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A RESIDUAL DUALITY OVER GORENSTEIN RINGS WITH APPLICATION TO LOGARITHMIC DIFFERENTIAL FORMS

MATHIAS SCHULZE AND LAURA TOZZO

Dedicated with admiration to Egbert Brieskorn

Abstract. Kyoji Saito's notion of a free divisor was generalized by the first author to reduced Gorenstein spaces and by Delphine Pol to reduced Cohen–Macaulay spaces. Starting point is the Aleksandrov–Terao theorem: A hypersurface is free if and only if its Jacobian ideal is maximal Cohen–Macaulay. Pol obtains a generalized Jacobian ideal as a cokernel by dualizing Aleksandrov's multi-logarithmic residue sequence. Notably it is essentially a suitably chosen complete intersection ideal that is used for dualizing. Pol shows that this generalized Jacobian ideal is maximal Cohen–Macaulay if and only if the module of Aleksandrov's multi-logarithmic differential k-forms has (minimal) projective dimension $k-1$, where k is the codimension in a smooth ambient space. This equivalent characterization reduces to Saito's definition of freeness in case $k=1$. In this article we translate Pol's duality result in terms of general commutative algebra. It yields a more conceptual proof of Pol's result and a generalization involving higher multi-logarithmic forms and generalized Jacobian modules.

1. Introduction

Logarithmic differential forms along hypersurfaces and their residues were introduced by Kyoji Saito (see [22]). They are part of his theory of primitive forms and period mappings where the hypersurface is the discriminant of a universal unfolding of a function with isolated critical point (see [23, 24]). The special case of normal crossing divisors appeared earlier in Deligne's construction of mixed Hodge structures (see [8]). Here the logarithmic differential 1-forms form a locally free sheaf. In general a divisor with this property is called a free divisor. Further examples include plane curves (see [22, (1.7)]), unitary reflection arrangements and their discriminants (see [29, Thm. C]) and discriminants of versal deformations of isolated complete intersection singularities and space curves (see [17, (6.13)] and [30]). Free divisors also occur as discriminants in prehomogeneous vector spaces (see [10]). In case of hyperplane arrangements the study of freeness attracted a lot of attention (see [31]).

Let D be a germ of reduced hypersurface in $Y \cong (\mathbb{C}^n, 0)$ defined by $h \in \mathscr{O}_Y$. The $\mathscr{O}_Y$-modules $\Omega^q(\log D)$ of logarithmic differential q-forms along D and the $\mathscr{O}_D$-modules ω_D^p of regular meromorphic differential p-forms on D fit into a short exact logarithmic residue sequence (see [22, §2] and [2, §4])

$$0 \longrightarrow \Omega_Y^q \longrightarrow \Omega^q(\log D) \xrightarrow{\operatorname{res}_D^q} \omega_D^{q-1} \longrightarrow 0.$$

Denoting by $\nu_D \colon \widetilde{D} \to D$ the normalization of D, $(\nu_D)_*\mathscr{O}_{\widetilde{D}} \subseteq \omega_D^0$ (see [22, (2.8)]). For plane curves Saito showed that equality holds exactly for normal crossing curves (see [22, (2.11)]).

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Granger and the first author (see [11]) generalized this fact and thus extended the Lê–Saito Theorem (see [16]) by an equivalent algebraic property. They showed that $(\nu_D)_*\mathscr{O}_{\widetilde{D}} = \omega_D^0$ if and only if D is normal crossing in codimension one, that is, outside of an analytic subset of Y of codimension at least 3. The proof uses the short exact sequence

$$0 \longleftarrow \mathscr{J}_D \xleftarrow{\langle -,dh\rangle} \Theta_Y \longleftarrow \mathrm{Der}(-\log D) \longleftarrow 0$$

obtained as the $\mathscr{O}_Y$-dual of the logarithmic residue sequence. It involves the Jacobian ideal $\mathscr{J}_D$ of D, the $\mathscr{O}_Y$-module $\Theta_Y := \mathrm{Der}_{\mathbb{C}}(\mathscr{O}_Y) \cong (\Omega_Y^1)^*$ of vector fields on Y and its submodule $\mathrm{Der}(-\log D) \cong \Omega^1(\log D)^*$ of logarithmic vector fields along D. It is shown that $\omega_D^0 = \mathscr{J}_D^*$ and that $\mathscr{J}_D = (\omega_D^0)^*$ if D is a free divisor. In fact freeness of D is equivalent to $\mathscr{J}_D$ being a Cohen–Macaulay ideal by the Aleksandrov–Terao theorem (see [2, §2] and [28, §2]).

As observed by first author (see [27]) the inclusion $(\nu_D)_*\mathscr{O}_{\widetilde{D}} \subseteq \omega_D^0$ can be seen as

$$(\nu_D)_*\omega_{\widetilde{D}}^0 \hookrightarrow \omega_D^0.$$

He showed that $(\nu_X)_*\omega_{\widetilde{X}}^0 = \omega_X^0$ is equivalent to X being normal crossing in codimension one for reduced equidimensional spaces X which are free in codimension one. Here freeness means Gorenstein with Cohen–Macaulay ω-Jacobian ideal. As the latter coincides with the Jacobian ideal for complete intersections (see [19, Prop. 1]), this generalizes the classical freeness of divisors which holds true in codimension one.

Multi-logarithmic differential forms generalize Saito's logarithmic differential forms replacing hypersurfaces $D \subseteq Y$ by subspaces $X \subseteq Y$ of codimension $k \geq 2$. They were first introduced with meromorphic poles along reduced complete intersections by Aleksandrov and Tsikh (see [5, 6]), later with simple poles by Aleksandrov (see [3, §3]) and recently along reduced Cohen–Macaulay and reduced equidimensional spaces by Aleksandrov (see [4, §10]) and by Pol (see [21, §4.1]). The precise relation of the forms with simple and meromorphic poles was clarified by Pol (see [21, Prop. 3.1.33]). Here we consider only multi-logarithmic forms with simple poles.

The $\mathscr{O}_Y$-modules $\Omega^q(\log X/C)$ of multi-logarithmic q-forms on Y along X depend on the choice of divisors $D_1, \dots, D_k$ defining a reduced complete intersection $C = D_1 \cap \cdots \cap D_k \subseteq Y$ such that $X \subseteq C$. Consider the divisor $D = D_1 \cup \cdots \cup D_k$ defined by $h = h_1 \cdots h_k \in \mathscr{O}_Y$. Due to Aleksandrov and Pol there is a multi-logarithmic residue sequence

$$0 \longrightarrow \Sigma\Omega_Y^q \longrightarrow \Omega^q(\log X/C) \xrightarrow{\mathrm{res}^q_{X/C}} \omega_X^{q-k} \longrightarrow 0 \tag{1.1}$$

where $\Sigma = \mathscr{I}_C(D)$ is obtained from the ideal $\mathscr{I}_C$ of $C \subseteq Y$ and ω_X^p is the $\mathscr{O}_X$-module of regular meromorphic p-forms on X (see [4, §10] and [21, §4.1.3]). Pol introduced an $\mathscr{O}_Y$-module $\mathrm{Der}^k(-\log X/C)$ of logarithmic k-vector fields on Y along X and a kind of Jacobian ideal $\mathscr{J}_{X/C}$ of X that fit into the short exact sequence dual to (1.1) for $q = k$

$$0 \longleftarrow \mathscr{J}_{X/C} \xleftarrow{\langle -,\alpha_X\rangle} \Theta_Y^k \longleftarrow \mathrm{Der}^k(-\log X/C) \longleftarrow 0 \tag{1.2}$$

where $\Theta_Y^q = \bigwedge_{\mathscr{O}_Y}^q \Theta_Y$ and $\begin{bmatrix} \alpha_X \\ h_1, \dots, h_k \end{bmatrix} \in \omega_X^0$ is a fundamental form of X (see [21, §4.2.2-3]). Notably the duality applied here is $-^\Sigma = \mathrm{Hom}_{\mathscr{O}_Y}(-, \Sigma)$. Pol showed that Cohen–Macaulayness of $\mathscr{J}_{X/C}$ serves as a further generalization of freeness. In fact the property is independent of C (see [21, Prop. 4.2.21]) and $\mathscr{J}_{X/C}$ coincides with the ω-Jacobian ideal in case X is Gorenstein (see [21, §4.2.5]). By relating Σ- and $\mathscr{O}_Y$-duality Pol established the following major result (see [21, Thm. 4.2.22] or [20]). In particular it generalizes Saito's original definition of freeness to the case $k > 1$.

Theorem 1.1 (Pol)**.** *Let $X \subseteq C \subseteq Y \cong (\mathbb{C}^n, 0)$ where X is a reduced Cohen–Macaulay germ and C a complete intersection germ, both of codimension $k \geq 1$ in Y. Then*

$$\operatorname{pdim}(\Omega^k(\log X/C)) \geq k - 1$$

with equality equivalent to freeness of X.

In §2 we pursue the main objective of this article: a translation of Theorem 1.1 in terms of general commutative algebra. The role of $\mathscr{O}_Y \twoheadrightarrow \mathscr{O}_C = \mathscr{O}_Y/\mathcal{I}_C$ is played by a map of Gorenstein rings $R \to \overline{R} = R/I$ of codimension $k \geq 2$. For dualizing we use

$$-^I = \operatorname{Hom}_R(-, I), \quad -^\vee = \operatorname{Hom}_R(-, \omega_R), \quad -^{\underline{\vee}} = \operatorname{Hom}_{\overline{R}}(-, \overline{\omega}_R)$$

where ω_R is a canonical module for R and $\overline{\omega}_R = \overline{R} \otimes_R \omega_R$, which is a canonical module for $\overline{R}$ due to the Gorenstein hypothesis (see Notation 2.1). Modelled after the multi-logarithmic residue sequence (1.1) along $X = C$ we define an I-free approximation of a finitely generated R-module M as a short exact sequence

$$0 \longrightarrow IF \xrightarrow{\iota} M \longrightarrow W \longrightarrow 0$$

where F is free and W is an $\overline{R}$-module. More precisely M plays the role of $\Omega^q(\log X/C)(-D)$ which, as opposed to $\Omega^q(\log X/C)$, is independent of the choice of D. The I-dual sequence

$$0 \longleftarrow V \xleftarrow{\alpha} F^\vee \xleftarrow{\lambda} M^I \longleftarrow 0$$

plays the role of the Σ-dual sequence (1.2) for $X = C$. In Proposition 2.13 we show that M is I-reflexive if and only if W is the $\overline{R}$-dual of V. Our main result is

Theorem 1.2. *Let R be a Gorenstein local ring and let I be an ideal of R of height $k \geq 2$ such that $\overline{R} = R/I$ is Gorenstein. Consider an I-free approximation*

$$0 \longrightarrow IF \xrightarrow{\iota} M \xrightarrow{\rho} W \longrightarrow 0$$

of an I-reflexive finitely generated R-module M with $W \neq 0$ and the corresponding I-dual

$$0 \longleftarrow V \xleftarrow{\alpha} F^\vee \xleftarrow{\lambda} M^I \longleftarrow 0.$$

Then $W = V^{\underline{\vee}}$ and V is a maximal Cohen–Macaulay $\overline{R}$-module if and only if $\operatorname{G-dim}(M) \leq k-1$. *In this latter case $V = W^{\underline{\vee}}$ is ($\overline{\omega}_R$-)reflexive. Unless $\overline{\alpha} := \overline{R} \otimes \alpha$ is injective,* $\operatorname{G-dim}(M) \geq k-1$.

Due to the Gorenstein hypothesis, Theorem 1.2 applies to the complete intersection ring $\overline{R} = \mathscr{O}_C$, but in general not to $\overline{R} = \mathscr{O}_X$. In §2.5 we describe a construction to restrict the support of an I-free approximation to the locus defined by an ideal $J \trianglelefteq R$ with $I \subseteq J$. Lemma 3.15 shows that it is made in a way such that the multi-logarithmic residue sequence along X is obtained from that along C by restricting with $J = \mathcal{I}_X$. Corollary 2.29 extends Theorem 1.2 to this generalized setup.

In §3 we apply our results to multi-logarithmic forms. We define $\mathscr{O}_Y$-submodules

$$\operatorname{Der}^q(-\log X) \subseteq \Theta_Y^q$$

of logarithmic q-vector fields on Y along X independent of C and show that

$$\operatorname{Der}^k(-\log X) = \operatorname{Der}^k(-\log X/C).$$

We further define Jacobian $\mathscr{O}_X$-modules $\mathcal{J}_X^{n-q} \subseteq \mathscr{O}_X \otimes_{\mathscr{O}_Y} \Theta_Y^{q-k}$ of X independent of C and Y such that $\mathcal{J}_X^{\dim X} = \mathcal{J}_{X/C}$. The Σ-dual of the multi-logarithmic residue sequence reads

$$0 \longleftarrow \mathcal{J}_X^{n-q} \xleftarrow{\alpha^X} \Theta_Y^q \longleftarrow \operatorname{Der}^q(-\log X) \longleftarrow 0$$

where α^X is contraction by α_X. As a consequence of Corollary 2.29 we obtain the following result which is due to Pol in case $q = k$ (see [21, Prop. 4.2.17, Thm. 4.2.22]).

Theorem 1.3. *Let $X \subseteq C \subseteq Y \cong (\mathbb{C}^n, 0)$ where X is a reduced Cohen–Macaulay germ and C a complete intersection germ, both of codimension $k \geq 2$ in Y. For $k \leq q < n$, $\omega_X^{q-k} = \operatorname{Hom}_{\mathscr{O}_X}(\mathscr{J}_X^{n-q}, \omega_X)$ where $\omega_X = \operatorname{Hom}_{\mathscr{O}_C}(\mathscr{O}_X, \mathscr{O}_C)(D)$ and $\operatorname{pdim}(\Omega^q(\log X/C)) \geq k-1$. Equality holds if and only if $\mathscr{J}_X^{n-q}$ is maximal Cohen–Macaulay. In this latter case $\mathscr{J}_X^{n-q} = \operatorname{Hom}_{\mathscr{O}_X}(\omega_X^{q-k}, \omega_X)$ is ω_X-reflexive.*

The analogy with the hypersurface case (see [22, (1.8)]) now raises the question whether $\mathscr{J}_X^{n-q}$ being maximal Cohen–Macaulay for $q = k$ implies the same for all $q > k$. An explicit description of the Jacobian modules is given in Remark 3.25.

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2. Residual duality over Gorenstein rings

For this section we fix a Cohen–Macaulay local ring R with $n := \dim(R)$ and an ideal $I \trianglelefteq R$ with $k := \operatorname{height}(I) \geq 2$ defining a Cohen–Macaulay factor ring $\overline{R} := R/I$. These fit into a short exact sequence

$$0 \longrightarrow I \longrightarrow R \xrightarrow{\pi} \overline{R} \longrightarrow 0. \tag{2.1}$$

Note that (see [7, Thm. 2.1.2.(b), Cor. 2.1.4])

$$n - \dim(\overline{R}) = \operatorname{grade}(I) = \operatorname{height}(I) = k \geq 2.$$

In particular I is a regular ideal of R and hence any $\overline{R}$-module is R-torsion.

