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SURGERY OF REAL SYMPLECTIC FOURFOLDS AND WELSCHINGER INVARIANTS

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ABSTRACT. A surgery of a real symplectic manifold $X_{\mathbb{R}}$ along a real Lagrangian sphere S is a modification of the symplectic and real structure on $X_{\mathbb{R}}$ in a neighborhood of S. Genus 0 Welschinger invariants of two real symplectic 4-manifolds differing by such a surgery have been related in [BP15]. In the present paper, we explore some particular situations where general formulas from [BP15] greatly simplify. As an application, we reduce the computation of genus 0 Welschinger invariants of all del Pezzo surfaces to the cases covered by [Bru15], and of all $\mathbb{R}$-minimal real conic bundles to the cases covered by [HS12]. As a by-product, we establish the existence of some new relative Welschinger invariants. We also generalize results from [BP15] to the enumeration of curves of higher genus, and give relations between hypothetical invariants defined in the same vein as [Shu14].

Let Λ be either $\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}$, or $\mathbb{Q}$. Given a (oriented if $\Lambda = \mathbb{Z}$ or $\mathbb{Q}$) smooth compact manifold X of dimension 4, the intersection product of two elements $d_1, d_2 \in H_2(X;\Lambda)$ is denoted by $d_1 \cdot d_2 \in \Lambda$. The class realized in $H_2(X;\Lambda)$ by a 2-cycle C in X is denoted by $[C]$. The subgroup orthogonal to a class $\delta \in H_2(X;\Lambda)$ for the intersection form is denoted by $\delta^\perp$.

A *real symplectic manifold* $X_{\mathbb{R}} = (X, \omega_X, \tau_X)$ is a symplectic manifold (X, ω_X) equipped with an anti-symplectic involution τ_X. The *real part* of (X, ω_X, τ_X), denoted by $\mathbb{R}X$, is by definition the fixed point set of τ_X. A projective real algebraic variety is always implicitly assumed to be equipped with some Kähler form which turns it into a real symplectic manifold. Two symplectic forms ω_X and ω'_X (resp. two real symplectic structures (ω_X, τ_X) and (ω'_X, τ'_X)) on a manifold X are said to be *deformation equivalent* if there exists a smooth family of symplectic forms connecting ω_X and ω'_X (resp. a smooth family of real symplectic structures connecting (ω_X, τ_X) to (ω'_X, τ'_X)). Two symplectic manifolds (X, ω_X) and (Y, ω_Y) (resp. two real symplectic manifolds (X, ω_X, τ_X) and (Y, ω_Y, τ_Y)) are said to be *deformation equivalent* if one can pass from one to the other by a finite sequence of deformations of symplectic form and symplectomorphisms (resp. deformations of real symplectic structure and equivariant symplectomorphisms).

In this text, the manifold X will always be 4-dimensional, and we denote by $H_2^{\tau_X}(X;\Lambda)$ the space of τ_X-invariant classes, and by $H_2^{-\tau_X}(X;\Lambda)$ the space of τ_X-anti-invariant classes.

1. Introduction

Beside blow-up, surgery along a real Lagrangian sphere is a natural and elementary operation on *real* algebraic or symplectic manifolds. For example, there exists only three real rational algebraic surfaces up to deformation, blow-up, and surgery along a real Lagrangian sphere: $\mathbb{C}P^2$ equipped with its standard real structure, and $\mathbb{C}P^1 \times \mathbb{C}P^1$ equipped with any of its two non-equivalent real structures with an empty real part[1]. Welschinger invariants are invariant

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[1]The classification up to deformation and blow-up is given in [DK02]; the classification up to surgery along a real Lagrangian sphere follows then from the rigid isotopy classifications of plane real quartics, of real cubic sections of the quadratic cone in $\mathbb{C}P^3$, and of real quadrics in $\mathbb{C}P^3$, see for example [DK00].

under deformation, hence understanding how they behave under blow-up and surgery along a real Lagrangian sphere would allow for reducing significantly the basic ambient real symplectic manifolds in which to perform actual computations.

In this paper we relate, under mild assumptions, Welschinger invariants of two real symplectic 4-manifolds differing by such surgery. We start by describing informally this operation, and we refer to Section 2 for precise definitions. Let $X_{\mathbb{R}} = (X, \omega_X, \tau_X)$ be a real compact symplectic manifold of dimension 4, and let $S \subset X$ be a real Lagrangian sphere, i.e. a Lagrangian sphere globally invariant under τ_X. It follows from Weinstein Lagrangian neighborhood Theorem (see for example [MS98, Theorem 3.33]) that there exists a real open symplectic embedding of a neighborhood V of S to the real affine quadric (Q, ω_Q, τ) in $\mathbb{C}^3$ given by the equation

$$(-1)^{\varepsilon_1}x^2 + (-1)^{\varepsilon_2}y^2 + (-1)^{\varepsilon_3}z^2 = 1 \qquad \text{with } \varepsilon_i \in \{0,1\},$$

and S to the sphere S_Q in $i^{\varepsilon_1}\mathbb{R} \times i^{\varepsilon_2}\mathbb{R} \times i^{\varepsilon_3}\mathbb{R}$ with equation

$$x^2 + y^2 + z^2 = 1.$$

Observe that the automorphism $(x, y, z) \mapsto (-x, -y, -z)$ of $\mathbb{C}^3$ provides another real structure τ' on (Q, ω_Q) that coincide with τ at infinity, and for which S_Q remains globally invariant. As a consequence, one can modify the symplectic and real structure of $X_{\mathbb{R}}$ in V so that V now becomes equivariantly symplectomorphic to a bounded open subset of the real affine quadric in $\mathbb{C}^3$ with equation

$$(-1)^{\varepsilon_1}x^2 + (-1)^{\varepsilon_2}y^2 + (-1)^{\varepsilon_3}z^2 = -1.$$

The resulting real symplectic manifold $Y_{\mathbb{R}}$ is called a *surgery of* $X_{\mathbb{R}}$ *along* S. From this local description, we see that (with the convention that $\chi(\emptyset) = 0$)

$$\chi(\mathbb{R}Y) = \chi(\mathbb{R}X) \pm 2,$$

and that the class $[S]$ in $H_2(X; \mathbb{Z})$ is τ_X-anti-invariant if and only if it is τ_Y-invariant (in which case we have $\chi(\mathbb{R}Y) = \chi(\mathbb{R}X) + 2$). Note that $X_{\mathbb{R}}$ and $Y_{\mathbb{R}}$ have the same underlying smooth manifold, and that $Y_{\mathbb{R}}$ only depends, up to deformation, on $X_{\mathbb{R}}$ and on the class realized by S in $H_2(X; \mathbb{Z})$. Since we are interested in this paper in invariants under deformation of real symplectic manifolds, we say that $Y_{\mathbb{R}}$ is *the* surgery of $X_{\mathbb{R}}$ along S rather than *a* surgery.

The two above real quadrics can be put into the real family Q_t of quadrics with equation

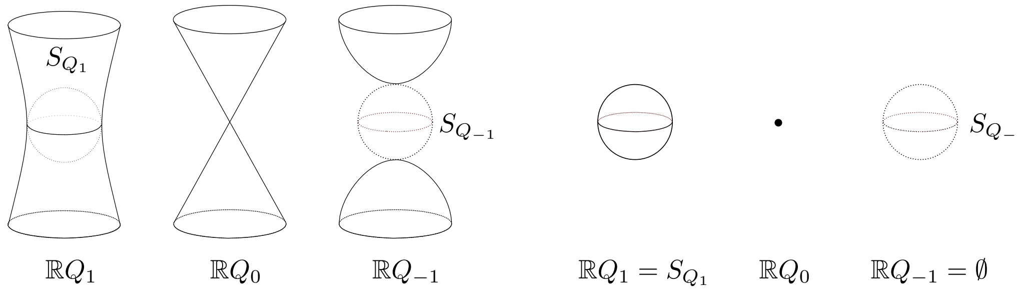
$$(-1)^{\varepsilon_1}x^2 + (-1)^{\varepsilon_2}y^2 + (-1)^{\varepsilon_3}z^2 = t \qquad \text{with } |t| \leq 1.$$

The quadric Q_0 is the unique singular quadric of the family. Hence $X_{\mathbb{R}}$ and $Y_{\mathbb{R}}$ can be represented as two different real fibers of a real Lefschetz fibration of over a disk, having a unique singular fiber for which S realizes precisely the vanishing cycle (see Figure 1).

Now we specialize our main statement, Theorem 3.8, for a particular type of Welschinger invariants that are easy to define. Choose the following:

- a connected component L of $\mathbb{R}X$,
- R which is either the empty set or the set $\mathbb{R}X \setminus L$; denote by F the class realized by R in $H_2^{\tau_X}(X \setminus L; \mathbb{Z}/2\mathbb{Z})$;
- a class $d \in H_2^{-\tau_X}(X; \mathbb{Z})$,
- $r, s \in \mathbb{Z}_{\geq 0}$ such that $c_1(X) \cdot d - 1 = r + 2s$,
- a configuration $\underline{x}$ made of r points in L and s pairs of τ_X-conjugated points in $X \setminus \mathbb{R}X$.

Given an almost complex structure J tamed by ω_X for which τ_X is J-antiholomorphic (i.e. $J \circ d\tau_X = -d\tau_X \circ J$), we denote by $\mathcal{C}(d, \underline{x}, L, J)$ the set of real rational J-holomorphic curves $f : \mathbb{C}P^1 \to X$, with $f_*[\mathbb{C}P^1] = d$, passing through $\underline{x}$, and such that $f(\mathbb{R}P^1) \subset L$. For a generic choice of J, the set $\mathcal{C}(d, \underline{x}, L, J)$ is finite and composed of immersions. Given an element



a) Q_t with equation $x^2 + y^2 - z^2 = t$ b) Q_t with equation $x^2 + y^2 + z^2 = t$

FIGURE 1. Q_t with equation $x^2 + y^2 \pm z^2 = t$.

$f : \mathbb{C}P^1 \to X$ of $\mathcal{C}(d, \underline{x}, L, J)$, we define the (L, F)-mass $m_{L,F}(f)$ of f as the number of elliptic real nodes of $f(\mathbb{C}P^1)$ (i.e. real nodes with two τ_X-conjugated branches) contained in $L \cup R$. The number

$$W_{X_\mathbb{R},L,F}(d; s) = \sum_{f \in \mathcal{C}(d,\underline{x},L,J)} (-1)^{m_{L,F}(f)}$$

only depends on the choices of L, F, d, s, and on the deformation class of $X_\mathbb{R}$ [Wel05b, Wel15], and is called a genus 0 *Welschinger invariant* of $X_\mathbb{R}$. Following [IKS13a], one can generalize the previous definition of Welschinger invariants to any class $F \in H_2^{\tau_X}(X \setminus L; \mathbb{Z}/2\mathbb{Z})$, see Section 3.2. Note nevertheless that among all possible choices of F in $H_2^{\tau_X}(X \setminus L; \mathbb{Z}/2\mathbb{Z})$, the two classes 0 and $[\mathbb{R}X \setminus L]$ seem to play a special role, see Remark 3.7. Hence it seems worthwhile to specialize Theorem 3.8 in these two special cases.

Theorem 1.1. *Let $X_\mathbb{R}$ be a compact real symplectic manifold of dimension 4. Let S be a real Lagrangian sphere in $X_\mathbb{R}$, and L be a connected components of $\mathbb{R}X$ disjoint from S. We denote by $Y_\mathbb{R}$ the surgery of $X_\mathbb{R}$ along S, and we assume that $\chi(\mathbb{R}Y) = \chi(\mathbb{R}X) + 2$.*

Then for any class $d \in H_2^{-\tau_Y}(X; \mathbb{Z})$, the two following identities hold:

$$W_{Y_\mathbb{R},L,0}(d; s) = W_{X_\mathbb{R},L,0}(d; s) + 2 \sum_{k \geq 1} W_{X_\mathbb{R},L,0}(d - k[S]; s),$$

and

$$W_{Y_\mathbb{R},L,[\mathbb{R}Y \setminus L]}(d; s) = W_{X_\mathbb{R},L,[\mathbb{R}X \setminus L]}(d; s) + 2 \sum_{k \geq 1} (-1)^k \, W_{X_\mathbb{R},L,[\mathbb{R}X \setminus L]}(d - k[S]; s).$$

The case when $F = 0$ follows immediately from Theorem 3.8. The case when $F = [\mathbb{R}X \setminus L]$ follows from Theorem 3.8 combined with the identity

$$[\mathbb{R}Y \setminus L] = [\mathbb{R}X \setminus L] + [S] \quad \text{in} \quad H_2^{\tau_Y}(X \setminus L; \mathbb{Z}/2\mathbb{Z}).$$

As mentioned in the beginning, one may reasonably expect that a suitable combination of Theorem 3.8 with Solomon's real WDVV equations [HS12, Sol] reduces the computation of genus 0 Welschinger invariants of all real rational algebraic surfaces to the cases of $\mathbb{C}P^2$ and $\mathbb{C}P^1 \times \mathbb{C}P^1$. (Recall that thanks to complex WDVV equations [KM94], the computation of genus 0 Gromov-Witten invariants of all rational symplectic 4-manifolds can be reduced to computations in $\mathbb{C}P^2$ and $\mathbb{C}P^1 \times \mathbb{C}P^1$ [GP98, McD90].)

Remark 1.2. Although the formulas from Theorem 1.1 are surprisingly simple, our proof goes by tedious computations involving binomial coefficients. There might exists a simpler and more transparent (geometrical) proof of Theorems 1.1 and 3.8.

Remark 1.3. Given a Lagrangian sphere S in a symplectic manifold (X, ω), one can define a symplectomorphism of (X, ω) called a *Dehn twist* along S, see [Arn95]. If X is four dimensional, this automorphism acts on $H_2(X;\mathbb{Z})$ by the involution $d \mapsto d + (d\cdot[S])[S]$. Hence similarly to the complex setting, if S is a real Lagrangian sphere in a real symplectic fourfold $X_{\mathbb{R}}$, one thus obtains the following relation for Welschinger invariants:

$$W_{X_{\mathbb{R}},L,F}(d;s) = W_{X_{\mathbb{R}},L,F}(d+(d\cdot[S])[S];s) \qquad \forall d \in H_2^{-\tau_X}(X;\mathbb{Z}). \tag{1}$$

(Note that by Lemma 2.6, this relation can be non-trivial only when $[S] \in H_2^{-\tau_X}(X;\mathbb{Z})$.) Equation (1) allows to rewrite the two identities from Theorem 1.1 in a more symmetric form[2]

$$W_{Y_{\mathbb{R}},L,0}(d;s) = \sum_{k\in\mathbb{Z}} W_{X_{\mathbb{R}},L,0}(d+k[S];s),$$

$$W_{Y_{\mathbb{R}},L,[\mathbb{R}Y\setminus L]}(d;s) = \sum_{k\in\mathbb{Z}} (-1)^k\ W_{X_{\mathbb{R}},L,[\mathbb{R}X\setminus L]}(d+k[S];s),$$

which may be useful in the perspective of Remark 1.2.

