

GEODESICS IN GENERALIZED FINSLER SPACES: SINGULARITIES IN DIMENSION TWO

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ABSTRACT. We study singularities of geodesics flows in two-dimensional generalized Finsler spaces (pseudo-Finsler spaces). Geodesics are defined as extremals of a certain auxiliary functional whose non-isotropic extremals coincide with extremals of the action functional. This allows us to consider isotropic lines as (unparametrized) geodesics.

INTRODUCTION

This paper is a study of singularities of geodesics flows in generalized Finsler spaces (pseudo-Finsler spaces). This is a natural development of ongoing research on understanding the geometry of surfaces endowed with a signature-varying pseudo-Riemannian metric; see [8, 10, 11, 16, 19, 20, 21] and the references therein. One of the purposes of this paper is to compare singularities of geodesics flows in pseudo-Finsler and pseudo-Riemannian metrics. On the other hand, the interest in pseudo-Finsler spaces is motivated by physical applications; see, e.g., [1, 5].

Following [23], by a pseudo-Finsler space, we mean a manifold M , $\dim M = m$, with coordinates (x_i) endowed with a metric function $\bar{f}(x_i; \dot{x}_i) = \bar{F}(x_i; \dot{x}_i)^{\frac{1}{n}}$, where $\bar{F}: TM \rightarrow \mathbb{R}$ is positively homogeneous in (\dot{x}_i) of degree n and smooth on the complement of the zero section of TM (a more detailed definition is given in Sections 1.1, 1.2). A well-known example is *Berwald-Moor* space (M, \bar{f}) , where $\bar{f}(x_i; \dot{x}_i) = (\dot{x}_1 \cdots \dot{x}_n)^{\frac{1}{n}}$, $n = m$; see, e.g., [6, 14, 23].

This paper starts with a discussion of the notion of geodesics in Finsler and pseudo-Finsler spaces with $n \geq 3$ (Section 1). Here, we use the variational definitions of geodesics [7, 23]. In contrast to pseudo-Riemannian spaces ($n = 2$), where naturally parametrized geodesics of all types (including isotropic) can be defined as extremals of the action functional, in pseudo-Finsler spaces a similar definition is not correct for isotropic lines. The solution of this problem is either to exclude isotropic lines from consideration or to find a natural extension of the definition of geodesics.

In the present paper, we choose the second way. Based on a simple variational property, we define geodesics as extremals of a certain auxiliary functional whose non-isotropic extremals coincide with extremals of the action functional. In this direction, we have the following result: in the case $m = 2$, all isotropic lines are (unparametrized) geodesics.

In Section 2, we consider singularities of the geodesic flows in pseudo-Finsler spaces (M, \bar{f}) , where $m = 2$ and \bar{F} is a polynomial in (\dot{x}_i) of degree $n \geq 3$. The main results are presented in Section 2.2, where we consider the case $n = 3$ in detail. It is proved that singularities of the geodesic flow are connected with the degeneracy of the isotropic lines net. Namely, if the function \bar{F} is generic, the manifold M contains two open domains M_+ and M_- separated by a curve M_0 so that, at every point $q \in M_+$ (resp., M_-), there exist 3 (resp., 1) different isotropic

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directions and isotropic lines which are tangent at $q \in M_0$. Singularities of the geodesic flow appear in the domain M_- and on the curve M_0 .

Section 3 is devoted to a special case: pseudo-Finsler spaces (M, \bar{f}) , where M is a surface in n -dimensional Berwald–Moor space. The corresponding function \bar{F} is a non-generic polynomial in (\dot{x}_i) of degree n . In this case, the domain $M_- = \emptyset$, and singularities of the geodesic flow appear on the curve M_0 only.

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1. VARIATIONAL DEFINITION OF GEODESICS

1.1. **Finsler spaces.** Consider a smooth (here and below, by smooth, we mean C^∞ unless otherwise stated) manifold M , $\dim M = m$, with coordinates (x_i) and a function $\bar{F}(x_i; \dot{x}_i): TM \rightarrow \mathbb{R}$ that is positively homogeneous of degree $n \geq 2$ in (\dot{x}_i) and smooth on the complement of the zero section of TM .

Define the function $\bar{f}(x_i; \dot{x}_i) = \bar{F}^{\frac{1}{n}}(x_i; \dot{x}_i)$, which is positively homogeneous of degree 1 in (\dot{x}_i) . The pair (M, \bar{f}) or, equivalently, (M, \bar{F}) is a *Finsler space* (in the classic sense), if the following conditions hold:

B. $\bar{F}(x_i; \dot{x}_i) > 0$ if $|\dot{x}_1| + \dots + |\dot{x}_m| \neq 0$.

C. The Hessian of the function \bar{f}^2 with respect to (\dot{x}_i) is positive definite, that is,

$$(1.1) \quad \sum_{i,j=1}^m \frac{\partial^2(\bar{f}^2)}{\partial \dot{x}_i \partial \dot{x}_j} \xi_i \xi_j > 0 \quad \text{if} \quad \sum_{i=1}^m |\xi_i| \neq 0.$$

Here, we use the letters B and C to preserve the notations from the book [23], to which we shall refer. The quadratic form (1.1) is called *the fundamental tensor*, and \bar{f} (positive and smooth on the complement of the zero section of TM) is called the *metric function* on M .

The metric function $\bar{f}(x_i; \cdot)$ defines a Minkowski norm on each tangent space $T_x M$. For a curve $\gamma: I \rightarrow M$, it allows us to define the length and the action functionals similarly to Riemannian metrics:

$$(1.2) \quad J^{(\nu)}(\gamma) = \int_I \bar{f}^\nu(x_i; \dot{x}_i) dt = \int_I \bar{F}^{\frac{\nu}{n}}(x_i; \dot{x}_i) dt, \quad \dot{x}_i = \frac{dx_i}{dt},$$

with $\nu = 1$ (length) and $\nu = 2$ (action), see, e.g., [13, 23]. As in the Riemannian case, the length functional $J^{(1)}$ is invariant with respect to reparametrizations of γ , while $J^{(2)}$ is not.

Parametrized geodesics can be defined as extremals of the action functional $J^{(2)}$, the corresponding parametrization is called *natural* or *canonical* (it coincides with the arc-length parametrization, where $ds = \bar{f}$).

Non-parametrized geodesics can be defined as extremals of any one of the functionals $J^{(2)}$ and $J^{(1)}$. The difference between using $J^{(2)}$ and $J^{(1)}$ is the following. In the first case, we simply forget the natural parametrization of the extremals of $J^{(2)}$, while in the second case the Euler-Lagrange system with the Lagrangian $\bar{f}(x_i; \dot{x}_i)$ contains $m - 1$ independent equations only [23]. This reflects the fact that the length functional $J^{(1)}(\gamma)$ is invariant with respect to reparametrizations of γ . Using this degree of freedom and assuming that we deal with continuously differentiable geodesics with definite tangent directions at all points, one can set (at least, locally) the parameter t equal to one of the coordinates x_i , and consequently, reduce the Euler-Lagrange system for $J^{(1)}(\gamma)$.

From now on, we use the following general notation. Let $\bar{\Phi}(x_i; \dot{x}_i)$ be a function on TM which is positively homogeneous of degree k in (\dot{x}_i) , then the formula

$$(1.3) \quad \Phi = \frac{\bar{\Phi}}{\dot{x}_1^k}$$

defines a function on the projectivized tangent bundle PTM . For instance, put $x = x_1$ and $y_i = x_i, p_i = dy_i/dx$ for $i = 2, \dots, m$. This yields

$$(1.4) \quad \frac{d}{dx} \left(\frac{\partial f}{\partial p_i} \right) = \frac{\partial f}{\partial y_i}, \quad f(x, y_i, p_i) = \frac{\bar{f}(x_i; \dot{x}_i)}{\dot{x}_1}, \quad i = 2, \dots, m.$$

The passage to equation (1.4) is the standard projectivization $\Pi: TM \rightarrow PTM$ of the tangent bundle. Moreover, non-parametrized geodesics can be defined as extremals of $J^{(\nu)}$ with arbitrary $\nu \geq 1$, on the basis of the following simple property (see, e.g., [15]).

Lemma 1. *Let $\psi: \mathbb{R} \rightarrow \mathbb{R}$ be a function such that $\psi \circ \bar{F}$ is C^2 -smooth on the complement of the zero section of TM and $\psi'(s) \neq 0$ for all $s \neq 0$. Then non-parametrized extremals of the functional*

$$(1.5) \quad J_\psi(\gamma) = \int_I \psi \circ \bar{F}(x_i; \dot{x}_i) dt, \quad \dot{x} = \frac{dx}{dt}, \quad \dot{y} = \frac{dy}{dt},$$

coincide with non-parametrized extremals of $J_{\text{id}}(\gamma)$, where id is the identity map.

Proof. The Euler-Lagrange equation of $J_\psi(\gamma)$ reads

$$(1.6) \quad \frac{d}{dt} \left(\psi' \circ \bar{F}(x_i; \dot{x}_i) \cdot \frac{\partial \bar{F}}{\partial \dot{x}_i} \right) = \psi' \circ \bar{F}(x_i; \dot{x}_i) \cdot \frac{\partial \bar{F}}{\partial x_i}, \quad i = 1, \dots, m.$$

In light of the condition $\bar{F}(x_i; \dot{x}_i) > 0$, every curve γ admits the arc-length parametrization, that is, $\bar{F}(x_i; \dot{x}_i) \equiv c \neq 0$ along γ . Using the arc-length parametrization in (1.5), after reducing the constant factor $\psi' \circ \bar{F}(x_i; \dot{x}_i) = \psi'(c)$ in both sides of (1.6), we get the Euler-Lagrange equation of the functional $J_{\text{id}}(\gamma)$. \square

Thus non-parametrized geodesics can be defined as extremals of $J_\psi(\gamma)$ with arbitrary function ψ from Lemma 1, in particular, of the functional $J_{\text{id}}(\gamma)$, which is equal to $J^{(\nu)}(\gamma)$ with $\nu = n$ from (1.2). In classical Finsler spaces this extended definition of geodesics gives us nothing essentially new, but it can be useful for generalized Finsler spaces considered below.

1.2. Generalized Finsler spaces. A generalization of the notion of a Finsler space may be obtained if conditions B and C are dropped, such spaces are sometimes called *special Finsler* or *pseudo-Finsler*. Here we take the liberty to cite a passage from the classical book [23] (page 265):

Again, it should be remarked that very frequently the metric function which is given by a homogeneous Lagrangian function of a dynamical system does not always satisfy conditions B and C. The singularities which may occur as a result of the relaxation of condition C are usually ignored, but it is well possible that an investigation of these singularities in connection with physical applications cannot be avoided and might furthermore prove to be fruitful.

From now on, we will consider pseudo-Finsler spaces (M, \bar{F}) , where the function \bar{F} is not assumed to satisfy conditions B and C.

The absence of condition B brings us to the existence of the *isotropic hypersurface* \mathcal{F} given by the equation $\bar{F}(x_i; \dot{x}_i) = 0$ in TM or, equivalently, $F(x, y_i, p_i) = 0$ in PTM . The Euler-Lagrange equation for the functional $J^{(\nu)}(\gamma)$ with $\nu < n$ is not defined on \mathcal{F} , since the derivatives of

$\bar{F}^{\frac{\nu}{n}}(x_i; \dot{x}_i)$ are discontinuous on \mathcal{F} . This explains the advantage of the functional $J^{(n)}(\gamma)$ for the definition of geodesics compared to $J^{(\nu)}(\gamma)$ with $\nu < n$.

