Br]). Here all vertices correspond to rational curves of self-intersection number -2, the mutual intersection numbers are either 0 or 1, and two vertices are joined by an edge if and only if the intersection number of the corresponding rational curves is equal to 1. The values of the numbers $\alpha_1,\ldots,\alpha_m$ are indicated in Table 1. They are the *Dolgachev numbers* of the singularity, see [ET]. It turns out that these graphs are precisely the ordinary Coxeter-Dynkin diagrams of type ADE. (Note that the corresponding intersection matrix is the Cartan matrix multiplied by -1.)

Now let (X,x) be a Fuchsian hypersurface or complete intersection singularity. A natural compactification of X is given by $\overline{X}:=\mathrm{Proj}(A_f[t])$, where t has degree 1 for the grading of $A_f[t]$ (see [P]). This is a normal projective surface with a $\mathbb{C}^*$-action. The surface $\overline{X}$ may acquire additional singularities on the boundary $\overline{X}_\infty:=\overline{X}\setminus X=\mathrm{Proj}(A_f)$. Let $g=g(\overline{X}_\infty)$ be the genus of the boundary. We assume $g=0$. Let $\pi:S\to\overline{X}$ be the minimal normal crossing resolution of all singularities of $\overline{X}$. The preimage $\widetilde{X}_\infty$ of $\overline{X}_\infty$ under $\pi\colon S\to\overline{X}$ consists of the strict transform δ_0 of $\overline{X}_\infty$ and m chains $\delta_1^i,\ldots,\delta_{\alpha_i-1}^i$, $i=1,\ldots,m$, of rational curves of self-intersection -2 which intersect again according to the dual graph shown in Figure 1 (see, e.g., [D, E5]). By the adjunction formula and $g=0$, the self-intersection number of the rational curve δ_0 is also -2. The numbers $\alpha_1,\ldots,\alpha_m$ of the Fuchsian singularities corresponding to finite subgroups of $\mathrm{SL}_3(\mathbb{C})$ are indicated in Table 4. They are again the *Dolgachev numbers* of the singularity, see [ET, E3].

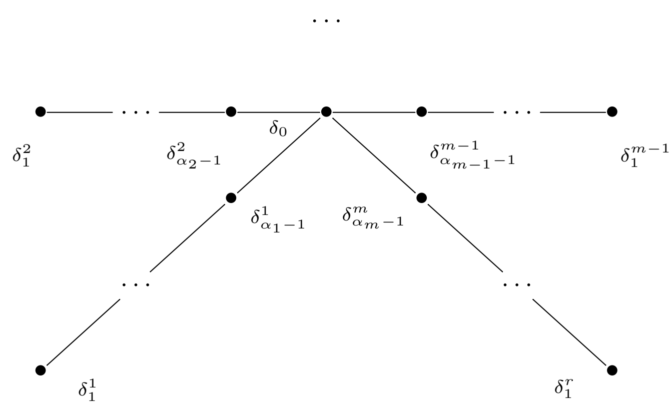

FIGURE 1. The graph $T^-_{\alpha_1,\ldots,\alpha_m}$

G	Name	Normal form	Weights	$\alpha_1,\ldots,\alpha_m$
(C): T	E_6	$z^2+y^3+x^4$	3,4,6;12	$2,3,3$
$\Delta(3\cdot 4^2)$	E_{14}	$z^2+y^3+x^8$	3,8,12;24	$3,3,4$
(D): O	E_7	$z^2+y^3+yx^3$	4,6,9;18	$2,3,4$
$\Delta(6\cdot 3^2)$	$\delta 1$	$\left\{\begin{matrix} xy+z^2 \\ x^3+y^3+z^3+w^2 \end{matrix}\right\}$	2,2,2,3;4,6	$2,2,2,2,2,2$
$\Delta(6\cdot 4^2)$	Z_{11}	$z^2+xy^3+x^5$	6,8,15;30	$2,3,8$
$\Delta(6\cdot 6^2)$	$Z_{1,0}$	$z^2+xy^3+x^7$	2,4,7;14	$2,2,2,4$
(E)	$K'_{1,0}$	$\left\{\begin{matrix} xu+y^2 \\ ax^4+xy^2+z^2+u^2, \\ a\neq 0,\frac{1}{4} \end{matrix}\right\}$	2,3,4,4;6,8	$2,2,4,4$
(F)	U_{12}	$z^3+y^3+x^4$	3,4,4;12	$4,4,4$
(G)	$U_{1,0}$	$z^3+yz^2+x^3y$	2,3,3;9	$2,3,3,3$
(H)=I	E_8	$z^2+y^3+x^5$	6,10,15;30	$2,3,5$
(I)	E_{12}	$z^2+y^3+x^7$	6,14,21;42	$2,3,7$
(J)	$Q_{2,0}$	$xz^2+y^3+x^4y$	2,4,5;12	$2,2,2,5$
(K)	Q_{11}	$xz^2+y^3+yx^3$	4,6,7;18	$2,4,7$
(L)	E_{13}	$z^2+y^3+x^5y$	4,10,15;30	$2,4,5$

TABLE 4. Normal forms, reduced weight systems, and Dolgachev numbers

We call the graph $T^-_{\alpha_1,\ldots,\alpha_m}$ a *Coxeter-Dynkin diagram*. Let V_- be the free $\mathbb{Z}$-module with the basis

$$\delta_1^1,\ldots,\delta_{\alpha_1-1}^1,\delta_1^2,\ldots,\delta_{\alpha_2-1}^2,\ldots,\delta_1^m,\ldots,\delta_{\alpha_m-1}^m,\delta_0$$

equipped with the symmetric bilinear form $\langle -,-\rangle$ given by the intersection matrix corresponding to Fig. 1. This defines a lattice $(V_-,\langle -,-\rangle)$.

We consider two extensions of this lattice. Let $V_0 = V_-\oplus\mathbb{Z}\delta_1$ with the symmetric bilinear form defined by Fig. 2. Here the double dashed line between δ_0 and δ_1 means $\langle\delta_0,\delta_1\rangle = -2$. Let $V_+ = V_0\oplus\mathbb{Z}\delta_2$ with the symmetric bilinear form defined by Fig. 3.

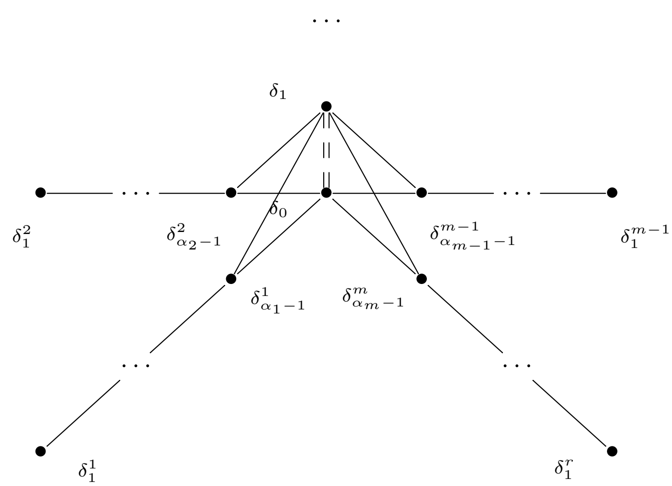

FIGURE 2. The graph $T_{\alpha_1,\ldots,\alpha_m}$

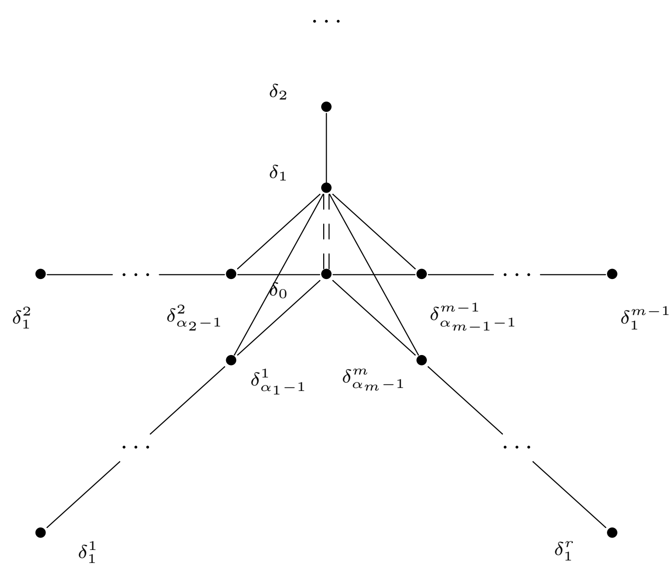

FIGURE 3. The graph $T^+_{\alpha_1,\ldots,\alpha_m}$

If $(V, \langle -, - \rangle)$ is an arbitrary lattice and $e \in V$ is a *root*, i.e. $\langle e, e \rangle = -2$, then the reflection corresponding to e is defined by

$$s_e(x) = x - \frac{2\langle x, e \rangle}{\langle e, e \rangle} e = x + \langle x, e \rangle e \quad \text{for } x \in V.$$

If $B = (e_1, \ldots, e_n)$ is an ordered basis consisting of roots, then the *Coxeter element* τ corresponding to B is defined by

$$\tau = s_{e_1} s_{e_2} \cdots s_{e_n}.$$

For a Coxeter element τ, let $\Delta(t) = \det(1 - \tau^{-1}t)$ be its characteristic polynomial, using a suitable normalization.

If D is a Coxeter-Dynkin diagram, then we denote by $\Delta_D(t)$ the characteristic polynomial of the Coxeter element corresponding to the graph D. These polynomials can be computed as in [E1] and one gets

$$\begin{aligned}
\Delta_{T^-_{\alpha_1,\ldots,\alpha_m}}(t) &= (1+t)\prod_{i=1}^m \frac{1-t^{\alpha_i}}{1-t} - t\sum_{i=1}^m \frac{1-t^{\alpha_i-1}}{1-t}\prod_{\substack{j=1\\ j\neq i}}^m \frac{1-t^{\alpha_j}}{1-t},\\
\Delta_{T_{\alpha_1,\ldots,\alpha_m}}(t) &= (1-t)^{2-m}(1-t^{\alpha_1})\cdots(1-t^{\alpha_m}),\\
\Delta_{T^+_{\alpha_1,\ldots,\alpha_m}}(t) &= (1-2t-2t^2+t^3)\prod_{i=1}^m \frac{1-t^{\alpha_i}}{1-t} + t^2\sum_{i=1}^m \frac{1-t^{\alpha_i-1}}{1-t}\prod_{\substack{j=1\\ j\neq i}}^m \frac{1-t^{\alpha_j}}{1-t}.
\end{aligned}$$

(The last two formulas can also be found in [E2, p. 98], but note that, unfortunately, there is a misprint in [E2, p. 98].)

Now we can state the main result of [EP].

Theorem 1. (i) *For a Kleinian singularity not of type A_{2n} we have*

$$p_f(t) = \frac{\Delta_{T^-_{\alpha_1,\ldots,\alpha_m}}(t)}{\Delta_{T_{\alpha_1,\ldots,\alpha_m}}(t)}.$$

(ii) *For a Fuchsian singularity with $g = 0$ we have*

$$p_f(t) = \frac{\Delta_{T^+_{\alpha_1,\ldots,\alpha_m}}(t)}{\Delta_{T_{\alpha_1,\ldots,\alpha_m}}(t)}.$$

Remark 2. Unfortunately, the exclusion of the case A_{2n} is only implicit in [EP] and was forgotten in the statement of [EP, Theorem 1].

Remark 3. Note that we have $T_{2,3,3} \sim T^-_{3,3,3}$, $T_{2,3,4} \sim T^-_{2,4,4}$, $T_{2,3,5} \sim T^-_{2,3,6}$, where $\sim$ means equivalence under the braid group action, see [E6]. Similarly, one can show that the graphs T_{2n-1}, $n \geq 1$, and $T_{2,2,n}$, $n \geq 2$, are equivalent under the braid group action to the extended Coxeter-Dynkin diagrams of type A_{2n-1} and D_{n+2} respectively.

3. Poincaré series of subgroups of $\mathrm{SL}_2(\mathbb{C})$ and $\mathrm{SL}_3(\mathbb{C})$

Let G be a finite subgroup of $\mathrm{SL}_n(\mathbb{C})$ for $n = 2, 3$. Consider the algebra of complex polynomials $\mathbb{C}[x_1, \ldots, x_n]$ graded by the degree for homogeneous ones. It is isomorphic to the symmetric algebra

$$S := S(\mathbb{C}^n) = \bigoplus_{k=0}^\infty S^k(\mathbb{C}^n),$$

where $S^k(\mathbb{C}^n)$ denotes the k-th symmetric power of $\mathbb{C}^n$. Let S^G be the algebra of invariant polynomials with respect to G.

For $n = 2$, it is generated by 3 elements x, y, z which satisfy a relation $R(x, y, z) = 0$. The elements x, y, z correspond to invariant polynomials and their degrees correspond to the weights of these variables. Let c_G denote the greatest common divisor of these weights. The weights of the variables x, y, z, the number c_G, and the polynomial $R(x, y, z)$ are indicated in Table 1.

Now let G be one of the finite subgroups of $\mathrm{SL}_3(\mathbb{C})$ of Table 3. Except in the cases (E) and $\Delta(6 \cdot 3^2)$, the algebra S^G is generated by 4 elements w, x, y, z which satisfy a relation $R(w, x, y, z)$. In the cases (E) and $\Delta(6 \cdot 3^2)$, S^G is generated by 5 elements w, x, y, z, u which satisfy two relations $R_1(w,x,y,z,u)=0$ and $R_2(w,x,y,z,u)=0$. The degrees of the invariants and the polynomials $R(w,x,y,z)$ and $R_1(w,x,y,z,u)$, $R_2(w,x,y,z,u)$ respectively can be found in [YY]. The degrees of the invariant polynomials and the restriction to the hyperplane $w=0$ of the polynomials $R(w,x,y,z)$ and $R_1(w,x,y,z,u)$, $R_2(w,x,y,z,u)$ respectively are indicated in Table 3. Let c_G be the greatest common divisor of the weights of the remaining variables $x,y,z(,u)$ (with the weight of w excluded). The number c_G is also indicated in Table 3. Note that, except in the case (G), c_G also divides the weight of w.

For $n=2$, the algebra $A_G := S^G = \mathbb{C}[x,y,z]/R(x,y,z)$ coincides with the coordinate algebra A_f of the corresponding singularity indicated in the last column of Table 1 up to the grading. The grading is shifted by c_G. For $n=3$ and G one of the cases of Table 3 except the cases (E) and $\Delta(6{\cdot}3^2)$, the algebra $A_G := \mathbb{C}[x,y,z]/R(0,x,y,z)$ coincides with the coordinate algebra A_f of the corresponding singularity indicated in the last column of Table 3 with the grading shifted by c_G. In the cases (E) and $\Delta(6\cdot 3^2)$, the algebra $A_G := \mathbb{C}[x,y,z,u]/(R_1(0,x,y,z,u), R_2(0,x,y,z,u))$ coincides with the coordinate algebra A_f of the complete intersection singularity $K'_{1,0}$ and $\delta 1$ respectively, again with the grading shifted by c_G. Let $p_G(t)$ be the Poincaré series of the algebra of A_G. Then we have

$$p_G(t) = p_f(t^{c_G}) \text{ for } G \subset \mathrm{SL}_2(\mathbb{C}), \quad p_G(t) = \frac{p_f(t^{c_G})}{(1-t^{\deg w})} \text{ for } G \subset \mathrm{SL}_3(\mathbb{C}).$$

The finite subgroups $G \subset \mathrm{SL}_n(\mathbb{C})$ for $n=2,3$ under consideration have a natural n-dimensional representation γ which is irreducible (except in the cases $G=\mathcal{C}_l$). Let γ^* be its contragredient representation. Let $\gamma_0,\ldots,\gamma_l$ be the irreducible representations of G, where γ_0 is the trivial representation. Let $B=(b_{ij})$ and $B^*=(b^*_{ij})$ be the $(l+1)\times(l+1)$-matrices defined by decomposing the tensor products

$$\gamma_j \otimes \gamma = \bigoplus_i b_{ij}\gamma_i \quad \text{and} \quad \gamma_j \otimes \gamma^* = \bigoplus_i b^*_{ij}\gamma_i$$

respectively into irreducible components.

For each integer $k \geq 0$, let ρ_k be the representation of G on $S^k(\mathbb{C}^n)$ induced by its natural action on $\mathbb{C}^n$. We have a decomposition $\rho_k = \sum_{i=0}^l v_{ki}\gamma_i$ with $v_{ki} \in \mathbb{Z}$. We associate to ρ_k the vector $v_m = (v_{m0},\ldots,v_{ml})^t \in \mathbb{Z}^{l+1}$. As in [K, p. 211] we define

$$P_G(t) := \sum_{m=0}^{\infty} v_m t^m.$$

This is a formal power series with coefficients in $\mathbb{Z}^{l+1}$. We also put $P_G(t)_i := \sum_{m=0}^{\infty} v_{mi}t^m$. Note that $P_G(t)_0$ is the usual Poincaré series $p_G(t)$ of the group G. Let V denote the set of all formal power series $x = \sum_{m=0}^{\infty} x_m t^m$ with $x_m \in \mathbb{Z}^{l+1}$. This is a free module of rank $l+1$ over the ring R of formal power series with integer coefficients.

Now let $n=2$ and $G \subset \mathrm{SL}_2(\mathbb{C})$ be a finite subgroup not of type $\mathcal{C}_{2n+1}$. Then $c_G=2$ and we have

$$p_f(t^2) = P_G(t)_0.$$

Moreover, we have $\gamma^* = \gamma$ and therefore $B^* = B$. The irreducible representations of $\mathrm{SL}_2(\mathbb{C})$ are of the form ρ_m, m a non-negative integer. The Clebsch-Gordon formula reads in this case

$$\rho_m \otimes \gamma = \rho_{m+1} \oplus \rho_{m-1}$$

setting $\rho_{-1}=0$ (cf., e.g., [FH, Exercise 11.11]). This yields the equation

$$Bv_m = v_{m+1} + v_{m-1}.$$

Following [K, p. 222], one can easily derive from this equation that $x = P_G(t)$ is a solution of the following linear equation in V:

$$((1+t^2)I - tB)x = v_0.$$

Let $M(t)$ be the matrix $(1+t^2)I - tB$ and $M_0(t)$ be the matrix obtained by replacing the first column of $M(t)$ by $v_0 = (1, 0, \ldots, 0)^t$. By Cramer's rule we can derive the following theorem [E4, Sect. 3] (see also [St]).

Theorem 4. *For a finite subgroup $G \subset \mathrm{SL}_2(\mathbb{C})$ not of type $\mathcal{C}_{2n+1}$ we have*

$$p_f(t^2) = P_G(t)_0 = \frac{\det M_0(t)}{\det M(t)} = \frac{\det(t^2 I - \tau)}{\det(t^2 I - \tau_a)},$$

where τ is the Coxeter element and τ_a the affine Coxeter element of the corresponding root system of type ADE associated to the singularity defined by the equation $f = 0$ with the same name.

Now let $n = 3$ and $G \subset \mathrm{SL}_3(\mathbb{C})$ be a finite subgroup. For a pair a, b of non-negative integers, let $\Gamma_{a,b}$ be the unique irreducible, finite-dimensional representation of $\mathrm{SL}_3(\mathbb{C})$ of [FH, Theorem 13.1]. By [FH, Proposition 15.25] and [FH, (13.5)], we have for a non-negative integer m (setting $\Gamma_{-1,b} = 0$) the following Clebsch-Gordon formulas:

$$\begin{aligned} \Gamma_{m,0} \otimes \gamma &= \Gamma_{m+1,0} \oplus \Gamma_{m-1,1}, \\ \Gamma_{m,0} \otimes \gamma^* &= \Gamma_{m-1,0} \oplus \Gamma_{m,1}. \end{aligned}$$

Since $\Gamma_{m,0} = \rho_m$, we can derive from these formulas

$$v_{m+2} = Bv_{m+1} - B^* v_m + v_{m-1}.$$

Therefore $x = P_G(t)$ is a solution of the following linear equation in V (see also [BI, BP]):

$$((1-t^3)I - tB + t^2 B^*)x = v_0.$$

Let $M(t)$ be the matrix $(1-t^3)I - tB + t^2 B^*$ and $M_0(t)$ be the matrix obtained by replacing the first column of $M(t)$ by $v_0 = (1, 0, \ldots, 0)^t$. Again Cramer's rule yields

$$P_G(t)_0 = \frac{\det M_0(t)}{\det M(t)}.$$

We have the following theorem:

Theorem 5. *Let $G \subset \mathrm{SL}_3(\mathbb{C})$ be one of the groups T, $\Delta(3 \cdot 4^2)$, O, $\Delta(6 \cdot 3^2)$, $\Delta(6 \cdot 4^2)$, $\Delta(6 \cdot 6^2)$, (E), (F), (G), (H)=I, (I), (J), (K), or (L), let c_G be the greatest common divisor of the weights of the variables $x, y, z(, u)$, and let $\alpha_1, \ldots, \alpha_m$ be the Dolgachev numbers of the singularity corresponding to G. Moreover, let $q^{(e)}_{a,b}(t) = (1-t)^a(1-t^e)^b$ for $a, b, e \in \mathbb{Z}$.*

(i) *For $G = T, O, I$ (E_6, E_7, E_8) we have $c_G = 1$ and*

$$\det M_0(t) = q^{(2)}_{a,b}(t)\Delta_{T^-_{\alpha_1,\alpha_2,\alpha_3}}(t), \quad \det M(t) = (1-t)q^{(2)}_{a,b}(t)\Delta_{T_{2,\alpha_1,\alpha_2,\alpha_3}}(t),$$

where $(a, b) = (3, 0), (3, 1), (4, 0)$ respectively.

(ii) *For G=(I), $\Delta(3 \cdot 4^2)$, $\Delta(6 \cdot 4^2)$ (E_{12}, E_{14}, Z_{11}) we have $c_G = 1$ and*

$$\det M_0(t) = q^{(4)}_{a,b}(t)\Delta_{T^+_{\alpha_1,\alpha_2,\alpha_3}}(t), \quad \det M(t) = (1-t)q^{(4)}_{a,b}(t)\Delta_{T_{4,\alpha_1,\alpha_2,\alpha_3}}(t).$$

where $(a, b) = (3, 0), (3, 2), (2, 3)$ respectively.

G	$\det M_0(t)$	$\det M(t)$
T	$(1-t)^3\Delta_{T^-_{2,3,3}}(t)$	$(1-t)^4\Delta_{T_{2,2,3,3}}(t)$
$\Delta(3\cdot 4^2)$	$(1-t)^3(1-t^4)^2\Delta_{T^+_{3,3,4}}(t)$	$(1-t)^4(1-t^4)^2\Delta_{T_{3,3,4,4}}(t)$
O	$(1-t)^3(1-t^2)\Delta_{T^-_{2,3,4}}(t)$	$(1-t)^4(1-t^2)\Delta_{T_{2,2,3,4}}(t)$
$\Delta(6\cdot 3^2)$	$\frac{(1-t^3)^{10}}{(1-t^6)^3}\Delta_{T^+_{2,2,2,2,2,2}}(t^3)$	$\frac{(1-t^3)^9}{(1-t^6)^3}\Delta_{T_{2,2,2,2,2}}(t^3)$
$\Delta(6\cdot 4^2)$	$(1-t)^2(1-t^4)^3\Delta_{T^+_{2,3,8}}(t)$	$(1-t)^3(1-t^4)^3\Delta_{T_{2,3,4,8}}(t)$
$\Delta(6\cdot 6^2)$	$(1-t^3)^7(1-t^6)\Delta_{T^+_{2,2,2,4}}(t^3)$	$(1-t^3)^8(1-t^6)\Delta_{T_{2,2,2,2,4}}(t^3)$
(E)	$\frac{(1-t^3)^8}{(1-t^6)^3}\Delta_{T^+_{2,2,4,4}}(t^3)$	$\frac{(1-t^3)^9}{(1-t^6)^3}\Delta_{T_{2,2,2,4,4}}(t^3)$
(F)	$\frac{(1-t^3)^7}{(1-t^6)^2}\Delta_{T^+_{4,4,4}}(t^3)$	$\frac{(1-t^3)^8}{(1-t^6)^2}\Delta_{T_{2,4,4,4}}(t^3)$
(G)	$\frac{(1-t^3)^4(1-t^6)(1-t^9)}{(1-t^{18})}\Delta_{T^+_{2,3,3,3}}(t^6)$	$\frac{(1-t^3)^4(1-t^6)(1-t^9)^2}{(1-t^{18})}\Delta_{T_{2,3,3,3}}(t^6)$
(H)=I	$(1-t)^4\Delta_{T^-_{2,3,5}}(t)$	$(1-t)^5\Delta_{T_{2,2,3,5}}(t)$
(I)	$(1-t)^3\Delta_{T^+_{2,3,7}}(t)$	$(1-t)^4\Delta_{T_{2,3,4,7}}(t)$
(J)	$\frac{(1-t^3)^8}{(1-t^6)^2}\Delta_{T^+_{2,2,2,5}}(t^3)$	$\frac{(1-t^3)^9}{(1-t^6)^2}\Delta_{T_{2,2,2,2,5}}(t^3)$
(K)	$\frac{(1-t^3)^6}{(1-t^6)}\Delta_{T^+_{2,4,7}}(t^3)$	$\frac{(1-t^3)^7}{(1-t^6)}\Delta_{T_{2,2,4,7}}(t^3)$
(L)	$\frac{(1-t^3)^7}{(1-t^6)}\Delta_{T^+_{2,4,5}}(t^3)$	$\frac{(1-t^3)^8}{(1-t^6)}\Delta_{T_{2,2,4,5}}(t^3)$

TABLE 5. Determinants of the matrices $M_0(t)$ and $M(t)$

(iii) *For G=(K), (L), (F), $\Delta(6\cdot 6^2)$, (J), (E) ($Q_{11}, E_{13}, U_{12}, Z_{1,0}, Q_{2,0}, K'_{1,0}$) we have $c_G = 3$ and*

$$\det M_0(t) = q^{(2)}_{a,b}(t^3)\Delta_{T^+_{\alpha_1,\ldots,\alpha_m}}(t^3), \quad \det M(t) = (1-t^3)q^{(2)}_{a,b}(t^3)\Delta_{T_{2,\alpha_1,\ldots,\alpha_m}}(t^3),$$

where $(a,b) = (6,-1), (7,-1), (7,-2), (7,1), (8,-2), (8,-3)$ *respectively.*

(iv) *For $G = \Delta(6\cdot 3^2)$ ($\delta 1$) we have $c_G = 3$, $m = 6$, $\alpha_i = 2$ for $i = 1,\ldots,m$, and*

$$\det M_0(t) = (1-t^3)q^{(2)}_{9,-3}(t^3)\Delta_{T^+_{2,\alpha_2,\ldots,\alpha_m}}(t^3), \quad \det M(t) = q^{(2)}_{9,-3}(t^3)\Delta_{T_{\alpha_2,\ldots,\alpha_m}}(t^3).$$

(v) *For G=(G) ($U_{1,0}$) we have $c_G = 6$ and*

$$\det M_0(t) = q(t^3)\Delta_{T^+_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}}(t^6), \quad \det M(t) = (1-t^9)q(t^3)\Delta_{T_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}}(t^6),$$

where $q(t) = \frac{(1-t)^4(1-t^2)(1-t^3)}{(1-t^6)}$.

Proof. The character tables of the tetrahedral and icosahedral group are given in [Art]. The character table of the octahedral group can be found, e.g., in [HH]. From these tables, one can calculate the matrices B. The matrices B for the remaining cases are given in [BP]. For the case (D), only one example is treated. More complete results for the cases $\Delta(3n^2)$ and $\Delta(6n^2)$ can be found in [LNR] and [EL] respectively. From these results, one can derive the corresponding matrices B. The proof of the theorem is then obtained by a direct calculation from these matrices using the computer algebra system SINGULAR [DGPS]. □

The results are summarized in Table 5.

Remark 6. Let G be one of the groups T, O, I. In this case, the matrix B is symmetric and we have $B^* = B$. Therefore

$$M(t) = (1-t^3)I - tB + t^2B^* = (1-t)((1+t+t^2)I - tB).$$

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ACCURACY OF NOISY SPIKE-TRAIN RECONSTRUCTION: A SINGULARITY THEORY POINT OF VIEW

GIL GOLDMAN, YEHONATAN SALMAN, AND YOSEF YOMDIN

To the Memory of Egbert Brieskorn. Among the most important events which inspired scientific interests of the third author was (in late 1960s) Brieskorn's discovery that 28 Milnor exotic spheres can be described by simple algebraic equations, and (in the early 1980s) participation in the Brieskorn Singularities Seminar in Bonn.

Abstract. This is a survey paper discussing one specific (and classical) system of algebraic equations - the so called "Prony system". We provide a short overview of its unusually wide connections with many different fields of Mathematics, stressing the role of Singularity Theory. We reformulate Prony System as the problem of reconstruction of "Spike-train" signals of the form $F(x)=\sum_{j=1}^{d} a_j\delta(x-x_j)$ from the noisy moment measurements. We provide an overview of some recent results of [1–3, 7, 8, 10, 11, 29, 53] on the "geometry of the error amplification" in the reconstruction process, in situations where the nodes x_j near-collide. Some algebraic-geometric structures, underlying the error amplification, are described (Prony, Vieta, and Hankel mappings, Prony varieties), as well as their connection with Vandermonde mappings and varieties. Our main goal is to present some promising fields of possible applications of Singularity Theory.

1. Introduction

In this paper we consider the classical *Prony system* of algebraic equations, with the real unknowns $a_j, x_j,\ j=1,\dots,d$, and with the right hand side formed by the known real "measurements" $m_0,\dots,m_{2d-1}$. This system has a form

$$\sum_{j=1}^{d} a_j x_j^k = m_k,\ k=0,1,\dots,2d-1. \tag{1.1}$$

We denote by $A=(a_1,\dots,a_d)\in\mathbb{R}^d$ and $X=(x_1,\dots,x_d)\in\mathbb{R}^d,\ x_1\le x_2\le\dots\le x_d$, the unknowns in system (1.1), and denote by $\mathcal{P}_d^A$ (resp. $\mathcal{P}_d^X$) the "parameter spaces" of the unknowns A and X, respectively. $\mathcal{P}_d=\mathcal{P}_d^A\times\mathcal{P}_d^X$ denotes the total parameter space of (A,X). The space (isomorphic to $\mathbb{R}^{2d}$) of the right-hand sides $\mu=(m_0,m_1,\dots,m_{2d-1})$ of (1.1) is denoted by $\mathcal{M}_d$.

In what follows we will usually identify (A,X) with a "spike-train signal"

$$F(x)=\sum_{j=1}^{d} a_j\delta(x-x_j). \tag{1.2}$$

Clearly, the moments $m_k(F)=\int x^k F(x)dx,\ k=0,1,\dots,$ are given by $m_k(F)=\sum_{j=1}^{d} a_j x_j^k$, so reconstructing F from its $2d-1$ initial moments is equivalent to solving (1.1), with $m_k=m_k(F)$.

Prony system appears in many classical theoretical and applied mathematical problems. In Section 2 we discuss some of these appearances. Explicit solution of (1.1) was given already in [48] (see Section 3 below).

There exists a vast literature on Prony and similar systems. In particular, the bibliography in [5] (1981) contains more than 50 pages. Most of recent applications are in Signal Processing. As a very partial sample we mention that in [14] and in many other publications a method, essentially equivalent to solving Prony system, was used in reconstructing signals with a "finite rate of innovation". In [45, 46] the applicability of Prony-type systems was extended to some new wide and important classes of signals. In [12, 21] multidimensional Prony systems were investigated via symmetric tensors, in particular, connecting them to the polynomial Waring problem. In [25] Prony system appears in a general context of Compressed Sensing. In [6, 9] Prony-like systems were used in reconstructing piecewise-smooth functions from their Fourier data. Finally, in [6] the same reconstruction accuracy as for smooth functions was demonstrated (thus confirming the Eckhoff conjecture).

Some applications of Prony system are of major practical importance, and various algorithms and numerical methods have been developed for its solution (see [47] and references therein). However, in a (very important) case when some of the nodes x_j nearly collide, while the measurements are noisy, these collision singularities lead to major mathematical and numerical difficulties. In particular, this happens in the context of the "super-resolution problem", which was investigated in many recent publications. See [1–3, 7, 8, 10, 11, 17, 18, 22, 24, 26, 41] as a small sample.

Notice that the Prony system (1.1) is linear (with the Vandermonde matrix on the "nodes" X) with respect to the "amplitudes" A, while it is highly nonlinear with respect to X. As the nodes collide (or near-collide), the Vandermonde determinant vanishes. Even knowing the position of the nodes, the reconstruction of the amplitudes is still ill-posed.

Thus singularities enter the solution process of the Prony system because of its geometric nature, no matter what solution method do we use. We believe that using the tools of Singularity Theory in this problem is well justified. In [11,53] we study the algebraic nature of nodes collision. In particular, we include into consideration the "confluent Prony systems", corresponding to signals with multiple nodes, and with the derivatives of the δ-function. We also introduce and study in [11] the "bases of finite differences" in the signal space $\mathcal{P}_d$, which behave coherently as the nodes collide.

In the present paper we give, following [1–3, 7, 8, 10, 11, 29, 53], a somewhat different point of view on the problem, stressing the role of Singularity Theory in understanding of Prony systems *with noisy right-hand side*. Below we discuss the following main topics:

1. In case of near-colliding nodes the initial measurements errors may be strongly amplified in the solution, making it unfeasible. However, the possible error-affected solutions are not distributed uniformly, but rather tightly concentrated along certain algebraic sets, known a priori ("Prony varieties" - see Sections 7 and 5 below).

Prony varieties are generalizations (via "making free" the amplitudes A) of the Vandermonde mappings and varieties, introduced and studied in Singularity Theory in [4, 32] and other publications (see Section 4).

2. A related notion is that of "Prony scenarios" (Section 8), which predict the error behavior along the Prony curves. *An important part here is the description of the combinatorics of real zeroes in polynomial pencils, actively studied in Singularity Theory - see [15, 33–35].*

3. In the presence of the measurement noise, statistical estimations of feasible solutions can be used. These methods are considered in the literature as superior in accuracy, but their practical

implementation is difficult, because of complicated nonlinear minimization problems involved. We expect that the tools developed in Singularity Theory for the study of "maxima of smooth functions", "cut-loci", and similar objects, can be useful here (Section 6.1).

4. In the case of the real nodes X (mainly presented in this paper) hyperbolic polynomials become a central topic in all the problems above. *Hyperbolic polynomials and related objects are actively studied in Singularity Theory (see [4, 15, 23, 32–35] as a partial sample), and we expect some of the available results to be directly applicable to Prony systems.*

Among other common topics with Singularity Theory we shortly discuss below rank stratification of the space of Hankel-type matrices, solving parametric linear systems, polynomial Waring problem, and finite differences. We hope that the connections presented will proof useful in both domains.

2. Some appearances of the Prony system

We outline here some prominent classical appearances of the Prony system.

2.1. Exponential Interpolation. This was the problem studied by Prony himself in [48]. We consider an interpolation problem for a given function $f(x)$ at the $2d$ consequent integer points $0, 1, \ldots, 2d-1$, with the interpolant being the sum of the exponents

$$\sum_{j=1}^{d} a_j e^{\zeta_j x}.$$

We can choose freely $2d$ parameters a_j, ζ_j, in order to fit the values

$$y_k = f(k),\ k = 0, \ldots, 2d-1.$$

Substituting $x = k$, and denoting e^{ζ_j} by x_j we get the Prony system of equations

$$\sum_{j=1}^{d} a_j e^{k\zeta_j} = \sum_{j=1}^{d} a_j x_j^k = y_k,\ k = 0, 1, \ldots, 2d-1.$$

2.2. Gauss quadratures. Let λ be a measure on the real line $\mathbb{R}$. For a given d we want to find d points $x_1, \ldots, x_d \in \mathbb{R}$, and d real coefficients $a_1, \ldots, a_d$ such that the quadrature formula

$$\int g(x) d\lambda \approx \sum_{j=1}^{d} a_j g(x_j) \tag{2.1}$$

be accurate for g being any polynomial of degree at most $2d-1$. By linearity, it is sufficient to get an equality in (2.1) only for g being the monomials $x^k,\ k = 0, 1, \ldots, 2d-1$, and this leads immediately to the Prony system

$$\sum_{j=1}^{d} a_j x_j^k = m_k(\lambda) := \int x^k d\lambda,\ k = 0, 1, \ldots, 2d-1, \tag{2.2}$$

with the right-hand side given by the consecutive moments $m_k(\lambda)$ of the measure λ.

Another interpretation is that we are looking for an atomic measure (a spike-train signal) $\tilde{\lambda} = \sum_{j=1}^{d} a_j \delta(x - x_j)$ satisfying $m_k(\tilde{\lambda}) = m_k(\lambda),\ k = 0, 1, \ldots, 2d-1$.

2.3. **Moment Theory and Padé approximations.** The classical Hamburger Moment problem consists in providing necessary and sufficient conditions for a sequence

$$m = \{m_0, m_1, \dots, m_k, \dots\}$$

to be the sequence of the consecutive moments $m_k = m_k(\lambda) = \int x^k d\lambda,\ k = 0, 1, \dots,$ of a non-atomic positive measure λ on the real line $\mathbb{R}$, and in reconstructing λ from m. The condition is that all the Hankel-type matrices

$$M_d(m) = \begin{bmatrix} m_0 & m_1 & \dots & m_{d-1} \\ m_1 & m_2 & \dots & m_d \\ \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & & \\ m_{d-1} & m_d & \dots & m_{2d-2} \end{bmatrix} \qquad d = 0, 1, ..., \tag{2.3}$$

are positive definite. The proof essentially consists in Gaussian quadrature approximation of the measure λ by positive atomic measures $\lambda_d = \sum_{j=1}^d a_{d,j}\delta(x - x_{d,j}),\ d = 0, 1, \dots,$ satisfying the condition $m_k(\lambda_d) = m_k,\ k = 0, 1, \dots, 2d - 1$, i.e. solving the Prony systems

$$\sum_{j=1}^d a_{d,j} x_{d,j}^k = m_k,\ k = 0, 1, \dots, 2d - 1,\ d = 0, 1, \dots, \tag{2.4}$$

with the right-hand side given by the input sequence $m = \{m_0, m_1, \dots, m_k, \dots\}$.

Another point of view is provided by the Padé approximation approach. For a sequence m as above consider a formal power series at infinity

$$f(z) = \sum_{k=0}^{\infty} m_k z^{-k-1}. \tag{2.5}$$

The d-th (diagonal) Padé approximant of $f(z)$ is a rational function $R_d(z) = \frac{P_d(z)}{Q_d(z)}$ with P_d, Q_d polynomials in z of the degrees $d - 1$ and d, respectively, such that the Taylor development of $R_d(z)$ at infinity has the form

$$R_d(z) = \sum_{k=0}^{2d-1} m_k z^{-k-1} + O(z^{-2d-1}). \tag{2.6}$$

In other words, the first $2d$ Taylor coefficients of $R_d(z)$ are $m_0, \dots, m_{2d-1}$.

Write $R_d(z)$ as the sum of elementary fractions, and develop at infinity:

$$R_d(z) = \sum_{j=0}^d \frac{a_{d,j}}{z - x_{d,j}} = \sum_{j=0}^d \frac{a_{d,j}}{z(1 - \frac{x_{d,j}}{z})} = \sum_{j=0}^d \frac{a_{d,j}}{z}(1 + \frac{x_{d,j}}{z} + (\frac{x_{d,j}}{z})^2 + \dots) =$$
$$= \sum_{k=0}^{\infty} \tilde{m}_k z^{-k-1},$$

where

$$\tilde{m}_k = \sum_{j=1}^d a_{d,j} x_{d,j}^k,\ k = 0, 1, \dots.$$

Thus condition (2.6) becomes the Prony system (2.4).

We do not discuss here other remarkable connections of the Prony system, provided by the classical Moment Theory, in particular, with continued fractions and orthogonal polynomials, see, for example, [42].

2.4. **Polynomial Waring problem.** We consider only the case of two variables (in more variables the calculations are, essentially, the same). Let $P(x,y)=\sum_{i=0}^m b_i x^{m-i} y^i$ be a homogeneous polynomial of degree m in (x,y). We look for a representation of P as a sum of m-th powers of d linear forms in (x,y):

$$P(x,y)=\sum_{j=1}^d (\eta_j x+\zeta_j y)^m, \tag{2.7}$$

within an attempt to minimize d in this expression. This problem is actively studied today. Many important results on generic and non-generic configurations in different degrees and dimensions are available. For details we refer the reader to [12, 20, 21, 36, 40], and references therein, as a very partial sample.

Let us put $x=1$ in (2.7). We get

$$P(1,y)=\sum_{i=0}^m b_i y^i=\sum_{j=1}^d (\eta_j+\zeta_j y)^m=\sum_{j=1}^d \eta_j (1+\frac{\zeta_j}{\eta_j} y)^m. \tag{2.8}$$

Denoting in (2.8) the fraction $\frac{\zeta_j}{\eta_j}$ by ξ_j we get

$$\sum_{i=0}^m b_i y^i=\sum_{j=1}^d \eta_j (1+\xi_j y)^m=\sum_{j=1}^d \eta_j \sum_{i=0}^m \binom{d}{i} \xi_j^i y^i=\sum_{i=0}^m y^i \sum_{j=1}^d \eta_j \binom{d}{i} \xi_j^i.$$

Comparing the coefficients of y^i on the two sides we obtain

$$\sum_{j=1}^d \eta_j \binom{d}{i} \xi_j^i=b_i,\ i=0,\dots,m.$$

Finally, dividing by $\binom{d}{i}$ and denoting $b_i/\binom{d}{i}$ by μ_i, we get the Prony system

$$\sum_{j=1}^d \eta_j \xi_j^i=\mu_i,\ i=0,\dots,m.$$

3. Explicit solution of the Prony system

From now on, and till Section 6, we allow complex nodes and amplitudes (A,X). In Section 6 we return to the real case, and explain the role of hyperbolic polynomials in the solution process.

In order to solve explicitly Prony system

$$\sum_{j=1}^d a_j x_j^k=m_k,\ k=0,1,\dots,2d-1, \tag{3.1}$$

consider the d-th diagonal Padé approximant $R_d(z)$ of the moment generating function, defined by (2.6) above.

Writing $R_d(z)$ as $R_d(z)=\frac{P_d(z)}{Q_d(z)}$ with

$$P_d(z)=b_0+b_1 z+\dots+b_{d-1} z^{d-1},\ Q_d(z)=c_0+c_1 z+\dots+c_{d-1} z^{d-1}+z^d,$$

substituting into (2.6), and comparing coefficients, we obtain the following linear system of equations for the coefficients $c = (c_0, \dots, c_{d-1})$ of the denominator Q:

$$\begin{bmatrix} m_0 & m_1 & \dots & m_{d-1} \\ m_1 & m_2 & \dots & m_d \\ \iddots & \iddots & & \\ m_{d-1} & m_d & \dots & m_{2d-2} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{d-1} \end{bmatrix} = - \begin{bmatrix} m_d \\ m_{d+1} \\ \vdots \\ m_{2d-1} \end{bmatrix}. \tag{3.2}$$

with the Hankel matrix $M_d(\mu),\ \mu = (m_0, \dots, m_{2d-1})$.

Finding c from (3.2), we then find the coefficients $b = (b_0, \dots, b_{d-1})$ of the numerator P as

$$b_0 = m_0 c_0, \quad b_1 = m_0 c_1 + m_1 c_0,\ \dots,\ \ b_{d-1} = m_0 c_{d-1} + \dots + m_d c_0.$$

This provides us explicitly the Padé approximant $R_d(z) = \frac{P(z)}{Q(z)}$. In order to find a_j, x_j it remains to represent R_d as the sum of the elementary fractions $R_d(z) = \sum_{j=0}^{d} \frac{a_j}{z - x_j}$. Essentially, this procedure appeared already in the Prony paper [48], and it remains a basis for most of recent algorithms.

3.1. **Solvability conditions.** Solvability conditions for (3.2) (and for the Prony system) are well known in the classical Moment Theory, in Padé approximations, and in other related fields, sometimes in quite different forms. One of possible formulations, convenient for our setting, was given in [11]. In order to present these conditions in a compact form, we allow complex nodes and amplitudes, as well as multiple nodes. (Including multiple nodes requires a rather accurate treatment, which we omit here. Details are given in [11]).

