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DECOMPOSITION THEOREM FOR SEMI-SIMPLES

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Abstract. We use standard constructions in algebraic geometry and homological algebra to extend the decomposition and hard Lefschetz theorems of T. Mochizuki and C. Sabbah so that they remains valid without the quasi-projectivity assumptions.

1. Introduction

M. Kashiwara [Ka] has put-forward a series of conjectures concerning the behavior of holonomic semi-simple D-modules on a complex algebraic variety under proper push-forward and under taking nearby/vanishing cycles.

Inspired by this conjecture, T. Mochizuki [Mo] has proved Kashiwara conjectures in the very important case where one assumes the holonomic D-modules to be regular. Mochizuki's work built on earlier work by C. Sabbah [Sa]. Because of the regularity assumptions (see [Sa, p.2-3, Remark 6]) for more context), part of their results can be expressed, via the Riemann-Hilbert correspondence, in the form of Theorem 2.1.1 below.

The methods employed in [Mo, Sa] are essentially analytic. Moreover, [Mo, Sa] are placed in the context of projective morphisms of quasi projective manifolds, so that Theorem 2.1.2 below, which generalizes Theorem 2.1.1, is not directly affordable by their methods: one would first need to extend aspects of their theory of polarizable pure twistor D-modules from projective manifolds to complex algebraic varieties. To my knowledge, this extension is not in the literature.

V. Drinfeld [Dr] has shown that an arithmetic conjecture by A. de Jong implies, rather surprisingly and again under the regularity assumption, Kashiwara's conjectures. Drinfeld's proof uses also algebraic geometry for varieties over finite fields. Note that [Dr] allows for arbitrary characteristic-zero coefficients. de Jong's conjecture has been proved by D. Gaitsgory [Ga] and by G. Böckle and C. Khare [Bo-Ka].

The combination of the work in [Dr, Ga, Bo-Ka] yields an arithmetic proof of Theorems 2.1.1 and of 2.1.2 below.

The purpose of this note is to provide a proof of Theorem 2.1.2 that stems directly from Theorem 2.1.1 and uses only simple reductions based on standard constructions in algebraic geometry.

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2. Decomposition and relative hard Lefschetz for semi-simples

2.1. **Statement.** A variety is a separated scheme of finite type over the field of complex numbers $\mathbb{C}$. For the necessary background concerning what follows, the reader may consult [dCM]. Given a variety Y, we work with the rational and complex constructible derived categories $D(Y,\mathbb{Q})$ and $D(Y,\mathbb{C})$ endowed with the middle-perversity t-structures, whose hearts, i.e. the respective categories of perverse sheaves on Y, are denoted by $P(Y,\mathbb{Q})$ and $P(Y,\mathbb{C})$, respectively. The simple objects in $P(Y,\mathbb{Q})$ and in $P(Y,\mathbb{C})$ have the form $IC_S(L)$, where S is an irreducible closed

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subvariety of Y, L is a simple (i.e. irreducible) complex/rational local system defined on some dense open subset of the regular part of S, and IC stands for intersection complex. We say that $K \in D(Y, \mathbb{Q})$ is semi-simple if it is isomorphic to the finite direct sum of shifted simple perverse sheaves as above: $K \cong \oplus_b {}^{\mathfrak{p}}\mathcal{H}^b(K)[-b] \cong \oplus_b \oplus_{(S,L)\in EV_b} IC_S(L)[-b]$, where ${}^{\mathfrak{p}}\mathcal{H}^b$ denotes the b-th perverse cohomology sheaf functor, and EV_b is a uniquely determined finite set of pairs (S, L) as above. Similarly, with $\mathbb{C}$-coefficients.

Our starting point is the following result of T. Mochizuki [Mo, §14.5 and §14.6], which generalizes one of C. Sabbah [Sa]. In fact, they both work in the more refined setting of polarized pure twistor D-modules and their results have immediate and evident counterparts in the setting of the constructible derived category, which is the one of this note.

Theorem 2.1.1. *Let $f : X \to Y$ be a projective map of irreducible quasi projective nonsingular varieties. If $K \in P(X, \mathbb{C})$ is semi-simple, then $f_*K \in D(Y, \mathbb{C})$ is semi-simple. The relative hard Lefschetz theorem holds.*

Even if the methods in [Mo] seem to require the smoothness and quasi projectivity assumptions, as well as $\mathbb{C}$-coefficients, one can deduce the following more general statement. We have nothing to say concerning the refined context of polarizable pure twistor D-modules.