The Euler-Lagrange equation for the functional $J^{(n)}(\gamma)$ reads

$$(1.7) \quad \sum_{j=1}^m \frac{\partial^2 \bar{F}}{\partial \dot{x}_i \partial \dot{x}_j} \ddot{x}_j + \sum_{j=1}^m \frac{\partial^2 \bar{F}}{\partial \dot{x}_i \partial x_j} \dot{x}_j = \frac{\partial \bar{F}}{\partial x_i}, \quad i = 1, \dots, m,$$

or, equivalently,

$$(1.8) \quad \ddot{x}_i = \frac{\bar{H}_i(x_i; \dot{x}_i)}{\bar{H}(x_i; \dot{x}_i)}, \quad \text{where } \bar{H} = \det\left(\frac{\partial^2 \bar{F}}{\partial \dot{x}_i \partial \dot{x}_j}\right), \quad i = 1, \dots, m,$$

and \bar{H}_i are the determinants defined from (1.7) by Cramer's rule. It is not hard to see that the functions \bar{H}_i are positively homogeneous of degree n in (\dot{x}_i) and \bar{H} is positively homogeneous of degree $n - 2$ in (\dot{x}_i) .

Similarly to (1.4), the projectivization $\Pi: TM \rightarrow PTM$ sends equation (1.8) to

$$(1.9) \quad p_i = \frac{dy_i}{dx}, \quad \frac{dp_i}{dx} = \frac{dp_i}{dt} \left(\frac{dx}{dt}\right)^{-1} = \frac{1}{\dot{x}} \frac{d}{dt} \left(\frac{\dot{y}_i}{\dot{x}}\right) = \frac{\ddot{y}_i \dot{x} - \dot{y}_i \ddot{x}}{\dot{x}^3} = \frac{1}{\dot{x}^2} \frac{\bar{H}_i - p_i \bar{H}_1}{\bar{H}} = \frac{H_i - p_i H_1}{H}, \quad i = 2, \dots, m,$$

where the functions H, H_i are obtained from \bar{H}, \bar{H}_i by (1.3). Recall that, by formula (1.3), the independent variable x is the coordinate x_1 . From Lemma 1, it follows that out of the isotropic hypersurface \mathcal{F} , integral curves of (1.9) coincide with integral curves of (1.4). However, equation (1.9) is defined in the whole space PTM .

Lemma 2. *\mathcal{F} is an invariant hypersurface of both equations (1.7) and (1.9). Moreover, in the case $m = 2$ all isotropic lines are non-parametrized extremals of the functional $J^{(n)}$.*¹

Proof. After straightforward transformations, equation (1.4) gives a direction field which is parallel to

$$(1.10) \quad F^{-\mu} X \left(\frac{\partial}{\partial x} + p \frac{\partial}{\partial y} \right) + F^{-\mu} \sum_{i=2}^m Y_i \frac{\partial}{\partial p_i},$$

where $\mu = (m - 1)(2 - \frac{1}{n})$ and X, Y_i are smooth functions on PTM .

Since the vector field (1.10) is derived directly from the Euler-Lagrange equation (1.4), it is divergence-free in PTM except for the hypersurface \mathcal{F} where the factor $F^{-\mu}$ is discontinuous, and the field is not defined. By Theorem 1 [8], \mathcal{F} is an invariant hypersurface of the vector field

$$(1.11) \quad X \left(\frac{\partial}{\partial x} + p \frac{\partial}{\partial y} \right) + \sum_{i=2}^m Y_i \frac{\partial}{\partial p_i},$$

which is obtained from (1.10) by eliminating the common factor $F^{-\mu}$. By Lemma 1, integral curves of (1.11) coincide with integral curves of (1.9); hence, $X = H, Y_i = H_i - p_i H_1$, and \mathcal{F} is an invariant hypersurface of (1.9). We remark that every solution of (1.7) is obtained from a solution of (1.9) by choosing an appropriate parametrization $x_1(t)$. Thus \mathcal{F} is an invariant hypersurface of equation (1.7) also.

Finally, consider the case $m = 2$. Then $\dim PTM = 3$ and $\dim \mathcal{F} = 2$. The field of contact planes $y_2 = p_2 dx$ cuts a direction field on \mathcal{F} , which coincides with the restriction of the field

¹ This statement holds true only for $m = 2$, while in the case $m > 2$ some isotropic lines are geodesics and some of them are not. An example is given for $m = 3, n = 2$ (pseudo-Riemannian metrics) in [20].

(1.9) to \mathcal{F} . Hence the projection of integral curves from the surface $\mathcal{F} \subset PTM$ to M are isotropic lines and non-parametrized extremals of the functional $J^{(n)}$ simultaneously. \square

In accordance with the previous reasoning, one can give the following definition.

Definition 1. *The projections of integral curves of equation (1.9) from PTM to M distinguished from a point are non-parametrized geodesics in the pseudo-Finsler space (M, \bar{F}) .*

By Lemma 2, in the case $m = 2$, all isotropic lines are geodesics in the sense of the given definition.

The natural parametrization of non-isotropic geodesics is defined by equation (1.6) with $\psi(s) = s^{\frac{2}{n}}$ and coincides with the arc-length parametrization. In the case $n > 2$, the natural parametrization of isotropic geodesics is not defined, while in the case $n = 2$ it is defined by equation (1.6) with $\psi = \text{id}$.

2. POLYNOMIAL PSEUDO-FINSLER METRICS ON 2-MANIFOLDS

From now on, we consider the case when $m = 2$ and the function \bar{F} is a homogeneous polynomial of degree $n \geq 2$ in (\dot{x}_i) . Denote the coordinates on the manifold M by (x, y) .

Consider a pseudo-Finsler space with the metric function $\bar{f} = \bar{F}^{\frac{1}{n}}$, where

$$(2.1) \quad \bar{F}(x, y; \dot{x}, \dot{y}) = \sum_{i=0}^n a_i(x, y) \dot{x}^{n-i} \dot{y}^i, \quad F(x, y; p) = \sum_{i=0}^n a_i(x, y) p^i,$$

the coefficients a_i smoothly depend on (x, y) . Then equation (1.8) reads

$$(2.2) \quad \ddot{x} = \frac{\bar{H}_1}{\bar{H}}, \quad \ddot{y} = \frac{\bar{H}_2}{\bar{H}}, \quad \bar{H} = \left| \frac{\bar{F}_{\dot{x}\dot{x}}}{\bar{F}_{\dot{x}\dot{y}}} \frac{\bar{F}_{\dot{x}\dot{y}}}{\bar{F}_{\dot{y}\dot{y}}} \right|, \quad \bar{H}_1 = \left| \frac{\bar{G}_1}{\bar{G}_2} \frac{\bar{F}_{\dot{x}\dot{y}}}{\bar{F}_{\dot{y}\dot{y}}} \right|, \quad \bar{H}_2 = \left| \frac{\bar{F}_{\dot{x}\dot{x}}}{\bar{F}_{\dot{x}\dot{y}}} \frac{\bar{G}_1}{\bar{G}_2} \right|,$$

where $\bar{G}_1 = \bar{F}_x - \dot{x}\bar{F}_{\dot{x}x} - \dot{y}\bar{F}_{\dot{x}y}$ and $\bar{G}_2 = \bar{F}_y - \dot{x}\bar{F}_{\dot{x}y} - \dot{y}\bar{F}_{\dot{y}y}$.

Lemma 3. *The projectivization $\Pi: TM \rightarrow PTM$ sends equation (2.2) to*

$$(2.3) \quad p = \frac{dy}{dx}, \quad \frac{dp}{dx} = \frac{H_2 - pH_1}{H} = \frac{P}{\Delta},$$

where

$$(2.4) \quad \begin{aligned} \Delta(x, y; p) &= nF F_{pp} - (n-1)F_p^2, \\ P(x, y; p) &= nF(F_y - F_{xp} - pF_{yp}) + (n-1)F_p(F_x + pF_y). \end{aligned}$$

Proof. Taking into account (1.9), it remains for us to establish the equality $\frac{H_2 - pH_1}{H} = \frac{P}{\Delta}$, where Δ, P are defined in (2.4). Let us prove that $H = (n-1)\Delta$ and $H_2 - pH_1 = (n-1)P$, i.e., $\bar{H} = \dot{x}^{2n-4}(n-1)\Delta$ and $\bar{H}_2 - p\bar{H}_1 = \dot{x}^{2n-2}(n-1)P$. Since both sides of the two last equalities can be treated as quadratic forms on a_0, \dots, a_n with coefficients depending on \dot{x}, \dot{y} , it suffices to compare the coefficients of the monomials $a_i a_j$ in the left- and right-hand sides.

Put $\varepsilon_{ij} = 1$ if $i \neq j$ and $\varepsilon_{ij} = \frac{1}{2}$ if $i = j$. Direct calculation shows that the coefficient of the monomial $a_i a_j$, $i + j = k$, in the expression $\bar{H} = \bar{F}_{\dot{x}\dot{x}} \bar{F}_{\dot{y}\dot{y}} - \bar{F}_{\dot{x}\dot{y}}^2$ is $\alpha_{ij} \varepsilon_{ij} \dot{x}^{2(n-1)-k} \dot{y}^{k-2}$, where

$$(2.5) \quad \begin{aligned} \alpha_{ij} &= (n-i)(n-1-i)j(j-1) + (n-j)(n-1-j)i(i-1) - 2ij(n-i)(n-j) = \\ &= (n-1)(n(k^2 - k - 4ij) + 2ij). \end{aligned}$$

On the other hand, the coefficient of the monomial $a_i a_j$, $i + j = k$, in the expression

$$\Delta = nF F_{pp} - (n-1)F_p^2$$

is $\beta_{ij}\varepsilon_{ij}p^{k-2}$, where

$$(2.6) \quad \beta_{ij} = n(i(i-1) + j(j-1)) - 2ij(n-1) = n(k^2 - k - 4ij) + 2ij.$$

From (2.5) and (2.6), we have $\alpha_{ij} = (n-1)\beta_{ij}$, that proves $\overline{H} = \dot{x}^{2n-4}(n-1)\Delta$. The proof of the equality $\overline{H}_2 - p\overline{H}_1 = \dot{x}^{2n-2}(n-1)P$ is similar. \square

Remark 1. From formula (2.4), it follows that Δ and P are polynomials in p of degrees not greater than $2n-4$ and $2n-1$, respectively. For instance,

$$\Delta(x, y; p) = (2na_n a_{n-2} - (n-1)a_{n-1}^2)p^{2n-4} + \dots + 2na_0 a_2 - (n-1)a_1^2.$$

For our further purposes, it is convenient to write equation (2.3) as the field

$$(2.7) \quad \Delta \left(\frac{\partial}{\partial x} + p \frac{\partial}{\partial y} \right) + P \frac{\partial}{\partial p}.$$

The field (2.7) is defined on the whole space PTM including the isotropic surface \mathcal{F} . The field of contact planes $dy = p dx$ defines on \mathcal{F} a direction field whose integral curves correspond to isotropic lines, while all remaining integral curves of the field (2.7) (that do not belong entirely to the isotropic surface) correspond to non-isotropic geodesics.

In accordance with Definition 1, non-parametrized geodesics in the pseudo-Finsler space (M, \overline{F}) are the projections of integral curves of the field (2.7) from PTM to M distinguished from a point. Singularities of geodesics occur at the points of PTM where $\Delta(x, y; p)$ vanishes. To describe the locus of such points, we use the following lemma.

