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Volume 13

Geometry and Topology of Singular Spaces
in honour of David Trotman
for his 60th birthday
Luminy- Marseille, France, Oct. 29-Nov. 2, 2012

Editors:
K. Bekka
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Journal *of* Singularities

David Trotman

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This volume contains the proceedings of the international workshop “Topology and Geometry of Singular Spaces”, held in honour of David Trotman in celebration of his sixtieth birthday. The workshop took place at the Centre International de Rencontres Mathématiques (CIRM), Marseilles, France from October 29 to November 2nd 2012. Its main theme was the singularity theory of spaces and maps.

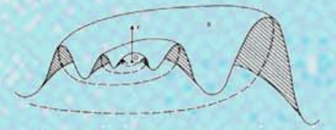
The meeting was attended by 74 participants from all over the world. 29 talks were given by major specialists, and 8 posters were presented by some younger mathematicians. The topics of the talks and posters were wide-ranging: stratification theory, stratified Morse theory, geometry of definable sets, singularities at infinity of polynomial maps, additive invariants of real algebraic varieties, applications of singularities to robotics, and topology of complex analytic singularities.

We thank all participants, especially the speakers, for making the meeting successful and fruitful, both socially and scientifically.

We are also very grateful to all the research bodies who contributed to the financing of the conference: the CIRM institution, the University of Aix-Marseille for Fonds FIR, the LABEX Archimède, and FRUMAM, the University of Rennes 1, the University of Savoy, the ANR SIRE, the city of Marseilles, the "Conseil Gènèral des Bouches du Rhône", the Ministry of Education via the ACCES program, the GDR (Groupement de Recherche) of the CNRS *Singularités et Applications* and the GDR-International *franco-japonais-vietnamien de singularités*.

The papers of this volume cover a variety of the subjects discussed at the workshop. All the manuscipts have been carefully peer-reviewed. We thank the authors for their valuable contributions, and the referees for their careful and conscientious work.

The Editors (who were also the organisers of the workshop),
April 2015.



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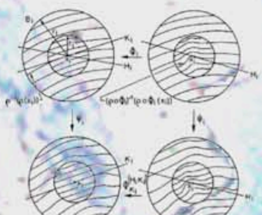

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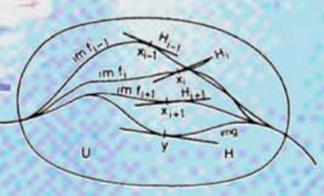

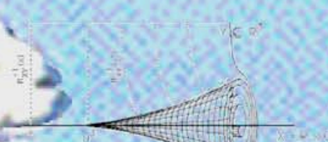

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David Trotman

David John Angelo Trotman was born on September 27th, 1951, in Plymouth, U.K..

As a boy he lived in Stourbridge, attending first Gig Mill School (1958-62) and then King Edward's School (1962-69). His interest in, and enjoyment of, mathematics was evident very early, but he had many other interests too - the end of his schooldays coincided with success in a competition involving walking 240 kilometers through mountainous terrain in western Turkey, undertaking a dozen set projects on the way!

He read mathematics as an undergraduate at the University of Cambridge (1969-72), where a prize-winning essay on *Plane Algebraic Curves* is already indicative of his mathematical focus. He went on to do post-graduate work at the University of Warwick - from where his M.Sc. dissertation *Classification of elementary catastrophes of codimension less than or equal to 5*, often cited and much used, comes - and later on at the Université Paris-Sud, at Orsay. His Ph.D. was awarded in 1978; his thesis was entitled *Whitney Stratifications: Faults and Detectors*. The list of advisors who encouraged this work is remarkable - Christopher Zeeman, Bernard Teissier, René Thom, and, less officially, Terry Wall.

He has held tenured positions at the University of Paris XI (Orsay) and at the University of Angers, but from 1988 he has been Professor of Mathematics at the University of Provence (Aix-Marseille I). Here he has played an important role, both administratively (for example, he was Director of the Graduate School in Mathematics and Computing of Marseilles from 1996 to 2004, and he was an elected member of the CNU (the National University Council in France) from 1999 to 2007), and especially in teaching and research. He has supervised ten Ph.D. students, with great success - all ten are active in teaching and/or research in mathematics. They are listed below.

David's extensive published research in singularity theory is described by Les Wilson elsewhere in this volume. An equally important part of David's contribution to the research milieu lies in the way he interacts with colleagues and students. He is very helpful, and very generous with his time - and his wide knowledge of, and intuition for, singularities in general and stratifications in particular has helped towards the success of many a research project, to the extent that René Thom, in an article in the Bulletin of the AMS, could write of the work of "Trotman and his school" on the theory of stratifications. David has always been good at asking interesting questions, and at finding, or helping to find, interesting answers!

I have known David since we were undergraduates at St. John's College, Cambridge, and over time have got to know him, and his family, very well. He has helped me very many times, both mathematically and practically. It is a privilege to have him as my colleague and my friend. I congratulate him on a most successful career so far, and wish for him - and for us, his colleagues, collaborators and students - many more years of interesting mathematics.

Andrew du Plessis

The Trotman School of Stratifications.

First generation: Patrice Orro (1984); Karim Bekka (1988); Stephane Simon (1994); Laurent Noirel (1996);, Claudio Murolo (1997); Georges Comte (1998); Didier D'Acunto (2001); Dwi Juniati (2002); Guillaume Valette (2003); Saurabh Trivedi (2013).

The next generation: via Orro: Mohammad Alcheikh, Abdelhak Berrabah, Si Tiep Dinh, Farah Farah, Sébastien Jacquet, Mayada Slayman; via Bekka: Nicolas Dutertre, Vincent Grandjean; via Comte: Lionel Alberti; via Juniati: Mustamin Anggo, M.J. Dewiyani, Sulis Janu, Jackson Mairing, Theresia Nugrahaningsih, Herry Susanto, Nurdin.

The coauthors: Kambouchner, Brodersen, Navarro Aznar, Orro, Bekka, Kuo, Li Pei Xin, Kwieciński, Risler, Wilson, Murolo, Noirel, Comte, Milman, Juniati, du Plessis, Gaffney, King, Plénat, Trivedi and Nguyen Nhan.

THE RESEARCH OF DAVID TROTMAN

Leslie Wilson

In order to analyze singular spaces (differentiable or analytic), Whitney and Thom in the 1950's and 1960's partitioned the spaces into disjoint unions of manifolds satisfying some conditions on how they approached each other; this was the beginning of Stratification Theory. Early work by them, Mather and others focused on proving topological equisingularity of the stratifications, or of stratified mappings. The theory has continued to develop, and has become an essential tool in Differential Topology, Algebraic Geometry and Global Analysis. Since his first publications in 1976, David Trotman has played a central role in Stratification Theory. I will give a brief presentation of his work. Citation numbers refer to the following Publication List. I will assume some familiarity with basic stratification theory on the reader's part; an excellent survey of real stratification theory is [C16].

Whitney's stratification condition (b) and Verdier's (w) were both early on proven to guarantee topological equisingularity. How are these conditions related? (b) is equivalent to (w) in the complex analytic case; (w) implies (b) in the real subanalytic case. The converse is not true: the first semialgebraic example appeared in Trotman [C2], the first algebraic example $y^4 = t^4x + x^3$ was due to Brodersen-Trotman [6]. In the differentiable case neither condition implies the other (the slow spiral satisfies (w) but not (b)).

Wall conjectured that condition (b) (and Whitney's weaker condition (a)) were equivalent to more geometric conditions (b_s) and (a_s): these conditions hold for strata X and Y with Y in the closure of X if for every C^1 tubular neighborhood T of Y (with C^1 projection π to Y and C^1 control function ρ to R with $Y = \rho^{-1}(0)$), $(\pi, \rho)|X$ is a submersion to $Y \times R$ (respectively, $\pi|X$ is a submersion to Y)— here π and ρ are assumed C^1-equivalent to orthogonal projection and the distance squared to Y function, respectively. Trotman in [4] showed that (b_s) implies (b) and (a_s) implies (a) (Thom having established the converse earlier). Trotman's Arcata paper [C5] is still a beautiful though no longer complete listing of known relationships between stratification conditions.

For some applications (for example the classification of topological stable mappings) it is necessary to consider stratification conditions weaker than (b) and (w), but which still guarantee topological equisingularity. One useful such condition is condition (C), introduced by Trotman's student Bekka; this involves replacing ρ with a generalized control function. Bekka and Trotman in [25] (see also [11]) study a notion of "locally-radial (C)-regular spaces": in addition to yielding stratifications which are topologically trivial, the stratifications are locally homeomorphic to a cone on a stratified space such that the rays of the cone have finite length and the volume is locally finite. In [C14], Bekka and Trotman define a notion of "weakly Whitney", which lies between (b)-regular and locally-radial (C)-regular; it has the additional property that the intersection of two weakly Whitney stratified spaces is weakly Whitney (see also [32]).

Condition (a) is weaker then (b), and doesn't imply topological triviality; why is it interesting? Trotman showed in [3] that (a) has the following important property: a locally finite stratification of a closed subset Z of a C^1 manifold M is (a)-regular iff for every C^1 manifold N, $\{f \in C^1(N, M)| f$ is transverse to the strata of $Z\ \}$ is an open set in the Whitney C^1 topology.

An important property for stratification conditions is invariance under transverse intersection. The following was proved in Orro-Trotman [C17]: if (Z, Σ) and (Z', Σ') are Whitney (b)-regular (resp. (a)-regular, resp. (w)-regular) and have transverse intersections in M, then $(Z \cap Z', \Sigma \cap \Sigma')$ is (b)-regular (resp. (a)-regular, resp. (w)-regular) (the (b) case was done earlier, the Orro-Trotman result includes other conditions we haven't looked at).

Similarly one would like to know which conditions are invariant under intersection with generic wings. Suppose X and Y are disjoint C^2 submanifolds of a C^2 manifold M, and $y \in Y \cap \overline{X}$. Suppose E is a regularity condition (like (b)). Then (X, Y) is said to be (E^*)-regular if for all k, $0 \leq k < codY$, there is an open, dense subset of the Grassmannian of codimension k subspaces of T_yM containing T_yY such that if W is a C^2 submanifold of M with $Y \subset W$ near y, and $T_yW \in U^k$, then W is transverse to X near y and $(X \cap W, Y)$ is (E)-regular at y (the W is a *generic wing*). From Navarro Aznar-Trotman [7]: for subanalytic stratifications, $(w) \implies (w^*)$, and if $\dim Y = 1$, $(b) \implies (b^*)$. This property plays an important role in the work of Goresky and MacPherson on existence of stratified Morse functions, and in Teissier's equisingularity results. More recently, it was shown by Juniati-Trotman-Valette [26]: for subanalytic stratifications, $(L) \implies (L^*)$ (where (L) is the condition of Mostowski guaranteeing Lipschitz equisingularity).

Another interesting stratification condition, due originally to Thom, is condition (t). Recall Whitney's example $Z = \{y^2 = t^2x^2 + x^3\}$, which satisfies (a) but not (b). The intersections of Z with planes through 0 transverse to the t-axis have constant topological type. A theorem by Kuo in 1978 states: if (X, Y) is (a)-regular at $y \in Y$ then (h^∞) holds, i.e. the germs at y of intersections $S \cap X$, where S is a C^∞ submanifold transverse to Y at y and $\dim S + \dim Y = \dim M$ (S is called a *direct transversal*) are homeomorphic. Trotman refined Thom's condition to be: (X, Y) is (t^k)-*regular* at $y \in Y$ if every C^k submanifold S transverse to Y at y is transverse to X nearby. He proved the following theorems.

Theorem (Trotman [1]): If Y is semianalytic, then (t^1) is equivalent to (a).

In the above result one needs non-direct transversals. In the results below, we always restrict to direct transversals.

Theorem (Trotman [9]): (t^1) is equivalent to (h^1).

Theorem (Trotman-Wilson [17], following Kuo-Trotman [12] and Kuo-Li-Trotman [13]): For subanalytic strata (t^k) is equivalent to the finiteness of the number of topological types of germs at y of $S \cap X$ for S a C^k transversal to Y $(1 \leq k \leq \infty)$.

The proofs use the"Grassmann blowup": like the regular blowup, but with lines through y replaced with all linear subspaces through y of dimension equal to the codimension of Y.

Theorem ([12] and [17]): (X, Y) is (t^k)-regular at $0 \in Y$ iff its Grassmann blowup $(\tilde{X}, \tilde{Y})$ is (t^{k-1})-regular at every point of $\tilde{Y}$ $(k \geq 1)$.

A definition of (t^k) is given in [17] so that (t^0) is equivalent to (w). So $(t^1) \implies (h^1)$ follows from blowup and then applying the Verdier Isotopy Theorem.

Consider the Koike-Kucharz example: let $Z = \{x^3 - 3xy^5 + ty^6 = 0\}$, with Y the t-axis and $X = Z - Y$. Then (X, Y) is (t^2), but not (t^1). There are two topological types of germs at 0 of intersections $S \cap X$ where S is a C^2 submanifold transverse to Y at 0. However the number of topological types of such germs for S of class C^1 is uncountable.

Also there is a theory in [17] of (t^{k-}) such that (t^{1-}) is essentially (a) holding for all sequences going to 0 not tangent to Y. The (t^k) and (t^{k-})-conditions were formulated for jets of transversals. The (t^k) and (t^{k-})-conditions were then used to characterize sufficiency of jets of functions, generalizing theorems of Bochnak, Kuo, Lu and others.

In Gaffney-Trotman-Wilson[30] condition (t^k) was expressed in terms of integral closure of modules, giving more algebraic techniques for computations. In the complex analytic case, (t^k) is characterized by the genericity of the multiplicity of a certain submodule.

If a subset Z of $\mathbf{R}^n$ or $\mathbf{C}^n$ contains a submanifold Y, and p is the local orthogonal projection to Y, then the normal cone $C_Y Z$ of Z to Y is the set of limits $t_i(z_i - p(z_i))$, where $z_i \in Z$ converge some $y \in Y$, and t_i is in $\mathbf{R}$ or $\mathbf{C}$ as appropriate.

Theorem (Hironaka in analytic (b) case, Orro-Trotman [22] generalize to smooth $(a) + (r^e)$): a stratification of Z satisfying the above regularity conditions is (npf) (= normally pseudo-flat, i.e. p is an open map), and (n) (= the fibre of the normal cone is the tangent cone of the fibre).

Orro-Trotman [22] show the Theorem fails for (a)-regularity. Trotman-Wilson [28] show that it also fails in the non-polynomial bounded o-minimal category for (b); our example is :
$z = f(x, y) = x - x \ln(y + \sqrt{x^2 + y^2})/\ln(x)$.

The Nash fiber of a singular space X at x is the set of limits of tangent spaces at regular points x_i of X as $x_i \to x$. Kwieciński-Trotman [15] show: every continuum can be realized as the Nash fiber of a Whitney stratified set.

The classical Poincaré-Hopf Theorem equates the index of a vector field with isolated zeros on a smooth compact manifold with the Euler characteristic of the manifold. Trotman (with King) proved a generalization to singular spaces satisfying fairly general stratification conditions; their manuscript has been influential in the field for many years, but has only recently been published in [33].

Finally, Trotman has made several contributions toward the proof of Zariski's Conjecture: the multiplicity of complex analytic hypersurface-germs with isolated singularity is invariant under homeomorphism. C^1-invariance was proven in [C11]; bi-Lipschitz invariance is proved in Risler-Trotman [16]. In Comte-Milman-Trotman [23] it is proven that multiplicity is preserved by homeomorphisms which preserve both $|z|$ and level sets of the moduli of our defining equations. More recently, Plenát and Trotman in [31] prove: if the family $F(z, t) = f(z) + tg_1(z) + t^2g_2(z) + t^3g_3(z) + \ldots$ has constant Milnor number at $z = 0$, then $mult(g_r) = mult(f) - r + 1$ for $r \geq 1$.

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2. (with Leslie Wilson) *Stratifications and topology of singular spaces*, Hermann - Travaux en Cours, 55, 1997.

3. (with Denis Chéniot, Nicolas Dutertre, Claudio Murolo and Anne Pichon) *Singularity Theory, Proceedings of the 2005 Marseille Singularity School and Conference, World Scientific*, 2007, 1065 pages.

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A Survey on the Trotman works, Les WILSON

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ON THE BI-LIPSCHITZ CONTACT EQUIVALENCE OF PLANE COMPLEX FUNCTION-GERMS

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To David Trotman for his sixtieth birthday.

Abstract. In this note, we consider the problem of bi-Lipschitz contact equivalence of complex analytic function-germs of two variables. Basically, it is inquiring about the infinitesimal sizes of such function-germs up to bi-Lipschitz changes of coordinates. We show that this problem is equivalent to right topological classification of such function-germs.

1. Contact equivalence

Two $\mathbb{K}$-analytic function-germs $f, g\colon (\mathbb{K}^n, \mathbf{0}) \to (\mathbb{K}, 0)$, at the origin $\mathbf{0}$ of $\mathbb{K}^n$, are ($\mathbb{K}$-analytically) *contact equivalent* if the ideals (in $\mathcal{O}_{\mathbb{K}^n,\mathbf{0}}$) generated by f and, respectively, generated by g are $\mathbb{K}$-analytically isomorphic. As is well known, this classical ($\mathbb{K}$-analytic) contact equivalence admits moduli. For a complete description and answer to Zariski *problème des modules pour les branches planes* in the uni-branch case, see [5], (see also [6] for an answer towards the general case). Over the years several generalizations of the notion of ($\mathbb{K}$-analytic) contact equivalence appeared, and for some rough ones moduli do not exist.

More precisely, we will say that two function-germs $f, g\colon (\mathbb{K}^n, \mathbf{0}) \to (\mathbb{K}, 0)$ at the origin $\mathbf{0}$ of $\mathbb{K}^n$ are *bi-Lipschitz contact equivalent* if there exists $H\colon (\mathbb{K}^n, \mathbf{0}) \to (\mathbb{K}^n, \mathbf{0})$ a bi-Lipschitz homeomorphism and there exist positive constants A and B, and $\sigma \in \{-1, +1\}$ such that

$$A|f(\mathbf{p})| \leq |g \circ H(\mathbf{p})| \leq B|f(\mathbf{p})| \text{ when } \mathbb{K} = \mathbb{C},$$
$$Af(\mathbf{p}) \leq \sigma \cdot (g \circ H(\mathbf{p})) \leq Bf(\mathbf{p}) \text{ when } \mathbb{K} = \mathbb{R},$$

for any point $\mathbf{p} \in \mathbb{K}^n$ close to $\mathbf{0}$.

When the bi-Lipschitz homeomorphism H is also subanalytic, we will say that the functions f and g are *subanalytically bi-Lipschitz contact equivalent*.

A consequence of the main result of [1] on bi-Lipschitz contact equivalence of Lipschitz function-germs is the following finiteness

Theorem ([1]). *For any given pair n and k of positive integers, the subspace of polynomial function-germs $(\mathbb{K}^n, \mathbf{0}) \to (\mathbb{K}, 0)$ of degree smaller than or equal to k has finitely many bi-Lipschitz contact equivalence classes.*

Later on, Ruas and Valette (see [10]) obtained for real mappings a result more general than that of [1], and which again ensures the finiteness of the bi-Lipschitz contact equivalent classes for

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polynomial function-germs $(\mathbb{K}^n, \mathbf{0}) \to (\mathbb{K}, 0)$ with given bounded degree. However, we observe that in the aforementioned papers [1, 10], the proofs of the finiteness theorems for bi-Lipschitz contact equivalence do not say anything about the corresponding recognition problem.

The preprint [2] completely solves the recognition problem of subanalytic contact bi-Lipschitz equivalence for continuous subanalytic function-germs $(\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}, 0)$ by providing an explicit combinatorial object which completely characterizes the corresponding orbit.

In the present note, we solve the recognition problem for the subanalytic bi-Lipschitz contact equivalence of complex analytic function-germs $(\mathbb{C}^2, \mathbf{0}) \to (\mathbb{C}, 0)$.

Our main result, Theorem 4.2, states that the subanalytic bi-Lipschitz contact equivalence class of a plane complex analytic function-germ $f\colon (\mathbb{C}^2, \mathbf{0}) \to (\mathbb{C}, 0)$ determines and is determined by purely numerical data, namely: *the Puiseux pairs of each branch of its zero locus, the multiplicities of its irreducible factors and the intersection numbers of pairs of branches of its zero locus.* It is a consequence of Theorem 3.6 which explicits the order of an irreducible function-germ g along real analytic half-branches at $\mathbf{0}$ as an affine function of the contact of the half-branch and the zero locus of g.

Last, combining the main result of [8] and our main result, we eventually get that two complex analytic function germs $f, g\colon (\mathbb{C}^2, \mathbf{0}) \to (\mathbb{C}, 0)$ are subanalytically bi-Lipschitz contact equivalent if, and only if, they are *right topologically equivalent,* i.e. there exists a homeomorphism $\Phi\colon (\mathbb{C}^2, \mathbf{0}) \to (\mathbb{C}^2, \mathbf{0})$ such that $f = g \circ \Phi$.

2. Preliminaries

We present below some well known material about complex analytic plane curve-germs. It will be used in the description and the proof of our main result.

2.1. Embedded topology of complex plane curves.

Let $f\colon (\mathbb{C}^2, \mathbf{0}) \to (\mathbb{C}, 0)$ be the germ at $\mathbf{0}$ of an irreducible analytic function. It admits a Puiseux parameterization of the following kind:

$$x \to (x^m, \Psi(x)) \text{ with } \Psi(x) = x^{\beta_1}\varphi_1(x^{e_1}) + \ldots + x^{\beta_s}\varphi_s(x^{e_s}), \tag{1}$$

where each function φ_i is a holomorphic unit at $x = 0$, the integer number m is the multiplicity of the function f at the origin and $(\beta_1, e_1), \ldots, (\beta_s, e_s)$ are the Puiseux pairs of f. Then we can write down,

$$f(x^m, y) = U(x, y)\Pi_{i=1}^m (y - \Psi(\omega^i x)), \tag{2}$$

where ω is a primitive m-th root of unity, the function U is a holomorphic unit at the origin, and Ψ is a function like in Equation (1).

The following relations determines the Puiseux pairs of f. Let us write $\Psi(x) = \sum_{j>m} a_j x^j$ and $e_0 := m$ and $\beta_{s+1} := +\infty$. We recall that

$$\beta_{i+1} = \min\{j : a_j \neq 0 \text{ and } e_i \not| j\} \text{ and } e_{i+1} := \gcd(e_i, \beta_{i+1})$$

for $i = 0, \ldots, s-1$. We deduce that there exists positive integers $m_1, \ldots, m_s$, such that for each $k = 1, \ldots, s$, we find

$$m = e_1 m_1 = e_2 m_2 m_1 = \ldots = e_k(m_k \cdots m_1) \tag{3}$$

We recall that the irreducibility of the function f implies that $e_s = 1$.

Remark 1. *Let $f\colon (\mathbb{C}^2,\mathbf{0}) \to (\mathbb{C},0)$ be an irreducible analytic function-germ and let X be its zero locus. The ideal I_X of $\mathbb{C}\{x,y\}$ consisting of all the functions vanishing on X is generated by f. If $g = \lambda f$ is any other generator of I_X, then the functions f and g have the same Puiseux pairs. Thus we will speak of the* Puiseux pairs of the branch X.

Let $f_1, f_2\colon (\mathbb{C}^2,\mathbf{0}) \to (\mathbb{C},0)$ be irreducible analytic function-germs, and let X_1 and X_2 be the respective zero sets of f_1 and f_2.

The *intersection number at* $\mathbf{0}$ of the branches X_1 and X_2 is defined as:

$$(X_1,X_2)_{\mathbf{0}} = \dim_{\mathbb{C}} \frac{\mathbb{C}\{x,y\}}{(f_1,f_2)}$$

where (f_1,f_2) denotes the ideal generated by f_1 and f_2.

Notation: Let $\Phi : (\mathbb{C}^2,\mathbf{0}) \to (\mathbb{C}^2,\mathbf{0})$ be a homeomorphism and let X be a subset germ of $(\mathbb{C}^2,\mathbf{0})$. We will write

$$\Phi : (\mathbb{C}^2,X,\mathbf{0}) \to (\mathbb{C}^2,Y,\mathbf{0})$$

to mean that the subset germ Y is the germ of the image $\Phi(X)$ of X.

The following classical result completely described the classification of embedded complex plane curve germs:

Theorem 2.1 ([3, 11])**.** *Let $f,g\colon (\mathbb{C}^2,\mathbf{0}) \to (\mathbb{C},0)$ be reduced analytic function-germs and let X and Y be the respective zero sets of f and g. Let $X = \bigcup_{i=1}^r X_i$ and $Y = \bigcup_{i=1}^s Y_i$ be the irreducible components of X and Y respectively. There exists a homeomorphism $\Phi\colon (\mathbb{C}^2,X,\mathbf{0}) \to (\mathbb{C}^2,Y,\mathbf{0})$ if and only if, up to a re-indexation of the branches of Y, the components X_i and Y_i have the same Puiseux pairs, and each pair of branches X_i and X_j have the same intersection numbers as the pair Y_i and Y_j.*

We end-up this subsection in recalling a recent result of Parusiński [8]. It is as much a generalization of Theorem 2.1 to the non reduced case, as it is an improvement in the sense that it provides a more rigid statement.

Theorem 2.2. *Let $f,g\colon (\mathbb{C}^2,\mathbf{0}) \to (\mathbb{C},0)$ be complex analytic function-germs (thus not necessarily reduced). There exists a germ of homeomorphism $\Phi : (\mathbb{C}^2,\mathbf{0}) \to (\mathbb{C}^2,\mathbf{0})$ such that $g\circ\Phi = f$ (the function-germs f and g are then said* topologically right-equivalent*) if, and only if, there exists a one-to-one correspondence between the irreducible factors of f and g which preserves the multiplicities of these factors, their Puiseux pairs and the intersection numbers of any pairs of distinct irreducible components of the respective zero loci of f and g.*

2.2. Lipschitz geometry of complex plane curve singularities.

The Lipschitz geometry of complex plane curve singularities we are interested in is the Lipschitz geometry which comes from being embedded in the plane. It is described in a collection of three articles over 40 years, initiated with the seminal paper [9], followed then by [4] and concluding for now with the recent preprint [7]. Those papers state that the Lipschitz geometry of complex plane curve singularities determines and is determined by the embedded topology of such singularities. The version of this result which we are going to use is the following one:

Theorem 2.3. *Let X and Y be germs of complex analytic plane curves at $\mathbf{0} \in \mathbb{C}^2$. Then, there exists a homeomorphism $\Phi\colon (\mathbb{C}^2,X,\mathbf{0}) \to (\mathbb{C}^2,Y,\mathbf{0})$ if, and only if, there exists a (subanalytic) bi-Lipschitz homeomorphism $H\colon (\mathbb{C}^2,X,\mathbf{0}) \to (\mathbb{C}^2,Y,\mathbf{0})$.*

The version stated above is almost Theorem 1.1 of [7]. The exact statement of Theorem 1.1 of [7] does not require the subanalyticity of the homeomorphism H. However, we observe that the proof presented there actually guarantees the subanalyticity of the mapping H.

3. On the irreducible functions case

This section is devoted to the relation between the order of a given irreducible plane complex function-germ f along any real analytic half-branch germ at the origin $\mathbf{0}$ of $\mathbb{C}^2$, and the contact (at the origin) between the half-branch and the zero locus X of f. (Both notions of order and contact will be recalled below.) Theorem 3.6 is the main result of the section and the key new ingredient to complete the subanalytic bi-Lipschitz contact classification. It states that the contact and the order satisfies an affine relation whose coefficients can be explicitly computed by means of the Puiseux data of X presented in sub-Section 2.1.

We suppose given some local coordinates (w, y) centered at the origin of $\mathbb{C}^2$.

Let Γ be a real-analytic half branch germ at the origin of $\mathbb{C}^2$, that is the image of (the restriction of) a real analytic map-germ $\gamma : (\mathbb{R}_+, 0) \to (\mathbb{C}^2, \mathbf{0})$ defined as $s \to \gamma(s) = (w(s), y(s))$. When Γ is not contained in the y-axis, we can assume that $\gamma(s) = (s^e \mathbf{u}(s), s^{e'} \mathbf{v}(s))$ for positive integers e, e' with $\mathbf{u}(z), \mathbf{v}(z) \in \mathcal{O}_1 := \mathbb{C}\{z\}$ and $\mathbf{u}(0), \mathbf{v}(0) \neq 0$.

When Γ is not contained in the y-axis, we want to find a holomorphic change of coordinates $w \to x(w)$ so that

$$x(z^e \mathbf{u}(z)) = z^e \iff \mathbf{u}(z) \cdot \mathbf{x}(z^e \mathbf{u}(z)) = 1 \tag{4}$$

writing x as $x(w) := w \cdot \mathbf{x}(w)$ for a local holomorphic unit $\mathbf{x}$. Thus Equation (4) admits a holomorphic solution. The mapping $\Theta : (w, y) \to (x(w), y) = (x, y)$ is bi-holomorphic in a neighbourhood of the origin. In the new coordinates (x, y), the mapping γ now writes as $s \to (s^e, s^{e'} \mathbf{v}(s))$.

Vocabulary. A map-germ $\phi : (\mathbb{R}_+, 0) \to (\mathbb{C}^2, \mathbf{0})$ is *ramified analytic* if there exists a function germ $\tilde{\phi} \in \mathcal{O}_1$ and (co-prime) positive integers p, q such that $\phi(t) = \tilde{\phi}(t^{p/q})$. We will further say that ϕ is *a ramified analytic unit* if $\tilde{\phi}$ is a holomorphic unit.

When Γ is not contained in the y-axis, we re-parameterize γ with $s(t) := t^{e/m}$ for $t \in \mathbb{R}_+$, so that $\gamma(t) := \gamma(s(t)) = (t^m, y(t))$ where y is ramified analytic with $y(0) = 0$ and m is the multiplicity of the function f at the origin.

If Γ is contained in the y-axis then we take $s = t$ and Θ is just the identity mapping.

We recall that the Puiseux pairs introduced in sub-Section 2.1 are bi-holomorphic invariant. We denote again $f = f(x, y)$ for $f \circ \Theta^{-1}$ and use the Puiseux decomposition for $f(x^m, y)$ given in Equation (2) to define for each $k = 0, \ldots, s$, the function germ $\Psi_k \in \mathcal{O}_1$ as

$$\begin{aligned} \Psi_0(x) &:= 0, \\ \Psi_k(x) &:= x^{\beta_1} \varphi_1(x^{e_1}) + \ldots + x^{\beta_k} \varphi_k(x^{e_k}) \text{ when } k \geq 1. \end{aligned}$$

Note that $\Psi_k(x) = \theta_k(x^{e_k})$ for some function germ $\theta_k \in \mathcal{O}_1$.

For each $l = 1, \ldots, m$, we can write

$$y(t) = \Psi(\omega^l t) + t^{\lambda_l} u_l(t)$$

where $\lambda_l \in \mathbb{Q}_{>0} \cup \{+\infty\}$ for u_l is a ramified analytic unit, and with the convention that we write the null function 0 as $0 = t^{+\infty} u_l(t)$. Thus the half-branch Γ is contained in X if and only if there exists l such that $\lambda_l = +\infty$.

Notation. *Let* $\lambda := \max_{l=1,\ldots,m} \lambda_l$.

Let $l \in \{1, \ldots, m\}$ so that $\lambda = \lambda_l$. When Γ is not contained in X (equivalently $\lambda < +\infty$) and convening further that $\beta_0 = 0$ and $\beta_{s+1} = +\infty$, there exists a unique integer $k \in \{0, \ldots, s\}$ such that

$$\beta_k \leq \lambda < \beta_{k+1},$$

and consequently we can write

$$y(t) = \Psi_k(\omega^l t) + t^\lambda u(t)$$

for u a ramified analytic unit. (Note that $\Psi = \Psi_k + R_k$ where $R_k(x) = (\Psi - \Psi_k)(x) = O(x^{\beta_{k+1}})$.)

Evaluating the function f along the parameterized arc $t \to \gamma(t)$ using Equation (2) gives

$$f(\gamma(t)) = f(t^m, y(t)) = f(t^m, \Psi_k(t) + t^\lambda u(t)) = U(t)\Pi_{i=1}^m[\Psi_k(\omega^l t) + t^\lambda u(t) - \Psi(\omega^i t)]$$

where $t \to U(t)$ is a ramified analytic unit. Since the function $t \to f(\gamma(t))$ is a ramified analytic function, there exist a ramified analytic unit V and a number $\nu \in \mathbb{Q}_{>0} \cup \{+\infty\}$ such that

$$f(\gamma(t)) = t^\nu V(t). \tag{5}$$

The number ν of Equation (5) is called the *order of the function f along the parameterized curve $t \to \gamma(t)$*.

Lemma 3.1. *1) Assume Γ is contained in the y-axis. The order of the function f along the parameterized curve $t \to \gamma(t) = (0, t^{e'}\mathbf{v}(t))$ is $\nu = m \cdot e'$.*

2) Assume Γ is not contained in the y-axis. The order of ν the function f along the parameterized curve $t \to \gamma(t)$ is given by

$$\nu = e_k\lambda + (e_0 - e_1)\beta_1 + \ldots + (e_{k-1} - e_k)\beta_k \in \mathbb{Q}_{>0} \cup \{+\infty\}.$$

Proof. If Γ is contained in the y-axis, then the order of f along $t \to (0, t^{e'}\mathbf{v}(t))$ is $m \cdot e'$.

We can assume that Γ is parameterized as $\mathbb{R}_+ \ni t \to \gamma(t) = (t^m, \psi_k(t) + t^\lambda u(t))$.

For $i \in \{1, \ldots, m\}$ such that $l-i$ is not a multiple of m_1, the order of $\Psi_k(\omega^l t) + t^\lambda u(t) - \Psi(\omega^i t)$ is β_1. There are $m - 1 - (e_1 - 1) = e_0 - e_1$ such indices i.

For any $0 < j < k$, when $i \in \{1, \ldots, m-1\}$ is such that $l - i$ is a multiple of $m_1 \ldots m_j$ but not a multiple of $m_1 \ldots m_{j+1}$, the order of $\Psi_k(\omega^l t) + t^\lambda u(t) - \Psi(\omega^i t)$ is β_j. There are $e_{j-1} - e_j$ such indices.

When $i \in \{1, \ldots, m\}$ is such that $l - i$ is a multiple of $m_1 \ldots m_k$, the order of

$$\Psi_k(\omega^l t) + t^\lambda u(t) - \Psi(\omega^i t)$$

is λ. There are e_k such indices.

We just add-up all these orders to get the desired number ν, once we have checked that this sum does not depend on the index l such that $\lambda = \lambda_l$. Let $r \in \{1, \ldots, m\}$ be an index such that $\lambda_r = \lambda$. Thus $y(t) = \Psi_k(w^r t) + t^\lambda u_r(t)$. If $l - r$ is not a multiple of $m_1 \cdots m_k$, then we check again that $0 = y(t) - y(t) = t^\lambda(u_l(t) - u_r(t)) + t^{\beta_j} W$ for a ramified analytic unit W and $\beta_j \leq \beta_{k-1} < \lambda$, which is impossible. Necessarily $l - r$ is a multiple of $m_1 \cdots m_k$ and thus $\Psi_k(w^r t) = \Psi_k(\omega^l t)$, so that ν is well defined. ☐

Now we can introduce a sort of normalized parameterization of real analytic half-branch germs in order to do bi-Lipschitz geometry. More precisely,

Definition 3.2. *A* (real) analytic arc *(at the origin of $\mathbb{C}^2$) is the germ at $0 \in \mathbb{R}_+$ of a mapping $\alpha : [0, \epsilon[\to \mathbb{C}^2$ defined as $t \to (x(t), y(t))$ such that:*

0) the mapping α is not constant and $\alpha(0) = \mathbf{0}$,

1) there exists a positive integer e such that $t \to \alpha(t^e)$ is (the restriction of) a real analytic mapping,

2) the arc is parameterized by the distance to the origin in the following sense: there exists positive constants $a < b$ such that for $0 \leq t \ll 1$ the following inequalities hold,

$$at \leq |\alpha(t)| \leq bt.$$

We will denote any analytic arc by its defining mapping α. Note that the semi-analyticity of the image of an analytic arc α implies a much better asymptotic than that proposed in the definition, namely we know that $|\alpha(t)| = \alpha_1 t + t\delta(t)$, with $\alpha_1 > 0$ and where δ is ramified analytic such that $\delta(0) = 0$.

Let α be a real analytic arc. The function $t \to f \circ \alpha(t)$ is ramified analytic, thus as already seen in Equation (5) can be written as $f \circ \alpha(t) = t^{\nu_f(\alpha)} V(t)$ for a ramified analytic function V and $\nu_f(\alpha) \in \mathbb{Q}_{>0} \cup \{+\infty\}$. The *order of the function f along the real analytic arc α* is the well defined rational number $\nu_f(\alpha)$.

Let C be a real-analytic half-branch germ at the origin of $\mathbb{C}^2$. Let α and β be two real analytic arcs parameterizing C. We check with an easy computation that $\nu_f(\alpha) = \nu_f(\beta)$. Thus we introduce the following

Definition 3.3. *The* order of the function f along the real analytic half-branch C *is the well defined number $\nu_f(C) := \nu_f(\delta)$ for any analytic arc δ parameterizing C.*

Let us denote $X(r) = \{\mathbf{p} \in X \ : |\mathbf{p}| = r\}$ for r a positive real number.

Let α be any analytic arc. The *contact* (*at the origin*) *between the analytic arc α and the complex curve-germ X* is the rational number defined as

$$c(\alpha, X) = \lim_{t \to 0+} \frac{\log(\operatorname{dist}(\alpha(t), X(|\alpha(t)|)))}{\log(t)}.$$

Let C be the image of the analytic arc α above. Given any other analytic arc β parameterizing C, it is a matter of elementary computations to check that $c(\alpha, X) = c(\beta, X)$. Thus we present the following

Definition 3.4. *The* contact between the real-analytic half-branch C and the curve X *is $c(C, X) := c(\delta, X)$ for any analytic arc δ parameterizing C.*

Let Γ be a real analytic half-branch at the origin of $\mathbb{C}^2$. Let γ be a parameterization of Γ of the form $\mathbb{R}_+ \ni t \to (0, y(t))$ when Γ is contained in the y-axis, where y is a ramified analytic function-germ. When Γ is not contained in the y-axis, possibly after a holomorphic change of coordinates at the origin of $\mathbb{C}^2$, we consider a parameterization of Γ of the form $\mathbb{R}_+ \ni t \to (t^m, y(t))$ for y ramified analytic.

When the half-branch Γ is not contained in X (and regardless of its position relatively to the y-axis), as already seen above, we can write $y(t)$ as $y(t) = \Psi_k(\omega^l t) + t^\lambda u(t)$ where $\beta_k \leq \lambda < \beta_{k+1}$ for some integer $k \in \{0, \ldots, s\}$, with u a ramified analytic unit and $l \in \{1, \ldots, m\}$. Let μ be the order of $|\gamma(t)|$ at $t = 0$, that is the positive rational number μ such that $|\gamma(t)| = Mt^\mu + o(t^\mu)$ for a positive constant M. Thus we find

Lemma 3.5. *The contact between Γ and X is $c(\Gamma, X) = \frac{\lambda}{\mu}$.*

Proof. Up to a linear change of coordinates we can assume that the tangent cone at the origin of the (irreducible) curve X is just the x-axis. Writing $\gamma(t) = (x(t), y(t))$, the half-branch is tangent to the x-axis if and only if $\lim_{t \to 0} x(t)^{-1} y(t) = 0$. When Γ is transverse to the x-axis, we have $k = 0$ in the writing of $y(t)$ above, so that $\mu = \lambda$ and thus $c(\Gamma, X) = 1$.

When the half-branch Γ is tangent to the x-axis, we deduce $\mu = m$ since the tangency hypothesis implies that $y(t) = o(t^m)$. Thus the mapping $t \to \gamma(t^{\frac{1}{m}}) = (t, y(t^{\frac{1}{m}}))$ is an analytic arc parameterizing Γ. In particular we must have $\lambda > m$.

Notation. Up to the end of this proof we will use the notation *Const* to mean a positive constant we do not want to precise further.

Let $\rho : (\mathbb{R}_+, 0) \to (\mathbb{R}_+, 0)$ be the function defined as $\rho(t) := \mathrm{dist}(\gamma(t^{\frac{1}{m}}), X)$. First, since γ is tangent to X and the function ρ is continuous and subanalytic, there exists a positive rational number c such that

$$\rho(t) = Const \cdot t^c + o(t^c). \tag{6}$$

Second, we obviously have for t positive and small enough $\rho(t) \leq t^{\frac{\lambda}{m}}|u(t)|$ so that we deduce from Equation (6) that $c \geq \frac{\lambda}{m}$.

Let $r(t) := |\gamma(t^{\frac{1}{m}})|$, so that we find $r(t) = t + o(t)$. Let $t \to \phi(t)$ be any analytic arc on X such that $\rho(t) = |\phi(t) - \gamma(t^{\frac{1}{m}})|$. From Equation (6) we get

$$||\phi(t)| - r(t)| \leq Const \cdot t^c. \tag{7}$$

Writing $\phi = (x_\phi, y_\phi)$, we see from Equation (7) that $x_\phi(t) = t + O(t^c)$. Let $\xi : (\mathbb{R}_+, 0) \to (\mathbb{C}, 0)$ be the ramified analytic function of the form $t \to \xi(t) := t^{\frac{1}{m}}[1 + O(t^{c-1})]$ and such that $\xi(t)$ is a m-th root of $x_\phi(t)$. Thus $y_\phi(t) = \Psi(\omega^i \xi(t))$ for some $i \in \{1, \ldots, m\}$ and we observe that $y_\phi(t) = \Psi(\omega^i t^{\frac{1}{m}}) + o(t^{\frac{\lambda}{m}})$. Since $y(t) = \Psi_k(\omega^l t^{\frac{1}{m}}) + t^{\frac{\lambda}{m}} u(t^{\frac{1}{m}})$, with u a ramified analytic function, and $|y_\phi(t) - y(t^{\frac{1}{m}})| \leq Const \cdot t^c$, we deduce that $\Psi_k(\omega^i T) = \Psi_k(\omega^l T)$. But this implies that $c \leq \frac{\lambda}{m}$, and thus $c = \frac{\lambda}{m}$.

From Equation (7) we deduce that

$$\rho(t) \leq \mathrm{dist}(\gamma(t^{\frac{1}{m}}), X(r(t))) \leq Const \cdot t^c. \tag{8}$$

Combining Equation (6) and Equation (8) we get the result. $\square$

The next result will be key for Theorem 4.2, the main result of this note, is indeed the new ingredient to the range of questions we are dealing with here. We recall that the Puiseux data notation convenes that $e_{-1} = \beta_0 = 0$, $e_0 = m$ and $\beta_{s+1} = +\infty$.

