
ON A DISCRIMINANT KNOT GROUP PROBLEM OF BRIESKORN

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ABSTRACT. Quite some time ago, at the singularity conference at Cargèse 1972, Brieskorn asked the following question:

Is the local fundamental group $\pi_1^s(S - D)$ of the discriminant complement inside the semi-universal unfolding S of an isolated hypersurface singularity constant for s in the μ -constant stratum Σ_E ?

We review this question and give an affirmative answer in case of singular plane curve germs of multiplicity at most 3.

1. INTRODUCTION

The question of Brieskorn was published in Astérisque 7-8, *Colloque sur les singularités en géométrie analytique*. In that article Brieskorn gives a summary of the problems and questions he considers central in the investigation of monodromy, and their answers which – as he writes – will help much to arrive at a more profound understanding, [Bri73].

In Brieskorn's view the local fundamental group of the discriminant complement – the discriminant knot group as it will be called in the present article – lies at the heart of the study of the algebraic monodromy and the intersection lattice of the Milnor fibre and should soon reveal to contain more or less the same amount of information.

This optimism probably resulted in the spectacular success in the study of simple hypersurface singularities where Brieskorn himself made important contributions, [Bri71a, Bri71b, BS72]. For the simple singularities the algebra, the geometry and the combinatorial group theory are most closely tied together and hope was widespread to get similar results for more general singularities under suitable forms of relaxation.

However, the topology of the discriminant complement remains a mystery up to the present day, and only little progress has been made on the problems Brieskorn addressed to it.

In this article we will review the problem stated in the abstract

Is the discriminant knot group $\pi_1^s(S - D)$ of an isolated hypersurface singularity constant for s in the μ -constant stratum?

At the time of writing the evidence in favour of a positive answer had two aspects. First in the case of simple singularities the answer is trivially positive. Second, the homomorphic image under algebraic monodromy is constant along the μ -constant stratum.

On the other hand an article of Pham [Pha73] presented at the very conference at Cargèse was interpreted by Brieskorn as evidence in favour of a negative answer: Pham showed that the topological type of the generic discriminant curve of certain plane curve singularities of multiplicity $m = 3$ is not constant along the μ -constant stratum.

2010 *Mathematics Subject Classification*. 32S50 primary, 14D05, 32S55, 20F36 secondary.

Key words and phrases. plane curve singularity, discriminant knot group, μ -constant stratum, braid monodromy.

The present work took place in the framework of the ERC Advanced grant n. 340258, 'TADMICAMT' .

In fact, Brieskorn proposes to study the discriminant knot group by the local Zariski hyperplane theorem as proved by Lê and Hamm [HL73]:

$$\pi_1(S - D) \cong \pi_1(H - H \cap D),$$

where H is a plane in S parallel to a generic plane $H_0 \neq H$ through the origin. $H_0 \cap D$ is called a *generic discriminant curve* and $H \cap D$ a corresponding *unfolded generic discriminant curve*. The topological type of the former is constant along the μ -constant stratum if and only if the topological type of the latter is constant along that stratum.

Therefore the result of Pham shows that the line of argument which Brieskorn had in mind cannot work.

In this article, however, we will follow Brieskorn's strategy and bridge the gap by using a stronger form of the Zariski van Kampen method applicable to more general plane sections of the discriminant.

We will turn the Pham examples into evidence for a positive answer to Brieskorn's problem by the following theorem.

Theorem 1. *Suppose f is a plane curve singularity of multiplicity at most 3, then the discriminant knot group is constant along the μ -constant stratum.*

As remarked before, in the case of simple singularities the claim trivially holds true. By classification this settles the case of multiplicity 2 and of plane curve singularities of Milnor number at most 8.

As a direct corollary we can sharpen the result of [Lön10]. Suppose f is topologically equivalent to a plane curve singularities of Brieskorn-Pham type of multiplicity 3

$$f \sim_{top} y^3 + x^{\nu+1} \text{ for some } \nu \geq 2,$$

then f is a μ -constant deformation of the Brieskorn-Pham singularity and has the same distinguished Dynkin diagram

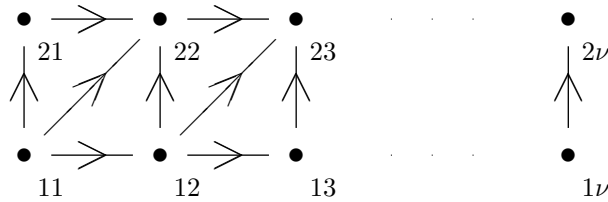


FIGURE 1. Dynkin diagram of $y^3 + x^{\nu+1}$

where the set V of vertices is ordered by the lexicographic order of their double indices and the set E of oriented edges contains the pair of corresponding vertices only in their proper order.