From the right hand side $\mu = (m_0, \dots, m_{2d-1}) \in \mathcal{M}_d$ we form the extended $d \times (d+1)$ Hankel matrix $\tilde{\mathcal{M}}_d(\mu)$.

Theorem 3.1. *(See [11]). Prony system (3.1) is solvable if and only if the following condition is satisfied: let the rank of $\tilde{\mathcal{M}}_d(\mu)$ be equal to $r \le d$. Then the left-upper $r \times r$ minor of $\tilde{\mathcal{M}}_d(\mu)$ is non-zero.*

Thus solvability of (3.1) can be read out from the right-hand side μ through the "rank stratification Σ" of the moment space $\mathcal{M}_d$.

Rank stratification for various classes of matrices is very important in Singularity Theory, and an extensive literature exists on this topic. Let us mention just [28, 38], which may be directly related to our study of Prony system. Specifically, J. Mather's theorem in [38] provides conditions for existence of smooth (in parameters) solutions of parametric families of linear systems (see also related results in [28]). We expect that Mather's theorem can be applied to the above system (3.2), providing a very important information on the behavior of solutions of (3.2) as μ approaches the low rank strata of Σ.

Let us mention also [31,37,43] where finite differences, and semi-simplicial resolutions, appear in study of Image singularities. They may be related to the study of the Prony mapping, via bases of finite differences in [11].

4. Prony, Vieta and Hankel mappings

In this section we suggest an algebraic-geometric picture capturing, to some extent, the mathematical structure of the solution procedure in Section 3. An important fact is that this picture appears as a natural extension of a construction, well known in Singularity Theory: that of

Vandermonde mapping and Vandermonde varieties, developed by Arnold, Givental, Kostov and others in the 1980's (see [4, 27, 32] and references therein).

Consider the following mappings:

1. The Prony map:

$$PM : \mathcal{P}_d \to \mathcal{M}_d,\ PM(F) = (m_0(F), \dots, m_{2d-1}(F)).$$

For each fixed amplitudes $A = (a_1, \dots, a_d)$ the restriction of the Prony map to $A \times \mathcal{P}_d^X$ coincides with the corresponding Vandermonde map, as defined in [4, 27, 32].

We call the space of all monic polynomials of degree d, $Q(z) = c_0 + c_1 z + \dots + c_{d-1} z^{d-1} + z^d$, the polynomial space $\mathcal{V}_d$.

2. The Vieta map:

$$VM : \mathcal{P}_d \to \mathcal{V}_d,\ VM(F) = Q_F(z) = z^d + \sigma_1(F) z^{d-1} + \dots + \sigma_1(F).$$

Here

$$\sigma_i(F) = \sigma_i(x_1, \dots, x_d)$$

is the i-th symmetric polynomial in the nodes X of F, and $Q(z) = Q_F(z)$ is the normalized polynomial with the roots $x_1, \dots, x_d$. Notice that the Vieta map depends only on the nodes X of F, but not on its amplitudes A.

3. The Hankel map:

$$HM : \mathcal{M}_d \to \mathcal{V}_d.$$

This map associates to any $\mu = (m_0, \dots, m_{2d-1}) \in \mathcal{M}_d$ the polynomial $Q \in \mathcal{V}_d$ obtained through solving a linear system (3.2)

Notice that in the coordinates A, X in the signal space $\mathcal{P}_d$ the mappings PM and VM are polynomial, while the mapping HM in the coordinates $\mu = (m_0, \dots, m_{2d-1})$ is rational, with the denominator $\Delta(\mu) = \det M_d(\mu)$, as provided by the Cramer rule.

We can put the mappings above into a mapping diagram D:

$$\begin{tikzcd}
& \mathcal{V}_d & \\
\mathcal{P}_d \arrow[ur, "VM"'] \arrow[rr, "PM"'] & & \mathcal{M}_d \arrow[ul, "HM"]
\end{tikzcd}$$

Now, a simple and basic fact, expressing the Prony solution algorithm, is the following:

Proposition 4.1. *The mapping diagram D is commutative, i.e.*

$$VM = HM \circ PM.$$

The proof was essentially given in Section 3 above.

The role of each of the three spaces in the solution process is different, and some important structures may look quite differently in these spaces. Below we give some examples.

5. Prony varieties

In this section we define, following [1–3], the "Prony varieties", which play an important role in the description of error amplification in solving Prony system.

Possessing the diagram D we can choose the easiest place to define the Prony varieties, which is the moment space $\mathcal{M}_d$. For each $d \le q \le 2d-1$ and for a given $\mu \in \mathcal{M}_d$, the "moment Prony variety" $S_q^{\mathcal{M}}(\mu)$ is the coordinate subspace in $\mathcal{M}_d$, passing through the point μ, where the first $q+1$ moments $m_0, \dots, m_q$ are constant.

The "signal Prony variety" $S_q^{\mathcal{P}}(\mu)$ is the preimage under the Prony mapping PM of the moment Prony variety $S_q^{\mathcal{M}}(\mu)$. Thus in $\mathcal{P}_d$ this variety is defined by the system of equations

$$\sum_{j=1}^{d} a_j x_j^k = m_k, \ k = 0, 1, \ldots, q, \tag{5.1}$$

which is formed by the first $q+1$ equations of the complete Prony system (3.1). This was the original definition of the "Prony leaves" in [2] and in later publications. For each fixed amplitudes $A = (a_1, \ldots, a_d)$ the signal Prony variety, intersected with $A \times \mathcal{P}_d^X$, coincides with the corresponding Vandermonde variety, as defined in [4, 27, 32]. We believe that the results of these papers may be important in study of Prony varieties, and we give more detail in [29].

The "polynomial Prony variety" $S_q^{\mathcal{V}}(\mu) \subset \mathcal{V}_d$ is the image under the Hankel map HM of the moment Prony variety $S_q^{\mathcal{M}}(\mu)$. We have the following fact:

Proposition 5.1. *([29]) For $q \geq d$ the polynomial Prony varieties $S_q^{\mathcal{V}}(\mu)$ are affine subspaces in $\mathcal{V}_d$, defined by the linear equations*

$$\begin{aligned} \mu_{d-1}c_1 + \mu_{d-2}c_2 + \ldots + \mu_0 c_d &= -\mu_d \\ \mu_d c_1 + \mu_{d-1}c_2 + \ldots + \mu_1 c_d &= -\mu_{d+1} \\ &\ldots\ldots\ldots\ldots \\ \mu_{q-1}c_1 + \mu_{q-2}c_2 + \ldots + \mu_{q-d}c_d &= -\mu_q. \end{aligned} \tag{5.2}$$

In the signal space we obtain in [29] the following description of the (node projections) of the Prony varieties $S_q^{\mathcal{P}}, \ d \leq q \leq 2d-1$:

Theorem 5.1. *([29]) The projection $S_q^{\mathcal{P},X}(\mu)$ of the signal Prony variety $S_q^{\mathcal{P}}(\mu)$ to the nodes space $\mathcal{P}_d^X$ is defined in $\mathcal{P}_d^X$ by the equations (5.2), with $c_j, \ j = 1, \ldots, d$, replaced by the symmetric polynomials $\sigma_j(x_1, \ldots, x_d)$.*

In the real case, the Vieta map VM provides a diffeomorphism of the interior of $S_q^{\mathcal{P},X}(\mu)$ to the interior of the intersection of the polynomial Prony varieties $S_q^{\mathcal{V}}(\mu)$ with the set H_d of hyperbolic polynomials in $\mathcal{V}_d$. The inverse is given by the "root mapping" RM, which associates to a hyperbolic polynomial $Q \in H_d^\circ$ its ordered roots $x_1 < \ldots < x_d$.

For any q between d and $2d-2$ we can consider the parametrization of the polynomial Prony varieties $S_q^{\mathcal{V}}$ through the last "free" moments $m_{q+1}, \ldots, m_{2d-1}$ in the right hand side of (3.2). This is the restriction of the mapping HM to the the moment Prony varieties $S_q^{\mathcal{M}}(\mu)$, i.e., to the coordinate subspaces in $\mathcal{M}_d$, passing through the point μ, where the first q moments $m_0, \ldots, m_q$ are constant. We have:

Proposition 5.2. *([29]) The restriction of the mapping HM to the the moment Prony varieties $S_q^{\mathcal{M}}(\mu)$ provides a rational parametrization of the polynomial Prony variety. It is a rational mapping of degree $2d-q-1$.*

For the moment Prony curves $S^{\mathcal{M}} = S_{2d-2}^{\mathcal{M}}$, which are the straight lines in $\mathcal{M}_d$ parallel to the coordinate axis Om_{2d-2}, this restriction is linear in the last moment m_{2d-1}, and it is provided by the expression

$$c_i = C_1^i(\tilde{\mu})m_{2d-1} + C_2^i(\tilde{\mu}), \tag{5.3}$$

where $\tilde{\mu} = (m_0, \dots, m_{2d-2})$*, and* $C_1^i(\tilde{\mu})$ *and* $C_2^i(\tilde{\mu})$ *are constant along the moment Prony curves* $S^{\mathcal{M}}$*.*

An important fact is that the moment Hankel matrix $M_d(\mu) = M_d(\tilde{\mu})$ *is constant along the moment Prony curves* $S^{\mathcal{M}}(\mu)$.

6. Solvability over the reals

The requirement for all the amplitudes A and the nodes X of the reconstructed signal F to be real is equivalent to requiring that all the moments $\mu = \big(m_0(F), \dots, m_{2d-1}(F)\big) \in \mathcal{M}_d$ are real, and that all the roots of the reconstructed polynomial Q are real, i.e Q is hyperbolic. As above, we denote by $H_d \subset \mathcal{V}_d$ the set of hyperbolic polynomials.

We define the "moment hyperbolicity set" $\tilde{H}_d \subset \mathcal{M}_d$ as the set of all $\mu \in \mathcal{M}_d$ for which the Hankel image $HM(\mu)$ belongs to the hyperbolicity set $H_d \subset \mathcal{V}_d$. Equivalently,

$$\tilde{H}_d = HM^{-1}(H_d).$$

The following result is a partial case of the conditions of Prony solvability over the reals, obtained in [29]:

Theorem 6.1. *([29]). For a real moments vector* $\mu \in \mathcal{M}_d$*, with* $\det M_d(\mu)$ *nonzero, Prony system (3.1) is solvable over the reals if and only if* μ *belongs to the moment hyperbolicity set* $\tilde{H}_d \subset \mathcal{M}_d$*.*

6.1. **Some statistical estimations for Prony solutions.** For a real signal F, if its moments vector μ was corrupted by the noise to μ', some roots of the reconstructed polynomial $Q = HM(\mu')$ could become complex. This makes the corresponding solution F' unfeasible.

This situation is common in practice, and usually the complex roots of Q are just projected to the real line. (In fact, in most of publications instead of real roots, the roots on the unit circle in the complex plane $\mathbb{C}$ are considered).

The same problem arises with the additional a priori known constraints on the feasible solutions F. (In particular, in most of applications we have a priori upper bounds on the nodes and amplitudes). We will denote by $Z \subset \mathcal{M}_d$ the set consisting of the moments of all the feasible signals F.

One of the most common *statistical estimations methods* is the maximum likelihood one (see e.g. [55] and references therein). Consider, for example, a Gaussian noise model $\mu' \sim \mathcal{N}(\mu, \Sigma)$ where μ is unknown. Then the maximum likelihood estimator $\hat{\mu}(\mu')$ of μ is any point $z \in Z \subset \mathcal{M}_d$ that is nearest to the measurement μ'.

In Bayesian estimation, besides the assumed probability distribution for the noise (e.g. Gaussian), we also assume a prior probability distribution of the moments vectors (or of the feasible signals) with support on Z, and a fixed loss function $L(\hat{\mu}(\mu'), \mu)$. Here the optimal Bayes estimator $\hat{\mu}(\mu')$ is given by the minimizer of the posterior risk

$$\hat{\mu}(\mu') = \inf_{\hat{\mu} \in Z} E[L(\hat{\mu}, \mu)|\mu'] = \inf_{\hat{\mu} \in Z} \int_Z L(\hat{\mu}, \mu) f_{\mu|\mu'}(\mu) d\mu,$$

where $f_{\mu|\mu'}(\mu)$ is the conditional density of μ given the measurement μ'.

Notice that minimisation is performed on an a priori known (and usually semi-algebraic) set Z. In our initial example Z is the hyperbolicity domain $\tilde{H}_d \subset \mathcal{M}_d$. The study of such minimization problems is in the mainstream of Singularity Theory. Specifically, a rich geometric information on the hyperbolicity domain, available today, may be useful (see [4, 32, 34] and references therein). Another highly relevant topic in Singularity Theory is the study of singularities of maximal functions, cut loci, and related objects. Some "old" results are in [16, 39, 50–52, 54][1], and in references therein. Some recent results are in [19, 49].

7. Error amplification and Prony curves

In this section we give a survey of recent results of [1], describing the geometry of error amplification in the case where the nodes of a signal F form a cluster of size $h \ll 1$. The central notion here is that of the ϵ-*error set* $E_\epsilon(F)$.

Definition 7.1. *The error set $E_\epsilon(F) \subset \mathcal{P}_d$ is the set consisting of all the signals $F' \in \mathcal{P}_d$ with*

$$|m_k(F') - m_k(F)| \le \epsilon, \;\; k = 0, \ldots, 2d-1.$$

In other words, $E_\epsilon(F)$ comprises all the signals $F' \in \mathcal{P}_d$ which can appear in reconstruction of F from its moments $\mu = (m_0, \ldots, m_{2d-1})$, each moment m_k corrupted by noise bounded by ϵ.[2]

The goal here is a detailed understanding of the geometry of the error set $E_\epsilon(F)$, in the various cases where the nodes of F near-collide.

7.1. The model space.

For $F \in \mathcal{P}_d$, we denote by $I_F = [x_1, x_d]$, the minimal interval in $\mathbb{R}$ containing all the nodes $x_1, \ldots, x_d$. We put $h(F) = \frac{1}{2}(x_d - x_1)$ to be the half of the length of I_F, and put $\kappa(F) = \frac{1}{2}(x_1 + x_d)$ to be the central point of I_F.

In case that $h(F) \ll 1$, we say that the nodes of F form a cluster of size h or simply that F forms an h-cluster.

For such signals F, consider the following "normalization": shifting the interval I_F to have its center at the origin, and then rescaling I_F to the interval $[-1, 1]$. For this purpose we consider, for each $\kappa \in \mathbb{R}$ and $h > 0$ the transformation

$$\Psi_{\kappa,h} : \mathcal{P}_d \to \mathcal{P}_d, \tag{7.1}$$

defined by $(A, X) \to (A, \bar{X})$, with

$$\bar{X} = (\bar{x}_1, \ldots, \bar{x}_d), \quad \bar{x}_j = \frac{1}{h}(x_j - \kappa), \; j = 1, \ldots, d.$$

For a given signal F we put $h = h(F), \; \kappa = \kappa(F)$ and call the signal $G = \Psi_{\kappa,h}(F)$ the model signal for F. Clearly, $h(G) = 1$ and $\kappa(G) = 0$. Explicitly G is written as

$$G(x) = \sum_{j=1}^{d} a_j \delta(x - \bar{x}_j).$$

With a certain misuse of notations, we will denote the space $\mathcal{P}_d$ containing the model signals G by $\bar{\mathcal{P}}_d$, and call it "the model space". For $F \in \mathcal{P}_d$ and $G = \Psi_{\kappa,h}(F)$, the moments of G

$$\bar{m}_k(F) = m_k(G) = \sum_{j=1}^{d} a_j \bar{x}_j^k, \; k = 0, 1, \ldots \tag{7.2}$$

are called the model moments of F.

For a given $F \in \mathcal{P}_d$ with the model signal $G = \Psi_{\kappa,h}(F)$, we denote by $\bar{E}_\epsilon(F)$ the "normalized" error set:

$$\bar{E}_\epsilon(F) = \Psi_{\kappa,h}(E_\epsilon(F)).$$

[1] Let us notice that the proof of one of the main results in [50] was incorrect, so the question remained open. Very recently a partial confirmation of the claim of [50] was obtained in [13].

[2] In contrast with Section 6.1, in [1] and here we make no probabilistic assumptions on the noise.

The set $\bar{E}_\epsilon(F)$ represents the error set $E_\epsilon(F)$ of F in the model space $\bar{\mathcal{P}}_d$. Note that $\bar{E}_\epsilon(F)$ is simply a translated and rescaled version of $E_\epsilon(F)$.

The reason for mapping a general signal F into the model space is that in the case of the nodes X forming a cluster of size $h \ll 1$, the moment coordinates centered at F,

$$\big(m_0(F') - m_0(F), \ldots, m_{2d-1}(F') - m_{2d-1}(F)\big),$$

turn out to be "stretched" in some directions, up to the order $(\frac{1}{h})^{2d-1}$. *In contrast, in the model space $\bar{\mathcal{P}}_d$, the coordinates system*

$$\big(m_0(G') - m_0(G), \ldots, m_{2d-1}(G') - m_{2d-1}(G)\big)$$

is bi-Lipschitz equivalent to the standard coordinates $(A, \bar{X})$ of $\bar{\mathcal{P}}_d$, for all signals G with "well-separated nodes" (see Theorem 7.2 below).

Throughout this section we will always use the maximum norm $||\cdot||$ on $\mathcal{M}_d$ and on $\mathcal{P}_d$ and on the nodes and amplitudes subspaces, $\mathcal{P}_d^X$ and $\mathcal{P}_d^A$ respectively. Explicitly:
For $\mu = (\mu_0, \ldots, \mu_{2d-1}),\ \mu' = (\mu'_0, \ldots, \mu'_{2d-1}) \in \mathcal{M}_d$

$$||\mu' - \mu|| = \max{}_{k=0,1,\ldots,2d-1} |\mu'_k - \mu_k|.$$

For $F = (A, X), F' = (A', X') \in \mathcal{P}_d$,

$$||F - F'|| = \max\,(||A - A'||, ||X - X'||).$$

7.2. **Sketch of the results.** We show that if the nodes of F form a cluster of size $h \ll 1$ and ϵ is of order h^{2d-1} or less then:
The ϵ-error set $\bar{E}_\epsilon(F)$ is a "curvilinear parallelepiped" Π, which closely follows the shape of the appropriate Prony varieties passing through G. The width of Π in the direction of the model moment coordinate $m_k(G') - m_k(G)$ is of order ϵh^{-k}.

Define the worst case reconstruction error of F as

$$\rho(F, \epsilon) = \max{}_{F' \in E_\epsilon(F)} ||F' - F||.$$

In a similar way we define $\rho_A(F, \epsilon)$ and $\rho_X(F, \epsilon)$ as the worst case errors in reconstruction of the amplitudes and the nodes of F, respectively:

$$\rho_A(F, \epsilon) = \max{}_{F'=(A',X') \in E_\epsilon(F)} ||A' - A||,$$
$$\rho_X(F, \epsilon) = \max{}_{F'=(A',X') \in E_\epsilon(F)} ||X' - X||.$$

We show that the worst case reconstruction error of the amplitudes A and the signal F, $\rho_A(F, \epsilon)$ and $\rho(F, \epsilon)$, are of order ϵh^{-2d+1}, and, the worst case reconstruction error of the nodes X is of order ϵh^{-2d+2}.

The above is shown in the following three steps:

1. First we normalize the signal F into its model signal $G = \Psi_{\kappa,h}(F)$, and describe in Theorem 7.1 the effect of this normalization on the image of the error set in $\mathcal{M}_d$. This theorem provides a description of the error set in the "moment coordinates", which are not, in general, equivalent to the coordinates of the signal space, because of the discussed singularities of the Prony mapping.
2. The second step is to use a "Quantitative Inverse Function Theorem" in order to show that the moment coordinates are bi-Lipschitz equivalent to the standard coordinates in signal space, in a sufficiently large domain around G. To get accurate constants, we improve in [1] some estimates of the norm of the inverse Jacobian JPM of the Prony mapping, obtained in [10].

(3) Finally, in order to get accurate bounds for the worst case error *separately in the amplitudes A, and in the nodes X* of the reconstructed signal F, we provide in [1] accurate estimates of the norm of the inverse Jacobian JPM composed with the projections into the amplitudes and the nodes subspaces, $\mathcal{P}_d^A$ and $\mathcal{P}_d^X$, of $\mathcal{P}_d$.

7.3. **The error set in the model signal space.** For any $G \in \bar{\mathcal{P}}_d$ and $\epsilon, \alpha > 0$ we denote by $\Pi_{\epsilon,\alpha}(G)$ the "curvilinear parallelepiped" consisting of all $G' \in \bar{\mathcal{P}}_d$ satisfying

$$|m_k(G') - m_k(G)| \le \epsilon\alpha^k, \ k = 0, \dots, 2d-1.$$

Notice that the Prony variety $S_q^{\mathcal{P}}(G)$ passing through G is defined by the equations

$$m_k(G') = m_k(G), \ k = 0, \dots, q,$$

and therefore, in the moments coordinates $m_k(G')$ the parallelepiped $\Pi_{\epsilon,h}(G)$ is ϵh^{-q} close to the Prony variety $S_q^{\mathcal{P}}(G)$.

Theorem 7.1. *Let $F \in \mathcal{P}_d$ form a cluster of size $h = h(F)$ and let $\kappa = \kappa(F)$ be the center of the cluster. Let $G = \Psi_{\kappa,h}(F)$ be the model signal for F. Set $\epsilon' = (1+|\kappa|)^{-2d+1}\epsilon$ and $h' = \frac{h}{1+|\kappa|}$. Then for any $\epsilon > 0$, the error set $\bar{E}_\epsilon(F)$ is bounded between the following two parallelepipeds:*

$$\Pi_{\epsilon',\frac{1}{h}}(G) \subset \bar{E}_\epsilon(F) \subset \Pi_{\epsilon,\frac{1}{h'}}(G).$$

Specifically, for $\kappa = \kappa(F) = 0$,

$$\bar{E}_\epsilon(F) = \Pi_{\epsilon,h}(G).$$

Theorem 7.1 holds without any assumptions on the mutual relation of ϵ and h, or on the distances between the nodes of F. It implies the following fact: *the Prony varieties $S_q^{\mathcal{P}}(G)$ form a "skeleton" of the error set $\bar{E}_\epsilon(F)$, and, in case when ϵ and h tend to zero at a certain rate, $S_q^{\mathcal{P}}(G)$ are the limits of $\bar{E}_\epsilon(F)$.*

Figures 1 and 2 illustrate the case $d = 2, q = 2d-2 = 2$ of Theorem 7.1.

7.4. **Applying quantitative Inverse Function Theorem.** In order to apply this theorem, we have to make explicit assumptions on the separation of the nodes X of the signal G, and on the size of its amplitudes A:

Assume that the nodes $x_1, \dots, x_d$ of a signal $G \in \bar{\mathcal{P}}_d$ belong to the interval $I = [-1, 1]$, and for a certain η with $0 < \eta \le \frac{2}{d-1}$, $d > 1$, the distance between the neighboring nodes $x_j, x_{j+1}, \ j = 1, \dots, d-1$, is at least η. We also assume that for certain m, M with $0 < m < M$, the amplitudes $a_1, \dots, a_d$ satisfy $m \le |a_j| \le M, \ j = 1, \dots, d$. We call such signals (η, m, M)-regular.

We want to show that for an (η, m, M)-regular signal $G \in \bar{\mathcal{P}}_d$ the moment coordinates $m_0(G') - m_0(G), \dots, m_{2d-1}(G') - m_{2d-1}(G)$ indeed form a coordinate system near G, which agrees with the standard coordinates $A, \bar{X}$ on $\bar{\mathcal{P}}_d$.

Definition 7.2. *For G a regular signal as above, and G' denoting signals near G, the moment coordinates are the functions $f_k(G') = m_k(G') - m_k(G), \ k = 0, ..., 2d-1$. The moment metric $d(G', G'')$ on $\bar{\mathcal{P}}_d$ is defined through the moment coordinates as*

$$d(G', G'') = \max{}_{k=0}^{2d-1} |m_k(G'') - m_k(G')|.$$

For any $\nu \in \mathcal{M}_d$ and $R > 0$ denote by $Q_R(\nu) \subset \mathcal{M}_d$ the cube of radius R

$$Q_R(\nu) = \{\nu' = (\nu'_0, \dots, \nu'_{2d-1}) \in \mathcal{M}_d, \ |\nu'_k - \nu_k| \le R, \ k = 0, 1, \dots, 2d-1\}. \tag{7.3}$$

Theorem 7.2. *Let $G \in \bar{\mathcal{P}}_d$ be an (η, m, M) regular signal and $\nu = PM(G)$. Then there are constants R, C_1, C_2, depending only on d, η, m, M, given explicitly in [1], such that:*

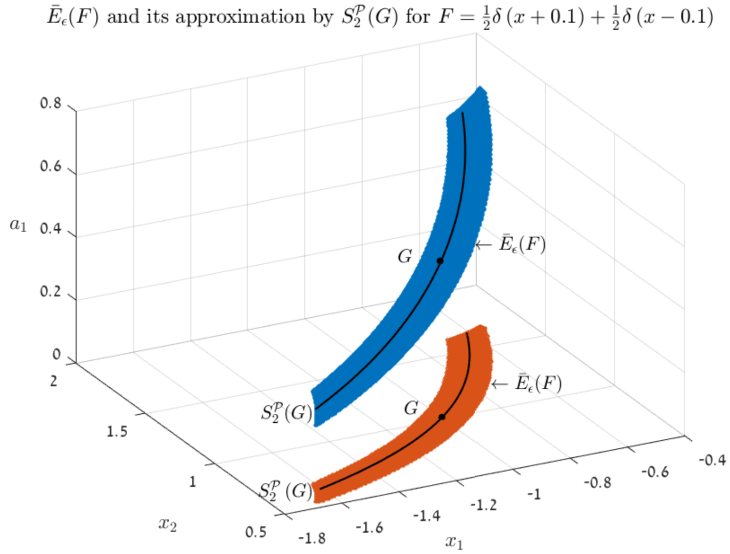

FIGURE 1. The projections of the error set $\bar{E}_\epsilon(F)$ and a section of the Prony curve $S_2^{\mathcal{P}}(G)$, for $h = 0.1$ and $\epsilon = h^3$.

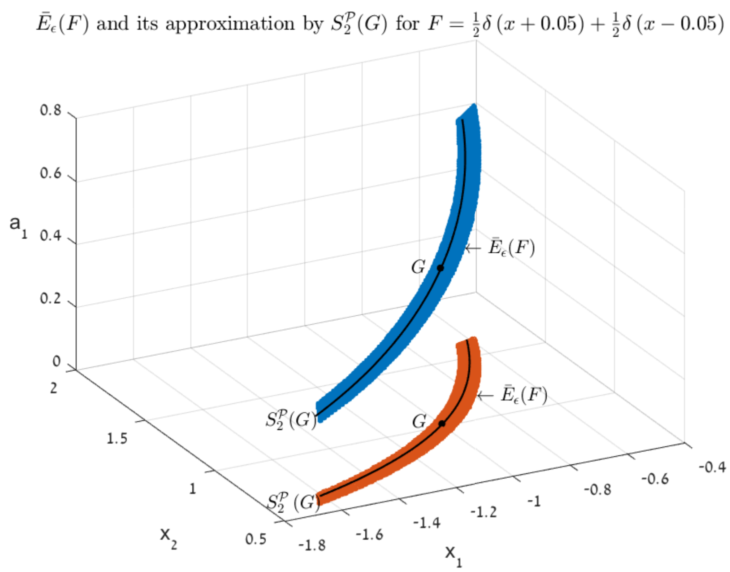

FIGURE 2. The error set $\bar{E}_\epsilon(F)$ and a section of $S_2^{\mathcal{P}}(G)$ for $h = 0.05$ and $\epsilon = h^3$. Note the convergence of $\bar{E}_\epsilon(F)$ into $S_2^{\mathcal{P}}(G)$.

(1) *The inverse mapping* PM^{-1} *exists on* $Q_R(\nu)$ *and provides a diffeomorphism of* $Q_R(\nu)$ *to* $\Omega_R(G) = PM^{-1}(Q_R(\nu))$.

(2) *The moment metric $d(G', G'')$ is bi-Lipschitz equivalent on $\Omega_R(G)$ to the maximum metric $||G'' - G'||$ in $\bar{\mathcal{P}}_d$:*

$$C_1 \; d(G', G'') \le ||G'' - G'|| \le C_2 \; d(G', G'').$$

Assume now that the measurement error $\epsilon \le Rh'^{2d-1}$, with $h' = \frac{h}{1+|\kappa|}$ as in Theorem 7.1. Then

$$PM(\bar{E}_\epsilon(F)) \subseteq PM(\Pi_{\epsilon, \frac{1}{h'}}(G)) \subset Q_R(PM(G)).$$

Combing Theorems 7.1 and 7.2 we obtain that the error set $\bar{E}_\epsilon(F)$ is a "deformed" paralelipiped in $\bar{\mathcal{P}}_d$ as illustrated in figures 1 and 2 above.

We use regular signals G as above, to model signals with a "regular cluster": For $F \in \mathcal{P}_d$ with $h = h(F)$ and $\kappa = \kappa(F)$, we say that F forms an (h, κ, η, m, M)-regular cluster if $G = \Psi_{\kappa,h}(F)$ is an (η, m, M)-regular signal.

The next theorem shows that the ϵ-error set is tightly concentrated around the Prony varieties.

Definition 7.3. *For each $0 \le q \le 2d-1$ denote by $S^{\mathcal{P}}_{q,\epsilon,\alpha}(G)$ the part of the Prony variety $S^{\mathcal{P}}_q(G)$, consisting of all signals $G' \in S^{\mathcal{P}}_q(G)$ with*

$$|m_k(G') - m_k(G)| \le \epsilon\alpha^k, \;\; k = q+1, \ldots, 2d-1.$$

Theorem 7.3. *Let $F \in \mathcal{P}_d$ form an (h, κ, η, m, M)-regular cluster and let $G = \Psi_{\kappa,h}(F)$ be the model signal for F. Set $h' = \frac{h}{1+|\kappa|}$. Then for any $\epsilon \le Rh'^{2d-1}$, the error set $\bar{E}_\epsilon(F)$ is contained within the Δ_q-neighborhood of the part of the Prony variety $S^{\mathcal{P}}_{q,\epsilon,\frac{1}{h'}}(G)$, for*

$$\Delta_q = C_2 \left(\frac{1}{h'}\right)^q \epsilon.$$

The constants R, C_2 are defined in Theorem 7.2 above.

7.5. **Worst case reconstruction error.** We now present lower and upper bounds, of the same order, for the worst case reconstruction error $\rho(F, \epsilon)$, defined, as above, by:

$$\rho(F, \epsilon) = \max{}_{F' \in E_\epsilon(F)} ||F' - F||.$$

We state separate bounds for $\rho_A(F, \epsilon)$ and $\rho_X(F, \epsilon)$ - the worst case errors in reconstruction of the amplitudes $A = (a_1, \ldots, a_d)$ and of the nodes $X = (x_1, \ldots, x_d)$ of F:

$$\rho_A(F, \epsilon) = \max{}_{F' \in E_\epsilon(F)} ||A' - A||, \;\; \rho_X(F, \epsilon) = \max{}_{F' \in E_\epsilon(F)} ||X' - X||.$$

Theorem 7.4. *[Upper bound] Let $F \in \mathcal{P}_d$ form an (h, κ, η, m, M)-regular cluster. Then for each positive $\epsilon \le \left(\frac{h}{1+|\kappa|}\right)^{2d-1} R$ the following bounds for the worst case reconstruction errors are valid:*

$$\rho(F, \epsilon), \; \rho_A(F, \epsilon) \le C_2 \left(\frac{1+|\kappa|}{h}\right)^{2d-1} \epsilon, \qquad \rho_X(F, \epsilon) \le C_2 h \left(\frac{1+|\kappa|}{h}\right)^{2d-1} \epsilon,$$

where C_2, R are the constants defined in Theorem 7.2.

Theorem 7.5. *[Lower bound] Let $F \in \mathcal{P}$ form an (h, κ, η, m, M)-regular cluster then:*

(1) *For each positive $\epsilon \le C_3 h^{2d-1}$ we have the following lower bound on the worst case reconstruction error of the nodes of F*

$$K_1 \epsilon \left(\frac{1}{h}\right)^{2d-2} \le \rho_X(F, \epsilon).$$

(2) *For each positive $\epsilon \le C_4 h^{2d-1}$ we have the following lower bound on the worst case reconstruction error of F and of the amplitudes of F*

$$K_2 \epsilon \left(\frac{1}{h}\right)^{2d-1} \le \rho(F,\epsilon),\ \rho_A(F,\epsilon).$$

Above, K_1, K_2, C_3, C_4 are constants not depending on h given explicitly in [1].

The lower and upper bounds given above are a special case of a more general result. In [1] (Theorem 4.4) it is shown that the Prony variety $S_q^{\mathcal{P}}(G)$ can be reconstructed from the moment measurements $\mu' \in \mathcal{M}_d$ with improved accuracy of order ϵh^{-q}.

8. Prony Scenarios

We keep the assumption that the nodes of our signal F form a regular cluster of a size $h \ll 1$. By Theorem 7.3, the signal Prony curve $S^{\mathcal{P}}(\mu)$ approximates the error set $E_\epsilon(F)$ with the accuracy of order ϵh^{-2d+2}. Note that the accuracy of point solution is of order ϵh^{-2d+1}. Thus, the Prony curve $S^{\mathcal{P}}(\mu)$ provides a rather accurate prediction of the possible behavior of all the noisy reconstructions of F.

In an actual solution procedure, the "true" Prony curve $S^{\mathcal{P}}(\mu)$ is not known. But from the noisy measurements $\mu' = (m'_0, \dots, m'_{2d-1})$ we can reconstruct the Prony curve $S^{\mathcal{P}}(\mu')$. This curve, by Theorem 4.4 in [1], approximates the "true curve" $S^{\mathcal{P}}(\mu)$ with an accuracy of the same (improved) order of ϵh^{-2d+2}, with which $S^{\mathcal{P}}(\mu)$ approximates $E_\epsilon(F)$. Therefore, we can consider this known curve $S^{\mathcal{P}}(\mu')$ as a prediction (or a "scenario") for all the noisy reconstructions of F.

Moreover, if we neglect possible errors of order ϵh^{-2d+2}, we can restrict the search of the optimal Prony solution (by any method, in particular, via statistical estimations) to the curve $S^{\mathcal{P}}(\mu')$.

We do not try to give here a rigorous definition of the "Prony scenario". Informally, this is a collection of data on the Prony curve $S^{\mathcal{P}}(\mu')$, which is necessary in order to find the optimal Prony solution on this curve, taking into account the available a priori constraints. Certainly we need an accurate description of the behavior of the nodes x_j and the amplitudes a_j along $S^{\mathcal{P}}(\mu')$ (or, better, along the polynomial Prony curve $S^{\mathcal{V}}(\mu') \subset \mathcal{V}_d$), including description of the intersection of $S^{\mathcal{V}}(\mu')$ with the hyperbolicity set H_d.

Some general results in this direction were obtained in [29]:

Theorem 8.1. *([29]) Assume that the matrix $M_d(\mu')$ is non-degenerate. Then in each case where the nodes x_i, x_j collide on $S^{\mathcal{P}}(\mu')$, the amplitudes a_i and a_j tend to infinity.*

Theorem 8.2. *([29]) Assume that the matrix $M_d(\mu')$ is non-degenerate, as well as its upper-left $(d-1)\times(d-1)$ minor. Then on each unbounded component of $S^{\mathcal{P},X}(\mu')$, for the coordinate m_{2d-1} on $S^{\mathcal{P}}(\mu')$ tending to infinity, exactly one node (x_1 or x_d) tends to infinity, while the rest of the nodes remain bounded.*

The polynomial Prony curves $S^{\mathcal{V}}$ can be considered as polynomials pencils. Some important results on the behavior of the real roots in polynomial pencils are provided in [15,35]. The result of [44] describing the behavior of roots in smooth 1-parametric families of polynomials may also be relevant. These results naturally enter the framework of the Prony scenarios, and in [29] we provide their more detailed treatment.

9. Some open questions

We would like to specify some open problems in the line of this paper. Mostly they concern the structure of the Prony varieties in the areas not covered by the inverse function theorem (Theorem 7.2 above).

1. Description of the global topology and geometry of the Prony varieties. In the topological study of the Vandermonde varieties in [4, 32] certain natural Morse functions were used. Can this method be extended to the Prony varieties?

On the other hand, an explicit parametrization of the Prony varieties, described in Section 5 above, reduces the problem to the study of the intersections of the affine subspaces in the polynomial space with the hyperbolic set H_d (which is motivated also by the considerations in Section 8 above). This study looks natural also from the point of view of Singularity Theory.

2. Understanding connections between the Prony and the Vandermonde varieties. The last are the fibers of a natural projection of the corresponding Prony varieties to the amplitudes. Is this projection regular? What topological information on the Prony varieties can be obtained from the known properties of the Vandermonde ones? Can information available on the Prony varieties (in particular, their explicit parametrization, see Section 5 above) be useful in study of the Vandermonde ones?

3. Behavior of the nodes $x_1, \dots, x_d$ on the Prony varieties $S_q^{\mathcal{P}}(\mu) \subset \mathcal{P}_d$ near the collision singularities. It would be important to describe an accurate asymptotic behavior of the distances between the colliding nodes as we approach the collision point. This question can be split into two: investigation of the intersection of the affine varieties $S_q^{\mathcal{V}}(\mu) \subset \mathcal{V}_d$ with the boundary of the hyperbolic set H_d, and investigation of the behavior of the root mapping RM near the boundary of H_d.

Already the case of the Prony curves is important and non-trivial.

4. Behavior of the amplitudes $a_1, \dots, a_d$ on the Prony varieties $S_q^{\mathcal{P}}(\mu) \subset \mathcal{P}_d$ near the collision singularities. In the case of the Prony curve, i.e. $q = 2d - 2$, Theorem 8.1 above gives conditions under which these amplitudes necessarily tend to infinity. It would be important to describe the accurate asymptotic behavior of the amplitudes as we approach the collision point. We expect that this question can be treated via methods from the classical Moment theory, combined with the techniques of "bases of finite differences" developed in [11, 53]. Also here the case of the Prony curves is important.

5. Extending the description of the Prony varieties, and of the error amplification patterns, to multi-cluster nodes configurations. This is a natural setting in robust inversion of the Prony system. Generalized Prony methods as well as other reconstruction methods typically reduce each cluster to a single node, thus forming a "reduced Prony system". It is important to estimate the accuracy of such an approximation (see [30] for some steps in this direction).

Because of the role of the Prony varieties in the analysis of the error amplification patterns, a natural question is: *To what extent do the Prony varieties of the reduced Prony system approximate the varieties of the "true" multi-cluster system?*

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NONCOMMUTATIVE DEFORMATIONS OF THICK POINTS

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Abstract. Any commutative algebra is of course also an associative algebra, and we may deform it as a non-commutative associative algebra. In particular this is of interest in singularity theory. It turns out that the versal base space of the non-commutative deformation functor of a thick point, in an affine three-dimensional variety, has properties that are rather astonishing. This base space is the main ingredient of a Toy Model for Quantum Theory, published in several books and papers, see [10], [11], [12], [2].

In this paper I shall describe the problems related to the computation of the local moduli suite, see [14], of the singularity consisting of an isolated point with a 3-dimensional tangent space.

1. Introduction

The last time I met with Egbert Brieskorn was, I think, in 2007 at Oberwohlfach. We talked a lot about his work on the legacy of Felix Hausdorff. We were both children during the 2nd World War, and we had both learned topology by reading the Grundzüge der Mengenlehre. I knew very little about Hausdorff's alter ego, Paul Mongré, and I was fascinated about yet another facet of this truly remarkable man, now well-documented by Brieskorn.

The reason why we sat down to talk, at the very end of a conference, was that Brieskorn had some questions. I had talked about my wild idea of modeling quantum theory and cosmology, using non-commutative deformation theory. We separated as good friends, even though I sensed the wise man's doubts about the endeavour.

It is therefore fitting that I, as a tribute to the always curious mathematician Egbert Brieskorn, one of the central workers in singularity theory during my lifetime, shall explain what I think he wanted to understand. It is a purely mathematical element of a Toy Model in physics; the computation of the versal base space of the non-commutative deformation functor of a thick point of imbedding dimension three.

It turns out that this base space is partitioned into a web of subspaces, the moduli suite of the singularity, see [14]. There is a maximal entropy subspace, equal to the Hilbert scheme $Hilb^{(2)}_{\mathbf{A}^3} = \underline{\tilde{H}}/Z_2$, of two points in affine 3-space, and there is a minimal non-trivial room, containing the quaternions $\mathfrak{Q}$

This $\underline{\tilde{H}}$, the blow up of $\mathbf{A}^3 \times \mathbf{A}^3$ in the diagonal $\underline{\Delta}$, turns out to be the base space of a canonical family of associative k -algebras in dimension 4. The study of the corresponding family of derivations leads to a natural way of introducing, on the tangent bundle of $\underline{\tilde{H}}$, an action of a Lie algebra containing the gauge group $\mathfrak{g}$, of the Standard Model. In particular $\mathfrak{g}/Rad = sl(2)$.

The fact that these results fit well with the set-up of the Standard Model, fusing our versions of General Relativity, Yang-Mills and Quantum Field Theory, is part of another story, see [12], [13], and [2].

2. Deformations and Noncommutativity

Non-commutativity comes up in algebraic geometry in many ways. In relation to deformation theory, there are two levels, implying four different mathematical tools.

First, we may ask wether deformations of an algebraic object must necessarily be parametrized by "commuting" parameters, i.e. wether the classifying objects should be restricted to commutative algebras, or should we accept non-commutative solutions?

Second, since any commutative algebra is also an associative algebra, should we restrict the deformations of a commutative algebra only to the commutative ones, or may we accept a non-commutative algebra as a deformation of a commutative one?

These questions comes up, in particular, in singularity theory. And here, in our story, it is related philosophically to some very central questions in physics.

Is our understanding, and the mathematical models we have made of the Universe, suggesting that not only Quantum Theory, but also our theory of gravity and space, the GRT, must be modelled by some sort of non-commutative algebraic geometry? The literature on these questions is huge, see [15] and later work of Majid, for a reasonably mathematical treatment, and references.

We shall first sketch the story of introducing noncommutative parameters. It turns out that this is the first important step in developing a noncommutative algebraic geometry. The very important fact, in the complex commutative case, that any finite-dimensional algebra is the sum of the local rings of its points, has a nice generalization that we have called the Generalised Burnside Theorem. This is a result that we shall use later, and that has been important in the study of the minimal model program in classical algebraic geometry, see [1]. As a tribute to the physics interested readers, we also add a short section on an algebraic geometric version of entropy, to prepare for the main subject of this paper.

Since the goal of this paper is limited to a strictly mathematical result in deformation theory of a thick point singularity, the main focus will be on the second question above. What can we learn by deforming a commutative singularity to associative algebras?

2.1. **Non-commutative Deformations of Modules.** In [7], [8] and [9], we introduced non-commutative deformations of families of modules of associative k-algebras, k a field. We shall here recall the definitions, and the main results that will be used in the sequel.

Let $\underline{a}_r$ denote the category of r-pointed not necessarily commutative k-algebras R. The objects are the diagrams of k-algebras

$$k^r \xrightarrow{\iota} R \xrightarrow{\pi} k^r$$

such that the composition of ι and π is the identity. Any such *r-pointed k-algebra* R is isomorphic to a k-algebra of $r \times r$-matrices $(R_{i,j})$. The radical of R is the bilateral ideal $Rad(R) := ker\pi$, such that $R/Rad(R) \simeq k^r$. The dual k-vector space of $Rad(R)/Rad(R)^2$ is called the tangent space of R.

The usual, category of commutative local Artinian k-algebras with residue field k, commonly denoted by $\underline{l}$, is of course the commutative part of $\underline{a}_1$. Fix a (not necessarily commutative) associative k-algebra A and consider a right A-module M. The ordinary deformation functor,

$$Def_M : \underline{l} \to \underline{Sets}$$

is then defined.

Assuming $Ext^i_A(M,M)$ has finite k-dimension for $i = 1,2$, it is well known, see [17], or [7], that Def_M has a pro-representing hull H, *the formal moduli of* M. Moreover, the tangent space of H is isomorphic to $Ext^1_A(M,M)$, and H can be computed in terms of $Ext^i_A(M,M),\ i = 1,2$ and their *matric* Massey products, see [7].