Theorem 3.6. *Let Γ be a real analytic half-branch at the origin of $\mathbb{C}^2$ as above. The order of the function f along Γ is given by*

$$\nu_f(\Gamma) = e_k \cdot c(\Gamma, X) + (e_0 - e_1)\frac{\beta_1}{m} + \ldots + (e_{k-1} - e_k)\frac{\beta_k}{m} \tag{9}$$

$$= e_k \left(c(\Gamma, X) - \frac{\beta_k}{m} \right) + \sum_{i=\min(k-1,0)}^{k-1} e_i \left(\frac{\beta_{i+1}}{m} - \frac{\beta_i}{m} \right), \tag{10}$$

where the integer number $k \in \{0, \ldots, s\}$ in Equations (9) and (10) is uniquely determined when $c(\Gamma, X) < +\infty$ by the following condition:

$$\beta_k \leq m \cdot c(\Gamma, X) < \beta_{k+1}.$$

Proof. It is just a rewriting of Lemma 3.1 in term of the size t of any arc parameterizing Γ and uses Lemma 3.5. $\square$

A direct consequence of the above result is the following result about bi-Lipschitz contact equivalence.

Proposition 3.7. *Let $(\mathbb{C}^2, X, \mathbf{0})$ and $(\mathbb{C}^2, Y, \mathbf{0})$ be two germs of irreducible complex plane curves defined by reduced function-germs f and g respectively. If there exists a subanalytic bi-Lipschitz homeomorphism $H : (\mathbb{C}^2, X, \mathbf{0}) \to (\mathbb{C}^2, Y, \mathbf{0})$ then there exist positive constants $0 < A < B < +\infty$ such that in a neighbourhood of the origin we find*

$$A|f| \leq |g \circ H| \leq B|f|.$$

Proof. If it is not true, it happens along a real-analytic half-branch C. Necessarily such a half-branch C must be tangent to the curve X. Taking a parameterization of C by an arc α, we can for instance assume that $(f\circ\alpha(t))^{-1}(g\circ H\circ\alpha(t))$ goes to 0 as t goes to 0. Let ν be the order of $f(\alpha(t))$ and ν' the order of $g(H(\alpha(t)))$. Theorem 3.6 provides

$$\begin{aligned}\nu &= (e_0-e_1)\frac{\beta_1}{m}+\ldots+(e_{k-1}-e_k)\frac{\beta_k}{m}+e_k\cdot c(C,X)\\ \nu' &= (e_0-e_1)\frac{\beta_1}{m}+\ldots+(e_{k'-1}-e_{k'})\frac{\beta_{k'}}{m}+e_{k'}\cdot c(H^{-1}(C),Y).\end{aligned}$$

From the proofs of Lemma 3.1 and Lemma 3.5 we know that

$$\beta_{k'}\le m\cdot c(H^{-1}(C),Y)<\beta_{k'+1} \text{ and } \beta_k\le m\cdot c(C,X)<\beta_{k+1}.$$

Since the contact is a bi-Lipschitz invariant we get $c(C,X)=c(H^{-1}(C),Y)$. Besides $\nu'>\nu$, thus we deduce $k'>k$. This latter inequality implies

$$m\cdot c(H^{-1}(C),Y)\ge\beta_{k'}\ge\beta_{k+1}>m\cdot c(C,X),$$

which is impossible. □

4. Main Result

Let $f\colon(\mathbb{C}^2,\mathbf{0})\to(\mathbb{C},0)$ be a germ of analytic function. Let $f=f_1^{\mathbf{m}_1}\cdots f_r^{\mathbf{m}_r}$ be the irreducible decomposition of the function, where $f_1,\ldots,f_r$ are irreducible function-germs and $\mathbf{m}_1,\ldots,\mathbf{m}_r$, the corresponding respective multiplicities, are positive integer numbers.

Let X_i be the zero locus of f_i, let m_i be the multiplicity of f_i at $\mathbf{0}$ and let $(\beta_j^{(i)},e_j^{(i)})_{j=1}^{s_i}$ be its Puiseux pairs. Let Γ be a real analytic half-branch at the origin. Let $c_i:=c(\gamma,X_i)$ be the contact of Γ with X_i and let $\nu_i=\nu_{f_i}(\Gamma)$ be the order of f_i along Γ.

Since we have defined in Section 3 the order of an irreducible function-germ along Γ, the order of f along Γ is defined as the sum of the order of each of its irreducible component weighted by the corresponding multiplicity (as a factor of the irreducible decomposition of f). From Theorem 3.6 we deduce straightforwardly the next

Lemma 4.1. *The order ν of the function f along Γ is*

$$\begin{aligned}\nu &:= \mathbf{m}_1\cdot\nu_1+\ldots+\mathbf{m}_r\cdot\nu_r\\ &= \sum_{i=1}^{r}\mathbf{m}_i\left[e_{k_i}^{(i)}\left(c_i-\frac{\beta_{k_i}^{(i)}}{m}\right)+\sum_{j=\min(k_i-1,0)}^{k_i-1}e_j^{(i)}\left(\frac{\beta_{j+1}^{(i)}}{m}-\frac{\beta_j^{(i)}}{m}\right)\right]\end{aligned}$$

where each of the integer $k_i\in\{0,\ldots,s_i\}$ is uniquely determined when $c_1\cdots c_r<+\infty$ by the condition

$$\beta_{k_i}^{(i)}\le m_i\cdot c_i<\beta_{k_i+1}^{(i)}.$$

The main result of this note is the following:

Theorem 4.2. *Let f and g be two analytic function-germs $(\mathbb{C}^2,\mathbf{0})\to(\mathbb{C},0)$. Let $f=f_1^{\mathbf{m}_1}\cdots f_r^{\mathbf{m}_r}$ and $g=g_1^{\mathbf{n}_1}\cdots g_s^{\mathbf{n}_s}$ be respectively the irreducible decompositions of the functions f and g. Let X_i be the zero locus of f_i and Y_j be the zero locus of g_j.*

The functions f and g are subanalytically bi-Lipschitz contact equivalent if, and only if, possibly up to a re-indexation of the irreducible factors f_i:

0) $r=s$,

1) the multiplicities of each corresponding factors are equal, that is $\mathbf{m}_i=\mathbf{n}_i$, *2) the Puiseux pairs of f_i and g_i are the same, and*

3) for any pair i, j, the intersection numbers $(X_i, X_j)_{\mathbf{0}}$ and $(Y_i, Y_j)_{\mathbf{0}}$ are equal.

In particular, f and g are subanalytically bi-Lipschitz contact equivalent if, and only if, they are right topologically equivalent.

Proof. First (and possibly after a re-indexation of the irreducible factors f_i) assume that,
- $r = s$,
- the intersection numbers $(X_i, X_j)_{\mathbf{0}}$ and $(Y_i, Y_j)_{\mathbf{0}}$ are equal for any $i \neq j$ and,
- the Puiseux pairs of the functions f_i and g_1 are equal and,
- the multiplicities $\mathbf{m}_i$ and $\mathbf{n}_i$ are equal, for $i = 1, \ldots r$.

From Theorem 2.3 we deduce there exists $H\colon (\mathbb{C}^2, \mathbf{0}) \to (\mathbb{C}^2, \mathbf{0})$ a subanalytic bi-Lipschitz homeomorphism such that $H(X_i) = Y_i$ for any $i = 1, \ldots r$. For each $i = 1, \ldots, r$, Proposition 3.7 implies there exist positive constants $0 < A_i < B_i < +\infty$ such that in a neighbourhood of the origin we find

$$A_i|f_i| \leq |g_i \circ H| \leq B_i|f_i|.$$

Thus the functions f and g are bi-Lipschitz contact equivalent (via h).

Conversely, we assume now that there exists $H\colon (\mathbb{C}^2, \mathbf{0}) \to (\mathbb{C}^2, \mathbf{0})$ a subanalytic bi-Lipschitz homeomorphism such that there exist positive constants $A < B$ such that in a neighbourhood of the origin the following inequalities hold true:

$$A|f| \leq |g \circ H| \leq B|f|. \tag{11}$$

We immediately find $H(X) = Y$ and $r = s$. Up to re-indexation of the branches Y_i, we also have $H(X_i) = Y_i$ for $i = 1, \ldots, r$. Using Theorem 2.3 again we deduce that the intersection numbers $(X_i, X_j)_{\mathbf{0}}$ and $(Y_i, Y_j)_{\mathbf{0}}$ are equal for any $i \neq j$ (let us denote each such number by $I_{i,j}$), the Puiseux pairs of the function-germs f_i and g_i are equal. It remains to prove that the multiplicities $\mathbf{m}_i$ and $\mathbf{n}_i$ are also equal, for $i = 1, \ldots, r$. In order to prove that $\mathbf{m}_1 = \mathbf{n}_1$, let C be any real-analytic half-branch such that the contact $c = c(C, X_1)$ is sufficiently large (and finite) and also such that the others contacts $c(C, X_i)$, for $i = 2, \ldots, r$, are equal to the intersection number $I_{i,1} := (X_i, X_1)_{\mathbf{0}}$ (see [4] for details). Since H is a subanalytic bi-Lipschitz homeomorphism such that $H(X_i) = Y_i$ for any $i = 1, \ldots, r$, the image $H(C)$ is still a real analytic half-branch. Since bi-Lipschitz homeomorphisms preserve the contact, we deduce that $c = c(H(C), Y_1)$ and each contact $c(H(C), Y_i)$ is equal to the contact $(Y_i, Y_1)_{\mathbf{0}}$, for $i = 2, \ldots, r$. In other words we see

$$\nu_g(H(C)) = c \cdot \mathbf{n}_1 + I_{2,1} \cdot \mathbf{n}_2 + \ldots + I_{r,1} \cdot \mathbf{n}_r \tag{12}$$

and

$$\nu_f(C) = c \cdot \mathbf{m}_1 + I_{2,1} \cdot \mathbf{m}_2 + \ldots + I_{r,1} \cdot \mathbf{m}_r. \tag{13}$$

Combining Equation (11) from the hypothesis, with Equations (12) and (13) we conclude that

$$c\mathbf{n}_1 + I_{2,1}\mathbf{n}_2 + \ldots + I_{r,1}\mathbf{n}_r = c\mathbf{m}_1 + I_{2,1}\mathbf{m}_2 + \cdots + I_{r,1}\mathbf{m}_r.$$

Since the half-branch C can be chosen asymptotically arbitrarily close to X_1, its contact c goes $+\infty$, and thus we find $\mathbf{m}_1 = \mathbf{n}_1$. The same procedure can be applied for each remaining $i = 2, \ldots, r$, substituting i for 1, thus we conclude that

$$\mathbf{m}_i = \mathbf{n}_i \ \text{ for } i = 1, \ldots, r,$$

thus proving what we wanted. $\square$

The first immediate consequence of our main result is the following:

Corollary 4.3. *Let f and g be two analytic function-germs $(\mathbb{C}^2, \mathbf{0}) \to (\mathbb{C}, 0)$. They are bi-Lipschitz contact equivalent if, and only if, they are subanalytically bi-Lipschitz contact equivalent.*

The second consequence is:

Corollary 4.4. *The subanalytic bi-Lipschitz contact equivalence classification of complex analytic plane function-germs has countably many equivalence classes.*

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DEFORMATION OF SINGULARITIES AND ADDITIVE INVARIANTS

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Abstract. In this survey on local additive invariants of real and complex definable singular germs we systematically present classical or more recent invariants of different nature as emerging from a tame degeneracy principle. For this goal, we associate to a given singular germ a specific deformation family whose geometry degenerates in such a way that it eventually gives rise to a list of invariants attached to this germ. Complex analytic invariants, real curvature invariants and motivic type invariants are encompassed under this point of view. We then explain how all these invariants are related to each other as well as we propose a general conjectural principle explaining why such invariants have to be related. This last principle may appear as the incarnation in definable geometry of deep finiteness results of convex geometry, according to which additive invariants in convex geometry are very few.

Introduction

A beautiful and fruitful principle occurring in several branches of mathematics consists in deforming the object under consideration in order to let appear some invariants attached to this object. In this deformation process, the object X_0 to study becomes the special fibre of a deformation family $(X_\varepsilon)_\varepsilon$ where each fibre X_ε approximates X_0, from a topological, metric or geometric point of view, depending on the nature of the invariant that one aims for X_0 through $(X_\varepsilon)_\varepsilon$.

For instance in Morse theory, where a smooth real valued function $f : M \to \mathbf{R}$ on a smooth manifold M is given, such that $f^{-1}([f(m)-\eta, f(m)+\eta]$ contains no critical point of f but m, the homotopy type of $f^{-1}(]-\infty, f(m)+\eta])$ is given by the homotopy type of $f^{-1}(]-\infty, f(m)-\varepsilon])$, for any ε with $0 < \varepsilon \leq \eta$, plus a discrete invariant attached to f at m, namely the index of f at m. In this case the family $(f^{-1}(]-\infty, f(m)-\varepsilon]))_{0<\varepsilon\leq\eta}$ has the same fibres, from the differential point of view, and approximates the special set $f^{-1}(]-\infty, f(m)+\eta])$, from the homotopy type point of view, up to some additional discrete topological invariant.

Another instance of this deformation principle can be found in tropical geometry, where a patchwork polynomial embeds a complex curve X_0 of the complex torus $(\mathbf{C}^*)^2$ into a family (X_ε) of complex curves. This family may be viewed as a curve $\mathscr{X}$ on the non archimedean valued field $\mathbf{C}((\varepsilon^{\mathbf{R}}))$ of Laurent series with exponents in $\mathbf{R}$. Then, by a result of Mikhalkin and Rullgård ([82] and [91]), the amoebas family $\mathscr{A}(X_\varepsilon)$ of (X_ε) has limit (in the Hausdorff metric) the non archimedean amoeba $\mathscr{A}(\mathscr{X})$ of $\mathscr{X}$.

In the theory of sufficiency of jets the aim is to approach a smooth map by its family of Taylor polynomials up to some sufficient degree depending on the kind of equivalence considered for maps (right, left, or V equivalence). Embedding a germ into a convenient deformation is a seminal way of thinking for R. Thom that has been successfully achieved in his cobordism theory or in his works on regular stratifications providing regular trivializations. We could multiply examples in this spirit, old ones as well as recent ones (from recent developments in general deformation theory itself for instance [1]), but in this introduction we will focus only on two specific examples, that will be developed thereafter: the Milnor fibration and the Lipschitz-Killing curvatures.

The Milnor fibre of a complex singularity. The first of these two examples is provided by the Milnor fibre of a complex singular analytic hypersurface germ $f : (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$. We will assume for simplicity that this singularity is an isolated one, that is to say that we will assume that $0 = f(0)$ is the only critical value of f, at least locally around 0. We denote $B_{(0,\eta)}$ the open ball of radius η, centred at 0 of the ambient space depending upon the context. Now, for $\eta > 0$ small enough and $0 < \varrho \ll \eta$, the family $(f^{-1}(\varepsilon) \cap B_{(0,\eta)})_{0<|\varepsilon|<\varrho}$ is a smooth bundle, with projection f, over the punctured disc $B_{(0,\varrho)} \setminus \{0\} \subset \mathbf{C}$. The topological type of a fibre

$$X_\varepsilon := f^{-1}(\varepsilon) \cap B_{(0,\eta)}$$

does not depend on the choice of ε, and the homotopy type of this fibre is the homotopy of a finite CW complex of dimension $n-1$, the one of a bouquet of μ spheres S^{n-1}, where μ is called the Milnor number of the fibration (see [86]). On the other hand, the special singular fibre

$$X_0 := f^{-1}(0) \cap B_{(0,\eta)}$$

is contractible, as a germ of a semialgebraic set. It follows that the family $(X_\varepsilon)_{0<|\varepsilon|<\eta}$ approximates the singular fibre X_0 up to μ cycles that vanish as ε goes to 0. The number μ of these cycles appears as an analytic invariant of the germ of the hypersurface f that is geometrically embodied on the nearby fibres X_ε of the deformation on the singular fibre (see also [23]). In [102], B. Teissier embedded the Milnor number μ in a finite sequence of integers in the following way. For a generic vector space V of $\mathbf{C}^n$ of dimension $n-i$, the Milnor number of the restriction of f to V does not depend on V and is denoted $\mu^{(n-i)}$. In particular $\mu = \mu^{(n)}$ and therefore the sequence $\mu^{(*)} := (\mu_0, \cdots, \mu^{(n-1)}, \mu^{(n)})$ gives a multidimensional version of μ.

We can consider other invariants attached to the Milnor fibre of f, also extending the simple invariant μ: the Lefschetz numbers of the iterates of the monodromy of the Milnor fibration, that we introduce now in order to fix notations in the sequel. The Milnor fibre X_ε may be endowed with an isomorphism M, the monodromy of the Milnor fibre, defined up to homotopy and that induces in an unambiguous way an automorphism, also denoted M, on the cohomology group $H^\ell(X_\varepsilon, \mathbf{C})$

$$M : H^\ell(X_\varepsilon, \mathbf{C}) \to H^\ell(X_\varepsilon, \mathbf{C}), \ell = 0, \cdots, n-1.$$

For the m-th iterate M^m of M, for any $m \geq 0$, one finally defines the Lefschetz number $\Lambda(M^m)$ of M^m by

$$\Lambda(M^m) := \sum_{i=0}^{n-1} (-1)^i tr(M^m, H^i(X_\varepsilon, \mathbf{C})),$$

where tr stands for the trace of endomorphisms. Note that $\Lambda(M^0) = \chi(X_0) = 1 + (-1)^{n-1}\mu$ and that the eigenvalues of M are roots of unity (see for instance [99]).

A more convenient deformation of $f^{-1}(0)$ than the family $(f^{-1}(\varepsilon) \cap B_{(0,\eta)})_\varepsilon$, at least for the practical computation of the topological invariants we just have introduced, is provided by an adapted resolution of the singularity of f at 0. To define such a resolution and fix the notations used in Section 3, let us consider $\sigma : (M, \sigma^{-1}(0)) \to (\mathbf{C}^n, 0)$ a proper birational map which is an isomorphism over the (germ of the) complement of $f^{-1}(0)$ in $(\mathbf{C}^n, 0)$, such that $f \circ \sigma$ and the jacobian determinant jac σ are normal crossings and $\sigma^{-1}(0)$ is a union of components of the divisor $(f \circ \sigma)^{-1}(0)$. We denote by E_j, for $j \in \mathscr{J}$, the irreducible components of $(f \circ \sigma)^{-1}(0)$

[1] As recalled by Kontsevich and Soibelman, Gelfand quoted that "any area of mathematics is a kind of deformation theory", see [68].

and assume that E_k are the irreducible components of $\sigma^{-1}(0)$ for $k \in \mathscr{K} \subset \mathscr{J}$. For $j \in \mathscr{J}$ we denote by N_j the multiplicity $mult_{E_j} f \circ \sigma$ of $f \circ \sigma$ along E_j and for $k \in \mathscr{K}$ by ν_k the number $\nu_k = 1 + mult_{E_k}$ jac σ. For any $I \subset \mathscr{J}$, we set $E_I^0 = (\bigcap_{i \in I} E_i) \setminus (\bigcup_{j \in \mathscr{J} \setminus I} E_j)$.

The collection $(E_I^0)_{I \subset \mathscr{J}}$ gives a canonical stratification of the divisor $f \circ \sigma = 0$, compatible with $\sigma = 0$ such that in some affine open subvariety U in M we have $f \circ \sigma(x) = u(x) \prod_{i \in I} x_i^{N_i}$, where u is a unit, that is to say a rational function which does not vanish on U, and $x = (x', (x_i)_{i \in I})$ are local coordinates. Now the nearby fibres X_ε are isomorphic to their lifting

$$\widetilde{X}_\varepsilon := \sigma^{-1}(f^{-1}(\varepsilon) \cap B_{(0,\eta)})$$

in M and the family $(\widetilde{X}_\varepsilon)_{0<|\varepsilon|<\eta}$ approximates the divisor $\widetilde{X}_0 := \sigma^{-1}(f^{-1}(0) \cap B_{(0,\eta)})$. Of course the geometry of $\widetilde{X}_0$ has apparently nothing to do with the geometry of our starting germ $(f^{-1}(0), 0)$, but as the topological information concerning the singularity of $f : (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ is carried by the nearby fibres family $(X_\varepsilon)_{0<|\varepsilon|<\eta}$, all this information is still encoded in the family $(\widetilde{X}_\varepsilon)_{0<|\varepsilon|<\eta}$, and the discrete data N_j, ν_k although depending on the choice of the resolution, may be combined in order to explicitly compute invariants of the singularity.

Not only μ, the most elementary of our invariants, may be computed in the resolution, but also more elaborated ones such as the Lefschetz numbers of the iterates of the monodromy of the singularity. Indeed, by [1] we have the celebrated A'Campo formulas

$$\Lambda(M^m) = \sum_{i \in \mathscr{K},\ N_i / m} N_i \cdot \chi(E_{\{i\}}^0),\ m \geq 0$$

and in particular

$$1 + (-1)^{n+1}\mu = \chi(X_0) = \Lambda(M^0) = \sum_{i \in \mathscr{K}} N_i \cdot \chi(E_{\{i\}}^0).$$

0.1. *Remark.* Denoting $\bar{X}_0$ the closure of the Milnor fibre of $f : (\mathbf{C}, 0) \to (\mathbf{C}, 0)$, since the boundary $\bar{X}_0 \backslash X_0$ is a compact smooth manifold with odd dimension, we have $\chi(\bar{X}_0 \backslash X_0) = 0$, and in particular $\chi(X_0) = \chi(\bar{X}_0)$. This is why, in the complex case and for topological considerations at the level of the Euler-Poincaré characteristic, the issue of the open or closed nature of balls is not so relevant. In contrast, in the real case, this issue really matters.

Metric invariants coming from convex geometry. The second main example of invariants arising from a deformation that we aim to emphasize and develop here, comes from convex geometry. In this case, starting from a compact convex set of $\mathbf{R}^n$, it is usual to approximate this set by its family of ε-tubular neighbourhoods, $\varepsilon > 0$, since those neighbourhoods remain convex and generally have a more regular shape than the original set. This method is notably used in [101] to generate a finite sequence of metric invariants attached to a compact convex polytope P (the convex hull of a finite number of points) in $\mathbf{R}^n$ (actually in $\mathbf{R}^2$ or $\mathbf{R}^3$ in [101]). It is established in [101] that the volume of the tubular neighbourhood of radius $\varepsilon \geq 0$ of P,

$$T_{P,\varepsilon} := \bigcup_{x \in P} \bar{B}_{(x,\varepsilon)},$$

where $\bar{B}_{(x,\varepsilon)}$ is the closed ball of $\mathbf{R}^n$ centred at x with radius ε, is a polynomial in ε with coefficients $\Lambda_0(P), \cdots, \Lambda_n(P)$ depending only on P and being invariant under isometries of $\mathbf{R}^n$. We have

$$\forall \varepsilon \geq 0,\ Vol_n(T_{P,\varepsilon}) = \sum_{i=0}^{n} \alpha_i \Lambda_{n-i}(P) \cdot \varepsilon^i, \tag{1}$$

It is convenient to normalize the coefficients $\Lambda_i(P)$ by the introduction, in the equality (1) defining them, of the i-volume α_i of the i-dimensional unit ball.

When $\varepsilon = 0$ in this formula, one gets $\Lambda_n(P) = Vol_n(P)$. On the other hand, denoting $\delta = \max\{|x-y|; x, y \in P\}$ the diameter of P, for any $x \in P$, the inclusions

$$B_{(x,\varepsilon)} \subset T_{P,\varepsilon} \subset B_{(x,\varepsilon+\delta)}$$

show that $Vol_n(T_{P,\varepsilon}) \underset{\varepsilon\to\infty}{\sim} \alpha_n \cdot \varepsilon^n$ and thus $\Lambda_0(P) = 1$. Denoting the Euler-Poincaré characteristic by χ and having in mind further generalizations, the relation $\Lambda_0(P) = 1$ has rather to be considered as $\Lambda_0(P) = \chi(P)$. A direct proof of (1) leads to an expression of the other coefficients $\Lambda_i(P)$ in terms of some geometrical data of P. To give this proof, we set now some notation.

For P a polytope in $\mathbf{R}^n$ of dimension n, generated by $n+1$ independent points, an affine hyperplane in $\mathbf{R}^n$ generated by n of these points is called a facet of P. The normal vector to a facet F of P is the unit vector orthogonal to F and pointing in the half-space defined by F not containing P. For $i \in \{0, \cdots, n-1\}$, a *i-face* of P is the intersection of P with $n-i$ distinct facets of P. We denote $\mathscr{F}_i(P)$ the set of i-faces of P. By convention $\mathscr{F}_n(P) = \{P\}$. For $x \in P$ one consider F_x, the unique face of P of minimal dimension containing x. If $x \in \partial P$ (the boundary of P), the *normal exterior cone* of P at x, denoted $C(x, P)$, is the $\mathbf{R}_+$-cone of $\mathbf{R}^n$ generated by the normal vectors to the facets of P containing x. By convention $C(x, P) = \{0\}$, for $x \in P \setminus \partial P$.

We note that for F_x of dimension $i \in \{0, \cdots, n-1\}$, $C(x, P)$ is a $\mathbf{R}_+^\times$-invariant cone of $\mathbf{R}^n$ of dimension $n-i$. Furthermore, for any $y \in F_x$, $C(x, P) = C(y, P)$. One thus defines $C(F, P)$, the exterior normal cone of P along a face F of P, by $C(x, P)$, where x is any point in F. One has $C(P, P) = \{0\}$.

For P a degenerated polytope of $\mathbf{R}^n$, that is to say that the affine subspace $[P]$ of $\mathbf{R}^n$ generated by P is of dimension $< n$, one denotes $C_{[P]}(x, P)$ the exterior normal cone of P at x in $[P]$, since P is of maximal dimension in $[P]$. With this notation, the exterior normal cone of P at x in $\mathbf{R}^n$, denoted $C_{\mathbf{R}^n}(x, P)$, or simply $C(x, P)$ when no confusion is possible, is defined by $C_{[P]}(x, P) \times [P]^\perp$. We finally define $C(F, P)$, for P general, as $C(x, P)$ for any $x \in F$. The exterior normal cone of P depends on the ambient space in which we embed P, but we now define an intrinsic measure attached to the normal exterior cone, the exterior angle.

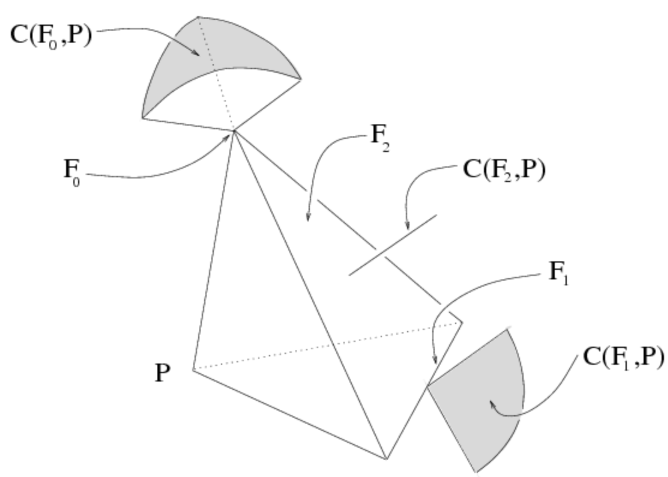

fig.1

0.2. **Definition.** Let P be a polytope of $\mathbf{R}^n$ and $F \in \mathscr{F}_i(P)$. One defines the exterior angle $\gamma(F,P)$ of P along F (see fig.1), by

$$\gamma(F,P) := \frac{1}{\alpha_{n-i}} \cdot Vol_{n-i}(C(F,P) \cap \bar{B}_{(0,1)}) = Vol_{n-i-1}(C(F,P) \cap S_{(0,1)}).$$

By convention $\gamma(P,P) = 1$.

With the definition of the exterior angle, the proof of (1) is trivial.

Proof of equality (1). We observe that

$$Vol_n(T_{P,\varepsilon}) = \sum_{i=0}^{n} \alpha_i \cdot \varepsilon^{n-i} \sum_{F \in \mathscr{F}_i(P)} Vol_i(F) \cdot \gamma(F,P).$$

In particular

$$\Lambda_i(P) = \sum_{F \in \mathscr{F}_i(P)} Vol_i(F) \cdot \gamma(F,P). \tag{2}$$

□

The equality (2) shows how the invariant $\Lambda_i(P)$ captures the concentration $\gamma(F,P)$ of the curvature of the family $(T_{P,\varepsilon})_{\varepsilon>0}$ along the i-dimensional faces of P as ε goes to 0.

In the general convex case and not only in the convex polyhedral case, the equality (1) still holds, defining invariants Λ_i on the set $\mathscr{K}^n$ of convex sets of $\mathbf{R}^n$. A proof of this equality by approximation of a convex set by a sequence of polytopes is given in [97], section 4.2. Another proof is indicated in [40] (3.2.35) and [74], using the Cauchy-Crofton formula, a classical formula in integral geometry, that we recall here.

0.3. **Cauchy-Crofton formula** ([40] 5.11, [41] 2.10.15, 3.2.26, [94] 14.69)**.** *Let $A \subset \mathbf{R}^n$ a $(\mathscr{H}^d, d)$-rectifiable set, where $\mathscr{H}^d$ is the d-dimensional Hausdorff measure. We have*

$$Vol_d(A) = \frac{1}{\beta(d,n)} \int_{\bar{P} \in \overline{G}(n-d,n)} Card(A \cap \bar{P}) \; d\overline{\gamma}_{n-d,n}(\bar{P}), \tag{$\mathscr{C}\mathscr{C}$}$$

with $\overline{G}(n-d,n)$ the Grassmannian of $(n-d)$-dimensional affine planes $\bar{P}$ of $\mathbf{R}^n$, $\overline{\gamma}_{n-d,n}$ its canonical measure and denoting Γ the Euler function, $\beta(d,n)$ the universal constant

$$\Gamma(\frac{n-d+1}{2})\Gamma(\frac{d+1}{2})/\Gamma(\frac{n+1}{2})\Gamma(\frac{1}{2}).$$

One can now prove equality (1) in the general compact convex case.

Proof of equality (1) *in the convex case.* We proceed by induction on the dimension of the ambient space is which our convex compact set K lies. If this dimension is 1, formula (1) is trivial, and if this dimension is $n > 1$, one has

$$Vol_n(T_{K,\varepsilon}) = Vol_n(K) + \int_{r=0}^{\varepsilon} Vol_{n-1}(K^r) \; dr,$$

where K^r is the set of points in $\mathbf{R}^n$ at distance r of K. We compute $Vol_{n-1}(K^r)$ using the Cauchy-Crofton formula.

Noting that $Card(\overline{L} \cap K^r) = 2$ or $Card(\overline{L} \cap K^r) = 0$, up to a $\overline{\gamma}_{1,n}$-null subset of $\overline{G}(1,n)$, we obtain by definition of $\bar{\gamma}_{1,n}$

$$Vol_n(T_{K,\varepsilon}) =$$
$$Vol_n(K) + \int_{r=0}^{\varepsilon} \frac{2}{\beta(1,n)} \int_{H \in G(n-1,n)} Vol_{n-1}(\pi_H(K^r)) \; d\gamma_{n-1,n}(H) \; dr,$$

where $G(n-1,n)$ is the Grassmannian of $(n-1)$-dimensional vector subspace of $\mathbf{R}^n$ equipped with its canonical measure $\gamma_{n-1,n}$ invariant under the action of $O_n(\mathbf{R})$ and π_H is the orthogonal projection onto $H \in G(n-1,n)$.

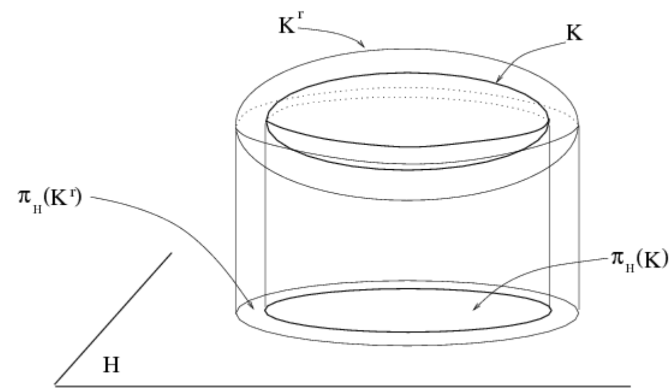

fig.2

By induction hypothesis, the expression of the volume of the tubular neighbourhood of radius r of the convex sets of $\mathbf{R}^{n-1}$ is a polynomial in r. Since $\pi_H(K^r)$ is $T_{\pi_H(K),r}$ in H, we have

$$Vol_n(T_{K,\varepsilon}) = Vol_n(K)$$

$$+\frac{2}{\beta(1,n)}\int_{r=0}^{\varepsilon}\int_{H\in G(n-1,n)}\sum_{i=0}^{n-1}\alpha_i\Lambda_{n-1-i}(\pi_H(K))\cdot r^i\ d\gamma_{n-1,n}(H)\ dr$$

$$= Vol_n(K) + \frac{2}{\beta(1,n)}\sum_{i=0}^{n-1}\frac{\alpha_i}{i+1}\cdot\varepsilon^{i+1}\int_{H\in G(n-1,n)}\Lambda_{n-1-i}(\pi_H(K))\ d\gamma_{n-1,n}(H).$$

□

In [101], the formula (1) is also proved for C^{2+} surfaces, giving a hint for a possible extension of this formula to the smooth case. This extension is due to H. Weyl, who proved in [110] the following statement (see also [74]).

0.4. **Theorem** (Weyl's tubes formula)**.** *Let X be a smooth compact submanifold of $\mathbf{R}^n$ of dimension d. Let $\eta_X > 0$ such that for any ε, $0 < \varepsilon \le \eta_X$, for any $y \in T_{X,\varepsilon}$, there exists a unique $x \in X$ such that $y \in x + (T_xX)^\perp$. Then for any $\varepsilon \le \eta_X$*

$$Vol_n(T_{X,\varepsilon}) = \sum_{i=0}^{[d/2]}\alpha_{n-d+2i}\Lambda_{d-2i}(X)\cdot\varepsilon^{n-d+2i},$$

where the $\Lambda_k(X)'s$ are invariant under isometric embeddings of X into Riemannian manifolds.

When, on the other hand, X is a non convex union of two polytopes P, Q, $Vol_n(T_{X,\varepsilon})$ is no more necessarily a polynomial in ε. For example for $X_1 = P \cup Q$ where $P = \{(0,0)\} \subset \mathbf{R}^2$ and $Q = \{(0,2)\} \subset \mathbf{R}^2$, and for $1 \le \varepsilon \le 2$. In the same way, when X is a singular set, for any $\varepsilon > 0$, $Vol_n(T_{X,\varepsilon})$ is not necessarily a polynomial in ε. For example for

$$X_2 = \{(x,y) \in \mathbf{R}^2; x \ge 0,\quad (x^2+y^2-1)(x^2+(y-2)^2-1) = 0\},$$

for any sufficiently small $\varepsilon > 0$, $Vol_2(T_{X,\varepsilon}) = (1+\varepsilon)^2 \arccos(\dfrac{1}{1+\varepsilon}) - \sqrt{\varepsilon^2 + 2\varepsilon}$ (see fig.3).

0.5. *Remark.* Nevertheless by [22] we know that for X a subanalytic subset of $\mathbf{R}^n$, $Vol(T_{X,\varepsilon})$ is a polynomial in subanalytic functions with variable ε and the logarithms of these functions, and thus that it is defined in some o-minimal structure over the reals.

0.6. *Remark.* The grey areas Σ_1 and Σ_2 in figure 3, counted with multiplicities 1 in $Vol_2(T_{X_i,\varepsilon})$ have non polynomial contributions. But when these areas are counted with multiplicity 2, on one hand, with this modified computation for $Vol_2(T_{X_1,\varepsilon})$, we obtain the sum of the areas of two discs of radius ε centred at P and Q and, on the other hand, with this modified computation for $Vol_2(T_{X_2,\varepsilon})$, we obtain twice the volume of the tubular neighbourhood of radius ε of half a circle minus the volume of a ball of radius ε.

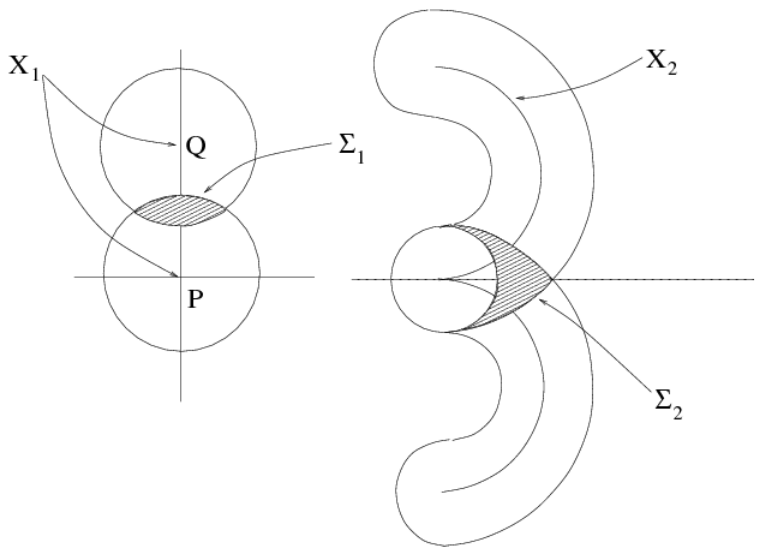

fig.3

In conclusion, a multiple contribution of the volume of the grey areas provides two polynomials in ε. Moreover, we observe that for $j = 1, 2$:

- $\forall x \in \Sigma_j$: $2 = \chi(X_j \cap \bar{B}_{(x,\varepsilon)})$,
- $\forall x \in T_{X_j,\varepsilon} \setminus \Sigma_j$: $1 = \chi(X_j \cap \bar{B}_{(x,\varepsilon)})$,
- $\forall x \in \mathbf{R}^2 \setminus T_{X_j,\varepsilon}$: $\chi(X_j \cap \bar{B}_{(x,\varepsilon)}) = 0$.

It follows that for j=1,2 $\displaystyle\int_{x\in\mathbb{R}^2} \chi(X_j \cap \bar{B}_{(x,\varepsilon)})\, dx = \int_{x\in T_{X_j,\varepsilon}} \chi(X_j \cap \bar{B}_{(x,\varepsilon)})\, dx$ is a polynomial in ε. These examples are in the scope of a general fact: the formula (1) can be generalized to compact sets definable in some o-minimal structure over the reals by the formula $(1')$ given below.

0.7. **Theorem** ([45], [46], [47], [48], [49], [50], [51], [8], [9], [11])**.** *Let X be a compact subset of $\mathbf{R}^n$ definable in some o-minimal structure over the ordered real field. There exist constants*

$\Lambda_0(X), \cdots, \Lambda_n(X)$ *such that for any* $\varepsilon \geq 0$

$$\int_{x \in T_{\varepsilon,X}} \chi(X \cap \bar{B}_{(x,\varepsilon)})\ dx = \sum_{i=0}^{n} \alpha_i \Lambda_{n-i}(X) \cdot \varepsilon^i. \qquad (1')$$

The real numbers $\Lambda_i(X), i = 0, \cdots, n$ *are called the Lipschitz-Killing curvatures of* X*, they only depend on definable isometric embeddings of* X *into euclidean spaces. Moreover, we have*

$$\Lambda_i(X) = \int_{\bar{P} \in \bar{G}(n-i,n)} \chi(X \cap \bar{P})\ \frac{d\bar{\gamma}_{n-i,n}(\bar{P})}{\beta(i,n)}. \qquad (2')$$

0.8. *Remarks about* $(1')$ *and* $(2')$. In Theorem 0.7 we assume the set X compact, although for X bounded but not compact the equality $(1')$ together with $(2')$ is still true with χ the Euler-Poincaré characteristic with compact support, usually considered for non-compact definable sets. This characteristic is additive and multiplicative and defined by any finite cell decomposition $\cup_i C_i$ of X by $\chi(X) = \sum_i (-1)^{\dim(C_i)}$ (see [39], p. 69). For simplicity, in what follows we will still consider the compact case.

0.9. *Remark.* The formula $(1')$ is clearly a generalization to the non convex case of the formula (1), since for X compact convex, for any $\varepsilon \geq 0$, for any $x \in T_{X,\varepsilon}$, $\chi(X \cap \bar{B}_{(x,\varepsilon)}) = 1$, thus $\int_{x\in T_{\varepsilon,X}} \chi(X \cap \bar{B}_{(x,\varepsilon)})\ dx = Vol_n(T_{X,\varepsilon})$. In the same way $(1')$ generalizes Weyl's tube formula to the singular case, since for X smooth, there exists $\eta_X > 0$ such that for any $\varepsilon, 0 < \varepsilon < \eta_X$, for any $x \in T_{X,\varepsilon}$, $\chi(X \cap \bar{B}_{(x,\varepsilon)}) = 1$ and $\int_{x\in T_{\varepsilon,X}} \chi(X \cap \bar{B}_{(x,\varepsilon)})\ dx = Vol_n(T_{X,\varepsilon})$.

The formula $(1')$ comes from a more general cinematic formula (see [11], [49]). For X and Y two definable sets of $\mathbf{R}^n$

$$\int_{g\in G} \Lambda_k(X \cap g \cdot Y)\ dg = \sum_{i+j=k+n} c_{n,i,j} \cdot \Lambda_i(X) \cdot \Lambda_j(Y)$$

with G the group of isometries of $\mathbf{R}^n$ and $c_{n,i,j}$ universal constants.

The expression of Λ_i given by $(2')$ generalizes to the definable case the representation formula (2) of Λ_i given in the polyhedral case. Furthermore, from $(2')$ we get the following characterization of Λ_0 and Λ_d, $d = \dim(X)$, already obtained from (1) in the compact convex case

$$\Lambda_0(X) = \chi(X)$$

and, using the Cauchy-Crofton formula,

$$\Lambda_d(X) = Vol_d(X).$$

Finally,

$$\Lambda_{d+1}(X) = \cdots = \Lambda_n(X) = 0,$$

for $d < n$.

The last remark made now here is a remark that, having in mind geometric measure theory, we are eager to address: the Euler-Poincaré characteristic being additive for definable sets (see [39]) the equality $(1')$ or $(2')$ shows that the Λ_i's are additive invariants of definable sets, in the following sense

$$\forall i \in \{0, \cdots, n\},\ \Lambda_i(X \cup Y) = \Lambda_i(X) + \Lambda_i(Y) - \Lambda_i(X \cap Y),$$

for any definable sets X and Y of $\mathbf{R}^n$.

We have now in hand two kinds of deformation of a singular set. When this set is an analytic isolated hypersurface singularity, we may consider its Milnor fibration, providing in particular as invariant the Milnor number of the singularity, and in the more general definable case, we have recalled in details the notion of Lipschitz-Killing curvatures, coming from the deformation family provided by the tubular neighbourhoods. It is worth noting that in these two cases, the deformations considered lead to additive invariants attached to the given germ.