We assume further that R admits a canonical module ω_R. Then also $\overline{R}$ admits a canonical module $\omega_{\overline{R}}$ (see [7, Thm. 3.3.7]).

Notation 2.1. Abbreviating $\overline{\omega}_R := \overline{R} \otimes_R \omega_R$ we deal with the following functors

$$-^* := \operatorname{Hom}_R(-, R), \qquad -^\vee := \operatorname{Hom}_R(-, \omega_R),$$
$$-^I := \operatorname{Hom}_R(-, I\omega_R), \quad -^{\overline{\vee}} := \operatorname{Hom}_R(-, \overline{\omega}_R).$$

In general $\overline{\omega}_R \not\cong \omega_{\overline{R}}$ and $-^{\overline{\vee}}$ is not the duality of $\overline{R}$-modules. For an $\overline{R}$-module N,

$$N^* = \operatorname{Hom}_{\overline{R}}(N, \overline{R})$$

but $N^\vee$ means either $\operatorname{Hom}_R(N, \omega_R)$ or $\operatorname{Hom}_{\overline{R}}(N, \omega_{\overline{R}})$, depending on the context. For R-modules M and N, we denote the canonical evaluation map by

$$\delta_{M,N} \colon M \to \operatorname{Hom}_R(\operatorname{Hom}_R(M, N), N), \quad m \mapsto (\varphi \mapsto \varphi(m)).$$

Whenever applicable we use an analogous notation for $\overline{R}$-modules. We denote canonical isomorphisms as equalities.

Lemma 2.2. *Let N be an $\overline{R}$-module. Then $\operatorname{Ext}^i_R(N, \omega_R) = 0$ for $i < k$ and $N^I = 0$.*

Proof. The first vanishing is due to Ischebeck's Lemma (see [12, Satz 1.9]), the second holds because ω_R and hence $I\omega_R$ is torsion free (see [7, Thm. 2.1.2.(c)]) whereas N is torsion. ☐

2.1. I-duality and I-free approximation.

Lemma 2.3. *There is a canonical identification $\omega_R = I^I$ and a canonical inclusion $I \hookrightarrow \omega_R^I$. They combine to the map $\delta_{I,I\omega_R}\colon I \to I^{II}$ which is an isomorphism if R is Gorenstein.*

Proof. Applying $-^\vee$ to (2.1) and $\mathrm{Hom}_R(I,-)$ to $I\omega_R \hookrightarrow \omega_R$ yields an exact sequence with a commutative triangle

$$
\begin{tikzcd}
\mathrm{Ext}^1_R(\overline{R},\omega_R) & I^\vee \arrow[l] & \omega_R \arrow[l] \arrow[dl, "\mu"] & \overline{R}^\vee \arrow[l] & 0 \arrow[l] \\
& I^I. \arrow[u, hook] & & &
\end{tikzcd}
\tag{2.2}
$$

The diagonal map sends $\varepsilon \in \omega_R$ to the multiplication map $\mu(\varepsilon)\colon I \to I\omega_R$, $x \mapsto x \cdot \varepsilon$. With Lemma 2.2 it follows that $\omega_R = I^\vee = I^I$.

There is an isomorphism $R \cong \mathrm{End}_R(\omega_R)$ sending each element to the corresponding multiplication map (see [7, Thm. 3.3.4.(d))]). Applying $\mathrm{Hom}_R(\omega_R,-)$ to $I\omega_R \hookrightarrow \omega_R$ yields a commutative square

$$
\begin{tikzcd}
R \arrow[r, "\cong"'] & \mathrm{End}_R(\omega_R) \\
I \arrow[u, hook] \arrow[r, "\delta'"] & \omega_R^I. \arrow[u, hook]
\end{tikzcd}
\tag{2.3}
$$

If R is Gorenstein, then $\omega_R^I = \mathrm{Hom}_R(R,I) = I$ and δ' is an isomorphism.

Combined with the above identification $\omega_R = I^I$, δ' defines a map $\delta\colon I \to I^{II}$. Since

$$\delta(x)(\mu(\varepsilon)) = \delta'(x)(\varepsilon) = x \cdot \varepsilon = \mu(\varepsilon)(x) = \delta_{I,I\omega_R}(x)(\mu(\varepsilon))$$

for all $x \in I$ and $\varepsilon \in \omega_R$, in fact $\delta = \delta_{I,I\omega_R}$. $\square$

Definition 2.4. If F is a free R-module, then we call $IF = I \otimes_R F$ an *I-free module.* An R-module M is called *I-reflexive* if $\delta_{M,I\omega_R}\colon M \to M^{II}$ is an isomorphism.

Proposition 2.5. *Let F be a free R-module F. Then $F^\vee = (IF)^I$ by restriction. The adjunction map $IF \to F^{\vee I}$ is induced by the isomorphism δ_{F,ω_R} and identifies with $\delta_{IF,I\omega_R}$. In case R is Gorenstein, IF is I-reflexive.*

Proof. Applying $\mathrm{Hom}_R(F,-)$ to μ in (2.2) yields $F^\vee = (IF)^I$ by Hom-tensor adjunction.

Applying $F \otimes_R -$ to (2.3) yields a commutative square

$$
\begin{tikzcd}
F \arrow[r, "\delta_{F,\omega_R}", "\cong"'] & F^{\vee\vee} \\
IF \arrow[u, hook] \arrow[r] & F^{\vee I} \arrow[u, hook]
\end{tikzcd}
$$

where the bottom row is adjunction. In fact, using Lemma 2.3,

$$\begin{aligned} IF = I \otimes_R F \to F \otimes_R \omega_R^I &= F \otimes_R \operatorname{Hom}_R(\omega_R, I\omega_R) \\ &= \operatorname{Hom}_R(F \otimes_R \omega_R, I\omega_R) \\ &= \operatorname{Hom}_R(F \otimes_R \operatorname{Hom}_R(R, \omega_R), I\omega_R) \\ &= \operatorname{Hom}_R(\operatorname{Hom}_R(F \otimes_R R, \omega_R), I\omega_R) \\ &= \operatorname{Hom}_R(\operatorname{Hom}_R(F, \omega_R), I\omega_R) = F^{\vee I}, \\ x \cdot e &\mapsto (\psi \mapsto x \cdot \psi(e)). \end{aligned}$$

Identifying $F^\vee = (IF)^I$ using Lemma 2.3 yields with the map μ in diagram (2.2)

$$\varepsilon = \psi(e) \leftrightarrow \mu(\varepsilon) \implies x \cdot \psi(e) = x \cdot \varepsilon = \mu(\varepsilon)(x).$$

Adjunction thus becomes identified with $\delta_{IF, I\omega_R}$. The last claim is due to Lemma 2.3. □

Definition 2.6. Let M be a finitely generated R-module. We call a short exact sequence

$$0 \longrightarrow IF \xrightarrow{\iota} M \xrightarrow{\rho} W \longrightarrow 0 \tag{2.4}$$

where F is free and $IW = 0$ an *I-free approximation of M* with *support* $\operatorname{Supp}(W)$. We consider W as an $\overline{R}$-module. The inclusion map $\iota\colon IF \hookrightarrow F = M$ defines the *trivial I-free approximation*

$$0 \longrightarrow IF \longrightarrow F \longrightarrow F/IF \longrightarrow 0.$$

A *morphism of I-free approximations* is a morphism of short exact sequences.

Lemma 2.7. *For any I-free approximation (2.4), ι fits into a unique commutative triangle*

$$\begin{tikzcd} & F \\ IF \arrow[ur, hook] \arrow[r, hook, "\iota"] & M. \arrow[u, "\kappa"'] \end{tikzcd} \tag{2.5}$$

If ι^{-1} denotes the choice of any preimage under ι, then $\kappa(m) = \iota^{-1}(xm)/x$ for any $x \in I \cap R^{\mathrm{reg}}$. If M is maximal Cohen–Macaulay, then κ is surjective. In particular, (2.4) becomes trivial if in addition κ injective.

Proof. Applying $\operatorname{Hom}_R(-, F)$ to (2.4) yields

$$\operatorname{Ext}^1_R(W, F) \longleftarrow \operatorname{Hom}_R(IF, F) \xleftarrow{\iota^*} \operatorname{Hom}_R(M, F) \longleftarrow \operatorname{Hom}_R(W, F) \longleftarrow 0.$$

By Ischebeck's Lemma (see [12, Satz 1.9]), $\operatorname{Ext}^1_R(W, F) = 0 = \operatorname{Hom}_R(W, F)$ making ι^* an isomorphism. Then κ is the preimage of the canonical inclusion $IF \hookrightarrow F$ under ι^*. The formula for κ follows immediately.

Since $\operatorname{coker}(\kappa)$ is a homomorphic image of F/IF, $\dim(\operatorname{coker}(\kappa)) \le n-k \le n-2$. If M is maximal Cohen–Macaulay, then $\operatorname{depth}(\operatorname{coker}(\kappa)) \ge n-1$ by the Depth Lemma (see [7, Prop. 1.2.9]). This forces $\operatorname{coker}(\kappa) = 0$ (see [7, Prop. 1.2.13]) and makes κ surjective. □

By functoriality of the cokernel, any $\varphi \in F^\vee$ gives rise to a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & I\omega_R \arrow[r] & \omega_R \arrow[r, "\pi_\omega"] & \overline{\omega}_R \arrow[r] & 0 \\
 & & F \arrow[u, "\varphi"'] & & \\
0 \arrow[r] & IF \arrow[r, "\iota"] \arrow[uu, "\varphi|_{IF}"] \arrow[ur, hook] & M \arrow[r, "\rho"] \arrow[u, "\kappa"'] & W \arrow[r] \arrow[uu, dashed, "\overline{\varphi}"'] & 0
\end{tikzcd} \tag{2.6}$$

with top exact row induced by (2.1) and bottom row (2.4). This defines a map

$$\begin{aligned} W^{\overline{\vee}} &\longleftarrow F^\vee \\ \overline{\varphi} &\longmapsfrom \varphi. \end{aligned} \tag{2.7}$$

Applying $\mathrm{Hom}_R(F, -)$ to the upper row of (2.6) yields a short exact sequence

$$0 \longrightarrow F^I \longrightarrow F^\vee \longrightarrow F^{\overline{\vee}} \longrightarrow 0. \tag{2.8}$$

By Lemma 2.2 applying $-^I$ to (2.4) and (2.5) yields the exact diagonal sequence and the triangle of inclusions with vertex F^I in the following commutative diagram.

$$\begin{tikzcd}
0 & V \arrow[l] \arrow[d, equal] & F^{\overline{\vee}} \arrow[l, "\overline{\alpha}"'] & M^I/F^I \arrow[l, "\overline{\lambda}"'] & 0 \arrow[l] \arrow[dl] \\
0 & V \arrow[l] \arrow[ddd, hook] & F^\vee \arrow[l, "\alpha"'] \arrow[u, two heads] \arrow[d, equal] & M^I \arrow[l, "\lambda"'] \arrow[u, two heads] \arrow[dl, "\iota^I"'] & 0 \arrow[l] \\
 & & (IF)^I \arrow[ddl] & F^I \arrow[l, hook'] \arrow[u, hook, "\kappa^I"'] & \\
 & & & & \\
 & \mathrm{Ext}^1_R(W, I\omega_R) & & &
\end{tikzcd} \tag{2.9}$$

By Proposition 2.5, the identification $F^\vee = (IF)^I$ in diagram (2.9) is given by

$$\varphi \leftrightarrow \varphi|_{IF} = \varphi \circ \kappa \circ \iota$$

in diagram (2.6). It defines the map λ with cokernel α. For $\psi \in M^I$, $\lambda(\psi)$ is defined by

$$\lambda(\psi)|_{IF} = \psi \circ \iota.$$

With $\mathrm{Ext}^1_R(W, I\omega_R)$ also V is an $\overline{R}$-module. Using (2.8) the Snake Lemma yields the short exact upper row of (2.9). By Lemma 2.2 the commutative square $\mathrm{Hom}_R(IF \hookrightarrow M, I\omega_R \hookrightarrow \omega_R)$ reads

$$\begin{tikzcd}
(IF)^I \arrow[d, hook] & M^I \arrow[l, "\iota^I"'] \arrow[d, hook] \\
(IF)^\vee & M^\vee. \arrow[l, "\iota^\vee"', "\cong"]
\end{tikzcd}$$

This allows one to check equalities of maps $M \to \omega_R$ after precomposing with ι. It follows that

$$\varphi \circ \kappa \in M^I \iff \varphi \in \lambda(M^I) \implies \varphi = \lambda(\varphi \circ \kappa) \tag{2.10}$$

for any $\varphi \in F^\vee$.

Definition 2.8. We call the middle row

$$0 \longleftarrow V \xleftarrow{\alpha} F^{\vee} \xleftarrow{\lambda} M^{I} \longleftarrow 0 \tag{2.11}$$

of diagram (2.9) the I-*dual* of the I-free approximation (2.4). We set

$$W' := \operatorname{Ext}^1_R(V, I\omega_R). \tag{2.12}$$

Lemma 2.9. *For any I-free approximation* (2.4) *the map* (2.7) *factors through the map α in* (2.9) *defining an inclusion $\nu\colon V \to W^{\triangledown}$, that is,*

$$\begin{tikzcd} W^{\triangledown} & V \arrow[l, hook', "\nu"'] & F^{\vee}, \arrow[l, two heads, "\alpha"'] \\ \overline{\varphi} & & \varphi. \arrow[ll, mapsto] \end{tikzcd}$$

Proof. By diagrams (2.6) and (2.9), equivalence (2.10) and exactness properties of Hom,

$$\overline{\varphi} = 0 \iff \overline{\varphi} \circ \rho = 0 \iff \varphi \circ \kappa \in M^I \iff \varphi \in \lambda(M^I) \iff \alpha(\varphi) = 0. \qquad \square$$

Remark 2.10. By Lemma 2.2 applying $\operatorname{Hom}_R(W, -)$ to the upper row of diagram (2.6) yields

$$W^{\triangledown} = \operatorname{coker} \operatorname{Hom}_R(W, \pi_\omega) \cong \operatorname{Ext}^1_R(W, I\omega_R).$$

The inclusion of V in the latter in diagram (2.9) uses $\operatorname{coker} \iota^I \hookrightarrow \operatorname{Ext}^1_R(W, I\omega_R)$. The relation with the inclusion ν in Lemma 2.9 is clarified by the double complex obtained by applying $\operatorname{Hom}_R(-,-)$ to (2.4) and the upper row of (2.6). By Lemma 2.2 it expands to a commutative diagram with exact rows and columns

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & & \\ \operatorname{Ext}^1_R(W, I\omega_R) \arrow[d] & (IF)^I \arrow[l] \arrow[d] & M^I \arrow[l, "\iota^I"'] \arrow[d] & 0 \arrow[l] \arrow[d] & \\ 0 & (IF)^{\vee} \arrow[l] \arrow[d] & M^{\vee} \arrow[l, "\iota^{\vee}"'] \arrow[d] & 0 \arrow[l] \arrow[d, "\operatorname{Hom}_R(W,\pi_\omega)"] & \\ & (IF)^{\triangledown} & M^{\triangledown} \arrow[l] & W^{\triangledown} \arrow[l] \arrow[d, "\cong"'] & 0 \arrow[l] \\ & & & \operatorname{Ext}^1_R(W, I\omega_R). & \end{tikzcd}$$

An element $\alpha(\varphi) \in V$ with $\varphi \in F^{\vee}$ maps to $\varphi|_{IF} \in (IF)^I$, to $\varphi \circ \kappa \in M^{\vee}$ and to $\overline{\varphi} \in W^{\triangledown}$.