Lemma 4. *Given a polynomial*

$$(2.8) \quad \Phi(p) = \prod_{i=1}^n (p + \gamma_i), \quad \gamma_i \in \mathbb{R}, \quad n \geq 2,$$

consider the polynomial

$$(2.9) \quad \Delta(p) = n\Phi(p)\Phi''(p) - (n-1)\Phi'(p)^2.$$

Then the following statements hold:

- (a) $\Delta \equiv 0$ if and only if $\gamma_1 = \dots = \gamma_n$.
- (b) Suppose that $\gamma_i \neq \gamma_j$ for at least one pair i, j . Then p is a real root of the polynomial Δ if and only if p is a multiple root of the polynomial Φ .
- (c) If p is a double root of Φ and $n \geq 3$, then p is a double root of Δ .

Proof. The implications $\gamma_1 = \dots = \gamma_n \Rightarrow \Delta \equiv 0$ in (a) and $\Phi(p) = \Phi'(p) = 0 \Rightarrow \Delta(p) = 0$ in (b) are trivial. The implication $\Delta \equiv 0 \Rightarrow \gamma_1 = \dots = \gamma_n$ in (a) follows from (b). Indeed, assume that $\Delta \equiv 0$ holds and any two of the numbers $\gamma_1, \dots, \gamma_n$ are not equal. By (b), $\Delta(p) = 0$ implies $\Phi(p) = 0$. Hence $\Phi \equiv 0$, which contradicts (2.8).

Statement (c) is also trivial: differentiating (2.9) twice, from $\Phi(p) = \Phi'(p) = 0$, we get

$$\Delta(p) = \Delta'(p) = 0 \text{ and } \Delta''(p) = (2-n)\Phi''(p)^2 \neq 0, \text{ if } \Phi''(p) \neq 0 \text{ and } n \neq 2.$$

It remains to prove the implication $\Delta(p) = 0$ and $p \in \mathbb{R} \Rightarrow \Phi(p) = \Phi'(p) = 0$ in statement (b). Assume that $\gamma_i \neq \gamma_j$ for at least one pair i, j and there exists $p_* \in \mathbb{R}$ such that $\Phi(p_*) \neq 0$. Making the change of variables $p \mapsto p - p_*$, without loss of generality we can assume that $p_* = 0$. Then $\Phi(0) = \gamma_1 \dots \gamma_n \neq 0$, and

$$\Phi'(0) = \sum_{i=1}^n \alpha_i \Phi(0), \quad \Phi''(0) = 2 \sum_{i < j} \alpha_i \alpha_j \Phi(0), \quad \alpha_i = \frac{1}{\gamma_i}.$$

Substituting the above formulae in (2.9), after straightforward transformations we get

$$(2.10) \quad \Delta(0) = \Phi^2(0) \left(2n \sum_{i < j} \alpha_i \alpha_j - (n-1) \left(\sum_{i=1}^n \alpha_i \right)^2 \right) = -\Phi^2(0) \varphi_n(\alpha_1, \dots, \alpha_n),$$

where

$$\varphi_n(\alpha_1, \dots, \alpha_n) = n \sum_{i=1}^n \alpha_i^2 - \left(\sum_{i=1}^n \alpha_i \right)^2.$$

Let us prove that for any $n \geq 2$ the form $\varphi_n(\alpha_1, \dots, \alpha_n) \geq 0$ and $\varphi_n(\alpha_1, \dots, \alpha_n) = 0$ if and only if $\alpha_1 = \dots = \alpha_n$. Indeed, consider the vectors $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (1, \dots, 1)$ in n -dimensional Euclidean space with the standard inner product. Then the Cauchy–Schwarz inequality $(\alpha, \beta)^2 \leq (\alpha, \alpha)(\beta, \beta)$ gives the required assertion.

By our assumption, $\alpha_i \neq \alpha_j$ for at least one pair i, j . Then $\varphi_n(\alpha_1, \dots, \alpha_n) > 0$, and equality (2.10) implies that $\Delta(0) = 0 \Leftrightarrow \Phi(0) = 0$. Moreover, from (2.9) it follows that

$$\Delta(p_*) = \Phi(p_*) = 0 \Rightarrow \Phi'(p_*) = 0,$$

i.e., $p_* = 0$ is a multiple root of Φ . The lemma is proved. \square

Remark 2. Obviously, the implication $\Phi = \Phi' = 0 \Rightarrow \Delta = 0$ holds true for all polynomial Φ , not necessarily (2.8). However, the inverse implication $\Delta = 0 \Rightarrow \Phi = 0$ is not valid if Φ has a complex root. The reason for this is easily ascertained: the inequality $\varphi_n(\alpha_1, \dots, \alpha_n) \geq 0$ is not valid if among the numbers α_i some are complex.

For example, consider the polynomial $\Phi = p^3 + p$ with the unique real root $p = 0$. Then the corresponding polynomial $\Delta = 2(3p^2 - 1)$ has two real roots, none of those coincides with 0. Moreover, the polynomial $\Phi = p^4 + 6p^2 + 1$ does not have real roots at all, while the corresponding polynomial $\Delta = 48(p^2 - 1)^2$ has two double roots $p = \pm 1$.

Lemma 4 gives a simple geometrical description of the singular locus of equation (2.3) for the domain $M' \subset M$, where the pseudo-Finsler space (M, \bar{F}) has m isotropic lines passing through every point of M' , i.e., the polynomial $F(p)$ has m real roots (taking into account the multiplicity and possibly including $p = \infty$). For $(x, y) \in M'$, the function $\Delta(x, y; p)$ vanishes if and only if at least two of m isotropic lines are tangent at (x, y) and p is the corresponding tangential direction. We remark that this statement is not valid for the complement of M' , where the polynomial $F(p)$ has complex roots.

This question will be considered in detail for $n = 3$.

2.1. Pseudo-Riemannian metrics. By Remark 1, in the case $n = 2$ (pseudo-Riemannian metrics) Δ is a zero degree polynomial in p , that is, Δ does not depend on p . Moreover, it is easy to check that $\Delta = -4D_{[F]}$, where $D_{[F]}$ means the discriminant of the quadratic polynomial F . Hence the locus of singularities of equation (2.3) coincides with the discriminant curve of the implicit differential equation $F(x, y; p) = 0$. It is not hard to see that the equation $\Delta(x, y) = 0$ defines an invariant surface of the field (2.7) filled with integral curves whose projections $PTM \rightarrow M$ are points forming the discriminant curve.

This property leads to a curious phenomenon: geodesics cannot pass through a point (x, y) of the discriminant curve in arbitrary tangential directions, but only in admissible directions p defined by the condition $P(x, y; p) = 0$. Generically, $P(x, y; p)$ is a cubic polynomial in p and the number of admissible directions is 1 or 3. Singularities of the geodesic flows in pseudo-Riemannian metrics are studied in detail (the interested reader is referred to the papers [8, 19, 20] devoted to 2-dimensional pseudo-Riemannian metrics; similar results for 3-dimensional pseudo-Riemannian metrics were announced in [16]).

It should be remarked that the case $n = 2$ is exceptional from the viewpoint of Finsler and pseudo-Finsler geometry ($n > 2$). In the case $n > 2$, Δ generically depends on p and the notion of admissible directions does not appear. Geodesics pass through every point of M in all possible directions, but some directions at some points are singular. In other words, only points of the space PTM may have the property of being singular.

In the rest of the paper, we consider the case $n = 3$ (cubic pseudo-Finsler metrics) in detail.

2.2. Cubic pseudo-Finsler metrics. Let $D_{[F]}$ and $D_{[\Delta]}$ be the discriminants of the cubic polynomial $F(x, y; p)$ and the quadratic polynomial $\Delta(x, y; p)$ in p , respectively. A direct calculation shows that $D_{[\Delta]} = -12D_{[F]}$.

Introduce the following stratification of the manifold M . The open domains M_+, M_- are defined by the conditions $D_{[F]} > 0, D_{[F]} < 0$, respectively. Generically, M_+, M_- are separated by the discriminant curve $M_0: D_{[F]} = 0$, which consists of regular points (the cubic polynomial F has one prime root and one double root) and cusps (F has a triple root). By $M_{0,1}$ denote the set of all regular points of M_0 , while $M_{0,0} = M_0 \setminus M_{0,1}$. The discriminant of the quadratic polynomial Δ is strictly negative in M_+ ; hence singularities of equation (2.3) occur only in M_- and M_0 . Further we exclude from consideration the stratum $M_{0,0}$ of dimension zero, and consider only M_- and $M_{0,1}$.

In a neighborhood of every point of $M \setminus M_{0,0}$, the cubic polynomial F has at least one prime real root $p_*(x, y)$ smoothly depending on x, y . To simplify calculations, choose local coordinates such that the integral curves of the vector field $\frac{dy}{dx} = p_*(x, y)$ (one of three families of isotropic lines) become $x = \text{const}$. This yields $a_3(x, y) \equiv 0$ and

$$(2.11) \quad \begin{aligned} F &= ap^2 + 2bp + c, \quad \Delta = -2(ap + b)^2 + 6(ac - b^2), \quad D_{[F]} = 4a^2(b^2 - ac), \\ M_{\pm} &= \{\pm(b^2 - ac) > 0, a \neq 0\}, \quad M_{0,1} = \{b^2 - ac = 0, a \neq 0\}, \\ P &= 3F(F_y - F_{xp} - pF_{yp}) + 2F_p(F_x + pF_y). \end{aligned}$$

2.2.1. Singularities in the stratum M_- . At every point in M_- , the quadratic equation $\Delta = 0$ has two prime real roots

$$(2.12) \quad p_{1,2} = \frac{\pm\delta - b}{a}, \quad \delta = \sqrt{3(ac - b^2)},$$

and the domain M_- is filled with two transverse families of integral curves of the binary implicit differential equation $\Delta = 0$, which we shall call *singular lines* of the metric.

Consider the curves $S_i \subset M_-$ defined by the equations $P(x, y; p_i) = 0, i = 1, 2$, where P is defined in (2.11). They can be also considered as the branches of the locus $\text{res}(\Delta, P) = 0$, where “res” means the resultant of two polynomials in p . In the space PTM , consider the corresponding curves

$$\bar{S}_i = \{(x, y; p_i) : (x, y) \in S_i\}, \quad i = 1, 2,$$

which consist of singular points of the field (2.7).

Let Γ_q denote the family of geodesics outgoing from a point $q = (x, y)$. The simplest type of singularities of the geodesic flow (codimension 0) is given in the following theorem.

Theorem 1. *Let $q \in M_-$ and $(q; p_i) \notin \bar{S}_i, i = 1, 2$. Then there exists a unique geodesic passing through the point q with tangential direction p_i : a semicubic parabola with the cusp at q . In particular, if $q \in M_- \setminus (S_1 \cup S_2)$, the family Γ_q contains two semicubic parabolas with tangential directions p_i , while geodesics with all remaining directions at q are smooth.*

Proof. If $P(q; p_i) \neq 0$, then, by the standard existence and uniqueness theorem, the field (2.7) has a unique integral curve γ_i passing through the point $(q; p_i)$. From the conditions $a \neq 0, ac - b^2 \neq 0$, it follows that Δ and Δ_p do not vanish simultaneously; see (2.11).

Hence the curve γ_i has first order tangency with the *vertical* direction (the vertical direction in the space PTM is called the p -direction, i.e., the kernel of the natural projection $PTM \rightarrow M$) at the point $(q; p_i)$, and the projection of the curve γ_i to M is a semicubic parabola with the cusp at q . \square

Example 1. Let F be given by formula (2.11) with $a = 1$ and $b = 0$. Then $\Delta = 2(3c - p^2)$ and $P = 7c_y p^2 + 4c_x p + 3cc_y$. The strata M_- and $M_{0,1}$ are defined by the conditions $c(x, y) > 0$ and $c(x, y) = 0$, respectively, and the curves S_i are $c_x = \pm 2c_y \sqrt{3c}$.