Remark 1.4. Several formulas involving Welschinger invariants, especially those based on degeneration formulas like symplectic sum formulas, are real counterparts of analogous formulas relating Gromov-Witten invariants, see for example [Mik05, IKS09, IKS15, BM07, BM08, BP13, BP15, Bru15, BG16b]. This led Göttsche to conjecture the existence of *quantum enumerative invariants* that would in particular contain both Gromov-Witten and Welschinger invariants as suitable specializations, see for example [GS14, BG16a, IM13, Mik15]. Theorems 1.1 and 3.8 might have an interpretation in this perspective.

An immediate consequence of Theorem 1.1 is that positivity and asymptotic results concerning Welschinger invariants of $X_{\mathbb{R}}$ transfer to $Y_{\mathbb{R}}$. Particular instances of such positivity and asymptotic results can be found in [IKS04, IKS13b, IKS13a, IKS15, Shu14, BM07, BM08, BM, Bru15].

Corollary 1.5. *Let $X_{\mathbb{R}}$, $Y_{\mathbb{R}}$, S, and L be as in Theorem 1.1. If $W_{X_{\mathbb{R}},L,0}(d;0) \geq 0$ for any $d \in H_2^{-\tau_X}(X;\mathbb{Z}/2\mathbb{Z})$, then $W_{Y_{\mathbb{R}},L,0}(d;0) \geq 0$ for any $d \in H_2^{-\tau_Y}(X;\mathbb{Z}/2\mathbb{Z})$.*

If furthermore for a given $d \in H_2^{-\tau_Y}(X;\mathbb{Z}/2\mathbb{Z})$, the sequence $(W_{X_{\mathbb{R}},L,0}(nd;0))_{n\geq 1}$ is logarithmically asymptotic to the sequence of the corresponding Gromov-Witten invariants of (X, ω_X), then so is the sequence $(W_{Y_{\mathbb{R}},L,0}(nd;0))_{n\geq 1}$.

Theorems 1.1 and 3.8 have several applications to the case of real rational algebraic surfaces, see Sections 4 and 5. In particular, combined with results from [Bru15], they complete the computation of genus 0 Welschinger invariants of all real del Pezzo surfaces.

Context and relation to other works. Using a real version of the symplectic sum formula (we refer for example to [IP04, LR01, EGH00] for complex versions, see also[Li02, Li04] for an analogous formula in the algebraic category), we related in [BP13, BP15] genus 0 Welschinger invariants of two real symplectic 4-manifolds $X_{\mathbb{R}}$ and $Y_{\mathbb{R}}$ differing by a surgery along a real Lagrangian sphere S (see also [IKS15, Shu14] for related works in the case of algebraic del Pezzo surfaces). In general, relations from [BP15] involve some quantities that depend on some choices additional to the choice of $X_{\mathbb{R}}$, $Y_{\mathbb{R}}$, and S. These quantities come from the enumeration of

[2] These two formulas also independently appeared in the February 2017 arXiv version of [IKS15, Corollary 4.3].

J-holomorphic curves in a real deformation of either $X_{\mathbb{R}}$ of $Y_{\mathbb{R}}$ for which S becomes symplectic, and with J chosen so that S is J-holomorphic. Since S has self-intersection -2, this almost complex structure J is not generic enough to ensure that counting real J-holomorphic curves with Welschinger signs give rise to an invariant. Still, relations from [BP15] have been applied in [BP15, Bru15] to obtain qualitative results and explicit computations of Welschinger invariants in a number of cases (see also the related works [IKS15, Shu14] in the case of algebraic del Pezzo surfaces of degree at least 2).

There are nevertheless particular situations where relations from [BP15] simplify so that only enumerative invariants of $X_{\mathbb{R}}$ and $Y_{\mathbb{R}}$ remain. Recall that the definition of genus 0 Welschinger invariants requires the choice of a connected component L of $\mathbb{R}X$, and of a class

$$F \in H_2^{\tau_X}(X \setminus L; \mathbb{Z}/2\mathbb{Z}).$$

In the case when L is disjoint from the real Lagrangian sphere S, and F is orthogonal to $[S]$, then [BP15, Theorem 2.5(1)] ultimately only involves genus 0 Welschinger invariants of $X_{\mathbb{R}}$ and $Y_{\mathbb{R}}$. Although this is an obvious consequence of the results exposed in [BP15], this remark is not explicitly made there (see for example [IKS15, Corollaries 4.2 and 4.3] for explicit similar remarks in the case of algebraic del Pezzo surfaces).

The aim of the present paper is to make explicit and to provide several applications of relations among Welschinger invariants of $X_{\mathbb{R}}$ and $Y_{\mathbb{R}}$. As a by-product, we obtain the existence of some new relative Welschinger invariants, see Theorem 3.9. Note that very few relative invariants are known in real enumerative geometry so far, see for example [Wel06, Wel07, IKS15, Shu17, IKS16]. The existence of absolute Welschinger invariants and Theorems 3.10 and 3.13 immediately imply the existence of real invariants relative to some collections of disjoint real embedded symplectic spheres with self-intersection -2, where only simple and non-fixed incidences to these spheres are prescribed. Furthermore we provide in Theorems 3.10 and 3.13 a mild generalization of [BP15, Theorem 2.5] to the case of hypothetical Welschinger invariants in positive genus defined in the same vein as in [Shu14].

Welschinger first proposed in [Wel07] an other, nevertheless related, treatment of Lagrangian spheres contained in $\mathbb{R}X$. Among other results, he proved there that all invariants $W_{X_{\mathbb{R}},S^2,F}$ can be expressed in terms of some relative Gromov-Witten invariants of the symplectic manifold (X, ω_X), and some real invariants of T^*S^2.

Several real enumerative invariants of higher dimensional symplectic manifolds have been defined in the last fifteen years, i.e. [Wel05c, Wel05a, Geo16, GZ15]. It could be interesting to generalize the methods of the present paper to the study of these invariants.

Organization of the paper. In Section 2 we describe in detail surgeries of real symplectic 4-manifolds along real Lagrangian spheres, and give several examples of such surgeries. Absolute and relative Welschinger invariants considered in this paper are defined in Section 3. We prove there Theorems 3.10 and 3.13 that relate such invariants for two real symplectic 4-manifolds differing by a surgery along a real Lagrangian sphere, from which we deduce Theorem 3.8. Qualitative applications of Section 3 to the case of real rational algebraic surfaces are discussed in Section 4. Finally, we illustrate in Section 5 the use of Theorem 3.8 with concrete computations in the case of real cubic surfaces, $\mathbb{R}$-minimal real conic bundles, and real del Pezzo surfaces of degree 1.

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2. Surgery along a real Lagrangian sphere

2.1. **Real structures on a quadric surface.** Here we recall some well-known facts about projective and affine quadrics. Any such quadric is always assumed to be equipped with the symplectic form ω_{FS} induced by the restriction of the Fubini-Study form on $\mathbb{C}P^3$.

A non-singular complex algebraic quadric surface Q in $\mathbb{C}P^3$ is biholomorphic to $\mathbb{C}P^1 \times \mathbb{C}P^1$. In particular the group $H_2(Q;\mathbb{Z})$ is isomorphic to $\mathbb{Z}^2$ and generated by the classes $l_1 = [\mathbb{C}P^1 \times \{p\}]$ and $l_2 = [\{p\} \times \mathbb{C}P^1]$. Clearly, these two classes are well defined in $H_2(Q;\mathbb{Z})$ only up to interchanging l_1 and l_2. A hyperplane section E of Q realizes the class $l_1 + l_2$. In what follows E is always assumed to be non-singular, which implies in particular that it is biholomorphic to $\mathbb{C}P^1$. In a suitable coordinate system, the complement $Q \setminus E$ is given in the corresponding affine chart of $\mathbb{C}P^3$ by the equation

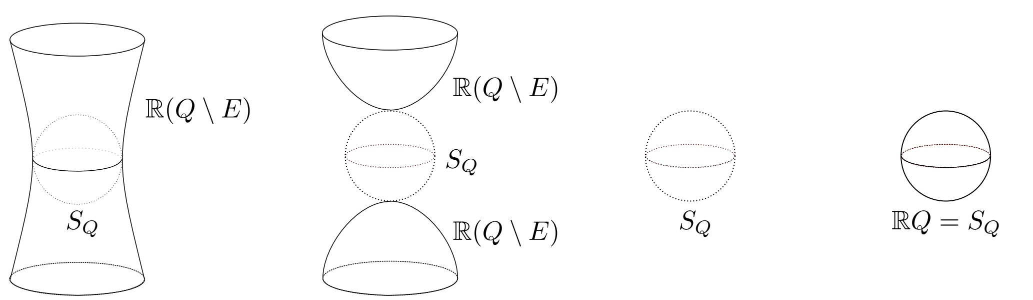
$$x^2 + y^2 + z^2 = 1.$$

When in addition both Q and E are real (i.e. stable under the standard complex conjugation on $\mathbb{C}P^3$), the affine quadric $Q \setminus E$ is given by the following equation in a suitable real coordinate system

$$(-1)^{\varepsilon_1}x^2 + (-1)^{\varepsilon_2}y^2 + (-1)^{\varepsilon_3}z^2 = 1 \qquad \text{with } \varepsilon_i \in \{0,1\}. \tag{2}$$

The trace S_Q of Q on $i^{\varepsilon_1}\mathbb{R} \times i^{\varepsilon_2}\mathbb{R} \times i^{\varepsilon_3}\mathbb{R}$ is the unit 2-sphere, and is real Lagrangian in $Q \setminus E$. Furthermore, it realizes the class $\pm(l_1 - l_2) \in H_2(Q;\mathbb{Z})$ when endowed with some orientation. Different choices of $\varepsilon_1, \varepsilon_2$, and ε_3 provide four different real structures on the pair (Q, E), see Figure 2:

- $\tau_{S^1,0}$: $\mathbb{R}E \neq \emptyset$, and $\mathbb{R}(Q \setminus E)$ is a one-sheeted hyperboloid;
- $\tau_{S^1,2}$: $\mathbb{R}E \neq \emptyset$, and $\mathbb{R}(Q \setminus E)$ is a two-sheeted hyperboloid;
- $\tau_{\emptyset,0}$: $\mathbb{R}E = \emptyset$, and $\mathbb{R}Q = \emptyset$;
- $\tau_{\emptyset,2}$: $\mathbb{R}E = \emptyset$, and $\mathbb{R}Q = S^2$ is an ellipsoid.



$x^2 + y^2 - z^2 = 1$ $\qquad$ $-x^2 - y^2 + z^2 = 1$ $\qquad$ $-x^2 - y^2 - z^2 = 1$ $\qquad$ $x^2 + y^2 + z^2 = 1$

a) $\tau_{S^1,0}$ $\qquad$ b) $\tau_{S^1,2}$ $\qquad$ c) $\tau_{\emptyset,0}$ $\qquad$ d) $\tau_{\emptyset,2}$

Figure 2. Real structures on (Q, E)

For each real structure $\tau_{A,a}$ we have

$$\mathbb{R}E = A \qquad \text{and} \qquad \chi(\mathbb{R}Q) = a.$$

Note that the two real structures $\tau_{A,0}$ and $\tau_{A,2}$ on the pair (Q, E) differ one from the other by the composition with the automorphism $(x, y, z) \mapsto (-x, -y, -z)$ of $\mathbb{C}^3$, i.e. the two corresponding equations of the form (2) are obtained one from the other by the change of variables $(x, y, z) \mapsto (ix, iy, iz)$.

2.2. **Surgery along a real Lagrangian sphere.** Let $X_{\mathbb{R}}$ be a real symplectic 4-manifold containing a real Lagrangian sphere S. Recall that this means that S is globally invariant under τ_X. It is proved in [Teh13, Proposition 2.1 and Lemma 2.14] that $X_{\mathbb{R}}$ is deformation equivalent to the equivariant symplectic sum of two real symplectic 4-manifolds $Z_{\mathbb{R}} = (Z, \omega_Z, \tau_Z)$ and a real quadric $(Q, \omega_{FS}, \tau_{A,a})$ along an embedded symplectic sphere E of self-intersection -2 in Z (hence of self-intersection 2 in Q) where:

- $Z_{\mathbb{R}}$ is a symplectic reduction of X with a small neighborhood of S removed;
- E is a real hyperplane section of Q;
- S is a Lagrangian deformation of a real Lagrangian sphere S_Q in $(Q \setminus E, \omega_{FS}, \tau_{A,a})$.

Remark 2.1. Audin proposed in [Aud07] an earlier non-equivariant version of this construction. Welschinger proposed in [Wel07] an equivalent approach using symplectic field theory.

From an algebraic geometric perspective, the degeneration of $X_{\mathbb{R}}$ to the union of $Z_{\mathbb{R}}$ and $(Q, \omega_{FS}, \tau_{A,a})$ can be thought as a degeneration of $X_{\mathbb{R}}$ to a real nodal symplectic manifold for which $[S]$ is precisely the vanishing cycle, see Example 2.4 below. From this point of view, one may think of $Z_{\mathbb{R}}$ as obtained from $X_{\mathbb{R}}$ by contracting S to a point and then blowin-up the resulting ordinary double point.

Definition 2.2. *We say that a real symplectic manifold $Y_{\mathbb{R}}$ is a* surgery of $X_{\mathbb{R}}$ along the real Lagrangian sphere S *if it is deformation equivalent to the equivariant symplectic sum of*

$$Z_{\mathbb{R}} = (Z, \omega_Z, \tau_Z) \quad \text{and} \quad (Q, \omega_{FS}, \tau_{A,2-a}).$$

Phrased differently, the real manifold $Y_{\mathbb{R}}$ is obtained from $X_{\mathbb{R}}$ by changing the symplectic and real structures of $X_{\mathbb{R}}$ in a neighborhood of S. Since all surgeries of $X_{\mathbb{R}}$ along the real Lagrangian sphere S are deformation equivalent, and since we are interested in properties that are invariants under deformation, we say that $Y_{\mathbb{R}}$ is *the* surgery of $X_{\mathbb{R}}$ along the real Lagrangian sphere S rather than *a* surgery. By extension, we will also say that $Y_{\mathbb{R}}$ is obtained from $Z_{\mathbb{R}}$ by a surgery along E. The notation $Y_{\mathbb{R}} \xrightarrow{S} X_{\mathbb{R}}$ means that $X_{\mathbb{R}}$ and $Y_{\mathbb{R}}$ are related by a surgery along the real Lagrangian sphere S, and that $\chi(\mathbb{R}Y) = \chi(\mathbb{R}X) + 2$. In this case, the real part $\mathbb{R}Y$ is obtained from $\mathbb{R}X$ by one of the following topological operations:

- if $A = S^1$: cut $\mathbb{R}X$ along $\mathbb{R}S = S^1$ and glue a disk to each boundary circle (from Figure 2a to Figure 2b);
- if $A = \emptyset$: the sphere S which contains no real point of $\mathbb{R}X$ becomes a connected component of $\mathbb{R}Y$ (from Figure 2c to Figure 2d).

Note that both symplectic manifolds (X, ω_X) and (Y, ω_Y) are a deformation of (Z, ω_Z). However the real manifold $X_{\mathbb{R}}$ is a real deformation of $Z_{\mathbb{R}}$ if and only if $\chi(\mathbb{R}X) = \chi(\mathbb{R}Y) - 2$.