Since the discriminant knot group by Theorem 1 is the same for f and the Brieskorn-Pham singularity we get from [Lön10, Thm1.1]. (Does it extend to all cases in [Lön07]?)

Theorem 2. *Suppose f is topologically equivalent to a Brieskorn-Pham polynomial $y^3 + x^{\nu+1}$. Then its discriminant knot group is presented by*

$$\left\langle t_i, i \in V \mid \begin{array}{ll} t_i t_j = t_j t_i & (i, j), (j, i) \notin E \\ t_i t_j t_i = t_j t_i t_j & (i, j) \in E \\ t_i t_k t_j t_i = t_j t_i t_k t_j & (i, j), (i, k), (j, k) \in E \end{array} \right\rangle.$$

A step beyond the result of this article might address the case of unimodal hypersurface singularities. Possibly it is sufficient to look at the generic discriminant curve, since in the cases not covered by our result, Greuel [Gre77, Gre78] has shown that at least the number of cusps of the unfolded generic discriminant curve is constant along the μ -constant stratum.

2. REVIEW OF THE RESULTS OF PHAM

In his article [Pha73] Pham provides a careful analysis of the generic discriminant curve in case of a plane curve singularity of multiplicity 3

$$f = y^3 - P(x)y + Q(x).$$

While skipping his calculation which we will mimic in the next section, here we only want to introduce the minimum of notation to state his results and draw some first conclusions towards the proof of our main theorem.

In addition to the well-known Milnor number

$$\mu = \dim \mathbb{C}[X, Y] / \langle f_x, f_y \rangle,$$

Pham needs the analytic σ -invariant associated to the ideal generated by f and its derivatives up to second order

$$\sigma = 1 + \dim \mathbb{C}[X, Y] / \langle f, f_x, f_y, f_{xx}, f_{xy}, f_{yy} \rangle.$$

He also gives some useful formulas for calculations:

Lemma 3 ([Pha73] §1,p.366). *If f is a function germ as above, the analytic invariant σ is given by*

$$\sigma = \min\{\text{ord } P, \text{ord } Q'\}$$

and the Milnor number is given by

$$\mu = \text{ord}(3Q'^2 - PP'^2).$$

Instead of citing the main result in its full strength, which is a complete topological classification of generic discriminant curves, we distill the essence, what we will need below.

Proposition 4 (cf. [Pha73]). *The topological type of the generic discriminant curve only depends on the topological invariant μ and the analytic invariant σ .*

Corollary 5. *The topological type of the unfolded generic discriminant curve only depends on the topological invariant μ and the analytic invariant σ .*

Proof. The topological type of the generic discriminant curve determines its Milnor number $\tilde{\mu}$. The number $\mu + \tilde{\mu} - 1$ is the sum of three times the number of cusps and twice the number of nodes of any corresponding unfolded discriminant curve, cf. [Pha73]. Since both cardinalities are upper semi-continuous and the set with constant σ and μ is connected, they are both constant along this set, and so is the topological type of the unfolded generic discriminant curve. \square

Proposition 6. *If f is a plane curve singularity of multiplicity 3 and*

$$f \not\sim_{\text{top}} y^3 + x^{\nu+1} \quad \text{for all } \nu,$$

then the discriminant knot group is constant along the μ -constant stratum.

Proof. According to the classification by Arnol'd [Arn76] f is simple, of type $J_{k,i}$, $k \geq 2, i > 0$, or of type E_{6k+1} , $k \geq 2$. In the simple case the claim is trivially true as was remarked before.

In case of $f \in J_{k,i}$, $k \geq 2, i > 0$ Arnol'd has given a normal form which by an analytic equivalence – more precisely by a Tschirnhaus transformation – can be put in the form considered by Pham:

$$y^3 + y^2x^k + a(x)x^{3k+i}, \quad \text{ord } a = 0$$

$$\sim_{an} y^3 - \frac{1}{3}yx^{2k} + \frac{2}{27}x^{3k} + a(x)x^{3k+i}.$$

According to the lemma $\sigma = 2k$ and thus σ is independent of $a(x)$.