I. Put $c(x, y) = -x$ (Fig. 1, left). Then $S_1 = S_2 = \emptyset$ and the semiplane $x > 0$ (M_+) is filled with the net of isotropic lines $y = \pm \frac{2}{3}x^{\frac{3}{2}} + \text{const}$ (dashed curves), while the semiplane $x < 0$ (M_-) is filled with the net of singular lines $y = \pm \frac{2}{\sqrt{3}}(-x)^{\frac{3}{2}} + \text{const}$ (dotted curves). Cusps appear when geodesics (solid curves) are tangent to singular lines. We remark that geodesics pass from M_- to M_+ or vice versa through $M_{0,1}$ (the y -axis) without singularity if they intersect the y -axis with any non-isotropic tangential direction $p \neq 0$. Otherwise, equation (2.3) has a singularity. As we shall see in Section 2.2.2, at such points there exist a one-parameter family of geodesics outgoing in both domains M_+ and M_- with the common tangential direction $p = 0$, and the prolongation of geodesics through $M_{0,1}$ is not naturally defined.

II. Put $c(x, y) = \alpha y^2 - x$ with $\alpha \neq 0$. Then $S_i \neq \emptyset$, $i = 1, 2$, but both curves S_i do not pass through the origin. In a neighborhood of the origin that does not contain the curves S_i , geodesics are presented in Fig. 1 (right). Here, for definiteness, we assume $\alpha > 0$. All notations have the same meanings as before.

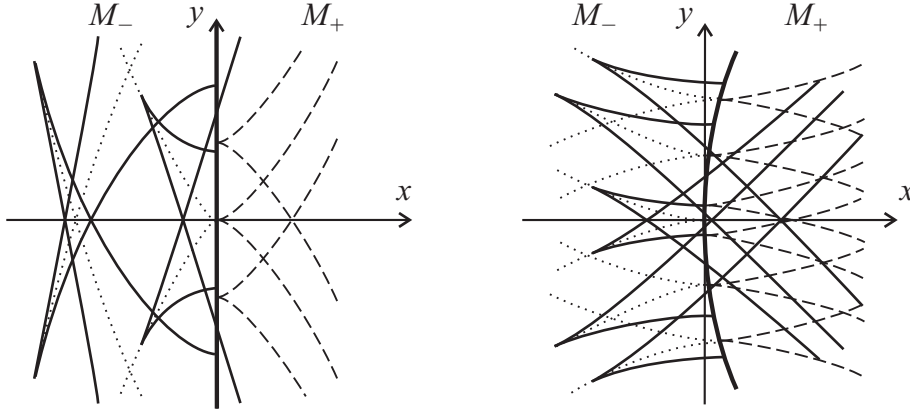


FIGURE 1. Example 1: I and II (left and right, resp.). The stratum $M_{0,1}$ (the y -axis on the left and the parabola on the right) is depicted with the bold solid line. Solid and dotted curves present geodesics and singular lines, respectively. Dashed curves present isotropic lines.

The next type of singularities of the geodesic flow in the domain M_- (codimension 1) is connected with vanishing of the field (2.7). This field belongs to a special class of vector fields whose singular points are not isolated, but form a manifold W of codimension 2 in the phase space. Yet such fields appear in many problems, see, e.g., [3, 8, 12, 17, 18, 22, 24]. It is convenient to express the above condition in the following algebraic form: the germs of all components of the field at every singular point belong to the ideal I (in the ring of smooth germs) generated by two of them.

The spectrum of the linear part of such a field (for brevity, we shall call it the spectrum of the field) contains only two non-zero eigenvalues $\lambda_{1,2}$, which play a prominent role in establishing the local normal form of the field. For instance, all components of the field (2.7) belong to the ideal $I = \langle \Delta, P \rangle$, and the set of singular points $W = \overline{S}_1 \cup \overline{S}_2$. The eigenvalues $\lambda_{1,2}$ and the corresponding eigenvectors are described by the following lemma.

Lemma 5.

1. The resonance $\lambda_1 + \lambda_2 = 0$ holds at all points $(q; p) \in \overline{S}_i$, i.e., $\lambda_{1,2}$ are real or pure imaginary numbers with opposite signs.
2. The following conditions are equivalent:
 - 2.1. The eigenvalues λ_1, λ_2 at $(q; p) \in \overline{S}_i$ are not equal to zero.
 - 2.2. \overline{S}_i is a regular curve transversal to the contact plane $pdx - dy = 0$ at $(q; p)$.
 - 2.3. S_i is a regular curve and the direction p_i is transversal to S_i at q .
3. Generically, at almost all points $(q; p) \in \overline{S}_i$, conditions 2.1–2.3 hold.

Proof. Without loss of generality, suppose that $q = 0$ (the origin) belongs to S_1 and choose local coordinates centered at 0 that preserve the lines $x = \text{const}$ and send integral curves of the vector field $\frac{dy}{dx} = p_1(x, y)$ to parallel lines $y = \text{const}$. The existence of such local coordinates follows from the general fact: if V_1 and V_2 are smooth vector fields on the plane transversal at the point 0, then in a neighborhood of 0 there exist local coordinates such that integral curves of V_1 and V_2 coincide with the coordinate lines.

Then the polynomials F , Δ , $D_{[F]}$ have the form (2.11) and the identities $p_1 \equiv 0$ and $b(x, y) \equiv \delta(x, y)$ hold. Note that the first of them implies $3ac \equiv 4b^2$. From $b^2 - ac < 0$, it follows that $ac > 0$. Taking into account $3ac \equiv 4b^2$, we conclude that none of the coefficients a, b, c vanishes at 0. Below, we present the proof for the stratum S_1 given by the equation $3c(c_y - 2b_x) + 4bc_x = 0$. The proof for the stratum S_2 is similar.

Let Λ be the matrix of the linear part of the field (2.7) and Λ_1 be the matrix of the Pfaffian system $d\Delta = 0$, $dP = 0$, $pdx - dy = 0$ considered at arbitrary point $(q; p) \in \overline{S}_1$, that is, for $q \in S_1$ and $p = 0$:

$$\Lambda = \begin{pmatrix} \Delta_x & \Delta_y & \Delta_p \\ 0 & 0 & 0 \\ P_x & P_y & P_p \end{pmatrix} \Big|_{p=0}, \quad \Lambda_1 = \begin{pmatrix} \Delta_x & \Delta_y & \Delta_p \\ P_x & P_y & P_p \\ 0 & -1 & 0 \end{pmatrix} \Big|_{p=0},$$

where

$$(2.13) \quad \begin{aligned} \Delta_x|_{p=0} &= 6(ac_x + a_xc) - 16bb_x, & \Delta_p|_{p=0} &= -4ab, \\ P_x|_{p=0} &= c_x(3c_y - 2b_x) + 4bc_{xx} + 3c(c_{xy} - 2b_{xx}), \\ P_p|_{p=0} &= 4ac_x - 6a_xc + 2b(5c_y - 2b_x). \end{aligned}$$

1. To prove the first statement, it suffices to show that $\text{tr } \Lambda = 0$. Taking into account the equality $3c(c_y - 2b_x) + 4bc_x = 0$ on S_1 and the identity $b \equiv \delta$ (that implies $3ac \equiv 4b^2$ in a neighborhood of the origin), from (2.13), we have

$$\text{tr } \Lambda = (\Delta_x + P_p)|_{p=0} = 10[ac_x + b(c_y - 2b_x)] = 10\left(ac_x - b\frac{4bc_x}{3c}\right) = \frac{10c_x}{3c}(3ac - 4b^2) = 0.$$

Hence the characteristic equation of the matrix Λ reads $\lambda(\lambda^2 + |\Lambda_1|) = 0$; this yields the equation $\lambda^2 + |\Lambda_1| = 0$ for $\lambda_{1,2}$.

2. Differentiating the identity $4b^2 \equiv 3ac$ by x , we get $8bb_x \equiv 3(a_xc + ac_x)$. Using these identities and (2.13), we have

$$\begin{aligned} \Delta_x|_{p=0} &= 6(ac_x + a_xc) - 16bb_x = 6(ac_x + a_xc) - 6(ac_x + a_xc) = 0, \\ |\Lambda_1| &= -\Delta_p P_x|_{p=0} = 16ab^2c_{xx} - 8abb_xc_x + 12ab[c_xc_y + cc_{xy} - 2b_{xx}c] = \\ &= 12a^2cc_{xx} - 3ac_x(a_xc + ac_x) + 12ab[c_xc_y + cc_{xy} - 2b_{xx}c]. \end{aligned}$$

Thus the condition $\lambda_{1,2} \neq 0$ is equivalent to $|\Lambda_1| \neq 0$ that, in turn, is equivalent to condition 2.2.

On the other hand, the curve S_1 is tangent to the direction $p = 0$ at the point $q = 0$ if and only if $[3c(c_y - 2b_x) + 4bc_x]'_x = 0$. Taking into account the equalities $4b^2 \equiv 3ac$ and $8bb_x \equiv 3(a_xc + ac_x)$, we get

$$\begin{aligned} [3c(c_y - 2b_x) + 4bc_x]'_x &= 3(c_xc_y + cc_{xy}) - 6b_{xx}c + \frac{4b^2c_{xx} - 2bb_xc_x}{b} = \\ &= 3(c_xc_y + cc_{xy}) - 6b_{xx}c + \frac{12acc_{xx} - 3(a_xc + ac_x)c_x}{4b} = \frac{|\Lambda_1|}{4ab}. \end{aligned}$$

This proves that $\lambda_{1,2} \neq 0$ is equivalent to condition 2.3.

3. Generically, at almost all points $(q; p) \in \bar{S}_1$, the determinants

$$\begin{vmatrix} \Delta_x & \Delta_y \\ P_x & P_y \end{vmatrix}, \quad |\Lambda_1| = \begin{vmatrix} \Delta_x & \Delta_p \\ P_x & P_p \end{vmatrix}$$

are not equal to zero. Hence S_1 and \bar{S}_1 are regular curves and, moreover, conditions 2.1–2.3 hold. \square

Theorem 2. *Let $(q; p_i) \in \bar{S}_i$ be a generic singular point of the field (2.7). Then the germ (2.7) at $(q; p_i)$ is smoothly orbitally equivalent to*

$$(2.14) \quad \xi \frac{\partial}{\partial \xi} - \eta \frac{\partial}{\partial \eta} + \xi \eta \frac{\partial}{\partial \zeta}, \quad \text{if } \lambda_{1,2} \in \mathbb{R} \setminus 0,$$

$$(2.15) \quad \eta \frac{\partial}{\partial \xi} - \xi \frac{\partial}{\partial \eta} + (\xi^2 + \eta^2) \frac{\partial}{\partial \zeta}, \quad \text{if } \lambda_{1,2} \in \mathbb{I} \setminus 0,$$

where \mathbb{R}, \mathbb{I} are real and imaginary axes, respectively.

In the first case, there exist two geodesics passing through the point $q \in S_i$ with the tangential direction p_i , both of them smooth. In the second case, there are no geodesics passing through the point $q \in S_i$ with the tangential direction p_i .