In what follows, we explain how to localize the Lipschitz-Killing curvatures in order to attach to a singular germ a finite sequence of additive invariants (Section 1). We will then explain how all our local invariants are related and how such kind of relation illustrates, in the very general context of definable sets over the reals, the emergence of a well-known principle in convex geometry wherein additive invariants (with some additional properties) may not be so numerous (Section 2). Finally, we will stress the fact that the additive nature of an invariant coming from a deformation allows us to compute this invariant in some adapted scissors ring via some generating zeta function capturing the nature of the deformation. This is the point of view which underlies the work of Denef and Loeser (Section 3).

0.10. *Notation.* As well as in this introduction, in the sequel, $B_{(x,r)}$, $\bar{B}_{(x,r)}$ and $S_{(x,r)}$ are respectively the open ball, the closed ball and the sphere centred at x and with radius r of the real or the complex vector spaces $\mathbf{R}^n$ or $\mathbf{C}^n$. If necessary, to avoid confusion, we emphasize the dimension d of the ambient space to which the ball belongs by denoting $B^d_{(x,r)}$. Definable means definable in some given o-minimal structure expanding the ordered real field $(\mathbf{R}, +, -, \cdot, 0, 1, <)$ (see [14], [39]).

1. Local invariants from the tubular neighbourhoods deformation

The invariants $\Lambda_0, \cdots, \Lambda_n$ defined in the introduction for compact definable sets (or at least bounded definable sets) may be extended to non bounded definable sets as well as they may be localized in order to be attached to any definable germ $(X, 0)$. The extension of the invariants Λ_i to non bounded definable sets has been proposed in [36]. These two possible extensions are similar; they essentially use the fact that near a given point or near infinity the topological types of affine sections of a definable set are finite in number. As we are mainly interested in local singularities, we explain in this section how to localize the sequence $(\Lambda_0, \cdots, \Lambda_n)$ at a given point.

For this goal, let us consider $X \subset \mathbf{R}^n$ a compact definable set. We assume that $0 \in X$ and we denote by X_0 the germ of X at 0, d its dimension. Representing elements $\bar{P}$ of the Grassmann manifold $\bar{G}(n-i, n)$ of $(n-i)$-dimensional affine subspaces of $\mathbf{R}^n$ by pairs $(x, P) \in \mathbf{R}^n \times G(i, n)$, where $x \in P$, and $\bar{P}$ is the affine subspace of $\mathbf{R}^n$ orthogonal to P at x, the measure $\bar{\gamma}_{n-i,n}$ on $\bar{G}(n-i, n)$ is the image through this representation of the product $m \otimes \gamma_{i,n}$, where m is the Lebesgue measure on P and P is identified with $\mathbf{R}^i$. It follows by formula (2') that Λ_i is i-homogeneous, that is to say $\Lambda_i(\lambda \cdot X) = \lambda^i \Lambda_i(X)$, for any $\lambda \in \mathbf{R}^*_+$. In consequence, it is natural to consider the asymptotic behaviour of $\dfrac{1}{\alpha_i}\Lambda_i(\dfrac{1}{\varrho} \cdot (X \cap \bar{B}_{(0,\varrho)})) = \dfrac{1}{\alpha_i \varrho^i}\Lambda_i(X \cap \bar{B}_{(0,\varrho)})$, as $\varrho \to 0$, in order to obtain invariants attached to the germ X_0 of X at 0.

Using standard arguments for the definable family

$$(\frac{1}{\varrho} \cdot (X \cap \bar{B}_{(0,\varrho)}) \cap \bar{P}))_{(\varrho, \bar{P}) \in \mathbf{R}^*_+ \times \bar{G}(n-i,n)}$$

such as Thom-Mather's isotopy lemma or cell decomposition theorem, one knows that, for any fixed $\bar{P} \in \bar{G}(n-i, n)$, the topological type of the family $(\frac{1}{\varrho} \cdot (X \cap \bar{B}_{(0,\varrho)}) \cap \bar{P}))_{\varrho) \in \mathbf{R}^*_+}$ is constant for ϱ small enough and therefore the limit of $\chi(\frac{1}{\varrho} \cdot (X \cap \bar{B}_{(0,\varrho)}) \cap \bar{P})$ for $\varrho \to 0$ does exist. Furthermore, still by finiteness arguments proper to definable sets, the family $(\chi(\frac{1}{\varrho}\cdot(X\cap\bar{B}_{(0,\varrho)})\cap\bar{P}))_{\bar{P}\in\bar{G}(n-i,n)}$ is bounded with respect to $\bar{P}$.

The next definition follows from these observations (see [21], Theorem 1.3).

1.1. **Definition** (Local Lipschitz-Killing invariants, see [21])**.** Let X be a (compact) definable set of $\mathbf{R}^n$, representing the germ X_0 at $0 \in X$. The limit

$$\Lambda_i^{\ell oc}(X_0) := \lim_{\varrho\to 0} \frac{1}{\alpha_i.\varrho^i}\Lambda_i(X\cap\bar{B}(0,\varrho)) \tag{3}$$

exists and the finite sequence of real numbers $(\Lambda_i^{\ell oc}(X_0))_{i\in\{0,\dots,n\}}$ is called the sequence of local Lipschitz-Killing invariants of the germ X_0.

1.2. *Remark.* Another kind of localization of the invariants Λ_i have been obtained and studied in [9], by considering the family $(X\cap S_{(0,\varrho)})_{\varrho>0}$ instead of the family $(X\cap\bar{B}_{(0,\varrho)})_{\varrho>0}$.

1.3. *Remarks.* For any $i\in\{0,\cdots,n\}$, just as Λ_i, $\Lambda_i^{\ell oc}$ is invariant under isometries of $\mathbf{R}^n$ and defines an additive function on the set of definable germs at the origin of $\mathbf{R}^n$. Moreover $\Lambda_i^{\ell oc}(X_0)=0$, for $i>d$, since $\Lambda_i=0$ for definable sets of dimension $<i$ and for any definable compact germ X_0, $\Lambda_0^{\ell oc}(X_0)=1$, since $\Lambda_0=\chi$ and a definable germ is contractible. Finally, since by the Cauchy-Crofton formula $(\mathscr{C}\mathscr{C})$ and $(2')$ we have

$$\Lambda_d^{\ell oc}(X_0) = \lim_{\varrho\to 0}\frac{Vol_d(X\cap B_{(0,\varrho)})}{Vol_d(B^d_{(0,\varrho)})}, \tag{4}$$

we observe that $\Lambda_d^{\ell oc}(X_0)$ is by definition the local density $\Theta_d(X_0)$ of X_0, and thus we have obtained, by finiteness arguments leading to Definition 1.1, the following theorem of Kurdyka and Raby.

1.4. **Corollary** ([69], [70], [78])**.** *The local density of definable sets of $\mathbf{R}^n$ exists at each point of $\mathbf{R}^n$.*

On figure 4 are represented the data taken into account in the computation of $\Lambda_i^{\ell oc}(X)$.

For $P\in G(i,n)$, we denote by $K_\ell^{P,\varrho}$ the domains of P above which the Euler-Poincaré characteristic of the fibres of $\pi_{P|X\cap\bar{B}(0,\varrho)}$ is constant and equals $\chi_\ell^{P,\varrho}\in\mathbf{Z}$. The quantity $\Lambda_i(X\cap\bar{B}^n_{(0,\varrho)})$ is then obtained as the mean value over the vector planes P of the sum $\sum_{\ell=1}^{\ell_P}\chi_\ell^{P,\varrho}\cdot Vol_i(K_\ell^{P,\varrho})$. In particular we'd like to stress the fact that are considered in this sum the volumes of the domains $K_\ell^{P,\varrho}$ (in green on figure 4) defined by the critical values of π_P coming from the link $X\cap S_{(0,\varrho)}$. We draw attention to these green domains, far from the origin, in view of other local invariants, the polar invariants, that will be defined in the next section, and for which only the domains $K_\ell^{P,\varrho}$ close to the origin will be considered.

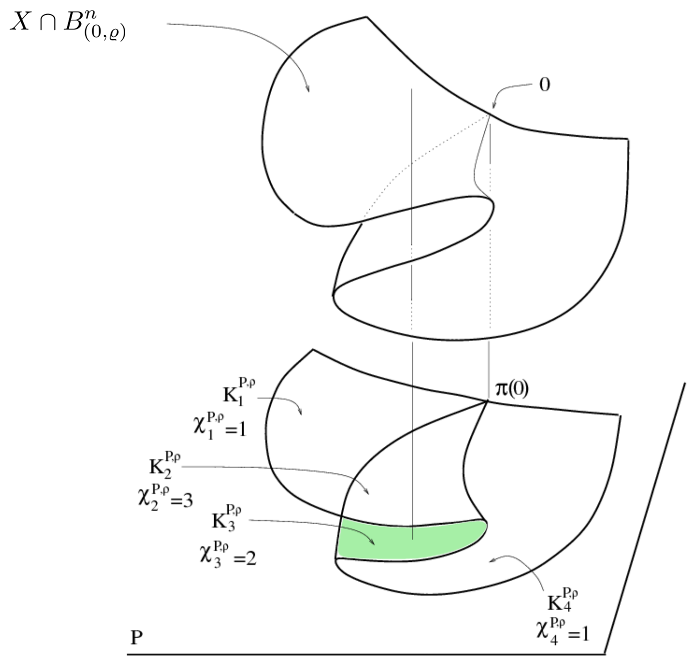

fig.4

2. Additive invariants of singularities and Hadwiger principle

In the previous section we have localized the Lipschitz-Killing invariants. We'd like now to investigate the question of how these invariants are related to classical local invariants of singularities, such as Milnor number, or Milnor numbers of generic plane sections of the singularity (for the complex case). The question of the correspondence of invariants coming from differential geometry and invariants of singularities has been tackled by several authors. In the first section 2.1, we briefly recall some of these works. We then introduce in section 2.2 a sequence of invariants of real singularities that is the real counterpart of classical invariants of complex singularities and we finally relate the localized Lipschitz-Killing invariants to the polar invariants in section 2.3. In section 2.4 we recall results from convex geometry and convex valuation theory giving a strong hint, called here Hadwiger principle, of the reason why such invariants have to be linearly related.

2.1. **Differential geometry of complex and real hypersurface singularities.** The first example we'd like to recall of such a correspondence may be found in [71] (see also [72] and [73]). In [71] R. Langevin relates the concentration of the curvature of the Milnor fibre

$$F^\eta_\varepsilon = f^{-1}(\varepsilon) \cap B_{(0,\eta)}, \quad 0 < \varepsilon \ll \eta \ll 1,$$

as ε and η go to 0 to the Milnor numbers μ and $\mu^{(n-1)}$ of the isolated hypersurface singularity $f : (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$. To present this relation, we need some definitions.

For a given real smooth oriented hypersurface H of $\mathbf{R}^n$ and for $x \in H$, we classically define the Gauss curvature $K(x)$ of H at x by $K(x) := \mathrm{jac}(\nu)$, where $\nu : H \to \mathbf{S}^{n-1}$ is the mapping giving the unit normal vector to H induced by the canonical orientation of $\mathbf{R}^n$ and the given orientation of H. The curvature $K(x)$ can be generalized in the following way to any submanifold

M of $\mathbf{R}^n$ of dimension $d, d \in \{0, \cdots, n-1\}$ (see [42]). Let x be a point of M and denote by $N(x) \simeq \mathbf{S}^{n-d-1}$ the manifold of normal vectors to M at x, and, for $\nu \in N(x)$, by $K(x,\nu)$ the Gauss curvature at x of the projection M_ν of M to $\mathrm{T}_x M \oplus \nu$. Note that the projection M_ν defines at x a smooth hypersurface of $\mathrm{T}_x M \oplus \nu$ oriented by ν. The mean value of $K(x,\nu)$ over $N(x)$ define the desired generalized Gauss curvature.

2.1.1. **Definition** (see [42]). With the above notation, the curvature $K(x)$ of M at x is defined by

$$K(x) := \int_{\nu \in \mathbf{P}N(x)} K(x,\nu)\ d\nu$$

2.1.2. *Remarks.* The curvature $K(x,\nu)$ is ε^{d-n} times the Gauss curvature of the boundary $\partial T_{M,\varepsilon}$ of the ε-neighbourhood of M at $x+\varepsilon\nu$ (see [15]).

In [71], following Milnor, it is observed that for M a smooth complex hypersurface of $\mathbf{C}^n$, $K(x) = (-1)^{n-1}\pi|\mathrm{jac}\gamma_{\mathbf{C}}(x)|^2$, where $\gamma_{\mathbf{C}}$ is the complex Gauss map sending $x \in M$ to the normal complex line $\gamma_{\mathbf{C}}(x) \in \mathbf{PC}^{n-1}$ to M at x.

In case M is a compact submanifold of $\mathbf{R}^n$, using a so-called exchange formula ([72] Theorem II.1, [74], [35]) relating $\int_M K(x)\ dx$ and the mean value over generic lines L in $\mathbf{R}^n$ of the total index of the projection of M on L, we obtain the Gauss-Bonnet theorem

$$\int_M K(x)\ dx = c(n,d)\chi(M).$$

Applying Definition 2.1.1 to the Milnor fibre F_ε^η of an isolated hypersurface singularity $f : (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ and using again the exchange formula, we can estimate the concentration of the curvature $K(x)$ of the Milnor fibre F_ε^η as ε and η go to 0. This value is related to the invariants μ and $\mu^{(n-1)}$ of the singularity, thanks to a result of [103], by the following formula

2.1.3. **Theorem** ([71], [72]). *The curvature K of a complex Milnor fibre satisfies*

$$\lim_{\eta\to 0}\lim_{\varepsilon\to 0}\ c(n)\int_{F_\varepsilon^\eta}(-1)^{n-1}K(x)\ dx = \mu + \mu^{(n-1)},$$

where $c(n)$ is a constant depending only on n.

This formula has been generalized to the other terms of the sequence $\mu^{(*)}$ by Loeser in [79] in the following way.

2.1.4. **Theorem** ([79]). *For $k \in \{1, \cdots, n-1\}$, we have*

$$\lim_{\eta\to 0}\lim_{\varepsilon\to 0}\frac{(-1)^{n-k}c(n,k)}{\eta^{2k}}\int_{F_\varepsilon^\eta} c_{n-1-k}(\Omega_{f^{-1}(\varepsilon)})\wedge\Phi^k = \mu^{(n-k)} + \mu^{(n-k-1)},$$

where $c_{n-1-k}(\Omega_{f^{-1}(\varepsilon)})$ is the $(n-1-k)$-th Chern form of $f^{-1}(\varepsilon)$, Φ the Kähler form of $\mathbf{C}^n$ and as usual $c(n,k)$ a constant depending only on n and k.

A real version of these two last statements has been given by N. Dutertre, in [35] for the real version of Theorem 2.1.3 and in [36] for the real version of Theorem 2.1.4. In [35] (see Theorem 5.6), a real polynomial germ $f : (\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$ having an isolated singularity at 0 is considered and the following equalities are given for the asymptotic behaviour of the Gauss curvature on the real Milnor fibre $F_\varepsilon^\eta = f^{-1}(\varepsilon) \cap B_{(0,\eta)}$.

2.1.5. **Theorem** ([35], Theorem 5.6). *The Gauss curvature K of the real Milnor fibre F_ε^η have the following asymptotic behaviour*

$$\lim_{\eta\to 0}\lim_{\varepsilon\to 0^+}\int_{F_\varepsilon^\eta} K(x)\ dx = \frac{Vol(S^{n-1})}{2}deg_0\nabla f + \frac{1}{2}\int_{G(n-1,n)} deg_0\nabla(f_{|P})\ dP$$

$$\lim_{\eta\to 0}\lim_{\varepsilon\to 0^-}\int_{F_\varepsilon^\eta} K(x)\ dx = -\frac{Vol(S^{n-1})}{2} deg_0 \nabla f + \frac{1}{2}\int_{G(n-1,n)} deg_0 \nabla(f_{|P})\ dP$$

In [36], the asymptotic behaviour of the symmetric functions $s_0, \cdots, s_{n-1}$ of the curvature of the real Milnor fibre are studied. For a given smooth hypersurface H of $\mathbf{R}^n$, the s_i's are defined by

$$\det(Id + tD\nu(x)) = \sum_{i=0}^{n-1} s_i(x)\cdot t^i = \prod_{i=1}^{n-1}(1+k_i(x)t),$$

where the k_i's are the principal curvatures of H, that is to say, the eigenvalues of the symmetric morphism $D\nu(x)$. The limits

$$\lim_{\eta\to 0}\lim_{\varepsilon\to 0}\frac{1}{\eta^k}\int_{F_\varepsilon^\eta} s_{n-k}(x)\ dx$$

are then given in terms of the mean value of $\deg_0 \nabla(f_{|P})$ for $P\in G(n-k+1,n)$ and for $P\in G(n-k-1,n)$ (see [36], Theorem 7.1). In particular, the asymptotic behaviour of the symmetric functions $s_0, \cdots, s_{n-1}$ of the curvature of the real Milnor fibre is related to the Euler-Poincaré characteristic of the real Milnor fibre by the following statement.

2.1.6. **Theorem** ([36], Corollary 7.2)**.** *For n odd,*

$$\chi(F_\varepsilon^\eta) = \sum_{k=0}^{(n-1)/2} c(n,k)\lim_{\eta\to 0}\lim_{\varepsilon\to 0}\frac{1}{\eta^{2k}}\int_{F_\varepsilon^\eta} s_{n-1-2k}(x)\ dx,$$

and for n even

$$\chi(F_\varepsilon^\eta) = \frac{1}{2}\chi(f^{-1}(0)\cap S^{n-1}_{(0,\eta)}) = \sum_{k=0}^{(n-2)/2} c'(n,k)\lim_{\eta\to 0}\lim_{\varepsilon\to 0}\frac{1}{\eta^{2k}}\int_{F_\varepsilon^\eta} s_{n-2-2k}(x)\ dx,$$

2.2. **Local invariants of definable singular germs.** In the present survey we aim to relate the local Lipschitz-Killing invariants $\Lambda_i^{\ell oc}, i=0,\cdots,n$, coming from the tubular neighbourhoods deformation, to local invariants of definable singular germs of $\mathbf{R}^n$. These germs have not necessarily to be of codimension 1 in $\mathbf{R}^n$, as it is the case in the complex and real statements recalled above in Section 2.1. Therefore we have to define local invariants of singularities attached to definable germs of $\mathbf{R}^n$ of any dimension and try to relate them to the sequence $\Lambda_*^{\ell oc}$. Furthermore those invariants have to extend, to the real setting, classical invariants of complex singularities, such as the sequence $\mu^{(*)}$ in the hypersurface case or the sequence of the local multiplicity of polar varieties in the general case. For this purpose we introduce now a new sequence σ_* of local invariants, called the sequence of polar invariants.

Let, as before, $X\subset\mathbf{R}^n$ be a closed definable set, and assume that X contains the origin of $\mathbf{R}^n$ and that d is the dimension of X at 0. We denote by $\mathscr{C}(X)$ the group of definable constructible functions on X, that is to say the group of definable $\mathbf{Z}$-valued functions on X. These functions f are characterized by the existence of a finite definable partition (X_i) of X (depending on f) such that $f_{|X_i}$ is a constant integer $n_i\in\mathbf{Z}$, for any i. We denote by $\mathscr{C}(X_0)$ the group of germs at the origin of functions of $\mathscr{C}(X)$. For $Y\subset\mathbf{R}^m$ a definable set, $f: X\to Y$ a definable mapping, a definable set $Z\subset X$ and $y\in Y$, we introduce the notation $f_*(\mathbf{1}_Z)(y) := \chi(f^{-1}(y)\cap Z)$ and we then define the following functor from the category of definable sets to the category of groups

$$\begin{array}{rcr} X & \rightsquigarrow & \mathscr{C}(X) \\ f\downarrow & & \downarrow f_* \\ Y & \rightsquigarrow & \mathscr{C}(Y) \end{array}$$

In [21], Theorem 2.6, it is stated that, for $f = \pi_P$ the (orthogonal) projection onto a generic i-dimensional vector subspace P of $\mathbf{R}^n$, this diagram leads to the following diagram for germs

$$\begin{array}{rcl} X_0 & \rightsquigarrow & \mathscr{C}(X_0) \\ \pi_{P_0} \downarrow & & \downarrow \pi_{P_0 *} \\ P_0 & \rightsquigarrow & \mathscr{C}(P_0) \end{array} \tag{5}$$

where for $Z_0 \subset X_0$ and $y \in P$, $\pi_{P_0 *}(\mathbf{1}_{Z_0})(y)$ is defined by $\chi(\pi_P^{-1}(y) \cap Z \cap \bar{B}_{(0,\varrho)})$, ϱ being sufficiently small and $0 < \|y\| \ll \varrho$. The existence of such a diagram for germs simply amounts to prove that a generic projection of a germ defines a germ [2] and that, for such a projection and for any $c \in \mathbf{Z}$, the germ at 0 of the definable set $\{y \in P; \chi(\pi_P^{-1}(y) \cap Z \cap \bar{B}_{(0,\varrho)}) = c\}$ does not depend on ϱ.

Denoting by $\theta_i(\varphi)$ the integral with respect to the local density Θ_i at $0 \in \mathbf{R}^i$ of a germ $\varphi : P_0 \to \mathbf{Z}$ of constructible function, that is to say

$$\theta_i(\varphi) := \sum_{j=1}^{N} n_j \cdot \Theta_i(K_0^j),$$

when $\varphi = \sum_{j=1}^N n_j \cdot \mathbf{1}_{K_0^j}$, for some definable germs $K_0^j \subset P_0$ partitioning P_0, we can define the desired polar invariants $\sigma_i(X_0)$ of X_0.

2.2.1. **Definition** (Polar invariants)**.** With the previous notation, the polar invariants of the definable germ X_0 are

$$\sigma_i(X_0) := \int_{P \in G(i,n)} \theta_i(\pi_{P_0 *}(\mathbf{1}_{X_0}))\, d\gamma_{i,n}(P), \quad i = 0, \cdots, n$$

2.2.2. *Remarks.* Since they are defined as mean values over generic projections, the σ_i's are invariant under the action of isometries of $\mathbf{R}^n$. On the other hand the σ_i's define additive invariants (as well as the $\Lambda_i^{\ell oc}$'s do), since they are defined through the Euler-Poincaré characteristic χ and the local density Θ_i, two additive invariants.

Observe that $\sigma_i(X_0) = 0$, for $i > d$, since a general k-dimensional affine subspace of $\mathbf{R}^n$ does not encounter a definable set of codimension $> k$. We also have $\sigma_0(X_0) = \Lambda_0^{\ell oc}(X_0) = 1$, again by the local conic structure of definable sets and because X_0 is closed. Finally, for $i = d$, one shows that

$$\sigma_d(X_0) = \Theta_d(X_0) = \Lambda_d^{\ell oc}(X_0)$$

(we recall that by the Cauchy-Crofton formula $(\mathscr{C}\mathscr{C})$ and by definition $(2')$ and (3) of Λ_d and of $\Lambda_d^{\ell oc}$, we have $\Theta_d(X_0) = \Lambda_d^{\ell oc}(X_0)$, as already observed for Corollary 1.4). Since the relation

$$\sigma_d(X_0) = \Theta_d(X_0)$$

asserts that the localization Θ_d of the d-volume is σ_d, that is to say, by definition of σ_d, that the localization of the volume may be computed by the mean value over (generic) d-dimensional vector subspaces $P \subset \mathbf{R}^n$ of the number of points in the fibre of the projections of the germ X_0 onto the germ P_0, this relation appears as the local version of the global Cauchy-Crofton formula $(\mathscr{C}\mathscr{C})$. We state it as follows.

---

[2] Let us for instance denote X the blowing-up of $\mathbf{R}^2$ at the origin and $x \in X$ a point of the exceptional divisor of X. We then note that the projection of the germ $(X)_x$ on $\mathbf{R}^2$, along the exceptional divisor of X, does not define a germ of $\mathbf{R}^2$. Indeed, the projection of $X \cap U$, for U a neighbourhood of x in X, defines a germ at the origin of $\mathbf{R}^2$ that depends of U.

2.2.3. **Local Cauchy-Crofton formula** ([18], [19] 1.16, [21] 3.1)**.** *Let X be a definable subset of $\mathbf{R}^n$ of dimension d (containing the origin), let $\mathscr{G}$ be a definable subset of $G(d,n)$ on which transitively acts a subgroup G of $O_n(\mathbf{R})$ and let m be a G-invariant measure on $\mathscr{G}$, such that*

- *the tangent spaces to the tangent cone of X_0 are in $\mathscr{G}$,*

- *There exists $P^0 \in \mathscr{G}$ such that $\{g \in G; g \cdot P^0 = P^0\}$, the isotropy group of P^0, transitively acts on the d-dimensional vector subspace P^0 and $m(\mathscr{G}) = m(\mathscr{G} \cap \mathscr{E}_X) = 1$, where $\mathscr{E}_X$ is the generic set of $G(d,n)$ for which the localization* (5) *is possible.*

Then, we have

$$\sigma_d^{\mathscr{G}}(X_0) = \Theta_d(X_0), \qquad (\mathscr{C}\mathscr{C}^{\ell oc})$$

where $\sigma_d^{\mathscr{G}}$ is defined as in Definition 2.2.1, but relatively to $\mathscr{G}$ and m.

In the case $\mathscr{G} = G(d,n)$ and $G = O_n(\mathbf{R})$, the formula $(\mathscr{C}\mathscr{C}^{\ell oc})$ is just

$$\sigma_d(X_0) = \Theta_d(X_0) = \Lambda_d^{\ell oc}(X_0).$$

In the case X is a complex analytic subset of $\mathbf{C}^n$, $\mathscr{G} = \widetilde{G}(d/2,n)$ (the $d/2$-dimensional complex vector subspaces of $\mathbf{C}^n$) and $G = U_n(\mathbf{C})$, since by definition the number of points in the fibre of a projection of the germ X_0 onto a generic $d/2$-dimensional complex vector subspace of $\mathbf{C}^n$ is the local multiplicity $e(X,0)$ of X_0, formula $(\mathscr{C}\mathscr{C}^{\ell oc})$ gives

$$e(X,0) = \sigma_d^{\tilde{G}(d/2,n)}(X_0) = \Theta_d(X_0).$$

Obtaining the equality $e(X,0) = \Theta_d(X_0)$ as a by-product of the formula $(\mathscr{C}\mathscr{C}^{\ell oc})$ provides a new proof of Draper's result (see [32]).

2.2.4. *Remark.* When $(X^j)_{j\in\{0,\cdots,k\}}$ is a Whitney stratification of the closed set X (see for instance [105] and [106] for a survey on regularity conditions for stratifications) and $0 \in X^0$, $\sigma_i(X_0) = 1$, for $i \le \dim(X^0)$ (see [21], Remark 2.9). Therefore, to sum up, when $(X^j)_{j\in\{0,\cdots,k\}}$ is a Whitney stratification of the closed definable set X and d_0 is the dimension of the stratum containing 0, one has

$$\sigma_*(X_0) = (1,\cdots,1,\sigma_{d_0+1}(X_0),\cdots,\sigma_{d-1}(X_0),\Lambda_d^{\ell oc}(X_0)(X_0) = \Theta_d(X_0),0,\cdots,0).$$

On figure 5 we represent the data taken into account in the computation of the invariant $\sigma_i(X_0)$. Here, contrary to the computation of the $\Lambda_i^{\ell oc}(X_0)$ where all the domains $K_\ell^{P,\varrho}$ matter (see figure 4), only the domains $K_\ell^{P,\varrho}$ having the origin in their adherence (these domains are coloured in red in figure 5) are considered, since only these domains appear as

$$\chi(\pi_P^{-1}(y) \cap Z \cap \bar{B}_{(0,\varrho)}) \cdot \Theta_i((K_\ell^{P,\varrho})_0)$$

in the computation of $\theta(\pi_{P_0 *}(\mathbf{1}_{X_0}))$.

In particular the domains $K_\ell^{P,\varrho}$ (in green on figure 5) defined by the critical values of the projection π_P restricted to the link $X \cap S_{(0,\varrho)}$ are not considered in the definition of $\theta(\pi_{P_0 *}(\mathbf{1}_{X_0}))$. One can indeed prove (see [21], Proposition 2.5) that for any generic projection π_P exists $r_P > 0$, such that for all ϱ, $0 < \varrho < r_P$, the discriminant of the restriction of π_P to the link $X \cap S_{(0,\varrho)}$ is at a positive distance from $0 = \pi_P(0)$.

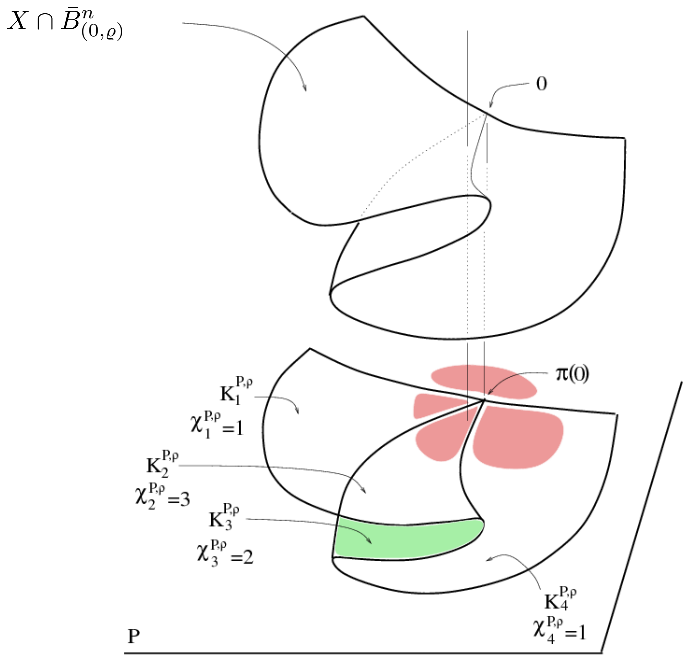

fig.5

Let us now deal with the question of what kind of invariants of complex singularities the sequence σ_* of invariants of real singularities generalizes. For this goal, we consider that X is a complex analytic subset of $\mathbf{C}^n$ of complex dimension d. One may define, like in the real case, the polar invariants of the germ X_0, denoted $\tilde{\sigma}_i$, $i = 0, \cdots, n$. These invariants are defined by generic projections on i-dimensional complex vector subspaces of $\mathbf{C}^n$. In the complex case, assuming $0 \in X$, there exists $r > 0$, such that for y generic in a generic i-dimensional vector space P of $\mathbf{C}^n$ and y sufficiently closed to 0

$$\tilde{\sigma}_i(X_0) = \chi(\pi_P^{-1}(y) \cap X \cap \bar{B}_{(0,r)}).$$

In particular, as already observed, $\tilde{\sigma}_d(X_0)$ is $e(X, 0)$, the local multiplicity of X at 0.

In the case where X is a complex hypersurface $f^{-1}(0)$, given by an analytic function $f : (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ having at 0 an isolated singularity, one has for a generic y in the germ at 0 of a generic i-dimensional vector space P_0 of $\mathbf{C}^n$

$$\chi(\pi_P^{-1}(y) \cap X \cap \bar{B}_{(0,\eta)}) = \chi(\pi_P^{-1}(0) \cap f^{-1}(\varepsilon) \cap \bar{B}_{(0,\eta)}),$$

where ε is generic in $\mathbf{C}$, sufficiently close to 0 and $0 < |\varepsilon| \ll \eta \ll 1$.

Therefore, for $0 < |\varepsilon| \ll \eta \ll 1$, the integer $\chi(\pi_P^{-1}(0) \cap f^{-1}(\varepsilon) \cap \bar{B}_{(0,\eta)})$ is the Euler-Poincaré characteristic of the Milnor fibre of f restricted to $P^\perp$, that is to say $1 + (-1)^{n-i-1}\mu^{(n-i)}$. In the case where X is a complex analytic hypersurface of $\mathbf{C}^n$ with an isolated singularity at 0, we thus have

$$\widetilde{\sigma}_i(X_0) = 1 + (-1)^{n-i-1}\mu^{(n-i)}.$$

For X a complex analytic subset of $\mathbf{C}^n$ of dimension d, d being not necessarily $n-1$, the complex invariants $\widetilde{\sigma}_i(X_0)$ have been first considered by Kashiwara in [65] (where the balls are open and not closed as it is the case here). An invariant $E^0_{X_0}$ is then defined in [65] by induction on the dimension of X_0 using $\tilde{\sigma}_i$. This invariant is studied in [33], [34] and in [12] where a multidimensional version $E^k_{X_0}$ of $E^0_{X_0}$ is given (see also [85]). The definition is the following

$$E^k_{X_0} = \sum_{X^{j_0} \subset \bar{X}^j \setminus X^j,\ \dim(X^j) < \dim(X_0)} E^k_{\bar{X}^j} \cdot \tilde{\sigma}_{k+\dim(X^j)+1}(X_0),$$

where (X^j) is a Whitney stratification of X_0, X^{j_0} the stratum containing 0 and $E^k_{\{0\}} = 1$. The authors then remark that (see also [33], [34])

$$E^k_{X_0} = Eu^k_{X_0},$$

where $Eu^k_{X_0} = Eu_{(X_0 \cap H)}$, H is a general vector subspace of dimension k of $\mathbf{C}^n$ and where Eu is the local Euler obstruction of X at 0, introduced by MacPherson in [84]. In particular,

$$E^0_{X_0} = Eu_{X_0}.$$

Let us now denote $\mathscr{P}^i(X_0)$, $i = 0, \cdots, d$, the codimension i polar variety of X_0, that is to say the closure of the critical locus of the projection of the regular part of X_0 to a generic vector space of $\mathbf{C}^n$ of dimension $d - i + 1$. The following relation between invariants $E^k_{X_0}$ and the local multiplicity of the polar varieties $\mathscr{P}^i(X_0)$ is obtained in [12]

$$(-1)^i (E^{\dim(X_0)-i-1}_{X_0} - E^{\dim(X_0)-i}_{X_0}) = e(\mathscr{P}^i(X_0), 0),$$

which in turn gives (see also [85], [75], [76], [77], [34])

$$Eu_{X_0} = \sum_{i=0}^{d-1} (-1)^i e(\mathscr{P}^i(X_0), 0),$$

where $e(\mathscr{P}^i(X_0), 0)$ is, as before, the local multiplicity at 0 of the codimension i polar variety $\mathscr{P}^i(X_0)$ of X_0 at 0.

All the invariants $\tilde{\sigma}_i(X_y)$, $E^i_{X_y}$, $e(\mathscr{P}^i(X_y), y)$, viewed as functions of the base-point y, enjoy the same remarkable property: they can detect subtle variations of the geometry of an analytic family (X_y), in the sense that the family (X_y) may be Whitney stratified with y staying in the same stratum if and only if these invariants are constant with respect to the parameter y. Without proof, it is actually stated in [33] Proposition 1, [34] Theorem II.2.7 page 30, and [12], that the invariants $\tilde{\sigma}_i(X_y)$ are constant as y varies in a stratum of a Whitney stratification of X_0 (see also [21] Corollary 4.5). And in [58], [87], [104] it is proved that the constancy of the multiplicities $e(\mathscr{P}^i(X_y), y)$ as y varies in a stratum of a stratification of X_0, is equivalent to the Whitney regularity of this stratification, giving also a proof, considering the relations between $e(\mathscr{P}^i(X_y), y)$ and $\tilde{\sigma}_i(X_y)$ stated above, of the constancy of $y \mapsto \tilde{\sigma}_i(X_y)$ along Whitney strata.

We sum-up these results in the following theorem, where $e(\Delta^i(X_y), y)$ is the local multiplicity at y of the discriminant $\Delta^i(X_y)$ associated to $\mathscr{P}^i(X_y)$, that is the image of $\mathscr{P}^i(X_y)$ under the generic projection that gives rise to $\mathscr{P}^i(X_y)$.

2.2.5. **Theorem** ([58], [87], [77], [104])**.** *Let X_0 be a complex analytic germ at 0 of $\mathbf{C}^n$ endowed with a stratification (X^j). The following statements are equivalent*

1. *The stratification (X^j) is a Whitney stratification.*
2. *The functions $X^j \ni y \mapsto e(\mathscr{P}^i(X^\ell_y), y)$, for $i = 0, \cdots, d-1$ and any pairs (X^j, X^ℓ) such that $X^j \subset \overline{X^\ell}$, are constant.*
3. *The functions $X^j \ni y \mapsto e(\Delta^i(X^\ell_y), y)$, for $i = 0, \cdots, d-1$ and for any pairs (X^j, X^ℓ) such that $X^j \subset \overline{X^\ell}$, are constant.*
4. *The functions $y \ni X^j \mapsto \tilde{\sigma}_i(X_y)$, for $i = 1, \cdots, d$ are constant.*

2.2.6. *Remark.* In the real case the functions $y \mapsto \sigma_i(X_y)$ are not $\mathbf{Z}$-valued functions as in the complex case, but $\mathbf{R}$-valued functions and in general one can not stratify a compact definable set in such a way that the restriction of these functions to the strata are constant. However, it is proved in [21] Theorems 4.9 and 4.10, that Verdier regularity for a stratification implies continuity of the restriction of $y \mapsto \sigma_i(X_y)$ to the strata of this stratification. Since, in the complex setting, Verdier regularity is the same as Whitney regularity, this result is the real counterpart of Theorem 2.2.5. Note that in the real case one can not expect that the continuity or even the constancy of the functions $y \mapsto \sigma_i(X_y)$ in restriction to the strata of a given stratification implies a convenient regularity condition for this stratification (see the introduction of [21]).

As a conclusion of this section, the complex version $\tilde{\sigma}_i$ of the real polar invariants σ_i of definable singularities plays a central role in singularity theory since they let us compute classical invariants of singularities and since their constancy, with respect to the parameter of an analytic family, as well as the constancy of other classical invariants related to them, means that the family does not change its geometry. Since our polar invariants σ_i appear now as the real counterpart of classical complex invariants, we'd like to understand in the sequel how they are related to the local Lipschitz-Killing invariants $\Lambda_i^{\ell oc}(X_0)$ coming from the differential geometry of the deformation of the germ X_0 through its tubular neighbourhoods family.

This is the goal of the next section.

2.3. **Multidimensional local Cauchy-Crofton formula.** The local Cauchy-Crofton formula $(\mathscr{C}\mathscr{C}^{\ell oc})$ given at 2.2.3 already equals σ_d and $\Lambda_d^{\ell oc}$ over definable germs. This relation suggests a more general relation between the $\Lambda_i^{\ell oc}$'s and the σ_j's. We actually can prove that each invariant of one family is a linear combination of the invariants of the other family. The precise statement is given by the following formula $(\mathscr{C}\mathscr{C}_{mult}^{\ell oc})$.

2.3.1. **Multidimensional local Cauchy-Crofton formula** ([21] Theorem 3.1)**.** *There exist real numbers $(m_i^j)_{1\le i,j\le n, i<j}$ such that, for any definable germ X_0, one has*

$$\begin{pmatrix} \Lambda_1^{\ell oc}(X_0) \\ \vdots \\ \Lambda_n^{\ell oc}(X_0) \end{pmatrix} = \begin{pmatrix} 1 & m_1^2 & \dots & m_1^{n-1} & m_1^n \\ 0 & 1 & \dots & m_2^{n-1} & m_2^n \\ \vdots & & & & \vdots \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \sigma_1(X_0) \\ \vdots \\ \sigma_n(X_0) \end{pmatrix} \qquad (\mathscr{C}\mathscr{C}_{mult}^{\ell oc})$$

These constant real numbers are given by

$$m_i^j = \frac{\alpha_j}{\alpha_{j-i}\cdot\alpha_i}\binom{i}{j} - \frac{\alpha_{j-1}}{\alpha_{j-1-i}\cdot\alpha_i}\binom{i}{j-1},$$

for $i+1 \le j \le n$.

2.3.2. *Remark.* Applied to a d-dimensional definable germ X_0, the last *a priori* non-trivial equality provided by formula $(\mathscr{C}\mathscr{C}_{mult}^{\ell oc})$, involving the d-th line of the matrices, is

$$\Theta_d(X_0) = \Lambda_d^{\ell oc}(X_0) = \sigma_d(X_0),$$

which is the local Cauchy-Crofton formula $(\mathscr{C}\mathscr{C}^{\ell oc})$. The local Cauchy-Crofton formula $(\mathscr{C}\mathscr{C}^{\ell oc})$ expresses the d-density of a d-dimensional germ as the mean value of the number of points in the intersection of this germ with a $(n-d)$-dimensional affine space of $\mathbf{R}^n$. This number of points may be viewed as the Euler-Poincaré characteristic of this intersection. Now, since for d-dimensional germs the d-density is the last invariant of the sequence $\Lambda_*^{\ell oc}$ and since formula

($\mathscr{C}\mathscr{C}^{\ell oc}_{mult}$) expresses all invariants $\Lambda_i^{\ell oc}$ in terms of the mean values of the Euler-Poincaré characteristics of the multidimensional plane sections of our germ, we see in formula ($\mathscr{C}\mathscr{C}^{\ell oc}_{mult}$) a multimensional version of the local Cauchy-Crofton formula.

2.4. **Valuations theory and Hadwiger principle.** In the previous section 2.3, with formula ($\mathscr{C}\mathscr{C}^{\ell oc}_{mult}$), we have answered the question: *how are the local Lipschitz-Killing invariants and the polar invariants related?* In this section we would like to risk some speculative and maybe prospective insights about formulas similar to ($\mathscr{C}\mathscr{C}^{\ell oc}_{mult}$), that is to say formulas linearly relating two families of local invariants of singularities. We include in this scope the complex formulas presented in Section 2.1. For this goal we first recall some definitions and celebrated statements from convex geometry, since it appears that from the theory of valuations on convex bodies one can draw precious lessons on the question: *why our additive invariants are linearly dependent?*

We have already observed that the additive functions Λ_i, $\Lambda_i^{\ell oc}$ and σ_j are invariant under isometries of $\mathbf{R}^n$. Therefore the first question we would like to address to convex geometry is the following: to what extend those invariants are models of additive and rigid motion invariants?

The systematic study of additive invariants (of compact convex sets of $\mathbf{R}^n$) has been inaugurated by Hadwiger and his school and motivated by Hilbert's third problem (solved by Dehn by introducing the so-called Dehn invariants) consisting in classifying scissors invariants of polytopes (see for instance [13] for a quick introduction to Hilbert's third problem). One of the most striking results in this field is Hadwiger's theorem that characterises the set of additive and rigid motion invariant functions (on the set of compact convex subsets of $\mathbf{R}^n$) as the vector space spanned by the Λ_i's. We give now the needful definitions to state Hadwiger's theorem and the still-open question of its extension to the spherical case (for more details one can refer to [83], [97] or [98]).