In case of $f \in E_{6k+1}$ the normal form of Arnol'd is in the form considered by Pham, so from

$$y^3 + yx^{2k+1} + a(x)x^{3k+2},$$

$\sigma = 2k + 1$ independent of $a(x)$ is immediate by the lemma again.

In both cases we conclude with the corollary that the topological type of the unfolded generic discriminant is constant along the μ -constant stratum. Therefore the fundamental groups of their complements also do not change. The local Zariski theorem on hyperplane sections [HL73] identifies these groups with the discriminant knot groups which are thus shown to be constant along the μ -constant stratum. \square

3. EXISTENCE OF SUITABLE NON-GENERIC DISCRIMINANT CURVES

In this section we follow the path traced by Pham to obtain a non-generic reduced discriminant curve which does not change its topological type under a small deformation along the μ -constant stratum, although the analytic invariant σ changes.

In fact, as the last section will prove, it will suffice to do so for the Brieskorn-Pham polynomials.

We recall from [Pha73] the construction of the discriminant curve in direction of a linear perturbation by a polynomial $p(x)y + q(x)$. The critical set of the unfolding of

$$f = f(x, y) = y^3 - P_0(x)y + Q_0(x)$$

by $-u + t(p(x)y + q(x))$ is a curve in 4-space and the corresponding discriminant curve is obtained by projection along the coordinates x, y , algebraically by elimination of x, y from

$$(1) \quad u = y^3 - Py + Q$$

$$(2) \quad 0 = -P'y + Q'$$

$$(3) \quad 0 = 3y^2 - P,$$

where $P = P_0 + tp$, $Q = Q_0 + tq$.

But as Pham does, we take the detour by the projection along u and y which is easier to obtain. The parameter u is eliminated by the sole use of (1) and from (2) and (3) we can eliminate y to get

$$(4) \quad 3Q'^2 - PP'^2 = 0.$$

First we consider for an additional parameter $s = 0$ or s which is sufficiently small the case

$$Q = Q_0 = x^{\nu+1}, \quad q = 0, \quad P = P_0 + tp = sx^\sigma + tx.$$

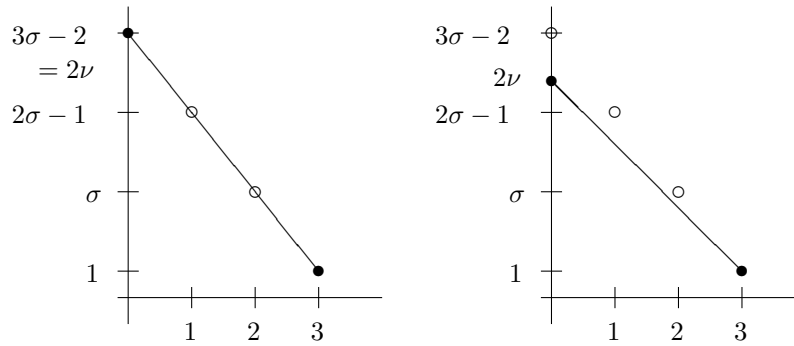
The first step according to Pham is to compute the branches $x(t)$. Recall the expansion of (4) in terms of the variable t according to

$$3(Q'_0 + tq')^2 - (P_0 + tp)(P'_0 + tp')^2 = A_0 + A_1t + A_2t^2 + A_3t^3.$$

In the current situation we get the following vanishing orders of the A_i under the assumption of s sufficiently small.

expansion	vanishing order
$A_0 = 3Q_0'^2 - P_0P_0'^2 = 3(\nu + 1)^2x^{2\nu} - s^3\sigma^2x^{3\sigma-2}$	$\mu = \min\{2\nu, 3\sigma - 2\}$
$A_1 = -xP_0'^2 - 2P_0P_0' = -s^2\sigma(\sigma + 2)x^{2\sigma-1}$	$2\sigma - 1$ for $s \neq 0$
$A_2 = -P_0 - 2xP_0' = -s(2\sigma + 1)x^\sigma$	σ for $s \neq 0$
$A_3 = -x$	1

Under the assumption $3\sigma - 2 \geq 2\nu$ the Newton Polygon looks as below depending on whether equality holds or not. (The \circ are only present for $s \neq 0$.)



The leading term corresponding to the compact face has no multiple root. This is obvious in case of $3\sigma - 2 > 2\nu$ and for $s = 0$, therefore it is true also for s sufficiently small.