Proof. Since $(q; p_i) \in \bar{S}_i$ is a generic singular point, $|\Lambda_1| \neq 0$. By Lemma 5, the eigenvalues $\lambda_{1,2} \neq 0$ and \bar{S}_i is a regular curve consisting of singular points of the field (2.7). The linear part of the germ (2.7) at $(q; p_i)$ (and every singular point sufficiently close to $(q; p_i)$) is orbitally equivalent to the linear part of the field (2.14) or (2.15) if $|\Lambda_1| < 0$ or $|\Lambda_1| > 0$, respectively. Here we use the following terminology: two vector fields are called orbitally smoothly (resp., topologically) equivalent, if there exists a diffeomorphism (resp., homeomorphism) that conjugates their integral curves, i.e., orbits of their phase flows.²

Recall that the field (2.7) belongs to the class of vector fields whose singular points are not isolated, but form a manifold W of codimension 2 in the phase space (in our case, $W = \bar{S}_i$). Local normal forms of such fields were studied by many authors. In [21] (Appendix A), we present a brief survey of such results, which covers all cases with $\text{Re } \lambda_{1,2} \neq 0$. This condition

² It slightly differs from the generally accepted definition of the orbital equivalence, where coincidence of the orientation of integral curves is also required. Our definition is naturally related to directions fields, whose integral curves do not have an orientation a priori.

is equivalent to the assumption that $W = \overline{S}_i$ is the local center manifold, and consequently, the phase portrait of (2.7) has a simple topological structure (we shall discuss it later on, in the proof of Theorem 4). For instance, in the case $\lambda_{1,2} \in \mathbb{R} \setminus 0$, the germ (2.7) with generic quadratic part is smoothly orbitally equivalent to (2.14). This result belongs to Roussarie [22]. The genericity condition is determined explicitly in [21] (Theorem 5.7). The case $\lambda_{1,2} \in \mathbb{I} \setminus 0$ is more complicated. However, in [3] (Chapter 2, Section 1.2) and [12] it is claimed that in this case the germ (2.7) with generic quadratic part is smoothly orbitally equivalent to (2.15).

We remark that the diffeomorphism that brings the germ (2.7) to the normal form (2.14) or (2.15) does not give a normal form of equation (2.3), since it does not preserve the contact structure $dy = pdx$. However, we need not a normal form of (2.3).

To prove the last statement of the theorem, we only need to consider the possible mutual relationships between the phase portrait of the germ (2.7) and the (x, y) -plane in the space PTM . Geodesics are obtained from those integral curves of the field (2.7) whose projection on the (x, y) -plane are distinguished from points. Moreover, isotropic geodesics correspond to those integral curves that belong to the isotropic hypersurface \mathcal{F} (by Lemma 2, \mathcal{F} is an invariant hypersurface of the field (2.7)).

We consider the real and imaginary cases separately.

The real case. The field (2.14) has the first integral $\xi\eta$. The invariant foliation $\xi\eta = \text{const}$ contains only two leaves $\xi = 0$ and $\eta = 0$ that pass through singular points of the field, while all remaining invariant leaves are hyperbolic cylinders $\xi\eta = \text{const} \neq 0$, which do not intersect the set of singular points. It is easy to see that, for every singular point of the field (2.14), there are only two integral curves passing through this point: the straight lines parallel to the ξ -axis and the η -axis, respectively.

We prove now that the eigenvectors with the eigenvalues $\lambda_{1,2} \neq 0$ are not vertical. Let e be an eigenvector of the matrix Λ with λ_i . Then $\Lambda e = \lambda_i e$, and $e = \alpha \partial_x + \beta \partial_p$, where $(\Delta_x|_{p=0} - \lambda_i)\alpha + \Delta_p|_{p=0}\beta = 0$. If the eigenvector e is vertical, i.e., $\alpha = 0, \beta \neq 0$, this equality yields $\Delta_p|_{p=0} = 0$. From (2.13), we have $a(0) = 0$ or $b(0) = 0$. This contradicts the fact (established in the proof of Lemma 5) that none of the coefficients a, b, c vanishes at 0.

From the considerations above, it follows that the field (2.7) has only two integral curves passing through the given point $(q; p_i)$, both of them smooth and having non-vertical tangential directions. Projecting these integral curves from PTM to M , we get two smooth geodesics passing through the point $q \in S_i$ with the tangential direction p_i ; see Fig. 2 (left).

The imaginary case. The field (2.15) has the first integral $\xi^2 + \eta^2$. The invariant foliation $\xi^2 + \eta^2 = \text{const}$ contains a one-dimensional degenerate leaf $\xi = \eta = 0$, which consists of singular points of the field (2.15) and a one-parameter family of two-dimensional leaves (elliptic cylinders $\xi^2 + \eta^2 = \text{const} \neq 0$), which do not intersect the set of singular points. The elliptic cylinders are filled with helix-like integral curves, whose projections to M have cusps; see Fig. 2 (right).

To complete the proof, observe that, in both the real and imaginary cases, the curve S_i itself is not a geodesic, since \overline{S}_i is transversal to the contact plane $pdx - dy = 0$ (statement 2.2 in Lemma 5). Consequently, \overline{S}_i is not a lift of a curve on M . \square

Example 2. Let F be given by formula (2.11) with $a = 1, b = 0, c(x, y) = \alpha y^2 - x$. Then $\Delta = 2(3c - p^2), P = 12\alpha y p^2 - 4p + 6\alpha y c$, and the curves S_i are the connected components of the graph $x = \alpha y^2 - \frac{1}{48}/\alpha^2 y^2$ lying in the upper and lower semiplanes. A straightforward calculation shows that $p_i|_{S_i} = 12\alpha y(\alpha y^2 - x)$; hence the direction p_i is tangent to the curve S_i at $x = \frac{47}{48}/\sqrt{\alpha}$ only. By statement 2.3 in Lemma 5, the eigenvalues $\lambda_{1,2} \neq 0$ at all points of \overline{S}_i if $\alpha < 0$ and at all points of \overline{S}_i with $x \neq \frac{47}{48}/\sqrt{\alpha}$ if $\alpha \geq 0$.

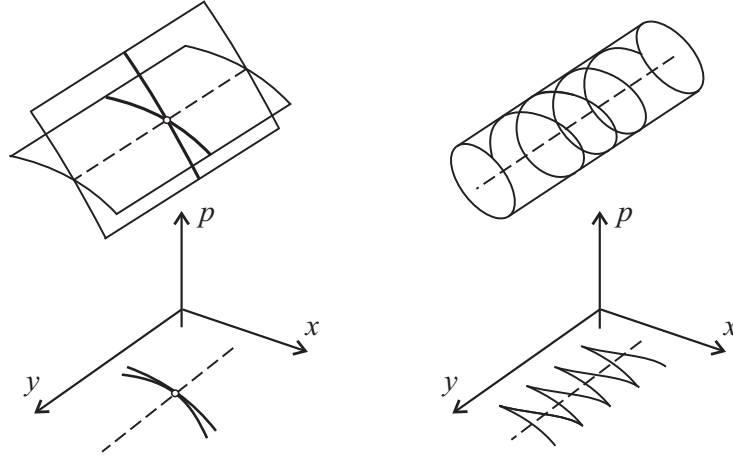


FIGURE 2. The phase portraits of the field (2.7) with the normal form (2.14) or (2.15) (left and right, resp.) and the projections of its integral curves to M . Dashed curves present \bar{S}_i (up) and S_i (down).

In Fig. 3 (left and center), we present geodesics in the case $\alpha > 0$. Here both real and imaginary eigenvalues exist. The parts of S_i with real (imaginary) eigenvalues $\lambda_{1,2} \neq 0$ are presented as short-dashed (resp., long-dashed) lines. The dots represent the points of S_i with $x = \frac{47}{48}/\sqrt{\alpha}$, where $\lambda_{1,2} = 0$. In Fig. 3 (right), we present geodesics in the case $\alpha < 0$. Here only real eigenvalues exist, and the curves S_i are presented as short-dashed lines. The dots represent the points where geodesics intersect the curves S_i with the singular tangential direction p_i .

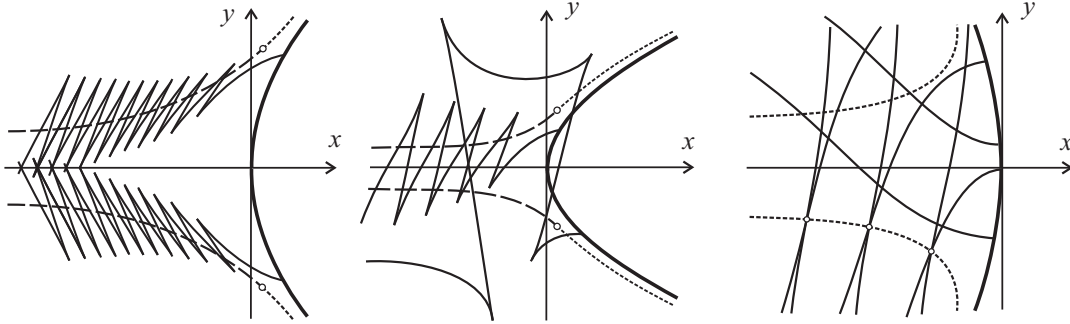


FIGURE 3. Example 2: $F = p^2 + c$, where $c = \alpha y^2 - x$ with $\alpha > 0$ (left, center) and $\alpha < 0$ (right). The solid lines present geodesics. $M_{0,1}$ is depicted with a bold solid line, S_i are depicted with short-dashed (long-dashed) lines if $\lambda_{1,2} \in \mathbb{R} \setminus 0$ ($\lambda_{1,2} \in \mathbb{I} \setminus 0$, respectively).

2.2.2. *Singularities in the stratum $M_{0,1}$.* In this section as before, we proceed in the local coordinates where F , Δ and $D_{[F]}$ have the form (2.11). At every point $q \in M_{0,1}$, the polynomial F has the double root $p_0 = -b/a$. It is easy to see that p_0 is also a double root of the polynomial Δ at q (this follows from Lemma 4 also). Thus $(q; p_0)$, $q \in M_{0,1}$, are singular points of both

implicit differential equations $F = 0$ and $\Delta = 0$. From (2.4), it follows that $P(q; p_0) = 0$; hence $(q; p_0)$, $q \in M_{0,1}$, are singular points of the field (2.7).

Furthermore, we restrict ourselves to generic points $q \in M_{0,1}$ where $M_{0,1}$ is a regular curve and the isotropic direction p_0 is transverse to $M_{0,1}$. Then both implicit differential equations $F = 0$ and $\Delta = 0$ have Cibrario normal forms at such a point and their integral curves are semicubic parabolas lying on opposite sides of $M_{0,1}$ (in the domains M_+ and M_- , resp.), as is presented in Fig. 1.

Theorem 3. *Suppose that the isotropic direction p_0 is transverse to $M_{0,1}$ at $q \in M_{0,1}$. Then the germ (2.7) at its singular point $(q; p_0)$ is smoothly orbitally equivalent to*

$$(2.16) \quad 3\xi \frac{\partial}{\partial \xi} + 2\eta \frac{\partial}{\partial \eta} + 0 \frac{\partial}{\partial \zeta},$$

and to p_0 corresponds a one-parameter family of geodesics outgoing from q into M_+ and M_- . There exist smooth local coordinates centered at q such that this family is

$$(2.17) \quad x = \alpha|\eta|^{\frac{3}{2}} + \eta^2 + \alpha\bar{X}_\alpha(\eta), \quad y = \alpha\eta|\eta|^{\frac{3}{2}} + \varepsilon\eta^3 + \alpha\bar{Y}_\alpha(\eta), \quad \varepsilon \neq 0,$$

where $\bar{X}_\alpha(\eta) = o(|\eta|^{\frac{3}{2}})$ and $\bar{Y}_\alpha(\eta) = o(|\eta|^{\frac{5}{2}})$ are C^2 -smooth functions.

Here $\alpha > 0$ ($\alpha < 0$) corresponds to non-isotropic geodesics outgoing from q in M_+ (resp., M_-), while $\alpha = 0$ gives the isotropic geodesic, a semicubic parabola lying in M_+ . The limit case $\alpha \rightarrow \infty$ corresponds to a unique smooth geodesic passing through q with the direction p_0 . In a neighborhood of q , every non-isotropic geodesics outgoing from q in M_+ belongs to the curvilinear tongue-like sector bounded by the branches of the isotropic geodesic as it is presented in Figure 4 (left).