2.2. ***I*-reflexivity over Gorenstein rings.** In this subsection we assume that R is Gorenstein and study I-reflexivity of modules M in terms of an I-free approximation (2.4). With the Gorenstein hypothesis $F^{\vee}$ is free and hence

$$\operatorname{Ext}^1_R(F^{\vee}, -) = 0. \tag{2.13}$$

Proposition 2.11. *Assume that R is Gorenstein. For any I-free approximation* (2.4) *and W' as in* (2.12) *there is a commutative square*

$$\begin{tikzcd}
M \arrow[r, two heads, "\rho"] \arrow[d, "\delta_{M,I\omega_R}"] & W \arrow[d, "\overline{\delta}"] \\
M^{II} \arrow[r, two heads, "\rho'"] & W'
\end{tikzcd}$$

and $\overline{\delta}$ is an isomorphism if and only if M is I-reflexive.

Proof. Consider the following commutative diagram whose rows are (2.4) and obtained by applying $-^I$ to the triangle with vertex $F^\vee$ in diagram (2.9).

$$\begin{tikzcd}
 & & F \arrow[ddd, bend left=80, "\delta_{F,\omega_R}", "\cong"'] & & \\
0 \arrow[r] & IF \arrow[ur, hook] \arrow[r, "\iota"] \arrow[d, "\delta_{IF,I\omega_R}", "\cong"'] & M \arrow[u, "\kappa"] \arrow[r, "\rho"] \arrow[d, "\delta_{M,I\omega_R}"] & W \arrow[r] \arrow[d, dashed, "\overline{\delta}"] & 0 \\
0 \arrow[r] & (IF)^{II} \arrow[r, "\iota^{II}"] \arrow[d, equal] & M^{II} \arrow[r, "\rho'"] & W' \arrow[r] & 0 \\
 & F^{\vee I} \arrow[ur, "\lambda^I"] \arrow[r, hook] & F^{\vee\vee}. & &
\end{tikzcd} \tag{2.14}$$

The latter is a short exact sequence by Lemma 2.2 and (2.13). The commutative squares in diagram (2.14) are due to functoriality of δ and the cokernel. The claimed equivalence then follows from the Snake Lemma. Proposition 2.5 yields the part of diagram (2.14) involving δ_{F,ω_R}. This part is just added for clarification but not needed for the proof. ☐

Lemma 2.12. *Assume that R is Gorenstein and consider an I-free approximation* (2.4). *Then the maps ν from Lemma 2.9 and $\overline{\delta}$ from Proposition 2.11 fit into a commutative square*

$$\begin{tikzcd}
W \arrow[r, "\delta_{W,\overline{\omega}_R}"] \arrow[d, "\overline{\delta}"] & W^{\overline{\vee}\overline{\vee}} \arrow[d, "\nu^{\overline{\vee}}"] \\
W' & V^{\overline{\vee}}. \arrow[l, "\xi", "\cong"']
\end{tikzcd}$$

Proof. Consider the double complex obtained by applying $\mathrm{Hom}_R(-,-)$ to the middle and top rows of diagrams (2.9) and (2.6). By Lemma 2.2 and (2.13) it expands to a commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & & \\
 & 0 \arrow[r] \arrow[d] & F^{\vee I} \arrow[r, "\lambda^I"] \arrow[d] & M^{II} \arrow[r, "\rho'"] \arrow[d] & W' \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] \arrow[d] & F^{\vee\vee} \arrow[r, "\lambda^\vee"] \arrow[d, "(\pi_\omega)_*"] & M^{I\vee} \arrow[r] \arrow[d] & 0 & \\
0 \arrow[r] & V^{\triangledown} \arrow[r, "\alpha^{\triangledown}"] & F^{\vee\triangledown} \arrow[r, "\lambda^{\triangledown}"] \arrow[d] & M^{I\triangledown} & & \\
 & & 0. & & &
\end{tikzcd}$$

The Snake Lemma yields an isomorphism $\xi\colon V^{\triangledown} \to W'$. Attaching the square of Proposition 2.11, the relation $\overline{\delta}(w) = \xi(\widetilde{\psi})$ is given by the diagram chase

$$\begin{tikzcd}
 & & & m \arrow[r, mapsto] \arrow[d, mapsto] & w \arrow[d, mapsto] \\
 & & & \delta_{M,I\omega_R}(m) \arrow[r, mapsto] \arrow[d, mapsto] & \overline{\delta}(w) \\
 & \psi \arrow[r, mapsto] \arrow[d, mapsto] & \psi \circ \lambda = \delta_{M,I\omega_R}(m) & & \\
\widetilde{\psi} \arrow[r, mapsto] & \widetilde{\psi} \circ \alpha = \pi_\omega \circ \psi. & & &
\end{tikzcd}$$

Using implication (2.10), diagram (2.6) and Lemma 2.9, one deduces that, with $x \in I \cap R^{\mathrm{reg}}$ and $v = \alpha(\varphi)$,

$$\begin{aligned}
x\varphi \circ \kappa \in M^I \implies x\varphi &= \lambda(x\varphi \circ \kappa) \\
\implies x\psi(\varphi) &= \psi(x\varphi) = (\psi \circ \lambda)(x\varphi \circ \kappa) \\
&= \delta_{M,I\omega_R}(m)(x\varphi \circ \kappa) = x(\varphi \circ \kappa)(m) \\
\implies \psi(\varphi) &= (\varphi \circ \kappa)(m) \\
\implies \widetilde{\psi}(v) &= (\widetilde{\psi} \circ \alpha)(\varphi) = (\pi_\omega \circ \psi)(\varphi) = (\pi_\omega \circ \varphi \circ \kappa)(m) = \overline{\varphi}(w) \\
&= (\nu \circ \alpha)(\varphi)(w) = \nu(\alpha(\varphi))(w) = \nu(v)(w) \\
&= \delta_{W,\overline{\omega}_R}(w)(\nu(v)) = \nu^{\triangledown}(\delta_{W,\overline{\omega}_R}(w))(v) = (\nu^{\triangledown} \circ \delta_{W,\overline{\omega}_R})(w)(v) \\
\implies \widetilde{\psi} &= (\nu^{\triangledown} \circ \delta_{W,\overline{\omega}_R})(w) \\
\implies \overline{\delta}(w) &= \xi(\widetilde{\psi}) = (\xi \circ \nu^{\triangledown} \circ \delta_{W,\overline{\omega}_R})(w) \\
\implies \overline{\delta} &= \xi \circ \nu^{\triangledown} \circ \delta_{W,\overline{\omega}_R}.
\end{aligned}$$

□

Proposition 2.13. *Assume that R is Gorenstein and consider an I-free approximation* (2.4)*. Then M is I-reflexive if and only if the map $\nu^{\triangledown} \circ \delta_{W,\overline{\omega}_R}$ with ν from Lemma 2.9 identifies $W = V^{\triangledown}$.*

Proof. The claim follows from Proposition 2.11 and Lemma 2.12. □

Lemma 2.14. *Assume that R is Gorenstein and consider an I-free approximation* (2.4)*. Then the map ν from Lemma 2.9 fits into a commutative diagram*

$$\begin{tikzcd}
W^{\triangledown} & & V \arrow[ll, "\nu"'] \arrow[d, "\delta_{V,\overline{\omega}_R}"] \\
W^{\triangledown\triangledown\triangledown} \arrow[u, "\delta^{\triangledown}_{W,\overline{\omega}_R}"] & & V^{\triangledown\triangledown}. \arrow[ll, "\nu^{\triangledown\triangledown}"] \arrow[ull, "(\nu^{\triangledown}\circ\delta_{W,\overline{\omega}_R})^{\triangledown}"']
\end{tikzcd}$$

Proof. For any $v \in V$ and $w \in W$ we have

$$\begin{aligned}
(\delta^{\triangledown}_{W,\overline{\omega}_R} \circ \nu^{\triangledown\triangledown} \circ \delta_{V,\overline{\omega}_R})(v)(w) &= \delta^{\triangledown}_{W,\overline{\omega}_R}(\nu^{\triangledown\triangledown}(\delta_{V,\overline{\omega}_R}(v)))(w) \\
&= \delta^{\triangledown}_{W,\overline{\omega}_R}(\delta_{V,\overline{\omega}_R}(v) \circ \nu^{\triangledown})(w) \\
&= (\delta_{V,\overline{\omega}_R}(v) \circ \nu^{\triangledown})(\delta_{W,\overline{\omega}_R}(w)) \\
&= \delta_{V,\overline{\omega}_R}(v)(\delta_{W,\overline{\omega}_R}(w) \circ \nu) \\
&= \delta_{W,\overline{\omega}_R}(w)(\nu(v)) \\
&= \nu(v)(w)
\end{aligned}$$

and hence $\nu = \delta^{\triangledown}_{W,\overline{\omega}_R} \circ \nu^{\triangledown\triangledown} \circ \delta_{V,\overline{\omega}_R}$ as claimed. □

Corollary 2.15. *Assume that R is Gorenstein and consider an I-free approximation* (2.4) *of an I-reflexive R-module M. Then V in diagram* (2.9) *is ($\overline{\omega}_R$-)reflexive if and only if ν in Lemma 2.9 identifies $V = W^{\triangledown}$.*

Proof. The claim follows from Proposition 2.13 and Lemma 2.14. □

2.3. ***R*-dual *I*-free approximation.** In this subsection we consider the R-dual of an I-free approximation (2.4). The interesting part of the long exact Ext-sequence of $-^{\vee}$ applied to (2.4) turns out to be

$$0 \leftarrow \operatorname{Ext}^k_R(M, \omega_R) \leftarrow \operatorname{Ext}^k_R(W, \omega_R) \xleftarrow{\beta} \operatorname{Ext}^{k-1}_R(IF, \omega_R) \leftarrow \operatorname{Ext}^{k-1}_R(M, \omega_R) \leftarrow 0. \tag{2.15}$$

In fact, applying $-^{\vee}$ to (2.1) yields (see Lemma 2.17 and [7, Thm. 3.3.10.(c).(ii)])

$$\operatorname{Ext}^i_R(IF, \omega_R) = F^* \otimes_R \operatorname{Ext}^i_R(I, \omega_R) = F^* \otimes_R \operatorname{Ext}^{i+1}_R(\overline{R}, \omega_R) = 0 \text{ for } i \neq 0, k-1.$$

In case both R and $\overline{R}$ are Gorenstein, we will identify the map β to its image with the map $\overline{\alpha}$ in (2.9) (see Corollary 2.21). In §2.4 this fact will serve to relate the Gorenstein dimension of M to the depth of V.

In order to describe the map β in (2.15) we fix a canonical module ω_R of R with an injective resolution $(E^\bullet, \partial^\bullet)$,

$$0 \longrightarrow \omega_R \longrightarrow E^0 \xrightarrow{\partial^0} E^1 \xrightarrow{\partial^1} E^2 \xrightarrow{\partial^2} \cdots .$$

We use it to fix representatives

$$\operatorname{Ext}^i_R(-, \omega_R) := H^i \operatorname{Hom}_R(-, E^\bullet).$$

Then (see [7, Thms. 3.3.7.(b), 3.3.10.(c).(ii)])

$$H^i \operatorname{Ann}_{E^\bullet}(I) = H^i \operatorname{Hom}(\overline{R}, E^\bullet) = \operatorname{Ext}^i_R(\overline{R}, \omega_R) = \delta_{i,k} \cdot \omega_{\overline{R}} \tag{2.16}$$

where

$$\omega_{\overline{R}} := H^k \operatorname{Ann}_{E^\bullet}(I)$$

is a canonical module of $\overline{R}$.

In the sequel we explicit the maps of the following commutative diagram

$$\begin{tikzcd}
\operatorname{Ext}^k_R(W, \omega_R) \arrow[dd, "\gamma"', "\cong"] & & \operatorname{Ext}^{k-1}_R(IF, \omega_R) \arrow[ll, "\beta"'] \\
& & F^* \otimes_R \operatorname{Ext}^{k-1}_R(I, \omega_R) \arrow[u, "\cong"', "\chi"] \\
& & F^* \otimes_R H^{k-1}(E^\bullet / \operatorname{Ann}_{E^\bullet}(I)) \arrow[u, "\cong"', "F^* \otimes H^{k-1}(\tau^\bullet)"] \arrow[d, "\cong"', "F^*\otimes\zeta"] \\
\operatorname{Hom}_{\overline{R}}(W, \omega_{\overline{R}}) = W^\vee & V' \arrow[l, hook', "\nu'"'] & F^* \otimes_R \omega_{\overline{R}} \arrow[l, two heads, "\alpha'"']
\end{tikzcd} \tag{2.17}$$

which defines the map $\nu' \circ \alpha'$ and its image V'. The maps $\tau^\bullet$, χ, ζ, γ and α' are described in Lemmas 2.16, 2.17, 2.18, 2.19 and Proposition 2.20 respectively.

Lemma 2.16. *For any injective R-module E there is a canonical isomorphism*

$$\tau \colon E/\operatorname{Ann}_E(I) \to \operatorname{Hom}_R(I, E), \quad \overline{e} \mapsto - \cdot e = (x \mapsto x \cdot e).$$

In particular, there is a canonical isomorphism $\tau^\bullet \colon E^\bullet/\operatorname{Ann}_{E^\bullet}(I) \to \operatorname{Hom}_R(I, E^\bullet)$.

Proof. Applying the exact functor $\operatorname{Hom}_R(-, E)$ to (2.1) yields a short exact sequence

$$0 \leftarrow \operatorname{Hom}_R(I, E) \leftarrow \operatorname{Hom}_R(R, E) \leftarrow \operatorname{Hom}_R(\overline{R}, E) \leftarrow 0.$$

Identifying $E = \operatorname{Hom}_R(R, E)$, $e \mapsto - \cdot e$, and hence

$$\operatorname{Hom}_R(\overline{R}, E) = \operatorname{Ann}_E(I) \tag{2.18}$$

yields the claim. ∎

Lemma 2.17. *For any $i \in \mathbb{N}$ there is a canonical isomorphism*

$$\begin{array}{ccc}
F^* \otimes_R \operatorname{Ext}^i_R(I, \omega_R) & \xrightarrow[\cong]{\chi_i} & \operatorname{Ext}^i_R(IF, \omega_R) \\
\| & & \| \\
F^* \otimes_R H^i \operatorname{Hom}_R(I, E^\bullet) & \longrightarrow & H^i \operatorname{Hom}_R(IF, E^\bullet) \\
\varphi \otimes [\psi] & \longmapsto & [\varphi|_{IF} \cdot \widetilde{\psi}(1)] = [(\kappa \circ \iota)^*(\varphi) \cdot \widetilde{\psi}(1)]
\end{array}$$

where $\widetilde{\psi} \in \operatorname{Hom}_R(R, E^\bullet)$ extends $\psi \in \operatorname{Hom}_R(I, E^\bullet)$. We set $\chi := \chi_{k-1}$.