In the general case, consider a finite family $\mathbf{V} = \{V_i\}_{i=1}^r$ of right A-modules, and put $V := \oplus_{i=1}^r V_i$. Assume that $\dim_k Ext_A^1(V_i, V_j) < \infty$, and call any such family of A-modules a *swarm*. We shall define a deformation functor,

$$Def_{\mathbf{V}} : \underline{a}_r \to \underline{Sets},$$

generalising the functor Def_M above. Given an object $\pi : R = (R_{i,j}) \to k^r$ of $\underline{a}_r$, consider the k-vector space and left R-module $(R_{i,j} \otimes_k V_j)$. It is easy to see that

$$End_R((R_{i,j} \otimes_k V_j)) \simeq (R_{i,j} \otimes_k Hom_k(V_i, V_j)).$$

Clearly π defines a k-linear and left R-linear map

$$\pi(R) : (R_{i,j} \otimes_k V_j) \to \oplus_{i=1}^r V_i$$

inducing a homomorphism of R-endomorphism rings,

$$\tilde{\pi}(R) : (R_{i,j} \otimes_k Hom_k(V_i, V_j)) \to \oplus_{i=1}^r End_k(V_i).$$

The right A-module structure on the V_i's is defined by a homomorphism of k-algebras,

$$\eta_0 : A \to \oplus_{i=1}^r End_k(V_i) \subset (Hom_k(V_i, V_j)) =: End_k(V).$$

Notice that this homomorphism also provides each $Hom_k(V_i, V_j)$ with an A- bimodule structure. Let $Def_{\mathbf{V}}(R) \in \underline{Sets}$ be the set of isoclasses of homomorphisms of k-algebras,

$$\eta' : A \to (R_{i,j} \otimes_k Hom_k(V_i, V_j))$$

such that, $\tilde{\pi}(R) \circ \eta' = \eta_0$, where the equivalence relation is defined by inner automorphisms in the R-algebra $(R_{i,j} \otimes_k Hom_k(V_i, V_j))$ inducing the identity on $\oplus_{i=1}^r End_k(V_i)$. One easily proves that $Def_{\mathbf{V}}$ has the same properties as the ordinary deformation functor and we may prove the following, see [7]:

Theorem 2.1. *The functor $Def_{\mathbf{V}}$ has a pro-representable hull, i.e. an object $H := H(\mathbf{V})$ of the category of pro-objects $\hat{\underline{a}}_r$ of $\underline{a}_r$, together with a versal family*

$$\tilde{V} = (H_{i,j} \otimes V_j) \in \varprojlim_{n \geq 1} Def_{\mathbf{V}}(H/\mathfrak{m}^n)$$

where $\mathfrak{m} = Rad(H)$, such that the corresponding morphism of functors on $\underline{a}_r$

$$\kappa : Mor(H, -) \to Def_{\mathbf{V}}$$

defined for $\phi \in Mor(H, R)$ by $\kappa(\phi) = R \otimes_\phi \tilde{V}$, is smooth, and an isomorphism on the tangent level. H is uniquely determined by a set of matric Massey products defined on subspaces

$$D(n) \subseteq Ext^1(V_i, V_{j_1}) \otimes \cdots \otimes Ext^1(V_{j_{n-1}}, V_k)$$

with values in $Ext^2(V_i, V_k)$.

Moreover, the right action of A on $\tilde{V}$ defines a homomorphism of k-algebras,

$$\eta : A \longrightarrow O(\mathbf{V}) := End_H(\tilde{V}) = (H_{i,j} \otimes Hom_k(V_i, V_j)).$$

The k-algebra $O(\mathbf{V})$, called the ring of observables of $\mathbf{V}$, acts on the family of A-modules $\{V_i\}_{i=1}^r$, extending the action of A.

If $dim_k V_i < \infty$, for all $i = 1, \dots, r$, the operation of associating $(O(\mathbf{V}), \mathbf{V})$ to $(A, \mathbf{V})$ is a closure operation.

There is a very useful result, see [8], [9],[2],

Theorem 2.2 (A Generalised Burnside Theorem)**.** *Let A be a finite-dimensional k-algebra, k an algebraically closed field. Consider the family $\mathbf{V} = \{V_i\}_{i=1}^r$ of all simple A-modules, then*

$$\eta : A \longrightarrow O(\mathbf{V}) = (H_{i,j} \otimes Hom_k(V_i, V_j))$$

is an isomorphism. Moreover the k-algebras A and H are Morita-equivalent.

We also prove that there exists, in the non-commutative deformation theory, an obvious analogy to the notion of pro-representing (modular) substratum H_0 of the formal moduli H, see [5]. The tangent space of H_0 is determined by a family of subspaces

$$Ext^1_0(V_i, V_j) \subseteq Ext^1_A(V_i, V_j), \quad i \neq j$$

the elements of which should be called the almost split extensions (sequences) relative to the family $\mathbf{V}$, and by a subspace, that we denote,

$$T_0(\Delta) \subseteq \prod_i Ext^1_A(V_i, V_i)$$

which is the tangent space of the deformation functor of the full subcategory of the category of A-modules generated by the family $\mathbf{V} = \{V_i\}_{i=1}^r$, see [6]. If $\mathbf{V} = \{V_i\}_{i=1}^r$ is the set of all indecomposable's of some Artinian k-algebra A, we show that the above notion of *almost split sequence* coincides with that of Auslander, see [16].

2.2. **Local Moduli and Entropy.** Consider an algebraic geometric object X, and let $aut(X)$ be the Lie algebra of infinitesimal automorphisms of X. The sub-Lie algebra $aut_0(X)$ that lifts to the formal moduli of X, is a Lie ideal. Put $\mathfrak{a}(X) := aut(X)/aut_0(X)$, then if $X(t)$ is a deformation of some X along a parameter t, we find $dim_k\mathfrak{a}(X(t)) \leq dim_k\mathfrak{a}(X)$. One may phrase this saying that an object X can never gain *information* when deformed. Moreover, deformation is, obviously, not a reversible process, so information can get lost. This measure of information losses, is related, as we shall see, to the notion of gain of entropy (en-ergy and tropos=transform) coined by Clausius (1865) and generalised by Boltzmann and Shannon.

In [14], studying moduli problems of singularities in (classical) algebraic geometry, we were led to consider the notion of *Modular Suite*. This is a canonical partition $\{\mathbf{M}_\alpha\}$, of the versal base space, $\mathbf{M}$, of the deformation functor of an algebraic object, X. The different *rooms*, $\mathbf{M}_\alpha$, correspond to the subsets of equivalence classes of deformations in $\mathbf{M}$, along which the Lie algebra $\mathfrak{a} := aut/aut_0$ deforms as Lie-algebras, and therefore conserves its dimension. Working with Thermodynamics, it occurred to me that the notion of entropy has an interesting parallel in deformation theory. In fact I have proposed the following,

Definition 2.3. *Fix an object X, and let $X(\underline{t})$ corresponds to the point $\underline{t} \in \mathbf{M}_\alpha$, then we shall term Entropy, of the state $\underline{t}$, the integer,*

$$S(\underline{t}) := dim_k(\mathbf{M}_\alpha).$$

In this classical situation, assuming that the field is algebraically closed, and that $\mathbf{M}$ is of finite Krull dimension, the modular suite $\{\mathbf{M}_\alpha\}$ is finite, with an inner room, the *modular substratum* and an *ambiant* (open) *maximal entropy* stratum. But the structure of the modular suite may be very complex, even for simple singularities X, see the example of the quasi homogenous plane curve singularity $x_1^5 + x_2^{11}$, in [14].

It is also clear that for any *algebraic dynamics* in $\mathbf{M}$, the entropy will always stay or grow, see again [14]. To be able to construct situations where the entropy is lowered, or the information goes up, we must leave classical algebraic geometry, and venture into non-commutative algebraic geometry, see [2].

In the general situation, where our algebras of *observables* are associative but not necessarily commutative, the first interesting cases are deformations of associative algebras, see [13].

2.3. **Deformations of Associative Algebras.** Given an associative k-algebra A, The tangent space of the formal moduli of A, as an associative k-algebra is, by deformation theory, see [5], and [14],

$$T_\star := A^1(k, A; A) = Hom_F(ker\rho, A)/Der,$$

where, $\rho : F \to A$ is any surjective homomorphism of a free k-algebra F, onto A, Hom_F means the F-bilinear maps, and Der denotes the subset of the restrictions to $I := ker\rho$ of the k-derivations from F to A.

As an example, let $A = k[x_1, .., x_d]$ be the polynomial algebra, then we find,

$$A^1(k, A; A) = Hom_F(ker\rho, A)$$

where $F = k < x_1, .., x_d >$, and $ker\rho =< [x_i, x_j] >$, and any element in $A^1(k, A; A)$ is a generalised Poisson structure. The technique for this general deformation theory, can be found in loc.cit. [5], see also [4], and we prove the same type of theorems as for modules over an associative algebra,

Now, let us consider the rather innocent singularity,

$$U := k[x_1, x_2, x_3]/(x_1, x_2, x_3)^2,$$

as an associative algebra. U is, geometrically, an isolated point, with a 3-dimensional tangent space. We shall be interested in the versal base space for the deformation-functor of U, as associative algebra.

The tangent space of the formal moduli of the singularity

$$U := k[x_1, x_2, x_3]/(x_1, x_2, x_3)^2,$$

as an associative k-algebra is now,

$$T_\star := A^1(k, U; U) = Hom_F(ker\rho, U)/Der,$$

where, $\rho : F \to U$ is the obvious surjection of the free k-algebra $F = k < x_1, x_2, x_3 >$, with $ker\rho = (\underline{x})^2$, generated as F bi-module by the family $\{x_{i,j} := x_i x_j\}$.

Any F-bilinear morphism $\phi : (\underline{x})^2 \to U$, must be of the form,

$$\phi(x_{i,j}) = a^0_{i,j} + \sum_{l=1}^{3} a^l_{i,j} x_l$$

and the bilinearity is seen to imply that $a^0_{i,j} = 0$. Thus, the dimension of $Hom_F(I, U)$ is 27.

Any derivation $\delta \in Der$, must be given by,

$$\delta(x_i) = b^0_i + \sum_{l=1}^{3} b^l_i x_l$$

and the restriction of this map, to the generators of $I = (\underline{x})^2$, must have the form,

$$\delta(x_{i,j}) = b^0_j x_i + b^0_i x_j,$$

therefore determined by the $b^0_i s$. It follows that the tangent space $T_\star$ is of dimension 27-3=24.

Now, let $o, p \in \mathbf{A}^3$, be two points, $o = (o_1, o_2, o_3), p = (p_1, p_2, p_3)$, with respect to the coordinate system, $\underline{x}$, and put,

$$\phi_{o,p}(x_{i,j}) = p_j x_i + o_i x_j,$$

then it is easy to see that the maps $\{\phi_{o,p}\}$ generate a 6-dimensional sub vector subspace T_0 of $T_\star$. Notice that, if $o = p$ then $\phi_{o,p}$, is a derivation, thus 0 in $T_\star$.

Moreover, the rather unexpected happens. We may integrate the tangent subspace T_0, and obtain a family of flat deformations of U. In fact, it is easy to see that,

$$U(o,p) := k < x_1, x_2, x_3 > /(x_i x_j - o_i x_j - p_j x_i + o_i p_j),$$

is an associative k-algebra of dimension 4, and a deformation of U, in a direction of $\underline{H}$. This defines a family of associative $H := k\,[o,p]$-algebras,

$$\mathbf{U} := H < x_1, x_2, x_3 > /(x_i x_j - o_i x_j - p_j x_i + o_i p_j).$$

Let us put,

$$\mathbf{x}_{i,j} := (x_i - o_i)(x_j - p_j) = x_i x_j - o_i x_j - p_j x_i + o_i p_j, o := (o_1, o_2, o_3), p = (p_1, p_2, p_3) \in H^3,$$

Notice that if $o = p$ then $U(o,p)$ is isomorphic to U, as it should, and that, $U(o,p) \simeq U(-o,-p)$. Moreover, for any non-zero element $\kappa \in k$, and any 3-vector $c \in \mathbf{A}^3$, we have,

$$U(o,p) \simeq U(\kappa o, \kappa p),\ U(o,p) \simeq U(o-c, p-c).$$

Choosing $c = 1/2(p+o)$, we find $o' := o - c = -(p-c) =: -p'$, and it is easy to see that if $o' \neq 0$ the sub Lie algebra generated by $\{x_1, x_2, x_3\}$ in $U(o', p')$, is isomorphic to the standard 3-dimensional Lie algebra with relations, $[y_1, y_2] = y_2, [y_1, y_3] = y_3, [y_2, y_3] = 0$. Moreover, choosing $c = (p+o)$, we find an isomorphism,

$$U(o,p) \simeq U(-p,-o) \simeq U(p,o),$$

which should be related to the obvious action of Z_2 on $\underline{H}$, which again might be related to the CPT-equivalence in physics, see [12], (4.9).

Let $\epsilon_{i,j,k}$ and $\delta_{i,j}$ be the usual indices, the first one nonzero only for $\{i,j,k\} = \{1,2,3\}$, and the last one the usual delta function. Then the algebra,

$$\mathbf{Q} := k < x_1, x_2, x_3 > /(x_i x_j - \epsilon_{i,j,k} x_k + \delta_{i,j}),$$

is isomorphic to the quaternions, which therefore is another non-trivial deformation of U. Notice that we here have used the ordinary notation for summation, by repeating indexes. Notice, for eventually later use that the discoverer of the Quaternions, Hamilton, wrote about his algebra as the *science of pure time*, see [3].

Consider now the restriction to the subscheme $\underline{H} - \underline{\Delta}$, of the family $\mathbf{U}$, denoted by,

$$\nu' : \mathbf{U}' \to \underline{H} - \underline{\Delta}.$$

Let $\tilde{\underline{H}}$ be the blown up of $\underline{H}$, in $\underline{\Delta}$, and recall that,

$$\mathbf{H} := Hilb^{(2)}_{\mathbf{A}^3} = \tilde{\underline{H}}/Z_2.$$

Since for all non-zero $\kappa \in k$, we have $U(\lambda + \kappa u, -\kappa u + \lambda) \simeq U(\kappa u, -\kappa u) \simeq U(u, -u)$, this family extends uniquely to a family,

$$\nu : \tilde{\mathbf{U}} \to \tilde{\underline{H}},$$

compatible with the action of Z_2.

Let us compute the algebras $U(o,p)$, and their Lie algebras of derivations, $\mathfrak{g}(\underline{t}) := Der_k(U(\underline{t}))$.

First, the 4-dimensional k-algebras $U(o,p)$, with relation,

$$x_{i,j} = (x_i - o_i)(x_j - p_j),$$

with, $o \neq$p, are all isomorphic, since in this case there is an element $\alpha \in Gl_k(3)$ sending (o,p) onto any other pair (o', p'), with $o' \neq p'$. Let us see this, using the generalised Burnside theorem, see [12], (3.2). Obviously $U(o,p)$ has only two simple representations, of dimension 1, call them k_o and k_p. By the O-construction, there is an isomorphisme,

$$\eta : U(o,p) \to \begin{pmatrix} H_{1,1} \otimes End(k_o) & H_{1,2} \otimes Hom_k(k_o,k_p) \\ H_{2,1} \otimes Hom_k(k_p,k_o) & H_{2,2} \otimes End_k(k_p) \end{pmatrix},$$

where, $H_{1,1}$ is a formal algebra with tangent space $Ext^1_{U(o,p)}(k_o,k_o)$, $H_{2,2}$ is a formal algebra with tangent space $Ext^1_{U(o,p)}(k_p,k_p)$, and $H_{1,2}$, respectively $H_{1,2}$, is a bi-module generated by $Ext^1_{U(o,p)}(k_o,k_p)^\star$, respectively by $Ext^1_{U(o,p)}(k_p,k_o)^\star$. There are no problems computing the Ext-groups. Recall that

$$Ext^1_{U(o,p)}(V_1,V_2) = Der_k(U(o,p), Hom_k(V_1,V_2))/Triv,$$

and that $u \in U(o,p)$ operates on $\phi \in Hom_k(V_1,V_2)$, as,

$$(u\phi)(v_1) = u\phi(v_1), (\phi u)(v_1) = \phi(uv_1).$$

In the general case (one may test it in the interesting case, $o = (1,0,0), p = (0,0,0)$ above), we obtain,

$$Ext^1_{U(o,p)}(k_o,k_o) = Ext^1_{U(o,p)}(k_p,k_p) = Ext^1_{U(o,p)}(k_o,k_p) = 0, Ext^1_{U(o,p)}(k_p,k_o) = k^2.$$

Therefore,

$$\eta : U(o,p) \to \begin{pmatrix} k & 0 \\ <\xi_1,\xi_2> & k \end{pmatrix}$$

is an isomorphism. Here $\xi_i \cdot 1 = \xi_i$. We may pick generators of this algebra,

$$x_1 := \begin{pmatrix} 0 & 0 \\ \xi_1 & 0 \end{pmatrix}, x_2 := \begin{pmatrix} 0 & 0 \\ \xi_2 & 0 \end{pmatrix}, x_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and obtain the relations corresponding to the choice of $o = (0,0,-1), p = (0,0,1)$. We have therefore obtained an algebraic subspace $\underline{\tilde{H}}$, of the miniversal base space $\mathbf{M}$ of the algebra U, corresponding to the algebras $U(o,p)$ that are all isomorphic. This subspace is therefore a *trivialising section* of this miniversal base space.

Remark 2.4. *Deformations of U(o,p)*

Using the same technique as above, computing the deformations of one of these isomorphic algebras, we may show that the tangent space of $Def_{U(o,p)}$ is trivial. In fact, as above, this tangent space is given by,

$$A^1(U(o,p),U(o,p)) = Hom_F(J,U(o,p))/Der,$$

where $J = ker(\pi)$ and $\pi : F \to U(o,p)$ is a surjective homomorphism of the free k-algebra $F = k < x_1,x_2,x_3 >$ onto $U(o,p)$. Obviously $J = ker(\pi)$ is generated by the elements

$$\{x_{i,j} := x_ix_j - o_ix_j - p_jx_i + o_ip_j\},$$

and we have in J the relations,

$$x_{i,j}x_k + o_ix_{j,k} + p_jx_{i,k} = x_ix_{j,k} + o_jx_{j,k} + p_kx_{i,j}.$$

Let $o = (1,0,0), p = (0,1,0)$, then an easy, but quite lengthy computation shows that these relations implies that any bilinear homomorphism, $c \in Hom_F(J,U(o,p))$, is the restriction of a derivation, $\beta \in Der_k(F,U(o,p))$, proving that,

$$A^1(U(o,p),U(o,p)) = 0.$$

Notice that any automorphism of U is reduced to a substitution,

$$y_i := \sum_{k=1}^{3} \alpha_{i,k} x_k, \alpha := (\alpha_{i,k}) \in Gl_k(3).$$

If we change the coordinates, of the point pair (o, p), by the automorphism above, then with obvious indexes,

$$U_x(o,p) \simeq U_y(\alpha(o), \alpha(p)).$$

3. Local Gauge Group

Borrowing notions from quantum physics, we shall call the principal Lie algebra bundle on the space, $\underline{\tilde{H}}$,

$$\mathfrak{g} := Der_H(\mathbf{U})$$

the local gauge group of the H-representation Θ_H.

3.1. **Computation of $\mathfrak{g}$, and its Action.** Any element $\delta \in Der_H(\mathbf{U})$ must be given by its values on the coordinates,

$$\delta(x_i) = \delta_i^0 + \delta_i^1 x_1 + \delta_i^2 x_2 + \delta_i^3 x_3,\ \delta_i^j \in H.$$

Now, let us define,

$$\tilde{\Theta}_{\tilde{H}} := \{\kappa \in End_{\tilde{H}}(\mathbf{U}), \kappa(1) = 0\}.$$

Oviously,

$$\mathfrak{g} \subset \tilde{\Theta}.$$

Any $\kappa \in \tilde{\Theta}_{\tilde{H}}$ will correspond to $\kappa_i := \kappa(x_i) \in \mathbf{U}, i = 1, 2, 3,$ i.e. to a matrix of the type,

$$M := \begin{pmatrix} 0 & 0 & 0 & 0 \\ \kappa_1^0 & \kappa_1^1 & \kappa_1^2 & \kappa_1^3 \\ \kappa_2^0 & \kappa_2^1 & \kappa_2^2 & \kappa_2^3 \\ \kappa_3^0 & \kappa_3^1 & \kappa_3^2 & \kappa_3^3 \end{pmatrix},$$

where, $\kappa_i := (\kappa_i^0, \kappa_i^1, \kappa_i^2, \kappa_i^3), \kappa_i^j \in H$. Moreover, it is clear that $\tilde{\Theta}$ is a Lie algebra, and that $\mathfrak{g}$ is a natural sub-Lie algebra, of this matrix algebra.

Put,

$$\bar{o} = (1, o_1, o_2, o_3), \bar{p} = (1, p_1, p_2, p_3),$$

and consider now the 4-vectors,

$$\delta_i = (\delta_i^0, \delta_i^1, \delta_i^2, \delta_i^3), i = 1, 2, 3.$$

Suppose $\delta \in \mathfrak{g}$, then computing in $\mathbf{U}$, we find the formula,

$$\delta(x_i x_j - o_i x_j - p_j x_i + o_i p_j) = (\delta_i \cdot \bar{o}) x_j - (\delta_i \cdot \bar{o}) p_j + x_i(\delta_j \cdot \bar{p}) - o_i(\delta_j \cdot \bar{p})$$

which leads to,

$$\delta \in Der_H(\mathbf{U})$$

if and only if, $\delta(x_i x_j - o_i x_j - p_j x_i + o_i p_j) = 0$, therefore, if and only if,

$$\delta_i \cdot \bar{o} = \delta_i \cdot \bar{p} = 0, i = 1, 2, 3.$$

Given a point $\underline{t} = (o, p) \in \underline{H}$, let us compute the Lie algebra $\mathfrak{g}(\underline{t}) := Der_k(U(\underline{t}))$. Any element $\delta \in Der_k(U(\underline{t}))$ must have the form,

$$\delta(x_i) = \delta_i^0 + \delta_i^1 x_1 + \delta_i^2 x_2 + \delta_i^3 x_3,\ \delta_i^p \in k.$$

Put, as above, $\bar{o} = (1, o_1, o_2, o_3), \bar{p} = (1, p_1, p_2, p_3)$, and consider the 4-vectors

$$\delta_i = (\delta_i^0, \delta_i^1, \delta_i^2, \delta_i^3), i = 1, 2, 3.$$

As above, we find that $\delta \in Der_k(U(\underline{t}))$ if and only if $\delta_i \cdot \bar{o} = \delta_i \cdot \bar{p} = 0, i = 1, 2, 3$.

The tangent space $\Theta_{H,\underline{t}}$ of $\underline{H}$, at $\underline{t}$, is represented by the space of all pairs of 3-vectors, (ξ, ν) and we are interested in the action of $\mathfrak{g}(\underline{t})$ on this tangent space. Since all $U(o,p)$ are isomorphic, there must, for any tangent, (ξ, ν), exist an isomorphism of $k[\epsilon]$-algebras,

$$\eta : U(o,p) \otimes k[\epsilon] \to U(o + \xi\epsilon, p + \nu\epsilon),$$

commuting with the projection onto $U(o,p)$. It must be given by formulas,

$$\eta(x_i) = x_i + \kappa(x_i)\epsilon,\ \kappa(x_i) = \kappa_i^0 + \kappa_i^1 x_1 + \kappa_i^2 x_2 + \kappa_i^3 x_3 \in U(o,p), i = 1, 2, 3.$$

Put $\kappa_i := (\kappa_i^0, \kappa_i^1, \kappa_i^2, \kappa_i^3)$, then,

$$\kappa \in \tilde{\Theta}_k.$$

A little computation now shows that we must have,

$$\xi_i = \kappa_i \cdot \overline{o},\ \nu_i = \kappa_i \cdot \overline{p}, i = 1, 2, 3.$$

Therefore, given a point $\underline{t} = (o,p)$, and the corresponding generators $\{x_i, i = 1, 2, 3\}$ of $U(o,p)$, any $\kappa \in \tilde{\Theta}_k$ will correspond to $\kappa_i := \kappa(x_i) \in U(o,p), i = 1, 2, 3$, and therefore to a tangent of $\underline{H}$ at the point $\underline{t} = (o,p)$,

$$(\xi = \overline{\kappa} \cdot \overline{o},\ \nu = \overline{\kappa} \cdot \overline{p}) \in \Theta_{\underline{H},\underline{t}}.$$

We therefore find an exact sequence of bundles on $\underline{\tilde{H}}$,

$$0 \to \mathfrak{g} \to \tilde{\Theta}_{\tilde{H}} \to \Theta_{\tilde{H}} \to 0.$$

The Lie algebra $\mathfrak{g}$, is now seen to operate naturally on $\tilde{\Theta}_{\tilde{H}}$, corresponding to exactly the operation above, drawn from the deformation theory of algebras. Any $\delta \in \mathfrak{g}$, operates on $\kappa \in \tilde{\Theta}_{\tilde{H}}$ as $\delta(\kappa) = \delta \cdot \kappa - \kappa \cdot \delta$. Since $\delta \cdot \overline{o} = \delta \cdot \overline{p} = 0$, we find

$$\delta(\xi, \nu) = (\delta(\xi), \delta(\nu)) := (\delta(\kappa)\overline{o}, \delta(\kappa)\overline{p}).$$

Observe that, since $(\overline{o} - \overline{p}) = (o - p)$, the Lie algebra representation of $\mathfrak{g}$ on the tangent space $\Theta_{\tilde{H},\underline{t}}$, at the point $\underline{t} = (o,p)$, kills the subspace generated by the vectors $\{\xi = (o - p), \nu = (o - p))\}$.

If $o \neq p$, it follows that $\bar{o}$ and $\bar{p}$, are linearly independent, in a 4-dimensional vector space, therefore each vector $\delta_i, i = 1, 2, 3$ is confined to a 2-dimensional vector space. Consequently, $\mathfrak{g}(\underline{t}) := Der_k(U(\underline{t}))$ is of dimension 6. Using the isomorphism, $U(o,p) \simeq U(o - c, p - c)$, mentioned above, we may choose coordinates such that $o = (0,0,0), p = (1,0,0)$.

In fact, we may first put $c = o$, and reduce to the situation where $o = 0$, and p is a non-zero 3-vector. Any $\delta \in Der_k(U(o,p))$ will then be represented by a matrix of the form,

$$M := \begin{pmatrix} \delta_1^1 & \delta_1^2 & \delta_1^3 \\ \delta_2^1 & \delta_2^2 & \delta_2^3 \\ \delta_3^1 & \delta_3^2 & \delta_3^3 \end{pmatrix},$$

where $M(p) = 0$, and we know that the Lie structure is the ordinary matrix Lie-products. Now, clearly we may find a nonsingular matrix N such that $N(p) = (1,0,0)$, and the Lie algebra of matrices M, will be isomorphic to the Lie-algebra of the matrices, NMN^{-1}, which are those corresponding to $p = e_1 := (1,0,0)$, and we are working with $U(0, e_1)$. Notice that in this picture, the fundamental vector $\overline{op} = (1,0,0)$. With this we find that, $\delta \in \mathfrak{g}(\underline{t})$ imply,

$$\delta_i^0 = \delta_i^1 = 0, i = 1, 2, 3.$$

The following result is now easily seen.

Theorem 3.1. *The Lie algebra $\mathfrak{g}(\underline{t})$ is isomorphic to the Lie algebra of matrices of the form,*

$$\begin{pmatrix} 0 & \delta_1^2 & \delta_1^3 \\ 0 & \delta_2^2 & \delta_2^3 \\ 0 & \delta_3^2 & \delta_3^3 \end{pmatrix}.$$

The radical $\mathfrak{r}$, is generated by 3 elements, $\{u, r_1, r_2\}$, with

$$u = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, r_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, r_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $u \notin [\mathfrak{g}, \mathfrak{g}]$, $[u, r_i] = -r_i, [r_1, r_2] = 0$, and the quotient,

$$\mathfrak{g}(\underline{t})/\mathfrak{r} = \mathfrak{sl}(2).$$

with the usual generators h, e, f,

$$h = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, e = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}.$$

In particular, we find that $\mathfrak{sl}(2) \subset \mathfrak{g}(\underline{t})$.

Remark 3.2. *The tangent space at a point $\underline{t} \in \underline{H}$, decomposes into,*

$$\Theta_{H,\underline{t}} = \tilde{\Delta} \oplus \tilde{c}$$

where, $\tilde{\Delta} = \{(\xi, \xi)\}, \tilde{c} = \{(\nu, -\nu)\}, \xi, \nu \in k^3$. We know that the action of $\mathfrak{g}$ kills the tangent vectors of the type $((o-p), (p-o))$, or $((o-p), (o-p))$, and the operator h, generating the Cartan subalgebra of $\mathfrak{g}$, picks out two eigenvectors, together forming a unique homogenous coordinate system for $\Theta_{H,\underline{t}}$,

$$\{d_1, d_2, d_3\} \subset \tilde{\Delta}, \{c_1, c_2, c_3\} \subset \tilde{c},$$

where, $d_3 = (o - p, o - p), c_3 = (o - p, (p - o))$, d_1 and c_1 positive eigenvectors for h, and d_2 and c_2 negative eigenvectors for h.

With this done, we may write up the action of $\mathfrak{g}$ on Θ_H.

3.2. **Action of the Local Gauge Group in Canonical Coordinates.** If

$$o = (0, 0, 0), p = (1, 0, 0),$$

then we have seen that the Lie algebra $\mathfrak{g}(\underline{t})$ comes out isomorphic to the Lie algebra of matrices of the form,

$$\begin{pmatrix} 0 & \delta_1^2 & \delta_1^3 \\ 0 & \delta_2^2 & \delta_2^3 \\ 0 & \delta_3^2 & \delta_3^3 \end{pmatrix}.$$

The radical $\mathfrak{r}$, is generated by 3 elements, $\{u, r_1, r_2\}$, with

$$u = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, r_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, r_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $u \notin [\mathfrak{g}, \mathfrak{g}]$, and

$$[u, r_i] = -r_i, [r_1, r_2] = 0 \quad \text{and the quotient} \quad \mathfrak{g}(\underline{t})/\mathfrak{r} = \mathfrak{sl}(2).$$

with the usual generators h, e, f,

$$h = u_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, e = u_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, f = u_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

In particular, we find that $\mathfrak{sl}(2) \subset \mathfrak{g}(\underline{t})$.

Notice also that, in this case, the unique 0-tangent line at the point

$$\underline{t}_0 = (o, p), \ o = (0,0,0), \ p = (1,0,0),$$

killed by $\mathfrak{g}$, is represented by the pair $d_3 := ((1,0,0),(1,0,0))$, and the unique light-velocity line is represented by $c_3 := ((1,0,0),(-1,0,0))$.

Let $d_1 := ((0,1,0),(0,1,0))$, $d_2 := ((0,0,1),(0,0,1))$ and let $c_1 := ((0,1,0),(0,-1,0))$, $c_2 := ((0,0,1),(0,0,-1))$. Then $\{c_1, c_2, c_3, d_1, d_2, d_3\}$ is a basis for the tangent space $\Theta_{\underline{t}_0}$, and $\{d_1, d_2, d_3\}$ is a basis for $\tilde{\Delta}_{\underline{t}_0}$.

We observe that the generator h of the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ acts in this basis as,

$$h = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

which makes the choice of basis above canonical, i.e. determines $\{c_1, c_2, d_1, d_2\}$ as (± 1) eigenvectors of h, in $\tilde{c}$, resp. in $\tilde{\Delta}$. The actions of the gauge fields $\mathfrak{g}$ can then be given canonically: The generators, $h, e, f \in \mathfrak{sl}(2) \subset \mathfrak{g}$ act, in the above basis, like,

$$h = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$e = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$f = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The generators, $u, r_1, r_2 \in rad(\mathfrak{g})$ act, in the above basis, like,

$$u = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$r_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

$$r_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

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STRATA OF DISCRIMINANTAL ARRANGEMENTS

ANATOLY LIBGOBER AND SIMONA SETTEPANELLA

In memory of Brieskorn

Abstract. We give an explicit description of the multiplicities of codimension two strata of discriminantal arrangements introduced by Manin and Schechtman. As applications, we discuss the connection of these results with properties of Gale transform and we calculate the fundamental groups of the complements to discriminantal arrangements.

1. Introduction

In 1989, Manin and Schechtman ([13]) introduced a family of arrangements of hyperplanes generalizing classical braid arrangements, which they called the *discriminantal arrangements* (p.209 [13]). Such an arrangement $\mathcal{B}(n,k,\mathcal{A}^0), n,k \in \mathbf{N}$ for $k \geq 2$ depends on a choice $\mathcal{A}^0 = \{H_1^0,...,H_n^0\}$ of a collection of hyperplanes in the general position in $\mathbb{C}^k$, i.e., such that $\dim \bigcap_{i\in K, \text{Card } K=k} H_i^0 = 0$. It consists of parallel translates of $H_1^{t_1},...,H_n^{t_n}, (t_1,...,t_n) \in \mathbb{C}^n$ which fail to form a general position arrangement in $\mathbb{C}^k$. $\mathcal{B}(n,k,\mathcal{A}^0)$ can be viewed as a generalization of the pure braid group arrangement ([16]) with which $\mathcal{B}(n,1) = \mathcal{B}(n,1,\mathcal{A}^0)$ coincides. These arrangements have several beautiful relations with diverse problems in the areas such as combinatorics (see [13] and also [4], which is an earlier appearance of discriminantal arrangmements), the Zamolodchikov equation with its relation to higher category theory (see Kapranov-Voevodsky [8]), and the vanishing of cohomology of bundles on toric varieties ([17]).

The aim of this note is to study the dependence of $\mathcal{B}(n,k,\mathcal{A}^0)$ on the data $\mathcal{A}^0 = \{H_1^0,...,H_n^0\}$. Paper [13] concerns the arrangements $\mathcal{B}(n,k,\mathcal{A}^0)$ for which the intersection lattice is constant when $\mathcal{A}^0$ varies within a Zariski open set $\mathcal{Z}$ in the space of general position arrangements. However [13] does not describe the set $\mathcal{Z}$ explicitly. It was shown in [6] that, contrary to what was frequently stated (see for instance [15], sect. 8, [16] or [10]), the combinatorial type of $\mathcal{B}(n,k,\mathcal{A}^0)$ indeed depends on the arrangement $\mathcal{A}^0$. This was done by providing an example of a discriminantal arrangement with a combinatorial type distinct from the one which occurs when $\mathcal{A}^0$ varies within the Zariski open set $\mathcal{Z}$. Few years later, in [1], Athanasiadis provided a full description of combinatorics of $\mathcal{B}(n,k,\mathcal{A}^0)$ when $\mathcal{A}^0$ belongs to $\mathcal{Z}$. In particular, in this case, codimension 2 strata of $\mathcal{B}(n,k,\mathcal{A}^0)$ only have a multiplicity equal to 2 or $k+2$. Following [1], we call arrangements $\mathcal{A}^0$ in $\mathcal{Z}$ *very generic*.

Our main result describes a necessary and sufficient *geometric* condition on arrangement $\mathcal{A}^0$ assuring that $\mathcal{B}(n,k,\mathcal{A}^0)$ admits codimension 2 strata of multiplicity 3.

This condition is given in terms of a notion of *dependency* for the arrangement $\mathcal{A}_\infty$ in $\mathbb{P}^{k-1}$ of hyperplanes $H_{\infty,1},...H_{\infty,n}$ which are the intersections of projective closures of $H_1^0,...,H_n^0 \in \mathcal{A}^0$ with the hyperplane at infinity. Consider three groups of $s \in \mathbb{Z}_{\geq 1}$ hyperplanes in $\mathbb{P}^{2s-2}$ such that together these $3s$ hyperplanes are in general position in $\mathbb{P}^{2s-2}$. If the three subspaces corresponding to this split in groups, each being the intersection of hyperplanes in each group,

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span a hyperplane in $\mathbb{P}^{2s-2}$, we say that the arrangement of $3s$ hyperplanes in $\mathbb{P}^{2s-2}$ is *dependent* (Definition 3.3 in Section 3). This dependence condition defines a proper Zariski closed subset of the space of arrangements of $3s$ hyperplanes in $\mathbb{P}^{2s-2}$ in general position. Our main result (Theorem 3.9) shows that $\mathcal{B}(n,k,\mathcal{A}^0), k>1$ admits a codimenion two stratum of multiplicity 3 if and only if $\mathcal{A}_\infty$ is an arrangement in $\mathbb{P}^{k-1}$ admitting a restriction[1] which is a dependent arrangement .

Subsequently, in Section 4, we interpret this result in terms of the Gale transform. The relation between discriminantal arrangements and the Gale transform can be seen, at least implicitly, already in paper [6]. From this view point our result asserts an equivalence of certain types of collinearity: the dependency of $\mathcal{A}_\infty$ is equivalent to presence of dependencies in the Gale transform which in turn is equivalent to the presence of strata of multiplicity 3 in an arrangement $\mathcal{B}(n,k,\mathcal{A}^0)$. We shall give a direct verification of such equivalences using the interpretation of Gale transform of six-tuples of point in $\mathbb{P}^2$ in terms of del Pezzo surfaces given in [5]. More precisely, an arrangement $\mathcal{B}(6,3,\mathcal{A}^0)$ depends on arrangement at infinity $\mathcal{A}_\infty$, which in this case is a six-tuple of lines in $\mathbb{P}^2$, or equivalently, a six-tuple $(\mathcal{A}_\infty)^*$ of points in the dual plane. A general position arrangement $\mathcal{A}_\infty$ is dependent if and only if the del Pezzo surface, which is the blow up of $\mathbb{P}^2$ at six-tuple $(\mathcal{A}_\infty)^*$, admits an Eckardt point (cf. subsection 4.2). On the other hand, the interpretation of $\mathcal{B}(6,3,\mathcal{A}^0)$ via Gale transform, described in subsection 4.1, shows that presence in $\mathcal{B}(6,3,\mathcal{A}^0)$ of codimension two strata of multiplicity 3 is equivalent to the following: *the Gale transform of* $(\mathcal{A}_\infty)^*$ is a six-tuple $G(\mathcal{A}_\infty)^*$ such that blow up of $\mathbb{P}^2$ at $G(\mathcal{A}_\infty)^*$ is a del Pezzo surface admitting an Eckardt point. Hence the main result in the Theorem 3.9, in the case of discriminantal arrangments $\mathcal{B}(6,3,\mathcal{A}^0)$, becomes an invariance of existence of Eckardt points in the Gale transform. We show that this can be verified directly (see subsection 4.2).

Finally we supplement R.Lawrence's presentation ([10]) by giving a presentation of the fundamental group in the case of non very generic arrangements (i.e. for which $\mathcal{A}^0 \notin \mathcal{Z}$). In fact, we give calculations yielding the braid monodromy and hence a presentation of the fundamental group of the complement to a discriminantal arrangement in all cases.

Notice that in the case $k=1$, the complement to the discriminantal arrangement $\mathcal{B}(n,1)$ coincides with the configuration space of ordered n-tuples of points in $\mathbb{C}$. A natural generalization of this configuration space to the case $k \geq 2$ is the space of arrangements of hyperplanes in $\mathbb{C}^k$ in a general position. This is a Zariski open in the product of n copies of spaces of affine hyperplanes in $\mathbb{C}^k$. The fundamental group of this space is another natural candidate for a generalization of the pure braid group. Our result shows the difficulty with a calculation of this fundamental group: natural maps between spaces $\mathcal{B}(n,k,\mathcal{A}^0)$ for various n,k, which in the case $k=1$ lead to the presentation of the pure braid group fail to be locally trivial fibrations and hence fails to produce an exact sequence of fundamental groups. For example, intersections of projective closures of arrangements in $\mathbb{C}^k$ with the hyperplane at infinity, yields a map from the space of general position arrangements in $\mathbb{C}^k$ to the space of general position arrangements in $\mathbb{P}^{k-1}$. Our result shows that this map is a locally trivial fibration only over the space of very general position arrangements in $\mathbb{P}^{k-1}$. Calculation of the fundamental groups of spaces of general position arrangements of lines will be addressed elsewhere.

The content of the paper is the following. In Section 2, we introduce several notions used later and recall definitions from [13]. Section 3 contains one of the main results of this paper, Theorem 3.9, describing the codimension 2 strata of discriminantal arrangements having multiplicity 3 and showing an absence of codimension 2 strata having a multiplicity different from $2, 3$ and $k+2$. The Section 4 contains the interpretation of the results in Section 3 in terms of the Gale transform.

[1]Here restriction is the standard restriction of arrangements to subspaces as defined in [16] (see also equation (8) in this paper).

The last Section describes the braid mondromy and fundamental groups of the complements to discriminantal arrangements.

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2. Preliminaries

2.1. **Discriminantal arrangements.** Let $H_i^0, i = 1, ..., n$, be a general position arrangement in $\mathbb{C}^k, k < n$, i.e., a collection of hyperplanes such that $\dim \bigcap_{\substack{i \in K, \\ \mathrm{Card} K = k}} H_i^0 = 0$. The space of parallel translates $\mathbb{S}(H_1^0, ..., H_n^0)$ (or simply $\mathbb{S}$ when the dependence on H_i^0 is clear or not essential) is the space of n-tuples $H_1, ..., H_n$ such that either $H_i \cap H_i^0 = \emptyset$ or $H_i = H_i^0$ for any $i = 1, ..., n$. One can identify $\mathbb{S}$ with an n-dimensional affine space $\mathbb{C}^n$ in such a way that $(H_1^0, ..., H_n^0)$ corresponds to the origin.

We will use the compactification of the arrangement $(H_1^0, ..., H_n^0)$ obtained by viewing the ambient space $\mathbb{C}^k$ as $\mathbb{P}^k \setminus H_\infty$ endowed with a collection of hyperplanes $\bar{H}_i^0$ which are projective closures of affine hyperplanes H_i^0. The condition of genericity is equivalent to $\bigcup_i \bar{H}_i^0$ being a normal crossing divisor in $\mathbb{P}^k$. The space $\mathbb{S}$ can be identified with product $\mathcal{L}_1 \times \ldots \times \mathcal{L}_n$ where $\mathcal{L}_i \simeq \mathbb{C}$ is the pencil of hyperplanes spanned by H_∞ and H_i^0 parametrized by $\mathbb{P}^1$ with the deleted point. The latter corresponds to H_∞ and the origin to H_i^0. In particular, an ordering of hyperplanes in $\mathcal{A}$ determines the coordinate system in $\mathbb{S}$.

For a general position arrangement $\mathcal{A}$ in $\mathbb{C}^k$ formed by hyperplanes $H_i, i = 1, ..., n$, *the trace at infinity* (denoted by $\mathcal{A}_\infty$) is the arrangement formed by hyperplanes $H_{\infty,i} = \bar{H}_i^0 \cap H_\infty$.

An arrangement $\mathcal{A}$ (or its trace $\mathcal{A}_\infty$) determines the space of parallel translates $\mathbb{S}(H_1^0, ..., H_n^0)$ (as a subspace in the space of n-tuples of hyperplanes in $\mathbb{P}^k$).

For a general position arrangement $\mathcal{A}_\infty$, we consider the closed subset of $\mathbb{S}(H_1^0, ..., H_n^0)$ formed by those collections which fail to form a general position arrangement. This subset is a union of hyperplanes with each hyperplane corresponding to a subset $K = \{i_1, ..., i_{k+1}\} \subset \{1, ..., n\}$ and consisting of n-tuples of translates of hyperplanes $H_1^0, ..., H_n^0$ in which translates of $H_{i_1}^0, ..., H_{i_{k+1}}^0$ fail to form a general position arrangement (equations are given by (3) below). Such a hyperplane will be denoted D_K. The corresponding arrangement will be denoted $\mathcal{B}(n, k, \mathcal{A})$ and called *the discriminantal arrangement corresponding to* $\mathcal{A}$.

The cardinality of $\mathcal{B}(n, k, \mathcal{A})$ is equal to $\binom{n}{k+1}$. Each hyperplane D_K contains the k-dimensional subspace $\mathbb{T}$ of $\mathbb{S}(H_1^0, ..., H_n^0)$ formed by n-tuples of hyperplanes containing a fixed point in $\mathbb{C}^k$. Clearly, the essential rank, i.e. the dimension of the ambient space minus the dimension of intersection of the hyperlanes of the arrangement (cf. [20]), in the case of $\mathcal{B}(n, k, \mathcal{A})$ is $n - k$ and the arrangement induced by the arrangement of hyperplanes D_K in the quotient of $\mathbb{S}(H_1^0, ..., H_n^0)$ by $\mathbb{T}$ is essential. It is called *the essential part* of the discriminantal arrangement.