Example 2.3. A real quadric hyperboloid in $\mathbb{C}P^3$ is obtained from a real quadric ellipsoid by a surgery along a real Lagrangian sphere intersecting the real part in two points (see Figures 2a and 2b). A real empty quadric in $\mathbb{C}P^3$ is obtained from a real quadric ellipsoid by a surgery along the real part (see Figures 2c and 2d).

Example 2.4. More generally, let $D \subset \mathbb{C}$ be a small disk endowed with its standard real structure, and let $\pi : \mathcal{X} \to D$ be a flat real morphism from a non-singular real algebraic manifold of complex dimension 3. Suppose that the fiber $X_t = \pi^{-1}(t)$ is a non-singular projective algebraic surface when $t \neq 0$, and is a real algebraic surface with a single non-degenerate double point p as only singularity when $t = 0$. Suppose in addition that $\mathcal{X}$ is locally given at p by the equation

$$x^2 + y^2 \pm z^2 = t \qquad (x, y, z, t) \in \mathbb{C}^4,$$

and that π is locally given by $\pi(x, y, z, t) = t$. Let us perform respectively the base changes $t \to t^2$ and $t \to -t^2$. Then the blow-up at the node of the two obtained families realize the two symplectic sums described above. In particular the real algebraic surface $Z_\mathbb{R}$ is simply the blow-up of the singular surface X_0 at the node. Hence X_t and X_{-t} are obtained one from the other by a surgery along a real Lagrangian sphere S realizing the vanishing cycle of the degeneration of X_t to X_0.

Example 2.5. Let $X_\mathbb{R}$ be a non-singular real cubic surface in $\mathbb{C}P^3$ with a real part consisting of the disjoint union of a projective plane $\mathbb{R}P^2$ and a sphere S. It is classical that the underlying complex algebraic surface is the complex projective plane $\mathbb{C}P^2$ blown up at 6 points in general position (see for example [Dol12]). Note however that since $\mathbb{R}X$ has two connected components, the real surface $X_\mathbb{R}$ is not rational over $\mathbb{R}$, and is not obtained as a blow-up of the real projective plane.

By Example 2.4, the surgery of $X_\mathbb{R}$ along S is a real algebraic cubic surface whose real part is homeomorphic to $\mathbb{R}P^2$, and it is classical that it is the blow-up of the real projective plane at three pairs of complex conjugated points (see for example [Man86, Kol97, DK02]).

Lemma 2.6. *Suppose that* $Y_\mathbb{R} \xrightarrow{S} X_\mathbb{R}$. *Then the class* $[S]$ *is in* $H_2^{-\tau_X}(X;\mathbb{Z})$ *and in* $H_2^{\tau_Y}(X;\mathbb{Z})$. *Furthermore we have*

$$H_2^{-\tau_Y}(X;\mathbb{Z}) \subset [S]^\perp \qquad \textit{and} \qquad H_2^{-\tau_X}(X;\mathbb{Q}) = H_2^{-\tau_Y}(X;\mathbb{Q}) \oplus \mathbb{Q}[S].$$

Proof. The first claim follows from the fact that the analogous statement holds for affine quadrics. Let $\gamma \in H_2^{-\tau_Y}(X;\mathbb{Z})$. Since $[S]^2 = -2$, the sphere S realizes a non-trivial class in $H_2(X;\mathbb{Q})$ and we have

$$\gamma = \gamma_\perp + q[S] \quad \text{in } H_2(X;\mathbb{Q}) \quad \text{with } \gamma_\perp \in [S]^\perp \text{ and } q \in \mathbb{Q}.$$

Since $\tau_{Y*}(\gamma) = -\gamma$, we have

$$2q = -q[S]\cdot[S] = -\gamma\cdot[S] = \tau_{Y*}(\gamma)\cdot[S] = q[S]\cdot[S] = -2q,$$

from which we deduce that $q = 0$, that is to say $\gamma \in [S]^\perp$. Since the restrictions of τ_{X*} and τ_{Y*} coincide on $[S]^\perp$, we obtain that $H_2^{-\tau_X}(X;\mathbb{Q}) = H_2^{-\tau_Y}(X;\mathbb{Q}) \oplus \mathbb{Q}[S]$. ☺

Since the first Chern class of X is τ_X-anti-invariant for any real structure τ_X on (X, ω_X), we have in particular

$$c_1(X)\cdot[S] = 0.$$

(this also follows from the adjunction formula.)

Considering homology with coefficients in $\mathbb{Z}/2\mathbb{Z}$, the previous lemma can be weakened as follows.

Lemma 2.7. *Suppose that* $Y_\mathbb{R} \xrightarrow{S} X_\mathbb{R}$. *Then we have*

$$H_2^{\tau_X}(X;\mathbb{Z}/2\mathbb{Z}) \cap [S]^\perp = H_2^{\tau_Y}(X;\mathbb{Z}/2\mathbb{Z}) \cap [S]^\perp.$$

3. Welschinger invariants

In this section we define absolute and relative Welschinger invariants considered in this paper, and we prove Theorems 3.10 and 3.13 that relate such invariants for two real symplectic 4-manifolds differing by a surgery along a real Lagrangian sphere. Welschinger invariants of symplectic 4-manifolds are up to now only defined in the case of rational curves, nevertheless Shustin proposed in [Shu14] a partial generalisation to positive genus in the case of algebraic del Pezzo surfaces. Our proof of Theorems 3.10 and 3.13 extends to enumeration of curves of any genus, hence we decided to state both theorems for (hypothetical if $g > 0$) Welschinger invariants of any genus defined in the same vein as in [Shu14]. The proof from [Shu14] should be adaptable to the symplectic setting in the obvious way using the strategy proposed in [Wel05b] (including the correction from [Wel15]). Doing so would nevertheless bring us quite far from our original purposes, so we leave the existence of of Welschinger invariants of positive genus considered in this text as an hypothesis.

3.1. **Preliminaries.** We start by proving a simple adaptation of [BP15, Lemma 3.1 and Proposition 3.3] that we will use at several places in the rest of this section.

Let (X, ω_X) be a compact symplectic manifold of dimension 4, containing a finite union $W = E_1 \cup \ldots \cup E_\kappa$ of pairwise disjoint embedded symplectic spheres with $[E_i]^2 = -2$. Let also J be an almost complex structure on X tamed by ω_X for which all curves $E_1, \ldots, E_\kappa$ are J-holomorphic. It is classical that such J exists, see for example [Wen18, Propposition 2.2]. Given $d \in H_2(X;\mathbb{Z})$ and $g \in \mathbb{Z}_{\geq 0}$, let us choose a configuration $\underline{x}$ of $c_1(X) \cdot d + g - 1$ distinct points in $X \setminus \bigcup_{i=1}^{\kappa} E_i$. We define $\mathcal{C}^{\mathbb{C}}(d, g, \underline{x}, W, J)$ as the set of irreducible J-holomorphic curves $f : C \to X$ of genus g, with $f_*[C] = d$, passing through all points in $\underline{x}$, and whose image is not contained in $\bigcup_{i=1}^{\kappa} E_i$. Such a J-holomorphic curve $f : C \to X$ is said to be *nodal* if all singularities of $f(C)$, if any, are transverse self-intersections.

Lemma 3.1. *Suppose that $c_1(X) \cdot d > 0$ if $g = 1$. Then for a generic choice of J among almost complex structure J tamed by ω_X such that all symplectic curves $E_1, \ldots, E_\kappa$ are J-holomorphic, the set $\mathcal{C}^{\mathbb{C}}(d, g, \underline{x}, W, J)$ is finite and composed of simple maps that are all nodal immersions.*

Proof. The proof consists in two steps: first we prove that no element of $\mathcal{C}^{\mathbb{C}}(d, g, \underline{x}, W, J)$ factors through a non-trivial ramified covering, from which we deduce the finiteness of $\mathcal{C}^{\mathbb{C}}(d, g, \underline{x}, W, J)$. Then all maps in $\mathcal{C}^{\mathbb{C}}(d, g, \underline{x}, W, J)$ are nodal immersions by [Wen18, Corollaries 2.26, 2.30, and 2.32, and Remark 2.17]. By perturbing J in the complement of a small neighborhood of W if necessary, we may assume that for any class $d_0 \in H_2(X;\mathbb{Z})$ and any J-holomorphic simple map $f_0 : C_0 \to X$ such that $f_{0*}[C_0] = d_0$, the curve C_0 has genus g_0, and $\underline{x} \subset f(C_0)$, we have

$$c_1(X) \cdot d_0 + g_0 - 1 \geq c_1(X) \cdot d + g - 1, \tag{3}$$

see for example [Wen18, Corollary 2.23 and Remark 2.17].

Step 1. Let $f : C \to X$ be an element of $\mathcal{C}^{\mathbb{C}}(d, g, \underline{x}, W, J)$ that factors through a ramified covering of degree $\delta \geq 2$ of a simple map $f_0 : C_0 \to X$. Denoting by g_0 the genus of C_0, we obtain by the Riemann-Hurwitz formula that

$$g \geq \delta g_0 + 1 - \delta. \tag{4}$$

Let $d_0 \in H_2(X;\mathbb{Z})$ denotes the class $f_{0*}[C_0]$. Since $d = \delta d_0$, Inequality (3) becomes

$$(\delta - 1)c_1(X) \cdot d_0 + g - g_0 \leq 0.$$

Combining this with (4), we obtain

$$(\delta - 1)(c_1(X) \cdot d_0 + g_0 - 1) \leq 0,$$

and so

$$c_1(X) \cdot d_0 + g_0 - 1 \leq 0.$$

By [Wen18, Corollary 2.23 and Remark 2.17], we also have the opposite inequality, which alltogether gives

$$c_1(X) \cdot d_0 + g_0 - 1 = 0. \tag{5}$$

Furthermore, all inequalities above are in fact equalities. In particular, the covering $C \to C_0$ through which f factors is non-ramified, which is possible only if $g = g_0 = 1$. In this case (5) gives $c_1(X) \cdot d = c_1(X) \cdot d_0 = 0$ which is excluded by assumption.

Step 2. Suppose that $\mathcal{C}^{\mathbb{C}}(d, g, \underline{x}, W, J)$ contains infinitely many simple maps. By Gromov compactness Theorem, there exists a sequence $(f_n)_{n\geq 0}$ of distinct simple maps in $\mathcal{C}^{\mathbb{C}}(d, g, \underline{x}, W, J)$ which converges to some J-holomorphic map $\overline{f} : \overline{C} \to X$. The genericity of J implies that the set of simple maps in $\mathcal{C}^{\mathbb{C}}(d, g, \underline{x}, W, J)$ is a 0-dimensional manifold, and in particular is discrete (see [Wen18, Theorem 2.21 and Remark 2.17]). Hence either $\overline{C}$ is reducible, or $\overline{f}$ is non-simple. Let $\overline{C}_1, \ldots, \overline{C}_m, \overline{C}'_1, \ldots, \overline{C}'_{m'}$ be the irreducible components of $\overline{C}$, labeled in such a way that

- $\overline{f}(\overline{C}_i) \not\subset W$ for any $i \in \{1, \cdots, m\}$;
- $\overline{f}(\overline{C}'_i) \subset W$, and $\overline{f}_{|\overline{C}'_i}$ factors through a ramified covering of degree k_i, for any $i \in \{1, \cdots, m'\}$.

Define $k = \sum_{i=1}^{m'} k_i$, and denote by g_i the genus of C_i. The restriction of $\overline{f}$ to $\bigcup_{i=1}^{m} \overline{C}_i$ is subject to $c_1(X) \cdot d + g - 1$ points conditions, so we have

$$c_1(X) \cdot d_1 + \sum_{i=1}^{m} g_i - m \geq c_1(X) \cdot d + g - 1, \tag{6}$$

where d_1 is the class realized by the image of this restriction. Since an irreducible component E of W is an embedded sphere with self-intersection -2, the adjunction formula implies that $c_1(X) \cdot [E] = 0$. Hence we get $c_1(X) \cdot d_1 = c_1(X) \cdot d$. Combining with (6) we obtain

$$\sum_{i=1}^{m} g_i - g + 1 - m \geq 0.$$

If $m = 0$, then the image of $\overline{f}$ must be contained in a curve in W, implying that $c_1(X) \cdot d = 0$. Since furthermore in this case we would have $c_1(X) \cdot d + g - 1 = 0$, we deduce that $g = 1$ contrary to our assumptions. Hence $m \geq 1$, and since $\sum_{i=1}^{m} g_i \leq g$, we deduce that $m = 1$ and $g_1 = g$. In particular, each irreducible component $\overline{C}'_i$ of $\overline{C}$ is rational and intersects $\overline{C}_1$ in a single point.

If $k = m' = 0$, then the curve $\overline{C}$ is irreducible. Hence as explained above, the map $\overline{f}$ has to factorize through a non-trivial ramified covering of a simple map $f_0 : C_0 \to X$, which contradicts Step 1. Hence we have $k > 0$. By genericity of J, the curve $\overline{f}(\overline{C}_1)$ is fixed by the $c_1(X) \cdot d + g - 1$ point constraints in X. By perturbing J in a neighborhood of W if necessary, we may assume that $\overline{f}(\overline{C}_1)$ intersects the curve W transversely. Any intersection point of $\overline{f}(\overline{C}_1 \setminus (\overline{C}'_1 \cup \ldots \cup \overline{C}'_{m'}))$ and W deforms to an intersection point of the image of f_n and W for $n >> 1$. Since $d_1 \cdot [W] = d \cdot [W] + 2k$ and $m' \leq k$, at least $d \cdot [W] + k$ intersection points of $\overline{f}(\overline{C}_1)$ and W deform to an intersection point of the image of f_n and W for $n >> 1$. But this contradicts the fact that two J-holomorphic curves intersect positively. ☺

Remark 3.2. Lemma 3.1 has an obvious equivariant version when (X, ω) is equipped with an anti-symplectic involution τ for which W is τ-anti-invariant. The proof is by adapting the proof of Lemma 3.1 following the proof of [Wel05b, Theorem 1.10] in the genus 0 case.

3.2. **Absolute Welschinger invariants.** Recall that if C is an irreducible compact non-singular real algebraic curve of genus g, then the set $\mathbb{R}C$ has at most $g+1$ connected components by the Harnack-Klein inequality. The real curve C is called *maximal* when equality holds. In this case, the set $C \setminus \mathbb{R}C$ has two connected components. The following lemma is an immediate consequence of [Man17, Lemme 3.6.22] and the Smith exact sequence.

Lemma 3.3. *Let $X_{\mathbb{R}}$ be a connected real symplectic 4-manifold with $b_1(X; \mathbb{Z}/2\mathbb{Z}) = 0$ and $\mathbb{R}X \neq \emptyset$, and let $\Gamma \in H_2(X, \mathbb{R}X; \mathbb{Z}/2\mathbb{Z})$. Then the image of $\partial\Gamma \in H_1(\mathbb{R}X; \mathbb{Z}/2\mathbb{Z})$ only depends on the class $\Gamma + \tau_{X,*}(\Gamma) \in H_2(X; \mathbb{Z}/2\mathbb{Z})$. In particular, any class $d \in H_2^{\tau_X}(X; \mathbb{Z}/2\mathbb{Z})$ induces a class $l_d \in H_1(\mathbb{R}X; \mathbb{Z}/2\mathbb{Z})$.*

Given a connected component L of $\mathbb{R}X$, we denote by $l_{L,d}$ the natural projection of the class l_d to $H_1(L; \mathbb{Z}/2\mathbb{Z})$.