Proof. For any $i \in \mathbb{N}$ there is a sequence of canonical isomorphisms

$$\begin{aligned}
F^* \otimes_R H^i \operatorname{Hom}_R(I, E^\bullet) &= \operatorname{Hom}_R(F, H^i \operatorname{Hom}_R(I, E^\bullet)) \\
&= H^i \operatorname{Hom}_R(F, \operatorname{Hom}_R(I, E^\bullet)) \\
&= H^i \operatorname{Hom}_R(IF, E^\bullet),
\end{aligned}$$

the latter one being Hom-tensor adjunction, sending

$$\begin{aligned}\varphi\otimes[\psi] &\mapsto (f\mapsto\varphi(f)\cdot[\psi]=[\varphi(f)\cdot\psi])\\ &\mapsto [f\mapsto\varphi(f)\cdot\psi]\\ &\mapsto [x\cdot f\mapsto\varphi(f)\cdot\psi(x)=\varphi(x\cdot f)\cdot\widetilde{\psi}(1)]=[\varphi|_{IF}\cdot\widetilde{\psi}(1)]\end{aligned}$$

where $x\in I$ and $f\in F$. □

Lemma 2.18. *There is a connecting isomorphism*

$$\begin{aligned}\zeta\colon H^{k-1}(E^\bullet/\operatorname{Ann}_{E^\bullet}(I)) &\to H^k\operatorname{Ann}_{E^\bullet}(I)=\omega_{\overline{R}},\\ [\overline{e}] &\mapsto [\partial^{k-1}(e)].\end{aligned}$$

Proof. The connecting homomorphism ζ in degree k of the short exact sequence

$$0\to\operatorname{Ann}_{E^\bullet}(I)\to E^\bullet\to E^\bullet/\operatorname{Ann}_{E^\bullet}(I)\to 0$$

is an isomorphism since $E^\bullet$ is a resolution and hence $H^i(E^\bullet)=0$ for $i\geq k-1\geq 1$. □

Lemma 2.19. *For any $\overline{R}$-module N there is a canonical isomorphism*

$$\begin{aligned}\gamma\colon H^k\operatorname{Hom}_R(N,E^\bullet) &\to \operatorname{Hom}_{\overline{R}}(N,H^k\operatorname{Ann}_{E^\bullet}(I))=N^\vee,\\ [\phi] &\mapsto (n\mapsto[\phi(n)]).\end{aligned}$$

Proof. Fix an $\overline{R}$-projective resolution $(P_\star,\delta_\star)$ of N and consider the double complex

$$A^{\star,\bullet}:=\operatorname{Hom}_R(P_\star,E^\bullet)=\operatorname{Hom}_{\overline{R}}(P_\star,\operatorname{Hom}_R(\overline{R},E^\bullet))=\operatorname{Hom}_{\overline{R}}(P_\star,\operatorname{Ann}_{E^\bullet}(I))$$

whose alternate representation is due to Hom-tensor adjunction and (2.18). It yields two spectral sequences with the same limit. By exactness of $\operatorname{Hom}_{\overline{R}}(P_\star,-)$ and (2.16) and using the alternate representation the E_2-page of the first spectral sequence identifies with

$$'E_2^{p,q}=H^p(H^{\star,q}(A^{\star,\bullet}))=H^p\operatorname{Hom}_{\overline{R}}(P_\star,H^q\operatorname{Ann}_{E^\bullet}(I))=\delta_{k,q}\cdot H^p\operatorname{Hom}_{\overline{R}}(P_\star,\omega_{\overline{R}}).$$

By exactness of $\operatorname{Hom}_R(-,E^\bullet)$ the E_2-page of the second spectral sequence reads

$$''E_2^{p,q}=H^q(H^{p,\bullet}(A^{\star,\bullet}))=H^q\operatorname{Hom}_R(H^pP_\star,E^\bullet)=\delta_{p,0}\cdot H^q\operatorname{Hom}_R(N,E^\bullet).$$

So both spectral sequences degenerate. The resulting isomorphism $''E_2^{0,k}\to{}'E_2^{0,k}$ is γ. □

Proposition 2.20. *Assume that R is Gorenstein and consider an I-free approximation (2.4). Then the map α' in diagram (2.17) is induced by*

$$\begin{aligned}\nu'\circ\alpha'\colon F^*\otimes_R\omega_{\overline{R}}=F^*\otimes_R H^k\operatorname{Ann}_{E^\bullet}(I) &\to \operatorname{Hom}_{\overline{R}}(W,H^k\operatorname{Ann}_{E^\bullet}(I))=W^\vee,\\ \varphi\otimes[a] &\mapsto \overline{\varphi}\cdot[a],\end{aligned}$$

where $\varphi\mapsto\overline{\varphi}$ is (2.7) with $\omega_R=R$. In particular, $\operatorname{Ext}^k_R(M,R)=0$ if ν' is surjective.

Proof. The proof is done by chasing diagram (2.17) and the diagram

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Hom}_R(W,E^{k-1}) \arrow[r,"\rho^*"] \arrow[d,"(\partial^{k-1})_*"] & \operatorname{Hom}_R(M,E^{k-1}) \arrow[r,"\iota^*"] \arrow[d,"(\partial^{k-1})_*"] & \operatorname{Hom}_R(IF,E^{k-1}) \arrow[r] \arrow[d,"(\partial^{k-1})_*"] & 0\\ 0 \arrow[r] & \operatorname{Hom}_R(W,E^k) \arrow[r,"\rho^*"] & \operatorname{Hom}_R(M,E^k) \arrow[r,"\iota^*"] & \operatorname{Hom}_R(IF,E^k) \arrow[r] & 0. \end{tikzcd}$$

This latter defines the connecting homomorphism β in (2.15) on representatives as

$$(\rho^*)^{-1}\circ(\partial^{k-1})_*\circ(\iota^*)^{-1},$$

where $(\iota^*)^{-1}$ denotes the choice of any preimage under ι^*.

Let $\varphi \otimes [\overline{e}] \in F^* \otimes_R H^{k-1}(E^\bullet / \operatorname{Ann}_{E^\bullet}(I))$. Then by Lemmas 2.16, 2.17, 2.18 and 2.19, and diagram (2.6) with $\omega_R = R$

$$\begin{tikzcd}
 & {[\kappa^*(\varphi) \cdot e]} \arrow[r, mapsto, "H^{k-1}(\iota^*)"] \arrow[d, mapsto] & {[(\iota^* \circ \kappa^*)(\varphi) \cdot e]} \\
{[((\rho^{-1})^* \circ \kappa^*)(\varphi) \cdot \partial^{k-1}(e)]} \arrow[r, mapsto, "H^k(\rho^*)"] \arrow[ddd, mapsto, "\gamma"] & {[\kappa^*(\varphi) \cdot \partial^{k-1}(e)]} & \varphi \otimes [- \cdot e] \arrow[u, mapsto, "\chi"'] \\
 & & \varphi \otimes [\overline{e}] \arrow[u, mapsto, "F^* \otimes H^{k-1}(\tau^\bullet)"'] \arrow[d, mapsto, "F^* \otimes \zeta"'] \\
(\pi \circ \varphi \circ \kappa \circ \rho^{-1}) \cdot [\partial^{k-1}(e)] = \overline{\varphi} \cdot [\partial^{k-1}(e)] & & \varphi \otimes [\partial^{k-1}(e)] \arrow[ll, mapsto, "\nu' \circ \alpha'"']
\end{tikzcd}$$

where ρ^{-1} denotes the choice of any preimage under ρ. By diagram (2.6) and Lemma 2.18 the ambiguity of this choice is cancelled when multiplying $(\rho^{-1})^* \circ \kappa^*(\varphi) = \varphi \circ \kappa \circ \rho^{-1}$ with $\partial^{k-1}(e) \in \operatorname{Ann}_{E^\bullet}(I)$.

The particular claim follows from diagram (2.17) and the exact sequence (2.15). $\square$

Corollary 2.21. *Assume that both R and $\overline{R}$ are Gorenstein and consider an I-free approximation* (2.4)*. Then identifying $\overline{\omega}_R = \omega_{\overline{R}}$ (see diagrams* (2.9) *and* (2.17)*) makes*

$$\alpha' = \overline{\alpha}, \quad V' = V, \quad \operatorname{Ext}_R^{k-1}(M, R) \cong \ker(\overline{\alpha}) = M^I / F^I.$$

In particular, if M is I-reflexive, then $\operatorname{Ext}_R^k(M, R) = 0$ if and only if V is ($\overline{\omega}_R$-)reflexive.

Proof. Let $\varphi \mapsto \overline{\varphi}$ be (2.7) with $\omega_R = R$. Pick free generators $\varepsilon \in \omega_R$ and $\widetilde{\varepsilon} \in \omega_{\overline{R}}$ inducing the identification $\overline{\omega}_R = \omega_{\overline{R}}$ by sending $\overline{\varepsilon} = \pi_\omega(\varepsilon) \mapsto \widetilde{\varepsilon}$. Then

$$\begin{aligned}
F^\vee \otimes_R \overline{R} = F^* \otimes_R \overline{\omega}_R = F^* \otimes_R \omega_{\overline{R}}, &\quad W^{\overline{\vee}} = W^\vee, \\
(\varphi \cdot \varepsilon) \otimes \overline{1} \leftrightarrow \varphi \otimes \overline{\varepsilon} \leftrightarrow \varphi \otimes \widetilde{\varepsilon}, &\qquad \overline{\varphi} \cdot \overline{\varepsilon} \leftrightarrow \overline{\varphi} \cdot \widetilde{\varepsilon}.
\end{aligned}$$

By diagram (2.6) and Lemma 2.9 the map $F^\vee \otimes_R \overline{R} \to W^{\overline{\vee}}$ induced by $\nu \circ \alpha$ sends

$$(\varphi \cdot \varepsilon) \otimes \overline{1} \mapsto \overline{\varphi \cdot \varepsilon} = \pi_\omega \circ ((\varphi \circ \kappa \circ \rho^{-1}) \cdot \varepsilon) = (\pi \circ \varphi \circ \kappa \circ \rho^{-1}) \cdot \pi_\omega(\varepsilon) = \overline{\varphi} \cdot \overline{\varepsilon}.$$

By Proposition 2.20 this map coincides with $\nu' \circ \alpha'$ subject to the above identifications. This shows that $\alpha' = \overline{\alpha}$ and $V' = V$. By the exact sequence (2.15), the commutative diagram (2.17) and the exact upper row of diagram (2.9),

$$\begin{aligned}
\operatorname{Ext}_R^{k-1}(M, R) &= \ker(\beta) \cong \ker(\alpha') = \ker(\overline{\alpha}) = M^I / F^I, \\
\operatorname{Ext}_R^k(M, R) &= \operatorname{coker}(\beta) \cong \operatorname{coker}(\nu') = W^\vee / \nu'(V').
\end{aligned}$$

In particular $\operatorname{Ext}_R^k(M, R) = 0$ if and only if ν' identifies $V' = W^\vee$ or, equivalently, if ν identifies $V = W^{\overline{\vee}}$. The particular claim now follows with Corollary 2.15. $\square$

2.4. **Projective dimension and residual depth.** Assume that R is Gorenstein. Then every finitely generated R-module M has finite Gorenstein dimension $\operatorname{G-dim}(M) < \infty$ (see [18, Thm. 17]). Recall that if M has finite projective dimension $\operatorname{pdim}(M) < \infty$, then

$$\operatorname{G-dim}(M) = \operatorname{pdim}(M)$$

(see [18, Cor. 21]). Consider an I-free approximation (2.4) of an R-module M. In the following we relate the case of minimal Gorenstein dimension of M to Cohen–Macaulayness of V, proving our main result.

Lemma 2.22. *Assume that R is Gorenstein and consider an I-free approximation* (2.4) *with $W \neq 0$. Then W is a maximal Cohen–Macaulay $\overline{R}$-module if and only if $\operatorname{G-dim}(M) \leq k$. In this case $\operatorname{G-dim}(M) \leq k-1$ if and only if $\operatorname{Ext}_R^k(M,R) = 0$. If $\overline{R}$ is Gorenstein, then $\operatorname{G-dim}(M) \geq k-1$ unless $\overline{\alpha}$ in diagram* (2.9) *is injective.*

Proof. By hypothesis $M \neq 0$ is finitely generated over the Gorenstein ring R. It follows that (see [18, Thm. 17, Lem. 23.(c)])

$$\operatorname{G-dim}(M) = \max\left\{i \in \mathbb{N} \;\middle|\; \operatorname{Ext}_R^i(M,R) \neq 0\right\} < \infty. \tag{2.19}$$

The Auslander–Bridger Formula (see [18, Thm. 29]) then states that

$$\operatorname{depth}(M) = \operatorname{depth}(R) - \operatorname{G-dim}(M) = \dim(R) - \operatorname{G-dim}(M) = n - \operatorname{G-dim}(M). \tag{2.20}$$

By the Depth Lemma (see [7, Prop. 1.2.9]) applied to the short exact sequence (2.1)

$$\begin{aligned} n-k+1 = \operatorname{depth}(\overline{R}) + 1 &\geq \min\{\operatorname{depth}(R), \operatorname{depth}(I) - 1\} + 1 = \operatorname{depth}(I) \\ &\geq \min\left\{\operatorname{depth}(R), \operatorname{depth}(\overline{R}) + 1\right\} = n-k+1 \end{aligned}$$

and hence

$$\operatorname{depth}(IF) = \operatorname{depth}(I) = n-k+1. \tag{2.21}$$

($\Longrightarrow$) Using (2.21) and (2.20) the Depth Lemma applied to the short exact sequence (2.4) gives

$$\operatorname{G-dim}(M) = n - \operatorname{depth}(M) \leq n - \min\{\operatorname{depth}(IF), \operatorname{depth}(W)\} \leq n - (n-k) = k.$$

($\Longleftarrow$) Using (2.20) and (2.21) the Depth Lemma applied to the short exact sequence (2.4) gives

$$n - k = \dim(\overline{R}) \geq \dim(W) \geq \operatorname{depth}(W) \geq \min\{\operatorname{depth}(M), \operatorname{depth}(IF) - 1\} \geq n-k.$$

By (2.19) this latter inequality becomes $\operatorname{G-dim}(M) \leq k-1$ if and only if $\operatorname{Ext}_R^k(M,R) = 0$ (see [18, Lem. 23.(c)]).

If $\overline{R}$ is Gorenstein and $\overline{\alpha}$ is not injective, then $\operatorname{Ext}_R^{k-1}(M,R) \neq 0$ by Corollary 2.21 and hence $\operatorname{G-dim}(M) \geq k-1$ by (2.19). ☐

We can now conclude the proof of our main result.

Proof of Theorem 1.2. Since M is I-reflexive, $W = V^{\triangledown}$ by Proposition 2.13.