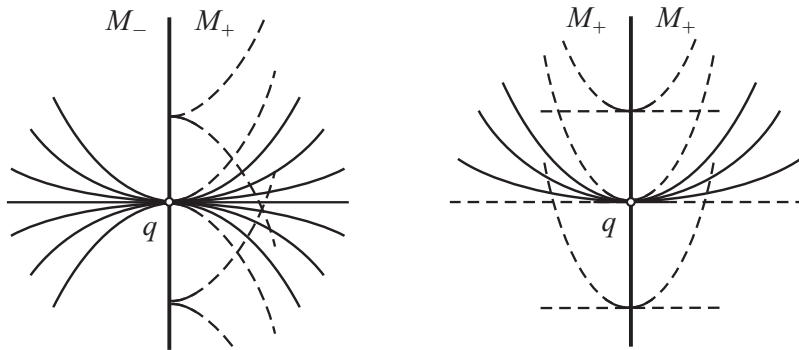


FIGURE 4. Illustrations of Theorem 3 (left) and Theorem 4 (right). The stratum $M_{0,1}$ is depicted with the bold solid line. Solid and dashed curves present non-isotropic and isotropic geodesics, respectively.

Proof. Without loss of generality, suppose that $q = 0$ (the origin of the (x, y) -plane) and choose local coordinates centered at 0 that preserve the lines $x = \text{const}$ and give $b(x, y) \equiv 0$. This can be done using an appropriate change of variables $y \mapsto yu(x, y)$, $u(0) \neq 0$. We remark that (unlike Lemma 5) we do not have the identity $p_1 \equiv 0$ nor $p_2 \equiv 0$ in a neighborhood of 0. Moreover, it is impossible to get any these identities using smooth changes of variables, since the integral curves of the implicit equation $\Delta = 0$ with roots p_1, p_2 have cusps on $M_{0,1}$.

In the local coordinates chosen above, we have

$$(2.18) \quad \begin{aligned} F &= ap^2 + c, \quad \Delta = -2(ap)^2 + 6ac, \quad D_{[F]} = -4a^3c, \\ P &= aa_y p^4 - 2aa_x p^3 + (7ac_y - 3a_y c)p^2 + (4ac_x - 6a_x c)p + 3cc_y. \end{aligned}$$

The curve $M_{0,1}$ is given by $c(x, y) = 0$ and the direction $p_0 = 0$ at every $q \in M_{0,1}$. Hence the condition “the direction p_0 is transverse to $M_{0,1}$ at 0” is equivalent to $c_x(0) \neq 0$.

Substituting Δ and P from (2.18) into (2.7), one can find that the spectrum of the field (2.7) at every point $(q; 0)$, $q \in M_{0,1}$, is $(\lambda_1, \lambda_2, 0)$, where $\lambda_1 = 6ac_x$, $\lambda_2 = 4ac_x$. A straightforward computation shows that the corresponding eigenvectors are

$$(2.19) \quad e_1 = 2a\partial_x + 3c_y\partial_p, \quad e_2 = \partial_p, \quad e_0 = c_y\partial_x - c_x\partial_y.$$

Note that $\lambda_1 : \lambda_2 \equiv 3 : 2$ at all points $(q; 0)$, $q \in M_{0,1}$, the pair (λ_1, λ_2) is non-resonant and belongs to the Poincaré domain. Therefore, the germ (2.7) at 0 is smoothly orbitally equivalent to the linear field (2.16) (Theorem 5.5 in [21]). Moreover, comparing (2.7) and (2.16), one can see that the conjugating diffeomorphism $(x, y, p) \mapsto (\xi, \eta, \zeta)$ can be chosen in the form

$$(2.20) \quad x = 2a\xi + c_y\zeta + f_1(\xi, \eta, \zeta), \quad p = 3c_y\xi + \eta + f_2(\xi, \eta, \zeta), \quad y = -c_x\zeta + f_3(\xi, \eta, \zeta),$$

where a, c_x, c_y are evaluated at 0 and $f_i \in \mathfrak{M}^1(\mathfrak{M}^k)$, $k \geq 0$, is the ideal of k -flat functions in the ring of smooth functions).

The field (2.16) has the invariant foliation $\zeta = \text{const}$. The set of integral curves of (2.16) passing through the origin consists of the ξ -axis and the one-parameter family

$$(2.21) \quad \{\xi = \alpha|\eta|^{\frac{3}{2}}, \zeta = 0\}, \quad \alpha \in \mathbb{R},$$

tending to the ξ -axis as $\alpha \rightarrow \infty$. Consider the possible mutual relationships between the phase portrait of the germ (2.7) at 0 and the (x, y) -plane in the space PTM using the eigenvectors (2.19). The integral curve of the field (2.7) corresponding to the ξ -axis in (2.16) has a non-vertical tangential direction at 0 (the eigenvector e_1); hence its projection to the (x, y) -plane is a smooth geodesic. On the contrary, the family (2.21) gives a family of integral curves of (2.7) with vertical tangential direction at 0 (the eigenvector e_2). The projections of these curves to the (x, y) -plane have a singularity at 0.

To establish the character of the singularity, substitute (2.21) in (2.20). This yields

$$x = 2a\alpha|\eta|^{\frac{3}{2}} + \bar{f}_{1,\alpha}(\eta), \quad p = \eta + \bar{f}_{2,\alpha}(\eta),$$

where $\bar{f}_{i,\alpha}(\eta) = f_i(\alpha|\eta|^{\frac{3}{2}}, \eta, 0)$. Observe that the functions $\bar{f}_{1,\alpha} = o(|\eta|^{\frac{3}{2}})$ and $\bar{f}_{2,\alpha} = o(\eta)$ are C^2 and C^1 , resp. Denote the sign of η by $s(\eta)$, then we have the equation

$$dy = pdx = (\eta + \bar{f}_{2,\alpha}(\eta))(3a\alpha|\eta|^{\frac{1}{2}}s(\eta) + \bar{f}'_{1,\alpha}(\eta))d\eta = (3a\alpha|\eta|^{\frac{3}{2}} + g_\alpha(\eta))d\eta,$$

where $g_\alpha = o(|\eta|^{\frac{3}{2}})$ is C^1 -smooth. Integrating, we get

$$y = \frac{6}{5}a\alpha|\eta|^{\frac{5}{2}}s(\eta) + h_\alpha(\eta) = \frac{6}{5}a\alpha\eta|\eta|^{\frac{3}{2}} + h_\alpha(\eta),$$

where $h_\alpha = o(|\eta|^{\frac{5}{2}})$ is C^2 -smooth. The scaling $y \mapsto \frac{5}{3}y$, $\alpha \mapsto \pm 2a\alpha$ yields

$$(2.22) \quad x = \alpha|\eta|^{\frac{3}{2}} + X_\alpha(\eta), \quad y = \alpha\eta|\eta|^{\frac{3}{2}} + Y_\alpha(\eta),$$

where $X_\alpha = o(|\eta|^{\frac{3}{2}})$ and $Y_\alpha = o(|\eta|^{\frac{5}{2}})$ are C^2 -smooth, and $\alpha > 0$ ($\alpha < 0$) corresponds to the domain M_+ (M_- , resp.). The asymptotic formula (2.22) makes sense for all real $\alpha \neq 0$.

In order to take care of the omitted case $\alpha = 0$, recall that the isotropic surface \mathcal{F} is an invariant surface of the field (2.7) (Lemma 2) and contains its singular points (Lemma 4). Hence, in the normal coordinates (ξ, η, ζ) , the surface \mathcal{F} contains the ζ -axis and intersects every invariant

leaf $\zeta = \text{const}$ in a certain integral curve of (2.16). For instance, \mathcal{F} intersects the leaf $\zeta = 0$ in an integral curve of the family (2.21), which corresponds to an isotropic geodesic passing through 0.

On the other hand, we know that the implicit differential equation $F = ap^2 + c = 0$, which described the isotropic lines in (M, \bar{F}) , has a Cibrario normal form at 0. Hence there exists a unique isotropic geodesic passing through 0, the semicubic parabola

$$(2.23) \quad x = \eta^2, \quad y = \eta^3 N(\eta), \quad N(0) \neq 0,$$

lying in the domain M_+ .

From the uniqueness of the isotropic geodesic passing through 0, it follows that the lift of (2.23) is the curve of the family (2.22) with $\alpha = 0$ and $Y_0(\eta) = \eta^3 N(\eta)$. Using the representation $N(\eta) = N_1(\eta^2) + \eta N_2(\eta^2)$ and the change of variables $y \mapsto N_1(0)(y - x^2 N_2(x))/N_1(x)$, we get $N(\eta) \equiv N(0)$. It is not hard to check that the number $\varepsilon = N(0)$ is a mutual invariant of the curves (2.22) and (2.23). \square

In Example 1, we considered $F = p^2 + c$ with $c = -x$ and $c = \alpha y^2 - x$. In both cases the isotropic direction $p_0 = 0$ is transverse to the curve $M_{0,1}$ given by the equation $x = 0$ and $x = \alpha y^2$, respectively, and the conditions of Theorem 3 hold true.

Example 3. We consider here the case $F = p^2 - x$ in more detail. The field (2.7) is

$$(2.24) \quad -2(3x + p^2) \left(\frac{\partial}{\partial x} + p \frac{\partial}{\partial y} \right) - 4p \frac{\partial}{\partial p}.$$

It is easy to check that the isotropic surface \mathcal{F} given by $p^2 = x$ is an invariant surface of the field (2.24) and the unique isotropic line passing through 0 is given by $x = p^2$, $y = \frac{2}{3}p^3$. Integrating the equation $dp/dx = 2p/(3x + p^2)$, we get the family $x = \alpha|p|^{\frac{3}{2}} + p^2$, where α is the constant of integration, and a single integral curve $p = 0$, which gives the smooth non-isotropic geodesic $y = 0$.

Integrating the relation $dy = p dx = p(\frac{3}{2}\alpha|p|^{\frac{1}{2}}s(p) + 2p)dp = (\frac{3}{2}\alpha|p|^{\frac{3}{2}} + 2p^2)dp$, we get $y = \frac{3}{5}\alpha p|p|^{\frac{3}{2}} + \frac{2}{3}p^3 + c_1$, where c_1 is the second constant of integration. The family of geodesics outgoing from $q = 0$ is characterized by $c_1 = 0$. The scaling $y \rightarrow \frac{5}{3}y$ brings this family to the form (2.17) with $\bar{X}_\alpha \equiv \bar{Y}_\alpha \equiv 0$ and $\eta = p$:

$$(2.25) \quad x = \alpha|\eta|^{\frac{3}{2}} + \eta^2, \quad y = \alpha\eta|\eta|^{\frac{3}{2}} + \frac{10}{9}\eta^3.$$

For $\alpha = 0$, formula (2.25) gives the isotropic geodesic. For $\alpha \rightarrow \infty$, the curves (2.25) tend to the smooth geodesic $y = 0$.

Remark 3. Theorem 3 shows that the extension of geodesics through the curve $M_{0,1}$ is not uniquely defined. Indeed, all geodesics of the family (2.17) have the same tangential direction at $q \in M_{0,1}$ and almost all of them have a singularity of the same type at q . So, a curve given by formula (2.17) with any $\alpha \neq 0$ does not have any advantages in comparison with the curve consisting of two bows (2.17) with α_1 if $(x, y) \in M_+$ and α_2 if $(x, y) \in M_-$.