For the rest of this section we fix once for all an integer $g \geq 0$, and $X_{\mathbb{R}} = (X, \omega_X, \tau_X)$ a real compact symplectic manifold of dimension 4. If $g > 0$, we furthermore assume that $b_1(X; \mathbb{Z}/2\mathbb{Z}) = 0$.

Suppose that $\mathbb{R}X$ contains $g+1$ connected components denoted by $L_1, \ldots, L_{g+1}$, and define $L = \bigcup_{i=1}^{g+1} L_i$. Note that $\mathbb{R}X$ might contain other connected components.

We say that an almost complex structure J tamed by ω_X is τ_X-compatible if τ_X is J-antiholomorphic, i.e. $J \circ d\tau_X = -d\tau_X \circ J$. Recall that there exists a well defined pairing

$$H_2(X, L; \mathbb{Z}/2\mathbb{Z}) \times H_2(X \setminus L; \mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$$

given by the intersection product modulo 2. Let C be a maximal irreducible real algebraic curve, and $f : C \to X$ be a real J-holomorphic nodal immersion such that $f(\mathbb{R}C) \subset L$, for some τ_X-compatible almost complex structure J on X. Denoting by C^+ the topological closure of one of the halves of $C \setminus \mathbb{R}C$, and given $F \in H_2^{\tau_X}(X \setminus L; \mathbb{Z}/2\mathbb{Z})$, we define the (L, F)-mass of f as

$$m_{L,F}(f) = m(f) + [f(C^+)] \cdot F,$$

where $m(f)$ is the number of elliptic real nodes of $f(C)$ (i.e. real nodes with two τ_X-conjugated branches) contained in L. Note that $m_{L,F}(f)$ does not depend on the chosen half of $C \setminus \mathbb{R}C$.

Example 3.4. If $F = [\mathbb{R}X \setminus L]$, then $m_{L,F}(f)$ is the total number of elliptic real nodes of $f(C)$.

Choose a class $d \in H_2^{-\tau_X}(X; \mathbb{Z})$, and $\underline{r} = (r_1, \ldots r_{g+1}) \in \mathbb{Z}_{\geq 0}^{g+1}$ and $s \in \mathbb{Z}_{\geq 0}$ such that

$$c_1(X) \cdot d + g - 1 = \sum_{i=1}^{g+1} r_i + 2s.$$

We furthermore assume that the following holds

$$\text{either} \qquad g = 0 \qquad \text{or} \qquad r_i = l_{L_i,d}^2 + 1 \mod 2 \qquad \forall i \in \{1, \ldots, g+1\}. \tag{7}$$

Choose a configuration $\underline{x}$ made of r_i points on each L_i for $i \in \{1, \ldots, g+1\}$, and s pairs of τ_X-conjugated points in $X \setminus \mathbb{R}X$. Given a τ_X-compatible almost complex structure J, we denote by $\mathcal{C}(d, \underline{x}, L, J)$ the set of irreducible real J-holomorphic curves $f : C \to X$ of genus g in X, with $f_*[C] = d$, passing through $\underline{x}$, and such that $f(\mathbb{R}C) \subset L$. It follows from (7) that given $f : C \to X \in \mathcal{C}(d, \underline{x}, L, J)$, each component L_i contains a connected component of $f(\mathbb{R}C)$, and so C is a maximal real curve. Furthermore according to Lemma 3.1, if $c_1(X)\cdot d>0$ when $g=1$, then the set $\mathcal{C}(d,\underline{x},L,J)$ is finite and composed of nodal immersions for a choice of J that is generic with respect to all choices made above.

Definition 3.5. *Let $F\in H_2^{\tau_X}(X\setminus L;\mathbb{Z}/2\mathbb{Z})$. We say that* Welschinger invariants *exist for the triple $(X_\mathbb{R},L,F)$ if the integer*

$$W_{X_\mathbb{R},L,F}(d;\underline{r},s)=\sum_{C\in\mathcal{C}(d,\underline{x},L,J)}(-1)^{m_{L,F}(C)}$$

depends neither on $\underline{x}$, J, nor on the deformation class of $X_\mathbb{R}$ as soon as $c_1(X)\cdot d\neq 0$ when $g=1$.

Note that the notation $W_{X_\mathbb{R},L,F}(d;\underline{r},s)$ contains the information about the genus g of the curves under enumeration: it is the number of connected components of L minus 1. When L is connected (i.e. $g=0$) we simply denote $W_{X_\mathbb{R},L,F}(d;s)$ rather than $W_{X_\mathbb{R},L,F}(d;\underline{r},s)$. In this case, Welschinger invariants always exist.

Theorem 3.6 ([Wel05b, Wel15, IKS17])**.** *Let $X_\mathbb{R}$ be a real compact symplectic manifold of dimension 4 with $\mathbb{R}X\neq\emptyset$. Then Welschinger invariants exist for* any *triple $(X_\mathbb{R},L,F)$ with L a connected component of $\mathbb{R}X$ and $F\in H_2^{\tau_X}(X\setminus L;\mathbb{Z}/2\mathbb{Z})$.*

These invariants were first defined and shown to exist by Welschinger in [Wel05b] for rational curves and when $F=[\mathbb{R}X\setminus L]$ (see also the correction from [Wel15] concerning the appearance of embedded J-holomorphic spheres with self-intersection -2 in the proof of [Wel05b, Theorem 0.1]). Welschinger's seminal work has been generalizsed by Itenberg, Kharlamov and Shustin to any F in [IKS17], and by Shustin to any g in [Shu14] for real algebraic del Pezzo surfaces. Thanks to Lemma 3.1, it should be possible to adapt in the obvious way the proof of [Shu14] in the strategy proposed in [Wel05b] (including the correction from [Wel15]) in order to prove the existence of Welschinger invariants for any triple $(X_\mathbb{R},L,F)$ (i.e. for curves of higher genus): the assumption on d prevent the appearance of non-trivial real ramified coverings, while condition (7) should prevent the appearance of real immersions that should be counted with multiplicity two or more, along a generic path of τ_X-compatible almost complex structures. Nevertheless, there is a certain amount of technical details to check that this is indeed the case, which is not the purpose of this paper.

Remark 3.7. Among all possible choices of F in $H_2^{\tau_Y}(X\setminus L;\mathbb{Z}/2\mathbb{Z})$, the two classes 0 and $[\mathbb{R}X\setminus L]$ seem to play a special role, at least in genus 0. In this case, any Welschinger invariant $W_{X_\mathbb{R},L,F}(d;s)$ either vanishes or is equal in absolute value to $W_{X_\mathbb{R},L,0}(d;s)$ as soon as $c_1(X)\cdot d-1-2s\geq 2$ and $X_\mathbb{R}$ is deformation equivalent to a real rational algebraic surface, see [BP15]. On the other hand, the invariant $W_{X_\mathbb{R},L,[\mathbb{R}X\setminus L]}(d;s)$ turns out to be sharp in many situation when $c_1(X)\cdot d-1-2s\leq 1$, see [Wel07]. Furthermore, we do not know yet any situation where $W_{X_\mathbb{R},L,[\mathbb{R}X\setminus L]}(d;s)$, with $c_1(X)\cdot d-1-2s\leq 1$, is not maximal in absolute value when F ranges over $H_2^{\tau_Y}(X\setminus L;\mathbb{Z}/2\mathbb{Z})$.

Next theorem, the main result of this paper, provides surprisingly very simple relations among Welschinger invariants of real symplectic 4-manifolds differing by a special kind of surgery. Its proof involves relative Welschinger invariants defined in next section, and is postponed until Section 3.5.4.

Theorem 3.8. *Let $X_\mathbb{R}$ be a compact real symplectic manifold of dimension 4, and let S be a real Lagrangian sphere in $X_\mathbb{R}$, endowed with some orientation, realizing a τ_X-anti-invariant class in $H_2(X;\mathbb{Z})$. We denote by $Y_\mathbb{R}$ the surgery of $X_\mathbb{R}$ along S.*

Let also L be the union of some connected components of $\mathbb{R}X$ that is disjoint from S, and $F \in H_2^{\tau_Y}(X \setminus L; \mathbb{Z}/2\mathbb{Z})$ be a class orthogonal to $[S]$ such that Welschinger invariants exist for both triples $(X_{\mathbb{R}}, L, F)$ and $(Y_{\mathbb{R}}, L, F)$. Then for any class $d \in H_2^{-\tau_Y}(X;\mathbb{Z})$, we have

$$W_{Y_{\mathbb{R}},L,F}(d;\underline{r},s) = W_{X_{\mathbb{R}},L,F}(d;\underline{r},s) + 2\sum_{k\geq 1} W_{X_{\mathbb{R}},L,F}(d-k[S];\underline{r},s)$$

whenever $\underline{r}$ and s are such that the invariant $W_{Y_{\mathbb{R}},L,F}(d;\underline{r},s)$ is defined.

The fact that F is orthogonal to $[S]$ in $H_2(X;\mathbb{Z}/2\mathbb{Z})$ ensures that all invariants

$$W_{X_{\mathbb{R}},L,F}(d-k[S];\underline{r},s)$$

are also defined as soon as $W_{Y_{\mathbb{R}},L,F}(d;\underline{r},s)$ is defined (which is always the case if $g=0$), see Lemma 2.7.

3.3. **Relative invariants.** As above, we choose $d \in H_2^{-\tau_X}(X;\mathbb{Z})$, and $\underline{r} = (r_1, \cdots, r_{g+1}) \in \mathbb{Z}_{\geq 0}^{g+1}$ and $s \in \mathbb{Z}_{\geq 0}$ such that

$$c_1(X)\cdot d + g - 1 = \sum_{i=1}^{g+1} r_i + 2s.$$

Let also $U = \{E_1, \dots E_\kappa\}$ and $V = \{E'_1, \dots E'_\lambda\}$ be two finite (maybe empty) sets of real embedded symplectic spheres in $X_{\mathbb{R}}$ such that for all $i,j \in \{1,\dots,\kappa\}$ and $u,v \in \{1,\dots,\lambda\}$, we have:

- $[E_i]^2 = [E'_u]^2 = -2$,
- $E_i \cap E_j = E'_u \cap E'_v = E_i \cap E'_u = \emptyset$ if $i \neq j$ and $u \neq v$,
- $\mathbb{R}E'_i \neq \emptyset$,
- $d \cdot [E'_u] = 0$,
- $\sum_{u=1}^{\lambda} [\mathbb{R}E'_u] = 0 \in H_1(\mathbb{R}X;\mathbb{Z}/2\mathbb{Z})$.

Let $L_1, \dots, L_{g+1}$ be the topological closure of $g+1$ connected components of $\mathbb{R}X \setminus \bigcup_{u=1}^{\lambda} \mathbb{R}E'_u$ such that

$$L_i \cap L_j = \emptyset \quad \text{if } i \neq j \qquad \text{and} \qquad \left(\bigcup_{i=1}^{g+1} L_i\right) \cap \left(\bigcup_{i=1}^{\kappa} \mathbb{R}E_i\right) = \emptyset.$$

Denote $L = \bigcup_{i=1}^{g+1} L_i$. Let also L_0 be the topological closure of the union of some connected components of $\mathbb{R}X \setminus \bigcup_{u=1}^{\lambda} \mathbb{R}E'_u$ such that

$$L_0 \cap L = \emptyset \qquad \text{and} \qquad \partial(L_0 \cup L) = \bigcup_{u=1}^{\lambda} \mathbb{R}E'_u.$$

In particular, the last condition implies that each circle $\mathbb{R}E'_v$ is contained in the boundary of a connected component of $L_0 \cup L$ and of a connected component of $\mathbb{R}X \setminus (L_0 \cup L)$. Given C a real symplectic curve and $f : C \to X$ a real symplectic immersion, we denote by $l_{L_i, f_*[C]}$ the image of $l_{\mathbb{R}X, f_*[C]}$ by the natural map $H_1(\mathbb{R}X;\mathbb{Z}/2\mathbb{Z}) \to H_1(L_i, \partial L_i;\mathbb{Z}/2\mathbb{Z})$. We still assume that (7) holds.

Choose a configuration $\underline{x}$ made of r_i points on each L_i for $i \in \{1, \cdots, g+1\}$, and s pairs of τ_X-conjugated points in $X \setminus \mathbb{R}X$. Given a τ_X-compatible almost complex structure J such that all symplectic curves in $U \cup V$ are J-holomorphic, we denote by $\mathcal{C}(d, \underline{x}, L, U, V, J)$ the set of irreducible real J-holomorphic curves $f : C \to X$ in X of genus g, with $f_*[C] = d$, whose image is not contained in $U \cup V$, passing through $\underline{x}$, and such that $f(\mathbb{R}C) \subset L$. Given $f : C \to X \in \mathcal{C}(d, \underline{x}, L, U, V, J)$, condition (7) forces each component L_i to contain a connected component of $f(\mathbb{R}C)$, and so C is a maximal real curve. Furthermore according to Lemma 3.1, if $c_1(X) \cdot d > 0$ in the case $g = 1$, then the set $\mathcal{C}(d, \underline{x}, L, U, V, J)$ is finite and composed of simple nodal immersions for a choice of J that is generic with respect to all choices made above.

Let $f : C \to X$ be an element of $\mathcal{C}(d, \underline{x}, L, U, V, J)$, and choose C^+ to be the topological closure of one of the halves of $C \setminus \mathbb{R}C$. As in the case of absolute invariants, given $F \in H_2^{\tau_X}(X \setminus L; \mathbb{Z}/2\mathbb{Z})$ we define the $(L \cup L_0, F)$-mass of f as

$$m_{L\cup L_0,F}(f) = m(f) + [f(C^+)] \cdot F,$$

where $m(f)$ is the number of real elliptic nodes of $f(C)$ in $L \cup L_0$. Again, $m_{L\cup L_0,F}(f)$ does not depend on the chosen half of $C \setminus \mathbb{R}C$.

Let $\overline{X}_{\mathbb{R}}$ be the successive real surgeries of $X_{\mathbb{R}}$ along the curves $E'_1, \dots, E'_\lambda$, and $\overline{L}$ and $\overline{L}_0$ be the union of the connected components of $\mathbb{R}\overline{X}$ obtained by gluing disks respectively to the boundary of L and L_0. Next theorem is a consequence of Corollary 3.11 and Theorem 3.13 that we prove in next sections.