2.2. **Hyperplanes in** $\mathcal{B}(n, k, \mathcal{A})$**.** Recall that an arbitrary arrangement $\mathcal{A}$ of hyperplanes $W_1, ..., W_N \subset \mathbb{C}^k$ defines the canonical stratification of $\mathbb{C}^k$ in which strata are defined as follows. Let $L(\mathcal{A})$ be the intersection poset of subspaces in $\mathbb{C}^k$, each being the intersection of a collection of hyperplanes chosen among $W_1, ..., W_N$, and for each $P \in L(\mathcal{A})$, let $\Sigma_P = \{i \in \{1, ..., N\} | P \in W_i\}$ be the set of indices of hyperplanes W_i such that $P = \cap_{i \in \Sigma_P} W_i$. Vice versa, given a subset $\Sigma \subset \{1, ..., N\}$, we denote by w_Σ the subspace $w_\Sigma = \cap_{i \in \Sigma} W_i$. The stratum of P is the

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submanifold of $\mathbb{C}^k$ defined as follows:

$$\mathcal{S}_P = P \setminus \bigcup_{\Sigma_P \subset \Sigma} w_\Sigma. \tag{1}$$

If an arrangement $\mathcal{A} = \{W_1, ..., W_N\}$ in $\mathbb{C}^k$ is *in the general position* then the finite subset in $\mathbb{C}^k$, consisting of 0-dimensional strata, has cardinality $\binom{N}{k}$ and its elements are in one to one correspondence with the subsets of $\{1, ..., N\}$ having cardinality k.

The multiplicity of a point $p \in \mathcal{S}_P$ considered as a point on the subvariety $\bigcup_{i=1,...,N} W_i$ in $\mathbb{C}^k$ is constant along the stratum. We call it *the multiplicity* of the stratum $\mathcal{S}_P$. It is equal to cardinality of the set Σ_P.

As we noted, the hyperplanes of $\mathcal{B}(n, k, \mathcal{A})$ correspond to subsets of cardinality $k+1$ in $\{1, \ldots, n\}$. Their equations can be obtained as follows. Let $K, \mathrm{Card}\ K = k+1$, be a subset in $\{1, ..., n\}$ and let

$$\alpha_1^j y_1 + \ldots + \alpha_k^j y_k = x_j^0, \quad j \in \{1, ..., n\} \tag{2}$$

be the equation of hyperplane H_j^0 of arrangement $\mathcal{A} = \{H_1^0, \ldots, H_n^0\} \in \mathbb{C}^n \setminus B(n, k, \mathcal{A})$ in selected coordinates $y_1, ..., y_k$ in $\mathbb{C}^k$. The hyperplanes H_j, $j \in K$, of an arrangement in $\mathbb{S}$ with equations $\alpha_1^j y_1 + \ldots + \alpha_k^j y_k = x_j, \quad j \in K$, will have non-empty intersection iff

$$det \begin{pmatrix} \alpha_1^1 & ... & \alpha_k^1 & x_1 \\ ... & ... & ... & ... \\ \alpha_1^{k+1} & ... & \alpha_k^{k+1} & x_{k+1} \end{pmatrix} = 0. \tag{3}$$

This provides a linear equation in $x_j, j = 1, ..., k+1$, for the hyperplane D_K corresponding to K.

Let J be a subset in $\{1, \ldots, n\}$ of cardinality a,

$$D_J = \{(H_1, \ldots, H_n) \in \mathbb{S} \text{ such that } \cap_{i \in J} H_i \neq \emptyset\} \tag{4}$$

and

$$\mathcal{P}_{k+1}(J) = \{K \subset J \text{ such that Card } K = k+1\}. \tag{5}$$

Then

$$D_J = \bigcap_{K \in \mathcal{P}_{k+1}(J)} D_K \tag{6}$$

is intersection of $\binom{a}{k+1}$ hyperplanes. In particular D_J, Card $J \geq k+1$, is a linear subspace and the multiplicity of $\bigcup_{\mathrm{Card}\ K=k+1} D_K$ at its generic point is $\binom{a}{k+1}$. Moreover, codim D_J is $a-k$.

2.3. **Projections of discriminantal arrangements.** Let $\Xi \subset \{1, ..., n\}$ be a subset of the set of indices and let $\mathbb{S}(\Xi) \subset \mathbb{S}$ be the subspace of the space of translates of hyperplanes of a general position arrangement $H_1^0, ..., H_n^0$ consisting of translates of hyperplanes with indices in Ξ. Let us consider the projection $p_\Xi : \mathbb{S} \to \mathbb{S}(\Xi)$ obtained by omitting from a collection of translates from $\mathbb{S}$, the translates of hyperplanes with indices outside of Ξ. The image of a subspace $D_J, J \subset \{1, ..., n\}$ is a proper subspace iff Card $J \cap \Xi \geq k+1$ and in fact $p_\Xi(D_J) = D_{J \cap \Xi}$. In particluar, if D_J is a hyperplane i.e. Card$J = k+1$ then $p_\Xi(D_J)$ is a hyperplane if and only if $J \subset \Xi$.

The maps p_Ξ restricted to the complement to the discriminantal arrangement $\mathbb{S} \setminus \mathcal{B}(n, k, \mathcal{A})$ for $n \geq k+3$ are locally trivial fibrations if and only if $k = 1$. Due to their local triviality they play a prominent role in the study of braid arrangements (cf. [2]). The failure of local triviality for $k \geq 2$ can be seen as follows. Consider, for example, the simplest case $k = 2$. Let $\mathcal{A} = \{l_1^0, .., l_4^0, l_5^0\}$ be a quintuple of lines in $\mathbb{C}^2$ and $\Xi = \{1, 2, 3, 4\} \subset \{1, 2, 3, 4, 5\}$. The fiber of $p_\Xi : \mathbb{C}^5 \to \mathbb{C}^4$ at a generic point $\{l_1, ..., l_4\}$ in the complement $\mathbb{C}^4 \setminus \mathcal{B}(4, 2, \mathcal{A} \setminus \{l_5^0\})$ is given by all general position arrangements $\{l_1, ..., l_4, l_5\}$ such that l_5 does not contain any of the 6 intersection points of $l_i \cap l_j$, $1 \le i < j \le 4$, that is $\mathbb{C}$ with deleted 6 points. On the other hand, one can select a generic point $\{l_1, ..., l_4\}$ in the complement $\mathbb{C}^4 \setminus \mathcal{B}(4, 2, \mathcal{A} \setminus \{l_5^0\})$ such that one of the diagonals of quadrangle formed by lines $l_1, ...l_4$ will be parallel to l_5^0. Hence the fiber of p_Ξ at such a point will be $\mathbb{C}$ with only 5 points deleted. Similar special configurations are inevitable for all $n \ge k + 3, k \ge 2$. This failure of local triviality brings serious complication in the study of the topology of the complement $\mathbb{S} \setminus \mathcal{B}(n, k, \mathcal{A})$ (see the last section for a description of the fundamental groups).

Note, that some recent works (see for example [7]) refer to discriminantal arrangements in a more narrow sense than used in this paper i.e. as the restriction arrangements to the fibers of p_Ξ given explicitly as

$$p^{-1}_{\{1,...,l\}}(t_1, ..., t_l) = \{(z_1, ..., z_{n-l}) | z_i = z_j \ \text{ or } \ z_i = t_k, \ k = 1, ..., l, \ i, j = 1, ..., n - l\}. \tag{7}$$

3. Codimension two strata having multiplicity 3

In this section we describe necessary and sufficient conditions which should be satisfied by the trace at infinity $\mathcal{A}_\infty$ in order that the corresponding discriminantal arrangement will have codimenion two strata having multiplicity 3. We shall start with the list of notations used throughout this section, some already introduced in the last section.

Notations 3.1. *Let's fix the following notations.*

- $\mathcal{A}^0$ *is a general position arrangement of* n *hyperplanes in* $\mathbb{C}^k$ *(we use* $\mathcal{A}^0$ *for the fixed arrangement to distinguish it from* $\mathcal{A}$ *which will denote a general translate of* $\mathcal{A}^0$*),*
- *for each* K *subset of* $\{1, ..., n\}$ *of* Card $K = k + 1$*,* $D_K \subset \mathbb{C}^n$ *will denote the hyperplane in* $\mathcal{B}(n, k, \mathcal{A}^0)$ *corresponding to the subset* K*.*
- *As in subsection 2.1, hyperplanes in the trace at infinity* $\mathcal{A}_\infty$ *are denoted by* $H_{\infty,i}$*.*
- *Let* $s \ge 2$*.* $K_i, i = 1, 2, 3$*, denote subsets of* $\{1, ..., n\}$ *such that*
$$\text{Card } K_i = 2s, \quad \text{Card } K_i \cap K_j = s, \quad i \ne j, \quad \bigcap_{i=1}^{i=3} K_i = \emptyset$$
(in particular Card $\bigcup K_i = 3s$*).*

Lemma 3.2. *Let* $s \ge 2$*,* $n = 3s, k = 2s - 1$*. Let* $\mathcal{A}^0$ *be a general position arrangement of* n *hyperplanes in* $\mathbb{C}^k$ *and let* $K_i, i = 1, 2, 3$ *be a triple of subsets of* $\{1, ..., n\}$ *as described in notations 3.1 above. Consider the triple of codimension* s **subspaces** *of the hyperplane at infinity* H_∞ *defined as follows:* $H_{\infty,i,j} = \cap_{s \in K_i \cap K_j} H_{\infty,s} \cap H_\infty, i \ne j$*. If subspaces* $H_{\infty,i,j} \subset H_\infty$ *span a proper subspace in* H_∞ *then* codim $\bigcap D_{K_i} = 2$*. Otherwise this codimension is equal to 3.*

This lemma suggests the following:

Definition 3.3. *A general position arrangement in* $\mathbb{P}^{2s-2}, s \ge 2$*, is called* **dependent** *if it is composed of* $3s$ *hyperplanes* W_i *which can be partitioned into 3 groups, each containing* s *hyperplanes, such that 3 subspaces of dimension* $s - 2$ *, each being intersection of hyperplanes in one group, span a proper subspace in* $\mathbb{P}^{2s-2}$*. We call these three* $s - 2$*-dimensional subspaces* **dependent**.

Remark that, with this terminology, the assumption of Lemma 3.2 is that the trace at infinity of $\mathcal{A}^0$ is a dependent general position arrangement.

If $s = 2$ in Lemma 3.2, then $H_{\infty,i,j}$ are points in the 2-dimensional space $\mathbb{P}^2$. The condition that these points span a proper subspace in H_∞, i.e., are collinear, corresponds to the case of

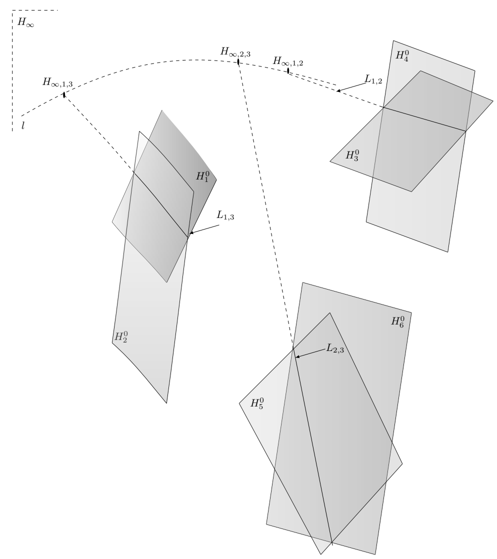

FIGURE 1.

Falk's example of the special discriminantal arrangement in [6]. We shall illustrate the argument in Lemma 3.2 by a discussion of this particular case since the argument for the proof of this lemma is a generalization of the argument used in Example 3.4.

Example 3.4. *Let us consider the case* $n = 6$ *and* $k = 3$*, that is a general position arrangement* $\mathcal{A}^0 = \{H_1^0, \dots, H_6^0\}$ *in* $\mathbb{C}^3$*. In Lemma 3.2, this corresponds to* $s = 2$ *and, after possible relabelling,* $K_1 = (1,2,3,4), K_2 = (3,4,5,6), K_3 = (1,2,5,6)$*. Then subspaces* $L_{i,j} = \bigcap_{s \in K_i \cap K_j} H_s^0$ *are lines* $L_{1,3}^0 = H_1^0 \cap H_2^0, L_{1,2}^0 = H_3^0 \cap H_4^0, L_{2,3}^0 = H_5^0 \cap H_6^0$ *with closures* $\bar{L}_{i,j}^0$*. In this case, (i.e., when* $\dim H_\infty = 2$*), the assertion of Lemma 3.2 is that the points* $H_{\infty,i,j} = \bar{L}_{i,j}^0 \cap H_\infty$ *span a line* l *in* H_∞*. In other words, the points* $H_{\infty,i,j}$ *are collinear if and only if codim* ${}_{i=1,2,3} \bigcap D_{K_i} = 2$ *(see Figure 1).*

Indeed, an arrangement $\mathcal{A} = \{H_1, \dots, H_6\}$ *of translates of planes in* $\mathcal{A}^0$ *is a point in* $D_{K_1} \cap D_{K_2}$ *iff pairwise intersections* $L_{1,3} \cap L_{1,2}$ *and* $L_{1,2} \cap L_{2,3}$ *in* $\mathbb{C}^3$ *of lines*

$$L_{1,3} = H_1 \cap H_2, \;\; L_{1,2} = H_3 \cap H_4 \;\;\text{ and }\;\; L_{2,3} = H_5 \cap H_6$$

are non-empty. We claim that the collinearity condition implies that two pairs of these three lines $L_{i,j}$ *are coplanar if and only if all the three are. Indeed, since* $\mathcal{A}$ *consists of translates of planes in* $\mathcal{A}^0$ *the line* $L_{i,j}$ *has the same point at infinity* $H_{\infty,i,j}$ *as does the line* $L_{i,j}^0$*. The condition that* $H_{\infty,i,j}$ *span a line* $l \in H_\infty$ *implies that the closure of any plane containing two lines* $L_{i,j}$ *intersects* H_∞ *in* l*. That is two planes containing respectively the pairs of lines* $L_{1,3}, L_{1,2}$ *and* $L_{1,2}, L_{2,3}$ *are coincident. This implies that lines* $L_{1,3}$ *and* $L_{2,3}$ *have a non-empty intersection i.e.* $\bigcap_{i=1,2,5,6} H_i \neq \emptyset$ *and hence* $\mathcal{A} \in D_{K_3}$*.*

Vice versa, if the points $H_{\infty,i,j}$ *aren't colinear, then it is possible to find configurations in which, for example,* $L_{1,3}$ *intersects both* $L_{1,2}$ *and* $L_{2,3}$*, but* $L_{1,2} \cap L_{2,3} = \emptyset$*, i.e.* $\mathcal{A} \in D_{K_1} \cap D_{K_3}$ *and* $\mathcal{A} \notin D_{K_2}$*.*

Proof of Lemma 3.2. Consider first the case when subspaces $H_{\infty,i,j}$ span a proper hyperplane in H_∞ which we shall denote $\mathcal{H}$. Note that dim $H_{\infty,i,j} = s-2$ and, as a consequence of $\mathcal{A}^0$ being in the general position, these subspaces do not intersect. In particular, the subspace which they span has a dimension greater than $2s-4$, i.e., either it is a hyperplane or it is all the space H_∞.

Let $\mathcal{A} = \{H_i\}$ be an arrangement in $\mathbb{C}^k = \mathbb{C}^{2s-1}$ which belongs to D_{K_1} and D_{K_2} (recall that hyperplanes H_i are translates of hyperplanes $H_i^0 \in \mathcal{A}^0$). Hence $\bigcap_{i\in K_1} H_i \neq \emptyset$ and $\bigcap_{i\in K_2} H_i \neq \emptyset$. We claim that $\bigcap_{i\in K_3} H_i \neq \emptyset$, which would imply that codim $\bigcap_{i=1,2,3} D_{K_i} =$ codim $\bigcap_{i=1,2} D_{K_i} = 2$. Let $L_{i,j} = \bigcap_{s\in K_i\cap K_j,} H_s, (i<j)$. Note that the codimension of each linear subspace $L_{i,j}$ of $\mathbb{C}^{2s-1}$ is equal to s and $L_{i,j}\cap H_\infty = H_{\infty,i,j}$.

Since $\mathcal{A} \in D_{K_1}$, the subspaces $L_{1,2}$ and $L_{1,3}$ have a non-empty intersection. Therefore they span in $\mathbb{C}^{2s-1}$ a hyperplane which we denote as $\mathcal{L}_1$. The intersection of $\mathcal{L}_1$ with H_∞ is the hyperplane $\mathcal{H}$ spanned by $H_{\infty,1,2}$ and $H_{\infty,1,3}$. The hyperplane $\mathcal{L}_1$ is spanned by the intersection point $L_{1,2}\cap L_{1,3}$ and the hyperplane $\mathcal{H}$.

Similarly, since $\mathcal{A} \in D_{K_2}$, both $L_{1,2}$ and $L_{2,3}$ have a point in common, they span the hyperplane $\mathcal{L}_2$ spanned by this point and the above hyperplane $\mathcal{H}$ which can be described as the plane spanned by $H_{\infty,1,2}$ and $H_{\infty,1,3}$. Both hyperplanes $\mathcal{L}_1$ and $\mathcal{L}_2$ contain $L_{1,2}$ and $\mathcal{H}$. Hence they coincide. Therefore $L_{1,3}$ and $L_{2,3}$, being both $(s-1)$-dimensional subspaces in $\mathcal{L}_1 = \mathcal{L}_2, \dim\mathcal{L}_1 = \dim\mathcal{L}_2 = 2s-2$, must have a point in common and hence $\mathcal{A} \in \bigcap_{i=1,2,3} D_{K_i}$.

Now assume that the triple $H_{\infty,i,j}$ spans H_∞. Let $\mathcal{A} \in D_{K_1}\cap D_{K_2}$ be sufficiently generic in this space. We show that it does not belong to D_{K_3}. Consider the family of s codimensional subspaces in $\mathbb{C}^{2s-1}$ which compactification intersects the hyperplane at infinity at $H_{\infty,2,3}$. The selection of $\mathcal{A}$ determines subspaces $L_{1,2}, L_{1,3} \subset \mathbb{C}^{2s-1}$ which have a common point and moreover the subspace $L_{2,3}$ which intersects $L_{1,2}$. Since triple $H_{\infty,i,j}$ is *not* in a hyperplane in H_∞, $\mathbb{P}^{2s-1}$, compactifying $\mathbb{C}^{2s-1}$ is spanned by $H_{\infty,2,3}$ and the closures of subspaces $L_{1,2}, L_{1,3}$. Hence the generic subspace L of codimension s containing $H_{\infty,2,3}$ and intersecting $L_{1,2}$ will have an empty intersection with $L_{1,3}$. The corresponding arrangement $\mathcal{A}'$ having L as the subspace $L_{2,3}$ will not belong to D_{K_3} but will be in $D_{K_1}\cap D_{K_2}$. This shows that $D_{K_1}\cap D_{K_2} \notin D_{K_3}$. ☐

Let's briefly recall here the basic notion of the restriction of an arrangement. For a subset $\mathcal{A}' \subseteq \mathcal{A}$, let us denote by $X_{\mathcal{A}'} = \bigcap_{H\in\mathcal{A}'} H$ the intersection of its hyperplanes. The arrangement

(8) $$\mathcal{A}^{X_{\mathcal{A}'}} = \{H\cap X_{\mathcal{A}'} \mid H \in \mathcal{A}\setminus\mathcal{A}', H\cap X_{\mathcal{A}'} \neq \emptyset\}$$

is called **a restriction** of $\mathcal{A}$ to $X_{\mathcal{A}'}$. Restrictions of $\mathcal{A}$ are in one-to-one correspondence with the splits $\mathcal{A} = \mathcal{A}'\bigcup\mathcal{A}''$ of the set of hyperplanes in $\mathcal{A}$ into a disjoint union. If $\mathcal{A}' = \emptyset$, then the restriction arrangement coincides with $\mathcal{A}$.

Via the restriction of arrangements, the Lemma 3.2 leads to other examples of discriminantal arrangements having codimension two strata with multiplicity 3.

Lemma 3.5. *Let $\mathcal{A}^0$ be a general position arrangement of n hyperplanes in $\mathbb{C}^k$ and $\mathcal{A}'$ be a subarrangement of t hyperplanes in $\mathcal{A}^0$. Assume that the trace at infinity of the restriction $\mathcal{A}^{X_{\mathcal{A}'}}$ of $\mathcal{A}^0$ to $X_{\mathcal{A}'}$ is a dependent arrangement of $3s = n-t$ hyperplanes (in the sense of Def. 3.3). Then $\mathcal{B}(n,k,\mathcal{A})$ admits a codimension two stratum of multiplicity 3.*

Proof. Assume that $\mathcal{A}' = \{H_1^0,...,H_t^0\} \subset \mathcal{A}^0 = \{H_1^0,...,H_n^0\}$ satisfies the conditions of lemma, i.e., the restriction $\mathcal{A}^{X_{\mathcal{A}'}}$ is an arrangement of $3s = n-t$ hyperplanes in $X_{\mathcal{A}'} \simeq \mathbb{C}^{n-t}$ and its trace at infinity $\mathcal{A}_\infty^{X_{\mathcal{A}'}}$ is dependent, i.e., the discriminantal arrangement $\mathcal{B}(n-t,k-t,\mathcal{A}^{X_{\mathcal{A}'}})$ admits a codimension 2 stratum having the multiplicity 3. The dimension of this stratum is $3s-2$ where $n-t = 3s$ and $k-t = 2s-1$. By Lemma 3.2, there are subsets

$$K_i, i=1,2,3, \text{ Card } K_i = 2s = k-t+1 \text{ of } \{t+1,...,n\}$$

such that $D_{K_i} \in \mathcal{B}(n-t,k-t,\mathcal{A}^{X_{\mathcal{A}'}})$ satisfy codim $\bigcap_{i=1,2,3} D_{K_i} = 2$. The above $(3s-2)$-dimensional stratum of the discriminantal arrangement of $n-t$ hyperplanes in $\mathbb{C}^{k-t}$ is the transversal intersection of two submanifolds (each being an open subset of a linear subspace) of $\mathbb{C}^n$. One is the stratum of discriminantal arrangement $\mathcal{B}(n,k,\mathcal{A}^0)$ having the dimension $3s-2+t$ formed by hyperplanes $D_{K_i\cup\{1,...,t\}}, i=1,2,3$, and another is the intersection of t hyperplanes in $\mathbb{S}(H_1^0,...,H_n^0)$ defined by the vanishing of coordinates corresponding to $H_1^0,...,H_t^0$. Hence the multiplicity of this stratum of $\mathcal{B}(n,k,\mathcal{A}^0)$ equals 3. This yields the lemma. □

Corollary 3.6. *If $k\geq 3$ and $n\geq k+3$, then there exists a general position arrangement of n hyperplanes in $\mathbb{C}^k$ such that the corresponding discriminantal arrangement admits a codimension two stratum of multiplicity 3.*

Proof. To apply Lemma 3.5, for a pair (n,k) such that there exist integers $t\geq 0, s\geq 2$ satisfying

$$n=3s+t \qquad k=2s-1+t \tag{9}$$

consider a general position arrangement $\mathcal{A}^0$ of n hyperplanes in $\mathbb{C}^k$ such that the restriction of trace $\mathcal{A}_\infty$ of $\mathcal{A}^0$ on intersection of its t hyperplanes is dependent. By Lemma 3.5, the discriminantal arrangement corresponding to such $\mathcal{A}^0$ will admit the required stratum. Given $(n,k)\in\mathbb{N}^2$, the relation (9) has a unique solution $s=n-k-1, t=3k-2n+3$ which satisfies $s\geq 2, t\geq 0$ iff

$$k+3\leq n\leq \frac{3}{2}(k+1), \qquad k\geq 3. \tag{10}$$

Note that given an arrangement $\mathcal{B}(n,k,\mathcal{A})$ admitting the codimension 2 strata of multiplicity 3, an extension of $\mathcal{A}$ to the arrangement of $N\geq n$ hyperplanes by adding sufficiently generic hyperplanes yields an arrangement $\mathcal{A}'$ such that $\mathcal{B}(n,k,\mathcal{A})$ is the intersection of $\mathcal{B}(N,k,\mathcal{A}')$ and the coordinate subspace. It follows that $\mathcal{B}(N,k,\mathcal{A}')$ admits strata of codimension 2 and the multiplicity 3 as well. On the other hand, for $n=k+2$, $\mathcal{B}(k+2,k,\mathcal{A})$ has only one stratum of multiplicity $k+2$ i.e., the inequality $n\neq k+3$ is sharp. □

The following example illustrates the above two lemmas.

Example 3.7. *Let $\mathcal{A}_\infty$ be a general position arrangement of 8 hyperplanes $H_{\infty,i}$ in $\mathbb{P}^4$ and $\mathcal{A}_\infty^X$ its restriction to the plane $X=H_{\infty,7}\cap H_{\infty,8}$. The restricted arrangement $\mathcal{A}_\infty^X$ is an arrangement in general position since $\mathcal{A}_\infty$ is in general position.*

Assume that the double points $H_{\infty,1}\cap H_{\infty,2}\cap H_{\infty,7}\cap H_{\infty,8}$, $H_{\infty,3}\cap H_{\infty,4}\cap H_{\infty,7}\cap H_{\infty,8}$, $H_{\infty,5}\cap H_{\infty,6}\cap H_{\infty,7}\cap H_{\infty,8}$ are co-linear.

Consider the hyperplanes

$$D_{1,2,3,4,7,8},\ D_{3,4,5,6,7,8},\ D_{1,2,5,6,7,8}$$

in a discriminantal arrangement $\mathcal{B}(8,5,\mathcal{A})$ corresponding to such $\mathcal{A}_\infty$ and the hyperplanes

$$D'_{1,2,3,4},\ D'_{3,4,5,6},\ D'_{1,2,5,6}$$

in the discriminantal arrangement in 3-space $H_7\cap H_8$ for a generic choice of hyperplanes H_7,H_8 intersecting the hyperplane at infinity at $H_{\infty,7},H_{\infty,8}$ respectively. Then the arrangement $\mathcal{A}$ of 8 hyperplanes in $\mathbb{C}^5$ including H_7,H_8 has a common point if and only if the arrangement of 6 planes in 3-space $H_7\cap H_8$ has a common point. Hence

$$\dim D_{1,2,3,4,7,8}\cap D_{3,4,5,6,7,8}\cap D_{1,2,5,6,7,8}=2+\dim D'_{1,2,3,4}\cap D'_{3,4,5,6}\cap D'_{1,2,5,6}=6 \tag{11}$$

(the last equality uses the Example 3.4). Hence the discriminantal arrangement $\mathcal{B}(8,5,\mathcal{A})$ has a codimension two stratum of multiplicity 3.

This case illustrates the case considered in Theorem 3.9 (2) below, corresponding to the dependent restriction arrangement of $\mathcal{A}_\infty$ given by hyperplanes $H_{\infty,i}\cap H_{\infty,7}\cap H_{\infty,8}$, $i=1,\dots,6$ and $s=2$.

The next Lemma will be useful in the proof below showing the absence of codimension 2 strata having the multiplicity 4.

Lemma 3.8. *For $s \geq 2$, there is no quadruple of subspaces $V_i \subset \mathbb{P}^{3s-2}, i = 1,2,3,4$ having dimension $2s-2$ such that intersections $P_{i,j} = V_i \cap V_j, i \neq j$ satisfy*
a) each $P_{i,j}$ has dimension $s-2$
b) any pair $P_{i,j}, P_{i,k}, i \neq j \neq k \neq i$ spans a hyperplane in V_i, and
c) all three, $P_{i,j}, P_{i,k}, P_{i,l}$, belong to a hyperplane in V_i.

Proof. We shall start with the case $s = 2$. Assume that a configuration as in Lemma 3.8 does exist and consider a quadruple of planes $V_i, i = 1, ..., 4$ in $\mathbb{P}^4$ such that
a) any two intersect at a single point,
b) all 6 points $P_{i,j} = V_i \cap V_j$, $i \neq j$, obtained in this way are distinct, and
c) all three points, $P_{i,j}, P_{i,k}, P_{i,l}$, are colinear, i.e., span a line L_i.

For a fixed k, the triple of points $P_{i,j}$,$i,j \neq k$, outside of V_k, determines the triple of lines $L_i \subset V_i, i \neq k$ spanned by points $P_{i,j}, P_{i,l}$, $i,j,l \neq k$. These lines $L_i, i \neq k$ by their definition are pairwise concurrent ($L_i \cap L_j = P_{i,j}$) and hence belong to a plane H. By assumption c), for each $i \neq k$, the three points $P_{k,i} = V_k \cap V_i$ are points on lines L_i distinct from $P_{i,j}, P_{i,l}$. Hence H and V_k have 3 distinct non-colinear points in common and therefore $H = V_k$, but this contradicts $\dim V_k \cap V_i = 0$.

Now consider the case $s > 2$. Similar to the above, $(s-2)$-dimensional subspaces $P_{i,j} = V_i \cap V_j$ of $\mathbb{P}^{3s-2}$ determine the subspaces $L_i \subset V_i, i \neq k$ (for a fixed k) each being spanned by pairs

$$P_{i,j}, P_{i,l}, i, j, l \neq k$$

which are outside of V_k. Each L_i is a hyperplane in V_i (i.e. $\dim L_i = 2s-3$). Moreover, the dimension of the subspace H of $\mathbb{P}^{3s-2}$ spanned by $L_{i,j}, L_{i,l}, i,j,l \neq k$, is $3s-4$. The subspace H can be described as the subspace of $\mathbb{P}^{3s-2}$ spanned by triple of subspaces $P_{i,j}, i, j \neq k$. Now by our assumption c), V_k contains an $(s-2)$-dimensional subspace of $L_{i,j}, i,j \neq k$, i.e., $P_{i,k}$. The subspace hence is also a subspace of H. This implies that $V_k \subset H$. The dimension of intersection $L_{i,j}$ and V_k, which are both subspaces of H, is equal to $(2s-3)+(2s-2)-(3s-4) = s-1$ and hence $\dim V_i \cap V_k = s-1$. This is a contradiction. ☐

Now we are ready for the main result of this section. It describes the codimension 2 strata of discriminantal arrangements having the multiplicity 3 and shows an absence of codimension 2 strata having the multiplicity 4 (with obvious exceptions).

Theorem 3.9. *Let $\mathcal{A}_\infty$ be a general position arrangement of hyperplanes in $\mathbb{P}^{k-1}$ which is the trace at infinity of a general position arrangement $\mathcal{A}^0$ in $\mathbb{C}^k$.*
1. The arrangement $\mathcal{B}(n,k,\mathcal{A}^0)$ has $\binom{n}{k+2}$ codimension 2 strata of multiplicity $k+2$.
2. There is a one-to-one correspondence between
a) the dependent restrictions of subarrangements of $\mathcal{A}_\infty$, and
b) triples of hyperplanes in $\mathcal{B}(n,k,\mathcal{A}^0)$ for which the codimension of their intersection is equal to 2.
3. There are no codimension 2 strata having the multiplicity 4 unless $k = 2$. All codimension 2 strata of $\mathcal{B}(n,k,\mathcal{A}^0)$ not mentioned in part 1, have a multiplicity which is either 2 or 3 (the latter corresponding to triples of hyperplanes in b).
4. The codimension 2 strata of $\mathcal{B}(n,2,\mathcal{A}^0)$ is independent of $\mathcal{A}^0$.

Proof. The statement (1) follows immediately from the discussion after (6) in section 2.2. If $J \subset \{1, ..., n\}$ is a subset of cardinality $k+2$, then D_J is a codimension 2 subspace in $\mathbb{C}^n$ and belongs to $k+2$ hyperplanes $D_K, K \subset J$.

Next we shall determine the conditions on three different sets of $k+1$ indices under which codim $D_{K_i} \cap D_{K_j} \cap D_{K_l} = 2$.

Consider first the case when sets K_i, K_j, K_l, each having the cardinality $k+1$, are such that for one of them, say K_i, one has $K_i \setminus (K_i \cap (K_j \cup K_l)) \neq \emptyset$, i.e., one of the set in this triple is not in the union of other two. If $r \in K_i \setminus (K_i \cap (K_j \cup K_l))$, then the hyperplanes in an arrangement $\mathcal{A} \in D_{K_j} \cap D_{K_l}$ with indices different from the indices in $K_j \cup K_l$ can be chosen as arbitrary parallel translates of hyperplanes in $\mathcal{A}^0$, while $H_r \in \mathcal{A}', \mathcal{A}' \in D_{K_i}$ is fixed by condition $\mathcal{A}'$ being in D_{K_i} and the selection of hyperplanes with indices different from r but in K_i. Hence $D_{K_i} \cap D_{K_j} \cap D_{K_l} \neq D_{K_j} \cap D_{K_l}$, i.e., codim $D_{K_i} \cap D_{K_j} \cap D_{K_l} = 3$.

Now let us consider the alternative to the case considered in the previous paragraph. Hence we have a triple K_i, K_j, K_l such that

$$K_i = (K_i \cap K_j) \bigcup (K_i \cap K_l) \tag{12}$$

for any permutation of (i, j, k). Condition (12) for $k = 2$ implies that either $\mathrm{Card}(K_i \cap K_j) = 2$ or $\mathrm{Card}(K_i \cap K_l) = 2$ that is either $D_{K_i} \cap D_{K_j} = D_{K_i \cup K_j}$ or respectively $D_{K_i} \cap D_{K_l} = D_{K_i \cup K_l}$. Since this imples that $\mathrm{Card}(K_i \cup K_j) = 4$ (resp. $\mathrm{Card}(K_i \cup K_l) = 4$), we obtain that $D_{K_i} \cap D_{K_j}$ (resp. $D_{K_i} \cap D_{K_l}$) is a codimension 2 subspace of multiplicity $4 = k + 2$ and part (4) follows.

Let $L_{\alpha,\beta} = (K_\alpha \cap K_\beta) \setminus \bigcap_{s=i,j,k} K_s, t = \mathrm{Card} \bigcap_{\alpha=i,j,k} K_\alpha, l_{\alpha,\beta} = \mathrm{Card}\ L_{\alpha,\beta}$. Then (12) implies that $K_\beta \setminus \bigcap_{\alpha=i,j,k} K_\alpha = L_{\alpha,\beta} \bigcup L_{\beta,\gamma}$ and since Card $K_i = k+1$ we have

$$l_{\alpha,\beta} + l_{\beta,\gamma} + t = k+1, \quad \alpha \neq \beta \neq \gamma. \tag{13}$$

Using these relations for allowable permutations of subscripts, yields:

$$l_{\alpha,\beta} = \frac{k+1-t}{2} \quad \alpha \neq \beta, \alpha, \beta \in \{i, j, k\}. \tag{14}$$

For a triple of subsets $K_i, K_j, K_l, \mathrm{Card}\ K_i \cap K_j \cap K_l = t$ and a fixed arrangement $\mathcal{A}$, consider the map of the spaces of translates:

$$\mathbb{S}(H_1^0, ..., H_n^0) \to \mathbb{C}^t = \mathbb{S}(..., H_r^0, ...) \quad r \in K_i \cap K_j \cap K_l$$

which assigns to a collection of n parallel translates $H_1^{t_1}, ..., H_n^{t_n}$ of $H_1^0, ..., H_n^0$ in $\mathbb{C}^k$, the intersections of the hyperplanes with indices outside of $K_i \cap K_j \cap K_l$ with the linear subspace which is the intersection of t hyperplanes with indices in $K_i \cap K_j \cap K_l$.

This map has as its fiber over the set of translates $H_\beta^{t_\beta}$, the space

$$\mathbb{S}(..., H_\alpha^0 \cap (\bigcap_{\beta \in K_i \cap K_j \cap K_l} H_\beta^{t_\beta}), ...), \qquad \alpha \in [1, ..., n] \setminus K_i \cap K_j \cap K_l$$

of translates in the $(k-t)$-dimenional space $\bigcap_{\beta \in K_i \cap K_j \cap K_l} H_\beta^{t_\beta}$. If s is the dimension of the family of arrangements which is the intersection of hyperplanes $D_{K_\alpha}, \alpha = i, j, l$ then the dimension of the family of restrictions of arrangements to $\mathbb{C}^{k-t}$ is $s - t$. Hence

$$\mathrm{codim}\ D_{K_i} \cap D_{K_j} \cap D_{K_l} = \mathrm{codim}\ D_{K_i \setminus \bigcap K_\alpha} \cap D_{K_j \setminus \bigcap K_\alpha} \cap D_{K_l \setminus \bigcap K_\alpha} \qquad \alpha = i, j, l \tag{15}$$

where the intersection on the right is taken in the space of parallel translates in $\bigcap_{j \in \bigcap K_i} H_j^0$.

Clearly $t < k$, and in the case when $t = k-1$, we have $l_{i,j} = 1$, i.e., Card $\bigcup K_i = k+2$ and we are in the case (1), i.e., the codimension 2 stratum has the multiplicity $k+2$. If $t = 0$, then we have the case considered in Lemma 3.2 and we also see from this lemma that the intersection of $D_{K_i}, i = 1, 2, 3$ has a codimension two stratum if and only if the assumptions of the theorem are fulfilled. The rest of the part (2) of the theorem follows from Lemma 3.5 applied to the restriction on $\bigcap_{\alpha \in K_i \cap K_j \cap K_l} H_\alpha^0$ and the relation (15) (with $s = l_{\alpha,\beta}$).

Now consider the existence of a codimension 2 strata of multiplicity 4. Suppose that such stratum exists and $K_i, i = 1, ..., 4$ are the corresponding subsets of $\{1, ..., n\}$. By the quadruples analog of restriction (15), it is enough to consider the case $\bigcap_{i=1,...,4} K_i = \emptyset$. Let

$$l_{i,j,m} = \text{Card } K_i \cap K_j \cap K_m.$$

Then for any i, Card $K_i \cup \bigcap_{j \neq i} K_j =$ Card $\bigcup K_i$, i.e., $l_{i,j,m} + k + 1$ is independent of (i, j, m). Hence one infers from (13) the relation $l_{i,j,m} = \frac{k+1}{3}$.

Note that codim $\bigcap_{i=1,...,4} D_{K_i} = 2$ if and only if codim $D_{K_{i_1}} \cap D_{K_{i_2}} \cap D_{K_{i_3}} = 2$ for all 4 triple $1 \leq i_j \leq 4$ of distinct integers. Applying part (2) of the theorem to each triple i_1, i_2, i_3, one infers the existence of a quadruple of subspaces as in Lemma 3.8. Hence this lemma implies part (3).

□

Corollary 3.10. *If a discriminantal arrangement $\mathcal{B}(n, k, \mathcal{A})$ satisfies $n > \frac{3}{2}(k + 1)$ and admits a codimension 2 stratum of multiplicity 3, then there exists a proper subarrangement $\mathcal{A}' \subset \mathcal{A}$ such that $\mathcal{B}(n, k, \mathcal{A}')$ admits a codimension 2 stratum of multiplicity 3.*

Proof. It follows immediately from the above theorem and inequality (10). □

3.1. **Numerology of singularities of generic plane sections.** Theorem 3.9 contains a complete description of combinatorics of codimension 2 strata of discriminantal arrangements. Indeed, the possible multiplicities of codimension two strata are $\binom{k+2}{k+1}$, 3 and 2. The number of points of multiplicity 3 is the number of triples of strata satisfying condition *2a)*. It is an interesting problem to determine the number of triple points which $\mathcal{B}(n, k, \mathcal{A})$ can have. It is clear from Theorem 3.9 that this number can be arbitrary large when $n \to \infty$, though even the precise asymptotic is not clear.

4. The Gale transform and codimension two strata

4.1. **The Gale transform and associated sets.** In this subsection we shall discuss interpretation of discriminantal arrangements using the Gale transform. Recall the following:

Definition 4.1. *Let V be a vector space over $\mathbb{C}$, $\dim V = k$, $l_i \in V^*, i = 1, ..., n$, be n vectors in the dual of the vector space V and let*

$$0 \to V \xrightarrow{L} \mathbb{C}^n \to W \to 0 \tag{16}$$

be the exact sequence in which $L(v) = (l_1(v), ..., l_n(v))$. The Gale transform of collection l_i is the collection $m_i \in W, i = 1, ..., n$, of images of the vectors e_i of the standard basis in $\mathbb{C}^n$.

The following is suggested by an argument in [6] (see also [17] and [3]).

Proposition 4.2. *Let $\mathcal{A}$ be a central arrangement of* Card $\mathcal{A} = n$ *in a k-dimensional vector space V such that the corresponding arrangement in $\mathbb{P}^{k-1}$ is in the general position. Let l_i be the elements in V^* corresponding to the hyperplanes in $\mathcal{A}$. The essential part of the discriminantal arrangement consists of hyperplanes in W spanned by collections of $n-k-1$ vectors of the Gale transform of vectors $l_i \in V^*$.*

Proof. Let $f_1, ..., f_k$ be a basis in V, $x_j, j = 1, ..., k$, be the coordinates in this basis, and let

$$l_i = \sum a^i_j x_j, j = 1, ..., k, i = 1, ..., n$$

be the equations of the hyperplanes of $\mathcal{A}$. Denote by $A = \{a^i_j\}$ the corresponding matrix. Translates of hyperplanes $l_{i_1} = 0, ..., l_{i_{k+1}} = 0$, by $c_{i_1}, ..., c_{i_{k+1}}$ respectively, have a non-empty intersection if and only if the system of equations $\sum a^{i_s}_j x_j = c_{i_s}, s = i_1, ..., i_{k+1}$, has a solution.

This takes place if and only if the projection $\pi_{i_1,...,i_{k+1}}(c)$ of the point $c = (c_1, \dots, c_n) \in \mathbb{C}^n$ on the subspace of $\mathbb{C}^n$ spanned by the vectors $e_{i_1}, ..., e_{i_{k+1}}$ belongs to the image of projection $\pi_{i_1,...,i_{k+1}}(L(V))$ of $L(V)$, which is equivalent to

$$c \in H_{\pi_{i_1,\dots,i_{k+1}}} \simeq Span(V, Ker_{\pi_{i_1,\dots,i_{k+1}}}) \tag{17}$$

(here by an abuse of notation, we identified V with its image $L(V)$ in $\mathbb{C}^n$). The hyperplanes $H_{\pi_{i_1,\dots,i_{k+1}}}$ form the discriminantal arrangement in $\mathbb{C}^n$ and the relation $V \subset H_{\pi_{i_1,\dots,i_{k+1}}}$ shows that the essential part of discriminantal arrangement is its restriction to $W = \mathbb{C}^n/V$. The inclusion (17) is equivalent to $c \in Ker_{\pi_{i_1,\dots,i_{k+1}}} mod\ V$. The image $Ker_{\pi_{i_1,\dots,i_{k+1}}} \in \mathbb{C}^n/V = W$ is spanned by the images of the Gale transform of $l_i, i = 1, ..., n$, and is the hyperplane in the essential part of discriminantal arrangement. ☐

Next recall the classical notion of associated sets (cf. [5], Ch.III):

Definition 4.3. *Let V,W be vector spaces such that $\dim V = k, \dim W = n-k$. Let $f_1, ..., f_k$ and $g_1, ..., g_{n-k}$ be the bases of V, W respectively. The set of vectors $l_1, ..., l_n$ in V and $m_1, ..., m_n$ in W are called associated if the matrices X and Y of coordinates of $l_i, i = 1, ..., n$, and $m_j, j = 1, ..., n$, satisfy:*

$$X \cdot \Lambda \cdot^t Y = 0, \tag{18}$$

where Λ is a diagonal matrix.

The sets in V and W are associated if and only if one is the Gale transform of another (see the discussion in [5] p.33 where the association is discussed in projective setting, for example).

4.2. **Discriminantal arrangements of planes in $\mathbb{C}^3$ and the Gale transform.** One can ask for the meaning of a codimension 2 strata with multiplicity three in discriminantal arrangements described in the Theorem 3.9 in terms of the Gale transform. In the case, $n = 6, k = 3$, one has a geometric interpretation (see [5] for geometric interpretations for some other values n, k) of the Gale transform which allows one to show the following:

Proposition 4.4. *The existence of a partition of 6-tuples of points in $\mathbb{P}^2$ into 3 pairs, each pair defining a line, and such that these lines are concurrent lines, is an invariant of the Gale transform.*

Remark 4.5. *After replacing hyperplanes by the points of a projective dual space, this proposition is equivalent to the case $n = 6, k = 3$ of the main theorem (cf. also Example 3.4). This equivalence follows from the classical description of the Gale transform recalled in the proof below.*

The more general case, considered in the Lemma 3.2, can be interpreted as the following property of the Gale transform $(\mathbb{P}^{2s-2})^{\times 3s} \to (\mathbb{P}^s)^{\times 3s}$.