3. SURFACES IN BERWALD–MOOR SPACES

Consider the space \mathbb{R}^n , $n \geq 3$, with the coordinates (x_1, \dots, x_n) equipped with the *Berwald–Moor metric* $ds = (dx_1 \cdots dx_n)^{\frac{1}{n}}$, and a smooth two-dimensional surface $M \subset \mathbb{R}^n$ parametrized by $x_i = f_i(x, y)$, $i = 1, \dots, n$. The Berwald–Moor metric of the ambient space defines a two-dimensional pseudo-Finsler space (M, \bar{F}) with the metric function $\bar{f} = \bar{F}^{\frac{1}{n}}$, where

$$(3.1) \quad \bar{F}(x, y; \dot{x}, \dot{y}) = \prod_{i=1}^n (f_{ix}(x, y)\dot{x} + f_{iy}(x, y)\dot{y}), \quad f_{ix} = \frac{\partial f_i}{\partial x}, \quad f_{iy} = \frac{\partial f_i}{\partial y},$$

and n families of isotropic lines

$$(3.2) \quad f_i(x, y) = \text{const}, \quad i = 1, \dots, n.$$

Given $q \in M$, the isotropic direction p is called *simple* (*double* or *multiple*) if there exist *only one* (*only two* or *more than one*, resp.) isotropic lines (3.2) passing through q with given direction p . By Lemma 4, singularities of the geodesic flow occur at the points $q \in M$ that have at least one multiple isotropic direction.

Remark 4. In the case $n = 3$, we have a cubic pseudo-Finsler space (M, \bar{F}) . But unlike Section 2.2, the function \bar{F} given by (3.1) is not generic. The corresponding cubic polynomial $F(x, y; p)$ at every point $q \in M$ has n real roots (taking into account the multiplicity and including the root $p = \infty$), and $M_- = \emptyset$. Hence singularities of geodesics appear only at the points where at least two of three isotropic lines (3.2) are tangent. Here the stratum $M_{0,1}$ consists of the points where two isotropic lines are tangent (the *double* isotropic direction) and the third one is transversal to them (the *simple* isotropic direction).

From now on, we assume that the functions $f_i(x, y)$ have non-degenerate differentials and every point $q \in M$ may have simple or double isotropic directions only (the number of double isotropic directions can vary from 0 to $\lfloor \frac{n}{2} \rfloor$). Moreover, assume that the tangency of isotropic lines with double isotropic directions has first order. Consider geodesics passing through a point q with a double isotropic direction p_0 satisfying the above conditions.

Without loss of generality assume that $q = 0$ (the origin in the (x, y) -plane) and p_0 corresponds to the isotropic lines $f_1(x, y) = 0$ and $f_2(x, y) = 0$, where $f_{2y}(0) \neq 0$. Making the change of variable $y \mapsto f_2(x, y)$, we transform the metric function (3.1) into a similar one with $f_2(x, y) = y$ and $f_{1x}(0) = 0$, $f_{1y}(0) \neq 0$, $f_{1xx}(0) \neq 0$. The double isotropic direction p_0 becomes $p = 0$ and, moreover, in a neighborhood of 0, $p = 0$ is the double isotropic direction at all points $q \in M_{0,1} = \{f_{1x}(x, y) = 0\}$.

By the condition $f_{1xx}(0) \neq 0$, $M_{0,1}$ is a smooth curve transversal to the x -axis. Making the change of variable $x \mapsto f_{1x}(x, y)$, we transform $M_{0,1}$ into $x = 0$ and the metric function (3.1) into a similar one with $f_{1x} = xa(x, y)$, $f_{1y} = b(x, y)$, $f_2(x, y) = y$, where a, b are smooth functions non-vanishing at 0. So, we get

$$(3.3) \quad F(x, y; p) = p(ax + bp)G, \quad G(x, y; p) = \prod_{i=3}^n (f_{ix} + f_{iy}p), \quad G(0, 0; 0) \neq 0.$$

Substituting (3.3) in (2.4), we get

$$(3.4) \quad \begin{aligned} \Delta &= ((1-n)(ax)^2 + 2(2-n)abxp + 2(2-n)(bp)^2) G^2 + \Delta_0, \\ P &= ap((n-2)bp - ax + P_0) G^2, \end{aligned}$$

where $\Delta_0 \in \langle x^3, x^2p, xp^2, p^3 \rangle$ and $P_0 \in \langle x^2, xp, p^2 \rangle$ (both ideals are in the ring of smooth functions on x, y, p). Formula (3.4) shows that all components of the field (2.7) vanish on the line $\{x = p = 0\}$ and the spectrum of (2.7) at any point of this line has three zero eigenvalues. This does not allow us to establish a normal form similar to Theorem 3.

To overcome this problem, consider the blowing up

$$(3.5) \quad \mathcal{B}: (x, y, u) \mapsto (x, y, p), \quad p = xu, \quad u \in \mathbb{R}P^1 = \mathbb{R} \cup \infty.$$

The mapping \mathcal{B} is one-to-one except on the plane $\Pi = \{(x, y, u): x = 0\}$, whose image is the line $\mathcal{B}(\Pi) = \{(x, y, p): x = p = 0\}$. The mapping \mathcal{B} is a local diffeomorphism at all points of the (x, y, u) -space except Π . It has an inverse defined on $\mathbb{R}^3 \setminus \mathcal{B}(\Pi)$ given by

$$\mathcal{B}^{-1}(x, y, p) = \left(x, y, \frac{p}{x}\right).$$

Observe that there is no geodesic that coincides with the line $\mathcal{B}(\Pi)$. A straightforward calculation shows that the field (2.7) corresponds to a smooth field in the (x, y, u) -space (away of Π) that, after dividing by the common factor xG^2 , is

$$(3.6) \quad x(A + \dots) \left(\frac{\partial}{\partial x} + xu \frac{\partial}{\partial y} \right) + u(n-2)(2(bu)^2 + 3abu + a^2 + \dots) \frac{\partial}{\partial u},$$

where

$$A(x, y, u) = (1-n)a^2 + 2(2-n)(abu + (bu)^2);$$

here and below the dots mean terms that belong to the ideal $\langle x \rangle$.

Remark 5. There exists $\varepsilon > 0$ such that $A(x, y, u) \neq 0$ for all u if $|x| + |y| < \varepsilon$. Indeed, consider $A(x, y, u)$ as a quadratic polynomial on u with the discriminant

$$D = (2-n)^2(ab)^2 - 2(1-n)(2-n)(ab)^2 = n(2-n)(ab)^2,$$

which is strictly negative if x, y are sufficiently close to zero.

Dividing the field (3.6) by $(A + \dots)$, we get

$$(3.7) \quad x \left(\frac{\partial}{\partial x} + xu \frac{\partial}{\partial y} \right) + u(n-2)(U + \dots) \frac{\partial}{\partial u}, \quad U(x, y, u) = \frac{2(bu)^2 + 3abu + a^2}{A(x, y, u)}.$$

Remark 6. The plane $x = 0$ is invariant for the fields (3.6) and (3.7). Moreover, it is filled with vertical (i.e., parallel to the u -direction) straight integral lines of these fields, whose projections to the (x, y) -plane along the u -axis are points on the y -axis.

Lemma 6. *Geodesics can pass through a point $q \in M$ lying on the y -axis with the direction $p = 0$ only with the following admissible values u :*

$$(3.8) \quad u_0 = 0, \quad u_1 = -a/2b, \quad u_2 = -a/b.$$

Proof. By the standard existence and uniqueness theorem, for every point (x, y, u) such that $x = 0$ and $U(x, y, u) \neq 0$, there exists a unique integral curve of the field (3.7) passing through this point. By Remark 6, it is a vertical straight line, whose projection to the (x, y) -plane is a point on the y -axis. Hence geodesics can pass through a point $q \in M$ lying on the y -axis with the direction $p = 0$ only with $u = 0$ or u such that $2(bu)^2 + 3abu + a^2 = 0$. This gives the three values in (3.8). \square

Lemma 7. *The set of singular points of the field (3.7) consists of three mutually disjoint curves*

$$W_i^c = \{(x, y, u) : x = 0, u = u_i(y)\}, \quad i = 0, 1, 2.$$

On every curve W_i^c , the linear part of the field (3.7) has the constant spectrum $(1, \lambda, 0)$, where $\lambda = \frac{n-2}{n}$ if $i = 1$ or $\lambda = \frac{n-2}{1-n}$ if $i \in \{0, 2\}$. In both cases, ∂_u is the eigenvector with λ .

Proof. The first statement is trivial. All other statements are by direct calculations. \square

Theorem 4. *Suppose that the functions $f_i(x, y)$ have non-degenerate differentials and p_0 is a double isotropic direction at $q \in M$ such that the corresponding isotropic lines have first order of tangency at q . Then the field (3.7) at its singular points $(q; u_i)$ has local orbital normal forms indicated in Table 1 and to p_0 corresponds a one-parameter family of geodesics outgoing from q . There exist smooth local coordinates centered at q such that this family consists of C^2 -smooth non-isotropic geodesics*

$$(3.9) \quad y = x^2 + Y(x, \alpha|x|^\lambda), \quad Y(x, \alpha|x|^\lambda) = o(x^2), \quad \lambda = \frac{n-2}{n}, \quad \alpha \in \mathbb{R},$$

where $Y(\cdot, \cdot)$ is a C^∞ -smooth function, together with two C^∞ -smooth isotropic geodesics

$$(3.10) \quad y = 0 \quad \text{and} \quad y = 2x^2 + o(x^2).$$

In a neighborhood of q , every geodesic of the family (3.9) belongs to the curvilinear tongue-like sector bounded by the curves (3.10) as it is presented in Figure 4 (right).

	Orbital normal form	
	topological	C^∞ -smooth
W_1^c	$\xi \frac{\partial}{\partial \xi} + \eta \frac{\partial}{\partial \eta} + 0 \frac{\partial}{\partial \zeta}$	$\xi \frac{\partial}{\partial \xi} + \lambda \eta \frac{\partial}{\partial \eta} + 0 \frac{\partial}{\partial \zeta}$, where $\lambda = \frac{n-2}{n}$
W_0^c and W_2^c	$\xi \frac{\partial}{\partial \xi} - \eta \frac{\partial}{\partial \eta} + 0 \frac{\partial}{\partial \zeta}$	$\xi(n-1 + \Phi_1(\rho, \zeta)) \frac{\partial}{\partial \xi} - \eta(n-2 + \Phi_2(\rho, \zeta)) \frac{\partial}{\partial \eta} + \rho \Psi(\rho, \zeta) \frac{\partial}{\partial \zeta}$, where $\rho = \xi^{n-2} \eta^{n-1}$; $(n-1)\xi \frac{\partial}{\partial \xi} - (n-2)\eta \frac{\partial}{\partial \eta} + \rho \frac{\partial}{\partial \zeta}$, if $\Psi(0,0) \neq 0$.

TABLE 1. Local orbital normal forms of the field (3.7).

Proof. Choose local coordinates so that $q \in M$ is the origin and consider the field (3.7) in a neighborhood of its singular points $(0, 0, u_i)$, $i = 0, 1, 2$, where u_i are given by formula (3.8). By Lemma 7, at all singular points the condition $\text{Re } \lambda_{1,2} \neq 0$ holds, and every curve W_i^c , $i = 0, 1, 2$, is the center manifold of this field. Moreover, there exist also 2-dimensional unstable manifolds if $i = 1$ and a pair of 1-dimensional stable and unstable manifolds if $i = 0, 2$. Hence all topological normal forms in Table 1 trivially follow from the reduction principle [2, 4, 9].

Indeed, the reduction principle asserts that the germ (3.7) is orbitally topologically equivalent to the direct product of the standard 2-dimensional node (if $i = 1$) or saddle (if $i = 0, 2$) and the restriction of the field to the center manifold W_i^c . Since the restriction of the field (3.7) to every center manifold W_i^c , $i = 0, 1, 2$, is identically zero, this gives us the topological normal forms in Table 1.