Theorem 3.9. *Let $F \in H_2^{\tau_X}(X \setminus L; \mathbb{Z}/2\mathbb{Z})$ orthogonal in $H_2(X; \mathbb{Z}/2\mathbb{Z})$ to all classes realized by the curves in V. If Welschinger invariants exist for the triple $(\overline{X}_{\mathbb{R}}, \overline{L}, F + [\overline{L}_0])$, then the integer*

$$W^{U,V}_{X_{\mathbb{R}},L,L_0,F}(d; \underline{r}, s) = \sum_{C \in \mathcal{C}(d,\underline{x},L,U,V,J)} (-1)^{m_{L\cup L_0,F}(C)}$$

depends neither on $\underline{x}$, J, nor on the deformation class of the 5-tuple $(X_{\mathbb{R}}, L, L_0, U, V)$.

When the conclusion of Theorem 3.9 holds, we call the numbers $W^{U,V}_{X_{\mathbb{R}},L,L_0,F}$ *Welschinger invariants of $X_{\mathbb{R}}$ relative to the pair (U, V).*

3.4. **Surgery and enumerative geometry.** We keep using notations and choices introduced in Section 3.3.

3.4.1. Suppose that $U \neq \emptyset$, and let $E \in U$. We denote by $\widehat{U} = U \setminus E$, and by $Y_{\mathbb{R}}$ the surgery of $X_{\mathbb{R}}$ along E. If $\mathbb{R}E \neq \emptyset$, recall that $\mathbb{R}Y$ is obtained topologically by cutting $\mathbb{R}X$ along $\mathbb{R}E$ and gluing back a disk along each boundary circle. If $\mathbb{R}E \cap L_0 \neq \emptyset$, then we denote by $\widehat{L}_0$ the union of L_0 with the two glued disks. Otherwise we set $\widehat{L}_0 = L_0$. Next theorem is proved in Section 3.5.

Theorem 3.10. *We have*

$$W^{\widehat{U},V}_{X_{\mathbb{R}},L,L_0,F}(d; \underline{r}, s) = \sum_{k\geq 0} \binom{\frac{1}{2}d \cdot [E] + 2k}{k} W^{U,V}_{X_{\mathbb{R}},L,L_0,F}(d - 2k[E]; \underline{r}, s). \tag{8}$$

If furthermore $d \cdot [E] = 0$, and $F \in [E]^\perp$ in $H_2(X; \mathbb{Z}/2\mathbb{Z})$, then

$$W^{\widehat{U},V}_{Y_{\mathbb{R}},L,\widehat{L}_0,F}(d; \underline{r}, s) = \sum_{k\geq 0} 2^k\, W^{U,V}_{X_{\mathbb{R}},L,L_0,F}(d - k[E]; \underline{r}, s).$$

According to Lemma 2.7, we have $F \in H_2^{\tau_Y}(X \setminus L; \mathbb{Z}/2\mathbb{Z})$ if $F \in [E]^\perp$ in $H_2(X; \mathbb{Z}/2\mathbb{Z})$. In particular the number $W^{\widehat{U},V}_{Y_{\mathbb{R}},L,\widehat{L}_0,F}(d; \underline{r}, s)$ in Theorem 3.10 is well defined.

In next corollary, we use the convention that

$$\binom{-1}{0} = 0.$$

Corollary 3.11. *The number* $W^{U,V}_{X_{\mathbb{R}},L,L_0,F}(d;\underline{r},s)$ *can be expressed in terms of the numbers* $W^{\widehat{U},V}_{X_{\mathbb{R}},L,L_0,F}(d-2k[E];\underline{r},s)$ *with* $k\geq 0$*. More precisely, we have*

$$W^{U,V}_{X_{\mathbb{R}},L,L_0,F}(d;\underline{r},s)=\sum_{k\geq 0}(-1)^k\left(\binom{\frac{1}{2}d\cdot[E]+k}{\frac{1}{2}d\cdot[E]}+\binom{\frac{1}{2}d\cdot[E]+k-1}{\frac{1}{2}d\cdot[E]}\right)W^{\widehat{U},V}_{X_{\mathbb{R}},L,L_0,F}(d-2k[E];\underline{r},s).$$

Proof. Relation (8) expresses the numbers $W^{\widehat{U},V}_{X_{\mathbb{R}},L,L_0,F}(d-2k[E];\underline{r},s)$ in terms of the numbers $W^{U,V}_{X_{\mathbb{R}},L,L_0,F}(d-2(k+l)[E];\underline{r},s)$, $k,l\geq 0$. Since this is an upper triangular linear system with coefficients 1 on the diagonal, it is invertible. The exact expression of $W^{U,V}_{X_{\mathbb{R}},L,L_0,F}(d;\underline{r},s)$ in terms of the numbers $W^{\widehat{U},V}_{X_{\mathbb{R}},L,L_0,F}(d-2k[E];\underline{r},s)$ follows from Proposition 3.12. ☺

Proposition 3.12. *Let* $m\in\mathbb{Z}_{\geq 0}$ *and* $K\in\mathbb{Z}_{\geq 1}$*, and let* M *be the matrix*

$$M=\left(\binom{m+2(j-1)}{j-i}\right)_{1\leq i,j\leq K}.$$

Then the matrix M *is invertible and we have*

$$M^{-1}=\left((-1)^{i+j}\left(\binom{m+i+j-2}{m+2i-2}+\binom{m+i+j-3}{m+2i-2}\right)\right)_{1\leq i,j\leq K}.$$

Proof. Let $(\nu_{i,j}))_{1\leq i,j\leq K}$ be the product of the two above matrices. Then we have

$$\begin{aligned}\nu_{i,j}&=\sum_{k=1}^{K}(-1)^{k+j}\binom{m+2(k-1)}{k-i}\left(\binom{m+k+j-2}{m+2k-2}+\binom{m+k+j-3}{m+2k-2}\right)\\&=(-1)^j\ \sum_{k=i}^{j}(-1)^k\binom{m+2(k-1)}{k-i}\left(\binom{m+k+j-2}{m+2k-2}+\binom{m+k+j-3}{m+2k-2}\right).\end{aligned}$$

Defining $m'=m+2i-2$ and $I=j-i$, we get

$$\nu_{i,j}=(-1)^I\ \sum_{k=0}^{I}(-1)^k\binom{m'+2k}{k}\left(\binom{m'+I+k}{m'+2k}+\binom{m'+I+k-1}{m'+2k}\right).$$

Using the identity

$$\binom{u}{v}\binom{v}{w}=\binom{u}{w}\binom{u-w}{v-w}$$

we obtain

$$\nu_{i,j}=(-1)^I\ \sum_{k=0}^{I}(-1)^k\left(\binom{m'+I+k}{k}\binom{m'+I}{m'+k}+\binom{m'+I+k-1}{k}\binom{m'+I-1}{m'+k}\right).$$

In the case when $I=0$, this gives $\nu_{i,i}=1$. In the case when $I>0$ we have

$$\nu_{i,j}=(-1)^I\ \sum_{k=0}^{I}(-1)^k\left(\binom{m'+I+k}{m'+I}\binom{m'+I}{m'+k}+\binom{m'+I+k-1}{m'+I-1}\binom{m'+I-1}{m'+k}\right).$$

It follows from [GKP94, Identity 5.25] that

$$\sum_{k=0}^{I}(-1)^k\binom{m'+I+k}{m'+I}\binom{m'+I}{m'+k}=(-1)^I$$
$$=-\sum_{k=0}^{I}(-1)^k\binom{m'+I+k-1}{m'+I-1}\binom{m'+I-1}{m'+k}.$$

Hence $\nu_{i,j}=0$ if $i\neq j$, and the proposition is proved. ☺

3.4.2. Suppose that $V\neq\emptyset$, and let $E\in V$. We denote by $\widehat{V}=V\setminus E$, and by $Y_{\mathbb{R}}$ the surgery of $X_{\mathbb{R}}$ along E. Let L_i be the connected component of $L\cup L_0$ whose boundary contains $\mathbb{R}E$. Recall that $\mathbb{R}Y$ is obtained topologically by cutting $\mathbb{R}X$ along $\mathbb{R}E$ and gluing back a disk along each boundary circle. We define $\widehat{L}_j=L_j$ for $j\neq i$, and by $\widehat{L}_i$ the union of L_i with the corresponding glued disk. Accordingly, we define

$$\widehat{L}=\bigcup_{j=1}^{g+1}\widehat{L}_j.$$

Next theorem is proved in Section 3.5.

Theorem 3.13. *Let $F\in H_2^{\tau_X}(X\setminus L;\mathbb{Z}/2\mathbb{Z})$ orthogonal in $H_2(X;\mathbb{Z}/2\mathbb{Z})$ to all classes realized by the curves in V. Then we have*

$$W^{U,\widehat{V}}_{Y_{\mathbb{R}},\widehat{L},\widehat{L}_0,F}(d;\underline{r},s)=W^{U,V}_{X_{\mathbb{R}},L,L_0,F}(d;\underline{r},s).$$

The assumption of Theorem 3.13 ensures that the number $W^{U,\widehat{V}}_{Y_{\mathbb{R}},\widehat{L},\widehat{L}_0,F}(d;\underline{r},s)$ is well defined by Lemma 2.7.

3.5. Proof of Theorems 3.8, 3.9, 3.10, and 3.13.

3.5.1. *Symplectic sums.* Here we describe a very particular case of the symplectic sum formula from [IP04, TZ14]. Since we are working in this paper only with symplectic sums of 4-dimensional manifolds (which are in particular semi-positive) along spheres (for which the so-called S-matrix is the identity), none of the major issues with [IP04] raised in [TZ14] are relevant in our situation. Let (Z,ω_Z) be a compact and connected symplectic manifold of dimension 4 containing an embedded symplectic sphere E with $[E]^2=-2$. We furthermore assume the existence of a symplectomorphism ϕ from E to a symplectic curve realizing the class l_1+l_2 in $(\mathbb{C}P^1\times\mathbb{C}P^1,\omega_{FS}\oplus\omega_{FS})$. By abuse, we still denote by E the image $\phi(E)$ in $\mathbb{C}P^1\times\mathbb{C}P^1$. Since the self-intersection of E in $\mathbb{C}P^1\times\mathbb{C}P^1$ and Z are opposite, there exists a symplectic bundle isomorphism ψ between the normal bundle of E in $\mathbb{C}P^1\times\mathbb{C}P^1$ and the dual of the normal bundle of E in Z. Out of these data, one produces a family of symplectic 4-manifolds $(\mathcal{Z}_t,\omega_t)$ parametrized by a small complex number t in $\mathbb{C}^*$, see [Gom95]. All these manifolds are deformation equivalent, and are called *symplectic sums of* $(\mathbb{C}P^1\times\mathbb{C}P^1,\omega_{FS}\oplus\omega_{FS})$ *and* (Z,ω_Z) *along* E. This family can be seen as a symplectic deformation of the singular symplectic manifold $X_\sharp=Z\cup_E\left(\mathbb{C}P^1\times\mathbb{C}P^1\right)$ obtained by gluing (Z,ω_Z) and $(\mathbb{C}P^1\times\mathbb{C}P^1,\omega_{FS}\oplus\omega_{FS})$ along E.

Proposition 3.14 ([IP04, Theorem 2.1],[TZ14, Proposition 3.1])**.** *There exists a symplectic 6-manifold $(\mathcal{Z},\omega_{\mathcal{Z}})$ and a symplectic fibration $\pi:\mathcal{Z}\to D$ over a disk $D\subset\mathbb{C}$ such that the central fiber $\pi^{-1}(0)$ is the singular symplectic manifold $X_\sharp$, and $\pi^{-1}(t)=(\mathcal{Z}_t,\omega_t)$ for $t\neq 0$.*

Topologically, $\mathcal{Z}_t$ is simply the connected sum along E of Z with $\mathbb{C}P^1 \times \mathbb{C}P^1$. Since

$$\mathbb{C}P^1 \times \mathbb{C}P^1 \setminus E$$

is the normal bundle of E in Z, this connected sum is trivial and we have $\mathcal{Z}_t = Z$. Furthermore, without loss of generality we may assume that the homology class realized by E in its normal bundle is the restriction of the class $l_1 - l_2$ in $H_2(\mathbb{C}P^1 \times \mathbb{C}P^1; \mathbb{Z})$.

In addition to E, suppose that Z contains a collection $W = \{E_1, \ldots, E_\kappa\}$ of pairwise disjoint embedded symplectic spheres with $[E_i]^2 = -2$ that are all disjoint from E. Let $d \in H_2(\mathcal{Z}_t; \mathbb{Z})$, and choose $\underline{x}(t)$ a set of $c_1(X) \cdot d + g - 1$ symplectic sections

$$D \to \mathcal{Z} \setminus \bigcup_{i=1}^{\kappa} E_i$$

such that $\underline{x}(0) \cap \left(E \bigcup_{i=1}^{\kappa} E_i \right) = \emptyset$. Choose an almost complex structure J on $\mathcal{Z}$ tamed by $\omega_{\mathcal{Z}}$, which restricts to an almost complex structure J_t tamed by ω_t on each fiber $\mathcal{Z}_t$ and for which all curves $E_1, \ldots, E_\kappa$ are J_t-holomorphic. We assume that J is generic with respect to all choices we made. Recall that the set $\mathcal{C}^{\mathbb{C}}(d, g, \underline{x}(t), W, J_t)$ for $t \neq 0$ has been defined in Section 3.1. We define $\mathcal{C}^{\mathbb{C}}(d, g, \underline{x}(0), W, J_0)$ to be the set $\left\{\overline{f} : \overline{C} \to X_\sharp\right\}$ of limits, as stable maps, of maps in $\mathcal{C}^{\mathbb{C}}(d, g, \underline{x}(W, t), J_t)$ as t goes to 0. Recall (see [IP04, Section 3], or comments following [TZ14, Theorem 1.1]) that $\overline{C}$ is a connected nodal curve of arithmetic genus g such that:

- $\underline{x}(0) \subset \overline{f}(\overline{C})$;
- any point $p \in \overline{f}^{\ -1}(E)$ is a node of $\overline{C}$ which is the intersection of two irreducible components $\overline{C}'$ and $\overline{C}''$ of $\overline{C}$, with $\overline{f}(\overline{C}') \subset Z$ and $\overline{f}(\overline{C}'') \subset \mathbb{C}P^1 \times \mathbb{C}P^1$;
- if in addition neither $\overline{f}(\overline{C}')$ nor $\overline{f}(\overline{C}'')$ is entirely mapped to E, then the multiplicity of intersection of both $\overline{f}(\overline{C}')$ and $\overline{f}(\overline{C}'')$ with E are equal.

Formally, [IP04, Section 3] only deals with the case $\kappa = 0$, however the value of κ plays no role there. Given an element $\overline{f} : \overline{C} \to X_\sharp$ of $\mathcal{C}^{\mathbb{C}}(d, g, \underline{x}(0), W, J_0)$, we denote by C_1 (resp. C_0) the union of the irreducible components of $\overline{C}$ mapped to Z (resp. $\mathbb{C}P^1 \times \mathbb{C}P^1$). The next three statements are proved in [BP15, Section 3.2]. Again formally, [BP15, Lemma 3.6, Propositions 3.7, and Corollary 3.8] are stated for $g = 0$ and $\kappa = 0$, however neither g nor κ plays no role in their proof once we replace [BP15, Proposition 3.2] by Lemma 3.1 above.