The condition that there is a partition of $3s$ points in $\mathbb{P}^s$ into 3 groups of s points, each set spanning a hyperplane in $\mathbb{P}^s$ and that, moreover, such that these hyperplanes belong to a pencil, is equivalent to the condition that the Gale transform of this set of points in $\mathbb{P}^{2s-2}$ admits a partition into 3 groups of cardinality s such that the triple of $(s-1)$-dimensional subspaces, each spanned by an s-tuple in $\mathbb{P}^{2s-2}$, have a non-empty intersection.

This restatement follows immediately from the dualization of hyperplanes of the general position arrangement. Indeed, $3s$ hyperplanes of the general position arrangement in $\mathbb{C}^{2s-1}$ considered in Lemma 3.2 define $3s$ hyperplanes $\mathbb{P}^{2s-2}$ or equivalently $3s$ points in the dual projective space. The assumption of the dependency of $3s$ hyperplanes in $\mathbb{P}^{2s-2}$, after the dualization is equivalent to requiring that 3 $s-1$-dimensional subspaces $\eta_1^{s-1}, \eta_2^{s-1}, \eta_3^{s-1} \subset \mathbb{P}^{2s-2}$ each spanned by one of 3 subsets of cardinality s (i.e. subsets $K_i \cap K_j$ in notations of definition 3.3) have non-empty intersection. Since $n - k - 1 = 3s - (2s - 1) - 1 = s$, by Proposition 4.2, the hyperplanes *of the essential part of the discriminant arrangment are spanned by s -subsets of the set of $3s$ points in $\mathbb{P}^s$. Lemma 3.2 states that the dependency condition is equivalent to the existence of the triple of hyperplanes in the discriminantal arrangement belonging to a pencil of hyperplanes which gives our claim.*

It would be interesting to have a geometric description of the Gale transform allowing one to show this directly for $s > 2$.

Proof. We shall use the projective setting which, in this case, relates 6-tuples of points in $\mathbb{P}^2$ to another 6-tuples in another copy of $\mathbb{P}^2$. Recall that the smooth cubic surfaces in $\mathbb{P}^3$ (i.e., the del Pezzo surfaces of degree 3) can be viewed as blow ups of a 6-tuples of points in $\mathbb{P}^2$ and the classes of projective equivalence of 6-tuples of points in $\mathbb{P}^2$ correspond to isomorphism classes of cubic surfaces. The 6-tuple of points in $\mathbb{P}^2$ is obtained by contracting 6 of 27 lines having pairwise empty intersections. In terms of the blow up of 6 points in $\mathbb{P}^2$, each of 27 lines is one of the following:

1. 6 exceptional curves of the blow up;
2. proper preimages of 15 lines defined by pairs of points;
3. proper preimages of 6 quadrics determined by a 5 points subset of the blown up 6-tuple.

6-tuples of lines as above on a cubic surface V correspond to the following homology classes in $H^2(V, \mathbb{Z})$:

$$h_i, i = 1, ..., 6, \qquad (h_i, h_j) = -\delta^i_j. \tag{19}$$

Given such 6-tuple h_i, one has a unique additional 6-tuple h'_i characterized by the following: together with h_i the collection h'_i form a double six, i.e., the following relations are satisfied:

$$h_i h_j = h'_i h'_j = -\delta^i_j, \quad h_i h'_j = 1 - \delta^i_j. \tag{20}$$

Using the description of 27 lines above in terms of lines and quadrics on $\mathbb{P}^2$ corresponding to the lines on a del Pezzo surface, the second component h'_i of a double six, in which the first component h_i is formed by the 6-tuple of exceptional curves, can be described as follows. The 6-tuple h'_i consists of the proper preimages of quadrics labeled in the way which assigns to (the class of) exceptional curve h_P contracted to a point $P \in \mathbb{P}^2$ (the class of) the quadric h'_P passing through points of the 6-tuple of points in $\mathbb{P}^2$ distinct from P.

Now the existence of partition of 6-tuples as in Proposition 4.4 is equivalent to existence of Eckardt point (i.e., a point common to a triple of lines on cubic surface) not involving the exceptional curves and to show Proposition 4.4 one needs to show that such Eckardt point exists also for the second component of a double six. But each line containing a pair of points P, P' on the plane $\mathbf{P}$ obtained by contraction of a 6-tuples of disjoint exceptional curves on del Pezzo surface will be passing through a pair of 6 points on the plane $\mathbf{P}'$ obtained by contracting proper preimages of 6 quadrics on $\mathbf{P}$. Indeed, such a line through $P \in \mathbf{P}$ will intersect the proper preimage on the blow up of $\mathbf{P}$ of the quadric not containing P at exactly one point (corresponding to the intersection point with this quadric distinct from the blown up point). Since the blow up of 6 points and contracting proper preimages of 6 quadrics determined by these 6 points is an isomorphism on the complement to quadrics which contains the concurrency point of the triple of lines, the claim follows. □

5. Fundamental groups of the complements to discriminantal arrangements

5.1. **Nilpotent completion of the fundamental group.** In this section we shall describe the nilpotent completion of $\pi_1(\mathcal{B}(n, k, \mathcal{A}))$ in the case when $\mathcal{A}$ is not very generic and the corresponding discriminantal arrangement admits a codimension two strata of multiplicity 3. This is a direct consequence of [9] Prop.2.2 (see also [13]).

Proposition 5.1. *Completion of the group ring* $\mathbb{C}[\pi_1(\mathbb{C}^n \setminus \mathcal{B}(n,k,\mathcal{A}))]$ *with respect to the powers of the augmentation ideals is the quotient of the algebra of non-commutative power series*

$$\mathbb{C} << X_J >>, \ J \in \mathcal{P}_{k+1}(\{1,\dots,n\})$$

by the two-sided ideal generated by relations

(i) $[X_J, \sum_I X_I]$ *for a pair of subsets* $J \in \mathcal{P}_{k+1}(K)$, *with summation over*

$$I \in \mathcal{P}_{k+1}(K), \ K \subset \{1,...,n\}, \text{Card } K = k+2.$$

(ii) $[X_J, X_I + X_J + X_K]$ *where* I, J, K *are subscripts corresponding to triples of hyperplanes in the discriminantal arrangement, such that corresponding hyperplanes in* $\mathcal{A}_\infty$ *satisfy dependency condition of Theorem 3.9 (2a).*

(iii) $[X_J, X_K]$ *for any pair of sets with* Card $J, K \geq k+3$ *and such that there does not exist subset* I *such that triple* I, J, K *satisfies the conditions of Theorem 3.9 (2a).*

5.2. **Braid monodromy of discriminantal arrangements and** $\pi_1(\mathbb{S} \setminus \mathcal{B}(n,k,\mathcal{A}))$**.** We shall describe the fundamental group of the complement to a discriminantal arrangement. In fact, we shall obtain the braid monodromy of the generic plane section of discriminantal arrangement, which by the classical van Kampen procedure yields the presentation of the fundamental group.

We describe the braid monodromy of the generic section of $\mathcal{B}(n,k,\mathcal{A})$ in terms of a collection of orderings of hyperplanes of $\mathcal{B}(n,k,\mathcal{A})$ constructed in terms of equations (2) of arrangement $\mathcal{A}$ of hyperplanes in the general position $H^0_j, j = 1,...,n$ as follows. The generic plane section of $\mathcal{B}(n,k,\mathcal{A})$ can be described as subset of $\mathbb{C}^2$ with coordinates (s,t) depending on a choice of generic a^n, b^n, c^n (specifying the plane section) consisting of points (s,t) such that the rank of the $(k+1) \times n$ matrix:

$$\begin{pmatrix} \alpha^1_1 & ... & \alpha^1_k & a^1 t + b^1 s + c^1 \\ ... & ... & ... & ... \\ \alpha^n_1 & ... & \alpha^n_k & a^n t + b^n s + c^n \end{pmatrix} \tag{21}$$

is maximal. This plane is given in $\mathbb{S}$ by

$$x_i = a^i t + b^i s + c^i. \tag{22}$$

For fixed $\alpha^i_j, a^i, b^i, c^i, i = 1,...,n, j = 1,...,k$, and generic (t,s), the rank of this matrix is $k+1$. For a generic fixed s, there is a finite collection $t_1(s) < ... < t_{\binom{n}{k+1}}(s)$ of real numbers such that the rank of (21) is k: each $t_i(s)$ corresponds to a $k+1$ subset of $\{1,...,n\}$ labeling a hyperplane in $\mathcal{B}(n,k,\mathcal{A})$. Moreover there will be finite collection of real numbers $s_1 < ... < s_N, N \geq \binom{n}{k+2}$ such that for these s there will be strictly less than $\binom{n}{k+1}$ constants t for which the rank of (21) is less than $k+1$. In fact, these values s correspond to projections on the s-coordinate of multiple points of the arrangement of lines restriction of $\mathcal{B}(n,k,\mathcal{A})$ to the (s,t) plane. In particular, to each s_i corresponds a subset P_i in the sequence $1,..., \binom{n}{k+1}$ corresponding to the set of $(k+1)$-subsets yielding the same value $t(s_i)$. The cardinality of the subset P_i is either $k+2$, 3 or 2 (according to the multiplicity of the singular point corresponding to s_i).

Recall (cf. for example [14] or [11]) that the real line $Im(s) = 0$ in the complex s-line $\mathbb{C}_s \simeq \mathbb{C}$ can be used to define in a canonical way the generators of the fundamental group $\pi_1(\mathbb{C}_s \setminus \bigcup_{i=1}^N s_i)$ of the complement of N points in $\mathbb{C}$. In details, the generator corresponding to the point $s_i, i = 1, ..., N$ is the loop from a base point $s_0, Im(s_0) = 0, s_0 << 0$, to the point s_i, circumventing each $s_j, j < i$ as a semi-circle into the halfplane $Im(s) < 0$ and returning back to s_0 after making the full circle around s_i. The braid monodromy for such a path is the product of factors corresponding to each $s_j, j \leq i$, i.e., the half twist β_{P_j} corresponding to P_j for $j < i$ and the full twist $\beta^2_{P_i}$.

Theorem 5.2. *1. The braid monodromy of a generic plane section corresponding to section (22) of $\mathcal{B}(n,k,\mathcal{A})$ is given by*

$$\Pi_{1\le k\le N}\Gamma_i \quad where \quad \Gamma_i=\beta_{P_1}^{-1}...\beta_{P_{k-1}}^{-1}\beta_{P_k}^2\beta_{P_{k-1}}...\beta_{P_2}\beta_{P_1}. \tag{23}$$

2. The fundamental group $\pi_1(\mathbb{C}^n\setminus\mathcal{B}(n,k,\mathcal{A}))$ has the following presentation:

$$\Gamma_i(\delta_j)=\delta_j \quad j=1,...,\binom{n}{k+1}, i=1,...,N. \tag{24}$$

These statements are the standard applications of the results from the theory of braid monodromy (cf., among others, [14],[11]). Different presentations can be obtained via Salvetti's presentation or Randell's presentation for complement of hyperplane arrangements (see [19], [18]). For $\mathcal{A}^0$ very generic, this yields a presentation equivalent to the one given in [10].

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ON A DISCRIMINANT KNOT GROUP PROBLEM OF BRIESKORN

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Abstract. Quite some time ago, at the singularity conference at Cargèse 1972, Brieskorn asked the following question:

> Is the local fundamental group $\pi_1^s(S-D)$ of the discriminant complement inside the semi-universal unfolding S of an isolated hypersurface singularity constant for s in the μ-constant stratum Σ_E?

We review this question and give an affirmative answer in case of singular plane curve germs of multiplicity at most 3.

1. Introduction

The question of Brieskorn was published in Astérisque 7-8, *Colloque sur les singularités en géometrie analytique*. In that article Brieskorn gives a summary of the problems and questions he considers central in the investigation of monodromy, and their answers which – as he writes – will help much to arrive at a more profound understanding, [Bri73].

In Brieskorn's view the local fundamental group of the discriminant complement – the discriminant knot group as it will be called in the present article – lies at the heart of the study of the algebraic monodromy and the intersection lattice of the Milnor fibre and should soon reveal to contain more or less the same amount of information.

This optimism probably resulted in the spectacular success in the study of simple hypersurface singularities where Brieskorn himself made important contributions, [Bri71a, Bri71b, BS72]. For the simple singularities the algebra, the geometry and the combinatorial group theory are most closely tied together and hope was widespread to get similar results for more general singularities under suitable forms of relaxation.

However, the topology of the discriminant complement remains a mystery up to the present day, and only little progress has been made on the problems Brieskorn addressed to it.

In this article we will review the problem stated in the abstract

> Is the discriminant knot group $\pi_1^s(S-D)$ of an isolated hypersurface singularity constant for s in the μ-constant stratum?

At the time of writing the evidence in favour of a positive answer had two aspects. First in the case of simple singularities the answer is trivially positive. Second, the homomorphic image under algebraic monodromy is constant along the μ-constant stratum.

On the other hand an article of Pham [Pha73] presented at the very conference at Cargèse was interpreted by Brieskorn as evidence in favour of a negative answer: Pham showed that the topological type of the generic discriminant curve of certain plane curve singularities of multiplicity $m = 3$ is not constant along the μ-constant stratum.

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In fact, Brieskorn proposes to study the discriminant knot group by the local Zariski hyperplane theorem as proved by Lê and Hamm [HL73]:

$$\pi_1(S - D) \quad \cong \quad \pi_1(H - H \cap D),$$

where H is a plane in S parallel to a generic plane $H_0 \neq H$ through the origin. $H_0 \cap D$ is called *a generic discriminant curve* and $H \cap D$ a corresponding *unfolded generic discriminant curve*. The topological type of the former is constant along the μ-constant stratum if and only if the topological type of the latter is constant along that stratum.

Therefore the result of Pham shows that the line of argument which Brieskorn had in mind cannot work.

In this article, however, we will follow Brieskorns strategy and bridge the gap by using a stronger form of the Zariski van Kampen method applicable to more general plane sections of the discriminant.

We will turn the Pham examples into evidence for a positive answer to Brieskorns problem by the following theorem.

Theorem 1. *Suppose f is a plane curve singularity of multiplicity at most 3, then the discriminant knot group is constant along the μ-constant stratum.*

As remarked before, in the case of simple singularities the claim trivially holds true. By classification this settles the case of multiplicity 2 and of plane curve singularities of Milnor number at most 8.

As a direct corollary we can sharpen the result of [Lön10]. Suppose f is topologically equivalent to a plane curve singularities of Brieskorn-Pham type of multiplicity 3

$$f \;\sim_{top}\; y^3 + x^{\nu+1} \;\text{ for some } \nu \geq 2,$$

then f is a μ-constant deformation of the Brieskorn-Pham singularity and has the same distinguished Dynkin diagram

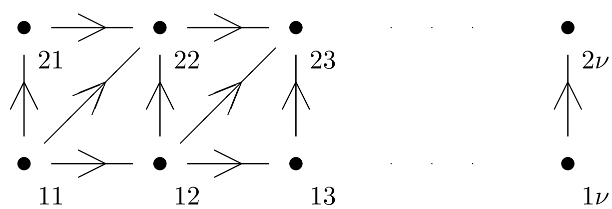


FIGURE 1. Dynkin diagram of $y^3 + x^{\nu+1}$

where the set V of vertices is ordered by the lexicographic order of their double indices and the set E of oriented edges contains the pair of corresponding vertices only in their proper order.

Since the discriminant knot group by Theorem 1 is the same for f and the Brieskorn-Pham singularity we get from [Lön10, Thm1.1]. (Does it extend to all cases in [Lön07]?)

Theorem 2. *Suppose f is topologically equivalent to a Brieskorn-Pham polynomial $y^3 + x^{\nu+1}$. Then its discriminant knot group is presented by*

$$\left\langle t_i,\ i \in V \;\middle|\; \begin{array}{cl} t_i t_j = t_j t_i & (i,j),(j,i) \notin E \\ t_i t_j t_i = t_j t_i t_j & (i,j) \in E \\ t_i t_k t_j t_i = t_j t_i t_k t_j & (i,j),(i,k),(j,k) \in E \end{array} \right\rangle.$$

A step beyond the result of this article might address the case of unimodal hypersurface singularities. Possibly it is sufficient to look at the generic discriminant curve, since in the cases not covered by our result, Greuel [Gre77, Gre78] has shown that at least the number of cusps of the unfolded generic discriminant curve is constant along the μ-constant stratum.

2. Review of the results of Pham

In his article [Pha73] Pham provides a careful analysis of the generic discriminant curve in case of a plane curve singularity of multiplicity 3

$$f \quad = \quad y^3 - P(x)y + Q(x).$$

While skipping his calculation which we will mimic in the next section, here we only want to introduce the minimum of notation to state his results and draw some first conclusions towards the proof of our main theorem.

In addition to the well-known Milnor number

$$\mu \quad = \quad \dim \mathbb{C}[X,Y]/\langle f_x, f_y\rangle,$$

Pham needs the analytic *σ-invariant* associated to the ideal generated by f and its derivatives up to second order

$$\sigma \quad = \quad 1 + \dim \mathbb{C}[X,Y]/\langle f, f_x, f_y, f_{xx}, f_{xy}, f_{yy}\rangle.$$

He also gives some useful formulas for calculations:

Lemma 3 ([Pha73] §1,p.366)**.** *If f is a function germ as above, the analytic invariant σ is given by*

$$\sigma \quad = \quad \min\{\operatorname{ord} P, \operatorname{ord} Q'\}$$

and the Milnor number is given by

$$\mu \quad = \quad \operatorname{ord}(3Q'^2 - PP'^2).$$

Instead of citing the main result in its full strength, which is a complete topological classification of generic discriminant curves, we distill the essence, what we will need below.

Proposition 4 (cf. [Pha73])**.** *The topological type of the generic discriminant curve only depends on the topological invariant μ and the analytic invariant σ.*

Corollary 5. *The topological type of the unfolded generic discriminant curve only depends on the topological invariant μ and the analytic invariant σ.*

Proof. The topological type of the generic discriminant curve determines its Milnor number $\tilde{\mu}$. The number $\mu + \tilde{\mu} - 1$ is the sum of three times the number of cusps and twice the number of nodes of any corresponding unfolded discriminant curve, cf. [Pha73]. Since both cardinalities are upper semi-continuous and the set with constant σ and μ is connected, they are both constant along this set, and so is the topological type of the unfolded generic discriminant curve. □

Proposition 6. *If f is a plane curve singularity of multiplicity* 3 *and*

$$f \quad \not\sim_{top} \quad y^3 + x^{\nu+1} \qquad \text{for all } \nu,$$

then the discriminant knot group is constant along the μ-constant stratum.

Proof. According to the classification by Arnol'd [Arn76] f is simple, of type $J_{k,i},\ k \geq 2, i > 0$, or of type $E_{6k+1},\ k \geq 2$. In the simple case the claim is trivially true as was remarked before.

In case of $f \in J_{k,i},\ k \geq 2, i > 0$ Arnol'd has given a normal form which by an analytic equivalence – more precisely by a Tschirnhaus transformation – can be put in the form considered by Pham:

$$\begin{aligned} & y^3 + y^2 x^k + a(x)x^{3k+i}, \quad \operatorname{ord} a = 0 \\ \sim_{an} \quad & y^3 - \frac{1}{3} y x^{2k} + \frac{2}{27} x^{3k} + a(x) x^{3k+i}. \end{aligned}$$

According to the lemma $\sigma = 2k$ and thus σ is independent of $a(x)$.

In case of $f \in E_{6k+1}$ the normal form of Arnol'd is in the form considered by Pham, so from

$$y^3 + yx^{2k+1} + a(x)x^{3k+2},$$

$\sigma = 2k+1$ independent of $a(x)$ is immediate by the lemma again.

In both cases we conclude with the corollary that the topological type of the unfolded generic discriminant is constant along the μ-constant stratum. Therefore the fundamental groups of their complements also do not change. The local Zariski theorem on hyperplane sections [HL73] identifies these groups with the discriminant knot groups which are thus shown to be constant along the μ-constant stratum. □

3. Existence of suitable non-generic discriminant curves

In this section we follow the path traced by Pham to obtain a non-generic reduced discriminant curve which does not change its topological type under a small deformation along the μ-constant stratum, although the analytic invariant σ changes.

In fact, as the last section will prove, it will suffice to do so for the Brieskorn-Pham polynomials.

We recall from [Pha73] the construction of the discriminant curve in direction of a linear perturbation by a polynomial $p(x)y + q(x)$. The critical set of the unfolding of

$$f \quad = \quad f(x,y) \quad = \quad y^3 - P_0(x)y + Q_0(x)$$

by $-u + t(p(x)y + q(x))$ is a curve in 4-space and the corresponding discriminant curve is obtained by projection along the coordinates x, y, algebraically by elimination of x, y from

$$u \;=\; y^3 - Py + Q \tag{1}$$

$$0 \;=\; -P'y + Q' \tag{2}$$

$$0 \;=\; 3y^2 - P, \tag{3}$$

where $P = P_0 + tp$, $Q = Q_0 + tq$.

But as Pham does, we take the detour by the projection along u and y which is easier to obtain. The parameter u is eliminated by the sole use of (1) and from (2) and (3) we can eliminate y to get

$$3Q'^2 - PP'^2 \;=\; 0. \tag{4}$$

First we consider for an additional parameter $s = 0$ or s which is sufficiently small the case

$$Q \;=\; Q_0 \;=\; x^{\nu+1}, \quad q = 0, \quad P \;=\; P_0 + tp \;=\; sx^\sigma + tx.$$

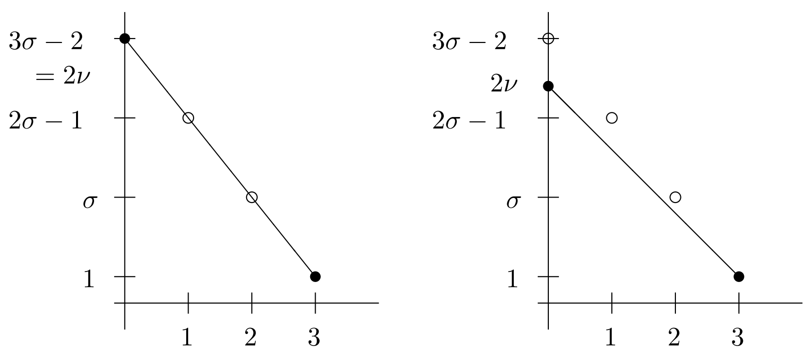
The first step according to Pham is to compute the branches $x(t)$. Recall the expansion of (4) in terms of the variable t according to

$$3(Q_0' + tq')^2 - (P_0 + tp)(P_0' + tp')^2 \;=\; A_0 + A_1 t + A_2 t^2 + A_3 t^3.$$

In the current situation we get the following vanishing orders of the A_i under the assumption of s sufficiently small.

$$\begin{array}{lclclc}
 & & \text{expansion} & & & \text{vanishing order} \\
A_0 & = & 3Q_0'^2 - P_0 P_0'^2 & = & 3(\nu+1)^2 x^{2\nu} - s^3\sigma^2 x^{3\sigma-2} & \mu = \min\{2\nu, 3\sigma-2\} \\
A_1 & = & -xP_0'^2 - 2P_0P_0' & = & -s^2\sigma(\sigma+2)x^{2\sigma-1} & 2\sigma-1 \text{ for } s \neq 0 \\
A_2 & = & -P_0 - 2xP_0' & = & -s(2\sigma+1)x^{\sigma} & \sigma \text{ for } s \neq 0 \\
A_3 & = & -x & & & 1
\end{array}$$

Under the assumption $3\sigma - 2 \geq 2\nu$ the Newton Polygon looks as below depending on whether equality holds or not. (The $\circ$ are only present for $s \neq 0$.)



The leading term corresponding to the compact face has no multiple root. This is obvious in case of $3\sigma - 2 > 2\nu$ and for $s = 0$, therefore it is true also for s sufficiently small.

In particular, for s sufficiently small, the number of branches is constant and the leading term of each branch has a non-vanishing coefficient which varies continuously with s.

We consider now the case $s = 0$ in detail (but claims hold true also for s small up to continuous changes of the coefficients) and distinguish the following cases

- (a) $\gcd(2\nu-1,3) = \gcd(2\nu+2,3) = \gcd(\nu+1,3) = 1$
- (b) $\gcd(2\nu-1,3) = \gcd(\nu+1,3) = 3$,
  $\gcd(3\nu+3, 2\nu-1) = \gcd(6\nu+6, 2\nu-1) = \gcd(2\nu-1, 9) = 3$
- (c) $\gcd(2\nu-1,3) = \gcd(\nu+1,3) = 3$, $\gcd(3\nu+3,2\nu-1) = \gcd(2\nu-1,9) = 9$

In case (a) there are two branches

$$x(t) = 0, \quad x(t) = c_0 t^{\frac{3}{2\nu-1}} + h.o.t. \tag{5}$$

in cases (b) and (c) there are four branches

$$x(t) = 0, \quad x(t) = c_0\omega^i t^{\frac{3}{2\nu-1}} + h.o.t., \quad i = 0,1,2. \tag{6}$$

where $c_0 \neq 0$ is a numerical constant and ω a primitive root of unity of order $2\nu - 1$.

To continue along the lines of [Pha73] we check first that the hypothesis

$$P'(x(t),t) \;=\; P_0'(x(t)) + tp'(x(t)) \;=\; P_0'(x(t)) + t \;\neq\; 0 \;\in\; \mathbb{C}\{t^{\frac{1}{2\nu-1}}\}$$

holds true for every possible branch $x(t)$.

Therefore the following formula derived by Pham is valid in the field of fractions $\mathbb{C}\big(\big(t^{\frac{1}{2\nu-1}}\big)\big)$ associated to the integral domain $\mathbb{C}\big\{t^{\frac{1}{2\nu-1}}\big\}$.

$$\begin{aligned} u &= -\frac{2}{3}\frac{P}{P'}Q' + Q \\ &= \Big(-\frac{2}{3}(\nu+1)+1\Big)\, x(t)^{\nu+1}. \end{aligned} \tag{7}$$

In case (a) we plug in the expansions (5) to get

$$u(t) = 0, \quad u(t) = \Big(-\frac{2}{3}(\nu+1)+1\Big)c_0^{\nu+1}t^{\frac{3\nu+3}{2\nu-1}} + h.o.t.$$

The corresponding branches are reduced and not equal. Moreover the second expansion does not have further essential summands, since the exponent of t is in its reduced form and has the maximal possible denominator.

In case (b) we write $2\nu-1 = 3e$ with e coprime to 3 and get the expansions

$$u(t) = 0, \quad u(t) = \Big(-\frac{2}{3}(\nu+1)+1\Big)c_0^{\nu+1}\omega^{(\nu+1)i}t^{\frac{3\nu+3}{2\nu-1}} + h.o.t., \quad i = 0,1,2.$$

Again the corresponding branches are reduced and pairwise not equal. This time the reduced form of the exponent has denominator e. Again this is the maximal possible denominator, since the u-degree of the Weierstrass polynomial of the first branch is 1 and that of the other three branches is the maximal denominator, but their sum is equal to the Milnor number which is $\mu = 3e+1$.

Thus in case (a) and case (b) we have found a perturbation such that the topological type of corresponding discriminant curve does not vary for s sufficiently small.

In case (c) we write $\nu - 5 = 9\rho$, but we fail to argue as above. In fact, for $s = 0$ we get expansions which parametrize the branches of the corresponding discriminant curve by a $3:1$ map so this curve is non-reduced.

Hence we rerun the method of Pham with the modified perturbation

$$t(xy + x^{3\rho+4}), \quad \text{i.e.} \quad p = x,\ q = x^{3\rho+4}.$$

The essential expansion of x in terms of t remains the same as before, since the new perturbation only adds the points $(1, 12\rho+8)$ and $(2, 6\rho+6)$ to the support, which both lie above the Newton polygon.

$$x(t) = 0, \quad x(t) = c_0\omega^i t^{\frac{3}{2\nu-1}} + h.o.t., \quad i = 0,1,2.$$

The reduced form of the exponent is the inverse of $6\rho+3$.

The formula (7) now gives (using c_ν, c_ρ for the obvious constants)

$$\begin{aligned} u &= \Big(-\frac{2}{3}(\nu+1)+1\Big)\, x(t)^{\nu+1} + \Big(-\frac{2}{3}(3\rho+4)+1\Big)\, tx(t)^{3\rho+4} \\ &= c_\nu c_0^{\nu+1}\omega^{(\nu+1)i}t^{\frac{\nu+1}{6\rho+3}} + c_\rho c_0^{3\rho+4}\omega^{(3\rho+4)i}t^{\frac{\nu+2}{6\rho+3}} + h.o.t. \end{aligned}$$

We can now argue as before, that $6\rho+3$ is the maximal possible denominator. Therefore no further essential summand occurs, and we get reduced, pairwise distinct branches also in the remaining case (c).

Let us summarize the results of the present section as follows.

Proposition 7. *Suppose $f = y^3 + x^{\nu+1}$ and m is an integer with*

$$2\nu \le 3m - 2, \quad m \le \nu.$$

Then there exists a 3-parameter unfolding $F(x, y; u, t, s)$, such that

(1) *along $u = t = 0$ the unfolding is μ-constant,*

(2) *for fix s sufficiently small, the discriminant curve of the unfolding F_s by the parameter t is reduced and topologically equivalent to that of F_0 and*
(3) *the analytic invariant σ is ν for $s = 0$ and m for $s \neq 0$ sufficiently small.*

4. The Zariski theorem

In this final section we have to revisit the local Zariski and van Kampen theorem which avoids the use of generic hyperplane sections, cf. the more extended exposition in [Lön10, Lön11].

Our main interest lies in the discriminant complement, so let us recall the basic setting: Given a holomorphic function germ $f = f(x, y)$ on the germ $\mathbb{C}^2, 0$ of the affine plane with coordinates x, y, we consider a versal unfolding, which can be given by a function germ on the affine space germ $(\mathbb{C}^2, 0) \times (\mathbb{C}, 0) \times (\mathbb{C}^k, 0)$

$$F(x, y, u, v) \quad = \quad f(x, y) - u + \sum_{i=1}^{k} v_i g_i(x, y),$$

where the g_i generate, as a vector space, the local ideal of function germs on $\mathbb{C}^2, 0$ vanishing at the origin up to elements in the Jacobian ideal of f. They are typically taken to be non-constant monomials.

We get a diagram

$$\begin{array}{ccccccc} (u, v_1, ..., v_k) & \in & \mathbb{C}^{k+1}, 0 & \supset & \mathcal{D} & = & \{(u, v) \,|\, F_{u,v}^{-1}(0) \text{ is singular}\} \\ \downarrow & & {}^{p}\!\downarrow & & & & \\ (v_1, ..., v_k) & \in & \mathbb{C}^k, 0 & \supset & \mathcal{B} & = & \{u \,|\, F_{0,v} \text{ is not Morse}\} \end{array}$$

The restriction $p|_{\mathcal{D}}$ of the projection to the *discriminant* $\mathcal{D}$ is a finite map, such that the branch set coincides with the *bifurcation set* $\mathcal{B}$ and the critical points are contained in the pre-image $\tilde{\mathcal{B}} = p^{-1}(\mathcal{B})$. In particular, the origin is an isolated point in the intersection of $\mathcal{D}$ with the fibre $p^{-1}(0)$. If a hypersurface germ has this property we call it *horizontal* for the projection p.

The key observation is, that a suitable representative of the complement of $\tilde{\mathcal{B}}$ is a trivial disc bundle by p into which $\mathcal{D}$ is embedded as a smooth submanifold, which is a connected topological cover by p. This situation, which can be treated also in the language of polynomial covers, cf. [Han89], naturally gives rise to a *braid monodromy homomorphism*: The domain is the fundamental group of the complement of $\mathcal{B}$, its target is the group of mapping classes of the punctured fibre, the image is called the *braid monodromy group*.

It coincides with the map of fundamental groups induced by the map of Lyashko Looijenga under the natural identification of the mapping class group with the fundamental group of the space of monic simple univariate polynomials of degree n at the corresponding base point:

$$\begin{array}{ccc} \mathbb{C}^k - \mathcal{B} & \longrightarrow & \mathbb{C}[x], \\ v & \mapsto & p_v, \end{array}$$

which maps to monic univariate polynomials of degree μ with simple roots only, where these roots correspond to the points of $\mathcal{D}$ which project to v.

To use the braid monodromy group of the fundamental group of the discriminant complement we employ the argument of Zariski and van Kampen [vK33]. It relies on a choice of a geometric basis in the fibre over the base point which is the customary tool to identify the action of the group of isotopy classes of diffeomorphisms on the fundamental group of the fibre with the right Artin action of the abstract braid group on the free generators $t_1, \dots, t_n$ given by

$$(t_j)\sigma_j = t_j t_{j+1} t_j^{-1}, \quad (t_{j+1})\sigma_j = t_j, \quad (t_i)\sigma_j = t_i, \text{if } i \neq j, j+1.$$

Theorem 8 (van Kampen). *Given a horizontal hypersurface germ with braid monodromy group generated by braids $\{\beta_s\}$ in Br_n, the local fundamental group of the complement is finitely presented as*

$$\pi_1 = \langle t_1, \dots, t_n \,|\, t_i^{-1} t_i^{\beta_s}, 1 \le i \le n, \mathit{all}\ \beta_s \rangle.$$

The consideration above applies again to the hypersurface germ $\mathcal{B}$ in the affine space germ $\mathbb{C}^k, 0$ provided we find a projection for which $\mathcal{B}$ is horizontal. In fact this puts a constraint on a discriminant curve as we will see in the following proof.

Proposition 9. *Let g_1 be a bivariate polynomial germ vanishing at 0 such that the discriminant curve of the unfolding*

$$f - u + tg_1$$

is reduced. Then the fundamental group of the complement of a corresponding unfolded discriminant curve is equal to the discriminant knot group of f.

Proof. Without loss of generality we may assume that g_1 is the first of the functions in the versal unfolding of f we consider. Hence the complement of the unfolded discriminant curve is a vertical plane section of the discriminant. (At this point we could conclude with the local Zariski hyperplane section theorem, if this vertical plane were known to be generic.)

By the van Kampen theorem, it suffices to show that the two braid monodromy groups are equal. They in turn are homomorphic images of the corresponding fundamental groups of complements to the bifurcation set.

If the discriminant curve is reduced, then the corresponding curve in the affine space germ $\mathbb{C}^k, 0$ does not belong to the bifurcation set, otherwise, the discriminant curve has less than μ points over every t and is non-reduced.

We deduce that the bifurcation set is horizontal for the projection along the coordinate corresponding to g_1, since the 0-fibre of that projection was just shown not to be in the bifurcation set.

In particular the fundamental group in a generic vertical line is generated by elements corresponding to a geometric basis. They also generate the fundamental group of the complement to the bifurcation set by the van Kampen theorem.

Put differently the fundamental group the smaller set surjects onto the fundamental group of the complement to the bifurcation set. Hence both fundamental groups map to same braid monodromy group.

Since a generic vertical line is the image under p of an unfolded discriminant curve associated to g_1 as in the beginning of the proof, we have precisely shown what was needed. □

Proof of the main Theorem. Thanks to Prop.6 it suffices to show that the discriminant knot group is constant along each μ-constant stratum which contains a Brieskorn-Pham polynomial $y^3 + x^{\nu+1}$.

Let f be any function in this stratum and σ_f its σ-invariant. Since the analytic equivalence class of f has a representative of the form

$$y^3 - P(x)y + x^{\nu+1} \quad \text{with} \quad \tfrac{2}{3}(\nu+1) \le \operatorname{ord} P, \quad \deg P \le \nu - 1,$$

we deduce $2\nu \le 3\sigma_f - 2$ and $\sigma_f \le \nu$.

Therefore by Prop.7 we can unfold the Brieskorn-Pham polynomial by a parameter s such that the σ-invariant is σ_f for $s \ne 0$, and there exists an associated family of discriminant curves of constant topological type.

Because they are reduced, we can apply the previous proposition to see that corresponding unfolded discriminant curves have a complement with fundamental group isomorphic to the respective discriminant knot groups. So by the same argument as in the proof of Prop.6 the two discriminant knot groups are isomorphic.

Since f and any deformation of the Brieskorn-Pham polynomial with s small share the same σ-invariant, we may invoke Cor.5 to have topologically equivalent complements of unfolded generic discriminant curves. Thus again the discriminant knot groups are isomorphic and therefore constant along the μ-constant stratum of each Brieskorn-Pham polynomial. $\square$

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LINEAR SUBSPACE ARRANGEMENTS ASSOCIATED WITH NORMAL SURFACE SINGULARITIES

ANDRÁS NÉMETHI

To the memory of Egbert Brieskorn

Abstract. Let us fix a normal surface singularity with rational homology sphere link and one of its good resolutions. It is known that each coefficient of the analytic Poincaré series associated with the multivariable divisorial filtration is the topological Euler characteristic of the complement of a certain linear subspace arrangement (determined by the divisorial filtration). In this note we construct the topological analogue valid for the multivariable topological series (zeta function) associated with the resolution graph. In this way the motivic version of this topological series can also be considered.

1. Introduction

Let us fix a good resolution ϕ of a normal surface singularity (X, o) with irreducible exceptional divisors $\{E_v\}_{v\in\mathcal{V}}$. (For details and notations see 2.1.) We assume that the link is a rational homology sphere. It is known that the coefficients of the Poincaré series $P(\mathbf{t})$ associated with the multivariable $\{E_v\}_v$–divisorial filtration can be identified with the topological Euler characteristic of certain projectivized linear subspace arrangements [CDGZ04, CDGZ08, N07, N08, N12], see section 2 and Corollary 3.1.3 here. More precisely, if $P(\mathbf{t}) = \sum_{l'} \mathfrak{p}(l')\mathbf{t}^{l'}$ (where the sum is over the dual lattice L' of ϕ), then for each l' there exists a finite-dimensional vector space $A(l')$ and for each $v \in \mathcal{V}$ a linear subspace $A_v(l') \subset A(l')$ such that $\mathfrak{p}(l') = \chi_{top}(\mathbb{P}(A(l') \setminus \cup_v A_v(l')))$. Let us denote this linear subspace arrangement $\{A_v(l')\}_{v\in\mathcal{V}}$ of $A(l')$ by $\mathcal{A}_{\rm an}(l')$.

In this note we prove the existence of the topological analogue of this fact, valid for the multivariable topological series (zeta function) $Z(\mathbf{t}) = \sum_{l'} \mathfrak{z}(l')\mathbf{t}^{l'}$. (For its definition see 4.1.) Namely, for each l' we construct a finite-dimensional vector space $T(l')$ and linear subspace arrangement $\{T_v(l')\}_{v\in\mathcal{V}}$ of $T(l')$, such that the following facts hold:

(1) the arrangement $\mathcal{A}_{\rm top}(l') := \{T_v(l')\}_{v\in\mathcal{V}}$ of $T(l')$ depends only on the resolution graph;
(2) $\mathfrak{z}(l') = \chi_{top}(\mathbb{P}(T(l') \setminus \cup_v T_v(l')))$;
(3) for each l' the vector space $A(l')$ embeds linearly into $T(l')$ such that

$$A_v(l') = A(l') \cap T_v(l') \text{ for every } v \in \mathcal{V}.$$

These facts can be interpreted as follows. Each topological type (of normal surface singularities with rational homology sphere link) with fixed dual resolution graph (or fixed lattice L and dual lattice L') determines for any $l' \in L'$ a 'topological linear subspace arrangement' $\mathcal{A}_{\rm top}(l')$. Furthermore, any analytic structure supported on this topological type determines canonically for any $l' \in L'$ an 'analytic linear subspace arrangement' $\mathcal{A}_{\rm an}(l')$ inside $T(l')$ and cut out from

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$\mathcal{A}_{\rm top}(l')$ by a subspace $A(l')\subset T(l')$. The analytic linear subspace arrangement might depend essentially on the analytic structure (since we know topologically equivalent pairs of analytical types, for which one type satisfies $Z(\mathbf{t})=P(\mathbf{t})$ while the other not, see e.g. [N08]).

In this way, $(A(l'),\{A_v(l')\}_v)\subset (T(l'),\{T_v(l')\}_v)$ looks a natural pairing. It immediately induces (by taking the Euler characteristic of the corresponding spaces) the pairing of the two series $Z(\mathbf{t})$ and $P(\mathbf{t})$. Though these two series looked apparently artificially paired in the earlier articles, now, after the present setup, this fact is totally motivated and justified. Furthermore, taking the periodic constants of the series Z and P, we get the pairing of the Seiberg–Witten invariants of the link, respectively the equivariant geometric genera of (X,o), cf. [N12], hence the pairing predicted by the Seiberg–Witten Invariant Conjecture is indeed very natural and totally justified.

In particular, these steps provide a totally conceptual explanation for the appearance of the Seiberg–Witten invariant in the theory of complex surface singularities.

Examples show that usually the identity $Z(\mathbf{t})=P(\mathbf{t})$ can happen even if $\mathcal{A}_{\rm top}(l')\neq\mathcal{A}_{\rm an}(l')$.

For each l' having a quasiprojective space $T(l')\setminus\cup_v T_v(l')$, instead of its topological Euler characteristic, has big advantages. Indeed, this provides a new source of invariants: one can replace $\chi_{top}(T(l')\setminus\cup_v T_v(l'))$ by several stronger invariants of this space, e.g. cohomology ring, the mixed Hodge structures, or the class in the Grothendieck ring of varieties. For the details of this last version see subsection 4.4.

2. Preliminaries. Divisorial filtration and its multivariable series

2.1. **Notations regarding a resolution.** Let (X,o) be the germ of a complex analytic normal surface singularity, and let us fix a good resolution $\phi:\widetilde{X}\to X$ of (X,o). We denote the exceptional curve $\phi^{-1}(o)$ by E, and let $\cup_{v\in\mathcal{V}}E_v$ be its irreducible components. Set also $E_I:=\sum_{v\in I}E_v$ for any subset $I\subset\mathcal{V}$. For more details see [N07, N12, N99b].

Let Γ be the dual resolution graph associated with ϕ; it is a connected graph. Then $M:=\partial\widetilde{X}$ can be identified with the link of (X,o), it is also an oriented plumbed 3–manifold associated with Γ. It is known that (X,o) locally is homeomorphic with the real cone over M, and M contains the same information as Γ. We will assume that M is a rational homology sphere, or, equivalently, Γ is a tree and all genus decorations of Γ are zero. We use the same notation $\mathcal{V}$ for the set of vertices, and δ_v for the valency of a vertex v.

$L:=H_2(\widetilde{X},\mathbb{Z})$, endowed with its negative definite intersection form $I=(\,,\,)$, is a lattice. It is freely generated by the classes of 2–spheres $\{E_v\}_{v\in\mathcal{V}}$. The dual lattice $L':=H^2(\widetilde{X},\mathbb{Z})$ is generated by the (anti)dual classes $\{E^*_v\}_{v\in\mathcal{V}}$ defined by $(E^*_v,E_w)=-\delta_{vw}$ (where δ_{vw} stays for the Kronecker symbol). The intersection form embeds L into L'. Then $H_1(M,\mathbb{Z})\simeq L'/L$, abridged by H. Usually one also identifies L' with those rational cycles $l'\in L\otimes\mathbb{Q}$ for which $(l',L)\in\mathbb{Z}$, or, $L'=\mathrm{Hom}_{\mathbb{Z}}(L,\mathbb{Z})$.

There is a natural (partial) ordering of L' and L: we write $l'_1\geq l'_2$ if $l'_1-l'_2=\sum_v r_vE_v$ with all $r_v\geq 0$. We set $L_{\geq 0}=\{l\in L\,:\,l\geq 0\}$ and $L_{>0}=L_{\geq 0}\setminus\{0\}$.

Set $\mathfrak{C}:=\{\sum l'_vE_v\in L',\ 0\leq l'_v<1\}$. For any $l'\in L'$ write its class in H by $[l']$, and or any $h\in H$ let $r_h\in L'$ be its unique representative in $\mathfrak{C}$.

All the E_v–coordinates of any E^*_u are strict positive. We define the Lipman cone as

$$\mathcal{S}':=\{l'\in L'\,:\,(l',E_v)\leq 0 \text{ for all } v\}.$$

It is generated over $\mathbb{Z}_{\geq 0}$ by $\{E^*_v\}_v$.

Finally, let $\theta:H\to\widehat{H}$ denote the isomorphism $[l']\mapsto e^{2\pi i(l',\cdot)}$ between H and its Pontrjagin dual $\widehat{H}$.