In particular, for s sufficiently small, the number of branches is constant and the leading term of each branch has a non-vanishing coefficient which varies continuously with s .

We consider now the case $s = 0$ in detail (but claims hold true also for s small up to continuous changes of the coefficients) and distinguish the following cases

- (a) $\gcd(2\nu - 1, 3) = \gcd(2\nu + 2, 3) = \gcd(\nu + 1, 3) = 1$
- (b) $\gcd(2\nu - 1, 3) = \gcd(\nu + 1, 3) = 3,$
 $\gcd(3\nu + 3, 2\nu - 1) = \gcd(6\nu + 6, 2\nu - 1) = \gcd(2\nu - 1, 9) = 3$
- (c) $\gcd(2\nu - 1, 3) = \gcd(\nu + 1, 3) = 3, \gcd(3\nu + 3, 2\nu - 1) = \gcd(2\nu - 1, 9) = 9$

In case (a) there are two branches

$$(5) \quad x(t) = 0, \quad x(t) = c_0 t^{\frac{3}{2\nu-1}} + h.o.t.$$

in cases (b) and (c) there are four branches

$$(6) \quad x(t) = 0, \quad x(t) = c_0 \omega^i t^{\frac{3}{2\nu-1}} + h.o.t., \quad i = 0, 1, 2.$$

where $c_0 \neq 0$ is a numerical constant and ω a primitive root of unity of order $2\nu - 1$.

To continue along the lines of [Pha73] we check first that the hypothesis

$$P'(x(t), t) = P_0'(x(t)) + tP'(x(t)) = P_0'(x(t)) + t \neq 0 \in \mathbb{C}\{t^{\frac{1}{2\nu-1}}\}$$

holds true for every possible branch $x(t)$.

Therefore the following formula derived by Pham is valid in the field of fractions $\mathbb{C}((t^{\frac{1}{2\nu-1}}))$ associated to the integral domain $\mathbb{C}\{t^{\frac{1}{2\nu-1}}\}$.

$$(7) \quad \begin{aligned} u &= -\frac{2}{3} \frac{P}{P'} Q' + Q \\ &= \left(-\frac{2}{3}(\nu + 1) + 1\right) x(t)^{\nu+1}. \end{aligned}$$

In case (a) we plug in the expansions (5) to get

$$u(t) = 0, \quad u(t) = \left(-\frac{2}{3}(\nu + 1) + 1\right) c_0^{\nu+1} t^{\frac{3\nu+3}{2\nu-1}} + h.o.t.$$

The corresponding branches are reduced and not equal. Moreover the second expansion does not have further essential summands, since the exponent of t is in its reduced form and has the maximal possible denominator.

In case (b) we write $2\nu - 1 = 3e$ with e coprime to 3 and get the expansions

$$u(t) = 0, \quad u(t) = \left(-\frac{2}{3}(\nu + 1) + 1\right) c_0^{\nu+1} \omega^{(\nu+1)i} t^{\frac{3\nu+3}{2\nu-1}} + h.o.t., \quad i = 0, 1, 2.$$

Again the corresponding branches are reduced and pairwise not equal. This time the reduced form of the exponent has denominator e . Again this is the maximal possible denominator, since the u -degree of the Weierstrass polynomial of the first branch is 1 and that of the other three branches is the maximal denominator, but their sum is equal to the Milnor number which is $\mu = 3e + 1$.

Thus in case (a) and case (b) we have found a perturbation such that the topological type of corresponding discriminant curve does not vary for s sufficiently small.

In case (c) we write $\nu - 5 = 9\rho$, but we fail to argue as above. In fact, for $s = 0$ we get expansions which parametrize the branches of the corresponding discriminant curve by a $3 : 1$ map so this curve is non-reduced.

Hence we rerun the method of Pham with the modified perturbation

$$t(xy + x^{3\rho+4}), \quad \text{i.e. } p = x, q = x^{3\rho+4}.$$

The essential expansion of x in terms of t remains the same as before, since the new perturbation only adds the points $(1, 12\rho + 8)$ and $(2, 6\rho + 6)$ to the support, which both lie above the Newton polygon.

$$x(t) = 0, \quad x(t) = c_0 \omega^i t^{\frac{3}{2\nu-1}} + h.o.t., \quad i = 0, 1, 2.$$

The reduced form of the exponent is the inverse of $6\rho + 3$.