Lemma 3.15. *Given an element $\overline{f} : \overline{C} \to X_\sharp$ of $\mathcal{C}^{\mathbb{C}}(d, g, \underline{x}(0), W, J_0)$, there exists $k \in \mathbb{Z}_{\geq 0}$ such that*

$$\overline{f}_*[C_1] = d - k[E] \quad \textit{and} \quad \overline{f}_*[C_0] = kl_1 + (d \cdot [E] + k)l_2.$$

Moreover $c_1(Z) \cdot \overline{f}_*[C_1] = c_1(\mathcal{Z}_t) \cdot d$.

Proposition 3.16. *Assume that $\underline{x}(0) \subset Z$. Then for a generic J_0, the set $\mathcal{C}^{\mathbb{C}}(d, g, \underline{x}(0), W, J_0)$ is finite, and only depends on $\underline{x}(0)$ and J_0. Given $\overline{f} : \overline{C} \to X_\sharp$ an element of $\mathcal{C}^{\mathbb{C}}(d, g, \underline{x}(0), W, J_0)$, the restriction of $\overline{f}$ to any component of $\overline{C}$ is a simple map, and no irreducible component of $\overline{C}$ is entirely mapped to E. Moreover the curve C_1 is irreducible, and the image of any irreducible component of C_0 realizes a class l_i. The map $\overline{f}$ is the limit of a unique element of $\mathcal{C}^{\mathbb{C}}(d, g, \underline{x}(t), W, J_t)$ as t goes to 0.*

Next Corollary generalizes [BP15, Corollary 3.8] and Abramovich-Bertram-Vakil formula [AB01, Theorem 3.1.1] and [Vak00, Theorem 4.2].

Corollary 3.17. *Suppose that $\underline{x}(0) \subset Z$. Let $\overline{f} : \overline{C} \to X_\sharp$ be an element of $\mathcal{C}^{\mathbb{C}}(d, g, \underline{x}(0), W, J_0)$. Define $\mathcal{C}_{\overline{f}}$ to be the set of elements $\overline{f}' : \overline{C}' \to X_\sharp$ in $\mathcal{C}^{\mathbb{C}}(d, g, \underline{x}(0), W, J_0)$ such that $\overline{f}_{|C_1} = \overline{f}'_{|C_1'}$. If $\overline{f}_*[C_1] = d - k[E]$, then $\mathcal{C}_{\overline{f}}$ contains exactly $\binom{d\cdot[E]+2k}{k}$ elements.*

3.5.2. *Proof of Theorems 3.10 and 3.13.* We proved these theorems using the obvious equivariant version of Section 3.5.1, which exists by [Teh13, Lemma 2.14] and Remark 3.2. In the case when $g = 0$, the curve E is the only element of U, and V is empty, Theorem 3.10 is exactly [BP15, Theorem 2.5(1)]. One simply has to note that since $\mathbb{R}E \cap L = \emptyset$, all intersection points of $\overline{f}(\overline{C}_1)$ and E are τ_X-conjugated for $\overline{f} \in \mathcal{C}^{\mathbb{C}}(d, 0, \underline{x}(0), W, J_0)$. In particular we must have

$$\overline{f}_*[C_1] = d - 2k[E] \quad \text{and} \quad \overline{f}_*[C_0] = 2kl_1 + (d \cdot [E] + 2k)l_2.$$

with $k \in \mathbb{Z}_{\geq 0}$.

In the case when $g = 0$, the curve E is the only element of V, and U is empty, Theorem 3.13 is exactly [BP15, Theorem 2.5(2)].

In general, Theorems 3.10 and 3.13 are obtained by the straightforward adaptation of the proof of [BP15, Theorem 2.5] replacing [BP15, Section 3.2] by Section 3.5.1. ☺

3.5.3. *Proof of Theorem 3.9.* By induction, it follows from Corollary 3.11 and Theorem 3.13 that the numbers $W^{U,V}_{X_{\mathbb{R}},L,L_0,F}(d; \underline{r}, s)$ can be expressed in terms of the numbers $W^{\emptyset,\emptyset}_{\overline{X}_{\mathbb{R}},\overline{L},\overline{L}_0,F}(d; \underline{r}, s)$. By definition we have

$$W^{\emptyset,\emptyset}_{\overline{X}_{\mathbb{R}},\overline{L},\overline{L}_0,F}(d; \underline{r}, s) = W_{\overline{X}_{\mathbb{R}},\overline{L},F+[\overline{L}_0]}(d; \underline{r}, s),$$

so Theorem 3.9 now follows from our hypothesis. ☺

3.5.4. *Proof of Theorem 3.8.* According to Lemma 2.6, we have $[S] \cdot d = 0$, hence Theorem 3.10 and Corollary 3.11 give

$$W_{Y_{\mathbb{R}},L,F}(d; \underline{r}, s) = \sum_{k,l \geq 0} (-1)^l \, 2^k \, \left(\binom{k+l}{k} + \binom{k+l-1}{k} \right) W_{X_{\mathbb{R}},L,F}(d - (k+2l)[S]; \underline{r}, s).$$

The total coefficient of the invariant $W_{X_{\mathbb{R}},L,F}(d - i[S]; \underline{r}, s)$ appearing in this sum is equal to 1 if $i = 0$, and is equal to

$$\begin{aligned}\sum_{k+2l=i} (-1)^l \, 2^k \, \left(\binom{k+l}{k} + \binom{k+l-1}{k} \right) &= \sum_{k+2l=i} (-1)^l \, 2^k \binom{k+l}{k} - \sum_{k+2l=i-2} (-1)^l \, 2^k \binom{k+l}{k} \\ &= u_i - u_{i-2}\end{aligned}$$

otherwise, where

$$u_i = \sum_{k+2l=i} (-1)^l \, 2^k \binom{k+l}{k}.$$

By Pascal's rule, we have

$$u_{i+2} = 2u_{i+1} - u_i,$$

hence we deduce from the values $u_0 = 1$ and $u_1 = 2$ that

$$u_i = i + 1 \qquad \forall i \geq 0,$$

and the theorem is proved. ☺

4. Applications to real algebraic rational surfaces

From now on, we restrict ourselves to the study of real symplectic manifolds that are deformation equivalent to a real algebraic rational surface. We refer for example to [Sil89, Kol97, DK00, DK02] for an account on real algebraic rational surfaces. Their classification up to deformation has been established in [DK02]. In particular, any two real algebraic $\mathbb{R}$-minimal rational surfaces with a non-empty real part are deformation equivalent if and only if their are deformation equivalent as complex algebraic surfaces and if their real part are homeomorphic. Furthermore, it follows from this classification and Example 2.4 that the surgery of a real algebraic rational surfaces along a real Lagrangian sphere contained in $\mathbb{R}X$ remains deformation equivalent to a real algebraic rational surface. Recall also that it follows from [BP15, Corollary 2.6 and Section 4] that any genus 0 Welschinger invariant $W_{X_\mathbb{R},L,F}(d;s)$ of a real rational algebraic surface $X_\mathbb{R}$ is equal, up to a well defined sign, to $W_{X_\mathbb{R},L,[R]}(d;s)$, where $R \subset \mathbb{R}X \setminus L$ only depends on L and F.

By the blow-up of a real algebraic rational surface $X_\mathbb{R}$, we mean the blow-up of $X_\mathbb{R}$ at a finite number of real points and pairs of complex conjugated points, equipped with the real structure that turns the blowing down map into a real map.

Proposition 4.1. *Let $X_\mathbb{R}$ be a real algebraic rational surface with a non-empty real part, and L be a connected component of $\mathbb{R}X$ such that $\mathbb{R}X \setminus L$ is a disjoint union of spheres. Then by finitely many successive applications of Theorem 3.8, all genus 0 Welschinger invariants $W_{X_\mathbb{R},L,F}$ can be computed out of genus 0 Welschinger invariants of either a blow-up of the real projective plane or a blow-up of a real quadric in $\mathbb{C}P^3$.*

Proof. As mentioned above, we may suppose that $F = [R]$ where R is the union of some connected components of $\mathbb{R}X \setminus L$. Since a connected component of $\mathbb{R}X \setminus L$ is a real Lagrangian sphere, it is orthogonal to $[R]$ in $H_2(X \setminus L; \mathbb{Z}/2\mathbb{Z})$. Hence by applying Theorem 3.8 to $W_{X_\mathbb{R},L,[R]}(d;s)$ and by successive surgeries along all spheres in $\mathbb{R}X \setminus L$, we are reduced to the case when $\mathbb{R}X$ is connected. It follows from [DK02, Main Theorem] that a real algebraic rational surface with a connected real part is deformation equivalent to either a blown-up real projective plane or a blown-up real quadric in $\mathbb{C}P^3$, so the statement is proved. ☺

Genus 0 Welschinger invariants of any blow-up of the real projective plane, and of any blow-up of a real quadric in $\mathbb{C}P^3$ are computed in [HS12]. Hence combining Proposition 4.1 and [HS12], one can compute any genus 0 Welschinger invariant $W_{X_\mathbb{R},L,F}$ as soon as $\mathbb{R}X \setminus L$ is a disjoint union of spheres. For example, genus 0 Welschinger invariants of $\mathbb{R}$-minimal real conic bundles can be deduced from Welschinger invariants of a quadric ellipsoid in $\mathbb{C}P^3$ blown-up at several pairs of complex conjugated points.

Analogously, one can reduce the computation of genus 0 Welschinger invariants of del Pezzo surfaces to the case when the real part of the surface has at most two connected components. These latter cases have been covered in [Bru15].

Proposition 4.2. *Let $X_\mathbb{R}$ a real algebraic del Pezzo surface, and L be a connected component of $\mathbb{R}X$. Then by finitely many successive applications of Theorem 3.8, all genus 0 Welschinger invariants $W_{X_\mathbb{R},L,F}$ can be computed out of genus 0 Welschinger invariants of a real algebraic del Pezzo surface of the same degree with a real part consisting of at most two connected components.*

Proof. It follows from the classification of real algebraic del Pezzo surfaces that if $\mathbb{R}X$ has three or more connected components, then at most one of them is not homeomorphic to a sphere. So the proof is analogous to the proof of Proposition 4.1. ☺

We can generalize Propositions 4.1 and 4.2 to relative Welschinger invariants as follows.

Theorem 4.3. *Let $X_\mathbb{R}$ be a real algebraic rational surface with a disconnected real part, and let L be a connected component of $\mathbb{R}X$. Then by finitely many successive applications of Theorems 3.8 and 3.13, all absolute genus 0 Welschinger invariants $W_{X_\mathbb{R},L,F}$ can be computed out of relative genus 0 Welschinger invariants of a blow-up $Y_\mathbb{R}$ of the $\mathbb{R}$-minimal real conic bundle with 2 spheres as real components.*

Furthermore by finitely many successive applications of Theorem 3.8, all absolute genus 0 Welschinger invariants $W_{X_\mathbb{R},L,0}$ and $W_{X_\mathbb{R},L,[\mathbb{R}X\setminus L]}$ can be respectively computed out of the absolute genus 0 Welschinger invariants $W_{Y_\mathbb{R},L,0}$ and $W_{Y_\mathbb{R},L,[\mathbb{R}Y\setminus L]}$.

Proof. The case when $\mathbb{R}X$ has two connected components follows from the classification of real algebraic rational surfaces up to deformations [DK02, Main Theorem]. Hence we suppose now that $\mathbb{R}X$ has a least three connected components. As above, it is enough to consider the case when F is realized by the union R of some connected components of $\mathbb{R}X \setminus L$. It follows again from [DK02, Main Theorem] that $\mathbb{R}X$ can be degenerated to a real algebraic rational nodal surface $Z_\mathbb{R}$ such that L is contained in the non-singular locus of $Z_\mathbb{R}$, and such that $\mathbb{R}Z \setminus L$ is connected. The non-singular real algebraic rational surface $Y_\mathbb{R}$ obtained by blowing up all nodes of $Z_\mathbb{R}$ has a real part consisting of exactly two connected components. Hence it follows from the classification of real algebraic rational surfaces that $Y_\mathbb{R}$ is, up to deformation, a blow-up of the $\mathbb{R}$-minimal real conic bundle with 2 spheres as real components. Thanks to Theorems 3.8 and 3.13 and Example 2.4, all genus 0 Welschinger invariants $W_{X_\mathbb{R},L,[R]}$ can be expressed in terms of genus 0 Welschinger invariants of $Y_\mathbb{R}$ relative to the vanishing cycles of the degeneration of $X_\mathbb{R}$ to $Z_\mathbb{R}$, so the first statement is proved. The statement about $W_{X_\mathbb{R},L,0}$ and $W_{X_\mathbb{R},L,[\mathbb{R}X\setminus L]}$ now follows from Theorem 1.1. ☺

Remark 4.4. Theorem 4.3 and its proof can of course be generalized to Welschinger invariants of higher genus. We decided nevertheless to restrict to the genus 0 case since it makes the statement much shorter.

Corollary 4.5. *Let $X_\mathbb{R}$ be a real algebraic rational surface with a disconnected real part, let L be a connected component of $\mathbb{R}X$, and let $Z_\mathbb{R}$ be the real algebraic rational surface obtained by blowing up $X_\mathbb{R}$ at some point $p \in \mathbb{R}X \setminus L$. Then the genus 0 Welschinger invariants $W_{Z_\mathbb{R},L,0}$ and $W_{Z_\mathbb{R},L,[\mathbb{R}Z\setminus L]}$ do not depend on the connected component of $\mathbb{R}X \setminus L$ containing p.*

5. A few concrete computations

Here we illustrate Theorem 3.8 with some explicit computations of absolute genus 0 Welschinger invariants. Again, we refer for example to [Sil89, Kol97, DK00, DK02]) for the classification of real algebraic rational surfaces.

5.1. **Real cubic surface with two real components.**

Let $Y_\mathbb{R}$ be a non-singular real algebraic cubic surface in $\mathbb{C}P^3$ whose real part is homeomorphic to the disjoint union of a sphere S and a real projective plane $\mathbb{R}P^2$, and let $X_\mathbb{R}$ be the surgery of $Y_\mathbb{R}$ along S. As a first and easy example of application of Theorem 3.8, we show how to compute absolute Welschinger invariants of $Y_\mathbb{R}$ out of those to $X_\mathbb{R}$.