We denote by $\mathscr{K}^n$ (resp. $\mathscr{K}\mathbf{S}^{n-1}$) the set of compact convex sets of $\mathbf{R}^n$ (resp. of $\mathbf{S}^{n-1}$, that is to say the intersection of the sphere $\mathbf{S}^{n-1}$ and conic compact convex sets of $\mathscr{K}^n$ with vertex the origin of $\mathbf{R}^n$). A function $v : \mathscr{K}^n \to \mathbf{R}$ (resp. $v : \mathscr{K}\mathbf{S}^{n-1} \to \mathbf{R}$) is called a valuation (resp. a spherical valuation) when $v(\emptyset) = 0$ and for any $K, L \in \mathscr{K}^n$ (resp. $K, L \in \mathscr{K}\mathbf{S}^{n-1}$) such that $K \cup L \in \mathscr{K}^n$ (resp. $K \cup L \in \mathscr{K}\mathbf{S}^{n-1}$), one has the additivity property

$$v(K \cup L) = v(K) + v(L) - v(K \cap L).$$

One says that a valuation v on $\mathscr{K}^n$ (resp. $\mathscr{K}\mathbf{S}^{n-1}$) is continuous when it is continuous with respect to the Hausdorff metric on $\mathscr{K}^n$ (resp. on $\mathscr{K}\mathbf{S}^{n-1}$). A valuation v on $\mathscr{K}^n$ (resp. $\mathscr{K}\mathbf{S}^{n-1}$) is called simple when the restriction of v to convex sets with empty interior is zero. Let G be a subgroup of the orthogonal group $O_n(\mathbf{R})$. A valuation v on $\mathscr{K}^n$ (resp. $\mathscr{K}\mathbf{S}^{n-1}$) is *$G$-invariante* when it is invariant under the action of translations of $\mathbf{R}^n$ and the action of G on $\mathscr{K}^n$ (resp. the action of G on $\mathscr{K}\mathbf{S}^{n-1}$).

The Hadwiger theorem emphasizes the central role played in convex geometry by the Lipschitz-Killing invariants as additive rigid motion invariants.

2.4.1. **Theorem** ([55], [66]). *A basis of the vector space of $SO_n(\mathbf{R})$-invariant and continuous valuations on $\mathscr{K}^n$ is $(\Lambda_0 = 1, \Lambda_1, \cdots, \Lambda_n = Vol_n)$.*

Equivalently (by an easy induction argument) a basis of the vector space of continuous and $SO_n(\mathbf{R})$-invariant simple valuations on $\mathscr{K}^n$ is $\Lambda_n = Vol_n$.

This statement forces a family of $n+2$ additive, continuous and $SO_n(\mathbf{R})$-invariant functions on euclidean convex bodies to be linearly dependent. The second formulation of Hadwiger's theorem, concerning the space of simple valuations, enables to address the question of such a rigid structure of the space of valuations in the setting of spherical convex geometry. This question of a spherical version of Hadwiger's result has been address by Gruber and Schneider.

2.4.2. *Question* ([53] Problem 74, [83] Problem 14.3). Is a simple, continuous and $O_n(\mathbf{R})$-invariant valuation on $\mathscr{K}\mathbf{S}^{n-1}$ a multiple of the $(n-1)$-volume on S^{n-1}?

2.4.3. *Remark.* In the case $n \leq 3$ a positive answer to this question given in [83], Theorem 14.4, and in the easy case where the simple valuation has constant sign one also has a positive answer given in [95] Theorem 6.2, and [96]. Note that in this last case the continuity is not required and that the valuation is *a priori* defined only on convex spherical polytopes.

This difficult and still unsolved problem naturally appears as soon as one consider the localizations $\Lambda_i^{\ell oc}$ of Λ_i and their relation with other classical additive invariants such as the σ_j's. Indeed, the question of why and how such invariants are related falls within the framework of Question 2.4.2. Let's clarify this principled position.

The invariants $(\Lambda_i^{\ell oc})_{i\in\{0,\cdots,n\}}$ define spherical $O_n(\mathbf{R})$-invariant and continuous valuations $(\widehat{\Lambda}_i)_{i\in\{0,\cdots,n\}}$ on the convex sets of $\mathbf{S}^{n-1}$ by the formula

$$\widehat{\Lambda}_i(K) := \Lambda_i^{\ell oc}(\widehat{K}_0) = \frac{1}{\alpha_i}\Lambda_i(\widehat{K} \cap \bar{B}_{(0,1)}), \tag{$\widehat{8}$}$$

where K is a convex set of $\mathbf{S}^{n-1}$, that is to say the trace in $\mathbf{S}^{n-1}$ of the cone $\widehat{K} = \mathbf{R}_+ \cdot K$ with vertex the origin of $\mathbf{R}^n$. Another possible finite sequence of continuous and $O_n(\mathbf{R})$-invariant spherical valuations on convex polytopes of $\mathbf{S}^{n-1}$ is

$$\Xi_i(P) := \sum_{F\in\mathscr{F}_i(P)} Vol_i(F)\cdot\gamma(\widehat{F},\widehat{P}) = Vol_i(S^i(0,1)) \sum_{F\in\mathscr{F}_i(P)} \Theta_i(\widehat{F}_0)\cdot\gamma(\widehat{F},\widehat{P}), \tag{$\widehat{9}$}$$

where $P \subset S^{n-1}$ is a spherical polytope, that is to say that $\mathbf{R}_+ \cdot P = \widehat{P}$ is the intersection of a finite number of closed half vector spaces of $\mathbf{R}^n$, $\mathscr{F}_i(P)$ the set of all i-dimensional faces of P (the $(i+1)$-dimensional faces of $\widehat{P}$) and $\gamma(\widehat{F},\widehat{P})$ the external angle of $\widehat{P}$ along $\widehat{F}$. The valuations Ξ_i are the natural spherical substitutes of the euclidean Lipschitz-Killing curvatures Λ_i according to formula (2).

Finally the polar invariants $(\sigma_i)_{i\in\{0,\cdots,n\}}$ also define continuous and $O_n(\mathbf{R})$-invariants spherical valuations $(\widehat{\sigma}_i)_{i\in\{0,\cdots,n\}}$ on the convex sets of $\mathbf{S}^{n-1}$, according to a formula of the same type that formula $(\widehat{8})$

$$\widehat{\sigma}_i(K) := \sigma_i(\widehat{K}_0). \tag{$\widehat{10}$}$$

These three families of continuous and $O_n(\mathbf{R})$-invariant spherical valuations

$$(\widehat{\Lambda}_i)_{i\in\{0,\cdots,n\}}, (\Xi_i)_{i\in\{0,\cdots,n\}}, (\widehat{\sigma}_i)_{i\in\{0,\cdots,n\}}$$

being linearly independent families in the space of spherical valuations, a positive answer to Question 2.4.2 would have for direct consequence that each element of one family is a linear combination of elements of any of the other two families.

Therefore, in restriction to polyhedral cones, each element of the family $(\Lambda_i^{\ell oc})_{i\in\{0,\cdots,n\}}$ could be expressed as a linear combination (with universal coefficients) of elements of the family $(\sigma_i)_{i\in\{0,\cdots,n\}}$ and conversely. Despite the absence of any positive answer to Question 2.4.2 for $n > 3$, this linear dependence is proved in [21] (Theorem A4 and A5) over the set of convex polytopes (and in [21], section 3.1, even over the set of definable cones). It is actually shown that each invariant $\Lambda_i^{\ell oc}$ and each invariant σ_j may be expressed as a linear combination (with universal coefficients) of elements of the family $(\Xi_i)_{i\in\{0,\cdots,n\}}$. The coefficients involved in such linear combinations may be explicitly computed by considering the case of polytopes.

In conclusion, an anticipating positive answer to this question would imply the existence of a finite number of independent models for additive, continuous and $O_n(\mathbf{R})$-invariant functions on convex and conic germs. Consequently, solely following this principle and restricted at least to convex cones, our local invariants σ_j and $\Lambda_i^{\ell oc}$ would be automatically linearly dependent. In the next step, in order to extend a relation formula involving some valuations from the set of finite union of convex conic polytopes to the set of general conic definable set one has to prove some general statement according to which the normal cycle of a definable conic set may be approximate by the normal cycles of a family of finite union of convex conic polytopes. We do not want to go more into technical details and even define the notion of normal cycle introduced by Fu; we just point out that this issue has been tackled recently in [44]. Finally to extend a relation formula from the set of conic definable set to the set of all definable germs one just has to use the local conic structure of definable germs and the deformation on the tangent cone (see [21], [38]).

To finish to shed light on convex geometry as an area from which some strong relations between singularity invariants may be understood, let us remark that the following generalization of Hadwiger's theorem 2.4.1 has been obtained by Alesker.

2.4.4. **Theorem.** *Let G be a compact subgroup of $O_n(\mathbf{R})$.*

(1) *The vector space $Val^G(\mathscr{K}^n)$ of continuous, translation and G-invariant valuations on $\mathscr{K}^n$ has finite dimension if and only if G acts transitively on $\mathbf{S}^{n-1}$ (see [2] Theorem 8.1, [4] Proposition 2.6).*
(2) *One can endowed the vector space $Val^G(\mathscr{K}^n)$ with a product (see [3], [5]) providing a graded algebra structure (the graduation coming from the homogeneity degree of the valuations) and*

$$\begin{array}{rcl} \mathbf{R}[x]/(x^{n+1}) & \to & Val^{O_n(\mathbf{R})}(\mathscr{K}^n) = Val^{SO_n(\mathbf{R})}(\mathscr{K}^n) \\ x & \mapsto & \Lambda_1 \end{array}$$

is an isomorphism of graded algebras (see [3], Theorem 2.6).

3. Generating additive invariants via generating functions

The possibility of generating invariants from a deformation of a singular set into a family of approximating and less complicated sets is perfectly illustrated by the work developed by Denef and Loeser consisting in stating that some generating series attached to a singular germ are rational. Such generating series have their coefficients in some convenient ring reflecting the special properties of the invariants to highlight, such as additivity, multiplicativity, analytic invariance, the relation with some specific group action and so forth, and on the other hand each of these coefficients is attached to a single element of the deformation family. It follows that such a generating series captures the geometrical aspect that one aims to focus on through the deformation family as well as its rationality indicates that asymptotically this geometry specializes on the geometry of the special fibre approximated by the deformation family. Indeed, being rational strongly expresses that a series is encoded by a finite amount of data concentrated in its higher coefficients.

To be more explicit we now roughly describe how Denef and Loeser define the notion of motivic Milnor fibre (for far more complete and precise introductions to motivic integration which is the central tool of the theory, and to motivic invariants in general, the reader may refer to [10], [16], [17], [25], [28], [29], [30], [31], [52], [54], [56], [57], [80], [81], [107]).

3.1. **The complex case.** A possible starting point of the theory of motivic invariants may be attributed to the works of Igusa (see [62], [63], [64]) on zeta functions introduced by Weil (see [109]). In the works of Igusa the rationality of some generating Poincaré series is proved. These series, called zeta functions, have for coefficients the number of points in $\mathscr{O}/\mathfrak{M}^m$ of $f=0$ mod $\mathfrak{M}^m$, for f a n-ary polynomial with coefficients in the valuation ring $\mathscr{O}$ of some discrete valuation field of characteristic zero, with maximal ideal $\mathfrak{M}$ and finite residue field of cardinal q. This result amounts to prove the rationality (as a function of q^{-s}, $s \in \mathbf{C}$, $\Re e(s) > 0$) of an integral of type

$$\int_{\mathscr{O}^n} |f|^s |ds| \tag{11}$$

(the Igusa local zeta function) which is achieve using a convenient resolution of singularities of f, as described in the introduction (see also [24] for comparable statements on Serre's series and a strategy based on Macintyre's proof of elimination of quantifiers as an alternative to Hironaka's resolution of singularities). In the opposite direction, the rational function, expressed in terms of the data of a resolution of f, associated to a polynomial germ $f : (\mathbf{C}, 0) \to (\mathbf{C}, 0)$ by the expression provided by the computation of Igusa's integral in the discrete valuation field case let Denef and Loeser define intrinsic invariants attached to the complex germ $(f^{-1}(0), 0)$, called topological zeta functions (see [26]).

Another key milestone in the systematic use of discrete valuation fields (here with finite residue field and more explicitely in the p-adic context) have been reached in Batyrev's paper [7], where it is shown that two birationally equivalent Calabi-Yau manifolds over $\mathbf{C}$ have the same Betti numbers. Indeed, by Weil's conjectures, these Betti numbers are obtained from the rational expression of the local zeta functions having for coefficients the number of points in the reductions modulo p^m of the manifolds into consideration (viewed as defined over $\mathbf{Q}_p$ when they are defined over $\mathbf{Q} \subset \mathbf{C}$) and on the other hand, these local zeta functions may be computed by Igusa's integrals over these manifolds. Being birationally equivalent, these manifolds provide the same integrals.

Kontsevich, in his seminal talk [67], extended this method (consisting in shifting a complex geometric problem in a discrete valuation field setting) in the equicharacteristic setting by developing an integration theory in particular over $\mathbf{C}[[t]]$. Note that the theory may be developed in great generality and not only in equicharacteristic zero (see [81], [16]). The idea of Kontsevich was to define an integration theory over arc spaces, say $\mathbf{C}[[t]]$, by considering a measure with values in the Grothendieck ring $K_0(\mathrm{Var}_{\mathbf{C}})$ of algebraic varieties over $\mathbf{C}$ (localized by the multiplicative set generated by the class $\mathbb{L}$ of $\mathbf{A}^1$ in $K_0(\mathrm{Var}_{\mathbf{C}})$). The main tool in this context being a change of variables formula that allows computation of integrals through morphisms, and in particular through a morphism given by a resolution of singularities. Formally the ring $K_0(\mathrm{Var}_{\mathbf{C}})$ is the free abelian group generated by isomorphism classes $[X]$ of varieties X over $\mathbf{C}$, with the relations

$$[X \setminus Y] = [X] - [Y],$$

for Y closed in X, the product of ring being given by the product of varieties (see for instance [88]). Denoting $\mathbb{L}$ the class of $\mathbf{A}^1$ in $K_0(\mathrm{Var}_{\mathbf{C}})$, we then denote $\mathscr{M}_{\mathbf{C}}$ the localization $K_0(\mathrm{Var}_{\mathbf{C}})[\mathbb{L}^{-1}]$. Any additive and multiplicative invariant on $\mathrm{Var}_{\mathbf{C}}$ with non zero value at $\mathbf{A}^1$, such as the Euler-Poincaré characteristic or the Hodge characteristic (both with compact support), factorizes through the universal additive and multiplicative map $\mathrm{Var}_{\mathbf{C}} \ni X \mapsto [X] \in \mathscr{M}_{\mathbf{C}}$. Now we equip the space $\mathscr{L}(\mathbf{C}^n, 0)$ of formal arcs of $\mathbf{C}^n$ passing through 0 at 0 with the abovementioned measure that provides a σ-additive measure, with values in a completion $\hat{\mathscr{M}}_{\mathbf{C}}$ of $\mathscr{M}_{\mathbf{C}}$, for sets of the boolean algebra of the so-called constructible sets of $\mathscr{L}(\mathbf{C}^n, 0)$. Finally denoting $\mathscr{L}_m(\mathbf{C}^n, 0)$, $m \geq 0$, the set of polynomial arcs of $\mathbf{C}^n$ of degree $\leq m$, passing through 0 at 0, and for $f : (\mathbf{A}^n, 0) \to (\mathbf{A}^1, 0)$ a morphism having a (isolated) singularity at the origin, inducing the morphism $f_m : \mathscr{L}_m(\mathbf{C}^n, 0) \to \mathscr{L}(\mathbf{C}, 0)$, we denote

$$\mathscr{X}_{m,0,1} := \{\varphi \in \mathscr{L}_m(\mathbf{C}^n, 0); (f_m \circ \varphi)(t) = t^m + \text{high order terms}\},$$

and we define (see for instance [31]) the motivic zeta function of f with formal variable T by

$$Z_f(T) := \sum_{m\geq 1} [\mathscr{X}_{m,0,1}]\ \mathbb{L}^{-mn} T^m.$$

This generating series appears as the $\mathbf{C}[[t]]$-substitute of p-adic zeta functions introduced by Weil, and whose rationality, following Igusa, amounts to compute an integral of type (11). Inspired by the analogy of $Z_f(T)$ with Igusa integrals, and also using a resolution of singularities of f as presented in the introduction and using the Kontsevich change of variables formula for this resolution applied to the coefficients of Z_f viewed as measures (in the ring $\hat{\mathscr{M}}_{\mathbf{C}}$) of the constructible sets $\mathscr{X}_{m,0,1}$, Denef and Loeser proved (see [27], [28], [31]) the rationality of $Z_f(T)$. With the notation given in the introduction, we then have

$$Z_f(T) = \sum_{I\cap\mathscr{K}\neq\emptyset} (\mathbb{L}-1)^{|I|-1}[\widetilde{E}_I^0]\prod_{i\in I}\frac{\mathbb{L}^{-\nu_i}T^{N_i}}{1-\mathbb{L}^{-\nu_i}T^{N_i}} \tag{12}$$

where $\widetilde{E}_I^0$ is a covering of E_I^0 defined in the following way. Let U be some affine open subset of M such that on U, $f\circ\sigma(x) = u(x)\prod_{i\in I} x_i^{N_i}$, with u a unit. Then $\widetilde{E}_I^0$ is obtained by gluing along $E_I^0\cap U$ the sets

$$\{(x,z)\in(E_I^0\cap U)\times\mathbf{A}^1; z^{m_I}\cdot u(x) = 1\},$$

where $m_I = \gcd(N_i)_{i\in I}$.

3.1.1. *Question* (Monodromy conjecture). We do not know how the poles (in some sense) of the rational expression of Z_f relates on the eigenvalues of the monodromy function M associated to the singular germ $f : (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$. The Monodromy Conjecture of Igusa, that has been stated in many different forms after Igusa, asserts that when $\mathbb{L}^\nu - T^N$ indeed appears as denominator of the rational expression of Z_f viewed as an element of the ring generated by $\hat{\mathscr{M}}_{\mathbf{C}}$ and $T^N/(\mathbb{L}^\nu - T^N)$, $\nu, N > 0$, then $e^{2i\pi\nu/N}$ is an eigenvalue of M (see for instance [25] and the references given in this article for additional classical references).

3.1.2. *Remark.* The rationality of Z_f illustrates again a deformation principle; the family $(\mathscr{X}_{m,0,1})_{m\geq 1}$ may be considered as a family of tubular neighbourhoods in $\mathscr{L}(\mathbf{C}^n, 0)$ around the singular fibre $\mathscr{X}_0 = \{f = 0\}$ and in a neighbourhood of the origin, with respect to the ultrametric distance given by the order of arcs. Now the rationality of Z_f expresses the regularity of the degeneracy of the geometry of $\mathscr{X}_{m,0,1}$ onto the geometry of $\mathscr{X}_0$. Following this principle, the rational expression (12) of Z_f is supposed to concentrate the part of the geometrical information encoded in $(\mathscr{X}_{m,0,1})_{m\geq 1}$ that accumulates at infinity in the series Z_f.

This is achieved in particular by the following observation (see [27], [30], [31]): the negative of the constant term of the formal expansion as a power series in $1/T$ of the rational expression of Z_f given by formula (12) defines the following element in $\hat{\mathscr{M}}_{\mathbf{C}}$

$$S_f := \sum_{I\cap\mathscr{K}\neq\emptyset} (\mathbb{L}-1)^{|I|-1}\ [\widetilde{E}_I^0],$$

called the motivic Milnor fibre of $f = 0$ at the singular point 0 of f. Taking the realization of S_f under the morphism $\chi : \hat{\mathscr{M}}_{\mathbf{C}} \to \mathbf{Z}$ (note that $\chi(\mathbb{L}) = 1$) gives, in particular, by the A'Campo formula recalled in the introduction,

$$\chi(S_f) = \sum_{i\in\mathscr{K}} N_i\cdot\chi(E_{\{i\}}^0) = \chi(X_0) = 1 + (-1)^{n-1}\mu. \tag{13}$$

3.1.3. *Remark.* Generally speaking, taking the constant term in the expansion of a rational function Z as a power series in $1/T$, amounts to consider $\lim_{T\to\infty} Z(T)$ (in a setting where this makes sense). A process that gives an increasing importance to the m-th coefficient of Z as m itself increases. On the other hand, the coefficients of Z_f may be directly interpreted at the level of the Euler-Poincaré characteristic, which could be seen as the first topological degree of realization of $K_0(\mathrm{Var}_{\mathbf{C}})$, and it turns out that the sequence $(\chi(\mathscr{X}_{m,0,1}))_{m\geq 1}$ has a strong regularity since it is in fact periodic. Indeed, one has by [31] Theorem 1.1 (we recall that M is the monodromy map and Λ the Lefschetz number)

$$\chi(\mathscr{X}_{m,0,1}) = \Lambda(M^m),\ \forall m \geq 1 \tag{14}$$

and by quasi-unipotence of M (see [99] I.1.2) there exists $N > 1$ such that the order of the eigenvalues of M divides N. It follows from (14) that $\chi(\mathscr{X}_{m+N,0,1}) = \chi(\mathscr{X}_{m,0,1}), m \geq 1$.

Now formula (13), showing that the motivic Milnor fibre has a realization, via the Euler-Poincaré characteristic, on the Euler-Poincaré characteristic of the set-theoretic Milnor fibre, is a direct consequence of formulas (14) (note in fact that the proofs of (13) and (14), using A'Campo's formulas and a resolution of singularities, are essentially the same and thus gives comparable statements). Indeed, as noticed by Loeser (personal communication), working with χ instead of formal classes of $K_0(\mathrm{Var}_{\mathbf{C}})$, on gets $\mathbb{L} = \chi(\mathbf{A}^1) = 1$ and thus by definition of Z_f, the series $\chi(Z_f)$, realization of the class Z_f under the Euler-Poincaré characteristic is

$$\chi(Z_f) = \sum_{m\geq 1} \chi(\mathscr{X}_{m,0,1}) T^m$$

that gives in turn, by formula (14),

$$\chi(Z_f) = \sum_{m\geq 1} \Lambda(M^m) T^m = \sum_{m=1}^{N} \Lambda(M^m) \sum_{k\geq 0} T^{m+kN} = \sum_{m=1}^{N} \Lambda(M^m) \frac{T^m}{1 - T^N}.$$

Since $\chi(S_f) = -\lim_{T\to\infty} \chi(Z_f)$, on finally find again that

$$\chi(S_f) = \Lambda(M^N) = \Lambda(Id) = \chi(X_0) = \chi(\bar{X}_0) = 1 + (-1)^{n-1}\mu.$$

3.1.4. *Remark.* One may consider a more specific Grothendieck ring, that is to say a ring with more relations, in order to take into account the monodromy action on the Milnor fibre. In this equivariant and more pertinent ring equalities (12) and (14) are still true (see [30], [31] Section 2.9)

3.1.5. *Remark.* In [61], Hrushovski and Loeser gave a proof of equality (14) without using a resolution of singularity, and therefore without using A'Campo's formulas. Since a computation of Z_f in terms of the data associated to a particular resolution of the singularities of f leads to the simple observation that one computes in this way an expression already provided by A'Campo's formulas, the original proof of (14) may, in some sense, appear as a not direct proof. The proof proposed in [61] uses étale cohomology of non-archimedean spaces and motivic integration in the model theoretic version of [59] and [60].

3.1.6. *Remark.* To finish with the complex case, let us note that in [93] and [100] a mixed Hodge structure on the Milnor fibre $f^{-1}(t)$ at infinity ($|t| \gg 1$) has been defined by a deformation process, letting t goes to infinity (see also [92]). In [89] and [90] a corresponding motivic Milnor fibre S_f^∞ has then be defined.

3.2. **The real case.** A real version of (12), giving rise to a real version of (13) has been obtained in [20] (see also [43]). In the real case, a singular germ

$$f : (\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$$

defines two smooth bundles $(f^{-1}(\varepsilon) \cap B_{(0,\eta)})_{0<-\varepsilon\ll\eta\ll 1}$ and $(f^{-1}(\varepsilon) \cap B_{(0,\eta)})_{0<\varepsilon\ll\eta\ll 1}$ and as well as $\mathscr{X}_{m,0,-1}$ and $\mathscr{X}_{m,0,1}$, it is natural to consider the two sets

$$\mathscr{X}_{m,0,>} := \{\varphi \in \mathscr{L}_m(\mathbf{R}^n, 0); f_m \circ \varphi = at^m + \text{high order terms}, a > 0\},$$

and

$$\mathscr{X}_{m,0,<} := \{\varphi \in \mathscr{L}_m(\mathbf{R}^n, 0); f_m \circ \varphi = at^m + \text{high order terms}, a < 0\}.$$

Let us denote X_0^{-1} and X_0^{+1} the fibre $f^{-1}(\varepsilon) \cap B_{(0,\eta)}$ for respectively $\varepsilon < 0$ and $\varepsilon > 0$. While $\mathscr{X}_{m,0,a}$, for $a \in \mathbf{C}^\times$, is a constructible set having a class in $K_0(\mathrm{Var}_{\mathbf{C}})$, the sets $\mathscr{X}_{m,0,<}$ and $\mathscr{X}_{m,0,>}$ are real semialgebraic sets and unfortunately, the Grothendieck ring of real semialgebraic sets is the trivial ring $\mathbf{Z}$, since semialgebraic sets admit semialgebraic cells decomposition.

Therefore, in the real case, since we have to deal with two signed Milnor fibres, we cannot mimic the construction of $K_0(\mathrm{Var}_{\mathbf{C}})$. To overcome this issue, in [43] we proposed to work in the Grothendieck ring of real (basic) semialgebraic formulas, $K_0(\mathrm{BS}_{\mathbf{R}})$. In this ring no semialgebraic isomorphism relations between semialgebraic sets, but algebraic isomorphim relations between sets given by algebraic formulas, are imposed and distinct real semialgebraic formulas having the same set of real points in $\mathbf{R}^n$ may have different classes. In particular, a first order basic formula in the language of ordered rings with parameters from $\mathbf{R}$ may have a nonzero class in $K_0(\mathrm{BS}_{\mathbf{R}})$ whereas no real point satisfies it. The ring $K_0(\mathrm{BS}_{\mathbf{R}})$ may be sent to the more convenient ring $K_0(\mathrm{Var}_{\mathbf{R}}) \otimes \mathbb{Z}[\frac{1}{2}]$, where explicit computations of classes of basic semialgebraic formula are possible as long as computations of classes of real algebraic formulas in the classical Grothendieck ring of real algebraic varieties $K_0(\mathrm{Var}_{\mathbf{R}})$ are possible.

In this setting, since $\mathscr{X}_{m,0,>}$ and $\mathscr{X}_{m,0,<}$ are given by explicit basic semialgebraic formulas, they do have natural classes in $K_0(\mathrm{BS}_{\mathbf{R}})$ and this allows the consideration of the associated zeta series

$$Z_f^? = \sum_{m\geq 1} [\mathscr{X}_{m,0,?}]\ \mathbb{L}^{-mn} T^m \in (K_0(\mathrm{Var}_{\mathbf{R}}) \otimes \mathbb{Z}[\frac{1}{2}])[\mathbb{L}^{-1}][[T]],\ \ ? \in \{-1, +1, <, >\}.$$

It is then proved, with the same strategy as in the complex case (using a resolution of singularities of f and the Kontsevich change of variables in motivic integration) that the real zeta function $Z_f^?$ is a rational function that can be expressed as

$$Z_f^?(T) = \sum_{I\cap\mathscr{K}\neq\emptyset} (\mathbb{L}-1)^{|I|-1} [\widetilde{E}_I^{0,?}] \prod_{i\in I} \frac{\mathbb{L}^{-\nu_i} T^{N_i}}{1-\mathbb{L}^{-\nu_i} T^{N_i}} \tag{12'}$$

for ? being $-1, +1, >$ or $<$, where $\widetilde{E}_I^{0,\epsilon}$ is defined as the gluing along $E_I^0 \cap U$ of the sets

$$\{(x,t) \in (E_I^0 \cap U) \times \mathbb{R};\ t^m \cdot u(x)\ !_?\ \},$$

where $!_?$ is $= -1$, $= 1$, > 0 or < 0 in case ? is respectively $-1, +1, >$ or $<$. The real motivic Milnor ?-fibre $S_f^?$ of f may finally be defined as

$$S_f^? := -\lim_{T\to\infty} Z_f^\epsilon(T) := -\sum_{I\cap\mathscr{K}\neq\emptyset} (-1)^{|I|} [\widetilde{E}_I^{0,?}] (\mathbb{L}-1)^{|I|-1} \in K_0(\mathrm{Var}_{\mathbb{R}}) \otimes \mathbb{Z}\left[\frac{1}{2}\right].$$

3.2.1. *Remark.* The class $S_f^?$, although having an expression in terms of the data coming from a chosen resolution of f, does not depend of such a choice, since the definition of $Z_f^?$ as nothing to do with any choice of a resolution.

3.2.2. *Remark.* There is no *a priori* obvious reason, from the definition of $Z_f^?(T)$, that the constant term $S_f^?$ in the power series in T^{-1} induced by the rational expression of $Z_f^?(T)$ could be accurately related to the topology of the corresponding set-theoretic Milnor fibre $X_0^?$, that is to say that $S_f^?$ could be the motivic version of the signed Milnor fibre $X_0^?$ of f. In the complex case, it has just been observed that $\chi(S_f)$ is the expression of $\chi(X_0)$ provided by the A'Campo formula. In the real case, taking into account that $\chi(\mathbf{R}) = -1$, the expression of $\chi(S_f^?)$ is

$$\chi(S_f^?) = \sum_{I\cap\mathscr{K}\neq\emptyset} (-2)^{|I|-1}\chi(\widetilde{E}_I^{0,?}),$$

showing a greater complexity than in the complex case where only strata $E_{\{i\}}$ of maximal dimension in the exceptional divisor $\sigma^{-1}(0)$ appear. Despite this increased complexity, in the real case the correspondence still holds, since it is proved in [20] that $\chi(S_f^?)$ is still $\chi(\bar{X}_0^?)$, where $? \in \{-1, +1\}$. This justifies the terminology of motivic real semialgebraic Milnor fibre of f at 0 for $S_f^?$, at least at the first topological level represented by the morphism

$$\chi : K_0(\mathrm{Var}_{\mathbb{R}}) \otimes \mathbb{Z}\left[\frac{1}{2}\right] \to \mathbf{Z}.$$

In order to accurately state the correspondence between the motivic real semialgebraic Milnor fibre and the set-theoretic Milnor fibre we set now the following notation.

3.2.3. *Notation.* Let us denote $Lk(f)$ the link $f^{-1}(0) \cap S(0, \eta)$ of f at the origin, $0 < \eta \ll 1$. We recall that the topology of $Lk(f)$ is the same as the topology of the boundary $f^{-1}(\varepsilon) \cap S(0, \eta)$, $0 < \varepsilon \ll \eta$, of the Milnor fibre $f^{-1}(\varepsilon) \cap B_{(0,\eta)}$, when f has an isolated singularity at 0.

- Let us denote, for $? \in \{<, >\}$, the topological type of $f^{-1}(]0, c_?[) \cap B(0, \eta)$ by $X_0^?$, and the topological type of $f^{-1}(]0, c_?[) \cap \bar{B}(0, \eta)$ by $\bar{X}_0^?$, where $c_< \in]-\eta, 0[$ and $c_> \in]0, \eta[$.

- Let us denote, for $? \in \{<, >\}$, the topological type of $\{f \ \bar{?}\ 0\} \cap S(0, \eta)$ by $G_0^?$, where $\bar{?}$ is $\leq$ when $?$ is $<$ and $\bar{?}$ is $\geq$ when $?$ is $>$.

3.2.4. *Remark.* When n is odd, $Lk(f)$ is a smooth odd-dimensional submanifold of $\mathbf{R}^n$ and consequently $\chi(Lk(f)) = 0$. For $? \in \{-1, +1, <, >\}$, we thus have in this situation, that $\chi(X_0^?) = \chi(\bar{X}_0^?)$. This is the situation in the complex setting. When n is even, since $\bar{X}_0^?$ is a compact manifold with boundary $Lk(f)$, one knows from general algebraic topology that

$$\chi(\bar{X}_0^?) = -\chi(X_0^?) = \frac{1}{2}\chi(Lk(f)),$$

for $? \in \{-1, +1, <, >\}$. For general $n \in \mathbb{N}$ and for $? \in \{-1, +1, <, >\}$, we thus have

$$\chi(\bar{X}_0^?) = (-1)^{n+1}\chi(X_0^?).$$

On the other hand we recall that for $? \in \{<, >\}$

$$\chi(G_0^?) = \chi(\bar{X}_0^{\delta_?}),$$

where $\delta_>$ is $+$ and $\delta_<$ is $-$ (see [6], [108]).

We can now state the real version of (13). We have, for $? \in \{-1, +1, <, >\}$

$$\chi(S_f^?) = \sum_{I\cap\mathscr{K}\neq\emptyset} (-2)^{|I|-1}\chi(\widetilde{E}_I^{0,?}) = \chi(\bar{X}_0^?) = (-1)^{n+1}\chi(X_0^?), \tag{13'}$$

and for $? \in \{<, >\}$

$$\chi(S_f^?) = \sum_{I\cap\mathscr{K}\neq\emptyset} (-2)^{|I|-1}\chi(\widetilde{E}_I^{0,?}) = -\chi(G_0^?). \tag{13''}$$

The formula (13′) below is the real analogue of the A'Campo-Denef-Loeser formula (13) for complex hypersurface singularities and thus appears as the extension to the reals of this complex formula, or, in other words, the complex formula is the notably first level of complexity of the more general real formula (13′).

3.2.5. *Remark.* In [111], following the construction of Hrushovski and Kazhdan (see [59], [60]), Yin develops a theory of motivic integration for polynomial bounded T-convex valued fields and studies, in this setting, topological zeta functions attached to a function germ, showing that they are rational. This a first step towards a real version of Hrushovski and Loeser work [61], where no resolution of singularities is used, in contrast with [20].

3.2.6. *Questions.* The question of finding a real analogue of the complex monodromy with real analogues of the invariants $\Lambda(M^m)$ is open. Similarly the question of defining a convenient zeta function with coefficients in an adapted Grothendieck ring in order to let appear invariants of type $\Lambda_i^{\ell oc}$ or σ_j ($e(\mathscr{P}^i)$ in the complex case) from a rational expression of this zeta function also naturally arises.

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SEMI-COHERENCE FOR SEMIANALYTIC SETS AND STRATIFICATIONS AND SINGULARITY THEORY OF MAPPINGS ON STRATIFICATIONS

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To David Trotman on his Sixtieth Birthday
With wishes of many more Happy and Successful Years in Mathematics

Abstract. We consider the conditions on a local stratification $\mathcal{V}$ which ensure that the local singularity theory in the sense of Thom-Mather, such as finite determinacy, versal unfolding, and classification theorems and their topological versions apply either to mappings on the stratified set $\mathcal{V}$ or for an equivalence of mappings which preserve $\mathcal{V}$ in source or target for any of the categories: complex analytic, real analytic, or smooth. For such a stratification $\mathcal{V}$, it is sufficient that the equivalence group be a "geometric subgroup of $\mathcal{A}$ or $\mathcal{K}$", and this reduces to the structure of the module $\mathrm{Derlog}(\mathcal{V})$ of germs of vector fields on the ambient space which are tangent to $\mathcal{V}$. In the holomorphic or real analytic categories, with holomorphic, resp. real analytic stratifications, we show the necessary conditions are satisfied.

However, in the smooth category the general question is open for smooth stratifications. We introduce a restricted class of "semi-coherent"semianalytic stratifications $(\mathcal{V}, 0)$ and semianalytic set germs $(V, 0)$ (and their diffeomorphic images). This notion generalizes Malgrange's notion of "real coherence"for real analytic sets. It is defined in terms of both $\mathrm{Derlog}(\mathcal{V})$ and $I(V)$ (the ideal of smooth function germs vanishing on $(V, 0)$) being finitely generated modulo infinitely flat vector fields, resp. functions. This class includes the *special semianalytic stratifications and sets* in [DGH], and semianalytic sets such as Maxwell sets, "medial axes/central sets", and the discriminants of C^∞-stable germs in the nice dimensions. We further show that the equivalence groups in the smooth category for these stratifications are then geometric subgroups of $\mathcal{A}$ or $\mathcal{K}$.

Introduction

For a stratification $\mathcal{V}$ of a germ $(V, 0)$, we consider singularity theory in the Thom-Mather sense for mappings $f : \mathbf{k}^n, 0 \to \mathbf{k}^p, 0$ either on $\mathcal{V}$ or by an equivalence preserving $\mathcal{V}$. in any of the categories: holomorphic (with $\mathbf{k} = \mathbb{C}$), real analytic, or smooth (for $\mathbf{k} = \mathbb{R}$). Traditionally, the main interests in stratifications $\mathcal{V}$ has involved their properties and the consequences for equisingularity of varieties and mappings as a result of the work of many people beginning with Whitney[Wh], Thom [Th], Hironaka [H1, H2] Lojasiewicz [Lo], Mather [M1] and further built upon by David Trotman with his many coworkers and students, e.g. [Tr1, Tr2, BTr, NTr, OTr, MPT, TrW], along with the important contributions by Verdier [Ve], Mostowski [Ms], Hardt [Ht], and many others. By contrast, singularity theory on a given stratified variety V has concentrated on the topological properties of V, either computed via stratified Morse functions on V, using Stratified Morse Theory of Goresky-MacPherson [GM] or generic projections of Lê and Teissier [LeT].

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For mappings on varieties $(V, 0)$ or equivalences preserving varieties, singularity-theoretic results have concerned: infinitesimal stability implies stability for a holomorphic germs on holomorphic $(V, 0)$, Galligo [Ga]; finite determinacy modulo an ideal (= I(V)), DuPlessis-Gaffney [DPG]; and the classification of function germs under $\mathcal{R}$-equivalence preserving a hypersurfaces $(V, 0)$ in several specific cases, Arnold [A] and Lyashko [Ly]. Also, a classification of low dimensional smooth germs has been carried out with $(V, 0)$ denoting either a smooth curve on a surface (or surface with boundary) Bruce-Giblin [BG] and Goryunov [Go], or "creases and corners" Tari [Ta1, Ta2].

These latter results fit into the general framework where for any of the three categories, a group of germs of diffeomorphisms of $(\mathbf{k}^n, 0)$, denoted by $\mathcal{D}_n$ is replaced by a group $\mathcal{D}_V$ which preserves a subspace $V, 0 \subset \mathbf{k}^n, 0$. In the holomorphic or real analytic categories, $(V, 0)$ can be the germ of any holomorphic, resp. real analytic set germ. However, in the smooth category, the results have been limited to $(V, 0)$ which are smooth diffeomorphic images of *real coherent analytic germs* in the sense of Malgrange [Mg]. Then, for example, for any of the standard equivalences in the Thom-Mather sense, $\mathcal{G} = \mathcal{R}$, $\mathcal{K}$, or $\mathcal{A}$, we may replace the group of diffeomorphisms in the source or target by the appropriate $\mathcal{D}_V$, and obtain the corresponding group $\mathcal{G}_V$ preserving $V, 0$ in the target, or ${}_V\mathcal{G}$ preserving $V, 0$ in the source. Second, we may further enlarge the equivalence group to yield equivalences $\mathcal{G}(V)$ capturing equivalence of germs on $V, 0$, and even allow both the variety $V, 0$ to vary along with the mappings.

The basic theorems of singularity theory are valid for these equivalences, because each of the groups $\mathcal{G}_V$, ${}_V\mathcal{G}$, or $\mathcal{G}_V$ are "geometric subgroup of $\mathcal{A}$ or $\mathcal{K}$"(with an adequately ordered system of algebras) in the sense of Damon [D2]. All of the four conditions to be such a group are naturally satisfied except for the tangent space condition which requires that the tangent space $T\mathcal{G}_e$ be finitely generated as a module over the system of algebras (and in the smooth case this can be relaxed to hold modulo infinitely flat vector fields, see [D1] and [D3, §8]). In the holomorphic or real analytic categories, the tangent space $\mathrm{Derlog}(V) = T\mathcal{D}_{V,e}$ (see §1) is finitely generated over the appropriate ring of germs, and in the smooth category for real coherent analytic germs $(V, 0)$, this is true (modulo infinitely flat vector fields, by [D1, Lemma 1.1]). As a consequence, the basic theorems of singularity theory are valid for these equivalences including: the finite determinacy theorem, versal unfolding theorem, and infinitesimal stability implies stability under deformations, and classification theorems.

Here we address two questions. First, in a number of situations of interest we wish to replace $(V, 0)$ by a stratification $(\mathcal{V}, 0)$ of a set germ $(V, 0)$ in the appropriate category; and furthermore, in the smooth category we would additionally like to allow the stratification $(\mathcal{V}, 0)$ and the set germ $(V, 0)$ to be semianalytic. Several examples where these conditions play a role involve: discriminants of stable germs, which in general are only (diffeomorphic to) semialgebraic sets; the Blum medial axis (or central set) for generic smooth regions in $\mathbb{R}^n$ are locally diffeomorphic to semialgebraic sets, and in computer vision, the stratifications which are needed to describe the geometric features of natural objects, and the refinements of these stratifications resulting from shade and shadows requires the consideration of semianalytic stratifications.

The first goal is to extend Malgrange's notion of real coherence for real analytic germs to a sufficently large class of semianalytic sets and stratifications. In the smooth category, A real coherent germ $(V, 0)$ in the sense of Malgrange has the property that the ideal $I(V)$ of smooth germs vanishing on $(V, 0)$ is finitely generated over the ring of smooth germs $\mathcal{E}_n$ by the generators of $I(V)^{an}$, the ideal of real analytic germs vanishing on $(V, 0)$ (see [Mg, Chap. VI, Theorem 3.10]). However, to be applicable to the equivalence groups described above, it was also necessary to have that the module $\mathrm{Derlog}(V)$ is finitely generated (modulo infinitely flat vector fields in the smooth category). We ask if there is a generalization of Malgrange's notion of being real coherent which will apply to these semianalytic sets and stratifications? Secondly, is this generalization useful to establish that the corresponding equivalence groups are geometric subgroups of $\mathcal{A}$ or $\mathcal{K}$?

We shall give a positive answer to both of these questions. We introduce a notion of *semi-coherence for semianalytic sets and stratifications*, which concerns the finite generation of both the ideal $I(V)$ and $\mathrm{Derlog}(V)$ (or the corresponding ideals and modules for a stratification $\mathcal{V}$) modulo infinitely flat vector fields. Besides having several naturality properties, this notion includes the three classes of semianalytic sets and stratifications described above, including the class of *special semianalytic sets and stratifications* introduced in [DGH]; and it establishes that the corresponding equivalence groups are geometric subgroups of $\mathcal{A}$ or $\mathcal{K}$ so that the basic theorems of singularity theory are valid for smooth mappings under such an equivalence preserving the stratification or for germs on the stratification. These results are used in [DGH] for the classification of local features of images of objects with geometric features inlcuding shade and shadows.