2.1.1. The module $\mathbb{Z}[[L']]$. We denote by $\mathbb{Z}[\mathbf{t}] := \mathbb{Z}[t_1,\dots,t_s]$, respectively by $\mathbb{Z}[[\mathbf{t}]] := \mathbb{Z}[[t_1,\dots,t_s]]$, the ring of polynomials, respectively the ring of formal power series, in variables $\{t_v\}_{v=1}^s$, where $s=|\mathcal{V}|$. Set also the ring of Laurent polynomials $\mathbb{Z}[\mathbf{t}][t_1^{-1},\dots,t_s^{-1}]$ too.

Then the formal Laurent series additive group $\mathbb{Z}[[\mathbf{t}^{\pm1}]] := \mathbb{Z}[[t_1^{\pm1},\dots,t_s^{\pm1}]]$ is a $\mathbb{Z}[\mathbf{t}][t_1^{-1},\dots,t_s^{-1}]$–module. It is contained in the larger module $\mathbb{Z}[[\mathbf{t}^{\pm1/d}]] = \mathbb{Z}[[t_1^{\pm1/d},\dots,t_s^{\pm1/d}]]$, the module of formal Laurent series in variables $t_v^{\pm1/d}$, where $d:=|H|$. $\mathbb{Z}[[L']]$ embeds into $\mathbb{Z}[[\mathbf{t}^{\pm1/d}]]$ as a submodule: it consists of the Laurent series with monomials of type

$$\mathbf{t}^{l'} = t_1^{l'_1}\cdots t_s^{l'_s}, \quad \text{where } \ l'=\textstyle\sum_v l'_vE_v\in L'.$$

In this way $\mathbb{Z}[[L]]$ identifies with $\mathbb{Z}[[\mathbf{t}^{\pm1}]]$. $\mathbb{Z}[[L']]$ also admits several $\mathbb{Z}$–submodules corresponding to different cones of L'; e.g. $\mathbb{Z}[[L'_{\geq0}]]$ and $\mathbb{Z}[[\mathcal{S}']]$, consisting of series with monomials of type $\mathbf{t}^{l'}$ with $l'\in L'_{\geq0}$, or $l'\in\mathcal{S}'$ respectively. Both $\mathbb{Z}[[L'_{\geq0}]]$ and $\mathbb{Z}[[\mathcal{S}']]$ have natural ring structure as well.

$\mathbb{Z}[[\mathcal{S}']]$ is a usual formal power series ring in variables $\mathbf{t}^{E_v^*}$: its elements are

$$\Phi(f)(\mathbf{t}) := f(\mathbf{t}^{E_1^*},\dots,\mathbf{t}^{E_s^*}), \quad \text{where } \ f(x_1,\dots,x_s)\in\mathbb{Z}[[\mathbf{x}]]=\mathbb{Z}[[x_1,\dots,x_s]]. \tag{2.1.2}$$

Definition 2.1.3. Any series $S(\mathbf{t})=\sum_{l'}a_{l'}\mathbf{t}^{l'}\in\mathbb{Z}[[L']]$ decomposes in a unique way as

$$S=\sum_{h\in H}S_h, \quad \text{where } \ S_h=\sum_{[l']=h}a_{l'}\mathbf{t}^{l'}. \tag{2.1.4}$$

S_h is called the h-component of S. In fact, if $F(\mathbf{t}):=\Phi(f)(\mathbf{t})$ for some $f\in\mathbb{Z}[[\mathbf{x}]]$ then

$$F_h(\mathbf{t})=\frac{1}{|H|}\cdot\sum_{\rho\in\widehat{H}}\rho(h)^{-1}\cdot f(\rho([E_1^*])\mathbf{t}^{E_1^*},\dots,\rho([E_s^*])\mathbf{t}^{E_s^*}). \tag{2.1.5}$$

Indeed, if $l':=\sum n_vE_v^*$ and $\prod x_v^{n_v}$ is a monomial of f, then $\Phi(\prod x_v^{n_v})(\mathbf{t})=\mathbf{t}^{l'}$ and the Fourier transform $(1/d)\sum_\rho\rho(h)^{-1}\rho([l'])\mathbf{t}^{l'}$ is $\mathbf{t}^{l'}$ if $[l']=h$ and it is zero otherwise.

2.2. **Natural line bundles.** Some line bundles on $\widetilde{X}$ are distinguished. They are provided by the splitting of the cohomological exponential exact sequence (see e.g. [N07, 4.2]):

$$0\to H^1(\widetilde{X},\mathcal{O}_{\widetilde{X}})\to \mathrm{Pic}(\widetilde{X})\xrightarrow{c_1}L'\to0.$$

The first Chern class c_1 has a natural section on the subgroup L, namely $l\mapsto\mathcal{O}_{\widetilde{X}}(l)$. One shows that this section has a unique extension $\mathcal{O}(\cdot)$ to L'. We call a line bundle *natural* if it is in the image of this section. Hence, by definition, a line bundle is natural if and only if one of its powers has the form $\mathcal{O}_{\widetilde{X}}(l)$ for some $l\in L$.

The natural line bundle associated with $l'\in L'$ will be denoted by $\mathcal{O}_{\widetilde{X}}(l')$.

2.3. **The universal abelian covering.** Let $c:(X_a,o)\to(X,o)$ be the universal abelian covering of (X,o), the unique normal singular germ corresponding to the regular covering of $X\setminus\{o\}$ associated with $\pi_1(X\setminus\{o\})\to H_1(X\setminus\{o\},\mathbb{Z})=H$. It has a natural $H=L'/L$–action. Since $\widetilde{X}\setminus E\approx X\setminus\{o\}$, $\pi_1(\widetilde{X}\setminus E)=\pi_1(X\setminus\{o\})\to H$ defines a regular Galois covering of $\widetilde{X}\setminus E$ as well. This has a unique extension $\widetilde{c}:Z\to\widetilde{X}$ with Z normal and $\widetilde{c}$ finite. In other words, $\widetilde{c}:Z\to\widetilde{X}$ is the normalized pullback of c via ϕ. The (reduced) branch locus of $\widetilde{c}$ is included in E, and the Galois action of H extends to Z as well. Since E is a normal crossing divisor, the only singularities what Z might have are cyclic quotient singularities. Let $r:\widetilde{Z}\to Z$ be a resolution of these singular points such that $(\widetilde{c}\circ r)^{-1}(E)$ is a normal crossing divisor. We have the following diagram:

$$\begin{array}{ccccc} \widetilde{Z} & \xrightarrow{r} & Z & \xrightarrow{\psi_a} & (X_a,o) \\ & & \Big\downarrow{\scriptstyle \widetilde{c}} & & \Big\downarrow{\scriptstyle c} \\ & & (\widetilde{X},o) & \xrightarrow{\phi} & (X,o) \end{array} \tag{2.3.1}$$

Set $\phi_a=\psi_a\circ r$ and $p=\widetilde{c}\circ r$. One verifies (see [N07, Lemma 4.2.3]) $p^*(l')$ is an integral cycle for any $l'\in L'$.

One can recover the natural line bundles via the universal abelian covering as follows.

$$p_*\mathcal{O}_{\widetilde{Z}}=\widetilde{c}_*\mathcal{O}_Z=\bigoplus_{l'\in\mathfrak{C}}\mathcal{O}_{\widetilde{X}}(-l') \quad (\mathcal{O}_{\widetilde{X}}(-l') \text{ being the } \theta([l'])\text{-eigenspace of } \widetilde{c}_*\mathcal{O}_Z). \tag{2.3.2}$$

2.4. **The divisorial filtration. The series $H(\mathbf{t})$ and $P(\mathbf{t})$.** We will define an L–filtration of the local ring of (X,o) and a compatible H–equivariant L'–filtration of the local ring of (X_a,o). Fore more see [N12].

Definition 2.4.1. The L'–filtration on $\mathcal{O}_{X_a,o}$ is defined as follows. For any $l'\in L'$, we set

$$\mathcal{F}(l'):=\{f\in\mathcal{O}_{X_a,o}\mid \mathrm{div}(f\circ\phi_a)\geq p^*(l')\}. \tag{2.4.2}$$

Notice that the natural action of H on (X_a,o) induces an action on $\mathcal{O}_{X_a,o}$, which keeps $\mathcal{F}(l')$ invariant. Therefore, H acts on $\mathcal{O}_{X_a,o}/\mathcal{F}(l')$ as well. For any $l'\in L'$, let $\mathfrak{h}(l')$ be the dimension of the $\theta([l'])$-eigenspace $(\mathcal{O}_{X_a,o}/\mathcal{F}(l'))_{\theta([l'])}$. Then one defines the Hilbert series $H(\mathbf{t})$ by

$$H(\mathbf{t}):=\sum_{l'\in L'}\mathfrak{h}(l')\cdot\mathbf{t}^{l'}\in\mathbb{Z}[[L']]. \tag{2.4.3}$$

By [N07, Prop. 4.3.3], for any $l'\in L'$ there exists a unique minimal $s(l')\in\mathcal{S}'$ such that $l'\leq s(l')$ and $[l']=[s(l')]$. Since for any $f\in\mathcal{O}_{X_a,o}$, that part of $div(f\circ\phi_a)$, which is supported by the exceptional divisor of ϕ_a, is in the Lipman cone of $\widetilde{Z}$, we get

$$\mathcal{F}(l')=\mathcal{F}(s(l')). \tag{2.4.4}$$

For a fixed l' we write $[l']=h$. If $l'>0$ one has the exact sequence

$$0\to\mathcal{O}_{\widetilde{Z}}(-p^*(l'))\to\mathcal{O}_{\widetilde{Z}}\to\mathcal{O}_{p^*(l')}\to 0. \tag{2.4.5}$$

The $\theta(h)$-eigenspace constitutes the exact sequence

$$0\to\mathcal{O}_{\widetilde{X}}(-l')\to\mathcal{O}_{\widetilde{X}}(-r_h)\to\mathcal{O}_{l'-r_h}(-r_h)\to 0. \tag{2.4.6}$$

In particular, for $l'>0$,

$$\mathfrak{h}(l')=\dim\Big(\frac{H^0(\widetilde{Z},\mathcal{O}_{\widetilde{Z}})}{H^0(\widetilde{Z},\mathcal{O}_{\widetilde{Z}}(-p^*(l')))}\Big)_{\theta(h)}=\dim\ \frac{H^0(\widetilde{X},\mathcal{O}_{\widetilde{X}}(-r_h))}{H^0(\widetilde{X},\mathcal{O}_{\widetilde{X}}(-l'))}. \tag{2.4.7}$$

Example 2.4.8. In (2.4.7) if $l'\in L$ then $r_h=0$. Hence the 0–component of $H(\mathbf{t})$ is

$$H_0(\mathbf{t})=\sum_{l\in L}\dim\frac{\mathcal{O}_{X,o}}{\{f\in\mathcal{O}_{X,o}:\mathrm{div}_E(f\circ\phi)\geq l\}}\ \mathbf{t}^l.$$

This is the Hilbert series of $\mathcal{O}_{X,o}$ associated with the divisorial filtration

$$L\ni l\mapsto\mathcal{F}_0(l)=\{f\in\mathcal{O}_{X,o}:\mathrm{div}_E(f\circ\phi)\geq l\}$$

of all irreducible exceptional divisors of ϕ.

2.4.9. Next, we define the Poincaré series $P(\mathbf{t})=\sum_{l'\in L'}\mathfrak{p}(l')\mathbf{t}^{l'}$ associated with the filtration $\{\mathcal{F}(l')\}_{l'}$ as

$$P(\mathbf{t})=-H(\mathbf{t})\cdot\prod_v(1-t_v^{-1}). \tag{2.4.10}$$

Using (2.4.7) one verifies that for any $l'\in\mathcal{S}'$ one has

$$\mathfrak{p}(l')=\sum_{I\subset\mathcal{V}}(-1)^{|I|+1}\dim\frac{H^0(\widetilde{X},\mathcal{O}_{\widetilde{X}}(-l'))}{H^0(\widetilde{X},\mathcal{O}_{\widetilde{X}}(-l'-E_I))}. \tag{2.4.11}$$

The series $P(\mathbf{t})$ is supported in $\mathcal{S}'$, and the following 'invertion identities' hold [N12, Prop. 3.2.4]:

$$\mathfrak{h}(l')=\sum_{l\in L,\, l\not\geq 0}\mathfrak{p}(l'+l). \tag{2.4.12}$$

2.4.13. A reformulation of $P(\mathbf{t})$. For $l'-r_h\in L_{>0}$ from (2.4.6) follows that

$$\mathfrak{h}(l')=\chi(\mathcal{O}_{l'-r_h}(-r_h))-h^1(\mathcal{O}_{\widetilde{X}}(-l'))+h^1(\mathcal{O}_{\widetilde{X}}(-r_h)).$$

Since $\chi(\mathcal{O}_{l'-r_h}(-r_h))=\chi(l')-\chi(r_h)$, this reads as

$$\mathfrak{h}(l')=\chi(l')-h^1(\mathcal{O}_{\widetilde{X}}(-l'))-\chi(r_h)+h^1(\mathcal{O}_{\widetilde{X}}(-r_h)).$$

Hence, using the definition of P, we get

$$P(\mathbf{t})=\sum_{l'\in\mathcal{S}'}\sum_{I\subset\mathcal{V}}(-1)^{|I|+1}\Big(\chi(l'+E_I)-h^1(\mathcal{O}_{\widetilde{X}}(-l'-E_I))\Big)\mathbf{t}^{l'}. \tag{2.4.14}$$

2.4.15. The definition of P by Campillo, Delgado and Gusein-Zade [CDGZ04, CDGZ08]**.** The infinite-dimensional arrangement $\{\mathcal{F}(l'+E_I)\}_I$ in the infinite-dimensional linear space $\mathcal{F}(l')$ can be reduced to a finite-dimensional arrangement as follows. Since all these subspaces contain $\mathcal{F}(l'+E)$, and $\mathcal{F}(l')/\mathcal{F}(l'+E)$ is finite-dimensional, it is natural to set the series

$$L(\mathbf{t}):=\sum_{l'\in L'}\dim\,(\mathcal{F}(l')/\mathcal{F}(l'+E))_{\theta([l'])}\cdot\mathbf{t}^{l'}\in\mathbb{Z}[[L']]. \tag{2.4.16}$$

Since $\mathcal{F}(l')=\mathcal{F}(l'+E_v)$ if $(l',E_v^*)>0$, one obtains that $L(\mathbf{t})\prod_v(t_v-1)$ is an element of $\mathbb{Z}[[L'_{\geq 0}]]$. Hence the next infinite power series in $\mathbb{Z}[[L'_{\geq 0}]]$ is well-defined:

$$P(\mathbf{t}):=-\frac{L(\mathbf{t})\prod_v(t_v-1)}{1-\mathbf{t}^E}=-L(\mathbf{t})\prod_v(t_v-1)\cdot\sum_{k\geq 0}\mathbf{t}^{kE}. \tag{2.4.17}$$

Since $\mathfrak{h}_h(l')=0$ for $l'\leq 0$ one has $L(\mathbf{t})=-H(\mathbf{t})(1-\mathbf{t}^{-E})$ and $P(\mathbf{t})=-H(\mathbf{t})\cdot\prod_v(1-t_v^{-1})$, cf. (2.4.10).

Example 2.4.18. Consider the cyclic quotient singularity whose minimal resolution ϕ has only one irreducible component E with self-intersection $-n$ ($n\geq 2$). Then $H=\mathbb{Z}_n$,$\mathcal{O}_{X_a,o}=\mathbb{C}\{z_1,z_2\}$. Moreover, $E^*=E/n$. The action of H is given by $h*z_i=\theta(E^*)(h)z_i$; hence, $z_1^iz_2^j$ is in the $\theta(E^*)^{i+j}$–eigenspace. Below, for a character χ of this action on $\mathbb{C}\{\mathbf{z}\}$ we denote the corresponding eigenspace by $\mathbb{C}\{\mathbf{z}\}^\chi$.

Therefore, the Poincaré series of the H–eigenspaces (with $\deg(z_i)=\frac{1}{n}\in\frac{1}{n}\mathbb{Z}$) is

$$P((\mathbb{C}\{\mathbf{z}\}^{\theta([qE^*])},t)=\sum_{k\geq 0}(1+q+nk)t^{k+\frac{q}{n}},\ \text{and}\ P(\mathbb{C}\{\mathbf{z}\},t)=\sum_{\ell\geq 0}(1+\ell)t^{\ell/n}.$$

$$\sum_{\rho\in\widehat{H}}P(\mathbb{C}\{\mathbf{z}\}^\rho,t)\cdot\rho=\frac{1}{(1-\theta(E^*)\cdot t^{E^*})^2}\in\mathbb{Z}[[t]][\widehat{H}]; \tag{2.4.19}$$

$\widetilde{Z} = Z$ is just the blow up ϕ_a of X_a at 0 with exceptional divisor $\widetilde{E}$ a (-1)-curve. Since $\widetilde{c}^*(E) = n\widetilde{E}$, we get that $\mathcal{F}(k'E)$ contains all the monomials of degree $\geq nk'$. We claim that this inclusion is, in fact, an isomorphism. Indeed, if $f = \sum_{i+j=nk'} c_i z_1^i z_2^j \in \mathbb{C}\{\mathbf{z}\}$, such that f is not identically zero, then $\phi_a^*(f)$ will have vanishing order exactly nk' (and never higher) along $\widetilde{E}$, independently of the choice of the coefficients $c_i \in \mathbb{C}$. Therefore, for $k' = k + q/n$ $(k \in \mathbb{Z})$, $(\mathcal{F}(k'E)/\mathcal{F}(k'E+E))_{\theta([qE^*])}$ can be identified with the vector space of monomials of degree $nk+q$ $(0 \leq q < p)$, and $P(\mathbf{t}) = 1/(1-t^{E^*})^2$. Its h–components are

$$\sum_{h\in H} P_h(t)\cdot h = \frac{1}{(1-[E^*]\cdot t^{E^*})^2} \in \mathbb{Z}[[t]][H]. \tag{2.4.20}$$

Note that $(1-[E^*]\cdot t^{E^*})^{-2}$ agrees exactly with the H–decomposition $\sum_{h\in H} Z_h(t)\cdot h$ of the topological series $Z(t)$, which will be considered in 4.1.

3. Linear subspace arrangements associated with the filtration

3.1. Fix a normal surface singularity as in 2.3, one of its resolutions and the filtration $\{\mathcal{F}(l')\}_{l'\in L'}$, $\mathcal{F}(l') \subset \mathcal{O}_{X_a,o}$, from 2.4.1. For any $l' \in L'$, the linear space

$$(\mathcal{F}(l')/\mathcal{F}(l'+E))_{\theta([l'])} = H^0(\mathcal{O}_{\widetilde{X}}(-l'))/H^0(\mathcal{O}_{\widetilde{X}}(-l'-E))$$

naturally embeds into

$$T(l') := H^0(\mathcal{O}_E(-l')).$$

Let its image be denoted by $A(l')$. Furthermore, for every $v \in \mathcal{V}$, consider the linear subspace $T_v(l')$ of $T(l')$ given by

$$T_v(l') := H^0(\mathcal{O}_{E-E_v}(-l'-E_v)) = \ker\big(H^0(\mathcal{O}_E(-l')) \to H^0(\mathcal{O}_{E_v}(-l'))\big) \subset T(l').$$

Then the image $A_v(l')$ of

$$H^0(\mathcal{O}_{\widetilde{X}}(-l'-E_v)/H^0(\mathcal{O}_{\widetilde{X}}(-l'-E))$$

in $T(l')$ satisfies $A_v(l') = A(l') \cap T_v(l')$. This fact follows from the following diagram:

$$\begin{array}{ccccccc}
 & & H^0(\mathcal{O}_{\widetilde{X}}(-l'-E)) & = & H^0(\mathcal{O}_{\widetilde{X}}(-l'-E)) & & \\
 & & \downarrow & & \downarrow & & \\
0 & \to & H^0(\mathcal{O}_{\widetilde{X}}(-l'-E_v)) & \to & H^0(\mathcal{O}_{\widetilde{X}}(-l')) & \to & H^0(\mathcal{O}_{E_v}(-l')) \\
 & & \downarrow & & \downarrow & & \| \\
0 & \to & H^0(\mathcal{O}_{E-E_v}(-l'-E_v)) & \to & H^0(\mathcal{O}_E(-l')) & \to & H^0(\mathcal{O}_{E_v}(-l')) \\
 & & \| & & \| & & \\
 & & T_v(l') & \hookrightarrow & T(l') & &
\end{array}$$

Definition 3.1.1. The (finite-dimensional) arrangement of linear subspaces $\mathcal{A}_{\rm top}(l') = \{T_v(l')\}_v$ in $T(l')$ is called the 'topological arrangement' at $l' \in L'$. The arrangement of linear subspaces $\mathcal{A}_{\rm an}(l') = \{A_v(l') = T_v(l') \cap A(l')\}_v$ in $A(l')$ is called the 'analytic arrangement' at $l' \in L'$. The corresponding projectivized arrangement complements will be denoted by $\mathbb{P}(T(l')\setminus \cup_v T_v(l'))$ and $\mathbb{P}(A(l')\setminus \cup_v A_v(l'))$ respectively.

If $l' \notin \mathcal{S}'$ then there exists v such that $(E_v, l') > 0$, that is $h^0(\mathcal{O}_{E_v}(-l')) = 0$, proving that $T_v(l') = T(l')$. Hence $A_v(l') = A(l')$ too. In particular, both arrangement complements are empty.

The connection with the series P is provided by the following topological Euler characteristic formula.

Lemma 3.1.2. *Assume that $\{V_\alpha\}_{\alpha\in\Lambda}$ is a finite family of linear subspaces of a finite-dimensional linear space V. For $I\subset\Lambda$ set $V_I := \cap_{\alpha\in I}V_\alpha$ (where $V_\emptyset = V$). Then*

$$\chi_{\rm top}(\mathbb{P}(V\setminus\cup_\alpha V_\alpha)) = \sum_{I\subset\Lambda}(-1)^{|I|}\dim V_I.$$

If $\Lambda\neq\emptyset$, then this also equals $\sum_I(-1)^{|I|+1}\mathrm{codim}(V_I\subset V)$.

Proof. Use the inclusion–exclusion principle and $\dim V_I = \chi_{\rm top}(\mathbb{P}V_I)$. □

Corollary 3.1.3. *For any $l'\in\mathcal{S}'$ one has*

$$\mathfrak{p}(l') = \chi_{\rm top}(\,\mathbb{P}(A(l')\setminus\cup_v A_v(l'))\,).$$

Proof. Use (2.4.11) and 3.1.2. □

The corresponding dimensions of the linear subspaces in $\mathcal{A}_{\rm an}(l')$ are as follows.

Lemma 3.1.4. *For any $l'\in L'$ one has:*

$$\dim\, A(l') = \mathfrak{h}(l'+E)-\mathfrak{h}(l'),\quad \dim\,\cap_{v\in I}A_v(l') = \mathfrak{h}(l'+E)-\mathfrak{h}(l'+E_I).$$

Thus, we can expect that the analytic arrangement is rather sensitive to the modification of the analytic structure, and in general, does not coincide with the topological arrangement.

Note that dim $A(l')$ is the l'–coefficient of the series $L(\mathbf{t})$, cf. paragraph 2.4.15.

Since the series $P(\mathbf{t})$ and $H(\mathbf{t})$ determine each other (see (2.4.12)), once the analytic Poincaré series $P(\mathbf{t})$ is fixed all the dimensions involved in $\mathcal{A}_{\rm an}(l')$ (for all l') are determined.

Example 3.1.5. We will write $Z_{min}\in L$ for the minimal (or fundamental) cycle, which is the minimal non–zero cycle of $\mathcal{S}'\cap L$ [A62, A66]. Yau's maximal ideal cycle $Z_{max}\in L$ defines the divisorial part of the pullback of the maximal ideal $\mathfrak{m}_{X,o}\subset\mathcal{O}_{X,o}$, i.e.

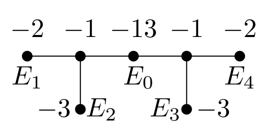
$$\phi^*\mathfrak{m}_{X,o}\cdot\mathcal{O}_{\widetilde{X}} = \mathcal{O}_{\widetilde{X}}(-Z_{max})\cdot\mathcal{I},$$

where $\mathcal{I}$ is an ideal sheaf with 0–dimensional support [Y80].

Consider the complete intersection singularity in $(\mathbb{C}^4,0)$ given by

$$z_1^2+z_2^3-z_3^2z_4 = z_4^2+z_3^3-z_2^2z_1 = 0.$$

Its graph is



One verifies that $\mathrm{div}(z_i)=E_i^*$ for $1\le i\le 4$; $Z_{min}=E_0^*$ is not the compact part of a divisor of an analytic function; $Z_{max}=2E_0^*=\min\{E_2^*,E_3^*\}$. We wish to find $\mathfrak{p}(Z_{max})$. Note that $T(Z_{max}) = H^0(\mathcal{O}_E(-Z_{max}))\simeq H^0(\mathcal{O}_{E_0}(-Z_{max}))\simeq\mathbb{C}^3$. On the other hand, $A(Z_{max})$ is the image of $H^0(\mathcal{O}(-Z_{max}))=\mathfrak{m}_{X,o}$. Since $z_1,\ z_4$ and $\mathfrak{m}_{X,o}^2$ are contained in $H^0(\mathcal{O}(-Z_{max}-E)$, $A(Z_{max})$ is 2–dimensional, generated by the classes of z_2 and z_3. (To see the linear independence of their classes, check their divisors.) Hence $A(Z_{max})\neq T(Z_{max})$.

Moreover, $\cup_v A_v(Z_{max})$ is the union of the two coordinate axes. Hence $\mathbb{P}(\mathbb{C}^2\setminus(\mathbb{C}\cup\mathbb{C}))=\mathbb{C}^*$ and $\mathfrak{p}(Z_{max})=0$.

Although $A(Z_{max}) \neq T(Z_{max})$, they still can be compared. Indeed, $T(Z_{max}) = H^0(\mathcal{O}_{E_0}(2))$ and $\cup_v T_v(Z_{max})$ is a union of two 2–planes (corresponding to global sections of $\mathcal{O}_{E_0}(2)$ vanishing at the two intersection points of E_0 with the other components). Hence

$$T \setminus \cup_v T_v = (A_v \setminus \cup_v A_v) \times \mathbb{C},$$

and $\chi_{\mathrm{top}}(\mathbb{P}(T \setminus \cup_v T_v)) = 0$ too.

Here $T_0 = 0$, which is contained in A, and A intersects all the other strata of $\{A_v\}_v$ generically. This quarantees that $\chi_{\mathrm{top}}(\mathbb{P}(A \setminus \cup_v A_v)) = \chi_{\mathrm{top}}(\mathbb{P}(T \setminus \cup_v T_v))$ holds.

3.1.6. Our next goal is to show that whenever the link of the singularity is a rational homology sphere the *topological arrangement* $\mathcal{A}_{\mathrm{top}}$ is indeed topological, it depends only on the combinatorics of the resolution graph.

We will need the following technical definition.

Lemma 3.1.7. *(1) For any $l' \in L'$ and subset $I \subset \mathcal{V}$ there exists a unique minimal subset $J(l', I) \subset \mathcal{V}$ which contains I, and has the following property:*

(3.1.8) *there is no $v \in \mathcal{V} \setminus J(l', I)$ with $(E_v, l' + E_{J(l',I)}) > 0$.*

(2) $J(l', I)$ can be found by the next algorithm: one constructs a sequence $\{I_m\}_{m=0}^k$ of subsets of $\mathcal{V}$, with $I_0 = I$, $I_{m+1} = I_m \cup \{v(m)\}$, where the index $v(m)$ is determined as follows. Assume that I_m is already constructed. If I_m satisfies (3.1.8) we stop and $m = k$. Otherwise, there exists at least one v with $(E_v, l' + E_{I_m}) > 0$. Take $v(m)$ one of them and continue the algorithm with I_{m+1}. Then $I_k = J(l', I)$.

Proof. For (1) notice that if J_1 and J_2 satisfies the wished requirement (3.1.8) of $J(l', I)$ then $J_1 \cap J_2$ satisfies too. Part (2) is a version of the usual Laufer type algorithm (see [La72, La77] or [N07, Prop. 4.3.3]). □

Proposition 3.1.9. *Assume that the resolution graph is a tree of rational curves. For any $l' \in L'$ and $I \subset \mathcal{V}$ write $J(I) := J(l', I)$. Then the following facts hold.*

(a) One has the following commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 \to & H^0(\mathcal{O}_{E-E_{J(I)}}(-l' - E_{J(I)})) & \to & H^0(\mathcal{O}_E(-l')) & \overset{k}{\twoheadrightarrow} & H^0(\mathcal{O}_{E_{J(I)}}(-l')) & \to 0 \\
 & \downarrow j & & \| & & \downarrow i & \\
0 \to & H^0(\mathcal{O}_{E-E_I}(-l' - E_I)) & \to & H^0(\mathcal{O}_E(-l')) & \to & H^0(\mathcal{O}_{E_I}(-l')) & \\
 & \| & & \| & & & \\
 & \cap_{v\in I} T_v(l') & \hookrightarrow & T(l') & & &
\end{array}$$

where j is an isomorphism (hence $\cap_{v\in I} T_v(l') = \cap_{v\in J(I)} T_v(l')$), i is injective and k is surjective.

(b) $\dim \cap_{v\in J(I)} T_v(l') = \chi(\mathcal{O}_{E-E_{J(I)}}(-l' - E_{J(I)})) = \chi(l' + E) - \chi(l' + E_{J(I)})$.

(c) In particular, if $J(I_1) = J(I_2)$ then $\cap_{v\in I_1} T_v(l') = \cap_{v\in I_2} T_v(l')$, and if $J(I_1) \subsetneq J(I_2)$ then $\cap_{v\in I_1} T_v(l') \supsetneq \cap_{v\in I_2} T_v(l')$. Therefore, $J(I)$ is the unique maximal subset $I_{max} \subset \mathcal{V}$, such that $I \subset I_{max}$, and $\cap_{v\in I} T_v(l') = \cap_{v\in I_{max}} T_v(l')$.

(d) Part (b) for $I = \emptyset$ reads as follows:

$$\dim T(l') = \dim \cap_{v\in J(\emptyset)} T_v(l') = \chi(l' + E) - \chi(l' + E_{J(\emptyset)}).$$

Hence, if $l' \in \mathcal{S}'$ then $\dim T(l') = -(l', E) + 1$.

(e) $\mathrm{codim}\,(\cap_{v\in I} T_v(l') \hookrightarrow T(l')) = \chi(l' + E_{J(I)}) - \chi(l' + E_{J(\emptyset)})$.

In particular, the arrangement complement is non–empty if and only if $J(\emptyset) = \emptyset$ (if and only if $l' \in \mathcal{S}'$).

Proof. First we prove the following fact: let $F \leq E$ be an effective non–zero cycle and we assume that for any $E_w \leq F$ one has $(E_w, l') \leq 0$. Then $h^1(\mathcal{O}_F(-l')) = 0$. The proof runs over induction: choose E_w from the support of F such that $(F - E_w, E_w) \leq 1$, then use the cohomological long exact sequence of $\mathcal{O}_F(-l') \to \mathcal{O}_{F-E_w}(-l')$.

From the definition, $\cap_{v\in I} T_v(l') = H^0(\mathcal{O}_{E-E_I}(-l' - E_I))$. To prove (a) note that this group is stable along the steps of the algorithm 3.1.7(2). Hence j is an isomorphism. Similarly, along these steps i is injective. Since $h^1(\mathcal{O}_{E-E_J}(-l' - E_J)) = 0$ by the above fact , k is onto and (b) follows too. For (c) use (b) and the fact that $\chi(l' + J(I_2)) > \chi(l' + J(I_1))$ whenever $J(I_1) \subsetneq J(I_2)$. $\square$

Corollary 3.1.10. *The arrangement $\mathcal{A}_{\rm top}(l')$ depends only on the combinatorial data of the graph.*

At topological Euler characteristic level one has:

Corollary 3.1.11. *If the graph is a tree of rational curves and $l' \in \mathcal{S}'$ then*

$$\chi_{\rm top}(\,\mathbb{P}(T(l') \setminus \cup_v T_v(l'))\,) = \sum_{I\subset \mathcal{V}} (-1)^{|I|+1} \chi(l' + E_{J(l',I)}).$$

Proof. Use Lemma 3.1.2 and Proposition 3.1.9(b). $\square$

Example 3.1.12. Consider the situation of the Example 3.1.5, and set $l' = Z_{min}$. Then $T(Z_{min}) = \mathbb{C}^2$ and $\cup_v T_v(Z_{min})$ consists of the union of two different lines. Therefore, at topological Euler characterisitc level, $\chi_{\rm top}(\,\mathbb{P}(T(Z_{min} \setminus \cup_v T_v(Z_{min}))\,) = 0$. At Z_{min} the complement of the analytic arrangement is empty.

Example 3.1.13. Using special vanishing theorems and computation sequences of rational and elliptic singularities (cf. [N99, N99b]) one can prove the following results as well (the details will be published elsewhere, see also [N]).

(I) Assume the following situations:

(a) either (X, o) is rational, ϕ is arbitrary resolution, and $l' \in \mathcal{S}'$ is arbitrary,

(b) or (X, o) is minimally elliptic singularity with $H^1(\widetilde{X}, \mathbb{Z}) = 0$, ϕ is a resolution whose elliptic cycle equals E, and we also assume that for the fixed $l' \in \mathcal{S}'$ there exists a computation sequence $\{x_i\}_i$ for Z_{min}, which contains E as one of its terms, and it jumps (that is, $(x_i, E_1) = 2$) at some E_1 with $(E_1, l') < 0$.

Then the topological and analytic arrangements at l' agree, $\mathcal{A}_{\rm top}(l') = \mathcal{A}_{\rm an}(l')$.

(II) For minimally elliptic singularities it can happen that $\mathcal{A}_{\rm top}(l') \neq \mathcal{A}_{\rm an}(l')$, even for the minimal resolution. E.g., in the case of the minimal good resolution of $\{x^2 + y^3 + z^7 = 0\}$, or in the case of minimal resolution of $\{x^2 + y^3 + z^{11} = 0\}$ (which is good), for $l = Z_{min}$ one has $\dim(T(Z_{min})) = 2$ and $\dim(A(Z_{min})) = 1$.

Remark 3.1.14. For any $l' \in \mathcal{S}'$ one has the exact sequence

$$0 \to A(l') \to T(l') \to H^1(\mathcal{O}_{\widetilde{X}}(-l' - E)) \to H^1(\mathcal{O}_{\widetilde{X}}(-l'))$$

Hence, $\mathcal{A}_{\rm an}(l') = \mathcal{A}_{\rm top}(l')$ whenever $H^1(\mathcal{O}_{\widetilde{X}}(-l' - E)) = 0$. This can occur, e.g. if $l' = \sum_v a_v E^*_v$ with $a_v \gg 0$, in which case $H^1(\mathcal{O}_{\widetilde{X}}(-l' - E)) = 0$ by the Grauert–Riemenschneider Vanishing Theorem.

4. The topological series $Z(\mathbf{t})$.

4.1. Using the notations of Subsection 2.1 (and under the above assumption $H^1(\widetilde{X}, \mathbb{Z}) = 0$) we define the following combinatorial/topological ‘candidate’ for $P(\mathbf{t})$. Sometimes we do not make distinction between a rational function and their Taylor expansion at the origin.

Definition 4.1.1. The series $Z(\mathbf{t})\in\mathbb{Z}[[\mathcal{S}']]$ is defined as the Taylor expansion at the origin of the rational function in variables $x_v=\mathbf{t}^{E_v^*}$ (cf. 2.1.2)

$$Z(\mathbf{t}):=\ \text{Taylor expansion at 0 of } \Phi(z)(\mathbf{t}),\quad \text{where}\ \ z(\mathbf{x}):=\prod_{v\in\mathcal{V}}(1-x_v)^{\delta_v-2}. \tag{4.1.2}$$

That is,

$$Z(\mathbf{t})=\ \text{Taylor expansion at 0 of } \prod_v(1-\mathbf{t}^{E_v^*})^{\delta_v-2}. \tag{4.1.3}$$

We call this form the **first appearance of** $Z(\mathbf{t})$.

By (2.1.5), its h-component $Z_h(\mathbf{t})$ is the expansion of

$$\frac{1}{|H|}\cdot\sum_{\rho\in\widehat{H}}\rho(h)^{-1}\cdot\prod_{v\in\mathcal{V}}\left(1-\rho([E_v^*])\mathbf{t}^{E_v^*}\right)^{\delta_v-2}. \tag{4.1.4}$$

4.2. We start the list of its properties by the following observation. If Σ is a topological space, let $S^a\Sigma$ ($a\geq 0$) denote its symmetric product $\Sigma^a/\mathfrak{S}_a$. For $a=0$, by convention, $S^0\Sigma$ is a point. Then, by Macdonald formula [Macd62],

$$\sum_{a\geq 0}\chi_{\rm top}(S^a\Sigma)\,x^a=(1-x)^{-\chi(\Sigma)}. \tag{4.2.1}$$

Let E_v° denote the regular part of $E_v\simeq\mathbb{P}^1$. Then $\chi_{\rm top}(E_v^\circ)=2-\delta_v$.

Corollary 4.2.2. The second appearance of $Z(\mathbf{t})$*, cf.* [CDGZ04, CDGZ08]. *With the notation* $\mathbf{x}^{\mathbf{a}}=x_1^{a_1}\cdots x_s^{a_s}$*,*

$$z(\mathbf{x})=\prod_v\sum_{a_v\geq 0}\chi_{\rm top}(S^{a_v}E_v^\circ)\,x_v^{a_v}=\sum_{\mathbf{a}\geq 0}\prod_v\chi_{\rm top}(S^{a_v}E_v^\circ)\mathbf{x}^{\mathbf{a}}. \tag{4.2.3}$$

4.3. In the next paragraphs we provide another interpretation of $Z(\mathbf{t})$. For the definition of the cycle $J(l',I)$ associated with $l'\in L'$ and $I\subset\mathcal{V}$ see 3.1.7.

Theorem 4.3.1. The third appearance of $Z(\mathbf{t})$**.**

$$Z(\mathbf{t})=\sum_{l'\in\mathcal{S}'}\sum_{I\subset\mathcal{V}}(-1)^{|I|+1}\chi(l'+E_{J(l',I)})\mathbf{t}^{l'}. \tag{4.3.2}$$

This formula can be compared with (2.4.14) valid for $P(\mathbf{t})$.

Proof. With the notation $l'=\sum_v a_vE_v^*$ set

$$y(\mathbf{x}):=\sum_{\mathbf{a}\geq 0}\sum_I(-1)^{|I|+1}\chi(l'+E_{J(l',I)})\mathbf{x}^{\mathbf{a}}. \tag{4.3.3}$$

We wish to show that $y(\mathbf{x})=z(\mathbf{x})$. In the proof we use induction over $|\mathcal{V}|$. The verification of the $|\mathcal{V}|=1$ case is left to the reader. Hence, we assume $|\mathcal{V}|>1$. Fix a vertex $w\in\mathcal{V}$ so that $\delta_w=1$. Let $\Gamma_0:=\Gamma\setminus\{w\}$, and let u be that vertex of Γ_0 which is adjacent to w in Γ. Let $\mathbf{x}_0$ be the $\mathbf{x}$–vector associated with $\mathcal{V}(\Gamma_0)$. Clearly, one has

$$z_\Gamma(\mathbf{x})=z_{\Gamma_0}(\mathbf{x}_0)\cdot(1-x_u)/(1-x_w).$$

We will establish similar identity for y_Γ. For this we write $l_0':=R(l')=\sum_{v\neq w}a_vE_v^{*,\Gamma_0}$ for any $l'=\sum_v a_vE_v^*$ (here E_v^{*,Γ_0} is the anti-dual of E_v in Γ_0). This is the restriction, the dual operator $L(\Gamma)'\to L(\Gamma_0)'$ of the inclusion $L((\Gamma_0)\to L(\Gamma)$. Hence, for $Z\in L(\Gamma_0)$

$$(l',Z)=(l_0',Z)\ \ \text{and}\ \ (-E_u^{*,\Gamma_0},Z)=(E_w,Z). \tag{4.3.4}$$

First, we fix some $l' \in \mathcal{S}'$ and a subset $I \subset \mathcal{V}$ with $w \notin I$. If $w \in J(l', I)$, we may assume (cf. the notations of 3.1.7) that $I_{k-1} = J(l', I) \backslash w$. Since $(l', E_w) \leq 0$, $\delta_w = 1$ and $(l' + E_{I_{k-1}}, E_w) > 0$, we get that, in fact, $(l' + E_{I_{k-1}}, E_w) = 1$. Hence

$$\chi(l' + E_{J(l',I)}) = \chi(l' + E_{J(l',I)\backslash w}).$$

Comparing the two algorithms on Γ and Γ_0 we get that $J(l', I) \setminus w = J^{\Gamma_0}(l'_0, I)$, and

$$\chi(l' + E_{J(l',I)\backslash w}) - \chi(l' + E_I) = \chi(l'_0 + E_{J^{\Gamma_0}(l'_0,I)}) - \chi(l'_0 + E_I).$$

By (4.3.4) one also has $\chi(l' + E_I) - \chi(l') = \chi(l'_0 + E_I) - \chi(l'_0)$. All these implies the next identity, where in the right hand side all invariants are considered in Γ_0:

$$\chi(l' + E_{J(l',I)}) - \chi(l') = \chi(l'_0 + E_{J^{\Gamma_0}(l'_0,I)}) - \chi(l'_0). \tag{4.3.5}$$

The same is true if $w \notin J(l', I)$. Next, fix again $l' \in \mathcal{S}'$ and take $I \subset \mathcal{V}$ with $w \in I$.

We need to distinguish two cases. In the first case we assume that $(l', E_u) = 0$. This happens exactly when $l'_0 - E_u^{*,\Gamma_0} \notin \mathcal{S}'(\Gamma_0)$. In this case, for any $K \subset \mathcal{V} \setminus \{u, w\}$ one has $J(l', K \cup w\}) = J(l', K \cup \{w, u\})$. Indeed, $(l' + E_{K\cup w}, E_u) \geq (E_w, E_u) = 1$, hence in the very first step of the algorithm of $J(l', K \cup w)$ we can add E_u. Thus,

$$\sum_{M \not\ni w} (-1)^{|M|} \chi(l' + E_{J(l', M\cup w)}) = 0. \tag{4.3.6}$$

Next, assume that $l'_0 - E_u^{*,\Gamma_0} \in \mathcal{S}'(\Gamma_0)$. Then, compared the two algorithms we get

$$J(l', I) = J^{\Gamma_0}(l'_0 - E_u^{*,\Gamma_0}, I \setminus w) \cup w,$$

$$\chi(l' + E_{J(l',I)}) - \chi(l' + E_I) = \chi(l'_0 - E_u^{*,\Gamma_0} + E_{J^{\Gamma_0}(l'_0 - E_u^{*,\Gamma_0}, I\backslash w)}) - \chi(l'_0 - E_u^{*,\Gamma_0} + E_{I\backslash w}).$$

Finally, (4.3.4) implies

$$\chi(l' + E_w + E_{I\backslash w}) - \chi(l' + E_w) = \chi(l'_0 - E_u^{*,\Gamma_0} + E_{I\backslash w}) - \chi(l'_0 - E_u^{*,\Gamma_0}).$$

Hence

$$\chi(l' + E_{J(l',I)}) - \chi(l' + E_w) = \chi(l'_0 - E_u^{*,\Gamma_0} + E_{J^{\Gamma_0}(l'_0 - E_u^{*,\Gamma_0}, I\backslash w)}) - \chi(l'_0 - E_u^{*,\Gamma_0}). \tag{4.3.7}$$

Since for any constant c, one has $\sum_{I: I \not\ni w} (-1)^{|I|+1} c = \sum_{I: I\ni w} (-1)^{|I|+1} c = 0$, the identities (4.3.5), (4.3.6) and (4.3.7) read as

$$\sum_{\mathbf{a}\geq 0} \sum_{I \not\ni w} (-1)^{|I|+1} \chi(l' + E_{J(l',I)}) \mathbf{x}^{\mathbf{a}} = y_{\Gamma_0}(\mathbf{x}_0) \cdot \sum_{n_w \geq 0} x_w^{a_w};$$

$$\sum_{\mathbf{a}\geq 0} \sum_{I \ni w} (-1)^{|I|+1} \chi(l' + E_{J(l',I)}) \mathbf{x}^{\mathbf{a}} = -y_{\Gamma_0}(\mathbf{x}_0) x_u \cdot \sum_{n_w \geq 0} x_w^{a_w}.$$

Hence $y_\Gamma(\mathbf{x}) = y_{\Gamma_0}(\mathbf{x}_0)(1 - x_u)/(1 - x_w)$. □

Corollary 4.3.8. The forth appearance of $Z(\mathbf{t})$.

$$Z(\mathbf{t}) = \sum_{l' \in \mathcal{S}'} \chi_{\mathrm{top}}(\, \mathbb{P}(T(l') \setminus \cup_v T_v(l'))\,) \cdot \mathbf{t}^{l'}.$$

Proof. Combine Corollary 3.1.11 and Proposition 4.3.1. □

This formula can be compared with the statement of Corollary 3.1.3 valid for $P(\mathbf{t})$.