By Example 2.5 and the classification of real cubic surfaces, we may assume that $X_\mathbb{R}$ is $\mathbb{C}P^2$ blown up in three pairs of complex conjugated points. Let us denote by $E_1, \dots, E_6$ the six corresponding exceptional divisors, and let D be the pull back of a line in $\mathbb{C}P^2$ not passing through the six blown up points. Without loss of generality, we may assume that the following identities hold in $H_2(X;\mathbb{Z})$:

$$c_1(X) = 3[D] - \sum_{i=1}^{6}[E_i] \qquad \text{and} \qquad [S] = 2[D] - \sum_{i=1}^{6}[E_i].$$

By Theorem 3.8 we have

$$W_{Y_\mathbb{R},\mathbb{R}P^2,F}(c_1(X);s) = W_{X_\mathbb{R},\mathbb{R}P^2,F}(c_1(X);s) + 2W_{X_\mathbb{R},\mathbb{R}P^2,F}([D];s).$$

It is well known that

$$W_{X_\mathbb{R},\mathbb{R}P^2,0}(c_1(X);s) = W_{\mathbb{C}P^2,\mathbb{R}P^2,0}(3[D];s+3) = 2-2s,$$
$$W_{X_\mathbb{R},\mathbb{R}P^2,0}([D];s) = W_{\mathbb{C}P^2,\mathbb{R}P^2,0}([D];s) = 1.$$

Combining with the identity

$$W_{X_\mathbb{R},\mathbb{R}P^2,[S]}(d;s) = (-1)^{\frac{d\cdot[S]}{2}} W_{X_\mathbb{R},\mathbb{R}P^2,0}(d;s), \tag{9}$$

we obtain

$$W_{Y_\mathbb{R},\mathbb{R}P^2,0}(c_1(X);s) = 4-2s \qquad \text{and} \qquad W_{Y_\mathbb{R},\mathbb{R}P^2,[S]}(c_1(X);s) = -2s.$$

Remark 5.1. By [BP15, Theorems 1.2(1) and 1.2], the vanishing of $W_{Y_\mathbb{R},\mathbb{R}P^2,[S]}(c_1(X);0)$ is actually a general fact. Hence we could also have used this general vanishing result to deduce the value of $W_{X_\mathbb{R},\mathbb{R}P^2,0}(c_1(X);0)$ out of the value $W_{\mathbb{C}P^2,\mathbb{R}P^2,0}([D];s) = 1$ and the equation

$$0 = W_{X_\mathbb{R},\mathbb{R}P^2,0}(c_1(X);0) - 2W_{\mathbb{C}P^2,\mathbb{R}P^2,0}([D];0)$$

given by the combination of (9), Theorem 3.8, and [BP15, Theorems 1.1(1) and 1.2].

By Theorem 3.10 we have

$$W_{Y_\mathbb{R},\mathbb{R}P^2,F}(2c_1(X);s) = W_{X_\mathbb{R},\mathbb{R}P^2,F}(2c_1(X);s) + 2W_{X_\mathbb{R},\mathbb{R}P^2,F}(4[D]-\sum_{i=1}^{6}[E_i];s) + 2W_{X_\mathbb{R},\mathbb{R}P^2,F}(2[D];s).$$

The following values are taken from [IKS04, ABLdM11] and [Bru15, Table 4].

s	$W_{X_\mathbb{R},\mathbb{R}P^2,0}(2c_1(X);s)$	$W_{X_\mathbb{R},\mathbb{R}P^2,0}(4[D]-\sum_{i=1}^{6}[E_i];s)$	$W_{X_\mathbb{R},\mathbb{R}P^2,0}(2[D];s)$
0	78	40	1
1	30	16	1
2	22	0	1

Combining this with (9) we obtain

s	$W_{Y_\mathbb{R},\mathbb{R}P^2,0}(2c_1(X);s)$	$W_{Y_\mathbb{R},\mathbb{R}P^2,[S]}(2c_1(X);s)$
0	160	0
1	64	0
2	24	24

in accordance with [Bru15, Table 4].

Remark 5.2. Again, one could have used [BP15, Theorems 1.1(1) and 1.2], to compute the invariants $W_{X_\mathbb{R},\mathbb{R}P^2,0}(2c_1(X);s)$ with $s<2$ out of the values

$$W_{\mathbb{C}P^2,\mathbb{R}P^2,0}(2[D];s), \quad W_{\mathbb{C}P^2,\mathbb{R}P^2,0}(4[D];s+3),$$

and the equation

$$0 = W_{X_\mathbb{R},\mathbb{R}P^2,0}(2c_1(X);0) - 2W_{\mathbb{C}P^2,\mathbb{R}P^2,0}(4[D];s+3) + 2W_{\mathbb{C}P^2,\mathbb{R}P^2,0}(2[D];s)$$

given by the combination of (9), Theorem 3.8, and [BP15, Theorems 1.1(1) and 1.2].

5.2. **$\mathbb{R}$-minimal conic bundles.** Given $n \in \mathbb{Z}_{>0}$, let X_n be the conic bundle given in an affine chart by the real equation

$$y^2 + z^2 = -\prod_{i=1}^{2n}(x - a_i)$$

where $a_1 < a_2 \ldots < a_{2n}$ are distinct real numbers. The conic bundle structure is given by the map $\rho : X_n \to \mathbb{C}P^1$ that forgets the (y, z) coordinates. Restricting the standard complex conjugation on $\mathbb{C}^3$ to X_n turns this latter into a real algebraic surface $X_{n,\mathbb{R}}$, and turns ρ into a real map. Note that $X_{n,\mathbb{R}}$ is $\mathbb{R}$-minimal if and only if $n \geq 2$. The real part of $X_{n,\mathbb{R}}$ is composed of the n spheres $S_i = \rho^{-1}([a_{2i-1}, a_{2i}]) \cap \mathbb{R}^3$. Next lemma is a straightforward combination of [BP15, Propositions 4.2, 4.3, and 4.5, and Corollary 2.6].

Lemma 5.3. *Given $F \in H_2^{\tau_X}(X_{n,\mathbb{R}} \setminus S_1; \mathbb{Z}/2\mathbb{Z})$, there exists $R \subset \mathbb{R}X_{n,\mathbb{R}} \setminus S_1$ a set of connected components of $\mathbb{R}X_{n,\mathbb{R}} \setminus S_1$ such that*

$$|W_{X_{n,\mathbb{R}},s_1,F}(d; s)| = |W_{X_{n,\mathbb{R}},S_1,[R]}(d; s)| \qquad \forall d \in H_2(X_{n,\mathbb{R}}; \mathbb{Z}) \text{ and } \forall s \in \mathbb{Z}_{\geq 0}.$$

That is to say, the computation of all Welschinger invariants of $X_{n,\mathbb{R}}$ can be reduced to the case when F is realized by the union of some connected components of $\mathbb{R}X_{n,\mathbb{R}} \setminus S_1$.

Now we describe how to construct recursively the real varieties $X_{n,\mathbb{R}}$, up to deformation, using real surgeries along Lagrangian spheres. Let $X_{0,\mathbb{R}} = (\mathbb{C}P^1 \times \mathbb{C}P^1, \omega_{FS} \oplus \omega_{FS}, \tau_{S^1,0})$ be the quadric hyperboloid in $\mathbb{C}P^3$, and let $\rho : X_{0,\mathbb{R}} \to \mathbb{C}P^1$ be the (real) projection to the first factor. Consider the blow-up of $X_{0,\mathbb{R}}$ at two complex conjugated points of a fiber Φ of ρ. Then $X_{1,\mathbb{R}}$ is the surgery of $X_{0,\mathbb{R}}$ along the strict transform of Φ. Suppose now that we have constructed $X_{n,\mathbb{R}}$, and let Φ be a real fiber of ρ with $\mathbb{R}\Phi = \emptyset$. Then $X_{n+1,\mathbb{R}}$ is the surgery of $X_{n,\mathbb{R}}$ along the strict transform of Φ under the blow-up of $X_{n,\mathbb{R}}$ at two complex conjugated points of Φ.

This recursive description of $X_{n,\mathbb{R}}$ exhibit the underlying complex surface as the blow-up of $X_{0,\mathbb{R}}$ at $2n$ points, and shows that one can denote the corresponding exceptional divisors $E_1, \ldots, E_{2n}$ such that

$$[S_i] = [\Phi] - [E_{2i-1}] - [E_{2i}] \in H_2(X_n; \mathbb{Z}) \tag{10}$$

for some orientation on S_i, where Φ is a fiber of ρ. Denoting by $\widetilde{c_1(X_0)}$ the pull back of $c_1(X_0)$ by this sequence of blow-ups, we have

$$c_1(X_n) = \widetilde{c_1(X_0)} - \sum_{i=1}^{2n}[E_i].$$

In Proposition 5.5 we compute the invariants $W_{X_{n,\mathbb{R}},S_1,[R]}(c_1(X_n) + b[\Phi]; s)$ recursively starting from the case $n = 1$, which is treated in next proposition.

Proposition 5.4. *Let $b \in \mathbb{Z}_{\geq -2}$. If $s \in \{0, \ldots, b+1\}$, then we have*

$$W_{X_{1,\mathbb{R}},S_1,0}(c_1(X_1) + b[\Phi]; s) = 2^{2b+2-s}.$$

If $s = b + 2$, then we have

$$W_{X_{1,\mathbb{R}},S_1,0}(c_1(X_1) + b[\Phi]; b+2) = \begin{cases} 2^{b+1} & \text{if } b \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The real manifold $X_{1,\mathbb{R}}$ is the quadric ellipsoid blown up at a pair of complex conjugated points. Hence one could use the methods described in [Bru15] to prove the proposition. We provide an alternative proof that illustrates applications of the method exposed in [BP15]. Here we use notations introduced in Sections 2.1 and 3.5.1. We also define $r = 2b + 5 - 2s$.

Let $\widetilde{X_{0,\mathbb{R}}}$ be the blow-up of $X_{0,\mathbb{R}}$ at two complex conjugated points on a fiber Φ of ρ, and let E be the strict transform of Φ. As explained above, $X_{1,\mathbb{R}}$ is, up to deformation, obtained as the real symplectic sum $\pi : \mathcal{Z} \to D$ of $X_{0,\mathbb{R}}$ and $(\mathbb{C}P^1 \times \mathbb{C}P^1, \omega_0, \tau_{S^1,2})$ along E. Now choose $\underline{x} : D \to \mathcal{Z}$ a set of r real sections and s pairs of conjugated real sections such that

$$|\underline{x}(0) \cap \widetilde{X_0}| = 2b + 4 \qquad \text{and} \qquad |\underline{x}(0) \cap (\mathbb{C}P^1 \times \mathbb{C}P^1)| = 1.$$

Let $\overline{f} : \overline{C} \to \widetilde{X_0} \cup \mathbb{C}P^1 \times \mathbb{C}P^1$ be an element of $\mathcal{C}^{\mathbb{C}}(c_1(X_1) + b[\Phi], 0, \underline{x}(0), \emptyset, J_0)$. According to [BP15, Proposition 3.7], we have

$$\overline{f}_*[C_1] = c_1(X_1) + b[\Phi] - [E] = \widetilde{c_1(X_0)} + (b-1)[\Phi] \in H_2(X_1; \mathbb{Z}).$$

According to [BP15, Proposition 3.7], we have the two following possibilities:

(i) C_1 is real and irreducible, and $\overline{f}(C_1)$ is tangent to E at a real point. In this case, it follows from [BP15, Proposition 3.10] that the contribution to $W_{X_{1,\mathbb{R}},S_1,0}(c_1(X_1)+b[\Phi]; s)$ of the two deformations of $\overline{f}$ cancel each other.

(ii) C_1 is real and has two irreducible components C' and C''. In this case, $\overline{f}$ deforms into a unique real curve in $\mathcal{C}^{\mathbb{C}}(c_1(X_1) + b[\Phi], \underline{x}(t), 0, \emptyset, J_t)$, and $\overline{f}(C_0)$ intersects E transversely in two points.

Hence it remains to estimate the contribution to $W_{X_{1,\mathbb{R}},S_1,0}(c_1(X_1)+b[\Phi]; s)$ of the maps in case (ii) above. None of the curves $\overline{f}(C')$ nor $\overline{f}(C'')$ intersects any of the two exceptional divisor coming from the blow-ups of $X_{0,\mathbb{R}}$, therefore there exists $b_1 \in \{0, \dots, b+1\}$ such that

$$\overline{f}(C') = l_1 + b_1 l_2 \qquad \text{and} \qquad \overline{f}(C'') = l_1 + (b+1-b_1) l_2$$

(we still denote by l_i the strict transform the class l_i under the blow-up map). The maps $\overline{f}_{|C'}$ and $\overline{f}_{|C''}$ can be constrained respectively by at most $2b_1 + 1$ and $2b - 2b_1 + 3$ points. Since $\overline{f}_{|C_1}$ is constrained by the $2b+4$ points in $\underline{x}(0) \cap \widetilde{X_{0,\mathbb{R}}}$, we deduce that $\overline{f}_{|C'}$ and $\overline{f}_{|C''}$ are constrained respectively by exactly $2b_1 + 1$ and $2b - 2b_1 + 3$ points.

Suppose that $r \geq 3$. In this case we have $\underline{x}(0) \cap \mathbb{R}\widetilde{X_{0,\mathbb{R}}} \neq \emptyset$, which in particular implies that both curves C' and C'' and both maps $\overline{f}_{|C'}$ and $\overline{f}_{|C''}$ are real. Let us choose $p \in \underline{x}(0) \cap \mathbb{R}\widetilde{X_{0,\mathbb{R}}}$. Without loss of generality, we may assume that $p \in \overline{f}(C')$. Since the Gromov-Witten invariant of $\mathbb{C}P^1 \times \mathbb{C}P^1$ for the class $l_1 + b_1 l_2$ is equal to 1 for any $b_1 \geq 0$, the map $\overline{f}$ is determined by the real subset of $\underline{x}(0) \setminus \{p_0, p\}$ that is contained in $\overline{f}(C')$. Let $\Im\underline{x}(0)$ be the set of pairs of conjugated points of $\underline{x}_0$. Given any sets $A \subset \mathbb{R}\underline{x}(0) \setminus \{p_0, p\}$ and $B \subset \Im\underline{x}(0)$, the pair among the two pairs (A, B) and $(\mathbb{R}\underline{x}(0) \setminus (A \cup \{p_0, p\}), \Im\underline{x}(0) \setminus B)$ for which the first component has an even cardinal corresponds to a real subset of $\underline{x}(0) \setminus \{p_0, p\}$ that is contained in $\overline{f}(C')$ for some map $\overline{f}$ as above. Hence there exist exactly

$$2^{r+s-3}$$

such maps. By the adjunction formula, any irreducible J-holomorphic curve in $\mathbb{C}P^1 \times \mathbb{C}P^1$ realizing the class $l_1 + b_1 l_2$ is smooth. Hence the curve $\overline{f}(C_1)$ has no elliptic real node, and the proposition is proved in the case when $r \geq 3$.