In §1 we recall Malgrange's notion of being real coherent and give several examples due to Malgrange and Whitney of analytic sets which do not satisfy the condition. Next, we introduce the more general notion of semi-coherence for semianalytic sets and explain how this condition includes the class of *special semianalytic sets* introduced in [DGH]. We also prove that the class of weighted homogeneous semianalytic germs are semi-coherent. This includes examples of analytic sets that are not real coherent and in addition the discriminants of stable germs in the nice dimensions. In §2, we extend the notion of semi-coherence to semianalytic stratifications and give several conditions that insure that a semianalytic stratification is semi-coherent, including the class of *special semianalytic stratifications* in [DGH]. In §3, we briefly indicate how the the resulting equivalence groups satisfy the conditions for being geometric subgroups. In §4, we give the proofs of several of the results and indicated how the others follow by slightly modifying the proofs in [DGH] for the special semianalytic stratifications.

1. Semi-coherent Semianalytic Sets

In this section we consider the smooth category, except we consider a semianalytic set $V, 0 \subset \mathbb{R}^n, 0$ with local analytic Zariski closure $(\tilde{V}, 0)$. We will simultaneously consider both the rings of smooth germs $\mathcal{E}_n$ with maximal ideal denoted by m_n, and real analytic germs $\mathcal{A}_n$. We let θ_n denote the module of germs of smooth vector fields on $(\mathbb{R}^n, 0)$. Then, we let $I(V)$ denote the ideal of smooth germs $f \in \mathcal{E}_n$ which vanish on V in a neighborhood of 0, and $I^{an}(V)$ the corresponding ideal of analytic germs. In general, it is not known when $I(V)$ is a finitely generated ideal in $\mathcal{E}_n$. Malgrange [Mg] introduced the notion of V being real coherent, which means that there is a set of generators $\{g_1, g_2, \dots, g_k\}$ for $I^{an}(V)$ and a neighborhood U of 0 on which they are defined so that for $x \in U$, the germs of the g_i at x generate the ideal of real analytic germs at x vanishing on (V, x). He then proves that for such a real coherent analytic germ $(V, 0)$, $I(V) = I^{an}(V) \cdot \mathcal{E}_n$, so in particular it is finite generated [Mg].

We let $\mathrm{Derlog}^{an}(\tilde{V})$ denote the module of real analytic vector fields ξ satisfying

$$\xi(I^{an}(\tilde{V})) \subset I^{an}(\tilde{V}).$$

It is a finitely generated $\mathcal{A}_n$-module. We let $\mathcal{V}$ denote the canonical Whitney stratification of $(V, 0)$. Then, we define

$$\mathrm{Derlog}(V) \;=\; \{\xi \in \theta_n : \xi \text{ is tangent to the strata of } \mathcal{V}\} \tag{1.1}$$

Remark 1.1. If $\xi \in \mathrm{Derlog}(V)$ and $g \in I(V)$, then as g vanishes on the strata of $\mathcal{V}$, $\xi(g)$ vanishes on the strata of $\mathcal{V}$, and hence on $(V, 0)$, so $\xi(g) \in I(V)$.

Moreover, if ξ is analytic and $g \in I^{an}(\tilde{V})$, then again g vanishes on the strata of $\mathcal{V}$, so $\xi(g)$ vanishes on $(V, 0)$ and hence on its local analytic Zariski closure $\tilde{V}$ so $\xi(g) \in I^{an}(\tilde{V})$ and $\xi \in \mathrm{Derlog}^{an}(\tilde{V})$.

Also, if $(V, 0)$ is real coherent in the sense of Malgrange, then by an argument in [D1, §1], if $\xi(I(V)) \subset I(V)$, then $\xi \in \mathrm{Derlog}(V)$ as defined in (1.1). Thus, $\mathrm{Derlog}(V)$ may be alternately be defined by the condition $\xi(I(V)) \subset I(V)$ as in [D1, §1], except there the notation θ_V was used.

The notation $\mathrm{Derlog}(V)$ is a variant of the notation introduced by Saito [Sa] for the module of "logarithmic vector fields" for a complex hypersurface singularity $V, 0$, reflecting the relation with logarithmic forms.

However, even for real coherent analytic germs it is generally unknown whether $\mathrm{Derlog}(V)$ is a finitely generated $\mathcal{E}_n$ module. A weaker result which is satisfactory for many applications in singularity theory is the following (see [D1, Lemma1.1]).

Proposition 1.2. *If $V, 0 \subset \mathbb{R}^n, 0$ is real coherent then*

$$\mathrm{Derlog}(V) \;\equiv \mathcal{E}^n\{\zeta_1, \dots, \zeta_r\} \quad \mathrm{mod}\ m_n^\infty \theta_n$$

where $\{\zeta_1, \dots, \zeta_r\}$ are a set of generators of $\mathrm{Derlog}^{an}(V)$*.*

Here m_n^∞ denotes the ideal of infinitely flat function germs.

By the result in [D3, §8], in the smooth category, for a real coherent analytic germ $V, 0 \subset \mathbb{R}^n$, we may replace $\mathcal{D}_n$ by $\mathcal{D}_V$ in any standard group of equivalences $\mathcal{G}$ and conclude they are geometric subgroups of $\mathcal{A}$ or $\mathcal{K}$. However, this places an excessive restriction even for real analytic $(V, 0)$, and does not address the case of semianalytic $V, 0$. We illustrate the issue with several examples due to Malgrange and Whitney.

Example 1.3 (Malgrange Umbrellas)**.** The following examples are generalizations of that given by Malgrange in [Mg, Example after Def. 3.9, Chap. VI]. We consider $V, 0 \subset \mathbb{R}^{n+1}, 0$ defined by

$$x_{n+1} \cdot \left(\sum_{i=1}^{n} x_i^2\right) \;=\; f(x_1, \dots, x_n)\,,$$

where f is homogeneous of degree $k \geq 3$. Then, the x_{n+1}-axis lies in V and is an isolated line, for if we consider any line $x_i = tb_i$ for $i = 1, \dots n$, with some $b_i \neq 0$, then

$$x_{n+1} \;=\; t^{k_2} \cdot \left(\frac{f(b_1, \dots, b_n)}{\sum_{i=1}^{n} b_i^2}\right)$$

Also, $(V, 0)$ is not real coherent as at a point $x' = (0, \dots, 0, x_{0,n+1})$ with $x_{0,n+1} \neq 0$, (V, x') is locally defined by $x_1 = \cdots = x_n = 0$, and is not generated by the single generator

$$G \;=\; x_{n+1} \cdot \left(\sum_{i=1}^{n} x_i^2\right) \;-\; f(x_1, \dots, x_n).$$

If $f(x_1, \dots, x_n) > 0$ when some $x_i \neq 0$, then we can remove the handle on the negative x_{n+1}-axis by adding the condition $x_{n+1} \geq 0$ and obtaining a germ of a semianalytic set whose Zariski closure is $(V, 0)$.

Example 1.4 (Generalized Whitney Umbrellas)**.** The standard Whitney umbrella is the image $V = D(F)$ of the stable map germ $F : \mathbb{R}^2, 0 \to \mathbb{R}^3, 0$, where

$$(y_1, y_2, y_3) = F(x_1, x_2) = (x_1, x_1 x_2, x_2^2).$$

It is semialgebraic with analytic Zariski closure $\tilde{V}, 0$ defined by $y_2^2 = y_3 y_1^2$. It has a handle consisting of the y_3 axis with $y_3 > 0$. As for the Malgrange umbrellas, $(\tilde{V}, 0)$ is not real coherent.

More generally we can define "generalized Whitney umbrellas"as images of maps

$$F : \mathbb{R}^{n+1}, 0 \to \mathbb{R}^{n+2}, 0$$

given by

$$(y_1, \ldots, y_{n+2}) \; = \; F(x_1, \ldots, x_{n+1}) \; = \; (x_1, \ldots, x_n, x_{n+1} \cdot f(x_1, \ldots, x_n, x_{n+1}^2), x_{n+1}^2)$$

where both f and $f(x_1, \ldots, x_n, 0)$ have isolated singularities. Such F are finitely $\mathcal{A}$-determined (see Mond [Mo] for the case $n = 1$); and such images are semialgebraic with Zariski closure $\tilde{V}$ defined by $G = y_{n+1}^2 - y_{n+2} f(y_1, \ldots, y_n, y_{n+1}) = 0$.

If $f(x_1, \ldots, x_n, x_{n+1}^2)$ is weighted homogeneous of weight c for positive weights $\mathrm{wt}\,(x_i) = b_i > 0$, then both F and G are weighted homogeneous (with $\mathrm{wt}\,(y_i) = b_i$ for $i \leq n$, $\mathrm{wt}\,(y_{n+1}) = b_{n+1} + c$ and $\mathrm{wt}\,(y_{n+2}) = b_{n+2}$ satisfying $b_{n+2} = 2b_{n+1} + c$. In the case that $f(x_1, \ldots, x_n, 0) > 0$ whenever some $x_i \neq 0$, then $\tilde{V}$ has a handle consisting of the negative y_{n+2}-axis. Again, it is not real coherent.

Next, we consider more generally $V, 0 \subset \mathbb{R}^n, 0$ a closed semianalytic set in the smooth category. We introduce a notion of $(V, 0)$ being *semi-coherent* which extends that of real coherence of Malgrange to closed semianalytic sets in a form which makes it sufficient for many applications in singularity theory. For $V, 0 \subset \mathbb{R}^n, 0$ which is closed and semianalytic, we let $(\tilde{V}, 0)$ denote its local analytic Zariski closure. We also define $\mathrm{Derlog}(V)$ for a semianalytic set $(V, 0)$ with canonical Whitney stratification $\mathcal{V}$, by (1.1). Then, we define

Definition 1.5. A closed semianalytic set germ $V, 0 \subset \mathbb{R}^n, 0$ will be said to be *semi-coherent* in the smooth category if the following two conditions are satisfied.

i) $I(V) \equiv \mathcal{E}_n\{g_1, \ldots g_s\} \mod m_n^\infty$,
where $\{g_1, \ldots g_s\}$ generate $I^{an}(\tilde{V})$; and

ii) $\mathrm{Derlog}(V) \; \equiv \mathcal{E}_n\{\zeta_1, \ldots, \zeta_r\} \mod m_n^\infty \theta_n$
where $\{\zeta_1, \ldots, \zeta_r\}$ are a set of germs in $\mathrm{Derlog}^{an}(\tilde{V})$ which are tangent to the strata of $\mathcal{V}$.

Here m_n^∞ denotes the ideal of infinitely flat smooth germs.

More generally a germ $V, 0 \subset \mathbb{R}^n, 0$ is *semi-coherent* if there is a germ of a smooth diffeomorphism $\varphi : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0$ and a semi-coherent semianalytic set $V', 0 \subset \mathbb{R}^n, 0$ such that $\varphi(V') = V$. We shall refer to the semi-coherent semianalytic set $(V', 0)$ as the *semianalytic model* for $(V, 0)$.

It follows by the same argument in [D3, §8], that $V, 0 \subset \mathbb{R}^n, 0$ being semi-coherent is sufficient to be able to conclude the unfolding and determinacy theorems and their consequences are valid for the equivalence groups in the smooth category preserve $(V, 0)$ or for equivalences of smooth germs on $(V, 0)$ (see also §3).

By the result of Malgrange and Proposition 1.2, real coherent analytic germs $(V, 0)$ are semi-coherent. A recent result Damon-Giblin-Haslinger [DGH] identifies a class of *special semianalytic germs* which are semi-coherent. A semianalytic set germ $V, 0 \subset \mathbb{R}^n, 0$ is a special semianalytic germ if its Zariski analytic closure $\tilde{V}, 0$ is real coherent and it satisfies conditions i) and ii) in definition 1.5. This allowed several important classes of semianalytic set germs which are semi-coherent to be identified using a *special semianalytic criteria* to be described in §2. However, for example, the discriminants of stable map germs and the classes of Malgrange and Whitney and umbrellas cannot satisfy the criterion for being special semianalytic set germs as their Zariski closures are not in general real coherent. This leads to the question.

Basic Question: When are semianalytic sets semi-coherent?

We give two distinct types of criteria for a semianalytic set to be semicoherent. The first simple criterion is given by the following.

Proposition 1.6. *Let $V, 0 \subset \mathbb{R}^n, 0$ be semianalytic with local analytic Zariski closure $\tilde{V}, 0$ in $\mathbb{R}^n, 0$. Suppose that $\tilde{V}, 0$ is weighted homogeneous (for positive weights) and that V is invariant under the corresponding $\mathbb{R}_+$-action. Then, $V, 0$ is semi-coherent.*

A consequence of Proposition 1.6 is that both the weighted homogeneous analytic and semianalytic Malgrange and Whitney umbrellas are semi-coherent, even though the analytic versions are not in general real coherent. Thus, the notion of semi-coherence is a more general notion than real coherence for analytic set germs $(V, 0)$. There follows a basic consequence for discriminants of C^∞ stable germs.

Theorem 1.7. *Let $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ be a simple C^∞ stable germ, which includes those in the nice range of dimensions. Then the discriminant $(D(f), 0)$ is semi-coherent.*

Proof of the Theorem. By Mather's classfication theorems for such simple stable germs (see [MIV], and [MVI]), f is $\mathcal{A}$-equivalent to a polynomial germ $g : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ which is weighted homogeneous of positive weights. Thus, there are germs of diffeomorphisms $\psi : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0$ and $\varphi : \mathbb{R}^p, 0 \to \mathbb{R}^p, 0$ so that $f = \varphi \circ g \circ \psi$. Hence, $\varphi(D(g)) = D(f)$, and it is sufficient to show that $(D(g), 0)$ is semi-coherent. However, as g is a polynomial mapping, it follows by the Tarski-Seidenberg theorem that the image $D(g) = g(\Sigma(g))$ of the singular set $\Sigma(g)$ is semialgebraic, so in particular, semianalytic.

Also, as g is weighted homogeneous for positive weights, so is the Zariski closure $\tilde{D}(g)$ (the complexification $g_{\mathbb{C}}$ has discriminant $D(g_{\mathbb{C}})$ which is weighted homogeneous for positive weights, and $D(g_{\mathbb{C}}) \cap \mathbb{R}^p$ is the Zariski closure of $D(g)$). Furthermore, if $y_0 = g(x_0) \in D(g)$ with $x_0 \in \Sigma(g)$, then by the weighted homogeneity of g, $\mathbb{R}_+ \cdot x_0 \subset \Sigma(g)$ and $g(\mathbb{R}_+ \cdot x_0) = \mathbb{R}_+ \cdot y_0$, so $\mathbb{R}_+ \cdot y_0 \subset D(g)$. Thus, by Proposition 1.6, $(D(g), 0)$, and hence $(D(f), 0)$, are semi-coherent. $\square$

Next, we illustrate that even for the simplest semianalytic germs that the equalities in Definition 1.5 are only true modulo infinitely flat functions and vector fields.

Example 1.8. Let $V, 0 \subset \mathbb{R}^n, 0$ denote the model for a k-corner. It is defined by $f = 0$ where $f(x_1, \ldots, x_k) = \prod_{i=1}^k x_i$ and the inequalities $x_i \geq 0$ for $i = 1, \ldots, k$. Its local analytic Zariski closure $\tilde{V}, 0$ is the germ defined by $f = 0$. The module $\mathrm{Derlog}^{an}(V)$ of germs of analytic vector fields tangent to V is generated by $x_i \dfrac{\partial}{\partial x_i}$, $i = 1, \ldots, k$ and $\dfrac{\partial}{\partial x_j}$, $j = k+1, \ldots, n$. We exhibit an infinitely flat smooth germ $g \in I(V)$, but not in the ideal $(f) \cdot \mathcal{E}_n$, and infinitely flat smooth germs of vector fields $g \dfrac{\partial}{\partial x_i} \in \mathrm{Derlog}(V)$, $i = 1, \ldots, k$, which are not in

$$\mathcal{E}_n\{x_i \frac{\partial}{\partial x_i}, i = 1, \ldots, k; \frac{\partial}{\partial x_j}, j = k+1, \ldots, n\}.$$

Let $\rho(x)$ be the infinitely flat germ

$$\rho(x) \;=\; \begin{cases} \exp(-\frac{1}{x^2}) & x < 0, \\ 0 & x \geq 0 \end{cases} .$$

Let $g(x_1, \ldots, x_n) = \sum_{i=1}^k \rho(x_i)^2$. Then, g vanishes on V. We claim it is not smoothly divisible by x_i for any $i = 1, \ldots, k$. For example, if g were smoothly divisible by x_1, then as $\rho(x_1)$ is smoothly divisible by x_1, so would be $g - \rho(x_1)^2 = \sum_{i=2}^k \rho(x_i)^2$. However, $\sum_{i=2}^k \rho(x_i)^2$ is not smoothly divisible by x_1. A similar argument works for not being smoothly divisible x_i for

$i = 2, \ldots, k$. Thus, $g \notin (f) \cdot \mathcal{E}_n$. Also, if $g\dfrac{\partial}{\partial x_1} \in \mathcal{E}_n\{x_i\dfrac{\partial}{\partial x_i}, i = 1, \ldots, k; \dfrac{\partial}{\partial x_j}, j = k+1, \ldots, n\}$, then $g\dfrac{\partial}{\partial x_1} = h \cdot x_1\dfrac{\partial}{\partial x_1}$. This would imply x_1 smoothly divides g, which, as we just saw, is impossible. There is an analogous argument for $i = 2, \ldots, k$.

We note that we could replace ρ by any infinitely flat function which vanishes for $x \geq 0$ but not identically on $\mathbb{R}$. Also, an analogous argument would work for more general semianalytic sets involving more than one inequality.

There is a second criterion, the *special semianalytic criterion* given in [DGH], which applies to semianalytic sets that are not necessarily weighted homogeneous and will yield *special semianalytic stratifications*. We describe it in §2.

There are also further properties of both semicoherent semianalytic sets and the special semianalytic sets. However, these properties are best described for the more general notion of semicoherent semianalytic stratifications to be introduced next.

2. Semi-coherent Semianalytic Stratifications

Let $V, 0 \subset \mathbb{R}^n, 0$ be a germ of a closed semianalytic set, and let $\tilde{V}, 0 \subset \mathbb{R}^n, 0$ be its real local analytic Zariski closure with $I^{an}(V) = I^{an}(\tilde{V})$ the ideal of real analytic germs vanishing on $(V, 0)$ and defining $\tilde{V}$. By a *semianalytic stratification* $\mathcal{V}$ of $(V, 0)$ we mean a decreasing sequence of closed semianalytic set germs $V = V_k \supset V_{k-1} \supset \cdots \supset V_1 \supset V_0 = \{0\}$ with $\dim V_j = j$ and $V_j \backslash V_{j-1}$ consisting of strata of dimension j. For the stratification $\mathcal{V}$, we define for the smooth category

$$\operatorname{Derlog}(\mathcal{V}) = \{\xi \in \theta_n : \xi \text{ is tangent to the strata } S_i \text{ of } \mathcal{V} \text{ for all } i\}. \tag{2.1}$$

We also consider $\operatorname{Derlog}^{an}(\tilde{V})$ in the real analytic category. Then, we define

Definition 2.1. The stratification $\mathcal{V}$ of the germ of the closed semianalytic set $V, 0 \subset \mathbb{R}^n, 0$ is a *semi-coherent stratification* if it satisfies the following two conditions:

i) if $\{g_1, \ldots, g_k\}$ generate $I^{an}(\tilde{V})$, then in the smooth category
$$I(V) \;\equiv\; \mathcal{E}_n\{g_1, \ldots, g_k\} \mod m_n^{\infty};$$
and

ii) there are $\xi_j \in \operatorname{Derlog}^{an}(\tilde{V})$, $j = 1, \ldots, m$ which are tangent to the strata S_i of $\mathcal{V}$ for all i such that
$$\operatorname{Derlog}(\mathcal{V}) \;\equiv\; \mathcal{E}_n\{\xi_1, \ldots, \xi_m\} \mod m_n^{\infty} \cdot \theta_n.$$

In general we say that a *stratification* $\mathcal{V}$ *of a germ* $V, 0 \subset \mathbb{R}^n, 0$ *is semi-coherent* if there is a germ of a diffeomorphism $\varphi : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0$ and a semi-coherent stratification $\mathcal{V}'$ of a semianalytic germ $(V', 0)$ such that $\varphi(V') = V$ and $\varphi(\mathcal{V}') = \mathcal{V}$.

If in Definition 2.1, we require the stronger condition that $\tilde{V}$ is real coherent, then the stratification is a *special semianalytic stratification* (SSA stratification) in the sense of [DGH].

Remark 2.2. If $(V, 0)$ is a semi-coherent semianalytic set, then the canonical Whitney stratification $\mathcal{V}$ of $(V, 0)$ is a semi-coherent semianalytic stratification in the sense of Definition 2.1. This follows since vector fields tangent to V are tangent to the canonical Whitney stratification of $(V, 0)$; and conversely by Remark 1.1, any analytic vector field ξ tangent to the Whitney stratification of $(V, 0)$, will satisfy $\xi(g) \in I^{an}(\tilde{V})$ for any $g \in I^{an}(\tilde{V})$. Hence, by property ii) for semi-coherent semianalytic sets, we have

$$\operatorname{Derlog}(\mathcal{V}) \;=\; \operatorname{Derlog}(V) \;\equiv\; \mathcal{E}_n\{\zeta_1, \ldots, \zeta_r\} \mod m_n^{\infty} \cdot \theta_n\,.$$

Hence, properties for semi-coherent stratifications will hold for semi-coherent semianalytic sets.

The definition of semi-coherent stratification depends upon an ambient space. We first note that the class of semi-coherent stratifications is preserved under two standard operations, which removes this restriction.

Proposition 2.3. *Let $\mathcal{V}$ be a semi-coherent stratification of a semianalytic set germ $V, 0 \subset \mathbb{R}^n, 0$.*

1. *If $\varphi : \mathbb{R}^n, 0 \to M, p$ is an analytic diffeomorphism to an analytic submanifold $M, p \subseteq \mathbb{R}^m, p$, then the stratification $\varphi(\mathcal{V})$ of $(\varphi(V), p)$ is a semi-coherent stratification.*
2. *Define a stratification $\mathcal{V}'$ of $V \times \mathbb{R}^k, 0 \subset \mathbb{R}^{n+k}, 0$ which has strata $S'_i = S_i \times \mathbb{R}^k$ for the strata S_i of $\mathcal{V}$. Then $\mathcal{V}'$ is a semi-coherent stratification of $V \times \mathbb{R}^k, 0 \subset \mathbb{R}^{n+k}, 0$.*

The proof of this proposition closely follows the proof of the corresponding result for special semianalytic stratifications [DGH, Prop. 5.4, Chap. 5]; see §4.

Second, we may refine a semi-coherent stratification by a series of semi-coherent stratifications in the following way. Let $\mathcal{V}_i$ be semi-coherent stratifications of closed semianalytic germs $V_i, 0$, $i = 1, \dots, k$, with $V_1, 0 \subset V_2, 0 \subset \dots V_k, 0 \subset \mathbb{R}^n, 0$ such that each stratum of $\mathcal{V}_i$ is contained in a stratum of $\mathcal{V}_{i+1}$ for each $i < k$. Then, we can define a stratification $\mathcal{V}$ of $(V, 0) = (V_k, 0)$ which is a refinement $\mathcal{V}_k$ with strata consisting of $S_i \backslash V_j$ for all S_i in $\mathcal{V}_{j+1}$ and all $1 \leq j < k$, together with the strata of $\mathcal{V}_1$.

Proposition 2.4. *In the preceding situation, the stratification $\mathcal{V}$ of the closed semianalytic germ $V, 0 \subset \mathbb{R}^n, 0$ is a semi-coherent semianalytic stratification.*

To accompany these results, we next give the second criterion for establishing semi-coherence of a stratification $\mathcal{V}$ of a germ of a closed semianalytic set $(V, 0)$, with Zariski closure $(\tilde{V}, 0)$. This is given by the following criterion from [DGH, Def 5.1, Chap 5].

Special Semianalytic Criterion:

Definition 2.5. A stratification $\mathcal{V}$ of $V, 0$ is said to satisfy the *special semianalytic criterion* (SSC) if $\tilde{V}$ is real coherent and the stratification satisfies the following conditions:

1. V and each of the irreducible components V_i are unions of connected components of the canonical Whitney stratification of $\tilde{V}$.
2. Each irreducible component $\tilde{V}_i$ of $\tilde{V}$ is smooth; and
3. For each i, the set of tangent lines $T_0\gamma$ to analytic curves γ in V_i with $\gamma(t) \in V_i$ for $t \geq 0$ and $\gamma(0) = 0$ form a Zariski dense subset of $\mathbb{P}T_0\tilde{V}_i$.

Then, the second criterion is the following given in [DGH, Prop. 5.3, Chap 5].

Proposition 2.6. *A stratification $\mathcal{V}$ of the closed semianalytic germ $V, 0 \subset \mathbb{R}^n, 0$ which satisfies the special semianalytic criterion is a special semianalytic stratification. Moreover,*

$$\mathrm{Derlog}(\mathcal{V}) \equiv \mathrm{Derlog}(\tilde{V}) \mod m_n^{\infty}\, \theta_n \tag{2.2}$$

In order to apply this result we use a simple criterion for an analytic set germ $(V, 0)$ being real coherent. This is given by the following (see [DGH, Chap. 5, Prop. 4.1]).

Proposition 2.7. *Let $V, 0 \subset \mathbb{R}^n, 0$ be a real analytic germ with complexification $V_{\mathbb{C}}, 0 \subset \mathbb{C}^n, 0$. Suppose that there is a neighborhood U of $0 \in \mathbb{R}^n$ such that for $x \in U$, the germ (V, x) is Zariski dense in $(V_{\mathbb{C}}, x)$ for the local analytic Zariski topology at x. Then, V is real coherent.*

We illustrate using these criterion for several examples that occur for natural images where stratifications defining generic geometric features of objects are refined by the stratification resulting from shade/shadow curves from a light source (see [DGH, Chap. 6, 7, 8]). The generic geometric features of objects are modeled by semianalytic sets which are "partial hyperplane arrangements".

Example 2.8 (Partial Hyperplane Arrangements)**.** Let $H_i \subset \mathbb{R}^n$, $i = 1, \dots, r$ denote a collection hyperplanes through 0 with defining equations $\alpha_i = 0$. Then $A = \cup_i H_i$ is a (central) real hyperplane arrangement. It has a canonical Whitney stratification given by the strata $\big(\cap_{i\in I} H_i \backslash (\cup_{j\notin I} H_j)\big)$ for each subset $I \subseteq \{1, \dots, r\}$.

For each hyperplane H_i, we let P_i denote the closure of a nonempty union of connected components of $H_i \backslash (\cup_{j\neq i} H_j)$. Then, $V = \cup_i P_i$ will be called a *partial hyperplane arrangement*. Such a partial hyperplane arrangement has Zariski closure the corresponding hyperplane arrangement, which is real coherent by Proposition 2.7. Hence, it is a special semianalytic set by Proposition 2.6. A sample of model semianalytic sets which model geometric features in [DGH] are given in Figure 1.

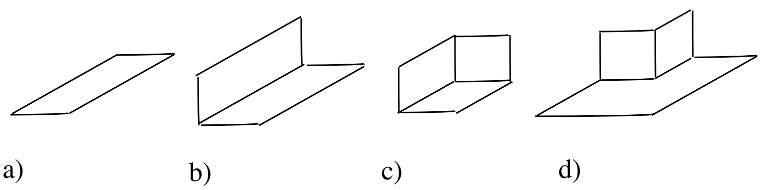

FIGURE 1. Examples of partial hyperplane arrangements which occur as models for feature stratifications: a) edge of surface; b) crease; c) convex or concave corner; and d) notch or saddle corner.

There are further examples which occur for generic structure of Blum medial axis which is the Maxwell set for the family of distance functions to the boundary hypersurface of a region, as in [M2] or [Y], are given in b) and c) in Figure 2.

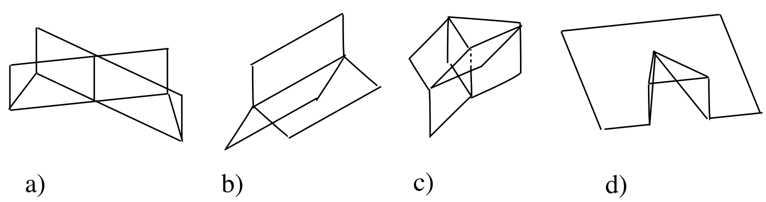

FIGURE 2. Examples of partial hyperplane arrangements which do not occur as models for feature stratifications: a) piecewise linear model of Whitney umbrella; b) and c) generic models for Blum medial axes; and d) nongeneric corner.

A second example involves 1-dimensional special semianalytic sets. First,

$$\mathbb{R}_+, 0 = \{x \in \mathbb{R} : x \geq 0\} \subset \mathbb{R}$$

with its Whitney stratification is immediately seen to satisfy SSC. Hence, by 1) of Proposition 2.3, the image of $\mathbb{R}_+, 0$ under an analytic diffeomorphism satisfies SSC. Hence, a half-branch of a smooth semianalytic curve in an analytic submanifold satisfies SSC. More generally, a germ of a 1–dimensional semianalytic set in an analytic manifold which consists of branches or half-branches of smooth analytic curves satisfies the condition SSC (see Example 5.5 and Proposition 5.6 of [DGH, Chap. 5]). This yields the following.

Proposition 2.9. *A 1-dimensional semianalytic set $V, 0 \subset \mathbb{R}^n, 0$ consisting of irreducible branches of real analytic curves and half-branches of smooth analytic curves has a special semianalytic stratification consisting of $\{V \backslash \{0\}, \{0\}\}$.*

Example 2.10 (Stratifications Refining Geometric Features by Shade/Shadows)**.** It follows from Proposition 2.4, that the refinement of a partial hyperplane arrangement by a 1-dimensional special semianalytic stratification is again a special semianalytic stratification, and hence semi-coherent. Using this result, it is proven in [DGH] that the stratifications resulting from the refinement of any stratification defining a generic geometric feature by the shade/shadow curves resulting from light in a generic direction is again a special semianalytic stratification $\mathcal{V}$ (and hence semi-coherent). This enabled the classification of (topologically) stable and (topological) codimension 1 germs for ${}_{\mathcal{V}}\mathcal{A}$-equivalence for each such stratification $\mathcal{V}$. The list of such stratifications and the corresponding classification of germs are given in Chapters 6, 7 and 8 of [DGH].

3. Equivalences of Mappings on Stratifications or Preserving Stratifications

We consider the groups of equivalences $\mathcal{G}_{\mathcal{V}}$ or ${}_{\mathcal{V}}\mathcal{G}$ preserving a stratification $\mathcal{V}$, defined by

$$V = V_k \supset V_{k-1} \supset \cdots \supset V_0 = \{0\},$$

where in the holomorphic or real analytic category the stratification is holomorphic (the $(V_i, 0)$ are holomorphic germs), resp. real analytic (the $(V_i, 0)$ are real analytic germs) and in the smooth category it is a semi-coherent semianalytic stratification. To speak of all three of these categories, we denote the corresponding ring of germs by $\mathcal{C}_n$. We also let θ_n denote the module of germs of vector fields on $(\mathbf{k}^n, 0)$ in the appropriate category. We explain how these groups satisfy the conditions for being geometric subgroups of $\mathcal{A}$ or $\mathcal{K}$ and hence the basic theorems of singularity theory are valid for them. The explanation follows the same form as that for the case for $\mathcal{G}_V$ or ${}_V\mathcal{G}$ given in [D3, §8] and [D4, §9, 10].

${}_{\mathcal{V}}\mathcal{A}$ as a geometric subgroup.

We now carry out the explanation for the case of ${}_{\mathcal{V}}\mathcal{A}$-equivalence, with that for the other groups being analogous. Then, ${}_{\mathcal{V}}\mathcal{A}$ consists of the group of pairs of diffeomorphisms (h, h') (in the appropriate category) where $h : \mathbf{k}^n, 0 \to \mathbf{k}^n, 0$ and $h' : \mathbf{k}^p, 0 \to \mathbf{k}^p, 0$ with h preserving the strata of $\mathcal{V}$. This group is a subgroup of $\mathcal{A}$ and acts on germs $f_0 : \mathbf{k}^n, 0 \to \mathbf{k}^p, 0$ in the appropriate category by $(h, h') \cdot f_0 = h' \circ f_0 \circ h^{-1}$. There are corresponding unfolding groups acting on unfoldings. ${}_{\mathcal{V}}\mathcal{A}_{un}(q)$ consists of unfoldings of diffeomorphisms on q parameters (H, H') acting on unfoldings F on q parameters by $(H, H') \cdot F = H' \circ F \circ H^{-1}$.

We let $\mathrm{Derlog}(\mathcal{V})$ be given by (2.1) for any of the three categories. In the holomorphic or real analytic categories, $\mathrm{Derlog}(\mathcal{V})$ is a finitely generate module over $\mathcal{C}_n$ (denoting the ring of holomorphic, resp. real analytic germs). In the smooth category, it is finitely generated over $\mathcal{E}_n$ modulo infinitely flat vector fields. If (h_t, t) is a one-parameter group of unfoldings in the unfolding group $\mathcal{D}_{\mathcal{V},un}(1)$, then as h_t preserves the strata of $\mathcal{V}$, it follows that $\zeta = \frac{\partial h_t}{\partial t}_{|t=0}$ is tangent to the strata of $\mathcal{V}$, so $\zeta \in \mathrm{Derlog}(\mathcal{V})$. If h_t fixes 0, then ζ vanishes on 0, and belongs to $\mathrm{Derlog}(\mathcal{V})^0$, the submodule of germs which vanish at 0. Conversely, the one-parameter subgroup h_t of germs of diffeomorphisms generated by some $\zeta \in \mathrm{Derlog}(\mathcal{V})$ will preserve the strata of $\mathcal{V}$. Hence, (h_t, t) is in the group of one-parameter unfoldings $\mathcal{D}_{\mathcal{V},un}(1)$. If in addition, h_t fixes 0, then ζ vanishes at 0, and conversely. Thus, the extended tangent space $T\mathcal{D}_{\mathcal{V},e} = \mathrm{Derlog}(\mathcal{V})$, with $T\mathcal{D}_{\mathcal{V}} = \mathrm{Derlog}(\mathcal{V})^0$ (the submodule of $\mathrm{Derlog}(\mathcal{V})$ consisting of vector fields vanishing at 0).

Thus, $T\,_{\mathcal{V}}\mathcal{A}_e$ can be written

$$T\,_{\mathcal{V}}\mathcal{A}_e \quad = \quad \mathrm{Derlog}(\mathcal{V}) \,\oplus\, \theta_p \tag{3.1}$$

Likewise, the tangent space $T\,_{\mathcal{V}}\mathcal{A}$ is given by

$$T\,_{\mathcal{V}}\mathcal{A} \quad = \quad \mathrm{Derlog}(\mathcal{V})^0 \,\oplus\, m_p \cdot \theta_p \tag{3.2}$$

For the smooth category, if $(\mathcal{V}, 0)$ is a semi-coherent semianalytic stratification of a closed semianalytic subset $V, 0 \subset \mathbb{R}^n, 0$, then by the results in §2, we may replace $\mathrm{Derlog}(\mathcal{V})$ by $\mathcal{E}_n\{\xi_1, \dots, \xi_m\}$ with ξ_j given in Definition 2.1. Then, the infinitesimal orbit map is the restriction of that for $\mathcal{A}$.

$$d\alpha_{f_0}(\xi, \eta) = \eta \circ f_0 \; - \xi(f_0) \qquad \text{for } \xi \in \mathrm{Derlog}(\mathcal{V}) \text{ and } \eta \in \theta_p \tag{3.3}$$

Then, just as for the case of $_V\mathcal{A}$, for f_0 in the appropriate category, $T\,_{\mathcal{V}}\mathcal{A}_e$ is a finitely generated module over the adequately ordered system of rings $f_0^* : \mathcal{C}_p \to \mathcal{C}_n$ (modulo infinitely flat vector fields in the smooth category), and $d\alpha_{f_0}$ would be a homomorphism of such modules. Hence, $_{\mathcal{V}}\mathcal{A}$ would satisfy the four conditions to be a geometric subgroup of $\mathcal{A}$ (the other three are easily seen to hold, using the modified version of the tangent space condition for the smooth category).

Hence, applying the results in [D2] and [D3], we conclude

Theorem 3.1. *Suppose $\mathcal{V}, 0$ is a stratification of $V, 0 \subset \mathbf{k}^n, 0$ of the corresponding type for each category of mappings: holomorphic, real analytic, or semi-coherent semianalytic stratification for the smooth category, then $_{\mathcal{V}}\mathcal{A}$ is a geometric subgroup of $\mathcal{A}$ (using (3.1) and (3.2)) for the adequately ordered system of rings $\{\mathcal{C}_n, \mathcal{C}_p\}$. Hence, both the finite determinacy and versal unfolding theorems and their consequences are valid for $_{\mathcal{V}}\mathcal{A}$.*

There is an analogous result for any $_{\mathcal{V}}\mathcal{G}$ or $\mathcal{G}_{\mathcal{V}}$ for $\mathcal{G} = \mathcal{A}, \mathcal{K}, \mathcal{R}$.

Example 3.2. The version of Theorem 3.1 for the case of special semianalytic stratifications is applied in [DGH] to the stratifications in $\mathbb{R}^3$ arising as refinements by shade/shadow curves of the stratifications by generic geometric features. The theorem together with application of classsification methods in [BKD], [BDW], and [Kr] and the topological methods in [D3] and [D4] yields the classification of both the (topologically) $_{\mathcal{V}}\mathcal{A}$-stable projections of the stratifications and the (topological) codimension 1 transitions given by Theorem 4.1 in Chap. 6 and Theorem 5.1 in Chap. 7 of [DGH].

$\mathcal{A}(\mathcal{V})$ as a geometric subgroup.

Let $\mathcal{V}$ be a stratification of a germ $(V, 0)$. Instead of $\mathcal{A}$-equivalence preserving a stratification $\mathcal{V}$, we may consider instead $\mathcal{A}$-equivalence for germs on $\mathcal{V}$, which we denote by the group $\mathcal{A}(\mathcal{V})$. For just the germ of a variety $(V, 0)$, the tangent space for the case of $\mathcal{A}(V)$ was determined in [D2, §8] and [D3, §9, 10]. To consider instead the germs on the stratification $\mathcal{V}$, the equivalence is defined via the group consisting of diffeomorphisms $H : \mathbf{k}^{n+p}, 0 \to \mathbf{k}^{n+p}, 0$, $h : \mathbf{k}^n, 0 \to \mathbf{k}^n, 0$, and $h' : \mathbf{k}^p, 0 \to \mathbf{k}^p, 0$, such that: i) $h \circ \pi_n = \pi_p \circ H$; ii) H preserves $V \times \mathbf{k}^p$; iii) $H|(V \times \mathbf{k}^p) = h \times h'$; and iv) h preserves the strata of $\mathcal{V}$. Then, $H \circ (h \times h')^{-1} \equiv id$ on $V \times \mathbf{k}^p$. A calculation then shows that

$$T\,\mathcal{A}(\mathcal{V})_e \quad = \quad \mathrm{Derlog}(\mathcal{V}) \,\oplus\, \theta_p \,\oplus\, I(V) \cdot \mathcal{C}_{n+p}\{\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_p}\}\,. \tag{3.4}$$

Likewise, the tangent space $T\,\mathcal{A}(\mathcal{V})$ is given by

$$T\,\mathcal{A}(\mathcal{V}) \quad = \quad \mathrm{Derlog}(\mathcal{V})^0 \oplus m_p \cdot \theta_p \oplus I(V) \cdot \mathcal{C}_{n+p}\{\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_p}\}\,. \tag{3.5}$$

Now the infinitesimal orbit map is defined by

$$d\alpha_{f_0}(\xi, \eta, \zeta) = \zeta \circ \tilde{f}_0 + \eta \circ f_0 \ - \xi(f_0) \tag{3.6}$$

where as above, $\xi \in \mathrm{Derlog}(\mathcal{V})$ and $\eta \in \theta_p$; in addition $\zeta \in I(V) \cdot \mathcal{C}_{n+p}\{\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_p}\}$, and $\tilde{f}_0(x) = (x, f_0(x))$.

Then, an analogous argument as above yields the following.

Theorem 3.3. *Suppose $\mathcal{V}, 0$ is a stratification of $V, 0 \subset \mathbf{k}^n, 0$ of the corresponding type for each category of mappings: holomorphic, real analytic, or semi-coherent semianalytic stratification for the smooth category, then $\mathcal{A}(\mathcal{V})$ is a geometric subgroup of $\mathcal{A}$ (using (3.4), (3.5)), and (3.5)) for the adequately ordered system of rings $\{\mathcal{C}_n, \mathcal{C}_p\}$. Hence, both the finite determinacy and versal unfolding theorems and their consequences are valid for $\mathcal{A}(\mathcal{V})$.*

Again there is an analogous result for $\mathcal{K}(\mathcal{V})$, and $\mathcal{R}(\mathcal{V})$.

Equivalences Allowing the Stratification to Deform.

Lastly, suppose that $(\mathcal{V}, 0)$ is defined as $g^{-1}(\mathcal{V}')$, for a stratification $\mathcal{V}'$ of a germ $V', 0 \subset \mathbf{k}^r, 0$, with the germ $g : \mathbf{k}^n, 0 \to \mathbf{k}^r, 0$ being finitely determined for $\mathcal{K}_{\mathcal{V}'}$-equivalence. Then, the equivalence of a germ $f : \mathbf{k}^n, 0 \to \mathbf{k}^p, 0$ on $(\mathcal{V}, 0)$, allowing both $\mathcal{V}$ and f to deform, is obtained by considering the action on the pair $(g, f) : \mathbf{k}^n, 0 \to \mathbf{k}^{r+p}, 0$ by $\mathcal{K}_{\mathcal{V}}$-equivalence on g and $\mathcal{A}$-equivalence on f, using a common diffeomorphism on $(\mathbf{k}^n, 0)$. Again, if the stratification $\mathcal{V}'$ is of the appropriate type for each category, then the equivalence group is a geometric subgroup of $\mathcal{A}$ or $\mathcal{K}$, and so the basic results of singularity theory apply for this equivalence.

Remark 3.4. We have concentrated on how the groups $\mathcal{G} = \mathcal{A}, \mathcal{K}, \mathcal{R}$ can be modified to allow an equivalence preserving a variety $(V, 0)$ or stratification $(\mathcal{V}, 0)$ for each of the three categories. In fact, for any geometric subgroup $\mathcal{G}$ which has a factor group $\mathcal{D}_r$, we can replace it by a subgroup $\mathcal{D}_V$ or $\mathcal{D}_{\mathcal{V}}$, for $V, 0 \subset \mathbf{k}^r, 0$ of $\mathcal{V}$ a stratification in $(\mathbf{k}^r, 0)$. Provided $(V, 0)$ or $(\mathcal{V}, 0)$ are appropriate for the category, the resulting group of equivalences will again be a geometric subgroup.

Concluding Remarks.