Theorem 2.1.2. *Let $f : X \to Y$ be a proper map of varieties. If $K \in P(X, \mathbb{Q})$ is semi-simple, then $f_*K \in D(Y, \mathbb{Q})$ is semi-simple. If f is projective, then the relative hard Lefschetz theorem holds.*

We first show how to deduce the $D(Y, \mathbb{C})$-version of Theorem 2.1.2 from Theorem 2.1.1. Then we show how the $D(Y, \mathbb{C})$-version implies formally the $D(Y, \mathbb{Q})$-version.

The reader should have no difficulty in replacing $\mathbb{Q}$ with any field of characteristic zero and proving the same result.

2.2. **Proof of Theorem 2.1.2 for $D(Y, \mathbb{C})$.** Theorem 2.1.1 is stated for $\mathbb{C}$-coefficients. In this section, we use this statement to deduce Theorem 2.1.2 for $\mathbb{C}$-coefficients, i.e. to deduce Corollary 2.2.1 below.

The theorem will be reduced to several special cases, where we progressively relax the hypotheses on f, from projective, to quasi projective, to proper, and on X and Y, from smooth quasi projective, to quasi projective, to arbitrary. These conditions will be denoted symbolically by $(f_{proj}, X_{qp}^{sm}, \ldots)$. For example, we summarize the hypotheses of Theorem 2.1.1 graphically as follows:

$$(f_{proj}, X_{qp}^{sm}, Y_{qp}^{sm}) \qquad (f \text{ projective, } X \text{ and } Y \text{ smooth and quasi projective}).$$

Our goal is to establish Corollary 2.2.1 as an immediate consequence of the five following claims.

(1) *Theorem 2.1.1 holds for $(f_{proj}, X_{qp}^{sm}, Y_{qp})$.*
   Choose any closed embedding $g : Y \to \mathbb{U}$ of Y into a Zariski-dense open subvariety $\mathbb{U} \subseteq \mathbb{P}$ of some projective space. Apply Theorem 2.1.1 to $h := g \circ f$ and observe that, modulo the natural identification of the objects in $D(Y, \mathbb{C})$ with the ones in $D(\mathbb{U}, \mathbb{C})$ supported on Y, we have $h_*K = f_*K$.

(2) *Theorem 2.1.1 holds for $(f_{proj}, X_{qp}, Y_{qp})$.*
   Pick a resolution of the singularities $g : Z \to X$ of X with g projective. Let $X^o \subseteq X_{reg} \subseteq X$ be a dense Zariski open subset on which the simple local system M is defined and over which g is an isomorphism. Let $IC_Z(M) \in P(Z, \mathbb{C})$ be the intersection complex on Z with coefficients in the local system M transplanted to $g^{-1}(U^o)$. Apply 1. to g and h. Observe that $IC_X(M)$ is a direct summand of $g_*IC_Z(M)$. Deduce

that $f_* IC_X(M)$ is a direct summand of $h_* IC_Z(M)$ so that the first part of Theorem 2.1.1 holds for $(f_{proj}, X_{qp}, Y_{qp})$. In order to prove the second part of Theorem 2.1.1, i.e. the relative hard Lefschetz theorem for f, we argue as in [dCM], Lemma 5.1.1: we do not need self-duality to conclude: the argument gives injectivity; by dualizing we get surjectivity for the dual of the hard Lefschetz maps; this dualized map is the hard Lefschetz map for f, $IC_X(M)^\vee$ and the f-ample $\eta \in H^2(X, \mathbb{C})$; by switching the roles of M and $M^\vee$, we see that the relative hard Lefschetz theorem maps are isomorphisms. (N.B.: we may impose self-duality artificially, by replacing M with $M \oplus M^\vee$ and reach the same conclusion.)