Suppose now that $r = 1$. In this case $\mathbb{R}\underline{x}(0) \cap \widetilde{X_0} = \emptyset$, and there is no possibilities to decompose $x(0) \cap \widetilde{X_0}$ into the disjoint union of two real subsets of odd cardinality. As a consequence the two curves C' and C'' and the two maps $\overline{f}_{|C'}$ and $\overline{f}_{|C''}$ are complex conjugated, and each map is determined by the point in each pair of $\underline{x}(0) \setminus p_0$ through which it passes. Hence there are exactly

$$2^{s-1} = 2^{b+1}$$

such maps $\overline{f}$ if $b+2$ is odd, and 0 such maps otherwise. Since $\mathbb{R}X_{0,\mathbb{R}}$ is null-homologous in $H_2(X_0;\mathbb{Z}/2\mathbb{Z})$, the curve $\overline{f}(C')$ has an even number of intersection points with $\mathbb{R}\widetilde{X_{0,\mathbb{R}}}$. Hence the proposition is proved as well when $r=1$. ☺

Proposition 5.5. *Let $n\geq 2$, $b\geq n-3$ and $s\leq b-n+3$ be three integer numbers, and let R be the union of some connected components of $\mathbb{R}X_n\setminus S_1$. Then we have*

$$W_{X_{n,\mathbb{R}},S_1,[R]}(c_1(X_n)+b[\Phi];s)=\begin{cases} 2^{2b+2-s} & \text{if } R=\emptyset;\\ (-1)^{b+1}\ 2^{b+n-1} & \text{if } R=\mathbb{R}X_n\setminus S_1 \text{ and } s=b-n+3;\\ 0 & \text{otherwise.}\end{cases}$$

Proof. The fact that $W_{X_{n,\mathbb{R}},S_1,[R]}(c_1(X_n)+b[\Phi];s)=0$ if $R\neq\emptyset$ and $s<b+n-3$ follows from [BP15, Theorems 1.1 and 1.2]. Given $n\geq 2$, Theorem 3.8 implies that

$$W_{X_{n,\mathbb{R}},S_1,F}(c_1(X_n)+b[\Phi];s)=W_{X_{n-1,\mathbb{R}},S_1,F}(c_1(X_{n-1})+b[\Phi];s+1)+2W_{X_{n-1,\mathbb{R}},S_1,F}(c_1(X_{n-1})+(b-1)[\Phi];s). \tag{11}$$

The proposition in the case when $R=\emptyset$ follows now by induction on n from (11) and Proposition 5.4.

Let us assume that $R=\mathbb{R}X_n\setminus S_1$ and $s=b-n+3$. According to Remark 3.7, identity (11) becomes

$$W_{X_{n,\mathbb{R}},S_1,[\mathbb{R}X_{n,\mathbb{R}}\setminus S_1]}(c_1(X_n)+b[\Phi];s)=W_{X_{n-1,\mathbb{R}},S_1,[\mathbb{R}X_{n-1,\mathbb{R}}\setminus S_1]}(c_1(X_{n-1})+b[\Phi];s+1)-2W_{X_{n-1,\mathbb{R}},S_1,[\mathbb{R}X_{n-1,\mathbb{R}}\setminus S_1]}(c_1(X_{n-1})+(b-1)[\Phi];s). \tag{12}$$

If $n=2$, then the proposition follows from (12) and Proposition 5.4. If $n\geq 3$, then the proposition follows by induction on n from (12).

The proof of the proposition when $\emptyset\neq R\subsetneq(\mathbb{R}X_n\setminus S_1)$ and $s=b-n+3$ is analogous: the case $n=3$ follows from (11) and the above computation of $W_{X_{2,\mathbb{R}},S_1,[\mathbb{R}X_{2,\mathbb{R}}\setminus S_1]}(c_1(X_2)+b[\Phi];s)$, and the case $n\geq 4$ follows by induction on n from (11). ☺

5.3. **Del Pezzo surfaces of degree 1.** Recall that there exist 11 deformation classes of real del Pezzo surfaces of degree 1. In [Bru15, Example 7.9], the values of $W_{X_{\mathbb{R}},L,[\mathbb{R}X\setminus L]}(2c_1(X);0)$ are computed for 6 of these deformation classes. Here we treat the 5 remaining cases.

Recall also that the real conic bundle $X_{n,\mathbb{R}}$ with $n\leq 3$ is a del Pezzo surface of degree $8-2n$, and that Φ denotes a generic fiber of $X_{n,\mathbb{R}}$. We denote by $\mathbb{R}P^2_k$ the real blow-up of $\mathbb{R}P^2$ at k points (i.e. we replace k disjoint disks in $\mathbb{R}P^2$ by k Mobius strips). By "$Y_{\mathbb{R}}$ is obtained by a surgery of $X_{\mathbb{R}}$ along the class γ", we mean that $Y_{\mathbb{R}}$ is obtained by a surgery of a deformation of $X_{\mathbb{R}}$ for which the class γ is realized by a real Lagrangian sphere. In what follows, the existence of such deformation is guaranteed by the classification up to deformations of real algebraic rational surfaces. We define the following real del Pezzo surfaces of degree 1:

- $Y_{1,\mathbb{R}}$ is the real blow-up of $X_{2,\mathbb{R}}$ at one real point on a connected component of $\mathbb{R}X_{2,\mathbb{R}}$ and at two real points on the other connected component. Denoting respectively by $\widetilde{E}_1,\widetilde{E}_2,\widetilde{E}_3$ the three corresponding exceptional divisors, we have

$$H_2^{-\tau_{Y_1}}(Y_{1,\mathbb{R}};\mathbb{Z})=\mathbb{Z}c_1(Y_1)\oplus\mathbb{Z}[\Phi]\oplus\mathbb{Z}[\widetilde{E}_1]\oplus\mathbb{Z}[\widetilde{E}_2]\oplus\mathbb{Z}[\widetilde{E}_3]\qquad\text{and}\qquad\mathbb{R}Y_{1,\mathbb{R}}=\mathbb{R}P^2\sqcup\mathbb{R}P^2_1.$$

- $Y'_{1,\mathbb{R}}$ is the real blow-up of $X_{2,\mathbb{R}}$ at three real points on a connected component of $\mathbb{R}X_{2,\mathbb{R}}$. Denoting by $\widetilde{E}_1,\widetilde{E}_2,\widetilde{E}_3$ the three corresponding exceptional divisors, we have

$$H_2^{-\tau_{Y'_1}}(Y'_{1,\mathbb{R}};\mathbb{Z})=\mathbb{Z}c_1(Y'_1)\oplus\mathbb{Z}[\Phi]\oplus\mathbb{Z}[\widetilde{E}_1]\oplus\mathbb{Z}[\widetilde{E}_2]\oplus\mathbb{Z}[\widetilde{E}_3]\qquad\text{and}\qquad\mathbb{R}Y'_{1,\mathbb{R}}=S^2\sqcup\mathbb{R}P^2_2.$$

- $Y''_{1,\mathbb{R}}$ is the real blow-up of $X_{1,\mathbb{R}}$ at a real point and two pairs of complex conjugated points. Denoting respectively by $\widetilde{E}_1, \dots, \widetilde{E}_5$ the five corresponding exceptional divisors, we have
$$H_2^{-\tau_{Y_1''}}(Y''_{1,\mathbb{R}};\mathbb{Z}) = \mathbb{Z}c_1(Y_1'') \oplus \mathbb{Z}[\Phi] \oplus \mathbb{Z}[\widetilde{E}_1] \oplus \mathbb{Z}([\widetilde{E}_2]+[\widetilde{E}_3]) \oplus \mathbb{Z}([\widetilde{E}_4]+[\widetilde{E}_5])$$
and
$$\mathbb{R}Y''_{1,\mathbb{R}} = \mathbb{R}P^2.$$
- $Y_{2,\mathbb{R}}$ is the real blow-up of $X_{2,\mathbb{R}}$ at a real point on a connected component of $\mathbb{R}X_{2,\mathbb{R}}$, and a pair of complex conjugated points. Denoting respectively by $\widetilde{E}_1, \widetilde{E}_6, \widetilde{E}_7$ the three corresponding exceptional divisors, we have
$$H_2^{-\tau_{Y_2}}(Y_{2,\mathbb{R}};\mathbb{Z}) = \mathbb{Z}c_1(Y_2) \oplus \mathbb{Z}[\Phi] \oplus \mathbb{Z}[\widetilde{E}_1] \oplus \mathbb{Z}([\widetilde{E}_6]+[\widetilde{E}_7]) \qquad \text{and} \qquad \mathbb{R}Y_{2,\mathbb{R}} = \mathbb{R}P^2 \sqcup S^2.$$
- $Y_{3,\mathbb{R}}$ is the surgery of $Y_{2,\mathbb{R}}$ along the class $[\Phi]-[\widetilde{E}_6]-[\widetilde{E}_7]$. We have
$$H_2^{-\tau_{Y_3}}(Y_{3,\mathbb{R}};\mathbb{Z}) = \mathbb{Z}c_1(Y_2) \oplus \mathbb{Z}[\Phi] \oplus \mathbb{Z}[\widetilde{E}_1] \qquad \text{and} \qquad \mathbb{R}Y_{3,\mathbb{R}} = \mathbb{R}P^2 \sqcup 2S^2.$$
Note that $Y_{3,\mathbb{R}}$ is the blow-up of $X_{3,\mathbb{R}}$ at a real point.
- $Y_{4,\mathbb{R}}$ is the surgery of $Y_{3,\mathbb{R}}$ along the class $c_1(Y_3)-[\Phi]+[\widetilde{E}_1]$. We have
$$H_2^{-\tau_{Y_4}}(Y_{4,\mathbb{R}};\mathbb{Z}) = \mathbb{Z}c_1(Y_4) \oplus \mathbb{Z}[\widetilde{E}_1] \qquad \text{and} \qquad \mathbb{R}Y_{4,\mathbb{R}} = \mathbb{R}P^2 \sqcup 3S^2.$$
Note that $Y_{4,\mathbb{R}}$ is the blow-up at a real point of an $\mathbb{R}$-minimal real del Pezzo surface of degree 2.
- $Y_{5,\mathbb{R}}$ is the surgery of $Y_{4,\mathbb{R}}$ along the class $c_1(Y_4)-[\widetilde{E}_1]$. We have
$$H_2^{-\tau_{Y_5}}(Y_{5,\mathbb{R}};\mathbb{Z}) = \mathbb{Z}c_1(Y_5) \qquad \text{and} \qquad \mathbb{R}Y_{5,\mathbb{R}} = \mathbb{R}P^2 \sqcup 4S^2.$$
Note that $Y_{5,\mathbb{R}}$ is the only (up to deformation) $\mathbb{R}$-minimal real del Pezzo surface of degree 1.

Note that $Y''_{1,\mathbb{R}}$ is also the real blow-up of $\mathbb{R}P^2$ at four pairs of complex conjugated points, and that $Y_{2,\mathbb{R}}$ can also be constructed as the surgery of $Y_{1,\mathbb{R}}$ or $Y'_{1,\mathbb{R}}$ along the class $[\Phi]-[\widetilde{E}_2]-[\widetilde{E}_3]$, as well as of $Y''_{1,\mathbb{R}}$ along the class $[\Phi]-[\widetilde{E}_4]-[\widetilde{E}_5]$. Hence we have the following diagram

$$\begin{tikzcd}
 & & & Y'_{1,\mathbb{R}} & \\
Y_{5,\mathbb{R}} \arrow[r, "c_1(Y_4)-[\widetilde{E}_1]"] & Y_{4,\mathbb{R}} \arrow[r, "c_1(Y_3)-[\Phi]+[\widetilde{E}_1]"] & Y_{3,\mathbb{R}} \arrow[r, "[\Phi]-[\widetilde{E}_6]-[\widetilde{E}_7]"] & Y_{2,\mathbb{R}} \arrow[u, "[\Phi]-[\widetilde{E}_2]-[\widetilde{E}_3]"'] \arrow[r, "[\Phi]-[\widetilde{E}_2]-[\widetilde{E}_3]"] \arrow[d, "[\Phi]-[\widetilde{E}_4]-[\widetilde{E}_5]"] & Y_{1,\mathbb{R}} \\
 & & & Y''_{1,\mathbb{R}} &
\end{tikzcd}$$

where $Y_\mathbb{R} \xrightarrow{\gamma} X_\mathbb{R}$ means that $X_\mathbb{R}$ and $Y_\mathbb{R}$ are related by a surgery along the class γ, and that $\chi(\mathbb{R}Y) = \chi(\mathbb{R}X)+2$.

Lemma 5.6. *Let $Y_{\mathbb{R}}$ be a real del Pezzo surface of degree 1, L be a connected component of $\mathbb{R}Y_{\mathbb{R}}$, and $\gamma \in H_2^{-\tau_Y}(Y;\mathbb{Z})$ be a class realized by an exceptional rational curve. Then*

$$W_{Y_{\mathbb{R}},L,[\mathbb{R}Y\setminus L]}(c_1(Y)+\gamma;0) = -\chi(\mathbb{R}Y)+1.$$

Proof. Let us denote by $Z_{\mathbb{R}}$ be the real del Pezzo surface of degree 2 obtained by blowing down the exceptional rational curve realizing the class γ. Then the class $c_1(Y)+\gamma$ is the pull-back of the class $c_1(Z)$, and we have

$$W_{Y_{\mathbb{R}},L,[\mathbb{R}Y\setminus L]}(c_1(Y)+\gamma;0) = W_{Z_{\mathbb{R}},L,[\mathbb{R}Z\setminus L]}(c_1(Z);0) = -\chi(\mathbb{R}Z)+2.$$

This latter equality can be proved for example by an Euler characteristic computation as in [DK00, Proposition 4.7.3]. Now the result follows since $\chi(\mathbb{R}Y)=\chi(\mathbb{R}Z)-1$. ☺

Proposition 5.7. *Let $Y_{\mathbb{R}}$ be a real del Pezzo surface of degree 1, and L be a connected component of $\mathbb{R}Y$. Then we have the following Welschinger invariants*

$Y_{\mathbb{R}}$	$Y_{5,\mathbb{R}}$	$Y_{4,\mathbb{R}}$	$Y_{3,\mathbb{R}}$	$Y_{2,\mathbb{R}}$	$Y_{1,\mathbb{R}}$	$Y'_{1,\mathbb{R}}$	$Y''_{1,\mathbb{R}}$
$W_{Y_{\mathbb{R}},L,[\mathbb{R}Y\setminus L]}(2c_1(Y);0)$	*30*	*18*	*10*	*6*	*6*	*6*	*6*

In particular, $W_{Y_{\mathbb{R}},L,[\mathbb{R}Y\setminus L]}(2c_1(Y);0)$ does not depend on the choice of L.

Proof. The cases $Y_{1,\mathbb{R}}$ and $Y'_{1,\mathbb{R}}$ have been computed in [Bru15][3]. Let $X_{\mathbb{R}}$ and $Y_{\mathbb{R}}$ be two real algebraic surfaces as in the proposition and such that $Y_{\mathbb{R}} \xrightarrow{\gamma} X_{\mathbb{R}}$. Suppose that there exists an immersed J-holomorphic curve C in X realizing the class $2c_1(X)-k\gamma$. Since the class γ satisfies

$$\gamma^2=-2 \qquad \text{and} \qquad \gamma\cdot c_1(X)=0,$$

it follows from the adjunction formula that C has genus at most $2-2k^2$. In particular we deduce that $k=0$ or 1. Hence by Theorem 3.8 we have

$$W_{X_{\mathbb{R}},L,[\mathbb{R}X\setminus L]}(2c_1(X);0) = W_{Y_{\mathbb{R}},L,[\mathbb{R}Y\setminus L]}(2c_1(Y);0) - 2W_{Y_{\mathbb{R}},L,[\mathbb{R}Y\setminus L]}(2c_1(Y)-\gamma;0).$$

Now the result follows from the values of

$$W_{Y_{1,\mathbb{R}},L,[\mathbb{R}Y_{1,\mathbb{R}}\setminus L]}(2c_1(Y_1);0) \quad \text{and} \quad W_{Y'_{1,\mathbb{R}},L,[\mathbb{R}Y'_{1,\mathbb{R}}\setminus L]}(2c_1(Y'_1);0),$$

and from Lemma 5.6. ☺

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