The local singularity-theoretic methods we have described apply to finite codimension germs for the appropriate equivalence group. The abundance of such germs will follow when the stratification $(\mathcal{V}, 0)$ or germ $(V, 0)$ is "holonomic" in the sense introduced by Saito [Sa]. By this we mean there is a neighborhood U of 0 such that for each $x \in U$, the generators $\{\xi_1, \dots, \xi_r\}$ of $\mathrm{Derlog}(\mathcal{V})$, resp. $\mathrm{Derlog}(V)$, span the tangent space $T_x S_i$ of the statum of $\mathcal{V}$, resp. the canonical Whitney stratification of $(V, 0)$, which contains x.

The special semianalytic stratifications which occur in [DGH] for the refinemments of the stratifications of geometric features by shade shadow curves are all holonomic. However, the classification shows that finite ${}_{\mathcal{V}}\mathcal{A}$-codimension germs of low codimension already are frequently multi-modal singularities; so that topological methods of [D3] and [D4] are needed to carry out the classification.

4. Proofs of the Results

It remains to prove the results concerning semi-coherence.

Proof of Proposition 1.6. First, for i), we let $f \in I(V)$. There exists a neighborhood $0 \in U \subset \mathbb{R}^n$ such that f is defined on U and vanishes on $V \cap U$. Also, we denote the weights of the coordinates on $\mathbb{R}^n$ by $\mathrm{wt}\,(x_i) = a_i > 0$ for $i = 1, \dots, n$. We expand the Taylor expansion of f in terms of weights $\hat{f}(x) = \sum_{j=1}^{\infty} f_j(x)$, where $\mathrm{wt}\,(f_j) = j$.

We claim that each $f_j \in I(V)$. If not, choose the smallest k for which this is not true. Suppose $x_0 \in V \cap U$ is such that $f_k(x_0) \neq 0$. Let $x_0 = (x_{0\,1}, \dots, x_{0\,n})$ and define $\gamma : \mathbb{R} \to \mathbb{R}^n$ by $\gamma(t) = (x_{0\,1}t^{a_1}, \dots, x_{0\,n}t^{a_n})$. By the weighted homogeneity of f_k, it follows $f_k \circ \gamma(t) = t^k f_k(x_0)$. Then, the Taylor expansion of $f \circ \gamma(t)$ is given by $\widehat{f \circ \gamma(t)} = \sum_{j=1}^{\infty} t^j f_j(x_0)$. On the one hand as $f \circ \gamma(t) = 0$ for $0 \leq t < \varepsilon$, the Taylor expansion of $f \circ \gamma(t)$ is zero. However, by assumption the coefficient of t^k is $f_k(x_0) \neq 0$, so it is the lowest nonzero term of the Taylor expansion, a contradiction. Thus, all $f_j \in I(V)$. As each f_j is analytic and $= 0$ on V, which has local analytic Zariski closure $\tilde{V}$, we conclude $f_j \in I^{an}(\tilde{V})$. Hence, we may write as a weighted homogeneous sum $f_j = \sum_{i=1}^{s} h_{i,j} g_i$, where g_i are a set of weighted homogeneous generators of $I^{an}(\tilde{V})$ with weights $\mathrm{wt}\,(g_i) = b_i > 0$. Hence, we may write as a formal sum

$$\hat{f} \;=\; \sum_{i=1}^{s} (\sum_{j=1}^{\infty} h_{i,j}) g_i \,.$$

As $\mathrm{wt}\,(h_{i,j}) = j - b_i$ the formal sum $\sum_{j=1}^{\infty} h_{i,j}$ defines an element $\hat{h}_i \in \mathbb{R}[[\mathbf{x}_n]]$, where $\mathbf{x}_n = (x_1, \dots, x_n)$.

Lastly, by Borel's Lemma, there is a germ $h_i \in \mathcal{E}_n$ with Taylor expansion $\hat{h}_i$. Thus, if we let $f' = \sum_{i=1}^{s} h_i g_i$, we have $\hat{f} = \hat{f}'$, or equivalently $f \equiv f' \mathrm{mod}\; m_n^{\infty}$. As this holds for all $f \in I(V)$, the result i) follows.

For ii) we follow an analogous line of reasoning and use the same notation as for i). Let $\xi \in \mathrm{Derlog}(V)$. There is a neighborhood $0 \in U \subset \mathbb{R}^n$ so that both ξ and the generators g_j of $I^{an}(\tilde{I})$ are defined on U and so that (by Remark 1.1) $\xi(g_j)$ vanishes on $V \cap U$ for $j = 1, \dots, s$. We again consider a weighted expansion of the Taylor series of ξ, $\hat{\xi} = \sum_{j=n_0}^{\infty} \xi_j$, where ξ_j is weighted homogeneous of weighted degree j. Here, as usual, we assign weights $\mathrm{wt}\,(\dfrac{\partial}{\partial x_i}) = -a_i$ and then we let $n_0 = -\max_i\{a_i\}$.

We claim that each $\xi_j \in \mathrm{Derlog}^{an}(\tilde{V})$. If not let the lowest j for which this fails be denoted by k and for this k there is an g_ℓ so that $\xi_k(g_\ell)$ does not vanish on V in a neighborhood of 0, otherwise as it is analytic, it also vanishes on $\tilde{V}$, so $\xi_k(g_\ell) \in I^{an}(\tilde{V})$. If this held for each i, then $\xi_k \in \mathrm{Derlog}^{an}(\tilde{V})$. Hence, there is an $x_0 \in V \cap U$ so that $\xi_k(g_\ell)(x_0) \neq 0$. We consider the curve $\gamma(t)$ as above. Then $\xi(g_\ell)$ vanishes on $V \cap U$, and hence on the curve $\gamma(t)$ for $0 \leq t < \varepsilon$. Thus, the Taylor expansion of $\xi(g_\ell) \circ \gamma(t)$ is 0.

Then $\xi_j(g_\ell)$ is a weighted homogeneous polynomial of weighted degree $j + b_\ell > 0$ (if it is a nonzero polynomial). As we assume it is nonzero, we also have $\xi_j(g_\ell) \circ \gamma(t) = \xi_j(g_\ell)(x_0)t^{j+b_\ell}$. We then compute the Taylor expansion of $\xi(g_\ell) \circ \gamma(t)$ by

$$\xi(\widehat{g_\ell) \circ \gamma}(t) \;=\; \sum_{j=n_0}^{\infty} \xi_j(g_\ell)(x_0)\, t^{j+b_\ell}$$

Again, this Taylor series has a lowest nonzero term t^{k+b_ℓ}, contradicting that it is zero. Thus, each $\xi_j \in \mathrm{Derlog}^{an}(\tilde{V})$.

If by [Lo], $V = V_k \supset V_{k-1} \supset \cdots \supset V_1 \supset V_0 = \{0\}$ defines the canonical Whitney stratification $\mathcal{V}$, consisting of semianalytic sets (also invariant under $\mathbb{R}_+$), then we may apply the preceding argument to each V_i to conclude $\xi_j \in \mathrm{Derlog}^{an}(\tilde{V}_i)$. As ξ_j is tangent to the regular strata of each V_i, $\xi_j \in \mathrm{Derlog}^{an}(\mathcal{V})$, the submodule of $\mathrm{Derlog}^{an}(\tilde{V})$ consisting of germs of analytic vector fields tangent to the strata of $\mathcal{V}$.

As $\mathcal{A}_n$ is Noetherian, $\mathrm{Derlog}^{an}(\mathcal{V})$ is a finitely generated $\mathcal{A}_n$-module. As $\tilde{V}$, V, and $\mathcal{V}$ are invariant under the $\mathbb{R}_+$-action, $\mathrm{Derlog}^{an}(\mathcal{V})$ has a set of weighted homogeneous generators $\{\zeta_1, \dots, \zeta_r\}$ of weights $\mathrm{wt}\,(\zeta_j) = c_j$. We may write $\xi_j = \sum_{i=1}^{r} h_{i,j}\zeta_i$, where $h_{i,j}$ is weighted homogeneous of weighted degree $j - c_i$ (and $h_{i,j} = 0$ if $j - c_i < 0$). Thus, we may define $\hat{h}_i = \sum_{i=n_0}^{\infty} h_{i,j} \in \mathbb{R}[[\mathbf{x}_n]]$ and obtain

$$\hat{\xi} \;=\; \sum_{i=1}^{r} \hat{h}_i \zeta_i$$

Again, using Borel's lemma, there are smooth germs h_i whose Taylor expansions are $\hat{h}_i$, and we let $\xi' = \sum_{i=1}^{r} h_i\zeta_i$. We conclude $\xi \equiv \xi' \mathrm{mod}\; m^{\infty}\theta_n$. As this holds for every $\xi \in \mathrm{Derlog}(V)$, we obtain ii). □

Propositions 2.7 and 2.6 were proven in [DGH, Chap. 5]. Also, Propositions 2.3 and 2.4 were proven for the case of special semianalytic stratifications in [DGH, Chap. 5, §6]; however, the conditions i) and ii) in Definition 2.1 directly follow from the arguments given in the proofs for the special semianalytic case.

We do remark that to deal with the lack of weighted homogeneity which was used heavily in the proof of Proposition 1.6, the arguments proceed by first reducing to the formal category, and using the Artin approximation theorem and the Artin-Rees Lemma to obtain the desired generators there. Then, Borel's Lemma gives the desired result. These ideas are used repeatedly in the proofs in [DGH].

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A LONG AND WINDING ROAD TO DEFINABLE SETS

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To David Trotman on the occasion of his 60th birthday.

Abstract. We survey the development of o-minimal structures from a geometric point of view and compare them with subanalytic sets insisting on the differences. The idea is to show the long way from semi-analytic to definable sets, from normal partitions to cell decompositions. Some recent results are discussed in the last section.

Introduction

This paper was conceived as a historical survey. In a sense it is a follow up of the book [DS1]. It does contain some recent results (mostly in the last section, e.g. on the Kuratowski convergence of definable sets from [DD]) and some results that are not new, but are not very well known; albeit, its aim is mostly didactical and historical. The younger author appreciated this historical insight as well as the intertwining of subanalytic geometry, Pfaffian geometry and o-minimal structures, and wishes to share it with others, as it proved useful to himself.

We have the feeling that definable sets and their cell decompositions have replaced nowadays every other kind of special sets and stratifications, especially in applications (for instance in control theory, cf. our later quotes). The cell decompositions have not necessarily the same proprieties as subanalytic stratifications (not only they may not be analytic, but even not $\mathcal{C}^\infty$-smooth cf. [LGR]). Other wrong beliefs are also quite popular (for instance that subanalytic sets form an o-minimal structure, which is not true). We spotted, as well, numerous omissions in various references by different authors. This is due partially to the fact that many important papers (especially those written in French) got forgotten.

This survey has two authors, which are (easily identifiable) mother and son. The older author worked in Łojasiewicz's group ever since 1967, presented Gabrielov's work [G] at Łojasiewicz's seminar (this was a starting point for the theory of subanalytic sets *à la polonaise*), wrote (with J. Stasica) the preprint [DS*] presenting the results obtained by Łojasiewicz's group and was even, by pure chance, present in Dijon when the Pfaffian sets were born there (in 1989, this was an idea of Robert Moussu developed this year by Claude Roche and Jean-Marie Lion and continued later cf. [L], [MR]...). The older author can be therefore considered as a witness to the development we describe here, which began in 1965, when Łojasiewicz published his IHES preprint on semi-analytic sets [Ł1], now accessible on line on the site of Michel Coste [CŁ]. Our survey will present the way that led from semianalytic to subanalytic, Pfaffian and definable sets (the order here is not as linear as most people tend to believe).

The younger author appreciated the historical knowledge that let him understand better definable sets and wishes to share it with others. He also contributed to the much modernized and completed book version [DS1] of the preprint [DS*].

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Let us remark that E. Bierstone and P. Milman, the authors of the very well written IHES text *Semi-analytic and subanalytic sets* [BM] were among the first to quote the preprint [DS*] that served them as a basis for their presentation of subanalytic sets (the *Fiber-Cutting Lemma* is, for instance, *lemme B* from the initial work [DŁS] of Denkowska, Łojasiewicz, Stasica). L. Van den Dries, who can be considered as a father of definable sets (cf. the book [vdD]) also knew the preprint [DS*].

As to our friend, David Trotman, we owe him a lot. We met very early in our careers and David, a world known specialist in singularities and in particular in stratifications, encouraged our work, asked questions that led to the writing of some of our papers, especially those concerning stratifications (like [DSW], [DW]) and, together with Bernard Teissier popularized the preprint [DS*]. Later, Trotman and Teissier played a very important role in the publication of its book version [DS1]. Many thanks to both of them.

The stratifications, a tool largely used by René Thom , were brought to Poland by Łojasiewicz, who was one of Thom's close friends. As we mention in the survey, Łojasiewicz had his own way of constructing different stratifications, to begin with normal partitions (they were a main ingredient used in Łojasiewicz's theory of subanalytic sets, as opposed to that of Hironaka, based on desingularization).

The so called 'Łojasiewicz group' in Kraków consisted of (in order in which they joined the group), the following Łojasiewicz's students: Krystyna Wachta, Zofia Denkowska, Jacek Stasica, Wiesław Pawłucki, Krzysztof Kurdyka and Zbigniew Hajto.

There are many sources of information about semi-analytic sets ([Ł1]), subanalytic sets ([H2], [DS1], [ŁZ]) and definable sets ([vdD], [vdDM], [C2]). In this paper we are only trying to put all this together in some order and in its historical context, with special interest given to stratifications. We also gathered in this survey a lot of information otherwise scattered in the literature (the bibliography is still far from being exhaustive, we included in it what we feel represents the different facets of the subject).

May it serve the younger!

1. A reminder

For a start, recall one of the (equivalent) definitions of an o-minimal structure (see [C2], [vdD]):

Definition 1.1. A *structure* on the field $(\mathbb{R}, +, \cdot)$ is a collection $\mathcal{S} = \{\mathcal{S}_n\}_{n\in\mathbb{N}}$, where each $\mathcal{S}_n$ is a family of subsets of $\mathbb{R}^n$ satisfying the following axioms:

(1) $\mathcal{S}_n$ contains all the algebraic subsets of $\mathbb{R}^n$;
(2) $\mathcal{S}_n$ is a Boolean algebra ([1]) of the powerset of $\mathbb{R}^n$;
(3) If $A \in \mathcal{S}_m$, $B \in \mathcal{S}_n$, then $A \times B \in \mathcal{S}_{m+n}$;
(4) If $\pi\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is the natural projection and $A \in \mathcal{S}_{n+1}$, then $\pi(A) \in \mathcal{S}_n$.

The elements of $\mathcal{S}_n$ are called *definable* (or *tame*) subsets of $\mathbb{R}^n$.
The structure $\mathcal{S}$ is *o-minimal* (*o* stands for *order*) if it satisfies the additional condition

(5) $\mathcal{S}_1$ is nothing else but all the finite unions of points and intervals of any type.

It is natural to introduce the following notion:

Definition 1.2. Given a structure $\mathcal{S}$, we call *definable* (in $\mathcal{S}$) any function $f\colon A \to \mathbb{R}^n$, where $A \subset \mathbb{R}^m$, such that its graph, again denoted f, belongs to $\mathcal{S}_{m+n}$.

[1] Recall that a family $\mathcal{S}$ of sets, subsets of $\mathbb{R}^n$ in our case, is a *Boolean algebra*, if $\varnothing \in \mathcal{S}$ and for every $A, B \in \mathcal{S}$, there is $A \cap B, A \cup B, \mathbb{R}^n \setminus A \in \mathcal{S}$.

Remark 1.3. Clearly, axiom (4) implies that if f is definable, its definition set $A \in \mathcal{S}_m$. The image, $f(A) \in \mathcal{S}_n$ since it coincides with $\pi(f \cap (A \times \mathbb{R}^n))$, where π is the natural projection onto $\mathbb{R}^n$, and $A \times \mathbb{R}^n \in \mathcal{S}_{m+n}$. Finally, the definability of $f = (f_1, \ldots, f_n)$ is equivalent to the definability of its components f_i.

Proposition 1.4. *Every o-minimal structure contains semi-algebraic sets. (cf. subsection 1.1)*

Proof. Indeed, by condition (1) it contains algebraic sets and thus it suffices to show that it contains all the sets of the form $\{x \in \mathbb{R}^n \mid P(x) > 0\}$ with P being a polynomial (axiom (2)). Any such set can be written as $\{x \in \mathbb{R}^n \mid \exists \varepsilon > 0 \colon P(x) = \varepsilon\}$ and thus it can be written as the projection $\pi(A)$ by $\pi(x,t) = x$ of the algebraic set $\{(x,t) \in \mathbb{R}^n \times \mathbb{R} \mid t^2 P(x) = 1\}$. Condition (4) yields $\pi(A) \in \mathcal{S}_n$. □

Remark 1.5. It is easy to see that if $A \in \mathcal{S}_{m+n}$ and $B \in \mathcal{S}_n$, then the set

$$\{x \in \mathbb{R}^m \mid \exists y \in B \colon (x,y) \in A\}$$

is in $\mathcal{S}_m$, this set being the projection onto $\mathbb{R}^m$ of $A \cap (\mathbb{R}^m \times B)$. Since taking the complement changes the quantifier $\forall$ to $\exists$, the same is true for $\{x \in \mathbb{R}^m \mid \forall y \in B, (x,y) \in B\}$, i.e., this set belongs to $\mathcal{S}_m$.

1.1. **Semi-algebraic geometry.** (See e.g. [C1] or [BCR]). The definition of semi-algebraic sets is global. In fact, Łojasiewicz [Ł1] used the notion of sets 'described by' the functions of a given subring $\mathcal{A}$ of the ring of continuous real functions defined in $\mathbb{R}^n$. These are the sets of the form

$$A = \bigcup_{i=1}^{p} \bigcap_{j=1}^{q} \{x \in \mathbb{R}^n \mid f_{ij}(x) * 0\}$$

where $*$ stands for any of the signs $>, <, =$. Such sets form a Boolean algebra denoted $S(\mathcal{A})$.

If $\mathcal{A}$ is the ring of polynomials of n variables, $S(\mathcal{A})$ is the Boolean algebra of semi-algebraic sets.

Clearly, semi-algebraic sets verify the conditions (1), (2), (3), (5) of o-minimal structures. It suffices to check the condition (4) (projection property), the others being easy. This condition is verified thanks to the following theorem of Tarski-Seidenberg:

Theorem 1.6 (Tarski-Seidenberg)**.** *Let $\pi \colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ be the natural projection and let $A \subset \mathbb{R}^n \times \mathbb{R}$ be a semi-algebraic set. Then $\pi(A)$ is semi-algebraic, too.*

The classical geometric approach to this theorem is based on the following lemma.

Lemma 1.7 (Cohen — *Lemme de saucissonage*)**.** ***Classical version:*** *Let $P(x,t)$ be a polynomial in $n+1$ variables. Then there exists a finite partition of $\mathbb{R}^n$: $\mathbb{R}^n = \bigcup_{i=1}^{p} A_j$ into semi-algebraic sets A_j such that for any $i = 1, \ldots, p$, either $P(x,t)$ has constant sign for $x \in A_i$ and all $t \in \mathbb{R}$, or there is a finite number of continuous semi-algebraic functions $\xi_1 < \ldots < \xi_{p_i}$ on A_i such that for $x \in A_i$, $\{P(x,t) = 0\} = \{\xi_j(x), j = 1, \ldots, p_i\}$ and the sign of $P(x,t)$ depends only on the signs of $t - \xi_j(x)$, $j = 1, \ldots, p_i$.*

Łojasiewicz's version: *Let $\mathcal{A}$ be a ring of real continuous functions defined on a topological space X. Assume that each set from $S(\mathcal{A})$ has only a finite number of connected components, each of them belonging to $S(\mathcal{A})$. Then for any $E \in S(\mathcal{A}[t])$ there exists a finite partition $X = \bigcup_{i=1}^{p} A_i$ with $A_i \in S(\mathcal{A})$ and real functions $\xi_{A_i,1} < \ldots < \xi_{A_i,p_i}$, continuous on A_i (it may happen that there are none for some i), such that E is the union of sets from $S(\mathcal{A}[t])$ of one of the two forms below:*

$$B_{ik} := \{(x,t) \in A_i \times \mathbb{R} \mid \xi_{A_i,k}(x) < t < \xi_{A_i,k+1}(x)\}, k = 0, \ldots, p_i + 1,$$
$$\text{or } C_{ik} := \{(x, \xi_{A_i,\ell}(x)) \mid x \in A_i\}, \ell = 1, \ldots, p_i,$$

where $\xi_{A_i,0} \equiv -\infty$ *and* $\xi_{A_i,p_i+1} \equiv +\infty$.

Clearly, Łojasiewicz's version implies the classical one, as the assumption on the finiteness of the number of connected components follows by induction. Below we quote the original Łojasiewicz's proof of his version:

Proof. The set E is described by some $f_i(x,t) = \sum_{j=0}^m a_{ij}(x)t^{m-j}$, $i = 1, \dots n$ with $a_{ij} \in \mathcal{A}$. Let φ_{ik} denote the kth derivative of f_i with respect to t, here $k = 1, \dots, m$. Put $f_J := \prod_{(i,k)\in J} \varphi_{ik}$, where $J \subset I := \{1, \dots, n\} \times \{1, \dots, m\}$. Define for $r = 1, \dots, m, \infty$,

$$A_{J,r} := \{x \in X \mid f_J(x,t) = 0 \quad \text{has exactly} \quad r \quad \text{complex roots} \quad t\}.$$

It is easy to check that each $A_{J,r} \in S(\mathcal{A})$. For any fixed J, the sets $A_{J,r}$, $r = 1, \dots, m, \infty$ form a partition of X, whence we recover a partition of X from the connected components of the intersections $\bigcap_J A_{J,r_J}$. We call them $A_1, \dots, A_p$.

It is easy to see by applying Rouché's Theorem (in fact, Hurwitz theorem, which is a corollary for analytic functions) that for any A_j and any $J = \{(i,k) \in I \mid \varphi_{ik} \neq 0 \,\text{on}\, A_j \times \mathbb{R}\}$ one can find continuous functions $\xi_{A_j,1}(x) < \dots < \xi_{A_j,p_j}(x)$ such that

$$\{x \in A_j \mid f_J(x,t) = 0\} = \bigcup_{i=1}^{p_j} \xi_{A_j,i},$$

the latter denoting the graphs of $\xi_{A_j,i}$.

Now, since $f_J \neq 0$ on B_{jk}, then on this set either $\varphi_{ik} \neq 0$, or $\varphi_{ik} \equiv 0$, depending on whether $(i,k) \in J$ or not. On the other hand, for C_{jk} either $\varphi_{ik} \equiv 0$ on $A_j \times \mathbb{R}$ which is the trivial case, or $\varphi_{ik} \not\equiv 0$ on it. If the latter occurs, then the roots of $\varphi_{ik}(x,t) = 0$ over A_j are continuous functions $\xi_1(x) < \ldots < \xi_r(x)$. Since each graph ξ_ρ is contained in $\bigcup_{\iota=1}^{p_j} \xi_{A_j,\iota}$ and the graphs $\xi_{A_j,\iota}$ are open-closed in this union, there is a unique ι_ρ such that $\xi_\rho = \xi_{A_j,\iota_\rho}$. Hence, on C_{jk} one has either $\varphi_{ik} \equiv 0$, or $\varphi_{ik} \neq 0$ depending on whether $k = \iota_\rho$ for some ρ or not.

Finally, we show that $B_{jk}, C_{jk} \in S(\mathcal{A}[t])$. Let D be one of these sets. Then

$$D \subset T := \bigcap_{i=1}^n \bigcap_{k=0}^m \{x \in A_j \mid \varphi_{ik} \in \Theta_{ik}\},$$

where Θ_{ik} is either $\{t < 0\}$, or $\{0\}$, or $\{t > 0\}$. It suffices to prove now that in fact $D = T$. If there were a point $(a,t) \in T \setminus D$, then for some t' there would be $(a,t') \in D$. By Thom's Lemma ([2]), the set $(\{a\} \times \mathbb{R}) \cap T$ is convex, whence $\{a\} \times [t,t'] \subset T$. Whatever the form of D (either B_{jk} or C_{jk}), there exists $t_1, t_2 \in [t,t']$ such that $f_J(t_1,a) = 0$ while $f_J(t_2,a) \neq 0$. That is a contradiction, since there must be either $f_J \equiv 0$, or $f_J \neq 0$ on T depending on whether $\Theta_{ik} = \{0\}$ for some $(i,k) \in J$, or not.

It remains to observe that the sets B_{ik}, C_{ik} form a partition of $X \times \mathbb{R}$ and on each of them one has either $f_i \equiv 0$, or $f_i \neq 0$, which implies that E is the union of some of them. □

Remark 1.8. Under the assumptions of the Lemma above on $\mathcal{A}$ we have:

(1) Each $E \in S(\mathcal{A}[t])$ has only a finite number of connected components, each of them belonging to $S(\mathcal{A}[t])$; therefore, by induction, the same is true in $S(\mathcal{A}[t_1, \dots, t_n])$.

(2) If $\pi \colon X \times \mathbb{R} \to X$ is the natural projection, then $\pi(E) \in S(\mathcal{A})$ for $E \in S(\mathcal{A}[t])$; therefore, by induction, the same is true for $\pi \colon X \times \mathbb{R}^n \to X$ and $S(\mathcal{A}[t_1, \dots, t_n])$.

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[2] Thom's Lemma: *Let $P(t)$ be a polynomial of degree n. Then each set $\Delta_P := \bigcap_{k=0}^n \{t \in \mathbb{R} \mid P^{(k)}(t) \in \Theta_k\}$, where Θ_k is either $\{t < 0\}$, or $\{0\}$, or $\{t > 0\}$, is connected: an open interval, a point, or possibly void.* Indeed, for $n = 0$ there is nothing to do. If the lemma holds for $n-1$ take n and apply the lemma to P'. Then $\Delta_P = \Delta_{P'} \cap \{P(t) \in \Theta_0\}$. If $\Delta_{P'}$ is an open interval, then $P'(t) \neq 0$ in it and thus P is strictly monotone on $\Delta_{P'}$ and the lemma follows.

Taking $X = \{0\}$ and $\mathcal{A} = \mathbb{R}$ the first remark above yields by induction:

Theorem 1.9. *Every semi-algebraic set has a finite number of connected components, each of them semi-algebraic.*

The second remark for $\mathcal{A} = \mathbb{R}[x_1, \ldots, x_m]$ implies the Tarski-Seidenberg Theorem, also by induction.

Remark 1.10. The theorem of Tarski-Seidenberg itself implies that the image of a semi-algebraic set under any semi-algebraic mapping is semi-algebraic as in Remark 1.3. It is clear that semi-algebraic sets form an o-minimal structure.

The theory of semi-algebraic sets is well exposed in [C1], [C2], [BR], [BCR]. We list here some of their basic properties:

Theorem 1.11. *The Euclidean distance to a nonempty semi-algebraic set is semi-algebraic (i.e., has semi-algebraic graph).*

The obvious proof follows from the description of the graph and we easily obtain the following corollary.

Corollary 1.12. *If A is semi-algebraic, then the closure $\overline{A}$, the interior* $\mathrm{int}A$ *and the border ∂A are semi-algebraic as well.*

Remark 1.13. The theorem and corollary above still hold true if one changes the words *semi-algebraic* to *definable* (partly due to Proposition 1.4).

The most striking property of semi-algebraic sets is the existence of *explicit* uniform bounds, for example on the number of connected components. These bounds are nicely gathered in the book [YC] by G. Comte and Y. Yomdin.

1.2. **Definable sets.** By 'definable sets' we always mean 'definable in some given o-minimal structure $\mathcal{S}$'. For this part we refer the reader to the works [vdD], [C3] and the survey [vdDM].

It is worth saying a few words about the point of view of mathematical logic: o-minimal structures can be introduced in the following way. Given a family of functions (the 'vocabulary' of a language) $\mathcal{F} = \{\mathcal{F}_n\}_{n\in\mathbb{N}}$, $\mathcal{F}_n \subset \mathbb{R}^{\mathbb{R}^n}$, one considers the sets described by first-order formulæ, or, in other words, by the 'operations' $=$, $<$, $+$, $\cdot$ and quantifiers applied to functions from $\mathcal{F}$ or real numbers. The collection of all the sets obtained in this way in the spaces $\mathbb{R}^n$ is the structure denoted by $\mathbb{R}_{\mathcal{F}}$. To be more precise, a subset of $\mathbb{R}^m$ is said to be definable in $\mathbb{R}_{\mathcal{F}}$, if it belongs to the smallest collection of subsets of $\mathbb{R}^n$, $n \in \mathbb{N}$, which

(1) contains the graphs of addition and multiplication, and all the graphs of functions in $\mathcal{F}$, and of constant maps;
(2) contains the graph of the order relation $<$, and of the equality;
(3) is closed under taking Cartesian products, finite unions or intersections, complements, and images under linear projections.

As earlier, a function $f\colon \mathbb{R}^m \to \mathbb{R}^n$ is said to be definable if its graph is definable. If each definable set has finitely many connected components, then $\mathbb{R}_{\mathcal{F}}$ is o-minimal.

The model theoretic notion of the structure $\mathbb{R}_{\mathcal{F}}$ generated by $\mathcal{F}$ provides useful information about the real geometry of the sets and functions obtained this way. The starting point of this approach is the question of how much we have to extend a given language in order to describe the solutions of systems of differential equations written in it, for instance: to what class does the solution of analytic differential equations belong?

Note that functions of one variable are particularly important since they carry most of the information about the structure (in some sense the whole structure is obtained through projections of graphs).

For $\mathcal{F} = \varnothing$, the structure $\mathbb{R}_{\varnothing}$ is just the class of semi-algebraic sets studied already by Tarski. The o-minimality of such a structure $\mathbb{R}_{\mathcal{F}}$ means precisely that all its sets have a finite number of connected components. This fact is important e.g. for differential equations as it excludes oscillations. If we take $\mathcal{F}$ to be the convergent power series in a given polidisc, extendable by zero outside it ([3]), then $\mathbb{R}_{\mathcal{F}}$ is usually denoted $\mathbb{R}_{\mathrm{an}}$ (*restricted analytic functions*). It is model complete (it follows from [G], see below for this notion) and contains all the *globally* subanalytic sets (of which we will speak later on). The structure $\mathbb{R}_{\mathrm{Pfaff}}$ generated by the so-called Pfaffian functions (see later on) is o-minimal as well (cf. [W2]). This implies the o-minimality of $\mathbb{R}_{\exp}$ which is the structure generated by the exponential function.

One more remark: among the first four axioms of a structure on $\mathbb{R}$ the difficulties arise mostly for two of them, namely the projection property (4) (or *elimination of quantifiers*) and the operation of taking the complement in (2). The projection property is what is missing for semi-analytic sets (see Example 4.1) and thus the larger class of subanalytic sets is needed, but when these were introduced, the problem with axiom (2) appeared: how to prove that the complement of a subanalytic set is again subanalytic? This was solved first by A. Gabrielov [G]. That property is called *model completeness* of the structure (notion introduced by A. Robinson). In other words, if in the definition of $\mathbb{R}_{\mathcal{F}}$ the operation of taking the complement is superfluous, the structure is said to be model complete.

The most important tool from the geometric point of view is the *cell decomposition*:

Definition 1.14. A set $C \subset \mathbb{R}^m$ is called a *definable cell* if
(1) for $m = 1$, C is a point or an open, nonempty interval;
(2) for $m > 1$,

- either $C = f$ is the graph of a continuous, definable function $f\colon C' \to \mathbb{R}$, where $C' \subset \mathbb{R}^{m-1}$ ($\mathbb{R}^{m-1}$ is the subspace of the first $m-1$ variables in $\mathbb{R}^m$) is a definable cell; such a cell we shall call *thin*;
- or $C = (f_1, f_2)$ is a definable *prism*, i.e. $(f_1, f_2) = \{(x,t) \in \mathbb{R}^{m-1} \times \mathbb{R} \mid x \in C',\ f_1(x) < t < f_2(x)\}$, where $C' \subset \mathbb{R}^{m-1}$ is a definable cell and both functions $f_j\colon C' \to \mathbb{R} \cup \{\pm\infty\}$ are continuous, definable and such that $f_1 < f_2$ on C' and each f_j either takes all values in $\mathbb{R}$, or is constant.

Definition 1.15. Let $C \subset \mathbb{R}^{n+1}$ be a definable cell over a cell $C' \subset \mathbb{R}^n$. Then its dimension $\dim C$ is defined to be either $\dim C'$, if C is thin, or $\dim C' + 1$ if C is a prism. Of course, in $\mathbb{R}$, $\dim\{a\} = 0$ and $\dim(a,b) = 1$.

It is easy to check that for a cell $C \subset \mathbb{R}^n$ one has $\dim C = n$ iff C is open and $\dim C < n$ iff C is nowhere-dense. Moreover, there is always a definable homeomorphism sending C, call it h_C, on an open cell in $\mathbb{R}^{\dim C}$.

Definition 1.16. A cell C defined over a cell C' is said to be of class $\mathscr{C}$ ([4]), if for the defining function f, or f_i respectively, the composition $f \circ h_{C'}^{-1}$ ($f_i \circ h_{C'}^{-1}$ respectively) is of that class ([5]).

Definition 1.17. A cylindrical cell decomposition of $\mathbb{R}^{n+1}$ is a finite decomposition of $\mathbb{R}^{n+1}$ into pairwise disjoint cells whose projections onto the first n coordinates yield a cylindrical cell

[3] To be more precise: $\mathcal{F}_n$ consists of functions $f\colon \mathbb{R}^n \to \mathbb{R}$ which are analytic in $[-1,1]^n$ and vanish off this cube.

[4] e.g. class $\mathscr{C}^k$ with $k = 1, 2, \ldots, \infty, \omega$ (where ω means analycity)

[5] In particular, a $\mathscr{C}^k$ cell is a $\mathscr{C}^k$ submanifold of dimension $\dim C$.

decomposition of $\mathbb{R}^n$. The cell decomposition is said to be of class $\mathscr{C}$ or $\mathscr{C}^k$, $k = 1, 2, \dots, \infty, \omega$, if all the cells are of that class.

A cell decomposition need not be a stratification in the sense of definition 2.23, since the frontier condition of the latter definition may fail to hold. To see this consider the decomposition of $\mathbb{R}^2$ into the following five cells: $C_1 = \{(x, y) \in \mathbb{R}^2 \mid x < 0\}$, $C_2 = \{0\} \times \mathbb{R}$, $C_3 = (0, +\infty) \times \{0\}$, $C_4 = \{x, y > 0\}$ and $C_5 = \{x > 0, y < 0\}$. Then $\overline{C_3} \setminus C_3$ cannot be obtained from the other cells. Turning a cell decomposition into a stratification requires a further refinement.

The following notion is identical with that of definition 2.24.

Definition 1.18. If $A_1, \dots A_n \in \mathcal{S}_n$, then a cell decomposition $\mathcal{C}$ is said to be *compatible* (or *adapted to*) with these sets if for any $C \in \mathcal{C}$ and any i, there is $C \cap A_i \neq \varnothing \Rightarrow C \subset A_i$. In that case each A_i is the union of some cells from $\mathcal{C}$.

Cohen's Lemma 1.7 provides a semi-algebraic cell decomposition of a given semi-algebraic set. The generalization of this to arbitrary o-minimal structure is the following theorem (compare to Theorem 2.25):

Theorem 1.19 (Cylindrical cell decomposition of class $\mathscr{C}^k$)**.** *Given a finite family of definable sets $A_1, \dots, A_n$ and a $k \in \mathbb{N}$ there is always a cylindrical cell decomposition of class $\mathscr{C}^k$ of $\mathbb{R}^n$ compatible with this family.*

Remark 1.20. Until quite recently it has been an open question whether an arbitrary o-minimal structure admits a $\mathscr{C}^\infty$ cell decomposition. The negative answer was given by O. Le Gal and J.-Ph. Rolin in [LGR], where an explicit example is given. Actually, most of the known o-minimal structures on the field $\mathbb{R}$ admit analytic cell decomposition. An earlier result — that the o-minimal structures generated by convenient quasianalytic Denjoy-Carleman classes admit $\mathscr{C}^\infty$ cell decomposition but no analytic cell decomposition was obtained in [RSW]. See also Remark 2.63.

Corollary 1.21. *A definable cell being connected, the theorem above implies that any definable set A has only finitely many connected components* ([6]) *and they all are definable, too (cf. Theorem 1.9). Moreover, they are open-closed in A.*

For a given set $E \subset \mathbb{R}^n$ let $cc(E)$ denote the family of its connected components. If $A \subset \mathbb{R}^m \times \mathbb{R}^n$, then we put $A_x := \{y \in \mathbb{R}^n \mid (x, y) \in A\}$. The following holds:

Theorem 1.22. *For any definable set $A \subset \mathbb{R}^m \times \mathbb{R}^n$ there is an N such that for all $x \in \mathbb{R}^m$, $\#cc(A_x) \leq N$.*

The possibility of obtaining a $\mathscr{C}^k$ cell decomposition for any k is based on the following:

Theorem 1.23. *Let $f\colon \Omega \to \mathbb{R}$ be a definable function on an open set $\Omega \subset \mathbb{R}^n$. Then for each $k \in \mathbb{N}$ there is a closed definable and nowhere-dense set $Z \subset \Omega$ apart from which f is of class $\mathscr{C}^k$.*

In particular:

Theorem 1.24. *For any definable $f\colon A \to \mathbb{R}$, $A \subset \mathbb{R}^n$, and any $k \in \mathbb{N}$, there is a $\mathscr{C}^k$ cell decomposition of $\mathbb{R}^n$, compatible with A and such that on any of its cells contained in A, f is of class $\mathscr{C}^k$.*

[6]Actually, they are even definably arcwise connected.

Remark 1.25. The Cell Decomposition Theorem provides also an interesting and useful observation:

Let $A \subset \mathbb{R}^n$ be definable and let $L \subset \mathbb{R}^n$ be a linear subspace. If for any $a \in \mathbb{R}^n$ the set $A \cap (L+a)$ is nowhere-dense in $L+a$, then A is nowhere-dense.

This clearly follows from the fact that A is nowhere-dense iff it does not contain an open cell and the trace of an open cell on $L+a$ is open.

Definition 1.26. One can define the dimension of a definable set to be

$$\dim A := \max\{\dim C \mid C \text{ is a cell}\colon C \subset A\}.$$

Proposition 1.27. *If $A \subset \mathbb{R}^n$ is definable, then $\dim A = n$ if and only if $\mathrm{int} A \neq \varnothing$ and $\dim A < n$ if and only if A is nowhere-dense. Moreover, for any definable $B \subset \mathbb{R}^m$ one has $\dim A \times B = \dim A + \dim B$; if $m = n$, then $\dim A \cup B = \max\{\dim A, \dim B\}$ and if $B \subset A$, then $\dim B \leq \dim A$. Finally, if $f\colon A \to \mathbb{R}^m$ is definable, then $\dim f(A) \leq \dim A$ ([7]).*

Remark 1.28. One can also prove that there is a definable bijection $f\colon A \to B$ between two given definable sets (in different ambient spaces), then $\dim A = \dim B$.

The next proposition shows how a cell decomposition induces a cell decomposition in subspaces:

Proposition 1.29. *Let $\mathcal{C}$ be a cell decomposition of $\mathbb{R}^m \times \mathbb{R}^n$ and, for $(x,y) \in \mathbb{R}^m \times \mathbb{R}^n$, let $\pi(x,y) = x$. Then*

(1) *$\tilde{\mathcal{C}} := \{\pi(C) \mid C \in \mathcal{C}\}$ is a cell decomposition of $\mathbb{R}^m$;*
(2) *Let $D \in \tilde{\mathcal{C}}$ and let $\mathcal{C}_D := \{C \in \mathcal{C} \mid \pi(C) = D\}$. Then for any $x \in D$ the sections $\{C_x \mid C \in \mathcal{C}_D\}$ are a cell decomposition of $\mathbb{R}^n$ and $\dim C_x = \dim C - \dim D$.*

Finally, o-minimal structures offer the possibility of triangulating definable sets:

Theorem 1.30. *Let $A \subset \mathbb{R}^n$ be a compact definable set and let $B_i \subset A$, $i = 1, \ldots, k$ be definable. Then there is a simplicial complex $\mathcal{K}$, with vertices in $\mathbb{Q}^n$, and a definable homeomorphism $\phi\colon |\mathcal{K}| \to A$ sich that each B_i is a union of images by ϕ of open simplices from $\mathcal{K}$.*

One important fact that excludes from o-minimal structures such an untame behaviour as that of the graph of $\sin 1/x$ is the following theorem:

Theorem 1.31. *Let $A \subset \mathbb{R}^n$ be definable. Then $\dim \overline{A} \setminus A < \dim A$.*

We end with the following useful lemma:

Lemma 1.32 (Curve Selecting Lemma)**.** *If $A \subset \mathbb{R}^n$ is definable and $a \in \overline{A \setminus \{a\}}$, then there is a definable curve $\gamma\colon [0,1) \to \mathbb{R}^n$, homeomorphic on its image and such that $\gamma(0) = a$, $\gamma((0,1)) \subset A$.*

2. Locally semi-algebraic, semi-analytic and subanalytic sets

The properties of locally semi-algebraic, semi-analytic and subanalytic sets are often richer than these of general o-minimal structures. We are now in the local situation. We will still have Boolean algebras with the properties (1), (2), (3) and (5) of the definition of o-minimal structures but the projection property is not satisfied in general without additional hypotheses like the set being bounded in the direction of the projection.

[7] This expresses well the tameness of the topology involved. No pathologies as that of the Peano curve are permitted.

Definition 2.1 (Łojasiewicz)**.** Let $E \subset M$ where M is a real analytic variety ([8]). Then $\dim E = -1$, if $E = \varnothing$, or, if E is nonempty,

$$\dim E = \max\{\dim \Gamma \mid \Gamma \text{ an analytic submanifold}\colon \Gamma \subset E\}.$$

Definition 2.2. A point $a \in E$ is called smooth or *regular*, if $E \cap U$ is an analytic submanifold for some neighbourhood U of a. Then we define $\dim_a E := \dim E \cap U$ (it does not depend on the choice of U).

Remark 2.3. Clearly $\dim E = \max\{\dim_a E \mid a \text{ regular in } E\}$.

In the case of the dimension of a definable set A, we have for any $k \in \mathbb{N}$,

$$\dim A = \max\{\dim \Gamma \mid \Gamma \text{ a definable } \mathscr{C}^k \text{ submanifold}\colon \Gamma \subset A\}.$$

Proposition 2.4. *In any of the classes of sets discussed in this part the assertions of Proposition 1.27 and of Theorem 1.31 remain true.*

2.1. **Semi-algebraic and locally semi-algebraic sets.** An important feature of semi-algebraic functions is that their smoothness implies analycity. Even more, the smoothness of a semi-algebraic function is equivalent to it being an *analytic-algebraic* or *Nash* function (see [Ł1]):

Definition 2.5. An analytic function $f\colon U \to \mathbb{R}$, where $U \subset \mathbb{R}^n$ is open, is called a *Nash function* if for any $x_0 \in U$ there is a neighbourhood $V \ni x_0$ and a non-zero polynomial $P(x,t)$ for which there is $P(x, f(x)) \equiv 0$ in V ([9]).