The formula (7) now gives (using c_ν, c_ρ for the obvious constants)

$$\begin{aligned} u &= \left(-\frac{2}{3}(\nu + 1) + 1\right) x(t)^{\nu+1} + \left(-\frac{2}{3}(3\rho + 4) + 1\right) tx(t)^{3\rho+4} \\ &= c_\nu c_0^{\nu+1} \omega^{(\nu+1)i} t^{\frac{\nu+1}{6\rho+3}} + c_\rho c_0^{3\rho+4} \omega^{(3\rho+4)i} t^{\frac{\nu+2}{6\rho+3}} + h.o.t. \end{aligned}$$

We can now argue as before, that $6\rho + 3$ is the maximal possible denominator. Therefore no further essential summand occurs, and we get reduced, pairwise distinct branches also in the remaining case (c).

Let us summarize the results of the present section as follows.

Proposition 7. *Suppose $f = y^3 + x^{\nu+1}$ and m is an integer with*

$$2\nu \leq 3m - 2, \quad m \leq \nu.$$

Then there exists a 3-parameter unfolding $F(x, y; u, t, s)$, such that

- (1) *along $u = t = 0$ the unfolding is μ -constant,*

- (2) for fix s sufficiently small, the discriminant curve of the unfolding F_s by the parameter t is reduced and topologically equivalent to that of F_0 and
- (3) the analytic invariant σ is ν for $s = 0$ and m for $s \neq 0$ sufficiently small.

4. THE ZARISKI THEOREM

In this final section we have to revisit the local Zariski and van Kampen theorem which avoids the use of generic hyperplane sections, cf. the more extended exposition in [Lön10, Lön11].

Our main interest lies in the discriminant complement, so let us recall the basic setting: Given a holomorphic function germ $f = f(x, y)$ on the germ $\mathbb{C}^2, 0$ of the affine plane with coordinates x, y , we consider a versal unfolding, which can be given by a function germ on the affine space germ $(\mathbb{C}^2, 0) \times (\mathbb{C}, 0) \times (\mathbb{C}^k, 0)$

$$F(x, y, u, v) = f(x, y) - u + \sum_{i=1}^k v_i g_i(x, y),$$

where the g_i generate, as a vector space, the local ideal of function germs on $\mathbb{C}^2, 0$ vanishing at the origin up to elements in the Jacobian ideal of f . They are typically taken to be non-constant monomials.

We get a diagram

$$\begin{array}{ccc} (u, v_1, \dots, v_k) \in \mathbb{C}^{k+1}, 0 & \supset \mathcal{D} = & \{(u, v) \mid F_{u,v}^{-1}(0) \text{ is singular}\} \\ \downarrow & p \downarrow & \\ (v_1, \dots, v_k) \in \mathbb{C}^k, 0 & \supset \mathcal{B} = & \{u \mid F_{0,v} \text{ is not Morse}\} \end{array}$$

The restriction $p|_{\mathcal{D}}$ of the projection to the *discriminant* \mathcal{D} is a finite map, such that the branch set coincides with the *bifurcation set* \mathcal{B} and the critical points are contained in the pre-image $\tilde{\mathcal{B}} = p^{-1}(\mathcal{B})$. In particular, the origin is an isolated point in the intersection of \mathcal{D} with the fibre $p^{-1}(0)$. If a hypersurface germ has this property we call it *horizontal* for the projection p .

The key observation is, that a suitable representative of the complement of $\tilde{\mathcal{B}}$ is a trivial disc bundle by p into which \mathcal{D} is embedded as a smooth submanifold, which is a connected topological cover by p . This situation, which can be treated also in the language of polynomial covers, cf. [Han89], naturally gives rise to a *braid monodromy homomorphism*: The domain is the fundamental group of the complement of \mathcal{B} , its target is the group of mapping classes of the punctured fibre, the image is called the *braid monodromy group*.

It coincides with the map of fundamental groups induced by the map of Lyashko Looijenga under the natural identification of the mapping class group with the fundamental group of the space of monic simple univariate polynomials of degree n at the corresponding base point:

$$\begin{array}{ccc} \mathbb{C}^k - \mathcal{B} & \longrightarrow & \mathbb{C}[x], \\ v & \longmapsto & p_v, \end{array}$$

which maps to monic univariate polynomials of degree μ with simple roots only, where these roots correspond to the points of \mathcal{D} which project to v .