(3) *Theorem 2.1.1 holds for* (f_{proj}, X_{qp}, Y).
Let $Y = \cup_i Y_i$ be an affine open covering. Let $f_i : X_i := f^{-1}(Y_i) \to Y_i$ be the obvious maps. By 2., the relative Hard Lefschetz holds for f_i. Since the relative hard Lefschetz maps are defined over Y and they are isomorphisms over the Y_i, the relative hard Lefschetz holds for f over Y. By the Deligne-Lefschetz criterion [De], we have $f_* K \cong \oplus_b {}^{\mathfrak{p}}\mathcal{H}^b(f_* K)[-b]$. It remains to show that the $P^b := {}^{\mathfrak{p}}\mathcal{H}^b(f_* K)$ are semi-simple. By 2., the $P^b_{|Y_i}$ are semi-simple after restriction to the open affine Y_i. By a repeated use of the the splitting criterion [dCM], Lemma 4.1.3[1] applied in the context of a Whitney stratification of Y w.r.t. which the P^b are cohomologically constructible, we deduce that the P^b split as direct sum of intersection complexes with coefficients in some local systems. (Note that [dCM], Assumption 4.1.1 is fulfilled in view of [dCM], Remark 4.1.2, because we already know that P_b splits as desired over the open Y_i.) We need to verify that these local systems are semi-simple. Since a local system on an integral normal variety is semisimple if and only if it is semisimple after restriction to a Zariski dense open subvariety, the desired semi-simplicity can be checked by restriction to the chosen affine covering of Y, where we can apply 2.

(4) *Theorem 2.1.1 holds for* (f_{proj}, X, Y).
As it was pointed out in 3., the relative hard Lefschetz can be verified on an affine covering $Y = \cup_i Y_i$. The resulting X_i are then quasi-projective and we can apply 3. For the semisimplicity of the direct image $f_* IC_X(M)$, we take a Chow envelope $g : Z \to X$ of X (Z quasi projective, g projective and birational); we produce $IC_Z(M)$ as above and we deduce the semisimplicity of $f_* IC_X(M)$ from the one –established in 3.– of $h_* IC_Z(M)$, as it was done in 2.

(5) *The semisimplicity statement in Theorem 2.1.1 holds for* (f_{proper}, X, Y).
Take a Chow envelope $g : Z \to X$ of f (g birational, g and $h := f \circ g$ projective). Produce $IC_Z(M)$ as above. Apply 4. and deduce that $f_* IC_X(M)$ is a direct summand of the semi-simple $h_* IC_Z(M)$.

The above, together with the obvious remark that it is enough to prove Theorem 2.1.2 in the case when X, Y are irreducible and $K = IC_X(M)$, yields the following

Corollary 2.2.1. *Theorem 2.1.2 holds for* $\mathbb{C}$*-coefficients.*

2.3. **Theorem 2.1.2 for $D(Y, \mathbb{C})$ implies the same for $D(Y, \mathbb{Q})$.** Let f be projective. Then we have the relative hard Lefschetz for $\mathbb{C}$-coefficients, hence for $\mathbb{Q}$-coefficients as well. By the Deligne-Lefschetz criterion, we have the isomorphism $f_* K \cong \oplus_b {}^{\mathfrak{p}}\mathcal{H}^b(f_* K)[-b]$ in $D(Y, \mathbb{Q})$. We

[1]Let P be a perverse sheaf on a variety Z; let $Z = U \coprod Z$ be Whitney-stratified in such a way that $U \subseteq Z$ is open and union of strata, $S \subseteq Z$ is a closed stratum, and P is cohomologically constructible with respect to the stratification; Lemma 4.1.3 in [dCM] is an iff criterion for the splitting of P into the intermediate extension $j_{!*}(P_{|U})$ to Z of the restriction $P_{|U}$ of P to U, direct sum a local system on S placed in cohomological degree minus the codimension of the stratum; the criterion is local in the classical and even in the Zariski topology

need to show that each $P^b := {}^{\mathfrak{p}}\mathcal{H}^b(f_*K)[-b]$ is semi-simple in $P(Y, \mathbb{Q})$. Note that extending the coefficients from $\mathbb{Q}$ to $\mathbb{C}$ is a t-exact functor $D(Y, \mathbb{Q}) \to D(Y, \mathbb{C})$. In particular, the formation of P^b is compatible with complexification. By arguing as in point 3. of the previous section, we see that each P^b is a direct sum of intersection complexes $IC_S(L)$, where the L are rational local systems (note that [dCM], Assumption 4.1.1 is now fulfilled in view of [dCM], Remark 4.1.2, because we already know that the complexification of P^b splits as desired over Y). We need to verify that each L is a semi-simple rational local system. We know its complexification is, hence so is L, in fact: let $0 \to L' \to L \to L'' \to 0$ be an extension of rational locally constant sheaves on S^o; it is classified by an element $e \in H^1(S^o, L''^* \otimes L')$; this element becomes trivial after complexification, hence it is trivial over $\mathbb{Q}$.

If f is proper, we take a Chow envelope $g : Z \to X$ of f, we set $h := f \circ g$ and we deduce semisimplicity of f_* from the semisimplicity of h_* (h is projective) as in point 5. of the previous section.

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