Example 2.6. The analycity assumption in the definition is better understood in view of the following example of a (semi-algebraic) function $f(t) = \sqrt[3]{t^2}$ for $t \in \mathbb{R}$. The polynomial

$$P(x,y) = y^3 - x^2$$

annihilates the graph, but f is not even differentiable at the origin.

Theorem 2.7 (see [BCR])**.** *Given a semi-algebraic open set $U \subset \mathbb{R}^n$ and a semi-algebraic function $f\colon U \to \mathbb{R}$ the following equivalence holds:*

$$f \text{ is of class } \mathscr{C}^\infty \Leftrightarrow f \text{ is a Nash function.}$$

For what follows we refer the reader to [Ł1] where locally semi-algebraic sets were introduced (later they were known as Nash sets).

Definition 2.8. A locally semi-algebraic set in an open set $\Omega \subset \mathbb{R}^n$ is a set which in a neighbourhood of any point $a \in \Omega$ can be described by a finite number of polynomial equations or inequalities.

Remark 2.9. In particular, any set $E \subset \Omega$ described by Nash functions in an open semi-algebraic set Ω is locally semi-algebraic. This implies that a *semi-Nash set*, i.e., a set described locally by Nash functions, is a locally semi-algebraic set (and vice versa).
Recall that a *Nash submanifold* is a submanifold admitting an atlas of Nash functions. Let us observe that a point of a locally semi-algebraic set is regular if and only if in a small neighbourhood of this point the set is a Nash submanifold.

Proposition 2.10. *For any semi-algebraic set $E \subset \mathbb{R}^n$ there exists an algebraic set $V \subset \mathbb{R}^n$ such that $V \supset E$ and $\dim V = \dim E$.*

[8] In this text 'variety' and 'manifold' mean the same.

[9] If U is connected, it is easy to check that the same polynomial is good at each point, i.e., $P(x, f(x)) \equiv 0$ in the whole of U.

Proposition 2.11. *Each connected Nash submanifold $N \subset \mathbb{R}^n$ which is closed in a semi-algebraic set is semi-algebraic. In particular, the frontier $\overline{N} \setminus N$ is semi-algebraic iff N is semi-algebraic.*

The following proposition provides a link between semi-algebraic and locally semi-algebraic sets:

Proposition 2.12. *If U is an affine chart of $\mathbb{P}_n$ and $A \subset U$, then A is semi-algebraic in U if and only if A is locally semi-algebraic in $\mathbb{P}_n$.*

It can be proved that any semialgebraic function $f\colon \Omega \to \mathbb{R}$, where $\Omega \subset \mathbb{R}^n$ open, is Nash ouside a nowhere-dense semialgebraic set $Z \subset \Omega$. This implies that the category of semi-algebraic sets admits Nash-analytic cell decomposition.

Two more facts about semi-algebraic functions, that we provide with Łojasiewicz's proofs:

Lemma 2.13. *Let $f\colon (a, +\infty) \to \mathbb{R}$ be semi-algebraic. Then for some $b, N > 0$, there is $|f(x)| \leq x^N$, when $x > b$.*

Proof. Write $f = \bigcup_i \bigcap_j \{P_i = 0, Q_{ij} > 0\}$ and observe that due to univalence of the graph, for each i there is $P_i \not\equiv 0$. Let $P = \prod_i P_i$. Since $P(x, f(x)) \equiv 0$, then $f(x)$ is the root of the polynomial $P(x, \cdot)$ with polynomial coefficients $a_i(x)$, $i = 1, \ldots, d$. If $a_0(x)$ is the leading coefficient, then for some $b > 0$ there is $a_0(x) \neq 0$, if $x > b$. Now, $f(x)$ being a root, one has

$$|f(x)| \leq 2 \max_{i=1}^{d} \left(\frac{|a_i(x)|}{|a_0(x)|}\right)^{1/j}, \quad x > b,$$

and the lemma follows. □

Theorem 2.14 (Łojasiewicz's inequality)**.** *If $f, g\colon K \to \mathbb{R}$ are continuous semi-algebraic functions on a compact semi-algebraic set K and $f^{-1}(0) \subset g^{-1}(0)$, then for some $C, N > 0$ there is*

$$|f(x)| \geq C|g(x)|^N, \quad x \in K.$$

Proof. For $t > 0$ let $G_t := \{x \in K \mid t|g(x)| = 1\}$. These are compact semi-algebraic sets. If $G_t \neq \varnothing$, then let $m(t) := \max_{G_t} 1/|f|$, otherwise put $m(t) = 0$. The function $m\colon (0, +\infty) \to \mathbb{R}$ is semi-algebraic and thus by the preceding lemma, $m(t) \leq t^N$ for $t > b$. This means that for all $x \in K$, $|g(x)| \in (0, 1/b)$ implies $|g(x)|^N \leq |f(x)|$. Finally let

$$M := \max\{|g(x)|^N/|f(x)| \mid x \in K\colon |g(x)| \geq 1/b\}$$

and $C := \max\{M, 1\}$. The assertion follows. □

Remark 2.15. Taking $g(x) := \mathrm{dist}(x, f^{-1}(0))$ we obtain the semi-algebraic version of the general Łojasiewicz inequality:

$$(\#) \qquad |f(x)| \geq \mathrm{const.dist}(x, f^{-1}(0))^N, \quad x \in K.$$

On the other hand, by applying the theorem to the functions G and F defined as

$$G\colon K \times K \ni (x, y) \mapsto |f(x) - f(y)|$$

and $F(x, y) = ||x - y||$ we obtain the Hölder continuity of f (with exponent $1/N$).

Corollary 2.16 (Regular separation)**.** *If A, B are compact nonempty semi-algebraic sets, then for some constants $C, N > 0$,*

$$\mathrm{dist}(x, A) \geq C\mathrm{dist}(x, A \cap B)^N, \quad x \in B.$$

Proof. Apply the preceding theorem to $f(x) = \mathrm{dist}(x, A)$ and $g(x) = \mathrm{dist}(x, A \cap B)$. □

Remark 2.17. Both inequalities exclude any kind of *flatness*. In particular regular separation means that the possible tangency of two sets at a common point is not of infinite order.

Example 2.18. The above properties may not be satisfied in general o-minimal structures — for instance, $\mathbb{R}_{\exp}$ contains $\exp(t)$ and $\exp(-1/t^2)$ as definable functions: the first one does not satisfy the inequality in the lemma above, the second one does not satisfy the Łojasiewicz inequality where g is the distance to the origin (neither is its graph regularly separated from its domain).

Let us also note the following theorem, whose direct and elegant proof is presented in [S]:

Theorem 2.19. *Let A be semi-algebraic and let $A^{(k)} = \{x \in A \mid A \cap U$ is a k-dimensional analytic (Nash) manifold for some neighbourhood $U \ni x\}$. Then $A^{(k)}$ is semi-algebraic. In particular, the set of singular (i.e., non regular) points is semi-algebraic of dimension $< \dim A$.*

Remark 2.20. Finally, observe that for $\mathbb{R}^n$ the semi-algebraic homeomorphism $h(x) = x/(1+||x||)$ sends any semialgebraic set onto a semi-algebraic bounded set. This remark is important in view of the fact that subanalytic sets form an o-minimal structure only if we restrict ourselves to those of them which are 'bounded at infinity'. In that case we have of course an analogy between that class of sets (considered already in [T]) and semi-algebraic sets. See Definition 2.59.

2.2. **Semi-analytic sets (Łojasiewicz 1965).**

Definition 2.21. A set $A \subset \mathbb{R}^n$ (or, more generally $A \subset M$, where M is an analytic variety) is called *semi-analytic*, if for any $x \in \mathbb{R}^n$, there are a neighbourhood $U \ni x$ and analytic functions f_i, g_{ij} in U such that

$$A \cap U = \bigcup_{i=1}^{p} \bigcap_{j=1}^{q} \{x \in U \mid f_i(x) = 0, g_{ij}(x) > 0\}.$$

A mapping $f\colon E \to \mathbb{R}^n$ with $E \subset \mathbb{R}^m$ is said to be semi-analytic if its graph is a semi-analytic set in $\mathbb{R}^{m+n}$.

Example 2.22. Note that the description is *local* but not in the sense that we are moving along the set in question. The difference is better understood on the following example:
the graph $G := \{(x, \sin(1/x)) \mid x > 0\}$ is semi-analytic in $\mathbb{R}_+ \times \mathbb{R}$ but not in the whole of $\mathbb{R}^2$ because no point $(0, y)$ with $|y| \leq 1$ has a neighbourhood in which G can be described by a finite number of analytic equations and inequalities.

It is easy to check that the sets semi-analytic in a given analytic manifold form a Boolean algebra. Moreover, the union of a locally finite family of semi-analytic sets and the pre-image of a semi-analytic set by a semi-analytic mapping are semi-analytic. Semi-analytic sets have almost all the nice properties of semi-algebraic sets except that they need not be stable under proper projections.

The theory of semi-analytic and subsequently subanalytic sets originates in Łojasiewicz's solution to Laurent Schwartz's famous Division Problem (1957), see [Ł2] for an account. S. Łojasiewicz was the first person who meticulously built the fully systematized theory of semi-analytic sets (as in his preprint [Ł1]), using normal partitions which are a very clever tool, being a particular instance of a stratification:

Definition 2.23. A family of submanifolds of a manifold M is called a *stratification* of M if

- M is the union of the sets of the family
- the family is locally finite,
- the sets of the family are pairwise disjoint,

- for any leaf (or stratum) Γ belonging to this family, its frontier $\overline{\Gamma}\setminus\Gamma$ is the union of some members of the family with dimensions strictly smaller than $\dim\Gamma$.

Definition 2.24. Let f be a function of nonvanishing germ at a, a point of a real analytic manifold M. A stratification of a neighbourhood of a is said to be *compatible with* f if on any leaf of the stratification either $f\equiv 0$, or $f\neq 0$.
Let $E\subset M$. A stratification $\mathcal{N}$ is *compatible with the set* E if, for any stratum $\Gamma\in\mathcal{N}$, either $\Gamma\subset E$, or $\Gamma\cap E=\varnothing$ ([10]).

In 1965, Łojasiewicz presented a construction of the so called *normal partitions* which are special stratifications of *normal neighbourhoods*. The normal neighbourhoods form a topological basis of neighbourhoods. The normal partition of a neighbourhood starts with choosing the direction that is good for the Weierstrass Preparation Theorem and replacing the zeroes of an analytic germ by the zeroes of a distinguished polynomial. Then the construction goes down. At each step a good direction must be chosen (this makes the construction non-explicit), the distinguished polynomials are complexified and their determinants are studied in order to control multiple zeroes. All this ends up as a very detailed stratification called normal partition. For a thorough construction, consult [Ł1] and [DS1].

Theorem 2.25 (Łojasiewicz)**.** *Let $f_1,\dots,f_r$ be analytic functions defined in a neighbourhood of the origin of a finite dimensional real vector space. Then there exists a normal partition $\mathcal{N}$ at 0 compatible with $f_1,\dots,f_r$. (The same is true on any real analytic manifold.)*

Normal partitions play a crucial role in the theory of semi-analytic sets. The striking fact about the normal partitions is that the existence of such a partition compatible with a given set is a necessary and sufficient condition for the set to be semi-analytic:

Theorem 2.26 ([Ł1])**.** *A set $E\subset M$ is semi-analytic if and only if at any point $a\in M$ there is a normal partition compatible with E.*

Remark 2.27. Of course, given a finite family of semi-analytic sets in a real analytic manifold we can always find a normal partition compatible with them, which is just a restatement of Theorem 2.25.

Normal partitions are also used to prove the semi-analytic version of the Bruhat-Cartan-Wallace *Curve Selecting Lemma*:

Lemma 2.28 (Semi-analytic curve selecting lemma)**.** *Let E be a semi-analytic set and suppose that $a\in\overline{E\setminus\{a\}}$. Then there exist an analytic function $\gamma\colon(0,1)\to E$ yielding a semi-analytic curve and such that $\lim_{t\to 0+}\gamma(t)=a$.*

As the construction of normal partitions is somehow tiring, this strong (but elementary) tool was used almost uniquely by Polish mathematicians, with one important exception: Pfaffian varieties, the theory of which started in Dijon (see section 3).

Although the distance to a semi-analytic set need not be semi-analytic (it is subanalytic — see last section Theorem 4.3) we have the following:

Theorem 2.29. *The statement of Corollary 1.12 is true in the semi-analytic category. Moreover, the Łojasiewicz inequalities 2.14 and (#) as well as the regular separation 2.16 and Hölder continuity hold for semi-analytic sets.*

To finish this part let us quote two important theorems:

[10] In other words, $\Gamma\cap E\neq\varnothing\Rightarrow\Gamma\subset E$, just as in Definition 1.18.

Theorem 2.30 (Łojasiewicz)**.** *For any semi-analytic set A, the family $cc(A)$ is locally finite and each component $C \in cc(A)$ is semi-analytic.*

Theorem 2.31 (Łojasiewicz)**.** *An obvious analogon of Theorem 2.19 holds for semi-analytic sets.*

2.3. **Subanalytic sets (1975).** For this part we refer the reader to [DS1] for the most detailed presentation. Otherwise, there are: [BM] (a much more concise presentation but including an elementary approach to uniformization), and still less detailed, Łojasiewicz's book [ŁZ] written in Spanish and Łojasiewicz's short survey [Ł2]. And of course there is the preprint of H. Hironaka presenting his approach via desingularization [H2].

After completing the theory of semi-analytic sets in 1965 S. Łojasiewicz tried to study the projections of relatively compact semi-analytic sets but was stopped by the difficulty of the theorem of the complement.

The theorem of the complement was finally proved, independently, by H. Hironaka and A. Gabrielov. For H. Hironaka the theory of subanalytic sets was a kind of by-product of his famous desingularization theorem (compare [H1]). Gabrielov in [G] proved the theorem in an elementary way, reducing it to the study of complements of the graphs of functions. S. Łojasiewicz decided to build the theory of subanalytic sets from a scratch, using normal partitions and an idea of René Thom, which was later given the name of *Fibre-Cutting Lemma*.

Many mathematicians proved very interesting subanalytic results using Hironaka's approach. Let us quote M. Tamm or R. Hardt and his very interesting stratification theorems [Ht1]. All theorems about subanalytic sets can be obtained by Łojasiewicz's methods, too. They are gathered in [DS1].

Definition 2.32. A set E in a real analytic variety M is called *subanalytic* if for any $x \in M$ there is a neighbourhood $U \ni x$ such that $E \cap U = \pi(A)$, where $\pi\colon M \times N \to M$ is the natural projection, N is a real variety and A is semi-analytic and relatively compact in $M \times N$.

Remark 2.33. Projections of semi-analytic sets need not be subanalytic even if the sets are relatively compact and the projections are *proper*([11]) — see Example 4.1. That is a major difference with the definable case that should be borne in mind.

Remark 2.34. The union of a locally finite family of subanalytic sets and the intersection of a finite family of subanalytic sets are subanalytic.

Let us speak now about a very useful concept of S. Łojasiewicz, namely *N-relatively compact* sets and their projections.

Definition 2.35 ([Ł1])**.** Let M, N be two analytic varieties and let $\pi\colon M \times N \to M$ the natural projection. A subset $E \subset M \times N$ is called *N-relatively compact* if for any $A \subset M$ relatively compact the set $(A \times N) \cap E$ is relatively compact, too.

Remark 2.36. If the set E in the definition above is subanalytic in $M \times N$, then $\pi(E)$ is subanalytic, too.

Definition 2.37. A map $f\colon E \to N$, where $E \subset M$ is a nonempty subanalytic set, is subanalytic iff its graph f is subanalytic in $M \times N$.

Note that the domain of a subanalytic map need not be subanalytic, especially if its graph is not N-relatively compact.

[11] The pre-image of any compact set is compact

Definition 2.38. A map $f \subset M \times N$ is said to be *h-relatively compact* if the pre-image of any relatively compact subset of N is relatively compact. The map f is called *v-relatively compact* if the image of any relatively compact subset of M is relatively compact ([12]).

Remark 2.39. f is h-relatively compact iff its graph est M-relatively compact and f is v-relatively compact iff its graph is N-relatively compact. If f is proper, then it is h-relatively compact and each continuous $f\colon M \to N$ with closed domain is v-relatively compact.

Proposition 2.40. *Let $f \subset M \times N$ be a map and E a subanalytic subset of N. Any of the following conditions guarantees that $f^{-1}(E)$ is subanalytic:*

(a) *f is subanalytic v-relatively compact ([13]),*
(b) *E is relatively compact and f is subanalytic.*

Proof. Observe that $f^{-1}(E) = \pi(f \cap (M \times E))$, where $\pi\colon M \times N \to M$ is the natural projection, and apply 2.36. □

Remark 2.41. Note that in o-minimal structures the assertion holds without any extra assumptions on the definable function f.

Proposition 2.42. *Let $f \subset M \times N$ be a map and H a subanalytic subset of M. Then any of the following conditions implies that $f(H)$ is subanalytic in N:*

(a) *f is subanalytic h-relatively compact ([14]);*
(b) *H is relatively compact and f is subanalytic;*
(c) *H is relatively compact and f is analytic in a neighbourhood of $\overline{H}$;*
(d) *f is analytic in a neighbourhood of $\overline{H}$ and $f|_{\overline{H}}$ is proper.*

Proof. It suffices to apply Remark 2.36 and observe that, if $\pi\colon M \times N \to N$ is the natural projection, then $f(H) = \pi(f \cap (H \times N))$. □

We give below three other definitions of subanalytic sets (they all are equivalent):

Definition 2.43. A subset E of a real analytic variety M is called *subanalytic* if for each $x \in M$ there is a neighbourhood V such that $E \cap V$ is the image of a semi-analytic set by a proper analytic mapping.

Proposition 2.42 implies that this definition is equivalent to the previous one.

Theorem 2.44 (Gabrielov)**.** *If $E \subset M$ is subanalytic, then so is $M \setminus E$.*

Proposition 2.45. *Basic properties of a subanalytic set $E \subset M$:*

- *The closure and thus the interior (cf. Gabrielov's Theorem) of a subanalytic set are subanalytic.*
- *The connected components $C \in cc(E)$ are all subanalytic.*
- *The family $cc(E)$ is locally finite in M.*
- *If E is relatively compact, then $\#cc(E) < \infty$.*
- *E is locally connected.*
- *If $F \subset E$ is open-closed in E, then it is subanalytic.*
- *The Curve Selecting Lemma holds for subanalytic sets: if $a \in \overline{E \setminus \{a\}}$, then there is an analytic function $\gamma\colon (-1, 1) \to M$ such that $\gamma(0) = a$ and $\gamma((0, 1)) \subset E$. Moreover, γ is a homeomorphism on its image $\Gamma_{\gamma|_{[0,1)}}$ which is a semi-analytic arc of class $\mathscr{C}^1$.*

[12] h comes from 'horizontally', while v stands for 'vertically', cf. one looks 'through' the graph.
[13] This is the case if e.g. f is analytic in M.
[14] This is the case if e.g. f is analytic in M and proper.

The proofs are based on the analoguous properties of semi-analytic sets and the Fibre-cutting Lemma (Lemmata A and B below).

Proposition 2.46. *Basic properties of subanalytic functions:*

- *The composition $g \circ f$ of subanalytic functions is subanalytic provided that either f is v-relatively compact, or g is h-relatively compact.*
- *If $f_i \dots, f_k \colon A \to N_i, i = 1, \dots, k$ are subanalytic, then the mapping*
$$(f_1, \dots, f_k) \colon A \to N_1 \times \dots \times N_k$$
is subanalytic, too.
- *The sum, the product and the quotient of real subanalytic functions defined on M is subanalytic, provided they are all locally bounded.*

Remark 2.47. Similar properties are satisfied by definable functions without extra assumptions. Note in particular that the composition of two subanalytic functions need not be subanalytic. The apparent analogy to semi-algebraic geometry or o-minimal structures is responsible for the fact that authors that use the subanalytic theory are often oblivious to that subtlety.

Definition 2.48. A *semi- or subanalytic leaf* in M is any analytic submanifold of M which is at the same time a semi- or, respectively, subanalytic set.

Example 2.49. The graph of $y = \sin 1/x$ is not subanalytic in the plane (note that the dimension of its frontier is again 1 which would be impossible for a subanalytic set, as Theorem 1.31 holds in the subanalytic category) although it is an analytic submanifold of it.

The following theorem of Łojasiewicz plays an important role in his theory of subanalytic sets without desingularization:

Theorem 2.50 (Łojasiewicz)**.** *Let Γ be a semi-analytic leaf in an affine space X. Denote by $G_k(X)$ the kth Grassmannian of X. Let $\tau \colon \Gamma \ni x \mapsto T_x\Gamma \in G_k(X)$, where $k = \dim \Gamma$, be the tangent mapping ($T_x\Gamma$ is the tangent space at x). Then for any semi-algebraic set $E \subset G_k(X)$, the pre-image $\tau^{-1}(E)$ is semi-analytic in X.*

For the subanalytic generalization see Theorem 4.23.

The key role in the subanalytic theory is played by the following lemmata suggested by René Thom (see [DLS]):

Lemma (A) (Decomposition)**.** *Let A be a semi-analytic, relatively compact subset of real, finite-dimensional vector space X. Assume that $X = U \oplus V$ is the direct sum of two vector spaces and let $\pi \colon X \to U$ be the projection parallel to V. Assume that $G_k(X)$ is decomposed into a finite number of open semi-algebraic sets: $G_k(X) = \bigcup G_i^{(k)}$. Then there exists a finite family of semi-analytic leaves $\{\Gamma_j\}$ such that $A = \bigcup \Gamma_j$ and*

(1) *the rank* rk π_{Γ_j} *is constant on each Γ_j,*
(2) *the Γ_j are members of some normal partitions,*
(3) *for any j there is an i such that $\tau(\Gamma_j) \subset G_i^{(k)}$ where $k = \dim \Gamma_j$.*

Lemma (B) (Replacement)**.** *Let A, X, U, V, π and $G_i^{(k)}$ be as in Lemma A. Then there is a finite family of semi-analytic leaves $\{\Gamma_j\}$ such that $\Gamma_j \subset A, \pi(A) = \pi(\bigcup \Gamma_j)$ and*

(1) *for any j, π_{Γ_j} is an immersion,*
(2) *the Γ_j are members of normal partitions,*
(3) *for any j there is an i such that $\tau(\Gamma_j) \subset G_i^{(k)}, k = \dim \Gamma_j$.*

Remark 2.51. If E is semi-analytic, then $\tau^{-1}(E)$ is only subanalytic (see 4.23), but in case where Γ is semi-algebraic, τ is semi-algebraic as well.

Hironaka started his theory with a different definition of subanalytic sets:

Definition 2.52. A set E is called *subanalytic* if for any point of M there is a neighbourhood V such that

$$E \cap V = \bigcup_{i=1}^{p} f_{i1}(A_{i1}) \setminus f_{i2}(A_{i2})$$

where f_{ij} are analytic and proper and A_{ij} are analytic sets.

The fourth definition of subanalytic sets is:

Definition 2.53. A subset $E \subset M$ is called *subanalytic in* M if for any point of M there is a neighbourhood V such that

$$E \cap V = \bigcup_{i=1}^{p} f_{i1}(M_{i1}) \setminus f_{i2}(M_{i2}),$$

with $f_{ij} \colon M_{ij} \to M$ analytic and proper, and this time M_{ij} analytic varieties.

Theorem 2.54 (see [DS1] for a proof)**.** *All four definitions of subanalytic sets are equivalent.*

Finally let us recall other important theorems:

Theorem 2.55 (Łojasiewicz)**.** *The Łojasiewicz inequality 2.14 and (#) as well as the regular separation 2.16 and Hölder continuity of functions hold for subanalytic sets.*

Theorem 2.56 (Gabrielov)**.** *Let $E \subset M \times N$ be a relatively compact subanalytic set, where M, N are analytic varieties. Then there is a constant N such that $\#cc(E_x) \leq N$ for all $x \in M$.*

A deep result of W. Pawłucki below is a subanalytic version with parameter of the well-known complex Puiseux Theorem:

Theorem 2.57 ([P1])**.** *Let X, Y be two real, finite-dimensional vector spaces, Γ a subanalytic leaf relatively compact in X, $\Theta \colon \Gamma \times (0,1) \to Y$ an analytic map which is subanalytic in $X \times \mathbb{R} \times Y$ and bounded.*

Then there exists a closed subanalytic set $E \subset \Gamma$, $\dim E < \dim \Gamma$ and $k \in \mathbb{N}$ such that: for all $a \in \Gamma \setminus E$ the map $(x,t) \mapsto \Theta(x, t^k)$ has an analytic extension to a neighbourhood of $(a, 0)$ in $\Gamma \times \mathbb{R}$.

Using this K. Wachta obtained an important version of the Curve Selecting Lemma 2.28 for open subanalytic sets:

Theorem 2.58 (Wachta)**.** *Let $E \subset \mathbb{R}^n$ be an open subanalytic set and $a \in \overline{E}$. Then the arc from the Curve Selecting Lemma can be chosen semi-algebraic (i.e., Nash).*

Of course, the openness assumption is unavoidable due to the existence of transcendental curves.

At this point we stress again the fact that subanalytic sets do not form an o-minimal structure ([15]). They will, if we restrict ourselves to the so-called *globally (or totally) subanalytic sets*:

[15]The difference in behaviour of subanalytic and definable sets may be illustrated by the main result of [Di], see the last section.

Definition 2.59. A subanalytic set $E \subset \mathbb{R}^n$ is called *globally subanalytic* if its image by the semialgebraic homeomorphism $h(x) = x/(1 + ||x||)$ sending it to the unit Euclidean ball is subanalytic.

Theorem 2.60. *Globally subanalytic sets form an o-minimal structure which coincides with* $\mathbb{R}_{\text{an}}$.

Remark 2.61. The same class of sets is obtained starting from functions *subanalytic at infinity* (see [T], see also [DS1]), i.e., such subanalytic functions $f\colon M \to \mathbb{R}$ which are subanalytic in $M \times \mathbb{S}^1$.

We end with a very useful lemma of K. Kurdyka, generalizing a result of M. Tamm (for $\mathscr{C}^k$), and its application:

Lemma 2.62 (Kurdyka)**.** *Let $f\colon U \to \mathbb{R}$ be a function subanalytic at infinity, $U \subset \mathbb{R}^n$ an open set. Then there is $k \in \mathbb{N}$ such that for any $x \in U$, if f is of Gâteaux class $\mathscr{G}^k$ ([16]) in a neighbourhood of x, then f is analytic at x.*

Remark 2.63. In connection with Remark 1.20 we may observe that this lemma readily implies that the structure $\mathbb{R}_{\text{an}}$ admits analytic cell decomposition (compare Theorem 1.19).

This was used by Kurdyka to obtain a desingularization-free proof of the following:

Theorem 2.64 (Tamm [T])**.** *For any subanalytic set E the set of singular points $E \setminus \mathrm{Reg}E$ is subanalytic of dimension strictly smaller than* $\dim E$.

Remark 2.65. There is no direct counterpart of the subanalytic Puiseux Theorem or the lemma above in general o-minimal structures (a necessary condition would be their polynomial boundedness, cf. Definition 4.5). Tamm's Lemma can be extended to the structure $\mathbb{R}_{\text{an},f_r,r\in\mathbb{R}}$ defined by the restricted analytic functions together with $f_r(t) = t^r$ for $t > 0$ and $f_r(t) = 0$ for $t \leq 0$. This implies analytic cell decomposition in the structure. See [vdDM] for details.

3. Pfaffian varieties and subpfaffian sets

This case is treated separately because it is much more recent than those dealt with in sections 1 and 2 and has an interesting history, often forgotten when Pfaffian sets are considered only on the ground of the model theory.

Subanalytic sets are insufficient for studying, for instance, the problems that arise in differential equations. Let us quote the following example from [MR]:

$$\begin{cases} \dot{x} = y, \\ \dot{y} = x^2. \end{cases}$$

The solutions of such a simple polynomial system are flat functions const.$\exp(-1/x)$ which are not subanalytic, but still quite regular, not to mention the fact that they arose from a simple polynomial dynamical system.

Outside France the history of Pfaffian varieties and the context in which they were born are totally unknown. And this despite the fact that [Ho2] contains a good historical introduction about how Pfaffian, semi-Pfaffian and sub-Pfaffian sets came into being. It all started with

[16] Recall that a function $f\colon U \to \mathbb{R}$ with $U \subset \mathbb{R}^n$ open is of class $\mathscr{G}^k$ in U if at any point $x \in U$ f possesses its kth Gâteaux derivative: for any $h \in \mathbb{R}^n$, the function $t \mapsto f(x + th)$ is k times differentiable at zero and the kth derivative is a homogeneous polynomial of degree k in h.

Hilbert XVIth problem and the works of Khovanskiĭ (see [Kh], [MR], [W2]). Hilbert XVIth problem deals with polynomial dynamical systems in the plane:

$$(PDS) \qquad \begin{cases} \dot{x} = P(x,y), \\ \dot{y} = Q(x,y), \end{cases}$$

and the question whether their *limit cycles* (closed trajectories that are isolated in the set of all closed trajectories of the system) can accumulate.

Extensive work was done on the subject in France and in Russia in the late 80's. Let us recall the names of Ilyashenko and Trifonov as well as those of Roussarie, Moussu, Ecalle and Ramis. Hilbert's question went further (Hilbert wanted to obtain a formula relating the maximal number of limit cycles to the degrees of P and Q above) but just their non-accumulation was a very difficult problem. As limit cycles can only accumulate on limit sets (cf. e.g. [DR]), it is possible to write a generalization of the classical Poincaré map, called the map of first return as it associates to the starting point (time t) the point of the first return to the curve we chose as transversal to the limit set, $\gamma(t)$. Back in 1988 R. Moussu started studying the properties of such mappings in order to show that $\gamma(t)-t$, even when it is not analytic, has no accumulation of zeroes. The map $\gamma(t)$ is seldom subanalytic — it often comes out infinitely flat. The idea was then to show it cannot oscillate.

Since solving (PDS) is equivalent to studying $\omega = 0$, where ω is the differential form (i.e., Pfaffian form) $\omega = -Q(x,y)dx + P(x,y)dy$, the notion of Pfaffian varieties was introduced by R. Moussu and C. Roche and studied, initially by Moussu, Roche and J.-M. Lion. There is a very good survey about that written by Moussu [M], based on [MR].

Definition 3.1. A *Pfaffian hypersurface* in $\mathbb{R}^n$ is a triplet (V,ω,M), where $M \subset \mathbb{R}^n$ is open and semi-analytic, ω is an analytic one-form defined on a neighbourhood of $\overline{M}$ and V is a maximal integral variety of $\omega = 0$ in M, smooth and of codimension 1 ([17]).

In other words we are given a codimension one foliation of a neighbourhood of $\overline{M}$ having V as one of its leaves and no singularities on M.

Definition 3.2. Let $X \subset \mathbb{R}^n$. (V,ω,M) is *of Rolle in* X (or just *of Rolle*, if $X = M$), if for any analytic $\gamma\colon [0,1] \to X \cap M$ there is a $t \in [0,1]$ such that $\gamma'(t) \in \mathrm{Ker}\omega(\gamma(t))$ ([18]).

In other words, any analytic path in $X \cap M$ connecting two points of V is tangent at some point to the field of hyperplanes defined by $\omega = 0$. In particular this excludes spiralling.

Definition 3.3. A Pfaffian hypersurface (V,ω,M) is *separating*, if the complement $M \setminus V$ has exactly two connected components whose common border in M is V.

By a theorem of Khovanskiĭ, a separating Pfaffian hypersurface is always of Rolle. The converse is not true as can be seen by considering $M = \mathbb{R}^2 \setminus \{0\}$ and $\omega = x^2dy - ydx$. Any integral curve of $\omega = 0$ is a Pfaffian hypersurface of Rolle and thus in particular the graphs of const.exp$(-1/x)$, $x > 0$. But their complement in M is connected. Besides, that example shows that in general V is just an analytic immersed submanifold which is not semi-analytic in $\mathbb{R}^n$.

Theorem 3.4 ([MR]). *Let $S(\omega) = \{x \mid \omega(x) = 0\}$ be the singular locus of ω. If $M \setminus S(\omega)$ is simply connected, then any Pfaffian hypersurface (V,ω,M) is of Rolle. If ω is integrable* ([19])*, then for any Pfaffian hypersurface (V,ω,M), V is a leaf of the foliation defined by ω.*

[17] That is to say: $\omega(x) \neq 0$ if $x \in V$, $\mathrm{Ker}\omega = T_xV$ and V is the maximal variety with this property among all the connected immersed subvarieties of M.

[18] In some sense that is an inverse approach to the classical Rolle Theorem: think of $\omega = dy$ in $\mathbb{R}^2$ and any differentiable function $y = \gamma(x)$ such that e.g. $\gamma(0) = \gamma(1) = 1$ — at some point t there is $\gamma'(t) = 0$.

[19] In the sense that $\omega \wedge d\omega = 0$ cf. the Frobenius Theorem.

Theorem 3.5 ([MR])**.** *Let $X \subset \mathbb{R}^n$ be semi-analytic and bounded and let $\omega_1, \ldots, \omega_k$ be analytic one-forms in a neighbourhood of $\overline{M}$, where $M \subset \mathbb{R}^n$ is an open semi-analytic set. Then there exists a natural number $b = b(M, X, \omega_1, \ldots, \omega_k)$ such that $\#cc(X \cap V_1 \cap \ldots \cap V_k) \leq b$, where (V_i, ω_i, M) are Pfaffian hypersurface of Rolle.*

Remark 3.6. The last theorem implies Łojasiewicz's Theorem bounding the number of connected components of the sections of a semi-analytic set.

The interesting point here is that this is the only case of applications of Łojasiewicz's normal partitions outside Poland. Despite the fact that Łojasiewicz did this work in France (his preprint was published in 1965 by IHES), the normal partitions were almost exclusively used in Poland. Applying them to study the sets that appear as solutions of differential equations was, indeed, a very original, ingenuous and unexpected way to use them.

This happened before the o-minimal structures were introduced.

Lion and Rolin [LR] proved that relatively compact Rolle (i.e., non-spiralling) leaves of a real analytic foliations belong to a class of stratifiable subsets of $\mathbb{R}^n$ which is stable under intersection, union, set difference, linear projections and closure. That means that Rolle leaves belong to an o-minimal structure.

The basic properties of Pfaffian hypersurfaces are all gathered (with proofs) in the article of R. Moussu and C. Roche. Later, numerous other extremely useful properties of Pfaffian sets were proved. For instance Lion [L] showed, (with the use of Łojasiewicz's normal partitions) that there is a semi-analytic stratification of a neighbourhood of each point $a \in \mathbb{R}^n$, compatible with an analytic differential one-form ω and a semi-analytic open set M. This stratification allows a local decomposition of every integral hypersurface V of $\omega = 0$ into 'plaques' . Every leaf is the graph of an analytic function and if a is in the closure of a leaf, then a Pfaffian curve ending in a with a tangent lies in V. Lion and Roche obtained a Pfaffian Curve Selecting Lemma and then Lion proved a Pfaffian version of the Łojasiewicz inequality.

A natural thing is to construct *subpfaffian* sets starting from *semipfaffian* sets defined using intersections of leaves of Pfaffian foliations with the strata of Łojasiewicz's normal partitions (just like it was done for subanalytic sets). This way of proceeding originates in a question asked by R. Moussu and M. Shiota — what do we obtain by adding to the class of subanalytic sets the solutions of Pfaffian equations? And this is how the whole theory is presented in the interesting paper [Ho1]. (In what follows we can replace $\mathbb{R}^n$ by an analytic manifold N.) Semipfaffian geometry was suggested already by [L] or [MR]. In [Ho1] Z. Hajto proved a kind of analog of Gabrielov theorem on the complement 2.44. We present it hereafter.

Definition 3.7. A normal partition $\mathcal{N}$ is said to be *strongly adapted* to a finite family of Pfaffian hypersurfaces $\mathcal{V} := \{(V_i, \omega_i, M_i),\}_{i\in I}$ if it is adapted to $\{M_i\}_i$ and any subfamily of $\{\omega_i\}$ in the sense that for any leaf $\Gamma \in \mathcal{N}$ there are $\omega_{i_1}, \ldots, \omega_{i_k}$ forming a base at each point $x \in \Gamma$ for the linear span (in $(\mathbb{R}^n)^*$) of $\{\omega_i(x)\}$.

Then by [L], for any leaf $\Gamma \in \mathcal{N}$ such that all the hypersurfaces from $\mathcal{V}$ is of Rolle for paths in Γ, the collection $\mathcal{V}_\Gamma := \{\bigcap_{i\in J} V_i \cap \Gamma\}_{J\subset I}$ is a finite family of analytic submanifolds with normal crossings in Γ; we call them *Pfaffian leaves.* These induce a stratification of Γ when we consider the connected components of $N_k \setminus N_{k-1}$ (with $N_{-1} = \varnothing$) where $N_k = \bigcup\{L \in \mathcal{V}_\Gamma \mid \dim L \leq k\}$, $k = 0, \ldots, \dim \Gamma$. These connected components are called *semi-pfaffian leaves.*

Definition 3.8. A subset $E \subset \mathbb{R}^n$ is *semi-pfaffian* (respectively: *basic semi-pfaffian*) if at every point $a \in \mathbb{R}^n$ there is a finite family of Pfaffian hypersurfaces $\mathcal{V}$ defined for neighbourhoods $M_i \ni a$ and a normal partition $\mathcal{N}$, defined in a normal neighbourhood $U \ni a$, strongly adapted to $\mathcal{V}$ and such that $E \cap U$ is a finite union of semipfaffian leaves. (respectively: of Pfaffian leaves defined by some strata of $\mathcal{N}$).

Locally finite unions and intersections and the Cartesian product of semipfaffian sets are semipfaffian. The family of connected components of a semipfaffian set is locally finite and the components are semipfaffian as well. However, there lacks the theorem on the closure of a semipfaffian set (and this is exactly a theorem one needs in the subanalytic category in order to prove the Gabrielov Theorem on the complement of a subanalytic set).

Definition 3.9. A subset $E \subset \mathbb{R}^n$ is *subpfaffian* if each point $a \in \mathbb{R}^n$ has a neighbourhood U such that $E \cap U = \pi(A)$ where $A \subset \mathbb{R}^n \times \mathbb{R}^k$ is a relatively compact *basic* semipfaffian set and $\pi\colon \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ is the natural projection.

Again locally finite unions and intersections remain in the category as well as the connected components which again form a locally finite family. Moreover, the projection on $\mathbb{R}^n$ of a $\mathbb{R}^k$-relatively compact subpfaffian set $E \subset \mathbb{R}^n \times \mathbb{R}^k$ is subpfaffian. In [Ho2] lemmata A and B are proved for subpfaffian sets. We remark that by a result of Cano, Lion and Moussu, the frontier of a Pfaffian hypersurface of Rolle is a subpfaffian set.

Definition 3.10. A semipfaffian set $E \subset \mathbb{R}^n$ is *subregular* if $\overline{E} \setminus E$ is contained in a closed subpfaffian set of dimension $< \dim E$ (the dimension being computed in the sense of Łojasiewicz 2.1).

Theorem 3.11 (Hajto)**.** *Any basic semipfaffian set is subregular.*

Remark 3.12. This theorem implies that the closure of any subpfaffian set is subpfaffian.

Theorem 3.13 (Hajto)**.** *The complement of a subpfaffian set is a subpfaffian set.*

All this is a good starting point for further study of the solutions of Pfaffian equations.

There is also another approach to *Pfaffian geometry* and we really do mean *another*, since until now nobody has compared $\mathbb{R}_{\mathrm{Pfaff}}$ with the following construction.

Definition 3.14. A $\mathscr{C}^1$ function $f\colon \mathbb{R}^n \to \mathbb{R}$ is called *Pfaffian* if there exist $\mathscr{C}^1$ functions $f_1, \ldots, f_k\colon \mathbb{R}^n \to \mathbb{R}$ with $f_k = f$, such that

$$\frac{\partial f_i}{\partial x_j}(x) = P_{ij}(x, f_1(x), \ldots, f_i(x)), \quad i = 1, \ldots, k, j = 1, \ldots, n,$$

for some polynomials P_{ij}.

The exponential function is clearly a Pfaffian function. Actually, any exponential polynomial

$$f(x_1, \ldots, x_n) := P(x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n}),$$

where P is a polynomial in $2n$ variables, is a Pfaffian function. By a theorem of Khovanskiĭ [Kh], any set of the form $f^{-1}(0)$ where f is Pfaffian, has only finitely many connected components. Using these functions one constructs the structure $\mathbb{R}_{\mathrm{Pfaff}}$. It has remained for long an open question whether this structure is o-minimal. In 1991, A. J. Wilkie [W1] proved the theorem of the complement (an analogous to the Gabrielov theorem for subanalytic sets) for geometric cathegories that include functions of the form $P(x_1, ..., x_n, \log x_1, ..., \log x_n)$ or again $P(x_1, ..., x_n, \exp(x_1), ..., \exp(x_n))$.
Finally, in 1999 it was proved by Wilkie that:

Theorem 3.15 ([W2])**.** *The structure* $\mathbb{R}_{\mathrm{Pfaff}}$ *is o-minimal.*

4. Relations and differences between the classes of sets introduced so far

We start with observing that the following inclusions of the Boolean algebras we were talking about hold:

semi-algebraic sets $\subset$ locally semi-algebraic sets $\subset$ semi-analytic sets $\subset$ subanalytic sets.

In other words we have an increasing chain of classes used as a model for introducing o-minimal structures.

The simplest example of a semi-analytic set whose projection is no longer semi-analytic was given by Łojasiewicz using the Osgood transcendental function $f(x,y) = (xy, xe^y)$.

Example 4.1. Let $A := \{(x,y,xy,xe^y) \mid x,y \in (0,1)\}$ and consider $\pi(x,y,u,v) = (x,u,v)$. Then $\pi(A) = \{((x,y,xe^{y/x}) \mid 0 < y < x < 1\}$ and this set is not semi-analytic at $0 \in \overline{\pi(A)}$. If this were the case, there would be a description

$$\pi(A) \cap U = \bigcup_{i=1}^{p} \bigcap_{j=1}^{q} \{f_i(x,y,z) = 0, g_{ij}(x,y,z) > 0\}$$

with f_i, g_{ij} analytic in the neighbourhood U of zero. The set $\pi(A)$ is the graph of an analytic function and so it is not open. This implies that for some i there is $f_i \not\equiv 0$ and f_i vanishes on some open subset of $\pi(A)$. By the identity principle, $f_i \equiv 0$ on $\pi(A) \cap V$ with some neighbourhood $V \subset U$ of zero, i.e., $f(x, xy, xe^y) = 0$ for $x \in (0,\varepsilon)$, $y \in (0,1)$.