To use the braid monodromy group of the fundamental group of the discriminant complement we employ the argument of Zariski and van Kampen [vK33]. It relies on a choice of a geometric basis in the fibre over the base point which is the customary tool to identify the action of the group of isotopy classes of diffeomorphisms on the fundamental group of the fibre with the right Artin action of the abstract braid group on the free generators t_1, \dots, t_n given by

$$(t_j)\sigma_j = t_j t_{j+1} t_j^{-1}, \quad (t_{j+1})\sigma_j = t_j, \quad (t_i)\sigma_j = t_i, \text{ if } i \neq j, j + 1.$$

Theorem 8 (van Kampen). *Given a horizontal hypersurface germ with braid monodromy group generated by braids $\{\beta_s\}$ in Br_n , the local fundamental group of the complement is finitely presented as*

$$\pi_1 = \langle t_1, \dots, t_n \mid t_i^{-1} t_i^{\beta_s}, 1 \leq i \leq n, \text{ all } \beta_s \rangle.$$

The consideration above applies again to the hypersurface germ \mathcal{B} in the affine space germ $\mathbb{C}^k, 0$ provided we find a projection for which \mathcal{B} is horizontal. In fact this puts a constraint on a discriminant curve as we will see in the following proof.

Proposition 9. *Let g_1 be a bivariate polynomial germ vanishing at 0 such that the discriminant curve of the unfolding*

$$f - u + tg_1$$

is reduced. Then the fundamental group of the complement of a corresponding unfolded discriminant curve is equal to the discriminant knot group of f .

Proof. Without loss of generality we may assume that g_1 is the first of the functions in the versal unfolding of f we consider. Hence the complement of the unfolded discriminant curve is a vertical plane section of the discriminant. (At this point we could conclude with the local Zariski hyperplane section theorem, if this vertical plane were known to be generic.)

By the van Kampen theorem, it suffices to show that the two braid monodromy groups are equal. They in turn are homomorphic images of the corresponding fundamental groups of complements to the bifurcation set.

If the discriminant curve is reduced, then the corresponding curve in the affine space germ $\mathbb{C}^k, 0$ does not belong to the bifurcation set, otherwise, the discriminant curve has less than μ points over every t and is non-reduced.

We deduce that the bifurcation set is horizontal for the projection along the coordinate corresponding to g_1 , since the 0-fibre of that projection was just shown not to be in the bifurcation set.

In particular the fundamental group in a generic vertical line is generated by elements corresponding to a geometric basis. They also generate the fundamental group of the complement to the bifurcation set by the van Kampen theorem.

Put differently the fundamental group the smaller set surjects onto the fundamental group of the complement to the bifurcation set. Hence both fundamental groups map to same braid monodromy group.

Since a generic vertical line is the image under p of an unfolded discriminant curve associated to g_1 as in the beginning of the proof, we have precisely shown what was needed. □

Proof of the main Theorem. Thanks to Prop.6 it suffices to show that the discriminant knot group is constant along each μ -constant stratum which contains a Brieskorn-Pham polynomial $y^3 + x^{\nu+1}$.

Let f be any function in this stratum and σ_f its σ -invariant. Since the analytic equivalence class of f has a representative of the form

$$y^3 - P(x)y + x^{\nu+1} \quad \text{with} \quad \frac{2}{3}(\nu + 1) \leq \text{ord } P, \quad \text{deg } P \leq \nu - 1,$$

we deduce $2\nu \leq 3\sigma_f - 2$ and $\sigma_f \leq \nu$.

Therefore by Prop.7 we can unfold the Brieskorn-Pham polynomial by a parameter s such that the σ -invariant is σ_f for $s \neq 0$, and there exists an associated family of discriminant curves of constant topological type.

Because they are reduced, we can apply the previous proposition to see that corresponding unfolded discriminant curves have a complement with fundamental group isomorphic to the

respective discriminant knot groups. So by the same argument as in the proof of Prop.6 the two discriminant knot groups are isomorphic.

Since f and any deformation of the Brieskorn-Pham polynomial with s small share the same σ -invariant, we may invoke Cor.5 to have topologically equivalent complements of unfolded generic discriminant curves. Thus again the discriminant knot groups are isomorphic and therefore constant along the μ -constant stratum of each Brieskorn-Pham polynomial. \square

Acknowledgement. The author wants to thank Patrick Popescu-Pampu for the invitation to give a talk at Lille which ultimately caused this project to find its shape. He gratefully acknowledges the financial support of the ERC 2013 Advanced Research Grant 340258-TADMICAMT.

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