($\Longrightarrow$) Suppose that V is maximal Cohen–Macaulay. Then also W is maximal Cohen–Macaulay and V is ($\overline{\omega}_R$-)reflexive (see [7, Prop. 3.3.3.(b).(ii), Thm. 3.3.10.(d).(iii)]). By Corollary 2.21 $\operatorname{Ext}_R^k(M,R) = 0$ and by Lemma 2.22 $\operatorname{G-dim}(M) = k-1$.

($\Longleftarrow$) Suppose that $\operatorname{G-dim}(M) \leq k-1$. By Lemma 2.22 W is maximal Cohen–Macaulay and $\operatorname{Ext}^k(M,R) = 0$. By Corollary 2.21 $V = W^{\triangledown}$ is ($\overline{\omega}_R$-)reflexive and maximal Cohen–Macaulay (see [7, Prop. 3.3.3.(b).(ii)]).

The last claim is due to Lemma 2.22. ☐

2.5. **Restricted I-free approximation.** In this subsection we describe a construction that reduces the support of an I-free approximation (2.4) and preserves I-reflexivity of M under suitable hypotheses. In §3.2 this will be related to the definition of multi-logarithmic differential forms and residues along Cohen–Macaulay spaces (see [4, §10] and [21, Ch. 4]).

Fix an ideal $J \trianglelefteq R$ with $I \subseteq J$ and set $S := \overline{R}$ and $T := R/J$. By hypothesis S is Cohen–Macaulay and hence (see [7, Prop.1.2.13])

$$\mathrm{Ass}(S) = \mathrm{Min}\,\mathrm{Spec}(S). \tag{2.22}$$

Lemma 2.23. *There is an inclusion*

$$\mathrm{Supp}_S(T) \cap \mathrm{Ass}(S) \subseteq \mathrm{Ass}_S(T).$$

In particular, equality in $\mathrm{Hom}_S(N, S)$ *for any* T*-module* N*, or in* $\mathrm{Hom}_S(N, T)$ *for any* S*-module* N*, can be checked at* $\mathrm{Ass}_S(T)$*.*

Proof. The inclusion follows from (2.22) and $\mathrm{Min}\,\mathrm{Supp}_S(T) \subseteq \mathrm{Ass}_S(T)$. For any T-module N (see [7, Exe. 1.2.27])

$$\mathrm{Ass}_S(\mathrm{Hom}_S(N, S)) = \mathrm{Supp}_S(N) \cap \mathrm{Ass}(S) \subseteq \mathrm{Supp}_S(T) \cap \mathrm{Ass}(S) \subseteq \mathrm{Ass}_S(T)$$

and the first particular claim follows, the second holds for a similar reason. $\square$

Definition 2.24. For any S-module N we consider the submodule supported on $V(J)$

$$N_T := \mathrm{Hom}_S(T, N) = \mathrm{Ann}_N(J) \subseteq N.$$

For an I-free approximation (2.4) its J*-restriction* is the I-free approximation

$$0 \longrightarrow IF \xrightarrow{\iota_J} M_J \xrightarrow{\rho_T} W_T \longrightarrow 0 \tag{2.23}$$

defined as its image under the map $\mathrm{Ext}^1_R(W, IF) \to \mathrm{Ext}^1_R(W_T, IF)$.

In explicit terms it is the source of a morphism of I-free approximations

$$\begin{tikzcd} 0 \arrow[r] & IF \arrow[r, "\iota"] & M \arrow[r, "\rho"] & W \arrow[r] & 0 \\ 0 \arrow[r] & IF \arrow[r, "\iota_J"] \arrow[u, equal] & M_J \arrow[r, "\rho_T"] \arrow[u, hook] & W_T \arrow[r] \arrow[u, hook] & 0. \end{tikzcd} \tag{2.24}$$

The right square is obtained as the pull-back of ρ and $W_T \hookrightarrow W$, whose universal property applied to ι and $0\colon IF \to W_T$ gives the left square. The analogue of κ in (2.5) for the J-restriction (2.23) is the composition

$$\kappa_J\colon M_J = IF :_M J \subseteq M \xrightarrow{\kappa} F. \tag{2.25}$$

By Lemma 2.2 and the Snake Lemma, applying $-^I$ to (2.24) yields (see Definition 2.8)

$$\begin{tikzcd} 0 & V \arrow[l] \arrow[d, two heads] & F^\vee \arrow[l, "\alpha"'] \arrow[d, equal] & M^I \arrow[l, "\lambda"'] \arrow[d, hook] & 0 \arrow[l] \\ 0 & V^T \arrow[l] & F^\vee \arrow[l, "\alpha^T"'] & M^I_J \arrow[l, "\lambda^J"'] & 0 \arrow[l] \end{tikzcd} \tag{2.26}$$

where the bottom row

$$0 \longleftarrow V^T \xleftarrow{\alpha^T} F^\vee \xleftarrow{\lambda^J} M^I_J \longleftarrow 0 \tag{2.27}$$

is the I-dual (2.11) of the J-restriction (2.23). In diagram (2.26), we denote

$$U := \ker(V \twoheadrightarrow V^T). \tag{2.28}$$

The J-restriction behaves well under the following hypothesis on T.

$$T_{\mathfrak{p}} = \begin{cases} S_{\mathfrak{p}} & \text{if } \mathfrak{p} \in \operatorname{Ass}_S(T), \\ 0 & \text{if } \mathfrak{p} \in \operatorname{Ass}(S) \setminus \operatorname{Ass}_S(T). \end{cases} \tag{2.29}$$

This is due to the following

Remark 2.25. Our constructions commute with localization. As special cases of the J-restriction and its I-dual we record

$$(\iota_J, \rho_T) = \begin{cases} (\iota, \rho) & \text{if } T = S, \\ (\mathrm{id}_{IF}, 0) & \text{if } T = 0, \end{cases} \quad (\lambda^J, \alpha^T) = \begin{cases} (\lambda, \alpha) & \text{if } T = S, \\ (\mathrm{id}_{F^\vee}, 0) & \text{if } T = 0. \end{cases}$$

Localizing (2.24) and (2.26) at the image of $\mathfrak{p} \in \operatorname{Ass}(S)$ under the map $\operatorname{Spec}(S) \to \operatorname{Spec}(R)$ yields these special cases under hypothesis (2.29).

In the setup of our applications in §3 condition (2.29) holds true due to the following

Lemma 2.26. *If S is reduced and T is unmixed with $\dim(T) = \dim(S)$, then condition (2.29) holds and $\operatorname{Ass}_S(T) \subseteq \operatorname{Ass}(S)$.*

Proof. By hypothesis on T and (2.22)

$$\operatorname{Ass}_S(T) = \operatorname{Min}\operatorname{Supp}_S(T) \subseteq \operatorname{Min}\operatorname{Spec}(S) = \operatorname{Ass}(S). \tag{2.30}$$

By hypothesis on S, for any $\mathfrak{p} \in \operatorname{Ass}(S)$, $S_{\mathfrak{p}}$ is a field with factor ring $T_{\mathfrak{p}}$. If $\mathfrak{p} \in \operatorname{Ass}_S(T)$, then $T_{\mathfrak{p}} \neq 0$ and hence $T_{\mathfrak{p}} = S_{\mathfrak{p}}$. Otherwise, $\mathfrak{p} \notin \operatorname{Supp}_S(T)$ by (2.30) and hence $T_{\mathfrak{p}} = 0$. $\square$

Lemma 2.27. *Assume that R is Gorenstein and consider the J-restriction (2.23) of an I-free approximation. If T satisfies condition (2.29), then for U as defined in (2.28)*

$$\alpha^{-1}(U) = \{\varphi \in F^\vee \mid \varphi \circ \kappa(M) \subseteq J\omega_R\}.$$

In particular, $JV \subseteq U$.

Proof. Let $\varphi \in F^\vee$ and denote by $\overline{\varphi}_T$ the map $\overline{\varphi}$ in diagram (2.6) for the J-restriction (2.23). Consider the map ψ defined by the commutative diagram

$$\begin{tikzcd} W \arrow[r, "\psi"] \arrow[dr, "\overline{\varphi}"] & T \otimes_R \omega_R \\ W_T \arrow[u, hook] \arrow[r, "\overline{\varphi}_T"] & S \otimes_R \omega_R. \arrow[u, two heads] \end{tikzcd} \tag{2.31}$$

By Lemma 2.23 and since $\omega_R \cong R$ both $\overline{\varphi}_T = 0$ and $\psi = 0$ can be checked at $\operatorname{Ass}_S(T)$. There the vertical maps in diagram (2.31) induce the identity by condition (2.29) and Remark 2.25. With diagram (2.26), Lemma 2.9 applied to (2.23) and diagram (2.6) it follows that

$$\alpha(\varphi) \in U \iff \alpha^T(\varphi) = 0 \iff \overline{\varphi}_T = 0 \iff \psi = 0 \iff \varphi \circ \kappa(M) \subseteq J\omega_R.$$

This proves the equality and the inclusion follows with $JV = J\alpha(F^\vee) = \alpha(JF^\vee)$. $\square$

Proposition 2.28. *Assume that R is Gorenstein and consider the J-restriction (2.23) of an I-free approximation. If T satisfies condition (2.29), then with M also M_J is I-reflexive.*

Proof. By Lemma 2.27 there is a short exact sequence

$$0 \to U/JV \to V/JV \to V^T \to 0. \tag{2.32}$$

By condition (2.29) and Remark 2.25

$$JS_{\mathfrak{p}} = \begin{cases} 0 & \text{if } \mathfrak{p} \in \operatorname{Ass}_S(T), \\ S_{\mathfrak{p}} & \text{if } \mathfrak{p} \in \operatorname{Ass}(S) \setminus \operatorname{Ass}_S(T), \end{cases}$$

$$(V \twoheadrightarrow V^T)_{\mathfrak{p}} = \begin{cases} \mathrm{id}_{V_{\mathfrak{p}}} & \text{if } \mathfrak{p} \in \operatorname{Ass}_S(T), \\ 0 & \text{if } \mathfrak{p} \in \operatorname{Ass}(S) \setminus \operatorname{Ass}_S(T), \end{cases}$$

and hence

$$\begin{aligned} \forall \mathfrak{p} \in \operatorname{Ass}(S)\colon (JV)_{\mathfrak{p}} = JS_{\mathfrak{p}}V_{\mathfrak{p}} = U_{\mathfrak{p}} &\implies (U/JV)_{\mathfrak{p}} = 0 \\ &\implies \dim(U/JV) < \dim(S) = \operatorname{depth}(\overline{\omega}_R). \end{aligned}$$

Then $(U/JV)^{\triangledown} = 0$ by Ischebeck's Lemma (see [12, Satz 1.9]). Using sequence (2.32) and Hom-tensor adjunction it follows that

$$(V^T)^{\triangledown} = (V/JV)^{\triangledown} = (T \otimes_S V)^{\triangledown} = (V^{\triangledown})_T.$$

Denote by ν_T the map ν from Lemma 2.9 applied to the J-restriction (2.23). We obtain a diagram

$$\begin{tikzcd} W_T \arrow[rr, "(\nu^{\triangledown} \circ \delta_{W,\overline{\omega}_R})_T"] \arrow[d, equal] & & (V^{\triangledown})_T \arrow[d, equal] \\ W_T \arrow[r, "\delta_{W_T,\overline{\omega}_R}"] & (W_T)^{\triangledown\triangledown} \arrow[r, "(\nu_T)^{\triangledown}"] & (V^T)^{\triangledown}. \end{tikzcd} \tag{2.33}$$

By Lemma 2.23 and since $\overline{\omega}_R \cong S$, its commutativity can be checked at $\operatorname{Ass}_S(T)$. By condition (2.29) and Remark 2.25 top and bottom horizontal maps in diagram (2.33) identify at $\operatorname{Ass}_S(T)$. Diagram (2.33) thus commutes and Proposition 2.13 yields the claim. □

The Cohen–Macaulay property is invariant under restriction of scalars $S \to T$ and by Hom-tensor adjunction $\operatorname{Hom}_S(-, \omega_S) = \operatorname{Hom}_T(-, \omega_T)$ on T-modules where (see [7, Thm. 3.3.7.(b)])

$$\omega_T = \operatorname{Hom}_S(T, \omega_S). \tag{2.34}$$

Combining Theorem 1.2 and Proposition 2.28 yields (see diagram (2.26))

Corollary 2.29. *In addition to the hypotheses of Theorem 1.2, let $J \trianglelefteq R$ with $J \subseteq I$ be such that $T = R/J$ satisfies condition (2.29) and $W_T \neq 0$. Consider the J-restriction (2.23) with I-dual (2.27). Then $W_T = \operatorname{Hom}_T(V^T, \omega_T)$ and V^T is a maximal Cohen–Macaulay T-module if and only if* $\operatorname{G-dim}(M_J) \leq k-1$. *In this latter case $V^T = \operatorname{Hom}_T(W_T, \omega_T)$ is ω_T-reflexive. Unless $T \otimes \alpha^T$ (and hence $\overline{\alpha}$) is injective* $\operatorname{G-dim}(M_J) \geq k-1$. □

Finally we mention a construction analogous to Definition 2.24 not used in the sequel.

Remark 2.30. Assume that J satisfies the hypotheses on I and consider an I-free approximation (2.4) where W is already a T-module. Then $W_T = W$ and $M_J = M$ and the image of (2.4) under the map $\operatorname{Ext}^1_R(W, IF) \to \operatorname{Ext}^1_R(W, JF)$ is a J-free approximation that fits into a commutative diagram with cartesian left square

$$\begin{tikzcd}
0 \arrow[r] & JF \arrow[r] & M^J \arrow[r] & W \arrow[r] & 0 \\
0 \arrow[r] & IF \arrow[r] \arrow[u, hook] & M \arrow[r] \arrow[u, hook] & W \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

where $M^J/M_J \cong JF/IF$. In particular, $M^J = M_J$ if and only if $I = J$.

3. Application to logarithmic forms

In this section results from §2 are used to give a more conceptual approach to and to generalize a duality of multi-logarithmic forms found by Pol [21] as a generalization of result by Granger and the first author [11].

Let Y be a germ of a smooth complex analytic space of dimension n. Then $Y \cong (\mathbb{C}^n, 0)$ and $\mathscr{O}_Y \cong \mathbb{C}\{x_1, \dots, x_n\}$ by a choice of coordinates $x_1, \dots, x_n$ on Y. We denote by

$$\mathscr{Q}_- := Q(\mathscr{O}_-)$$

the total ring of fractions of $\mathscr{O}_-$. In this section we set $-^* := \operatorname{Hom}_{\mathscr{O}_Y}(-, \mathscr{O}_Y)$.