We establish now the smooth normal forms in the cases $i = 1$ and $i = 0, 2$ separately.

The case $i = 1$. By Lemma 7, the linear part of the field (3.7) at any point on W_1^c has spectrum $(1, \lambda, 0)$ with $\lambda = \frac{n-2}{n}$. Then Theorem 5.5 in [21] asserts that the germ (3.7) at any point on W_1^c is orbitally C^∞ -smoothly equivalent to

$$(3.11) \quad (\xi + \varphi(\zeta)\eta^{1/\lambda}) \frac{\partial}{\partial \xi} + \lambda \eta \frac{\partial}{\partial \eta} + 0 \frac{\partial}{\partial \zeta},$$

where $\varphi(\zeta) \equiv 0$ if the number $1/\lambda$ is not integer (non-resonant case).

Assume $1/\lambda$ is an integer and prove that $\varphi(\zeta) \equiv 0$ iff for every point $\omega_* \in W_1^c$ the field (3.7) has a C^∞ -smooth integral curve passing through ω_* with the vertical tangential direction ∂_u . By Remark 6, such integral curves exist (the vertical straight lines); hence this establishes the equality $\varphi(\zeta) \equiv 0$ in the remaining cases $1/\lambda = 3$ ($n = 3$) and $1/\lambda = 2$ ($n = 4$).

For this, note that the field (3.11) has the invariant foliation $\zeta = \text{const}$. Every invariant leaf contains a single integral curve $\eta = 0$ corresponding to eigendirection with the eigenvalue 1 and the one-parameter family of integral curves

$$(3.12) \quad \xi = \eta^{1/\lambda}(\alpha + \varphi(\zeta) \ln |\eta|), \quad \alpha \in \mathbb{R},$$

corresponding to eigendirections with the eigenvalue λ . All curves (3.12) are $C^{1/\lambda-1}$ -smooth (but not $C^{1/\lambda}$ -smooth at zero) if $\varphi(\zeta) \neq 0$ and C^∞ -smooth if $\varphi(\zeta) = 0$. Without loss of generality, assume that the point $\omega_* \in W_1^c$ in the (x, y, u) -space corresponds to $(0, 0, \zeta_*)$ in the (ξ, η, ζ) -space. The equality $\varphi(\zeta_*) = 0$ is equivalent to the existence of at least one C^∞ -smooth integral curve of the field (3.11) with tangential direction ∂_η lying on the invariant leaf $\{\zeta = \zeta_*\}$. To

complete the proof, remark that the eigendirection ∂_η of (3.11) corresponds to the eigendirection ∂_u of (3.7).

The cases $i = 0, 2$. By Lemma 7, the linear part of the field (3.7) at all points on the curves W_0^c and W_2^c has spectrum $(\lambda_1, \lambda_2, 0)$, where $\lambda_1 = 1$ and $\lambda_2 = \frac{n-2}{1-n}$. This gives the resonance

$$(3.13) \quad \mu\lambda_1 + \nu\lambda_2 = 0$$

with the resonant monomial $\rho = \xi^\mu \eta^\nu$, where we set $\mu = n - 2$ and $\nu = n - 1$. Everything that we say below is true as well for arbitrary relatively prime $\mu, \nu \in \mathbb{N}$.

The resonance (3.13) does not allow us to get a normal form with one identically zero component (as we have in the case $i = 1$) even in the finite-smooth category; see the discussion in [8] (Section 3.2). Moreover, (3.13) generates two infinite series of resonances

$$(1 + l\mu)\lambda_1 + l\nu\lambda_2 = \lambda_1, \quad l\mu\lambda_1 + (1 + l\nu)\lambda_2 = \lambda_2, \quad l = 1, 2, \dots,$$

and consequently, an infinite number of resonant monomials in the corresponding (orbital) Poincaré–Dulac normal form:

$$(3.14) \quad \xi(\nu + \Phi_1(\rho, \zeta)) \frac{\partial}{\partial \xi} - \eta(\mu + \Phi_2(\rho, \zeta)) \frac{\partial}{\partial \eta} + \rho \Psi(\rho, \zeta) \frac{\partial}{\partial \zeta};$$

see, e.g., [8] (Section 3.2) or [17] (Section 5).

Moreover, if in addition, $\Psi \neq 0$ at a point ω_* , the germ (3.14) at ω_* is smoothly orbitally equivalent to

$$(3.15) \quad \nu \xi \frac{\partial}{\partial \xi} - \mu \eta \frac{\partial}{\partial \eta} + \rho \frac{\partial}{\partial \zeta}.$$

The normal form (3.15) was first established by Roussarie [22] in the partial case $\mu = \nu = 1$ in the C^∞ -smooth category. For arbitrary integers μ, ν , the proof (in the finite-smooth category) can be found in [17] (Section 5). Combining the methods from [22] and [17], one can establish the normal form (3.15) with arbitrary μ, ν in the C^∞ -smooth category also.

Completion of the proof. Integral curves of the field (3.7) passing through $(0, 0, u_1)$ correspond to integral curves of the field $\xi \frac{\partial}{\partial \xi} + \lambda \eta \frac{\partial}{\partial \eta}$ lying on the invariant leaf $\{\zeta = 0\}$: a single curve that coincides with the η -axis and one-parameter family $\{\eta = \alpha|\xi|^\lambda, \zeta = 0\}$, $\alpha \in \mathbb{R}$. Comparing the germ (3.7) at $(0, 0, u_1)$ with its normal form $\xi \frac{\partial}{\partial \xi} + \lambda \eta \frac{\partial}{\partial \eta}$, one can see that the conjugating diffeomorphism $(x, y, u) \mapsto (\xi, \eta, \zeta)$ can be chosen in the form

$$(3.16) \quad x = \xi, \quad u = u_1 + \eta + c_1 \xi + c_2 \zeta + \varphi(\xi, \eta, \zeta), \quad y = \zeta + \psi(\xi, \eta, \zeta), \quad \varphi, \psi \in \mathfrak{M}^2,$$

where \mathfrak{M}^k , $k \geq 0$, is the ideal of k -flat functions in the ring of smooth functions. Substituting $\xi = \zeta = 0$ in (3.16) and taking into account $p = ux$, we get $x = p = 0$. Hence the η -axis does not correspond to a geodesic.

Substituting $\eta = \alpha|\xi|^\lambda$ and $\zeta = 0$ in (3.16) and taking into account $x = \xi$ and $p = ux$, we get $p = x(u_1 + f(x, \alpha|x|^\lambda))$ with a certain smooth function $f \in \mathfrak{M}^0$. This gives the relation

$$dy = p dx = x(u_1 + f(x, \alpha|x|^\lambda)) dx,$$

where $xf(x, \alpha|x|^\lambda) = o(x)$ is a C^1 -smooth function. Integrating, we get $y = \frac{u_1}{2} x^2 + Y(x, \alpha|x|^\lambda)$. Here $Y(\cdot, \cdot)$ is a smooth function and $Y(x, \alpha|x|^\lambda) = o(x^2)$ is C^2 -smooth. After the scaling $y \rightarrow 2y/u_1$, we get the family (3.9).

The topological and smooth orbital normal forms in Table 1 show that the field (3.7) has only two integral curves passing through its singular point $(0, 0, u_i)$, where $i = 0$ or 2 . Moreover, one of these integral curves is a straight vertical line, whose projection to the (x, y) -plane is a point (see Remark 6 and Lemma 7). Another integral curve has a non-vertical tangential direction at $(0, 0, u_i)$; hence its projection to the (x, y) -plane is regular.

Thus all of the admissible values u_0 and u_2 give a smooth geodesic passing through the point q with tangential direction $p = 0$. It is not hard to see that these geodesics are isotropic lines, which are solutions of differential equations $p = 0$ and $ax + bp = 0$, respectively (see formula (3.3)). Taking into account (3.8), after the scaling $y \rightarrow 2y/u_1$ we get (3.10). \square

Remark 7. The normal forms $3\xi \frac{\partial}{\partial \xi} + 2\eta \frac{\partial}{\partial \eta}$ and $\xi \frac{\partial}{\partial \xi} + \frac{n-2}{n}\eta \frac{\partial}{\partial \eta}$ in Theorems 3, 4 are valid also in the analytic category; see, e.g., [25]. Therefore, in the analytic case, formulae (2.17) and (3.9) present Puiseux series for geodesics.

Example 4. Consider geodesics on the surface $z = y - 2x^2$ in the Berwald–Moor space (x, y, z) with the metric $ds = (dx dy dz)^{\frac{1}{3}}$. This yields

$$(3.17) \quad F(x, y; p) = p(p - 4x), \quad \Delta = -2(p^2 - 4xp + 16x^2), \quad P = -4p(p + 4x),$$

and the equation of geodesics (2.3) reads

$$(3.18) \quad \frac{dp}{dx} = \frac{2p(p + 4x)}{p^2 - 4xp + 16x^2}.$$

The isotropic lines are solutions of the differential equation $p(p - 4x) = 0$. This gives two families of isotropic lines $y = \text{const}$ and $y = 2x^2 + \text{const}$, which have first order tangency on the line $x = 0$. Substituting them into (3.18), one can see that they are geodesics.

Consider the geodesics outgoing from the point $q = 0$ with the double isotropic directions $p_0 = 0$. (Recall that, for every $p \neq 0$, there exists a unique geodesic passing through q with tangential direction p ; we exclude such geodesics from further consideration.) The isotropic geodesics $y = 0$ and $y = 2x^2$ (formula (3.10)) separate the (x, y) -plane into four parts: the upper domain $y > 2x^2$, the semiplane $y < 0$ and two tongue-like sectors between them. See Fig. 4 (right); the isotropic geodesics $y = 0$ and $y = 2x^2$ are depicted with dashed lines.

Theorem 4 claims that there exists a one-parameter family of geodesics outgoing from q with the double isotropic directions $p_0 = 0$ into the tongue-like sectors (non-isotropic family (3.9)) and there are no geodesics outgoing from q with the double isotropic directions $p_0 = 0$ into two remaining parts of the plane. Geodesics of the family (3.9) correspond the admissible value $u_1 = 2$ (compare formulae (3.3), (3.8) and (3.17)) and they can be presented as the Puiseux series

$$y = t^6 + 3t^6 \sum_{i \geq 4} \frac{a_i}{i+3} t^{i-3}, \quad p = \frac{dy}{dx} = \frac{1}{3t^2} \frac{dy}{dt} = 2t^3 + \sum_{i \geq 4} a_i t^i, \quad \text{where } x = t^3.$$

Substituting the above expression for p in (3.18), we obtain recurrence relations for the unknown coefficients a_i .

Namely, $a_i = 0$ for all odd i (this also follows from the fact that the surface $z = y - 2x^2$ is symmetric with respect to the plane $x = 0$). For even i , we have $24a_6 + 4a_4^3 = 0$, $48a_8 + 14a_4^2 a_6 = 0$, $72a_{10} + 16(a_4 a_6^2 + a_4^2 a_8) = 0$, etc. In general,

$$(3.19) \quad 12(2i - 4)a_{2i} + b_{2i} = 0, \quad i = 3, 4, 5, \dots,$$

where b_{2i} is a polynomial on a_{2j} with $j < i$ with zero free term. This shows that the coefficient a_4 is arbitrary, and all a_{2i} with $i \geq 3$ are uniquely defined by equations (3.19). This gives the one-parameter non-isotropic family (3.9). In particular, $a_4 = 0$ gives $a_{2i} = 0$ for all $i \geq 3$, and the corresponding solution $y = x^2$ presents the unique C^∞ -smooth geodesic of non-isotropic family (3.9) (the corresponding value of the parameter is $\alpha = 0$).

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