Example 4.3.9. Using Example 3.1.13 and Corollaries 3.1.3 and 4.3.8 we obtain that $P(\mathbf{t}) = Z(\mathbf{t})$ in the following cases (see also [N08]):

(a) (X, o) is rational, and ϕ is arbitrary resolution,

(b) or (X, o) is minimally elliptic singularity, and it satisfies the assumptions of Theorem 3.1.13.

More generally, the identity $Z(\mathbf{t}) = P(\mathbf{t})$ is true for any splice quotient singularity [N12]. Note that $Z(\mathbf{t}) = P(\mathbf{t})$ can happen even if $\mathcal{A}_{\rm an}(l') \neq \mathcal{A}_{\rm top}(l')$; see Example 3.1.5, which is a splice quotient singularity.

Remark 4.3.10. There is another incarnation of $Z(\mathbf{t})$, which uses **weighted cubes**. This realizes a connection with the lattice complex of the lattice cohomology, for details of this, see e.g. [N11].

The set of q–cubes (where $q \in \mathbb{Z}_{\geq 0}$) consists of pairs $(l', I) \in L' \times \mathcal{P}(\mathcal{V})$, $|I| = q$, where $\mathcal{P}(\mathcal{V})$ denotes the power set of $\mathcal{V}$. $\square_q = (l', I)$ can be identified with the 'vertices' $\{l' + \sum_{j\in I'} E_j)_{I'}$, where I' runs over all subsets of I, of a q–cube in $L' \otimes \mathbb{R}$. One defines the weight function

$$w : L' \to \mathbb{Q}, \qquad w(k) := \chi(l') = -(l', l' + K)/2. \tag{4.3.11}$$

This extends to a weight–function defined on the set of all q–cubes

$$w(\square_q) = w((l', I)) = \max_{I' \subset I} \{\, w(l' + \sum_{j\in I'} E_j)\,\}.$$

Then the **fifth appearance of** $Z(\mathbf{t})$ is

$$Z(\mathbf{t}) = \sum_{l'\in L'} \Big(\sum_{I\in \mathcal{P}(\mathcal{V})} (-1)^{|I|+1}\, w((l', I))\Big)\, \mathbf{t}^{l'}. \tag{4.3.12}$$

4.4. **The extension of $Z(\mathbf{t})$ in the Grothendieck ring.** The information contained in $Z(\mathbf{t})$ can be improved if we replace in the 'forth appearance'

$$Z(\mathbf{t}) = \sum_{l'\in \mathcal{S}'} \chi_{\rm top}(\mathbb{P}(T(l') \setminus \cup_v T_v(l')) \cdot \mathbf{t}^{l'}$$

the topological Euler characteristic of $\mathbb{P}(T(l') \setminus \cup_v T_v(l'))$ with the class of this space in the Grothendieck group of complex quasi–projective varieties. (In the analytic case the extension of $P(\mathbf{t})$ to the series $\sum_{l'\in\mathcal{S}'} [\mathbb{P}(A(l') \setminus \cup_v A_v(l'))] \cdot \mathbf{t}^{l'}$ with coefficients in the Grothendieck ring was already considered e.g. in [CDGZ07].)

Let $\mathbb{L}$ be the class of the 1–dimensional affine space. Then, by inclusion–exclusion principle (as the analogue of 3.1.2) one has the following. If $\{V_\alpha\}_{\alpha\in\Lambda}$ is a finite family of linear subspaces of a finite-dimensional linear space V, and for $I \subset \Lambda$ one writes $V_I := \cap_{\alpha\in I} V_\alpha$, then

$$[V \setminus \cup_\alpha V_\alpha] = \sum_I (-1)^{|I|} \mathbb{L}^{\dim(V_I)}, \quad [\mathbb{P}(V \setminus \cup_\alpha V_\alpha)] = (\sum_I (-1)^{|I|} \mathbb{L}^{\dim(V_I)})/(\mathbb{L}-1).$$

According to this, one defines

$$Z(\mathbb{L}, \mathbf{t}) = \sum_{l'\in\mathcal{S}'} [\mathbb{P}(T(l') \setminus \cup_v T_v(l'))] \cdot \mathbf{t}^{l'}, \tag{4.4.1}$$

which, using 3.1.9 reads as

$$\begin{aligned} Z(\mathbb{L}, \mathbf{t}) &= \frac{1}{\mathbb{L}-1} \cdot \sum_{l'\in\mathcal{S}'} \sum_{I\subset\mathcal{V}} (-1)^{|I|}\, \mathbb{L}^{\chi(l'+E) - \chi(l'+E_{J(l',I)})}\, \mathbf{t}^{l'} \\ &= \sum_{l'\in\mathcal{S}'} \sum_{I\subset\mathcal{V}} (-1)^{|I|} \cdot \frac{\mathbb{L}^{\chi(l'+E)-\chi(l'+E_{J(l',I)})} - 1}{\mathbb{L}-1}\, \mathbf{t}^{l'}. \end{aligned} \tag{4.4.2}$$

Note that $\lim_{\mathbb{L}\to 1} Z(\mathbb{L}, \mathbf{t}) = Z(\mathbf{t})$. The analogue of the topological/combinatorial identity (4.1.3) is the following.

Theorem 4.4.3.

$$Z(\mathbb{L},\mathbf{t})=\frac{\prod_{(u,v)\in\mathcal{E}}\,(1-\mathbf{t}^{E^*_u}-\mathbf{t}^{E^*_v}+\mathbb{L}\mathbf{t}^{E^*_u+E^*_v})}{\prod_{v\in\mathcal{V}}\,(1-\mathbf{t}^{E^*_v})(1-\mathbb{L}\mathbf{t}^{E^*_v})}. \tag{4.4.4}$$

Proof. Follow the steps and all the identities of the proof of 4.3.1. □

This formula was reproved in the Master Thesis of János Nagy as well [Nagy16]. In this Thesis also several cohomological properties of the linear subspace arrangement complements are studied.

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KULIKOV SINGULARITIES

JAN STEVENS

Dedicated to the memory of Egbert Brieskorn.

Abstract. In the study of normal surface singularities the relation between analytical and topological properties and invariants of the singularity is a very rich problem. This relation is particularly close for surface singularities constructed from families of curves. We use these Kulikov singularities to reexamine results of Némethi-Okuma and Tomaru.

Introduction

The first time I met Brieskorn was when I started my Ph.D. studies in Leiden and he was spending some months there. Horst Knörrer was then also working there. Through his students Brieskorn has influenced my career and work very much. And of course through his work, in the first place through his book with Knörrer on plane algebraic curves [2]. This is a most remarkable book, not only because of its value for money (Brieskorn negotiated a price below DM 50) and its white cover, but mainly because its style and contents. Ever since curve singularities and algebraic curves have been central in my work.

Trying to describe singularities one may ask the question:

Which discrete data are needed to know a singularity?

One interpretation of 'knowing a singularity' is that we can write down equations. As we only have discrete data, such equations necessarily describe an equisingular family of singularities.

For plane curve singularities there are very satisfactory answers to the question, which can be found in Brieskorn's book [2]. There is the link of the singularity, which gives the embedded topology (without the embedding one has only the number of components); another invariant is the resolution graph. Since Brieskorn's work on the exotic spheres as links of singularities it is realised that in high dimension the abstract link contains not enough information. In the surface case the situation is different. The topology of the link, encoded in the resolution graph, is a strong invariant. For rational and minimally elliptic singularities it determines the equisingularity class. For higher geometric genus this is no longer the case and the study of the relation between analytical and topological properties and invariants of singularities is a very rich problem.

To have a strong relation we have to look at special classes of singularities. In the work of Neumann and Wahl (for an overview see [18]) and of Némethi two kind of restrictions are imposed, an analytical one, that the singularity is $\mathbb{Q}$-Gorenstein, and a topological one, that the link is a rational homology sphere. Neumann and Wahl came even up with a way to write down equations from the resolution graph, provided certain special numerical conditions are satisfied. The so called splice type equations describe a complete intersection singularity in a particular simple form, however not for a singularity with the original graph, but for its universal abelian cover (which is a finite cover due to the rational homology sphere condition). In a recent paper Némethi and Okuma [12] study which analytic structures can occur for a specific resolution graph, giving details for an example already mentioned by Némethi [11]. One of the occurring structures is that of a Kodaira or Kulikov singularity.

Kodaira singularities were introduced by Karras [4], using a construction similar to the one earlier described by Kulikov [6]. In my thesis [16] I introduced the term *Kulikov singularities.* The construction starts from a (degenerating) 1-parameter family $\pi\colon W \to D$ of curves of genus g. Let $\sigma\colon \widetilde{W} \to W$ be the blow up of W in r points of the special fibre W_0, each point a smooth point on a component occurring with multiplicity 1. Then the strict transform of the special fibre can be blown down to a singular point $p \in \overline{W}$. By definition $(\overline{W}, p)$ is a Kulikov singularity. The study of properties of such singularities reduces in two ways to the study of curves. The morphism π descends to a function on the singularity, which defines a general hyperplane section. This curve singularity is more accessible and invariants like its multiplicity and embedding dimension determine the corresponding invariants of the surface singularity. The other occurrence of curves is by construction: the properties of the central fibre, considered as curve of arithmetic genus g, are essential.

Kulikov introduced his construction to give a uniform construction of the unimodal and bimodal singularities. These are the simplest types of minimally elliptic singularities. For higher genus Kulikov singularities should also be considered as simplest types. The generalisation of Laufer's minimally elliptic cycle [7] is the characteristic cycle, introduced by Karras for Kodaira singularities [5] and in [16] in general. Tomaru studied for which Brieskorn singularities the characteristic cycle is equal to the fundamental cycle [17] .

Karras' work on Kodaira singularities of higher genus [5] and my work on Kulikov singularities [16] was never published. When referred to, these singularities are mainly seen as singularities where there is a function defining the fundamental cycle Z, which is moreover reduced at components E_i with $E_i \cdot Z < 0$. In this paper I actually take this as definition (see Definition 2.1), being the shortest, but it is the construction using a family of curves which gives a good understanding of the singularity. As illustration I treat the results of Némethi and Okuma [12] and of Tomaru [17] from this point of view.

1. Invariants of surface singularities

The topological type of a normal complex surface singularity is determined by and determines the resolution graph of the minimal good resolution. But a resolution graph can be defined for any resolution, not necessarily good.

Definition 1.1. Let $\pi\colon (M, E) \to (V, p)$ be a resolution of a surface singularity with exceptional divisor $E = \bigcup_{i=1}^{r} E_i$. The *resolution graph* Γ is a weighted graph with vertices corresponding to the irreducible components E_i. Each vertex has two weights, the self-intersection $-b_i = E_i^2$, and the arithmetic genus $p_a(E_i)$, the second traditionally written in square brackets and omitted if zero. There is an edge between distinct vertices if the corresponding components E_i and E_j intersect, weighted with the intersection number $E_i \cdot E_j$ (only written out if larger than one).

Other definitions, which record more information, are possible: one variant is to have an edge for each intersection point $P \in E_i \cap E_j$, with weight the local intersection number $(E_i \cdot E_j)_P$. This is the more common definition in the case that the intersections are transverse.

The classes of the curves E_i form a preferred basis of $H := H_2(M, \mathbb{Z})$. Following algebro-geometric tradition the elements of H are called *cycles.* They are written as linear combinations of the E_i. The intersection form on M gives a negative definite quadratic form on H. Let $K \in H^2(M, \mathbb{Z})$ be the canonical class. It can be written as rational cycle in $H_{\mathbb{Q}} = H \otimes \mathbb{Q}$ by solving the adjunction equations $E_i \cdot (E_i + K) = 2p_a(E_i) - 2$. The function $-\chi(A) = \frac{1}{2} A \cdot (A + K)$, $A \in H$, makes H into a quadratic *quadratic lattice*, in the sense of [8, 1.4]. We prefer to work with the genus $p_a(A) = 1-\chi(A)$. Note that the genus function determines the intersection form, as

$$p_a(A+B) = p_a(A) + p_a(B) + A \cdot B - 1 \ .$$

The data (H, p_a) is equivalent to $(H, \{E_i \cdot E_j\}, \{p_a(E_i)\})$, encoded in the resolution graph Γ.

There are some important cycles on E, some of which only depend on the quadratic lattice, while others depend on the analytic structure.

Definition 1.2. The *fundamental cycle* Z is the is the smallest positive cycle such that $E_i \cdot Z \leq 0$ for all i. The *maximal ideal cycle* $Z_{\mathfrak{m}}$ is the smallest cycle occurring as compact part of the divisor of a function $f \in \mathfrak{m}_{(V,p)}$. The *canonical cycle* Z_K is the rational cycle on E, which is numerically equivalent to the anticanonical class of the resolution M.

We recall that the *geometric genus* $p_g(V,p)$ is the dimension of $R^1\pi_*\mathcal{O}_M$. This is equal to the maximal value of $h^1(\mathcal{O}_D)$ over all positive cycles. In fact, there is a unique minimal *cohomological cycle* with this maximal value (see [15, 4.8]). A topological lower bound for p_g is the arithmetic genus $p_a(V,p)$, which is the maximal value of $p_a(D)$ over all positive cycles. The genus $p_a(Z)$ of the fundamental cycle is also a topological invariant of the singularity, which is called the *fundamental genus* $p_f(V,p)$ [17].

Obviously $p_f \leq p_a \leq p_g$, and all inequalities can be strict; the easiest example with $p_a > p_f$ is the case of an irreducible exceptional curve E of genus $g > 1$ and self-intersection -1.

Definition 1.3. The *characteristic cycle* C of a nonrational singularity is the smallest cycle which realises the fundamental genus: it is the cycle $C \leq Z$ with $p_a(C) = p_a(Z)$ and $p_a(D) < p_a(C)$ for all cycles $0 < D < C$.

This cycle is a generalisation of Laufer's minimally elliptc cycle and its existence is proved in the same way. It was first introduced by Karras for Kodaira singularities [5]. The general definition is in [16]; Tomaru also introduced it under the name *minimal cycle* [17].

2. Kulikov singularities

In this section we introduce the Kulikov construction, give some properties and discuss when the resulting singularity is Gorenstein.

Definition 2.1. Let (V,p) be a normal surface singularity with fundamental cycle Z on the minimal resolution. It is called a *Kulikov singularity* if there exists a function $f\colon (V,p) \to (\mathbb{C},0)$ with $(X,p) = (f^{-1}(0),p)$ a reduced curve singularity with divisor on the minimal resolution of the form $Z + \widetilde{X}$, such that the strict (or proper) transform $\widetilde{X}$ of X intersects the exceptional set E transversally in smooth points on components having multiplicity one in the fundamental cycle Z.

Such singularities are the result of a construction first given by Kulikov [Kulikov], to describe the unimodal and bimodal singularities. He starts from a (degenerating) family $\pi\colon W \to D$ of curves of genus g. This is a proper morphism of a non-singular surface to a small disc. The special fibre $W_0 = \pi^{-1}(0)$ over $0 \in D$ can be written as $W_0 = n_1C_1 + \dots n_kC_k$, where the C_i are the irreducible components of this fibre. The intersection matrix $(C_i \cdot C_j)$ is negative semi-definite. Let $\sigma\colon \widetilde{W} \to W$ be the blow up of W in r points $q_1, \dots, q_r$, each a smooth point of a component C_i which has multiplicity $n_i = 1$ in W_0. We denote the strict transform of a component C_i by E_i. Then the special fibre $\widetilde{W_0}$ of $\tilde{\pi} = \pi \circ \sigma$ can be written as

$$\widetilde{W_0} = n_1E_1 + \dots n_kE_k + \widetilde{X}_1 + \dots + \widetilde{X}_r \ ,$$

where the $\widetilde{X}_j$ are (-1)-curves. Now the intersection matrix $(E_i \cdot E_j)$ is negative definite and $E = \bigcup E_i$ can be blown down to a singular point $p \in \overline{W}$.

Lemma 2.2. *Kulikov's construction results in a Kulikov singularity. Conversely, every Kulikov singularity can be obtained by this construction.*

Proof. The construction yields the minimal resolution if there are no (-1)-curves in the family $\pi\colon W \to D$ except possibly curves containing a point q_j. If there are other (-1)-curves we blow them down without changing the resulting singularity. So we may assume that $\widetilde{W} \to \overline{W}$ is the minimal resolution of the singularity $p \in \overline{W}$. We write $\widetilde{W}_0 = Y + \widetilde{X}$ and have to show that Y is the fundamental cycle of the singularity $(\overline{W}, p)$. We put $Y = Z + D$ with D an effective cycle supported on E. Then D does not intersect $\widetilde{X}$, as each $\widetilde{X}_i$ intersects Y in a component with multiplicity one. Now $0 = D \cdot \widetilde{W}_0 = D \cdot (Z + D + \widetilde{X}) = D \cdot Z + D \cdot D \le 0$, so $D \cdot D = 0$ and therefore $D = 0$.

Conversely, given a function $f\colon (V,p) \to (\mathbb{C}, 0)$ with divisor $Z + \widetilde{X}$ we compactify to a family of curves, following Karras [1980,Thm 2.9]: in each point $q \in E \cap \widetilde{X}$ there are local coordinates such that f is given by $xy = 0$, and $\widetilde{X}$ by $y = 0$. We glue the blow-up of the origin to it: with coordinates (u, y) we have two charts, given by $(u, y) = (u, u\eta) = (xy, y)$. The glueing is by identifying the (x, y) coordinates. Then $u = xy$ extends the function f. □

Kulikov singularities are a special case of Kodaira singularities, defined by Karras [4, 5]. In his construction it is allowed that points to be blown up coincide: one blows up consecutively, and it is allowed to blow up the strict transform of the fibre in a point of intersection with a previously blown up curve. Then the curve $(X, p) = (f^{-1}(0), p)$ is not necessarily a reduced curve.

The advantage of the more strict definition of Kulikov singularities is that the curve (X, p) is a general hyperplane section. The function $f\colon (V,p) \to (\mathbb{C}, 0)$ defines a smoothing of this curve with Milnor number $\mu = 2g + r - 1$. The structure of the hyperplane section is often much easier to describe than that of the singularity itself. It allows conclusion about the multiplicity and the embedding dimension of the singularity.

An alternative description of the construction starts from a *minimal family* $\pi\colon W \to D$, meaning that W does not contain (-1)-curves. One then blows up points consecutively, each time blowing up a point with multiplicity one in the special fibre. In each stage a (-1)-curve intersects only one other curve, so in the final surface the (-1)-curves are ends, and their complement is connected. Write as before $\widetilde{W}_0 = Y + \widetilde{X}$ with $\widetilde{X}$ the union of the (-1)-curves. Then the support of Y can be blown down.

We have the following properties.

Proposition 2.3.

1. *For a Kulikov singularity the maximal ideal cycle $Z_{\mathfrak{m}}$ is equal to the fundamental cycle Z.*
2. *The fundamental genus is equal to the genus of the curves in the family used in the construction: $p_f(V, p) = g$.*
3. *A rational singularity is Kulikov if and only if the fundamental cycle is reduced.*
4. *The characteristic cycle of a nonrational Kulikov singularity is the strict transform of the special fibre of the minimal family resulting in the singularity.*

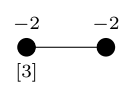
Proof. Only the last property needs a proof. It suffices to consider the case that the strict transform is the whole fundamental cycle. Suppose that $C < Z$ and choose a computation sequence $Z_j = Z_{j-1} + E_{i_j}$ from $Z_0 = C$ to $Z_k = Z$. As $p_a(Z_j) = p_a(Z)$ for all j, each E_{i_j} is a smooth rational curve with $E_{i_j} \cdot Z_{j-1} = 1$. This holds in particular for the last one and therefore $E_{i_k} \cdot Z = 1 + E_{i_k}^2 < 0$. This implies that E_{i_k} has multiplicity one in the fundamental cycle and

$E_{i_k} \cdot = -E_{i_k} \cdot \widetilde{X}$. After blowing down $\widetilde{X}$ the strict transform of E_{i_k} has self-intersection (-1), contradicting that the family we started from was a minimal family. $\square$

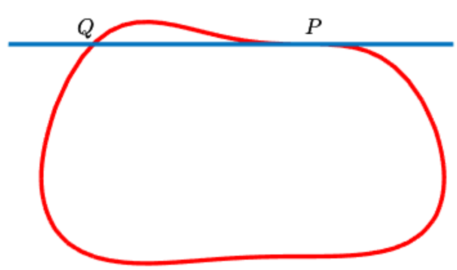
To obtain a Gorenstein Kulikov singularity we have to perform the construction in special points. Let $\pi\colon W \to D$ be a minimal family of curves of genus g. The relative dualising sheaf $\omega_{W/D}$ is isomorphic to Ω_W. Let (ω) be the divisor of a global section. It consists of an horizontal, non-compact part N and a divisor supported on the special fibre, determined up to a multiple of this fibre. Suppose that each component of N intersects the special fibre only transversally in components of multiplicity one. Now we perform the Kulikov construction starting from the minimal family, blowing up at least these intersection points, in such a way that in the final family $\tilde\pi\colon \widetilde{W} \to D$ the pull back of ω has the same multiplicity m along all (-1)-curves X_i, and that the horizontal part of its divisor intersects the special fibre only in $\widetilde{X}$. Let $f = \tilde\pi^*(t)$, with t a coordinate function on D. Then the meromorphic two-form $f^{-m}\omega$ is holomorphic and nowhere zero on $U \setminus E$, U a neighbourhood of E. Therefore the Kulikov singularity is Gorenstein.

Example 2.4. We give an example of a 1-parameter family of weighted homogeneous Gorenstein singularities V_a such that V_0 is not Kulikov but V_a is Kulikov for $a \neq 0$. It is the simplest of the series of examples of Briançon and Speder of a family which is μ-constant, but not μ^*-constant [1].

Consider
$$f_a(x,z,t) = z^3 + azx^3 + tx^4 + t^9\ .$$
The resolution graph is



The exceptional divisor on the minimal resolution is $E = E_1 + E_2$ with E_1 a curve of genus 3 with self-intersection -2, and E_2 a rational (-2)-curve. The canonical model of E_1 is the plane quartic $\eta\zeta^3 + a\zeta\xi^3 + \xi^4 + \eta^4$; this curve has a flex in $P = (0:0:1)$, and the tangent $\eta = 0$ intersects the curve in $Q = (-a:0:1)$, so for $a = 0$ there is a hyperflex. The normal bundle of E_1 has $P + Q$ as divisor, and E_2 intersects E_1 in Q. The general hyperplane section has two branches for $a \neq 0$; the strict transform of one branch passes through P, and the other intersects E_2 in a smooth point of E. For $a = 0$ the curve is irreducible, its strict transform passes through $P = Q = E_1 \cap E_2$.



To construct this singularity we start from the trivial family $W = E_1 \times D$. A canonical divisor is $3P \times D + Q \times D$. After blowing up in $P \times \{0\}$ the multiplicity along the newly introduced exceptional divisor is 4. Blowing up in $Q \times \{0\}$ gives multiplicity 2. We blow up again in intersection point of special fibre and strict transform of section $Q \times D$, resulting in multiplicity 4. By dividing with t^4 we see that the singularity is Gorenstein with $K = -4E_1 - 2E_2$. The functions t, $x = t^2\xi/\eta$ and $z = t^3\zeta/\eta$ are holomorphic on neighbourhood of E, giving $(t^3\zeta/\eta)^3 + a(t^3\zeta/\eta)(t^2\xi/\eta)^3 + t(t^2\xi/\eta)^4 + t^9 = 0$; this formula works also for $a = 0$. The blowing up can be done in family over a base $D \times A$, with a a coordinate on A. We first blow up in $P \times 0 \times A$, then in Q as lying on the strict transform of $C \times 0 \times A$ and then once again in the intersection point with the strict transform of the appropriate section. For $a = 0$ this means that we blow up in a double point of the special fibre, which is not allowed in the Kulikov construction.

3. The characteristic cycle of Brieskorn-Pham singularities

The simplest type of quasi-homogeneous hypersurface singularities has an equation, which is a sum of perfect powers, and is usually called a Brieskorn-Pham polynomial. We write in the surface case

$$x^a + y^b + t^c \tag{3.1}$$

with $a \le b \le c$. It is well known how to get the resolution of this surface singularity from the exponents a, b and c [14]. The precise form is not important for us now.

Lemma 3.1. *If $c \ge \operatorname{lcm}(a, b)$, the Brieskorn-Pham singularity (3.1) is a Kulikov singularity of genus $g = (\mu - r + 1)/2$, where $\mu = (a-1)(b-1)$ is the Milnor number of the curve singularity $x^a + y^b$ and $r = \gcd(a, b)$ is the number of branches.*

Proof. We construct the singularity with the Kulikov construction. We start with an affine family of curves, whose equation is in fact given by a Brieskorn-Pham polynomial, but with lower exponent c. Put $d = \operatorname{lcm}(a, b)$. Let $r = \gcd(a, b)$, then $d = \frac{ab}{r}$. Consider the family $\xi^a + \eta^b + t^{c-d} = 0$ as family of affine plane curves, parametrised by t, and complete it in the weighted projective space with weights $(\frac{d}{a}, \frac{d}{b}, 1)$. The homogeneous equation is then $\xi^a + \eta^b + t^{c-d}w^d = 0$. We resolve the singularity at the origin. We look at the chart $\xi = 1$. There the equation is $1 + \bar\eta^b + t^{c-d}\bar w^d = 0$, modulo the action $\frac{a}{d}(\frac{d}{b}, 1)$. For $t = 0$ we have $1 + \bar\eta^b = 0$, so there are indeed $\frac{b}{a/d} = r$ points on the compactification of the special fibre. The coordinate transformation from $(\xi, \eta, 1)$ coordinates to $(1, \bar\eta, \bar w)$ is $\xi = \bar w^{-\frac{d}{a}}$, $\eta = \bar\eta \bar w^{-\frac{d}{b}}$. We blow up in the r points at infinity on the special fibre. The functions $x := \xi t^{\frac{d}{a}}$, $y := \eta t^{\frac{d}{a}}$ and t are holomorphic in a neighbourhood of the strict transform of the special fibre, and generate the local ring of the Kulikov singularity. They satisfy $x^a + y^b + t^c = 0$. $\square$

It follows that the family of curves obtained by resolving the singularity of $\xi^a + \eta^b + t^{c-d}w^d = 0$ is not minimal if $c - d \ge d = \operatorname{lcm}(a, b)$. Furthermore the resolution graph of $x^a + y^b + t^{c-d}$ is a subgraph of the resolution graph of $x^a + y^b + t^c$.

Proposition 3.2. *Write $c = c_0 + c_1 d$ with $0 \le c_0 < d$. The characteristic cycle of the Brieskorn-Pham singularity (3.1) has support on the subgraph corresponding to the singularity $x^a + y^b + t^{c_0+d}$ and is the fundamental cycle of that singularity. In particular, the characteristic cycle is equal to the fundamental cycle if and only if $d \le c < 2d$.*

Proof. If the family used in the construction above is not minimal, one can blow down each component of the strict transform of the affine curve $\xi^a + \eta^b = 0$ and still have a family of the same type. So the family is minimal if and only $c - d < d$. The result now follows from Proposition 2.3.(4). $\square$

The Proposition was proved by Tomaru [17] using an explicit description of the resolution of the singularity. As to this resolution, we note that there are r chains of $c_1 - 1$ (-2)-curves from the characteristic cycle to the components of $\widetilde{X}$.

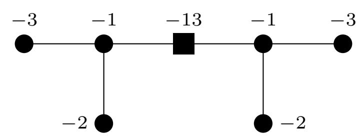
Remark 3.3. The above result extends with the same proof to the case of Brieskorn complete intersections. A proof in the style of [17] was given by Meng, Yuan and Wang [9].

4. Singularities with a specific resolution graph

A recent paper Némethi and Okuma [12] concerns the problem of determining upper and lower bounds for the geometric genus in terms of the resolution graph. The Authors study which analytic structures can occur for a specific resolution graph, giving details for an example already mentioned by Némethi [11]. Here we rederive their results from our point of view.

The main feature of the example is that the topological upper bound for p_g is not realised. The maximal p_g occurs for a non Gorenstein Kulikov singularity and for a Gorenstein splice type singularity.

The singularity considered has an integral homology sphere link. The resolution graph for the minimal good resolution is:

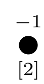


This graph satisfies the semigroup condition of Neumann and Wahl [13] so there exist singularities of splice type with this graph, with $p_g = 3$. The defining equations of this complete intersection singularity have 'leading' forms

$$z_1^2 z_2 + z_3^2 + z_4^3, \qquad z_1^3 + z_2^2 + z_4^2 z_3 \ . \tag{4.1}$$

On the minimal resolution the exceptional curve is an irreducible two-cuspidal rational curve, of self-intersection -1. Therefore the resolution graph for the minimal resolution is simply:

(4.2)



with a possibly singular central curve. This is the same graph as when the exceptional divisor is a smooth curve of genus two. We note that there exists a Gorenstein Kulikov singularity with this graph, namely the hypersurface $z^2 = y^5 + x^{10}$; it has the maximal geometric genus: $p_g = 4$.

We first analyse the Gorenstein condition. On the minimal resolution M adjunction gives for the exceptional curve that $\omega_E = \omega_M \otimes \mathcal{O}_E(E)$. The singularity is Gorenstein if and only if $\omega_M = \mathcal{O}_M(-3E)$. This happens if and only if $\omega_E = \mathcal{O}_E(-2E)$, that is, if the conormal bundle of E is a theta characteristic.

Lemma 4.1. *A singularity with resolution graph* (4.2) *satisfies* $2 \le p_g \le 4$. *If* $p_g = 4$ *then it is a Gorenstein Kulikov singularity. If* $p_g = 3$ *it is either non Gorenstein Kulikov of multiplicity 3 or a non Kulikov complete intersection.*

Proof. To analyse the possible values for p_g we look at a computation sequence. Here one compares the different $\mathcal{O}(-kE)$ via the short exact sequences

$$0 \longrightarrow \mathcal{O}(-(k+1)E) \longrightarrow \mathcal{O}(-kE) \longrightarrow \mathcal{O}_E(-kE) \longrightarrow 0$$

As $H^1(\widetilde{X}, \mathcal{O}(-3E)) = 0$ one gets the exact sequences

$$0 \longrightarrow H^1(\widetilde{X}, \mathcal{O}(-E)) \longrightarrow H^1(\widetilde{X}, \mathcal{O}) \longrightarrow H^1(E, \mathcal{O}_E) \longrightarrow 0$$

$$H^0(E,\mathcal{O}_E(-E)) \longrightarrow H^1(\widetilde{X},\mathcal{O}(-2E)) \longrightarrow H^1(\widetilde{X},\mathcal{O}(-E)) \longrightarrow H^1(E,\mathcal{O}_E(-E)) \longrightarrow 0$$

and the isomorphism $H^1(\widetilde{X},\mathcal{O}(-2E)) \cong H^1(E,\mathcal{O}_E(-2E))$.

This gives $2 \le p_g \le 4$. If $p_g = 4$ then $\mathcal{O}_E(-2E) = \omega_E$, so the singularity is Gorenstein. Moreover, the theta characteristic is odd. Indeed, on a smooth genus two curve the divisor of a Weierstrass point is an odd theta characteristic. The Kulikov construction starting from a trivial family and blowing just one Weierstrass point lying on the central fibre, yields the example $z^2 = y^5 + x^{10}$.

A two-cuspidal rational curve has only one theta characteristic, which is even [3]. This can also be seen from the description of the pencil with this special fibre in the list of Namikawa and Ueno [10]: their example is $y^2 = (x^3+t)((x-1)^3+t)$, and one sees that three Weierstrass points come together in cusp. This shows that there cannot be a singularity with this exceptional divisor with $p_g = 4$. But any computation with the quadratic lattice H cannot distinguish between such a curve and a smooth curve.

A non Gorenstein Kulikov singularity is obtained by blowing up one smooth point of the special fibre; for a smooth curve this point should not be a Weierstrass point. By construction the general hyperplane section is a curve with Milnor fibre of genus two, so $\delta = 2$. The only irreducible non Gorenstein curve singularity is the monomial curve (t^3,t^4,t^5). Therefore the surface singularity has multiplicity 3 and embedding dimension 4. In this case $H^0(E,\mathcal{O}_E(-E)) = \mathbb{C}$, so $p_g = 3$.

If the singularity is Gorenstein, but not Kulikov, then $p_g = 3$ and the curve E has an even theta characteristic. For a smooth E there exists a quasi-homogeneous singularity. Let $y^2 = f_6(x,\bar{x})$ be a hyperelliptic curve E, and write $f_6 = PQ$ with P, Q of degree 3. Consider the divisor $(P) = 2D$, with D a divisor of degree 3 on E, consisting of three Weierstrass points. Then $\mathcal{O}_E(D-K_E)$ is an even theta characteristic. The graded ring $\bigoplus H^0(E,\mathcal{O}_E(k(D-K_E)))$ is generated by $z = xP$, $\bar{z} = \bar{x}P$, $w = yP$ and $v = P^2$. The equations are then

$$w^2 = Q(z,\bar{z}), \qquad v^2 = P(z,\bar{z})\ .$$

The singularity with two-cuspidal curve as exceptional curve is a superisolated complete intersection singularity. The graded tangent cone is obtained in the same way as above, by taking $P = x^3$, $Q = \bar{x}^3$. We have to add terms of lowest degree to make the singularity isolated, resulting in splice diagram equations of the form (4.1):

$$w^2 = \bar{z}^3 + vz^2, \qquad v^2 = z^3 + w\bar{z}^2\ .$$

$\square$

Finally a quasi-homogeneous singularity with $p_g = 2$ is obtained from a divisor $D-K_E$ with$D$ a general effective divisor of degree 3 on a smooth curve E. The graded ring $\bigoplus H^0(E,\mathcal{O}_E(k(D-K_E)))$ has 7 generators. The same ring for the two-cuspidal rational curve gives a weighted tangent cone of a singularity in $\mathbb{C}^7$.

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REMARKS ON THE GAUDIN MODEL MODULO p

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In memory of Egbert Brieskorn (1936–2013)

Abstract. We discuss the Bethe ansatz in the Gaudin model on the tensor product of finite-dimensional $\mathfrak{sl}_2$-modules over the field $\mathbb{F}_p$ with p elements, where p is a prime number. We define the Bethe ansatz equations and show that if $(t_1^0, \dots, t_k^0)$ is a solution of the Bethe ansatz equations, then the corresponding Bethe vector is an eigenvector of the Gaudin Hamiltonians. We characterize solutions $(t_1^0, \dots, t_k^0)$ of the Bethe ansatz equations as certain two-dimensional subspaces of the space of polynomials $\mathbb{F}_p[x]$. We consider the case when the number of parameters k equals 1. In that case we show that the Bethe algebra, generated by the Gaudin Hamiltonians, is isomorphic to the algebra of functions on the scheme defined by the Bethe ansatz equation. If $k = 1$ and in addition the tensor product is the product of vector representations, then the Bethe algebra is also isomorphic to the algebra of functions on the fiber of a suitable Wronski map.

1. Introduction

The Gaudin model is a certain collection of commuting linear operators on the tensor product $V = \otimes_{i=1}^n V_i$ of representations of a Lie algebra $\mathfrak{g}$. The operators are called the Gaudin Hamiltonians. The Bethe ansatz is a method used to construct common eigenvectors and eigenvalues of the Gaudin operators. One looks for an eigenvector in a certain form $W(t)$, where $W(t)$ is a V-valued function of some parameters $t = (t_1, \dots, t_k)$. One introduces a system of equations on the parameters, called the Bethe ansatz equations, and shows that if t^0 is a solution of the system, then the vector $W(t^0)$ is an eigenvector of the Gaudin Hamiltonians, see for example [B, G, FFR, MV1, MV2, MTV1, MTV4, RV, SchV, SV1, V1, V2, V3]. The Gaudin model has strong relations with the Schubert calculus and real algebraic geometry, see for example [MTV2, MTV3, So].

All that is known in the case when the Lie algebra $\mathfrak{g}$ is defined over the field $\mathbb{C}$ of complex numbers. In the paper we consider the case of the field $\mathbb{F}_p$ with p elements, where p is a prime number, cf. [SV3]. We carry out the first steps of the Bethe ansatz, the deeper parts of the Gaudin model over a finite field remain to be developed. We consider the case of the Lie algeba $\mathfrak{sl}_2$, where the notations and constructions are shorter and simpler.

It is known that over $\mathbb{C}$, the Gaudin model is a semi-classical limit of the KZ differential equations of conformal field theory, and the construction of the multidimensional hypergeometric solutions of the KZ differential equations lead, in that limit, to the Bethe ansatz construction of eigenvectors of the Gaudin Hamiltonians, see [RV]. The $\mathbb{F}_p$-analogs of the hypergeometric solutions of the KZ differential equations were constructed recently in [SV3], see also [V5]. Thus the constructions of this paper may be thought of as a semi-classical limit of the constructions in [SV3].

In Section 2 we define the Bethe ansatz equations and show that if $(t_1^0, \dots, t_k^0)$ is a solution of the Bethe ansatz equations, then the corresponding Bethe vector is an eigenvector of the Gaudin

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Hamiltonians. In Section 3 we characterize solutions $(t_1^0, \dots, t_k^0)$ of the Bethe ansatz equations as certain two-dimensional subspaces of the space of polynomials $\mathbb{F}_p[x]$. In Section 4 we consider the case in which the number k of the parameters equals 1. In that case we show that the Bethe algebra, generated by the Gaudin Hamiltonians, is isomorphic to the algebra of functions on the scheme defined by the Bethe ansatz equation, see Theorem 4.2. If $k = 1$ and in addition the tensor product is the product of vector representations, then the Bethe algebra is also isomorphic to the algebra of functions on the fiber of a suitable Wronski map, see Corollary 4.9.

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2. $\mathfrak{sl}_2$ Gaudin model

2.1. $\mathfrak{sl}_2$ Gaudin model over $\mathbb{C}$. Let e, f, h be the standard basis of the complex Lie algeba $\mathfrak{sl}_2$ with $[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$. The element

$$\Omega = e \otimes f + f \otimes e + \frac{1}{2} h \otimes h \ \in \ \mathfrak{sl}_2 \otimes \mathfrak{sl}_2 \tag{2.1}$$

is called the Casimir element. Given n, for $1 \leqslant i < j \leqslant n$ let $\Omega^{(i,j)} \in (U(\mathfrak{sl}_2))^{\otimes n}$ be the element equal to Ω in the i-th and j-th factors and to 1 in other factors. Let $z^0 = (z_1^0, \dots, z_n^0) \in \mathbb{C}^n$ have distinct coordinates. For $s = 1, \dots, n$ introduce

$$H_s(z^0) = \sum_{l \neq s} \frac{\Omega^{(s,l)}}{z_s^0 - z_l^0} \ \in \ (U(\mathfrak{sl}_2))^{\otimes n}, \tag{2.2}$$

the *Gaudin Hamiltonians*, see [G]. For any s, l, we have

$$\big[H_s(z^0), H_l(z^0)\big] = 0, \tag{2.3}$$

and for any $x \in \mathfrak{sl}_2$ and s we have

$$[H_s(z^0), x \otimes 1 \otimes \dots \otimes 1 + \dots + 1 \otimes \dots \otimes 1 \otimes x] = 0. \tag{2.4}$$

Let $V = \otimes_{i=1}^n V_i$ be a tensor product of $\mathfrak{sl}_2$-modules. The commutative subalgebra of $\mathrm{End}(V)$ generated by the Gaudin Hamiltonians $H_i(z^0)$, $i = 1, \dots, n$, and the identity operator Id is called the *Bethe algebra* of V. If $W \subset V$ is a subspace invariant with respect to the Bethe algebra, then the restriction of the Bethe algebra to W is called the Bethe algebra of W, denoted by $\mathcal{B}(W)$.

The general problem is to describe the Bethe algebra, its common eigenvectors and eigenvalues.

2.2. Irreducible $\mathfrak{sl}_2$-modules. For a nonnegative integer i denote by L_i the irreducible $i+1$-dimensional module with basis $v_i, fv_i, \dots, f^i v_i$ and action $h.f^k v_i = (i-2k) f^k v_i$ for $k = 0, \dots, i$; $f.f^k v_i = f^{k+1} v_i$ for $k = 0, \dots, i-1$, $f.f^i v_i = 0$; $e.v_i = 0$, $e.f^k v_i = k(i-k+1) f^{k-1} v_i$ for $k = 1, \dots, i$.

For $m = (m_1, \dots, m_n) \in \mathbb{Z}_{\geqslant 0}^n$, denote $|m| = m_1 + \dots + m_n$ and $L^{\otimes m} = L_{m_1} \otimes \dots \otimes L_{m_n}$. For $J = (j_1, \dots, j_n) \in \mathbb{Z}_{\geqslant 0}^n$, with $j_s \leqslant m_s$ for $s = 1, \dots, n$, the vectors

$$f_J v_m := f^{j_1} v_{m_1} \otimes \dots \otimes f^{j_n} v_{m_n} \tag{2.5}$$

form a basis of $L^{\otimes m}$. We have

$$f.f_J v_m = \sum_{s=1}^n f_{J+1_s} v_m, \qquad h.f_J v_m = (|m| - 2|J|) f_J v_m,$$
$$e.f_J v_m = \sum_{s=1}^n j_s (m_s - j_s + 1) f_{J-1_s} v_m.$$

For $\lambda \in \mathbb{Z}$, introduce the weight subspace $L^{\otimes m}[\lambda] = \{\, v \in L^{\otimes m} \mid h.v = \lambda v\}$ and the singular weight subspace $\mathrm{Sing} L^{\otimes m}[\lambda] = \{\, v \in L^{\otimes m}[\lambda] \mid h.v = \lambda v,\ e.v = 0\}$. We have the weight decomposition $L^{\otimes m} = \oplus_{k=0}^{|m|} L^{\otimes m}[|m| - 2k]$. Denote

$$\mathcal{I}_k = \{J \in \mathbb{Z}^n_{\geqslant 0} \mid |J| = k,\ j_s \leqslant m_s,\ s = 1, \dots, n\}.$$

The vectors $(f_J v)_{J \in \mathcal{I}_k}$ form a basis of $L^{\otimes m}[|m| - 2k]$.

By (2.4), the Bethe algebra $\mathcal{B}(L^{\otimes m})$ preserves each of the subspaces $L^{\otimes m}[|m| - 2k]$ and $\mathrm{Sing} L^{\otimes m}[|m| - 2k]$. If $w \in L^{\otimes m}$ is a common eigenvector of the Bethe algebra, then for any $x \in \mathfrak{sl}_2$ the vector $x.w$ is also an eigenvector with the same eigenvalues. These observations show that in order to describe $\mathcal{B}(L^{\otimes m})$, its eigenvectors and eigenvalues it is enough to describe for all k the algebra $\mathcal{B}(\mathrm{Sing} L^{\otimes m}[|m| - 2k])$, its eigenvectors and eigenvalues.