Let $\Omega^\bullet_Y$ denote the *De Rham algebra on* Y, that is,

$$\mathscr{O}_Y \to \Omega^1_Y, \quad f \mapsto df,$$

is the universally finite $\mathbb{C}$-linear derivation of $\mathscr{O}_Y$ (see [25, §2] and [15, §11]) and $\Omega^q_Y = \bigwedge^q_{\mathscr{O}_Y} \Omega^1_Y$ for all $q \geq 0$. In terms of coordinates $\Omega^1_Y \cong \bigoplus_{i=1}^n \mathscr{O}_Y dx_i$ and hence

$$\Omega^q_Y = \bigwedge^q_{\mathscr{O}_Y} \Omega^1_Y \cong \bigoplus_{i_1 < \cdots < i_q} \mathscr{O}_Y dx_{i_1} \wedge \cdots \wedge dx_{i_q}$$

is a free $\mathscr{O}_Y$-module. By definition the dual

$$(\Omega^1_Y)^* = \operatorname{Der}_{\mathbb{C}}(\mathscr{O}_Y) =: \Theta_Y \cong \bigoplus_{i=1}^n \mathscr{O}_Y \frac{\partial}{\partial x_i}$$

is the module of $\mathbb{C}$-*linear derivations on* $\mathscr{O}_Y$, or of vector fields on Y. The module of q-*vector fields on* Y is then the free $\mathscr{O}_Y$-module

$$(\Omega^q_Y)^* = \bigwedge^q_{\mathscr{O}_Y} \Theta_Y =: \Theta^q_Y \cong \bigoplus_{i_1 < \cdots < i_q} \mathscr{O}_Y \frac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_q}}.$$

Notation 3.1. We set $N := \{1, \dots, n\}$ and $N^q_< := \left\{\underline{j} \in N^q \,\middle|\, j_1 < \cdots < j_q\right\}$. For $\underline{j} \in N^q$ and $\underline{f} = (f_1, \dots, f_\ell) \in \mathscr{O}_Y^\ell$ we abbreviate

$$dx_{\underline{j}} := dx_{j_1} \wedge \cdots \wedge dx_{j_q}, \quad \frac{\partial}{\partial x_{\underline{j}}} := \frac{\partial}{\partial x_{j_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{j_q}},$$
$$\underline{j}_{\hat{i}} := (j_1, \dots, \widehat{j_i}, \dots, j_q), \quad d\underline{f} = df_1 \wedge \cdots \wedge df_\ell.$$

The perfect pairing

$$\Theta^q_Y \times \Omega^q_Y \to \mathscr{O}_Y, \quad (\delta, \omega) \mapsto \langle \delta, \omega \rangle, \tag{3.1}$$

then satisfies

$$\left\langle \frac{\partial}{\partial x_{\underline{j}}}, dx_{\underline{k}} \right\rangle = \delta_{\underline{j},\underline{k}} := \delta_{j_1,k_1} \cdots \delta_{j_q,k_q}. \tag{3.2}$$

3.1. **Log forms along complete intersections.** Let $C \subseteq Y$ be a reduced complete intersection of codimension $k \geq 1$. Then $\mathscr{O}_C = \mathscr{O}_Y/\mathcal{I}_C$ where $\mathcal{I}_C = \mathcal{I}_{C/Y}$ is the ideal of $C \subseteq Y$. Let $\underline{h} = (h_1, \dots, h_k) \in \mathscr{O}_Y^k$ be any regular sequence such that $\mathcal{I}_C = \langle h_1, \dots, h_k \rangle$. Geometrically $C = D_1 \cap \cdots \cap D_k$ where $D_i := \{h_i = 0\}$ for $i = 1, \dots, k$.

Notation 3.2. We denote $D := D_1 \cup \cdots \cup D_k = \{h = 0\}$ where $h := h_1 \cdots h_k$,

$$-(D) := - \otimes_{\mathscr{O}_Y} \mathscr{O}_Y \frac{1}{h}, \qquad\qquad -(-D) := - \otimes_{\mathscr{O}_Y} \mathscr{O}_Y h,$$

$$\Sigma = \Sigma_{C/D/Y} := \mathcal{I}_C(D) = \sum_{i=1}^k \frac{h_i}{h} \mathscr{O}_Y \subseteq \mathscr{Q}_Y, \qquad -^\Sigma := \operatorname{Hom}_{\mathscr{O}_Y}(-, \Sigma).$$

Note that $\Sigma = \mathscr{O}_Y$ in case $k = 1$.

The following definition due to Aleksandrov (see [3, §3] and [21, Def. 3.1.4]) generalizes Saito's logarithmic differential forms (see [22]) from the hypersurface to the complete intersection case.

Definition 3.3. The module of *multi-logarithmic differential q-forms on Y along C* is defined by

$$\begin{aligned}\Omega^q(\log C) = \Omega_Y^q(\log C) &:= \left\{\omega \in \Omega_Y^q \;\middle|\; d\mathcal{I}_C \wedge \omega \subseteq \mathcal{I}_C \Omega_Y^{q+1}\right\}(D) \\ &= \left\{\omega \in \Omega_Y^q(D) \;\middle|\; \forall i = 1, \dots, k \colon dh_i \wedge \omega \in \Sigma\Omega_Y^{q+1}\right\}\end{aligned}$$

where the equality is due to the Leibniz rule. Observe that

$$\Sigma\Omega_Y^q \subseteq \Omega^q(\log C) \subseteq \mathscr{Q}_Y \otimes_{\mathscr{O}_Y} \Omega_Y^q$$

with $\Omega^q(\log C)(-D) \subseteq \mathscr{Q}_Y \otimes_{\mathscr{O}_Y} \Omega_Y^q$ independent of D (see [21, Prop. 3.1.10]).

Extending Saito's theory (see [22, §1-2]) Aleksandrov (see [3, §3-4,6]) gives an explicit description of multi-logarithmic differential forms and defines a multi-logarithmic residue map. We summarize his results.

Proposition 3.4. *An element $\omega \in \Omega_Y^q(D)$ lies in $\Omega^q(\log C)$ if and only if there exist $g \in \mathscr{O}_Y$ inducing a non zero-divisor in $\mathscr{O}_C$, $\xi \in \Omega_Y^{q-k}$ and $\eta \in \Sigma\Omega_Y^q$ such that*

$$g\omega = \frac{d\underline{h}}{h} \wedge \xi + \eta.$$

This representation defines a multi-logarithmic residue map

$$\operatorname{res}_C^q \colon \Omega^q(\log C) \to \mathscr{Q}_C \otimes_{\mathscr{O}_C} \Omega_C^{q-k}, \quad \omega \mapsto \frac{\xi}{g},$$

that fits into a short exact multi-logarithmic residue sequence

$$0 \longrightarrow \Sigma\Omega_Y^q \longrightarrow \Omega^q(\log C) \xrightarrow{\operatorname{res}_C^q} \omega_C^{q-k} \longrightarrow 0 \tag{3.3}$$

where ω_C^p is the module of regular meromorphic p-forms on C. □

Corollary 3.5. *For $q < k$, $\Omega^q(\log C) = \Sigma\Omega_Y^q$ and $\Omega^n(\log C) = \Omega_Y^n(D)$.* □

Remark 3.6. The multi-logarithmic residue map can be written in terms of residue symbols as $\operatorname{res}_C^q(\omega) = \begin{bmatrix} h\omega \\ \underline{h} \end{bmatrix}$ (see [27, §1.2][1]). In particular $\operatorname{res}_C^k(\frac{d\underline{h}}{h}) = \begin{bmatrix} d\underline{h} \\ \underline{h} \end{bmatrix} \in \omega_C^k$ is the fundamental form of C (see [13, §5]). □

[1]This remark was made in the first author's talk "Normal crossings in codimension one" at the 2012 Oberwolfach conference "Singularities" (see [26]).

Higher logarithmic derivation modules play a prominent role in arrangement theory (see for instance [1]). Here we extend the definitions of Granger and the first author (see [9, §5]) and by Pol (see [21, Def. 3.2.1]) as follows.

Definition 3.7. We define the module of *multi-logarithmic q-vector fields on Y along C* by

$$\begin{aligned}\mathrm{Der}^q(-\log C) = \mathrm{Der}^q_Y(-\log C) &:= \left\{\delta \in \Theta^q_Y \;\middle|\; \left\langle \delta, \wedge^k d\mathcal{I}_C \wedge \Omega^{q-k}_Y\right\rangle \subseteq \mathcal{I}_C\right\}\\ &= \left\{\delta \in \Theta^q_Y \;\middle|\; \left\langle \delta, d\underline{h} \wedge \Omega^{q-k}_Y\right\rangle \subseteq \mathcal{I}_C\right\}\end{aligned}$$

where the equality is due to the Leibniz rule. Observe that

$$\mathcal{I}_C\Theta^q_Y \subseteq \mathrm{Der}^q(-\log C).$$

Lemma 3.8. *We can identify the functors on $\mathscr{O}_Y$-modules (see Notation 2.1)*

$$-^{\Sigma} = -(-D)^{\mathcal{I}_C}, \quad (\Sigma \otimes_{\mathscr{O}_Y} -)^{\Sigma} = -^{*},$$

and hence $-^{\Sigma\Sigma} = -^{\mathcal{I}_C\mathcal{I}_C}$.

Proof. Since $\mathscr{O}_Y(D)$ is invertible and by Hom-tensor adjunction

$$-^{\Sigma} = \mathrm{Hom}_{\mathscr{O}_Y}(-, \mathcal{I}_C(D)) = \mathrm{Hom}_{\mathscr{O}_Y}(-, \mathrm{Hom}_{\mathscr{O}_Y}(\mathscr{O}_Y(-D), \mathcal{I}_C)) = -(-D)^{\mathcal{I}_C}$$

By Lemma 2.3 in case $k \geq 2$, $\mathscr{O}_Y = \mathcal{I}_C^{\mathcal{I}_C} = \Sigma^{\Sigma}$ and again by Hom-tensor adjunction

$$(\Sigma \otimes_{\mathscr{O}_Y} -)^{\Sigma} = \mathrm{Hom}_{\mathscr{O}_Y}(\Sigma \otimes_{\mathscr{O}_Y} -, \Sigma) = \mathrm{Hom}_{\mathscr{O}_Y}(-, \Sigma^{\Sigma}) = -^{*}. \qquad \square$$

Lemma 3.9. *Any elements $\delta \in \mathrm{Der}^q(-\log C)$ and $\omega \in \Omega^q(\log C)$ pair to $\langle \delta, \omega\rangle \in \Sigma$.*

Proof. Let g, ξ and η be as in Proposition 3.4. Then by definition

$$g\langle \delta, h\omega\rangle = \langle \delta, hg\omega\rangle = \langle \delta, d\underline{h} \wedge \xi + h\eta\rangle = \langle \delta, d\underline{h} \wedge \xi\rangle + h\langle \delta, \eta\rangle \in \mathcal{I}_C.$$

Since g induces a non zero-divisor in $\mathscr{O}_C = \mathscr{O}_Y/\mathcal{I}_C$ this implies that $\langle \delta, h\omega\rangle \in \mathcal{I}_C$ and hence $\langle \delta, \omega\rangle \in \frac{1}{h}\mathcal{I}_C = \Sigma$. $\square$

The following proofs for $q \geq k \geq 1$ proceed along the lines of Saito's base case $q = k = 1$ (see [22, (1.6)]) or Pol's generalization to $q = k \geq 1$ (see [21, Prop. 3.2.13]).

Lemma 3.10. *If $\omega \in \Omega^q_Y(D)$ with $\langle \mathrm{Der}^q(-\log C), \omega\rangle \subseteq \Sigma$, then $\omega \in \Omega^q(\log C)$.*

Proof. For every $\ell \in \{1, \ldots, k\}$ and $\underline{j} \in N^{q+1}_{<}$ consider

$$\delta^{\ell}_{\underline{j}} := \sum_{i=1}^{q+1} (-1)^{i+1} \frac{\partial h_\ell}{\partial x_{j_i}} \frac{\partial}{\partial x_{\underline{j}_{\hat{i}}}} \in \Theta^q_Y.$$

For every $\underline{i} \in N^{q-k}$

$$d\underline{h} \wedge dx_{\underline{i}} = \sum_{\underline{k} \in N^q_<} \frac{\partial(\underline{h}, x_{\underline{i}})}{\partial x_{\underline{k}}} dx_{\underline{k}},$$

where $\frac{\partial(\underline{h}, x_{\underline{i}})}{\partial x_{\underline{k}}}$ is the $q \times q$-minor of the Jacobian matrix of $(\underline{h}, x_{\underline{i}})$ with column indices $\underline{k}$, and hence using (3.2)

$$\begin{aligned}\left\langle \delta^{\ell}_{\underline{j}}, d\underline{h} \wedge dx_{\underline{i}}\right\rangle &= \sum_{i=1}^{q+1} (-1)^{i+1} \frac{\partial h_\ell}{\partial x_{j_i}} \sum_{\underline{k} \in N^q_<} \frac{\partial(\underline{h}, x_{\underline{i}})}{\partial x_{\underline{k}}} \left\langle \frac{\partial}{\partial x_{\underline{j}_{\hat{i}}}}, dx_{\underline{k}}\right\rangle\\ &= \sum_{i=1}^{q+1} (-1)^{i+1} \frac{\partial h_\ell}{\partial x_{j_i}} \frac{\partial(\underline{h}, x_{\underline{i}})}{\partial x_{\underline{j}_{\hat{i}}}} = \frac{\partial(h_\ell, \underline{h}, x_{\underline{i}})}{\partial x_{\underline{j}}} = 0.\end{aligned}$$

It follows that $\delta^\ell_{\underline{j}} \in \mathrm{Der}^q(-\log C)$ for all $\ell = 1, \ldots, k$ and $\underline{j} \in N^{q+1}_<$.

Now let $\omega = \sum_{\underline{k}\in N^q_<} \frac{a_{\underline{k}}}{h} dx_{\underline{k}} \in \Omega^q_Y(D)$ where $a_{\underline{k}} \in \mathscr{O}_Y$. For all $\ell = 1, \ldots, k$ and $\underline{j} \in N^{q+1}_<$

$$\left\langle \delta^\ell_{\underline{j}}, \omega \right\rangle = \sum_{i=1}^{q+1} (-1)^{i+1} \frac{\partial h_\ell}{\partial x_{j_i}} \sum_{\underline{k}\in N^q_<} \frac{a_{\underline{k}}}{h} \left\langle \frac{\partial}{\partial x_{\underline{j}_{\hat{i}}}}, dx_{\underline{k}} \right\rangle = \sum_{i=1}^{q+1} (-1)^{i+1} \frac{\partial h_\ell}{\partial x_{j_i}} \frac{a_{\underline{j}_{\hat{i}}}}{h}$$

by (3.2) and hence

$$\begin{aligned} dh_\ell \wedge \omega &= \sum_{j=1}^n \frac{\partial h_\ell}{\partial x_j} dx_j \wedge \sum_{\underline{k}\in N^q_<} \frac{a_{\underline{k}}}{h} dx_{\underline{k}} = \sum_{\underline{j}\in N^{q+1}_<} \sum_{i=1}^{q+1} \frac{\partial h_\ell}{\partial x_{j_i}} \frac{a_{\underline{j}_{\hat{i}}}}{h} dx_{j_i} \wedge dx_{\underline{j}_{\hat{i}}} \\ &= \sum_{\underline{j}\in N^{q+1}_<} \sum_{i=1}^{q+1} (-1)^{i+1} \frac{\partial h_\ell}{\partial x_{j_i}} \frac{a_{\underline{j}_{\hat{i}}}}{h} dx_{\underline{j}} = \sum_{\underline{j}\in N^{q+1}_<} \left\langle \delta^\ell_{\underline{j}}, \omega \right\rangle dx_{\underline{j}}. \end{aligned}$$

If $\langle \mathrm{Der}^q(-\log C), \omega\rangle \subseteq \Sigma$, then $dh_\ell \wedge \omega \in \Sigma\Omega^q_Y$ for all $\ell = 1, \ldots, k$ and hence $\omega \in \Omega^q(\log C)$. $\square$

Proposition 3.11. *There are chains of $\mathscr{O}_Y$-submodules of $\mathscr{Q}_Y \otimes_{\mathscr{O}_Y} \Omega^q_Y$ and $\mathscr{Q}_Y \otimes_{\mathscr{O}_Y} \Theta^q_Y$*

$$\Omega^q_Y \subseteq \Sigma\Omega^q_Y \subseteq \Omega^q(\log C) \subseteq \Omega^q_Y(D) \subseteq \Sigma\Omega^q_Y(D), \tag{3.4}$$

$$\Sigma\Theta^q_Y \supseteq \Theta^q_Y \supseteq \mathrm{Der}^q(-\log C) \supseteq \mathcal{I}_C\Theta^q_Y \supseteq \Theta^q_Y(-D) \tag{3.5}$$

that are Σ-duals of each other.

Proof. Tensoring with $\mathscr{Q}_Y$ makes both chains collapse. The cokernels of all inclusions are therefore torsion whereas Σ is torsion free. Applying $-^\Sigma$ thus results in a chain of $\mathscr{O}_Y$-modules again. In case of (3.4) this yields

$$(\Omega^q_Y)^\Sigma \supseteq (\Sigma\Omega^q_Y)^\Sigma \supseteq \Omega^q_Y(\log C)^\Sigma \supseteq \Omega^q_Y(D)^\Sigma \supseteq (\Sigma\Omega^q_Y(D))^\Sigma$$

and, with Lemma 3.8 and freeness of Ω^q_Y and Θ^q_Y, the chain of $\mathscr{O}_Y$-submodules of $\mathscr{Q}_Y \otimes_{\mathscr{O}_Y} \Theta^q_Y$

$$\Sigma\Theta^q_Y \supseteq \Theta^q_Y \supseteq \Omega^q_Y(\log C)^\Sigma \supseteq \mathcal{I}_C\Theta^q_Y \supseteq \Theta^q_Y(-D).$$

For every $\delta \in \Omega^q(\log C)^\Sigma$ and $\xi \in \Omega^{q-k}$, $\frac{d\underline{h}}{h} \wedge \xi \in \Omega^q(\log C)$ by Proposition 3.4, hence

$$\langle \delta, d\underline{h} \wedge \xi\rangle = h\left\langle \delta, \frac{d\underline{h}}{h} \wedge \xi \right\rangle \in h\Sigma = \mathcal{I}_C$$

and $\delta \in \mathrm{Der}^q(-\log C)$. With Lemma 3.9, it follows that $\Omega^q_Y(\log C)^\Sigma = \mathrm{Der}^q(-\log C)$.

By the same reasoning $-^\Sigma$ applied to (3.5) yields a chain of $\mathscr{O}_Y$-modules

$$(\Sigma\Theta^q_Y)^\Sigma \subseteq (\Theta^q_Y)^\Sigma \subseteq \mathrm{Der}^q(-\log C)^\Sigma \subseteq (\Sigma\Theta^q_Y)(-D)^\Sigma \subseteq \Theta^q_Y(-D)^\Sigma$$

that can be rewritten as the chain of $\mathscr{O}_Y$-submodules of $\mathscr{Q}_Y \otimes_{\mathscr{O}_Y} \Omega^q_Y$

$$\Omega^q_Y \subseteq \Sigma\Omega^q_Y \subseteq \mathrm{Der}^q(-\log C)^\Sigma \subseteq \Omega^q_Y(D) \subseteq \Sigma\Omega^q_Y(D).$$

The missing equality $\mathrm{Der}^q(-\log C)^\Sigma = \Omega^q(\log C)$ follows from Lemmas 3.9 and 3.10. $\square$

3.2. **Log forms along Cohen–Macaulay spaces.** Let $X \subseteq Y$ be a reduced Cohen-Macaulay germ of codimension $k \geq 2$. Then $\mathscr{O}_X = \mathscr{O}_Y/\mathcal{I}_X$ where $\mathcal{I}_X := \mathcal{I}_{X/Y}$ denotes the ideal $X \subseteq Y$. There is a reduced complete intersection $C \subseteq Y$ of codimension k such that $X \subseteq C$ and hence $\mathcal{I}_X \supseteq \mathcal{I}_C$ (see [21, Prop. 4.2.1]). Set $X' := \overline{C \setminus X}$ such that $C = X \cup X'$. The link with §2.5 is made by setting

$$S := \mathscr{O}_C, \quad T := \mathscr{O}_X.$$

By Lemma 2.26 condition (2.29) holds and

$$\mathscr{Q}_C = \prod_{\mathfrak{p} \in \operatorname{Ass}_{\mathscr{O}_C}(\mathscr{O}_X)} \mathscr{O}_{X,\mathfrak{p}} \times \prod_{\mathfrak{p} \in \operatorname{Ass}_{\mathscr{O}_C}(\mathscr{O}_{X'})} \mathscr{O}_{X',\mathfrak{p}} = \mathscr{Q}_X \times \mathscr{Q}_{X'}. \tag{3.6}$$

This decomposition extends to differential forms as follows.

Lemma 3.12. *We have* $\mathscr{Q}_X d\mathcal{I}_C = \mathscr{Q}_X d\mathcal{I}_X \subseteq \mathscr{Q}_X \otimes_{\mathscr{O}_Y} \Omega^1_Y$ *and hence*

$$\mathscr{Q}_C \otimes_{\mathscr{O}_C} \Omega^p_C = \mathscr{Q}_X \otimes_{\mathscr{O}_X} \Omega^p_X \oplus \mathscr{Q}_{X'} \otimes_{\mathscr{O}_{X'}} \Omega^p_{X'}.$$

Proof. By (3.6) we may localize at $\mathfrak{p} \in \operatorname{Ass}_{\mathscr{O}_C}(\mathscr{O}_X)$. We may further assume $p = 1$ since exterior product commutes with extension of scalars. Let $\mathfrak{p} \mapsto \mathfrak{q}$ under $\operatorname{Spec}(\mathscr{O}_C) \to \operatorname{Spec}(\mathscr{O}_Y)$. Then $\mathcal{I}_{C,\mathfrak{q}} = \mathcal{I}_{X,\mathfrak{q}}$ by (3.6) and hence $u\mathcal{I}_X \subseteq \mathcal{I}_C$ for some $u \in \mathscr{O}_Y \setminus \mathfrak{q}$. By the Leibniz rule $ud\mathcal{I}_X \subseteq d\mathcal{I}_C + \mathcal{I}_X du$ and hence the first claim. Since $\Omega^1_C = \Omega^1_Y/(\mathscr{O}_Y d\mathcal{I}_C + \mathcal{I}_C \Omega^1_Y)$ this yields $\Omega^1_{C,\mathfrak{p}} = \Omega^1_{X,\mathfrak{p}}$ and the second claim follows. $\square$

The following fact is well-known (see [27, (2.14)]); we only sketch a proof.

Lemma 3.13. *The modules of regular differential p-forms on X and C are related by*

$$\omega^p_X = \operatorname{Hom}_{\mathscr{O}_C}(\mathscr{O}_X, \omega^p_C) \subseteq \omega^p_C.$$

Proof. Kersken explicitly describes (see [14, (1.2)])

$$\omega^p_X = \left\{ \begin{bmatrix} \xi \\ \underline{h} \end{bmatrix} \;\middle|\; \xi \in \Omega^{p+k}_Y,\ \mathcal{I}_X \xi \subseteq \mathcal{I}_C \Omega^{p+k}_Y,\ d\mathcal{I}_X \wedge \xi \subseteq \mathcal{I}_C \Omega^{p+k+1}_Y \right\} \tag{3.7}$$

where $\begin{bmatrix} \xi \\ \underline{h} \end{bmatrix} = 0$ if and only if $\xi \in \mathcal{I}_C \Omega^{p+k}_Y$. In particular, $\omega^p_X \subseteq \operatorname{Hom}_{\mathscr{O}_C}(\mathscr{O}_X, \omega^p_C) \subseteq \omega^p_C$ and equality in ω^p_C can be checked at $\operatorname{Ass}(\mathscr{O}_C)$. Lemma 3.12 yields the claim. $\square$

The following modules of differential forms on Y due to Aleksandrov (see [4, Def. 10.1] and [21, Def. 4.1.3]) are defined by the relations in (3.7).

Definition 3.14. The module of *multi-logarithmic differential q-forms on Y along X relative to C* is defined by

$$\begin{aligned} \Omega^q(\log X/C) = \Omega^q_Y(\log X/C) &:= \left\{ \omega \in \Omega^q_Y \;\middle|\; \mathcal{I}_X \omega \subseteq \mathcal{I}_C \Omega^q_Y,\ d\mathcal{I}_X \wedge \omega \subseteq \mathcal{I}_C \Omega^{q+1}_Y \right\}(D) \\ &= \left\{ \omega \in \Omega^q_Y(D) \;\middle|\; \mathcal{I}_X \omega \subseteq \Sigma \Omega^q_Y,\ d\mathcal{I}_X \wedge \omega \subseteq \Sigma \Omega^{q+1}_Y \right\}. \end{aligned}$$

Observe that

$$\Sigma \Omega^q_Y \subseteq \Omega^q(\log X/C) \subseteq \Omega^q(\log C)$$

with $\Omega^q(\log X/C)(-D) \subseteq \mathscr{Q}_Y \otimes_{\mathscr{O}_Y} \Omega^q_Y$ independent of D (see [21, Prop. 4.1.5]).

Lemma 3.15. *There is an equality* $\Omega^q(\log X/C) = \Sigma \Omega^q_Y :_{\Omega^q(\log C)} \mathcal{I}_X$. *In other words,*

$$\Omega^q(\log X/C)(-D) = \mathcal{I}_X \Omega^q_Y :_{\Omega^q(\log C)} \mathcal{I}_X.$$

Proof. There are obvious inclusions

$$\Sigma\Omega^q_Y \subseteq \Omega^q(\log X/C) \subseteq \Sigma\Omega^q_Y :_{\Omega^q(\log C)} \mathcal{I}_X \subseteq \Omega^q(\log C).$$

By Proposition 3.4 and Lemma 3.12

$$\begin{aligned}\omega \in \Sigma\Omega^q_Y :_{\Omega^q(\log C)} \mathcal{I}_X &\implies \mathcal{I}_X \operatorname{res}^q_C(\omega) \subseteq \operatorname{res}^q_C(\Sigma\Omega^q_Y) = 0\\ &\implies \operatorname{res}^q_C(\omega) \in \mathscr{Q}_X \otimes_{\mathscr{O}_X} \Omega^{q-k}_X\\ &\implies 0 = d\mathcal{I}_X \wedge \operatorname{res}^q_C(\omega) = \operatorname{res}^{q+1}_C(d\mathcal{I}_X \wedge \omega)\\ &\implies d\mathcal{I}_X \wedge \omega \subseteq \Sigma\Omega^{q+1}_Y\\ &\implies \omega \in \Omega^q(\log X/C).\end{aligned}$$

□

The idea of Remark 3.6 is used by Aleksandrov (see [4, §10]) to define multi-logarithmic residues along X as the restriction of those along C. The bottom sequence of the diagram in the following Proposition 3.16 appears in his work (see [4, Thm. 10.2]); Pol proved exactness on the right (see [21, Prop. 4.1.21]). An alternative argument is suggested by §2.5. The following data

$$R := \mathscr{O}_Y, \quad I := \mathcal{I}_C, \quad J := \mathcal{I}_X, \quad F := \Omega^q_Y, \quad M := \Omega^q(\log C)(-D), \quad \rho := \frac{1}{h}\operatorname{res}^q_C \tag{3.8}$$

give rise to an I-free approximation (2.4) with J-restriction (2.23). By Corollary 3.5 $W = 0$ if $q < k$ and (2.4) is trivial for $q = n$. We are therefore concerned with the case $k \le q < n$. By Lemmas 3.13 and 3.15 (see Definition 2.24 and (2.25))

$$W_T = \omega^{q-k}_X, \quad M_J = \Omega^q(\log X/C)(-D). \tag{3.9}$$

Now twisting diagram (2.24) by D yields the following result.

Proposition 3.16. *Applying $\operatorname{Ext}^1_{\mathscr{O}_Y}(\omega^{q-k}_X \hookrightarrow \omega^{q-k}_C, \Sigma\Omega^q_Y)$ to the multi-logarithmic residue sequence (3.3) yields a commutative diagram with exact rows and cartesian right square*

$$\begin{tikzcd} 0 \arrow[r] & \Sigma\Omega^q_Y \arrow[r] & \Omega^q(\log C) \arrow[r, "\operatorname{res}^q_C"] & \omega^{q-k}_C \arrow[r] & 0 \\ 0 \arrow[r] & \Sigma\Omega^q_Y \arrow[r] \arrow[u, equal] & \Omega^q(\log X/C) \arrow[r, "\operatorname{res}^q_{X/C}"] \arrow[u, hook] & \omega^{q-k}_X \arrow[r] \arrow[u, hook] & 0 \end{tikzcd} \tag{3.10}$$

where ω^p_X is the module of regular meromorphic p-forms on X. □

3.3. **Higher log vector fields and Jacobian modules.** Pol gives a description of $\operatorname{res}^q_{X/C}$ preserving the analogy with the definition of res^q_C in Proposition 3.4 (see [21, §4.2.1]). As suggested by Remark 3.6 the role of $\frac{dh}{h} \in \Omega^k(\log C)$ is played by a preimage $\frac{\alpha_X}{h} \in \Omega^k(\log X/C)$ of the fundamental form $\begin{bmatrix}\alpha_X\\ \underline{h}\end{bmatrix} \in \omega^0_X$ of X (see [13, §5]).

Definition 3.17. Let $\mathbf{1}_X := (1, 0) \in \mathscr{Q}_X \times \mathscr{Q}_{X'} = \mathscr{Q}_C$ (see Lemma 3.12). A *fundamental form of X in Y* is an $\alpha_X = \alpha_{X/C/Y} \in \Omega^k_Y$ such that $\overline{\alpha_X} = \overline{\mathbf{1}_X d\underline{h}} \in \mathscr{Q}_C \otimes_{\mathscr{O}_Y} \Omega^k_Y$.

Such a fundamental form exists and the explicit description of multi-logarithmic differential forms in Proposition 3.4 generalizes verbatim (see [21, Prop. 4.2.6]).

Proposition 3.18. *An element $\omega \in \Omega^q_Y(D)$ lies in $\Omega^q(\log X/C)$ if and only if there exist $g \in \mathscr{O}_Y$ inducing a non zero-divisor in $\mathscr{O}_C$, $\xi \in \Omega^{q-k}_Y$ and $\eta \in \Sigma\Omega^q_Y$ such that*

$$g\omega = \frac{\alpha_X}{h} \wedge \xi + \eta$$

and the map $\mathrm{res}^q_{X/C}$ *in* (3.10) *is defined by* $\mathrm{res}^q_{X/C}(\omega) = \frac{\xi}{g}$. □

In the same spirit we extend Definition 3.7. We start with the first option as definition.