2.3. **Bethe ansatz on** $\mathrm{Sing} L^{\otimes m}\big[|m|-2k\big]$ **over** $\mathbb{C}$. Given $k, n \in \mathbb{Z}_{>0}$, $m = (m_1, \dots, m_n) \in \mathbb{Z}^n_{>0}$. Let $z^0 = (z^0_1, \dots, z^0_n) \in \mathbb{C}^n$ have distinct coordinates. The system of the *Bethe ansatz equations* is the system of equations

$$\sum_{j \neq i} \frac{2}{t_i - t_j} - \sum_{s=1}^{n} \frac{m_s}{t_i - z^0_s} = 0, \qquad i = 1, \dots, k, \tag{2.6}$$

on $t = (t_1, \dots, t_k) \in \mathbb{C}^k$. If $(t^0_1, \dots, t^0_k, z^0_1, \dots, z^0_n) \in \mathbb{C}^{k+n}_p$ has distinct coordinates, denote

$$\lambda_s(t^0, z^0) = \sum_{l \neq s} \frac{m_s m_l/2}{z^0_s - z^0_l} - \sum_{i=1}^{k} \frac{m_s}{z^0_s - t^0_i}, \qquad s = 1, \dots, n. \tag{2.7}$$

For any function or differential form $F(t_1, \dots, t_k)$, denote

$$\mathrm{Sym}_t[F(t_1, \dots, t_k)] = \sum_{\sigma \in S_k} F(t_{\sigma_1}, \dots, t_{\sigma_k}), \quad \mathrm{Ant}_t[F(t_1, \dots, t_k)] = \sum_{\sigma \in S_k} (-1)^{|\sigma|} F(t_{\sigma_1}, \dots, t_{\sigma_k}).$$

For $J = (j_1, \dots, j_n) \in \mathcal{I}_k$ define the *weight function*

$$W_J(t, z) = \frac{1}{j_1! \dots j_n!} \mathrm{Sym}_t \left[\prod_{s=1}^{n} \prod_{i=1}^{j_s} \frac{1}{t_{j_1 + \dots + j_{s-1} + i} - z_s} \right]. \tag{2.8}$$

For example,

$$W_{(1,0,\dots,0)} = \frac{1}{t_1 - z_1}, \qquad W_{(2,0,\dots,0)} = \frac{1}{t_1 - z_1} \frac{1}{t_2 - z_1},$$
$$W_{(1,1,0,\dots,0)} = \frac{1}{t_1 - z_1} \frac{1}{t_2 - z_2} + \frac{1}{t_2 - z_1} \frac{1}{t_1 - z_2}.$$

The function

$$W_{k,n,m}(t, z) = \sum_{J \in \mathcal{I}_k} W_J(t, z) f_J v_m \tag{2.9}$$

is the $L^{\otimes m}[|m| - 2k]$-valued *vector weight function*.

Theorem 2.1 ([RV, B], cf. [SV1])**.** *If* $(t^0, z^0) = (t^0_1, \dots, t^0_k, z^0_1, \dots, z^0_n)$ *is a solution of the Bethe ansatz equations* (2.6)*, then the vector* $W_{k,n,m}(t^0, z^0)$ *lies in* $\mathrm{Sing} L^{\otimes m}[|m| - 2k]$ *and is an eigenvector of the Gaudin Hamiltonians, moreover,*

$$H_i(z^0).W_{k,n,m}(t^0, z^0) = \lambda_i(t^0, z^0) W_{k,n,m}(t^0, z^0), \qquad i = 1, \dots, n. \tag{2.10}$$

The eigenvector $W_{k,n,m}(t^0,z^0)$ is called the *Bethe eigenvector*. On the Bethe eigenvectors see, for example, [SchV, MV1, MV2, V1, V2, V3].

The fact that $W_{k,n,m}(t^0,z^0)$ in Theorem 2.1 lies in $\mathrm{Sing}L^{\otimes m}[|m|-2k]$ may be reformulated as follows. For any $J\in\mathcal{I}_{k-1}$, we have

$$\sum_{s=1}^{n}(j_s+1)(m_s-j_s)W_{J+\mathbf{1}_s}(t^0,z^0)=0, \tag{2.11}$$

where we set $W_{J+\mathbf{1}_s}(t^0,z^0)=0$ if $J+\mathbf{1}_s\notin\mathcal{I}_k$.

2.4. **Proof of Theorem 2.1.** We sketch the proof following [SV1]. The intermediate statements in this proof will be used later when constructing eigenvectors of the Bethe algebra over $\mathbb{F}_p$. The proof is based on the following cohomological relations.

Given $k,n\in\mathbb{Z}_{>0}$ and a multi-index $J=(j_1,\dots,j_n)$ with $|J|\leqslant k$, introduce a differential form

$$\eta_J=\frac{1}{j_1!\cdots j_n!}\,\mathrm{Ant}_t\Big[\frac{d(t_1-z_1)}{t_1-z_1}\wedge\dots\wedge\frac{d(t_{j_1}-z_1)}{t_{j_1}-z_1}\wedge\frac{d(t_{j_1+1}-z_2)}{t_{j_1+1}-z_2}\wedge\dots$$
$$\wedge\frac{d(t_{j_1+\dots+j_{n-1}+1}-z_n)}{t_{j_1+\dots+j_{n-1}+1}-z_n}\wedge\dots\wedge\frac{d(t_{j_1+\dots+j_n}-z_n)}{t_{j_1+\dots+j_n}-z_n}\Big],$$

which is a logarithmic differential form on $\mathbb{C}^n\times\mathbb{C}^k$ with coordinates z,t. If $|J|=k$, then for any $z^0\in\mathbb{C}^n$ we have on $\{z^0\}\times\mathbb{C}^k$ the identity

$$\eta_J\big|_{\{z^0\}\times\mathbb{C}^k}=W_J(t,z^0)dt_1\wedge\dots\wedge dt_k. \tag{2.12}$$

Example 2.1. For $k=n=2$ we have

$$\begin{aligned}\eta_{(2,0)}&=\frac{d(t_1-z_1)}{t_1-z_1}\wedge\frac{d(t_2-z_1)}{t_2-z_1},\\ \eta_{(1,1)}&=\frac{d(t_1-z_1)}{t_1-z_1}\wedge\frac{d(t_2-z_2)}{t_2-z_2}-\frac{d(t_2-z_1)}{t_2-z_1}\wedge\frac{d(t_1-z_2)}{t_1-z_2}.\end{aligned}$$

Introduce the logarithmic differential 1-forms

$$\begin{aligned}\alpha&=\sum_{1\leqslant i<j\leqslant n}\frac{m_im_j}{2}\frac{d(z_i-z_j)}{z_i-z_j}+\sum_{1\leqslant i<j\leqslant k}2\frac{d(t_i-t_j)}{t_i-t_j}-\sum_{s=1}^{n}\sum_{i=1}^{k}m_s\frac{d(t_i-z_s)}{t_i-z_s},\\ \alpha'&=\sum_{1\leqslant i<j\leqslant k}2\frac{d(t_i-t_j)}{t_i-t_j}-\sum_{s=1}^{n}\sum_{i=1}^{k}m_s\frac{d(t_i-z_s)}{t_i-z_s}.\end{aligned}$$

We shall use the following algebraic identities for logarithmic differential forms.

Theorem 2.2 ([SV1]). *We have*

$$\alpha'\wedge\eta_J=\sum_{s=1}^{n}(j_s+1)(m_s-j_s)\eta_{J+\mathbf{1}_s}, \tag{2.13}$$

for any J with $|J|=k-1$, and

$$\alpha\wedge\sum_{J\in\mathcal{I}_k}\eta_Jf_Jv_m=\sum_{i<j}\Omega^{(i,j)}\frac{d(z_i-z_j)}{z_i-z_j}\wedge\sum_{|J|=k}\eta_Jf_Jv_m. \tag{2.14}$$

Proof. Identity (2.13) is Theorem 6.16.2 in [SV1] for the case of the Lie algebra $\mathfrak{sl}_2$. Identity (2.14) is Theorem 7.5.2" in [SV1] for the case of the Lie algebra $\mathfrak{sl}_2$. ☐

If (t^0, z^0) is a solution of the Bethe ansatz equations, then $\alpha'|_{(t^0,z^0)} = 0$ and formulas (2.13), (2.12) give (2.11). Similarly, if (t^0, z^0) is a solution of the Bethe ansatz equations, then

$$\alpha|_{(t^0,z^0)} = \sum_{s=1}^{n} \lambda_s(t^0, z^0) dz_s$$

and formulas (2.14) and (2.12) give (2.18). Theorem 2.1 is proved.

2.5. **Bethe ansatz on** $\mathrm{Sing}L^{\otimes m}\big[|m| - 2k\big]$ **over** $\mathbb{F}_p$**.** Given

$$k, n \in \mathbb{Z}_{>0}, \quad m = (m_1, \dots, m_n) \in \mathbb{Z}^n_{>0},$$

let p be a prime number. Consider the Lie algebra $\mathfrak{sl}_2$ as an algebra over the field $\mathbb{F}_p$ and the $\mathfrak{sl}_2$-modules L_{m_s}, $s = 1, \dots, n$, over $\mathbb{F}_p$. Let $z^0 = (z^0_1, \dots, z^0_n) \in \mathbb{F}^n_p$ have distinct coordinates. The Gaudin Hamiltonians $H_s(z^0)$ of formula (2.2) define commuting $\mathbb{F}_p$-linear operators on the $\mathbb{F}_p$-vector space $L^{\otimes m} = \otimes^n_{s=1} L_{m_s}$. By formula (2.4) the Gaudin Hamiltonians preserve the $\mathbb{F}_p$-subspaces $\mathrm{Sing}L^{\otimes m}[|m| - 2k]$ and we may study eigenvectors of the Gaudin Hamiltonians on a subspace $\mathrm{Sing}L^{\otimes m}[|m| - 2k]$.

Consider the *system of Bethe ansatz equations*

$$\sum_{j\neq i} \frac{2}{t_i - t_j} - \sum_{s=1}^{n} \frac{m_s}{t_i - z^0_s} = 0, \qquad i = 1, \dots, k, \tag{2.15}$$

as a system of equations on $t = (t_1, \dots, t_k) \in \mathbb{F}^k_p$. If $(t^0_1, \dots, t^0_k, z^0_1, \dots, z^0_n) \in \mathbb{F}^{k+n}_p$ has distinct coordinates, denote

$$\lambda_s(t^0, z^0) = \sum_{l\neq s} \frac{m_s m_l/2}{z^0_s - z^0_l} - \sum_{i=1}^{k} \frac{m_s}{z^0_s - t^0_i} \ \in \mathbb{F}_p, \qquad s = 1, \dots, n. \tag{2.16}$$

Theorem 2.3. *Let p be a prime number and $p > |m|$. Let $t^0 \in \mathbb{F}^k_p$ be a solution of the Bethe ansatz equations (2.15). Then the vector $W_{k,n,m}(t^0, z^0)$ is well-defined and lies in the subspace $\mathrm{Sing}L^{\otimes m}[|m| - 2k]$, that is, the equations*

$$\sum_{s=1}^{n} (j_s + 1)(m_s - j_s) W_{J+\mathbf{1}_s}(t^0, z^0) = 0, \tag{2.17}$$

hold, also the vector $W_{k,n,m}(t^0, z^0)$ satisfies the equations

$$H_s(z^0).W_{k,n,m}(t^0, z^0) = \lambda_s(t^0, z^0) W_{k,n,m}(t^0, z^0), \qquad s = 1, \dots, n. \tag{2.18}$$

Proof. The proof of Theorem 2.3 is the same as the proof of Theorem 2.1 since identities (2.13) and (2.14) hold over half integers and can be projected to $\mathbb{F}_p$. □

3. Two-dimensional spaces of polynomials

3.1. **Two-dimensional spaces of polynomials over** $\mathbb{C}$**.** For a function $g(x)$ denote $g' = \frac{dg}{dx}$. For functions $g(x), h(x)$ define the *Wronskian*

$$\mathrm{Wr}(g(x), h(x)) = g'(x)h(x) - g(x)h'(x).$$

Theorem 3.1 ([SchV], cf. [MV2])**.** *Let $k \in \mathbb{Z}_{>0}$, $m = (m_1, \dots, m_n) \in \mathbb{Z}^n_{>0}$. Let $(t^0, z^0) \in \mathbb{C}^{k+n}$ have distinct coordinates. Denote*

$$y(x) = \prod_{i=1}^{k} (x - t^0_i), \qquad T(x) = \prod_{s=1}^{n} (x - z^0_s)^{m_s}. \tag{3.1}$$

We have the following two statements.

(i) *If (t^0, z^0) is a solution of the Bethe ansatz equations (2.6), then $k \leqslant |m|+1$, $2k \neq |m|+1$, and there exists a polynomial $\tilde{y}(x) \in \mathbb{C}[x]$ of degree $|m|+1-k$ such that*

$$\mathrm{Wr}(\tilde{y}(x), y(x)) = T(x). \tag{3.2}$$

(ii) *If there exists a polynomial $\tilde{y}(x)$ satisfying equation (3.2), then $k \leqslant |m|+1$, $2k \neq |m|+1$ and (t^0, z^0) is a solution of the Bethe ansatz equations (2.6).*

Proof. We will use the proof below in the proof of the p-version of Theorem 3.1. Equation (3.2) is a first order differential equation with respect to $\tilde{y}(x)$. Then

$$\left(\frac{\tilde{y}(x)}{y(x)}\right)' = \frac{T(x)}{y(x)^2} \tag{3.3}$$

and

$$\tilde{y}(x) = y(x)\int \frac{T(x)}{y(x)^2}dx = y(x)\int \frac{T(x)}{\prod_{i=1}^k (x-t_i^0)^2}dx. \tag{3.4}$$

We have the unique presentation $T(x) = Q(x)\prod_{i=1}^k (x-t_i^0)^2 + R(x)$ with $P(x), Q(x) \in \mathbb{C}[x]$ such that $Q(x) = 0$ if $2k > |m|$ and $Q(x) = a_{|m|-2k}x^{|m|-2k} + \cdots + a_0$ is of degree $|m|-2k$ otherwise; $\deg R(x) < 2k$. We have the unique presentation

$$\frac{R(x)}{\prod_{i=1}^k (x-t_i^0)^2} = \sum_{i=1}^k \left(\frac{a_{i,2}}{(x-t_i^0)^2} + \frac{a_{i,1}}{x-t_i^0}\right), \tag{3.5}$$

where

$$a_{i,2} = \frac{T(x)}{\prod_{j\neq i}^k (x-t_j^0)^2}\bigg|_{x=t_i^0}, \qquad a_{i,1} = \frac{d}{dx}\left(\frac{T(x)}{\prod_{j\neq i}^k (x-t_j^0)^2}\right)\bigg|_{x=t_i^0}. \tag{3.6}$$

We have

$$\frac{d}{dx}\left(\frac{T(x)}{\prod_{j\neq i}^k (x-t_j^0)^2}\right)\bigg|_{x=t_i^0} = \left(\sum_{s=1}^n \frac{m_s}{t_i^0 - z_s^0} - \sum_{j\neq i}\frac{2}{t_i^0 - t_j^0}\right)\frac{T(t_i^0)}{\prod_{j\neq i}^k (t_i^0 - t_j^0)^2}.$$

Since (t^0, z^0) has distinct coordinates we conclude that $a_{i,1} = 0$ for $i = 1, \dots, k$, if and only if (t^0, z^0) is a solution of (2.6).

Let (t^0, z^0) be a solution of (2.6). By formula (3.4) we have

$$\tilde{y}(x) = \prod_{i=1}^k (x-t_i^0)\left(c - \sum_{i=1}^k \frac{a_{i,2}}{x-t_i^0}\right), \quad \text{if } 2k > |m|, \tag{3.7}$$

$$\tilde{y}(x) = \prod_{i=1}^k (x-t_i^0)\left(\frac{a_{|m|-2k}}{|m|-2k+1}x^{|m|-2k+1} + \cdots + a_0 x + c - \sum_{i=1}^k \frac{a_{i,2}}{x-t_i^0}\right),$$

if $2k \leqslant |m|$, where $c \in \mathbb{C}$ is an arbitrary number. In each of the two cases we may choose c so that $\deg \tilde{y}(x) \neq \deg y(x)$. Using the identity

$$\mathrm{Wr}(x^\alpha, x^\beta) = (\alpha - \beta)x^{\lambda+\beta-1} \tag{3.8}$$

we obtain in this case that

$$\deg \tilde{y}(x) + \deg y(x) = |m|+1. \tag{3.9}$$

Hence $k \leqslant |m|+1$ and $k \neq |m|+1-k$. The first part of the theorem is proved.

Let there exist a polynomial $\tilde{y}(x)$ satisfying equation (3.2). Adding to $\tilde{y}(x)$ the polynomial $y(x)$ with a suitable coefficient if necessary we may assume that $\deg \tilde{y}(x) \neq \deg y(x)$. Then (3.9) implies $k \leqslant |m|+1$ and $k \neq |m|+1-k$.

By formula (3.3) we have

$$\left(\frac{\tilde{y}(x)}{y(x)}\right)' = a_{|m|-2k}x^{|m|-2k} + \cdots + a_0 + \sum_{i=1}^{k}\Big(\frac{a_{i,2}}{(x-t_i^0)^2} + \frac{a_{i,1}}{x-t_i^0}\Big) \text{ if } 2k \leqslant |m| \tag{3.10}$$

and

$$\left(\frac{\tilde{y}(x)}{y(x)}\right)' = \sum_{i=1}^{k}\Big(\frac{a_{i,2}}{(x-t_i^0)^2} + \frac{a_{i,1}}{x-t_i^0}\Big) \text{ if } 2k > |m|. \tag{3.11}$$

The function $\frac{\tilde{y}(x)}{y(x)}$ has a unique decomposition into the sum of a polynomial and simple fractions. The term by term derivative of that decomposition equals the right-hand side of (3.10) or (3.11). Hence all of coefficients $a_{i,1}$ must be zero. Hence the roots of $y(x)$ satisfy the Bethe ansatz equations. $\square$

Remark. This construction assigns to a solution (t^0, z^0) of the Bethe ansatz equations the two-dimensional subspace $\langle \tilde{y}(x), y(x)\rangle$ of the space of polynomials $\mathbb{C}[x]$ such that $\deg y(x) = k, \deg \tilde{y}(x) = |m|-k+1$, $\mathrm{Wr}(y(x), \tilde{y}(x)) = T(x)$. That subspace is a point of the Grassmannian of two-dimensional subspaces of $\mathbb{C}[x]$.

3.2. **Two-dimensional spaces of polynomials over $\mathbb{F}_p$.**

Theorem 3.2. *Let $k \in \mathbb{Z}_{>0}$, $m = (m_1, \dots, m_n) \in \mathbb{Z}^n_{>0}$. Let $p > |m|+1$, $p > n+k$. Let $(t^0, z^0) \in \mathbb{F}_p^{k+n}$ have distinct coordinates. Denote*

$$y(x) = \prod_{i=1}^{k}(x-t_i^0), \qquad T(x) = \prod_{s=1}^{n}(x-z_s^0)^{m_s} \ \in \mathbb{F}_p[x]. \tag{3.12}$$

We have the following two statements.

(i) *If (t^0, z^0) is a solution of the Bethe ansatz equations (2.15), then $k \leqslant |m|+1$, $2k \neq |m|+1$, and there exists a polynomial $\tilde{y}(x) \in \mathbb{F}_p[x]$ of degree $|m|+1-k$ such that*

$$\mathrm{Wr}(\tilde{y}(x), y(x)) = T(x). \tag{3.13}$$

(ii) *If there exists a polynomial $\tilde{y}(x) \in \mathbb{F}_p[x]$ satisfying equation (3.13), then $k \leqslant |m|+1$, $2k \neq |m|+1$ and (t^0, z^0) is a solution of the Bethe ansatz equations (2.15).*

Proof.

Lemma 3.3. *Let p be a prime number. Let $d_1, \dots, d_k \in \mathbb{Z}_{>0}$ with $d_i \leqslant 2$ for all i. Let $t_1^0, \dots, t_k^0 \in \mathbb{F}_p$ be distinct and $T(x) \in \mathbb{F}_p[x]$. Then there exists a unique presentation*

$$\frac{T(x)}{\prod_{i=1}^{k}(x-t_i^0)^{d_i}} = Q(x) + \sum_{i=1}^{k}\sum_{j=1}^{d_i}\frac{a_{i,j}}{(x-t_i^0)^j}, \tag{3.14}$$

where $Q(x) \in \mathbb{F}_p[x]$ and

$$a_{i,j} = \frac{d^{j-1}}{dx^{j-1}}\left(\frac{T(x)}{\prod_{l\neq i}^{k}(x-t_l^0)^{d_l}}\right)\Big|_{x=t_i^0}. \tag{3.15}$$

Proof. The uniqueness is clear. Let us show the existence. Lift $t_1^0, \dots, t_k^0$, $T(x)$ to $t_1^1, \dots, t_k^1 \in \mathbb{Z}$, $T^1(x) \in \mathbb{Z}[x]$. We have

$$\frac{T^1(x)}{\prod_{i=1}^k (x - t_i^1)^{d_i}} = Q^1(x) + \sum_{i=1}^k \sum_{j=1}^{d_i} \frac{a_{i,j}^1}{(x - t_i^1)^j}, \tag{3.16}$$

where $Q^1(x) \in \mathbb{Z}[x]$ and

$$a_{i,j}^1 = \frac{d^{j-1}}{dx^{j-1}} \left(\frac{T^1(x)}{\prod_{j \neq i}^k (x - t_j^1)^{d_j}} \right)\Bigg|_{x = t_i^0}. \tag{3.17}$$

It is easy to see that for $j = 1, 2$ and all i the coefficient $a_{i,j}^1$ has a well-defined projection to $\mathbb{F}_p$. By projecting (3.16) to $\mathbb{F}_p$ we obtain a presentation of (3.14). □

The proof of Theorem 3.2 is based on Lemma 3.3 and is analogous to the proof of Theorem 3.1. If (t^0, z^0) is a solution of (2.15), then

$$\left(\frac{\tilde{y}(x)}{y(x)}\right)' = \frac{T(x)}{\prod_{i=1}^k (x - t^0)^2} = Q(x) + \sum_{i=1}^k \frac{a_{i,2}}{(x - t_i^0)^2},$$

where $a_{i,2}$ are given by (3.15); $Q(x) \in \mathbb{F}_p[x]$, $Q(x) = 0$ if $2k > |m|$ and

$$Q(x) = a_{|m|-2k} x^{|m|-2k} + \cdots + a_0$$

is of degree $|m| - 2k + 1$ if $2k \leqslant |m|$, see Section 3.1.

If $2k \leqslant |m|$, then

$$\tilde{y}(x) = \prod_{i=1}^k (x - t_i^0) \left(\frac{a_{|m|-2k}}{|m| - 2k + 1} x^{|m|-2k+1} + \cdots + a_0 x - \sum_{i=1}^k \frac{a_{i,2}}{x - t_i^0} \right)$$

is a polynomial of degree $|m| - k + 1$ satisfying (3.2). Notice that the polynomial in the brackets is well-defined since $p > |m| + 1$. If $2k > |m|$, then

$$\tilde{y}(x) = -\prod_{i=1}^k (x - t_i^0) \left(\sum_{i=1}^k \frac{a_{i,2}}{x - t_i^0} \right)$$

is a polynomial satisfying (3.13) of degree $< k$. Formula (3.8) and inequality $p > k + n$ imply (3.9). The first part of Theorem 3.2 is proved.

Let there exist a polynomial $\tilde{y}(x)$ satisfying equation (3.13). Adding to $\tilde{y}(x)$ a suitable polynomial of the form $c(x^p)y(x)$ for some $c(x) \in \mathbb{F}_p[x]$ if necessary, we may assume that $\deg \tilde{y}(x) - \deg y(x) \not\equiv 0 \bmod p$. Then (3.9) holds, $k \leqslant |m| + 1$ and $k \neq |m| + 1 - k$.

By formula (3.3) and Lemma 3.3 we have

$$\left(\frac{\tilde{y}(x)}{y(x)}\right)' = a_{|m|-2k} x^{|m|-2k} + \cdots + a_0 + \sum_{i=1}^k \left(\frac{a_{i,2}}{(x - t_i^0)^2} + \frac{a_{i,1}}{x - t_i^0} \right) \text{ if } 2k \leqslant |m| \tag{3.18}$$

and

$$\left(\frac{\tilde{y}(x)}{y(x)}\right)' = \sum_{i=1}^k \left(\frac{a_{i,2}}{(x - t_i^0)^2} + \frac{a_{i,1}}{x - t_i^0} \right) \text{ if } 2k > |m|. \tag{3.19}$$

The function $\frac{\tilde{y}(x)}{y(x)}$ has a unique decomposition into the sum of a polynomial and simple fractions. The term by term derivative of that decomposition equals the right-hand side of (3.18) or (3.19). Hence all of coefficients $a_{i,1}$ must be zero. Hence the roots of $y(x)$ satisfy the Bethe ansatz equations. □

Remark. This construction assigns to a solution (t^0, z^0) of the Bethe ansatz equations (2.15) the two-dimensional subspace $\langle \tilde{y}(x), y(x) \rangle$ of the space of polynomials $\mathbb{F}_p[x]$ such that

$$\deg y(x) = k, \deg \tilde{y}(x) = |m| - k + 1, \quad \mathrm{Wr}(y(x), \tilde{y}(x)) = T(x).$$

That subspace is a point of the Grassmannian of two-dimensional subspaces in $\mathbb{F}_p[x]$.

4. Example: the case $k = 1$

4.1. **Gaudin model on** $\mathrm{Sing} L^{\otimes m}[|m| - 2]$**.** Let $m = (m_1, \dots, m_n) \in \mathbb{Z}^n_{>0}$ and $p > |m| + 1$. Consider the Gaudin model on $\mathrm{Sing} L^{\otimes m}[|m| - 2]$ over $\mathbb{F}_p$. That means that $k = 1$ in the notations of the previous sections. A basis of $L^{\otimes m}[|m| - 2]$ is formed by the vectors

$$f^{(s)} = v_{m_1} \otimes \cdots \otimes v_{s-1} \otimes f v_{m_s} \otimes v_{s+1} \otimes \cdots \otimes v_{m_n}, \quad s = 1, \dots, n. \tag{4.1}$$

We have

$$\mathrm{Sing} L^{\otimes m}[|m| - 2] = \Big\{ \sum_{s=1}^{n} c_s f^{(s)} \mid c_s \in \mathbb{F}_p \ \text{ and } \ \sum_{s=1}^{n} m_s c_s = 0 \Big\}. \tag{4.2}$$

For $s = 1, \dots, n$, define the vectors $w_s \in \mathrm{Sing} L^{\otimes m}[|m| - 2]$ by the formula

$$w_s = f^{(s)} - \frac{m_s}{|m|} \sum_{l=1}^{n} f^{(l)}. \tag{4.3}$$

We have

$$w_1 + \cdots + w_n = 0. \tag{4.4}$$

By [V4, Lemma 4.2], any $n - 1$ of these vectors form a basis of $\mathrm{Sing} L^{\otimes m}[|m| - 2]$.

Let $z^0 = (z^0_1, \dots, z^0_n) \in \mathbb{F}^n_p$ have distinct coordinates. For $i = 1, \dots, n$, the Gaudin Hamiltonian $H_i(z^0)$ acts on $L^{\otimes m}[|m| - 2]$ by the formulas:

$$\begin{aligned} f^{(s)} &\mapsto \sum_{j \neq i} \frac{m_i m_j / 2}{z^0_i - z^0_j} f^{(s)} + \frac{1}{z^0_i - z^0_s} (m_s f^{(i)} - m_i f^{(s)}), \qquad s \neq i, \\ f^{(i)} &\mapsto \sum_{j \neq i} \frac{m_i m_j / 2}{z^0_i - z^0_j} f^{(i)} + \sum_{j \neq i} \frac{1}{z^0_i - z^0_j} (m_i f^{(j)} - m_j f^{(i)}). \end{aligned} \tag{4.5}$$

Hence

$$\begin{aligned} w_s &\mapsto \sum_{j \neq i} \frac{m_i m_j / 2}{z^0_i - z^0_j} w_s + \frac{1}{z^0_i - z^0_s} (m_s w_i - m_i w_s), \qquad s \neq i, \\ w_i &\mapsto \sum_{j \neq i} \frac{m_i m_j / 2}{z^0_i - z^0_j} w_i + \sum_{j \neq i} \frac{1}{z^0_i - z^0_j} (m_i w_j - m_j w_i). \end{aligned} \tag{4.6}$$

Recall that the Bethe algebra of $\mathrm{Sing} L^{\otimes m}[|m| - 2]$ is the subalgebra of $\mathrm{End}(\mathrm{Sing} L^{\otimes m}[|m| - 2])$ generated by the Gaudin Hamiltonians $H_i(z^0)$, $i = 1, \dots, n$, and the identity operator. We denote it by $\mathcal{B}(z^0, m)$.

4.2. **Bethe ansatz equation and algebra $\mathcal{A}(z^0, m)$.** Let $m = (m_1, \dots, m_n) \in \mathbb{Z}^n_{>0}$ and $p > |m| + 1$. Let $z^0 = (z^0_1, \dots, z^0_n)$ have distinct coordinates. The Bethe ansatz equations of $\mathrm{Sing} L^{\otimes m}[|m| - 2]$ is the single equation

$$\frac{m_1}{t - z^0_1} + \dots + \frac{m_n}{t - z^0_n} = 0. \tag{4.7}$$

Write

$$\frac{m_1}{t - z^0_1} + \dots + \frac{m_n}{t - z^0_n} = \frac{P(t)}{\prod_{s=1}^n (t - z^0_s)}, \tag{4.8}$$

where

$$P(t) = P(t, z^0, m) = \sum_{s=1}^n m_s \prod_{l \neq s} (t - z^0_l). \tag{4.9}$$

Let $\mathbb{A}_{\mathbb{F}_p}$ be the affine line over $\mathbb{F}_p$ with coordinate t. Denote $U = \mathbb{A}_{\mathbb{F}_p} - \{z^0_1, \dots, z^0_n\}$. Let $\mathcal{O}(U)$ be the ring of rational functions on the affine line $\mathbb{A}_{\mathbb{F}_p}$ regular on U. Introduce the algebra

$$\mathcal{A}(z^0, m) = \mathcal{O}(U)/(P(t)), \qquad \dim_{\mathbb{F}_p} \mathcal{A}(z^0, m) = n - 1. \tag{4.10}$$

Here $(P(t))$ is the ideal generated by $P(t)$. Let $u_s \in \mathcal{A}(z^0, m)$, $s = 1, \dots, n$, be the image of $\frac{m_s}{t - z^0_s}$ in $\mathcal{A}(z^0, m)$. The elements u_s span $\mathcal{A}(z^0, m)$ as a vector space and

$$u_1 + \dots + u_n = 0. \tag{4.11}$$

We have

$$\begin{aligned} u_i u_s &= \frac{1}{z^0_i - z^0_s} (m_s u_i - m_i u_s), \qquad s \neq i, \\ u_i u_i &= \sum_{j \neq i} \frac{1}{z^0_i - z^0_j} (m_i u_j - m_j u_i). \end{aligned} \tag{4.12}$$

For a function $g(t) \in \mathcal{O}(U)$ denote $[g(u)]$ its image in $\mathcal{A}(z^0, m)$. The elements $[1], [t], \dots, [t^{n-2}]$ form a basis of $\mathcal{A}(z^0, m)$ over $\mathbb{F}_p$. The defining relation in $\mathcal{A}(z^0, m)$ is $P([t]) = 0$. The following formulas express the elements $[t^i]$ in terms of the elements u_s.

Lemma 4.1. *We have*

$$\begin{aligned} [1] &= \frac{-1}{|m|} (z^0_1 u_1 + \dots + z^0_n u_n), \\ [t] &= \frac{1}{|m|^2} \left(\sum_{s=1}^n z^0_s m_s \right) \left(\sum_{s=1}^n z^0_s u_s \right) + \frac{-1}{|m|} \left(\sum_{s=1}^n (z^0_s)^2 u_s \right), \\ [t^i] &= \frac{-1}{|m|} \sum_{j=1}^i \sum_{s=1}^n (z^0_s)^j m_s [t^{i-j}] + \frac{-1}{|m|} \sum_{s=1}^n (z^0_s)^{i+1} u_s, \qquad i \geqslant 0. \end{aligned} \tag{4.13}$$

□

These formulas are related to formulas for the $\widehat{sl}_2$-action on tensor products of modules dual to Verma modules, see [SV2] and in particular to formula (11) in [SV2].

4.3. **Isomorphism of $\mathcal{A}(z^0,m)$ and $\mathcal{B}(z^0,m)$.** Define the isomorphism of vectors spaces

$$\alpha \;:\; \mathcal{A}(z^0,m) \;\to\; \mathrm{Sing}L^{\otimes m}[|m|-2], \qquad u_s \mapsto w_s, \qquad s=1,\dots,n, \tag{4.14}$$

in particular, we have

$$\langle 1\rangle := \alpha([1]) = \frac{-1}{|m|}(z_1^0 w_1 + \cdots + z_n^0 w_n). \tag{4.15}$$

Theorem 4.2. *The map*

$$[1] \mapsto \mathrm{Id}, \quad u_s \mapsto H_s(z^0) - \sum_{j\neq s}\frac{m_s m_j/2}{z_s^0 - z_j^0}\,\mathrm{Id}, \qquad s=1,\dots,n, \tag{4.16}$$

extends to an algebra isomorphism

$$\beta : \mathcal{A}(z^0,m) \to \mathcal{B}(z^0,m), \tag{4.17}$$

such that $\alpha(gh)=\beta(g).\alpha(h)$ for any $g,h\in\mathcal{A}(z^0,m)$.

Proof. The proof follows from comparing (4.6) and (4.12). □

Remark. Theorem 4.2 says that the isomorphism α of vector spaces and the isomorphism β of algebras establish an isomorphism between the $\mathcal{B}(z^0,m)$-module $\mathrm{Sing}L^{\otimes m}[|m|-2]$ and the regular representation of the algebra $\mathcal{A}(z^0,m)$.

Example 4.1. Theorem 4.2 in particular says that if $P(t)$ is irreducible then $\mathcal{B}(z^0,m)\cong\mathbb{F}_{p^{n-1}}$, where $\mathbb{F}_{p^{n-1}}$ is the field with p^{n-1} elements.

For example, if $n=3$, $m=(1,1,1)$, then

$$P(t,z^0) = 3t^2 - 2(z_1^0+z_2^0+z_3^0)t + z_1^0z_2^0 + z_1^0z_3^0 + z_2^0z_3^0.$$

If $p=5$, then $P(t,z^0)$ is irreducible in $\mathbb{F}_5[t]$ for all distinct $z_1^0,z_2^0,z_3^0\in\mathbb{F}_5$ and $\mathcal{B}(z^0,m)\cong\mathbb{F}_{25}$.

Corollary 4.3. *We have*

$$\dim_{\mathbb{F}_p}\mathcal{B}(z^0,m) = n-1. \tag{4.18}$$

Corollary 4.4. *The operators $\beta([1])=\mathrm{Id}$, $\beta([t^i])$, $i=1,\dots,n-2$, form a basis of the vector space $\mathcal{B}(z^0,m)$ over $\mathbb{F}_p$. The operator*

$$\begin{aligned}\{t\} := \beta([t]) \;&=\; \frac{1}{|m|^2}\left(\sum_{s=1}^n z_s^0 m_s\right)\left(\sum_{s=1}^n z_s^0\left(H_s(z^0) - \sum_{j\neq s}\frac{m_s m_j/2}{z_s^0-z_j^0}\,\mathrm{Id}\right)\right)\\ &+\; \frac{-1}{|m|}\left(\sum_{s=1}^n (z_s^0)^2\left(H_s(z^0) - \sum_{j\neq s}\frac{m_s m_j/2}{z_s^0-z_j^0}\,\mathrm{Id}\right)\right)\end{aligned} \tag{4.19}$$

generates $\mathcal{B}(z^0,m)$ as an algebra with defining relation $P(\{t\})=0$.

Corollary 4.5. *We have*

$$\left(H_s(z^0) - \sum_{j\neq s}\frac{m_s m_j/2}{z_s^0-z_j^0}\,\mathrm{Id}\right).\langle 1\rangle = w_s, \qquad s=1,\dots,n. \tag{4.20}$$

4.4. **Eigenvectors of $\mathcal{B}(z^0, m)$ and the polynomial $P(t)$.** The elements of the algebra $\mathcal{A}(z^0, m)$ have the form $Q([t])$, where $Q(t) \in \mathbb{F}_p[t]$, $\deg Q(t) < n-1$. An element $Q([t])$ is an eigenvector of all multiplication operators of $\mathcal{A}(z^0, m)$ if and only if $Q([t])$ is an eigenvector of the multiplication by $[t]$. If $t^0 \in \mathbb{F}_p$ is the eigenvalue, then $([t] - t^0)Q([t]) = 0$, that is,

$$(t - t^0)Q(t) = \text{const } P(t), \qquad \text{const} \in \mathbb{F}_p. \tag{4.21}$$

Hence the set of eigenlines of all multiplication operators of $\mathcal{A}(z^0, m)$ is in one-to-one correspondence with the set of distinct roots of the polynomial $P(t)$, namely, a root t^0 with decomposition $(t - t^0)Q(t) = P(t)$ corresponds to the line generated by the element $Q([t])$.

Corollary 4.6. *The set of eigenlines of $\mathcal{B}(z^0, m)$ are in one-to-one correspondence with the set of distinct roots of the polynomial $P(t)$, namely, a root t^0 with decomposition $(t-t^0)Q(t) = P(t)$ for some $Q(t) \in \mathbb{F}_p[t]$ corresponds to the line generated by the vector*

$$\omega(t^0, z^0) := Q(\{t\}).\langle 1\rangle \quad \in \ \text{Sing}L^{\otimes m}[|m| - 2]. \tag{4.22}$$

□

Thus we have two ways to construct the eigenlines of $\mathcal{B}(z^0, m)$ from roots t^0 of the polynomial $P(t)$. The first is given by Theorem 2.1 and the eigenline is generated by the vector

$$W_{1,n,m}(t^0, z^0) = \sum_{s=1}^{n} \frac{1}{t^0 - z_s} f^{(s)} = \sum_{s=1}^{n} \frac{1}{t^0 - z_s} w_s. \tag{4.23}$$

The second is given by Corollary 4.6 and the eigenline is generated by the vector $Q(\{t\}).\langle 1\rangle$.

Theorem 4.7. *The two eigenlines coincide, more precisely, we have*

$$W_{1,n,m}(t^0, z^0) = \text{const}\, Q(\{t\}).\langle 1\rangle, \qquad \text{const} \in \mathbb{F}_p. \tag{4.24}$$

Proof. We need to show that $([t] - t^0)\alpha^{-1}(W_{1,n,m}(t^0, z^0)) = 0$ in $\mathcal{A}(z^0, m)$. Indeed

$$([t] - t^0)\alpha^{-1}(W_{1,n,m}(t^0, z^0)) = \sum_{s=1}^{n} \frac{m_s}{t^0 - z_s}\left[\frac{t - t^0}{t - z_s}\right] = \sum_{s=1}^{n} \frac{m_s}{t^0 - z_s}[1] - \sum_{s=1}^{n}\left[\frac{m_s}{t - z_s}\right] = 0$$

due to the Bethe ansatz equation (4.7) and formula (4.11).

□

4.5. **Algebra $\mathcal{C}(T)$.** In this section, p is a prime number, $p > n + 1$. Fix a monic polynomial

$$T(x) = x^n + \sigma_1 x^{n-1} + \sigma_2 x^{n-2} + \cdots + \sigma_n \ \in \ \mathbb{F}_p[x]. \tag{4.25}$$

We consider the two-dimensional subspaces $V \subset \mathbb{F}_p[x]$ consisting of polynomials of degree n and 1 such that $\text{Wr}(g_1(x), g_2(x)) = \text{const}\, T(x)$, where $g_1(x), g_2(x)$ is any basis of V and const $\in \mathbb{F}_p$. Such a subspace V has a unique basis of the form

$$g_1(x) = x^n + a_1 x^{n-1} + \cdots + a_{n-2}x^2 + a_0, \qquad g_2 = x - t \tag{4.26}$$

with

$$\text{Wr}(g_1(x), g_2(x)) = (n-1)T(x). \tag{4.27}$$

Equation (4.27) is equivalent to the system of equations

$$\begin{aligned} &(n - r - 1)a_r - (n - r + 1)a_{r-1}t - (n-1)\sigma_r = 0, \qquad r = 1, \ldots, n-1, \\ &a_n - (n-1)\sigma_n = 0, \end{aligned} \tag{4.28}$$

where $a_0 = 1$. Expressing a_1 from the first equation in terms of t, then expressing a_2 from the first and second equations in terms of t and so on, we can reformulate system (4.28) as the system of equations

$$a_r - \frac{n-1}{2}(nt^r + (n-1)\sigma_1 t^{r-1} + \cdots + (n-r)\sigma_r) = 0, \qquad r = 1, \ldots, n-2, \tag{4.29}$$

$$nt^{n-1} + (n-2)\sigma_1 t^{n-2} + \cdots + 2\sigma_2 t + \sigma_1 = 0, \tag{4.30}$$

$$a_n + \sigma_n = 0. \tag{4.31}$$

Notice that equation (4.30) is the equation $\frac{dT}{dt}(t) = 0$, where $T(x)$ is defined in (4.25).

Let $I \subset \mathbb{F}_p[t, a_1, \ldots, a_{n-2}, a_n]$ be the ideal generated by n polynomials staying in the left-hand sides of the equations of the system (4.28). Define the algebra

$$\mathcal{C}(T) = \mathbb{F}_p[t, a_1, \ldots, a_{n-2}, a_n]/I. \tag{4.32}$$

Let $J \subset \mathbb{F}_p[t]$ be the ideal generated by $\frac{dT}{dt}(t)$. Define the algebra

$$\tilde{\mathcal{C}}(T) = \mathbb{F}_p[t]/J. \tag{4.33}$$

Lemma 4.8. *We have an isomorphism of algebras*

$$\tilde{\mathcal{C}}(T) \to \mathcal{C}(T), \quad [t] \mapsto [t]. \tag{4.34}$$

□

Let $m^0 = (1, \ldots, 1) \in \mathbb{Z}^n_{>0}$. Let $z^0 = (z^0_1, \ldots, z^0_n) \in \mathbb{F}^n_p$ be a point with distinct coordinates. The Bethe ansatz for $\mathrm{Sing} L^{\otimes m^0}[|m^0| - 2]$ has the form

$$\frac{1}{t - z^0_1} + \cdots + \frac{1}{t - z^0_n} = \frac{R(t)}{T(t)} = 0, \tag{4.35}$$

where

$$T(t) = \prod_{s=1}^{n}(t - z^0_s), \qquad R(t) = \frac{dT}{dt}(t). \tag{4.36}$$

Hence for this $T(x)$ we have

$$\tilde{\mathcal{C}}(T) = \mathcal{A}(z^0, m^0). \tag{4.37}$$

Corollary 4.9. *For $T(t)$ and $R(t)$ as in (4.36) we have*

$$\mathcal{A}(z^0, m^0) \cong \mathcal{B}(z^0, m^0) \cong \mathcal{C}(T) \tag{4.38}$$

and the $\mathcal{B}(z^0, m)$-module $\mathrm{Sing} L^{\otimes m^0}[|m^0| - 2]$ is isomorphic to the regular representation of the algebra $\mathcal{C}(T)$.

4.6. **Wronski map.** Let X_n be the affine space of all two-dimensional subspaces $V \subset \mathbb{F}_p[x]$, each of which consists of polynomials of degree n and 1. The space X_n is identified with the space of pairs of polynomials given by formula (4.26). Let $\mathbb{F}_p[x]_n \subset \mathbb{F}_p[x]$ be the affine subspace of monic polynomials of degree n. Introduce the *Wronski map*

$$W_n : X_n \to \mathbb{F}_p[x]_n, \quad \langle g_1(x), g_2(x)\rangle \mapsto \frac{1}{n-1}\,\mathrm{Wr}(g_1(x), g_2(x)), \tag{4.39}$$

cf. [MTV3]. The algebra $\mathcal{C}(T)$ is the algebra of functions on the fiber $W^{-1}(T)$ of the Wronski map.

Example 4.2. Let $n = 3$ and $T(x) = x^3+\sigma_1x^2+\sigma_2x+\sigma_3$. Then $W_3^{-1}(T)$ consists of one point if the discriminant $\sigma_1^2-3\sigma_2$ of $\frac{dT}{dx}(x)$ equals zero; $W_3^{-1}(T)$ consists of two points if the discriminant is a nonzero square, and is empty otherwise. Thus, p^2 points of X_3 have one preimage, $\frac{p-1}{2}p^2$ points have two preimages, and $\frac{p-1}{2}p^2$ points have none. Cf. Example 4.1.

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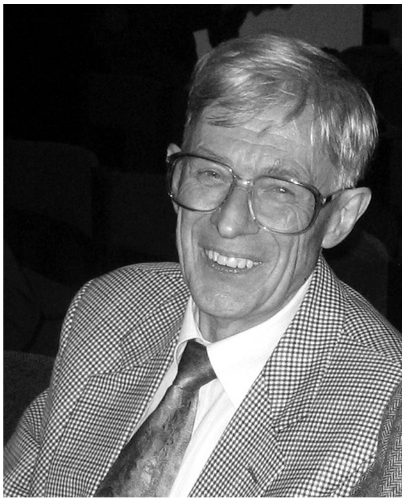

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