Definition 3.19. We define the module of *multi-logarithmic q-vector fields on Y along X* by

$$\mathrm{Der}^q(-\log X) = \mathrm{Der}^q_Y(-\log X) := \left\{\delta \in \Theta^q_Y \;\middle|\; \left\langle \delta, \wedge^k d\mathcal{I}_X \wedge \Omega^{q-k}_Y \right\rangle \subseteq \mathcal{I}_X \right\}.$$

The following result completes the analogy with Definition 3.7. In particular $\mathrm{Der}^k(-\log X)$ is Pol's module $\mathrm{Der}^k(-\log X/C)$ (see [21, Def. 4.2.8]) which is thus independent of C.

Lemma 3.20. *We have*

$$\begin{aligned}\mathrm{Der}^q(-\log C) \subseteq \left\{\delta \in \Theta^q_Y \;\middle|\; \left\langle \delta, \alpha_X \wedge \Omega^{q-k}_Y \right\rangle \subseteq \mathcal{I}_X \right\} &= \mathrm{Der}^q(-\log X) \\ = \left\{\delta \in \Theta^q_Y \;\middle|\; \left\langle \delta, \alpha_X \wedge \Omega^{q-k}_Y \right\rangle \subseteq \mathcal{I}_C \right\}&.\end{aligned}$$

Proof. By Definition 3.17 $\overline{\alpha_X} = \overline{\mathbf{1}_X d\underline{h}} = \overline{d\underline{h}} \in \mathscr{Q}_X \otimes_{\mathscr{O}_Y} \Omega^k_Y$. For $\delta \in \Theta^q_Y$ and $\xi \in \Omega^{q-k}_Y$

$$\begin{aligned}\langle \delta, \alpha_X \wedge \xi \rangle \in \mathcal{I}_X \iff 0 &= \overline{\langle \delta, \alpha_X \wedge \xi \rangle} = \left\langle \overline{\delta}, \overline{\alpha_X} \wedge \overline{\xi} \right\rangle \\ &= \left\langle \overline{\delta}, \overline{d\underline{h}} \wedge \overline{\xi} \right\rangle = \overline{\langle \delta, d\underline{h} \wedge \xi \rangle} \in \mathscr{Q}_X\end{aligned}$$

where $\overline{\delta} \in \mathscr{Q}_X \otimes_{\mathscr{O}_Y} \Theta^q_Y$ and $\overline{\xi} \in \mathscr{Q}_X \otimes_{\mathscr{O}_Y} \Omega^{q-k}_Y$. The claimed inclusion follows. Using the Leibniz rule and that $\mathscr{Q}_X d\mathcal{I}_C = \mathscr{Q}_X d\mathcal{I}_X \subseteq \mathscr{Q}_X \otimes_{\mathscr{O}_Y} \Omega^1_Y$ by Lemma 3.12

$$\begin{aligned}0 = \left\langle \overline{\delta}, \overline{d\underline{h}} \wedge \overline{\xi} \right\rangle \in \mathscr{Q}_X \iff 0 &= \left\langle \overline{\delta}, \wedge^k \overline{d\mathcal{I}_C} \wedge \overline{\xi} \right\rangle = \left\langle \overline{\delta}, \wedge^k \overline{d\mathcal{I}_X} \wedge \overline{\xi} \right\rangle \\ &= \overline{\left\langle \delta, \wedge^k d\mathcal{I}_X \wedge \xi \right\rangle} \subseteq \mathscr{Q}_X \iff \left\langle \delta, \wedge^k d\mathcal{I}_X \wedge \xi \right\rangle \subseteq \mathcal{I}_X.\end{aligned}$$

This proves the first equality. With $\mathcal{I}_C = \mathcal{I}_X \cap \mathcal{I}_{X'}$ the second equality follows from $\alpha_X \in \mathcal{I}_{X'}\Omega^k_Y$ (see [21, Prop. 4.2.5]). □

Using Proposition 3.18 and Lemma 3.20 we obtain the following analogue of Lemma 3.9 and of the equality $\mathrm{Der}^q(-\log C) = \Omega^q(\log C)^\Sigma$ from Proposition 3.11.

Lemma 3.21. *For* $\delta \in \mathrm{Der}^q(-\log X)$ *and* $\omega \in \Omega^q(\log X/C)$ *we have* $\langle \delta, \omega \rangle \in \Sigma$. □

Lemma 3.22. *There is an equality* $\mathrm{Der}^q(-\log X) = \Omega^q(\log X/C)^\Sigma$. □

The following proposition extends Proposition 3.11 and includes the counterpart of Lemma 3.10.

Proposition 3.23. *There are chains of* $\mathscr{O}_Y$*-submodules of* $\mathscr{Q}_Y \otimes_{\mathscr{O}_Y} \Omega^q_Y$ *and* $\mathscr{Q}_Y \otimes_{\mathscr{O}_Y} \Theta^q_Y$

$$\begin{gathered}\Omega^q_Y \subseteq \Sigma\Omega^q_Y \subseteq \Omega^q(\log X/C) \subseteq \Omega^q(\log C) \subseteq \Omega^q_Y(D) \subseteq \Sigma\Omega^q_Y(D), \\ \Sigma\Theta^q_Y \supseteq \Theta^q_Y \supseteq \mathrm{Der}^q(-\log X) \supseteq \mathrm{Der}^q(-\log C) \supseteq \mathcal{I}_C\Theta^q_Y \supseteq \Theta^q_Y(-D)\end{gathered}$$

that are Σ*-duals of each other.*

Proof. By Lemma 3.8 and Proposition 3.11 M in (3.8) is I-reflexive. By Proposition 2.28 and (3.9) $\Omega^q(\log X/C)(-D)$ is therefore $\mathcal{I}_C$-reflexive and, again by Lemma 3.8, $\Omega^q(\log X/C)$ Σ-reflexive. The claim follows with Proposition 3.11 and Lemmas 3.20 and 3.22. □

Definition 3.24. Contraction with α_X defines a map

$$\alpha^X \colon \Theta^q_Y \to \mathscr{O}_X \otimes_{\mathscr{O}_Y} \Theta^{q-k}_Y = \mathrm{Hom}_{\mathscr{O}_Y}(\Omega^{q-k}_Y, \mathscr{O}_X), \quad \delta \mapsto (\omega \mapsto \overline{\langle \delta, \alpha_X \wedge \omega \rangle}).$$

Taking $p + q = n$ we define the *pth Jacobian module of X* as the $\mathscr{O}_X$-module

$$\mathcal{J}^p_X := \alpha^X(\Theta^q_Y).$$

The Jacobian module $\mathscr{J}_X^{\dim X}$ agrees with Pol's Jacobian ideal $\mathscr{J}_{X/C}$ (see [21, Not. 4.2.14]) which coincides with the ω-Jacobian ideal if X is Gorenstein (see [21, Prop. 4.2.34]).

Remark 3.25. In explicit terms

$$\alpha^X\colon \Theta_Y^q \to \bigoplus_{\underline{i}\in N_<^{q-k}} \mathscr{O}_X dx_{\underline{i}}, \quad \delta \mapsto \sum_{\underline{i}\in N_<^{q-k}} \langle \delta, \alpha_X \wedge dx_{\underline{i}}\rangle dx_{\underline{i}}.$$

In case $X = C$, $\alpha_C = d\underline{h}$ and

$$\left\langle \delta, d\underline{h} \wedge dx_{\underline{i}} \right\rangle = \sum_{\underline{j}\in N_<^q} \frac{\partial(\underline{h}, x_{\underline{i}})}{\partial x_{\underline{j}}} \left\langle \delta, dx_{\underline{j}} \right\rangle.$$

In particular, $\mathscr{J}_C^{\dim C}$ is the Jacobian ideal of C.

Lemma 3.26. *If $k \le q \le n$, then $\omega_X^{q-k} \neq 0$ and, unless $q = n$, $\mathscr{O}_X \otimes \alpha^X$ is not injective.*

Proof. This can be checked at smooth points of $X = C$ where $\underline{h} = (x_1, \dots, x_k)$ and $\alpha_X = d\underline{h}$. Here $\omega_X^{q-k} = \Omega_X^{q-k} \neq 0$ and $0 \neq \frac{\partial}{\partial x_{\underline{j}}} \in \ker(\mathscr{O}_X \otimes \alpha^X)$ if $\{1, \dots, k\} \not\subseteq \{j_1, \dots, j_q\}$. $\square$

By Lemma 3.20 there is a short exact sequence (see [21, Prop. 4.2.16] for $q = k$)

$$0 \longleftarrow \mathscr{J}_X^{n-q} \xleftarrow{\alpha^X} \Theta_Y^q \longleftarrow \mathrm{Der}_Y^q(-\log X) \longleftarrow 0. \tag{3.11}$$

Lemma 3.27. *There is a pairing*

$$\mathscr{J}_X^{n-q} \otimes \omega_X^{q-k} \to \mathrm{Hom}_{\mathscr{O}_C}(\mathscr{O}_X, \mathscr{O}_C)(D) = \omega_X, \quad \left(\alpha^X(\delta), \mathrm{res}_{X/C}^q(\omega)\right) \mapsto \langle \delta, \omega\rangle.$$

Proof. By Lemma 3.21 the pairing $\Omega_Y^q(D) \times \Theta_Y^q \to \mathscr{O}_Y(D)$ obtained from (3.1) maps both $\Omega_Y^q(\log X/C) \times \mathrm{Der}_Y^q(-\log X)$ and $\Sigma\Omega_Y^q \otimes \Theta_Y^q$ to Σ. Using the bottom row of (3.10) and (3.11) this yields a pairing $\mathscr{J}_X^{n-q} \otimes \omega_X^{q-k} \to \mathscr{O}_Y(D)/\Sigma = \mathscr{O}_C(D) = \omega_C$. Both $\mathscr{J}_X^{n-q}$ and ω_X^{q-k} are supported on X and applying $\mathrm{Hom}_{\mathscr{O}_C}(\mathscr{O}_X, -)$ yields the claim (see (2.34)). $\square$

We can now prove our main application.

Proof of the Theorem 1.3. By Lemmas 3.8 and 3.22 sequence (3.11) in terms of (3.8) is the I-dual J restriction (2.27) twisted by D, that is, $V^T = \mathscr{J}_X^{n-q}$ and $\alpha^T = \alpha^X$ up to a twist by D. With (3.9) and Lemma 3.26 the claim now reduces to Corollary 2.29. The identifications are induced by the pairing in Lemma 3.27. $\square$

Proposition 3.28. *The $\mathscr{O}_X$-modules $\mathscr{J}_X^{n-q}$ depend only on X.*

Proof. We identify $\mathscr{J}_X^{n-q} = \Theta_Y^q / \mathrm{Der}_Y^q(-\log X)$ by the exact sequence (3.11). Any isomorphism $Y' \cong Y$ of minimal embeddings of X induces an isomorphism $\varphi\colon \mathscr{O}_Y \cong \mathscr{O}_{Y'}$ over $\mathscr{O}_X$ identifying $\mathcal{I}_{X/Y} \cong \mathcal{I}_{X/Y'}$. There are induced compatible isomorphisms $\Theta_Y^q \cong \Theta_{Y'}^q$ and $\Omega_Y^p \cong \Omega_{Y'}^p$ over φ resulting in an isomorphism over φ

$$\mathrm{Der}_Y^q(-\log X) \cong \mathrm{Der}_{Y'}^q(-\log X).$$

Any general embedding $X \subseteq Y'$ arises from a minimal embedding $X \subseteq Y$ up to isomorphism of the latter as $Y' = Y \times Z$ where $Z \cong (\mathbb{C}^m, 0)$ and hence

$$\mathcal{I}_{X/Y'} = \mathscr{O}_Y \hat{\otimes} \mathfrak{m}_Z + \mathcal{I}_{X/Y} \hat{\otimes} \mathscr{O}_Z.$$

Pick coordinates $z_1,\dots,z_m$ on Z and abbreviate $d\underline{z} := dz_1\wedge\cdots\wedge dz_m$ and $\frac{\partial}{\partial\underline{z}} := \frac{\partial}{\partial z_1}\wedge\cdots\wedge\frac{\partial}{\partial z_m}$. Then there are decompositions

$$\Omega^{q+m}_{Y'} = \mathscr{O}_Z\hat{\otimes}\Omega^q_Y\wedge d\underline{z}\oplus\widetilde{\Omega}^{q+m}_{Y'},\quad \Theta^{q+m}_{Y'} = \mathscr{O}_Z\hat{\otimes}\Theta^q_Y\wedge\frac{\partial}{\partial\underline{z}}\oplus\widetilde{\Theta}^{q+m}_{Y'}$$

where the modules with tilde are generated by basis elements not involving $d\underline{z}$ and $\frac{\partial}{\partial\underline{z}}$ respectively. Fundamental forms of X in Y' and Y can be chosen compatibly as

$$\alpha_{X/C/Y'} = \alpha_{X/C/Y}\wedge d\underline{z}\in\Omega^{k+m}_{Y'}.$$

With Lemma 3.20 this yields inclusions

$$\operatorname{Der}^q_Y(-\log X)\wedge\frac{\partial}{\partial\underline{z}}+\widetilde{\Theta}^{q+m}_{Y'}\subseteq\operatorname{Der}^{q+m}_{Y'}(-\log X)\supseteq\mathcal{I}_{X/Y'}\Theta^{q+m}_{Y'}\supseteq\mathfrak{m}_Z\hat{\otimes}\Theta^q_Y\wedge\frac{\partial}{\partial\underline{z}}$$

and a cartesian square

$$\begin{tikzcd}
\mathscr{O}_Z\hat{\otimes}\Theta^q_Y \arrow[r, hook, "-\wedge\frac{\partial}{\partial\underline{z}}"] & \Theta^{q+m}_{Y'} \\
\operatorname{Der}^q_Y(-\log X)+\mathfrak{m}_Z\hat{\otimes}\Theta^q_Y \arrow[u, hook] \arrow[r, hook] & \operatorname{Der}^{q+m}_{Y'}(-\log X). \arrow[u, hook]
\end{tikzcd}$$

It gives rise to an isomorphism of $\mathscr{O}_X$-modules

$$\Theta^{q+m}_{Y'}/\operatorname{Der}^{q+m}_{Y'}(-\log X)\ \cong\ \mathscr{O}_Z\hat{\otimes}\Theta^q_Y/(\operatorname{Der}^q_Y(-\log X)+\mathfrak{m}_Z\hat{\otimes}\Theta^q_Y\ \cong\ \Theta^q_Y/\operatorname{Der}^q_Y(-\log X).\quad\square$$

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M. Schulze, Department of Mathematics, TU Kaiserslautern, 67663 Kaiserslautern, Germany
Email address: `mschulze@mathematik.uni-kl.de`

L. Tozzo, Department of Mathematics, TU Kaiserslautern, 67663 Kaiserslautern, Germany
Email address: `tozzo@mathematik.uni-kl.de`