Expanding $f_i = \sum_{\nu \geq k} P_\nu$ into a series of homogeneous forms P_ν of degree ν, with $P_k \not\equiv 0$, yields then $P_\nu(1,y,e^y) \equiv 0$ for all ν and all $y \in (0,1)$, and thus for all $y \in \mathbb{R}$. Then

$$Q(y,z) := P_k(1,y,z)$$

is a non-zero polynomial vanishing on the graph of the exponential function which is a contradiction.

There are however two instances when the projection respects semi-analycity:

Theorem 4.2 ([Ł1])**.** *Let M, N be analytic varieties and $A \subset M \times N$ a semi-analytic set M-relatively compact. Let $\pi\colon M \times N \to N$ be the natural projection. If either $\dim A \leq 1$, or there is a semi-analytic set in N of dimension ≤ 2 containing $\pi(A)$, then $\pi(A)$ is semi-analytic. In particular, this is the case, if $\dim N \leq 2$.*

• Among the well-known and widely used results concerning subanalytic sets there is the fact that the Euclidean distance to a semi- or subanalytic set is subanalytic. As we have seen, this result is valid also in o-minimal structures: the distance to a definable set is definable. However, with subanalytic sets one has to be somewhat more cautious — the assertion stated above is not quite right (though one comes across it even in textbooks!).

Theorem 4.3 (Raby)**.** *Let E be subanalytic in an open set $U \subset \mathbb{R}^n$ and let $\delta(x) := \operatorname{dist}(x,E)$ denote the Euclidean distance. Then δ is subanalytic in some neighbourhood $V \subset U$ of E. Besides, if $U = \mathbb{R}^n$, then V can be taken to be $\mathbb{R}^n$, too.*

However, if $U \neq \mathbb{R}^n$, then in general $V \subsetneq U$ as is shown in the following example of Raby:

Example 4.4. The set $E = \{(1/n, 0) \mid n = 1,2,\dots\}$ is semi-analytic in $\mathbb{R}^2 \setminus \{0\}$. If δ were subanalytic in the whole of $\mathbb{R}^2 \setminus \{0\}$, one would have

$$\{x \in \mathbb{R}^2 \setminus \{0\} \mid \delta(x) = 1\} \cap (\mathbb{R} \times \{1\}) = \{(0,1), (1/n,1), n = 1,2,\dots\}$$

which clearly is not subanalytic, being discrete and accumulating in $\mathbb{R}^2 \setminus \{0\}$.

It is worth noting that for $\alpha \in \mathbb{R}$ the function t^α, $t > 0$ is subanalytic if and only if $\alpha \in \mathbb{Q}$. This is a consequence of Theorem 2.57. On the other hand, in the structure $\mathbb{R}_{\exp}$ any t^α is definable, because $\ln t$ is so (as the inverse of the exponential) and $t^\alpha = \exp(\alpha \ln t)$. Of course each t^α is definable in $\mathbb{R}_{\mathrm{an}, f_r, r\in\mathbb{R}}$.

On the other hand, as noted in Example 2.18, such nice properties as the Łojasiewicz inequalities do not hold in general o-minimal structures. They are satisfied in *polynomially bounded* o-minimal structures. These are defined by analogy to Lemma 2.13:

Definition 4.5. A structure is *polynomially bounded* if every function $f\colon \mathbb{R} \to \mathbb{R}$ definable in it satisfies for some N, $f = O(t^N)$ at infinity.

This property has a very nice characterization:

Theorem 4.6 (Miller [Mi])**.** *An o-minimal structure is not polynomially bounded iff the exponential function is definable in it.*

Theorem 4.7 (cf. [vdDM])**.** *In a polynomially bounded o-minimal structure, continuous definable functions on compact sets are Hölder continuous and they satisfy the Łojasiewicz inequality 2.14 (therefore also the property of regular separation 2.16 is satisfied in such structures).*

Nonetheless, there is a general definable counterpart of the Łojasiewicz inequality, namely:

Theorem 4.8 ([vdDM])**.** *If $f, g\colon A \to \mathbb{R}$ are continuous definable functions such that $f^{-1}(0) \subset g^{-1}(0)$ and $A \subset \mathbb{R}^n$ is compact, then there exists a $\mathscr{C}^p$ definable, strictly increasing bijection $\phi\colon \mathbb{R} \to \mathbb{R}$ which is p-flat at zero* ([20])*, such that $|\phi(g(t))| \leq |f(t)|$ on A.*

In [K1] Kurdyka showed in this spirit the general definable analogon of the Łojasiewicz gradient inequality, which is important due to its applications to the study of the gradient dynamics. We recall both versions:

Theorem 4.9. (Gradient inequality.)
(1) *Łojasiewicz's classical gradient inequality: Let $f\colon (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be an analytic germ* ([21])*. Then there exists $\theta \in (0, 1)$ such that in a neighbourhood of zero $||\mathrm{grad} f(x)|| \geq |f(x)|^\theta$.*
(2) *Kurdyka's definable version: Let $f\colon \Omega \to (0, +\infty)$ be a definable differentiable function on an open and bounded $\Omega \subset \mathbb{R}^n$. Then there exist positive constants $c, r > 0$ and a strictly increasing positive definable function $\phi\colon \mathbb{R}_+ \to \mathbb{R}$ of class $\mathscr{C}^1$ such that $||\mathrm{grad}(\phi \circ f)(x)|| \geq c$ whenever $f(x) \in (0, r)$.*

Remark 4.10. Of course the classical version cannot be applied to flat functions. Therefore it cannot hold e.g. in $\mathbb{R}_{\exp}$. Though it may not be apparent, Kurdyka's version is equivalent to the Kurdyka-Parusiński generalization of the classical Łojasiewicz's gradient inequality.

It may seem at first glance that the definable version consists only in avoiding the problem of possible existence of flat definable functions by composing f with a kind of 'desingularizing' function. However, even in this form the generalized gradient inequality has a great impact on the gradient dynamics (see [Ł3], [K1]):

Theorem 4.11. (1) *Łojasiewicz's gradient theorem: If $f\colon (0, \mathbb{R}^n) \to ([0, +\infty), 0)$ is analytic, then there is a neighbourhood U of zero such that each trajectory $y_x(t)$ of $x' = -\mathrm{grad} f(x)$ with $y_x(0) = x \in U$ satisfies:*

(1) *$y_x(t)$ is defined for all $t \geq 0$;*
(2) *the length $lg(y_x) = \int_0^{+\infty} ||y'_x(t)||\, dt$ is finite and uniformly bounded;*

[20] i.e., $\varphi^k(0) = 0, k = 0, \ldots, p$.
[21] It is still true for f just subanalytic $\mathscr{C}^1$.

(3) *there is an equilibrium point* $z \in \{\mathrm{grad} f = 0\}$ *for which there is* $\lim_{t\to+\infty} y_x(t) = z$ ([22]). *Moreover, the covergence is uniform with respect to* $x \in U$.

(2) *Kurdyka's gradient theorem: If* $f\colon U \to \mathbb{R}$ *is definable and* $\mathscr{C}^1$ *on a bounded open set* $U \subset \mathbb{R}^n$*, then:*

(1) *all the trajectories of* $-\mathrm{grad} f$ *have uniformly bounded length;*
(2) *the* ω*-limit set of any trajectory consists of only one point.*

Remark 4.12. For further information on applications in non-smooth analysis and optimization we refer the reader to [BDLM]. Note by the way, that the first applications in optimal control were done for subanalytic geometry, see e.g. [T], (or works of H. Sussmann, Łojasiewicz jr, Brunovsky in optimal control, some other applications by B. Teissier — cf. the most recent [BT] with J.-P. Brasselet — and J.-P. Françoise, Y. Yomdin, e.g. [FY] ...).

Another kind of application of subanalytic geometry, this time in approximation theory, was performed by Pawłucki and Pleśniak who introduced in [PP] *uniformly polynomially cuspidal* sets in connection with the Markov inequality for bounded subanalytic sets (here the Wachta's Curve Selecting Lemma 2.58 is useful). Their result was then carried over to the definable setting (some o-minimal structures generated by quasi-analytic functions) by R. Pierzchała [Pr] — polynomial boundedness of the structure is needed.

Another result that found direct applications:

Theorem 4.13 (Denkowska-Wachta)**.** *Let* V *and* W *be two finite-dimensional real vector spaces and* $\pi\colon V \times W \to V$ *the natural projection. If* $E \subset V \times W$ *is subanalytic and* $F = \pi(E)$*, then there exists a subanalytic function* $\varphi\colon F \to W$ *such that* $\varphi \subset E$.

Here E can be seen as a subanalytic multifunction:

$$F \ni v \mapsto E_v \subset W$$

and φ is what is called *a selection* for this multifunction. The theorem above has applications in optimization where subanalytic multifunctions appear most naturally (cf. the works of R. J. Aumann, H. Halkin and E. C. Hendricks or, more recently M. Quincampoix) and the problem of finding a subanalytic selection is often crucial.

Remark 4.14. There exists a natural definable counterpart of this theorem, see e.g. [vdD]. It may be used to obtain the Curve Selecting Lemma.

• Some more metric properties:

Definition 4.15. A set $E \subset \mathbb{R}^n$ has *Whitney property* (in the class $\mathscr{C}$) if any two points $x, y \in E$ can be joined in E by a rectifiable arc (in the class $\mathscr{C}$) γ of length $lg(\gamma) \leq c||x-y||^r$ for some $c, r > 0$.

The above notion is important. For instance if E is a fat set (i.e., $\overline{\mathrm{int} E} = E$) satisfying the Whitney property, then any $\mathscr{C}^\infty$ function in $\mathrm{int} E$ whose derivatives have continuous extensions onto E, has a $\mathscr{C}^\infty$ continuation to $\mathbb{R}^n \setminus E$.

Theorem 4.16 (Łojasiewicz-Stasica)**.** *The analytic Whitney property holds for semi- and subanalytic closed sets.*

Remark 4.17. Note that many properties of subanalytic sets hold in a 'parameter version', for instance regular separation (with a uniform exponent, Łojasiewicz-Wachta), Whitney property (uniform exponent, Denkowska), there is also a uniform bound on the lengths of arcs joining

[22] In other words, the ω-limit set of y_x consists of a single point.

points in the fibres of a bounded subanalytic set (Teissier and Denkowska-Kurdyka). See [DS1] for details.

On the other hand, Kurdyka in [K2] showed that any subanalytic set can be stratified into subanalytic leaves (regular in the sense of Mostowski-Parusiński) each of which satisfies the Whitney property with exponent 1. The same kind of result for definable sets, this time with parameter, has been obtained recently by B. Kocel-Cynk [KC].

The Whitney property is obviously involved in comparisons of the inner metric of a subanalytic or definable set ([23]) with the outer one and bi-Lipschitz equivalence problems. Here Kurdyka's *Pancake Lemma* from [K2] is the main ingredient: see the works of L. Birbrair and others e.g. [Bb]). We recall shortly the idea:
A definable or subanalytic set $X \subset \mathbb{R}^n$ is said to be *normally embedded*, if the identity map induces a bi-Lipschitz isomorphism between the metric spaces $(X; d_o)$ and $(X; d_i)$, d_o being the outer (Euclidean) metric, and d_i the inner one (this means precisely that the Łojasiewicz exponent of X is equal to 1).

Theorem 4.18 (Pancake Decomposition [K2])**.** *Let $X \subset \mathbb{R}^n$ be definable or subanalytic and bounded. Then there exists a finite collection of definable/subanalytic subsets $X_i \subset X$ such that*

(1) $\bigcup X_i = X$;
(2) *Each X_i is normally embedded in $\mathbb{R}^n$;*
(3) $\forall i \neq j,\ \dim(X_i \cap X_j) < \min\{\dim X_i, \dim X_j\}$.

The collection $\{X_i\}$ is called Pancake Decomposition.

A nice decomposition of subanalytic sets, crucial from the point of view of the Whitney property (both in the subanalytic as in the definable setting):

Definition 4.19. An *(L)-analytic leaf* is a semi- or subanalytic subset of $\mathbb{R}^n$ which can be written in appropriate coordinates as the graph of a function $f \subset \mathbb{R}^k \times \mathbb{R}^{n-k}$ with open domain and which is analytic with bounded differential.

Theorem 4.20 (Stasica)**.** *Any bounded subanalytic set in $\mathbb{R}^n$ is a finite union of (L)-analytic leaves.*

An analoguous theorem for semi-analytic sets is due to de Rham.

Just to stress once again the difference between the definable and subanalytic settings we quote part one of the results from [Di] where the following problem is considered. Let $M \subset \mathbb{R}^k_t \times \mathbb{R}^m_x$ be a set with closed t-sections M_t (not all empty) and let

$$m(t,x) = \{y \in M_t \mid ||x-y|| = \mathrm{dist}(x, M_t)\}.$$

Proposition 4.21 ([Di])**.** *If M is definable, then the set*

$$E := \{(t,x) \mid \#m(t,x) > 1\}$$

is definable, too.

Example 4.22. We have already observed that without an additional assumption (like that of M being x-relatively compact i.e., having proper projection onto $\mathbb{R}^k$) we cannot expect the function $(t,x) \mapsto \mathrm{dist}(x, M_t)$ to be subanalytic for a subanalytic M. Neither is the proposition true in the general subanalytic setting:

$$M = \{(x, 1/x) \mid x > 0\} \cup \bigcup_{n=1}^{+\infty} \{(1/n, -n)\} \subset \mathbb{R} \times \mathbb{R}$$

[23] i.e., the greatest lower bound of the lenghts of rectifiable curves joining two points in this set; triangulation theorems warrant this is well defined.

is subanalytic, but $E = \bigcup\{(1/n, 0)\}$ is not.

Nevertheless, the proposition above is true for subanalytic sets if we get rid of the parameter t. In this case we could be tempted to derive the proof from the definable case applied to the globally subanalytic sets $M_\nu = M \cap [-\nu, \nu]^n \subset \mathbb{R}^n$. However, the thing is more subtle than it seems and we do not have $E = \bigcup E_\nu$ where E_ν is constructed for M_ν. Indeed, take for instance M to be the union of semi-circles $\{x^2 + (y-\nu)^2 - (3/4)^2, y \leq \nu\}$. Then $(0, \nu) \in E_\nu \setminus E_{\nu+1}$ and in particular $(0, \nu) \notin E$.

• Many, though not all semi-analytic theorems have their subanalytic versions (cf. [DS1] for a thorough survey, e.g. each semi-analytic set germ admits an analytic germ of the same dimension as a superset which is no longer true for subanalytic germs cf. Example 4.1) and once again many, though not all, of these can be transposed to the definable setting. Here come some examples; first the theorem of the tangent mapping (compare Theorem 2.50):

Theorem 4.23. *Let $\Gamma \subset \mathbb{R}^n$ be a semi- or subanalytic leaf of dimension k. Then the tangent map $\tau\colon \Gamma \ni x \mapsto T_x\Gamma \in G_k(\mathbb{R}^n)$ is semi- or subanalytic (according to the case).*

Corollary 4.24. *If Γ is a subanalytic leaf, then for any subanalytic subset F of the Grassmannian $G_k(\mathbb{R}^n)$, $\tau^{-1}(E)$ is subanalytic, and for any bounded subanalytic set $E \subset \mathbb{R}^n$, $\tau(E)$ is subanalytic, too.*

Remark 4.25. The theorem above has a definable counterpart to be found in the articles by Ta Lê Loi.

The next result is a generalization of the Curve Selecting Lemma to higher dimensions:

Lemma 4.26 (Wings' Lemma). *Let $\Gamma \subset M$ be a subanalytic leaf and $E \subset \overline{\Gamma} \setminus \Gamma$ a subanalytic set. Then there exists a subanalytic leaf Λ of dimension* $\dim E + 1$ *and such that $\Lambda \subset \Gamma$ and* $\dim \overline{\Lambda} \cap E = \dim E$.

A definable counterpart of the result above is given in [Loi] ([24]).

• **Stratifications**

Stratifications are an important tool and they are often asked to satisfy some additional properties — we shall discuss this briefly. Let V be a finite-dimensional real vector space and denote by J the family of pairs (V', V'') of subspaces of V satisfying $V' \subset V''$.
Let N_0, N be two differentiable subvarieties of V of dimension k and l respectively, with $k < l$.

Definition 4.27. We say that the pair (N_0, N) *satisfies Whitney's condition (a)* at $c \in N_0 \cap N$ if $(T_c N_0, T_z N)$ tends to J in $G_k(V) \times G_l(V)$ when $z \in N$ tends to c.
We say that (N_0, N) *satisfies Whitney's condition (b)* at c if the pair $(\mathbb{R} \cdot (z-x), T_z N)$ tends to J in $G_1(V) \times G_l(V)$ when the point $(x, z) \in (N_0 \times N) \cap \{x \neq z\}$ tends to (c, c).

Remark 4.28. The convergence above is invariant with respect to diffeomorphisms, whence it can be formulated in the same way for a differentiable variety. Recall also that Whitney's condition (b) implies (a).

Theorem 4.29. *Let M be an affine space. Let $E_1, \dots, E_r$ be subanalytic in M. Then there exists a stratification $\mathcal{N}$ of M into subanalytic leaves, compatible with $E_1, \dots, E_r$ and such that for all pairs of strata $\Gamma_1, \Gamma_2 \in \mathcal{N}$ such that $\Gamma_1 \subset \overline{\Gamma_2} \setminus \Gamma_2$, the varieties Γ_1, Γ_2 satisfy Whitney's condition (b) at any point of Γ_1.*

[24] We thank the referee for pointing this out.

Definition 4.30. Let $f\colon M \to N$ be an analytic map, $\mathcal{T}$ a stratification of the analytic variety M, $\mathcal{S}$ a stratification of another analytic variety N. The pair $\mathcal{T}, \mathcal{S}$ is said to be *compatible with* f if

(i) for all $T \in \mathcal{T}$, $f(T) \in \mathcal{S}$,
(ii) for all $T \in \mathcal{T}$, $\mathrm{rk} f|_T \equiv \dim f(T)$,
(iii) if $\mathrm{rk} f|_T = \dim T$, then $f|_T$ is injective.

Theorem 4.31 (Hardt)**.** *Let $f\colon M \to N$ be analytic. Given two locally finite families $\mathcal{M}, \mathcal{N}$ of subanalytic sets in M, N, respectively, and an open subanalytic set K such that $f|_{\overline{K}}$ is proper, there exists a stratification $\mathcal{S}$ compatible with $\mathcal{N}$ and a stratification $\mathcal{T}$ compatible with $\mathcal{M}$ together with K, such that the pair $(\mathcal{T}_K, \mathcal{S})$ is compatible with f_K, where $\mathcal{T}_K = \{T \in \mathcal{T}\colon T \subset K\}$.*

Let X be a finite-dimensional real vector space and U, V its linear subspaces. We define after T.-C. Kuo the function $\delta(U,V) := \sup\{d(x,V)\colon x \in U, |x| = 1\}$ where d is the Euclidean distance. There is $\delta(U,V) = 0$ if and only if $U \subset V$.

Definition 4.32. Let M, N be two $\mathscr{C}^\infty$ subvarieties of X such that $\overline{M} \cap N \neq \varnothing$. We say that the pair (M,N) *satisfies the Verdier condition (w) at a point* $a \in \overline{M} \cap N$ if there is a neighbourhood V of a in X and a constant $C > 0$ such that

$$\delta(T_x M, T_y N) \leq C||x - y||, \quad \text{for any } x \in V \cap M, y \in V \cap N.$$

We say that (M,N) *satisfies the condition (w)* if it satisfies this condition at all points $a \in \overline{M} \cap N$.

Remark 4.33. Kuo in [Kuo] showed that condition (w) implies Whitney's condition (b) in the semi-analytic case. Since the Curve Selecting Lemma and the Tangent Mapping Theorem hold also in the subanalytic case, the same kind of argument as that used by Kuo works also in the subanalytic case. Nonetheless, condition (w) in general is not stronger than condition (b) (see [Vd]). For more informations see [DSW], [DW], [KT], [OTr], [Tr].

Theorem 4.34 (Verdier [Vd] ([25]))**.** *Let $\{E_i\}$ be a locally finite family of subanalytic subsets of X. Then there is a subanalytic stratification of X compatible with that family and such that any pair of its strata satisfies the Verdier condition (w).*

It is worth adding a few words about Łojasiewicz's approach to stratifications. Needless to say, unlike e.g. Verdier, he made no use of Hironaka's desingularization. Instead, his idea was to start with the following key-lemma:

If M and N are subanalytic varieties in an affine space X and $N \subset \overline{M} \setminus M$, then the set $\{x \in N \mid (M,N)$ verifies condition $(\#)\}$ where $(\#)$ stands for one of the conditions introduced so far, is subanalytic in the space X and dense in N.

To prove *dense in* N (subanalytic is easy), we use Whitney's Wings' Lemma 4.26. See [D], [DW], [DS2].

• A natural question is whether subanalytic sets admit triangulation (cf. Theorem 1.30). The positive answer was given by Goresky [Go] as well as Verona [V]. Independently of the general result, H. Hironaka [H3] and R. Hardt [H2] gave both explicit methods of triangulation for subanalytic sets. Their constructions are natural and geometric. As noted by H. Hironaka the method is close to that used by S. Łojasiewicz for semi-algebraic sets, for both classes of sets — semi-algebraic and subanalytic — are closed with respect to projections.

[25] There are other proofs by Łojasiewicz-Stasica-Wachta, Coste-Roy and historically the first one by Denkowska-Wachta [DW] as an answer to a question of D. Trotman for a desingularization-free proof, presented in [D].

Theorem 4.35 (Hironaka)**.** *Let $\{X_\alpha\}_{\alpha\in A}$ be a locally finite family of subanalytic subsets of $\mathbb{R}^n$. Then there exists a simplicial decomposition of $\mathbb{R}^n = \bigcup \sigma_\mu$ into open simplices and a subanalytic homeomorphism $\theta\colon \mathbb{R}^n \to \mathbb{R}^n$ such that*

(i) *each X_α is a locally finite union of some of the images $\theta(\sigma_\mu)$,*
(ii) *for any μ, $\theta(\sigma_\mu)$ is an analytic subvariety of $\mathbb{R}^n$ and $\theta|_{\sigma_\mu}\colon \sigma_\mu \to \theta(\sigma_\mu)$ is an analytic isomorphism.*

Theorem 4.36 (Hardt)**.** *Let $\{X_\alpha\}_{\alpha\in A}$ be a locally finite family of subanalytic subsets of $\mathbb{R}^n$. Then there exists a simplicial decomposition Σ of $\mathbb{R}^n$ and a subanalytic map $f\colon [0,1]\times\mathbb{R}^n \to \mathbb{R}^n$ such that*

(i) *for each $t \in [0,1]$ the map $f_t(x) = f(t,x)$ is a homeomorphism,*
(ii) *$f_0 = id$,*
(iii) *for any $\alpha \in A$, $f_1^{-1}(X_\alpha)$ is a subcomplex of Σ.*

Remark 4.37. In the case of semi-analytic sets, a class of sets without the projection property, the construction of a semi-analytic triangulation is much more delicate (see S. Łojasiewicz [Ł4]).

Remark 4.38. Semi-algebraic, semi-analytic and subanalytic sets admit triangulation.
Quite recently, a student of W. Pawłucki, M. Czapla, proved in her Ph. D. Thesis (using a description of the Lipschitz structure of definable sets by G. Valette [Val]) that every definable set has a definable triangulation which is locally Lipschitz and weakly bi-Lipschitz on the natural stratification of a simplicial complex. She also proved that such a stratification may be obtained with Whitney's (b) condition or Verdier's condition.

On the other hand, it is well-known that subanalytic sets admit Lipschitz stratification (see [Pa]). A direct method of constructing a Lipschitz cell decomposition (which must involve some coordinate changes) has been produced recently by Pawłucki in [P2].

We started with semi-algebraic sets and we will end with them. The following theorem, proved using simple stratifications, show how ubiquous they are:

Theorem 4.39 ([DD])**.** *Let $E \subset \mathbb{R}^m$ be a compact subanalytic or definable set. Then there exists a sequence $\{A_\nu\}$ of semi-algebraic sets such that*

(1) *$E = \lim A_\nu$;*
(2) *For each $a \in E$ and any neighbourhood U of a one has for ν large enough,*

$$\dim U \cap E = \dim U \cap A_\nu .$$

Moreover, for each such a sequence $\{A_\nu\}$ one has the following: for any $S \in cc(E)$ there is a sequence $\{S_\nu\}$ such that each S_ν is the union of some connected components of A_ν and (1) *and* (2) *holds for S and the sequence $\{S_\nu\}$.*

Here the convergence is understood in the following sense (*Kuratowski convergence of closed sets*):
$A = \lim A_n$ iff each point $a \in A$ is the limit of a sequence of points $a_n \in A_n$, $n \in \mathbb{N}$ and for each compact set K such that $K \cap A = \varnothing$ one has $K \cap A_n = \varnothing$ for almost all indices n.

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STRATIFIED CRITICAL POINTS ON THE REAL MILNOR FIBRE AND INTEGRAL-GEOMETRIC FORMULAS

NICOLAS DUTERTRE

Dedicated to professor David Trotman on his 60th birthday

Abstract. Let $(X,0) \subset (\mathbb{R}^n,0)$ be the germ of a closed subanalytic set and consider two subanalytic functions f and $g : (X,0) \to (\mathbb{R},0)$. Under some conditions, we relate the critical points of g on the real Milnor fibre $f^{-1}(\delta) \cap B_\epsilon$, $0 < |\delta| \ll \epsilon \ll 1$, to the topology of this fibre and other related subanalytic sets. As an application, when g is a generic linear function, we obtain an "asymptotic" Gauss-Bonnet formula for the real Milnor fibre of f. From this Gauss-Bonnet formula, we deduce "infinitesimal" linear kinematic formulas.

1. Introduction

Let $F = (f_1, \ldots, f_k) : (\mathbb{C}^n, 0) \to (\mathbb{C}^k, 0)$, $2 \le k \le n$, be a complete intersection with isolated singularity. The Lê-Greuel formula [21, 22] states that

$$\mu(F') + \mu(F) = \dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathbb{C}^n,0}}{I},$$

where $F' : (\mathbb{C}^n, 0) \to (\mathbb{C}^{k-1}, 0)$ is the map with components $f_1, \ldots, f_{k-1}$, I is the ideal generated by $f_1, \ldots, f_{k-1}$ and the $(k \times k)$-minors $\frac{\partial(f_1,\ldots,f_k)}{\partial(x_{i_1},\ldots,x_{i_k})}$ and $\mu(F)$ (resp. $\mu(F')$) is the Milnor number of F (resp. F'). Hence the Lê-Greuel formula gives an algebraic characterization of a topological data, namely the sum of two Milnor numbers. However, since the right-hand side of the above equality is equal to the number of critical points of f_k, counted with multiplicity, on the Milnor fibre of F', the Lê-Greuel formula can be also viewed as a topological characterization of this number of critical points.

Many works have been devoted to the search of a real version of the Lê-Greuel formula. Let us recall them briefly. We consider an analytic map-germ $F = (f_1, \ldots, f_k) : (\mathbb{R}^n, 0) \to (\mathbb{R}^k, 0)$, $2 \le k \le n$, and we denote by F' the map-germ $(f_1, \ldots, f_{k-1}) : (\mathbb{R}^n, 0) \to (\mathbb{R}^{k-1}, 0)$. Some authors investigated the following difference:

$$D_{\delta,\delta'} = \chi\big(F'^{-1}(\delta) \cap \{f_k \ge \delta'\} \cap B_\epsilon\big) - \chi\big(F'^{-1}(\delta) \cap \{f_k \le \delta'\} \cap B_\epsilon\big),$$

where (δ, δ') is a regular value of F such that $0 \le |\delta'| \ll |\delta| \ll \epsilon$.

In [12], we proved that

$$D_{\delta,\delta'} \equiv \dim_{\mathbb{R}} \frac{\mathcal{O}_{\mathbb{R}^n,0}}{I} \bmod 2,$$

where $\mathcal{O}_{\mathbb{R}^n,0}$ is the ring of analytic function-germs at the origin and I is the ideal generated by $f_1, \ldots, f_{k-1}$ and all the $k \times k$ minors $\frac{\partial(f_k,f_1,\ldots,f_{k-1})}{\partial(x_{i_1},\ldots,x_{i_k})}$. This is only a mod 2 relation and we may ask if it is possible to get a more precise relation.

When $k = n$ and $f_k = x_1^2 + \cdots + x_n^2$, according to Aoki et al. ([1], [3]),

$$D_{\delta,0} = \chi\big(F'^{-1}(\delta) \cap B_\varepsilon\big) = 2\deg_0 H$$

and $2\deg_0 H$ is the number of semi-branches of $F'^{-1}(0)$, where

$$H = (\frac{\partial(f_n, f_1, \ldots, f_{n-1})}{\partial(x_1, \ldots, x_n)}, f_1, \ldots, f_{n-1}).$$

They proved a similar formula in the case $f_k = x_n$ in [2] and Szafraniec generalized all these results to any f_k in [23].

When $k = 2$ and $f_2 = x_1$, Fukui [18] stated that

$$D_{\delta,0} = -\mathrm{sign}(-\delta)^n \deg_0 H,$$

where $H = (f_1, \frac{\partial f_1}{\partial x_2}, \ldots, \frac{\partial f_1}{\partial x_n})$. Several generalizations of Fukui's formula are given in [19], [11], [20] and [13].

In all these papers, the general idea is to count algebraically the critical points of a Morse perturbation of f_k on $F'^{-1}(\delta) \cap B_\epsilon$ and to express this sum in two ways: as a difference of Euler characteristics and as a topological degree. Using the Eisenbud-Levine formula [16], this latter degree can be expressed as a signature of a quadratic form and so, we obtain an algebraic expression for $D_{\delta,\delta'}$.

In this paper, we give a real and stratified version of the Lê-Greuel formula. We restrict ourselves to the topological aspect and relate a sum of indices of critical points on a real Milnor fibre to some Euler characteristics (this is also the point of view adopted in [7]). More precisely, we consider a germ of a closed subanalytic set $(X, 0) \subset (\mathbb{R}^n, 0)$ and a subanalytic function $f : (X, 0) \to (\mathbb{R}, 0)$. We assume that X is contained in a open set U of $\mathbb{R}^n$ and that f is the restriction to X of a C^2-subanalytic function $F : U \to \mathbb{R}$. We denote by X^f the set $f^{-1}(0)$ and we equip X with a Thom stratification adapted to X^f. If $0 < |\delta| \ll \epsilon \ll 1$ then the real Milnor fibre of f is defined by

$$M_f^{\delta,\epsilon} = f^{-1}(\delta) \cap X \cap B_\epsilon.$$

We consider another subanalytic function $g : (X, 0) \to (\mathbb{R}, 0)$ and we assume that it is the restriction to X of a C^2-subanalytic function $G : U \to \mathbb{R}$. We denote by X^g the set $g^{-1}(0)$. Under two conditions on g, we study the topological behaviour of $g_{|M_f^{\delta,\epsilon}}$.

We recall that if $Z \subset \mathbb{R}^n$ is a closed subanalytic set, equipped with a Whitney stratification and $p \in Z$ is an isolated critical point of a subanalytic function $\phi : Z \to \mathbb{R}$, restriction to Z of a C^2-subanalytic function Φ, then the index of ϕ at p is defined as follows:

$$\mathrm{ind}(\phi, Z, p) = 1 - \chi\big(Z \cap \{\phi = \phi(p) - \eta\} \cap B_\epsilon(p)\big),$$

where $0 < \eta \ll \epsilon \ll 1$ and $B_\epsilon(p)$ is the closed ball of radius ϵ centered at p. Let $p_1^{\delta,\epsilon}, \ldots, p_r^{\delta,\epsilon}$ be the critical points of g on $f^{-1}(\delta) \cap \mathring{B}_\epsilon$, where $\mathring{B}_\epsilon$ denotes the open ball of radius ϵ. We set

$$I(\delta, \epsilon, g) = \sum_{i=1}^{r} \mathrm{ind}(g, f^{-1}(\delta), p_i^{\delta,\epsilon}),$$

$$I(\delta, \epsilon, -g) = \sum_{i=1}^{r} \mathrm{ind}(-g, f^{-1}(\delta), p_i^{\delta,\epsilon}).$$

Our main theorem (Theorem 3.10) is the following:

$$I(\delta, \epsilon, g) + I(\delta, \epsilon, -g) = 2\chi(M_f^{\delta,\epsilon}) - \chi(f^{-1}(\delta) \cap S_\epsilon) - \chi(X^g \cap f^{-1}(\delta) \cap S_\epsilon).$$

As a corollary (Corollary 3.11), when $f : (X, 0) \to (\mathbb{R}, 0)$ has an isolated stratified critical point at 0, we obtain that

$$I(\delta, \epsilon, g) + I(\delta, \epsilon, -g) = 2\chi(M_f^{\delta,\epsilon}) - \chi(\mathrm{Lk}(X^f)) - \chi(\mathrm{Lk}(X^f \cap X^g)),$$

where $\mathrm{Lk}(-)$ denotes the link at the origin.

Then we apply these results when g is a generic linear form to get an asymptotic Gauss-Bonnet formula for $M_f^{\delta,\epsilon}$ (Theorem 4.5). In the last section, we use this asymptotic Gauss-Bonnet formula to prove infinitesimal linear kinematic formulas for closed subanalytic germs (Theorem 5.5), that generalize the Cauchy-Crofton formula for the density due to Comte [8].

The paper is organized as follows. In Section 2, we prove several lemmas about critical points on the link of a subanalytic set. Section 3 contains real stratified versions of the Lê-Greuel formula. In Section 4, we establish the asymptotic Gauss-Bonnet formula and in Section 5, the infinitesimal linear kinematic formulas.

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2. Lemmas on critical points on the link of a stratum

In this section, we study the behaviour of the critical points of a C^2-subanalytic function on the link of stratum that contains 0 in its closure, for a generic choice of a C^2-distance function to the origin.

Let $Y \subset \mathbb{R}^n$ be a C^2-subanalytic manifold such that 0 belongs to its closure $\overline{Y}$. Let $\theta : \mathbb{R}^n \to \mathbb{R}$ be a C^2-subanalytic function such that $\theta(0) = 0$. We will first study the behaviour of the critical points of $\theta_{|Y} : Y \to \mathbb{R}$ in the neighborhood of 0, and then the behaviour of the critical points of the restriction of θ to the link of 0 in Y.

Lemma 2.1. *The critical points of $\theta_{|Y}$ lie in $\{\theta = 0\}$ in a neighborhood of 0.*

Proof. By the Curve Selection Lemma, we can assume that there is a C^1-subanalytic curve $\gamma : [0, \nu[\to \overline{Y}$ such that $\gamma(0) = 0$ and $\gamma(t)$ is a critical point of $\theta_{|Y}$ for $t \in]0, \nu[$. Therefore, we have

$$(\theta \circ \gamma)'(t) = \langle \nabla \theta_{|Y}(\gamma(t)), \gamma'(t) \rangle = 0,$$

since $\gamma'(t)$ is tangent to Y at $\gamma(t)$. This implies that $\theta \circ \gamma(t) = \theta \circ \gamma(0) = 0$. $\square$

Let $\rho : \mathbb{R}^n \to \mathbb{R}$ be another C^2-subanalytic function such that a is a regular value of ρ and $\rho^{-1}(a)$ intersects Y transversally. Then the set $Y \cap \{\rho \leq a\}$ is a manifold with boundary. Let p be a critical point of $\theta_{|Y \cap \{\rho \leq a\}}$ which lies in $Y \cap \{\rho = a\}$ and which is not a critical point of $\theta_{|Y}$. This implies that

$$\nabla \theta_{|Y}(p) = \lambda(p) \nabla \rho_{|Y}(p),$$

with $\lambda(p) \neq 0$.

Definition 2.2. *We say that $p \in Y \cap \{\rho = a\}$ is an outwards-pointing (resp. inwards-pointing) critical point of $\theta_{|Y \cap \{\rho \leq a\}}$ if $\lambda(p) > 0$ (resp. $\lambda(p) < 0$).*

Now let us assume that $\rho : \mathbb{R}^n \to \mathbb{R}$ is a C^2-subanalytic function such that $\rho \geq 0$ and $\rho^{-1}(0) = \{0\}$ in a neighborhood of 0. We call ρ a C^2-distance function to the origin. By Lemma 2.1, we know that for $\epsilon > 0$ small enough, the level $\rho^{-1}(\epsilon)$ intersects Y transversally. Let p^ϵ be a critical point of $\theta_{|Y \cap \rho^{-1}(\epsilon)}$ such that $\theta(p^\epsilon) \neq 0$. This means that there exists $\lambda(p^\epsilon)$ such that

$$\nabla \theta_{|Y}(p^\epsilon) = \lambda(p^\epsilon) \nabla \rho_{|Y}(p^\epsilon).$$

Note that $\lambda(p^\epsilon) \neq 0$ because $\nabla \theta_{|Y}(p^\epsilon) \neq 0$ for $\theta(p^\epsilon) \neq 0$.

Lemma 2.3. *The point p^ϵ is an outwards-pointing (resp. inwards-pointing) for $\theta_{|Y\cap\{\rho\leq\epsilon\}}$ if and only if $\theta(p^\epsilon)>0$ (resp. $\theta(p^\epsilon)<0$).*

Proof. Let us assume that $\lambda(p^\epsilon)>0$. By the Curve Selection Lemma, there exists a C^1-subanalytic curve $\gamma:[0,\nu[\to\overline{Y}$ passing through p^ϵ such that $\gamma(0)=0$ and for $t\neq 0$, $\gamma(t)$ is a critical point of $\theta_{|Y\cap\{\rho=\rho(\gamma(t))\}}$ with $\lambda(\gamma(t))>0$. Therefore we have

$$(\theta\circ\gamma)'(t)=\langle\nabla\theta_{|Y}(\gamma(t)),\gamma'(t)\rangle=\lambda(\gamma(t))\langle\nabla\rho_{|Y}(\gamma(t)),\gamma'(t)\rangle.$$

But $(\rho\circ\gamma)'>0$ for otherwise $(\rho\circ\gamma)'\leq 0$ and $\rho\circ\gamma$ would be decreasing. Since $\rho(\gamma(t))$ tends to 0 as t tends to 0, this would imply that $\rho\circ\gamma(t)\leq 0$, which is impossible. We can conclude that $(\theta\circ\gamma)'>0$ and that $\theta\circ\gamma$ is strictly increasing. Since $\theta\circ\gamma(t)$ tends to 0 as t tends to 0, we see that $\theta\circ\gamma(t)>0$ for $t>0$. Similarly if $\lambda(p^\epsilon)<0$ then $\theta(p^\epsilon)<0$. $\square$

Now we will study these critical points for a generic choice of the C^2-distance function to the origin. We denote by $\mathrm{Sym}(\mathbb{R}^n)$ the set of symmetric $n\times n$-matrices with real entries, by $\mathrm{Sym}^*(\mathbb{R}^n)$ the open dense subset of such matrices with non-zero determinant and by $\mathrm{Sym}^{+,*}(\mathbb{R}^n)$ the open subset of these invertible matrices that are positive definite or negative definite. Note that these sets are semi-algebraic. For each $A\in\mathrm{Sym}^{+,*}(\mathbb{R}^n)$, we denote by ρ_A the following quadratic form:

$$\rho_A(x)=\langle Ax,x\rangle.$$

We denote by $\Gamma^Y_{\theta,A}$ the following subanalytic polar set:

$$\Gamma^Y_{\theta,A}=\left\{x\in Y\mid \mathrm{rank}\left[\nabla\theta_{|Y}(x),\nabla\rho_{A|Y}(x)\right]<2\right\},$$

and by Σ^Y_θ the set of critical points of $\theta_{|Y}$. Note that $\Sigma^Y_\theta\subset\{\theta=0\}$ by Lemma 2.1.

Lemma 2.4. *For almost all A in $\mathrm{Sym}^{+,*}(\mathbb{R}^n)$, $\Gamma^Y_{\theta,A}\setminus(\Sigma^Y_\theta\cup\{0\})$ is a C^1-subanalytic curve (possible empty) in a neighborhood of 0.*

Proof. We can assume that $\dim Y>1$. Let

$$Z=\left\{(x,A)\in\mathbb{R}^n\times\mathrm{Sym}^{+,*}(\mathbb{R}^n)\mid x\in Y\setminus(\Sigma^Y_\theta\cup\{0\})\text{ and rank }\left[\nabla\theta_{|Y}(x),\nabla\rho_{A|Y}(x)\right]<2\right\}.$$

Let (y,B) be a point in Z. We can suppose that around y, Y is defined by the vanishing of k subanalytic functions $f_1,\ldots,f_k$ of class C^2. Hence in a neighborhood of (y,B), Z is defined be the vanishing of $f_1,\ldots,f_k$ and the minors

$$\frac{\partial(f_1,\ldots,f_k,\theta,\rho_A)}{\partial(x_{i_1},\ldots,x_{i_{k+2}})}.$$

Furthermore, since y does not belong to Σ^Y_θ, we can assume that

$$\frac{\partial(f_1,\ldots,f_k,\theta)}{\partial(x_1,\ldots,x_k,x_{k+1})}\neq 0,$$

in a neighborhood of y. Therefore Z is locally defined by $f_1=\cdots=f_k=0$ and

$$\frac{\partial(f_1,\ldots,f_k,\theta,\rho_A)}{\partial(x_1,\ldots,x_{k+1},x_{k+2})}=\cdots=\frac{\partial(f_1,\ldots,f_k,\theta,\rho_A)}{\partial(x_1,\ldots,x_{k+1},x_n)}=0.$$

Let us write $M=\frac{\partial(f_1,\ldots,f_k,\theta)}{\partial(x_1,\ldots,x_k,x_{k+1})}$ and for $i\in\{k+2,\ldots,n\}$, $m_i=\frac{\partial(f_1,\ldots,f_k,\theta,\rho_A)}{\partial(x_1,\ldots,x_{k+1},x_i)}$. If $A=[a_{ij}]$ then

$$\rho_A(x)=\sum_{i=1}^n a_{ii}x_i^2+2\sum_{i\neq j}a_{ij}x_ix_j,$$

and so $\frac{\partial \rho_A}{\partial x_i}(x) = 2\sum_{j=1}^n a_{ij}x_j$. For $i \in \{k+1,\dots,n\}$ and $j \in \{1,\dots,n\}$, we have

$$\frac{\partial m_i}{\partial a_{ij}} = 2x_j M.$$

Since $y \neq 0$, one of the x_j's does not vanish in the neighborhood of y and we can conclude that the rank of

$$[\nabla f_1(x), \dots, \nabla f_k(x), \nabla m_{k+2}(x,A), \dots, \nabla m_n(x,A)]$$

is $n-1$ and that Z is a C^1-subanalytic manifold of dimension $\frac{n(n+1)}{2}+1$. Now let us consider the projection $\pi_2 : Z \to \mathrm{Sym}^{+,*}(\mathbb{R}^n)$, $(x,A) \mapsto A$. Bertini-Sard's theorem implies that the set D_{π_2} of critical values of π_2 is a subanalytic set of dimension strictly less than $\frac{n(n+1)}{2}$. Hence, for all $A \notin D_{\pi_2}$, $\pi_2^{-1}(A)$ is a C^1-subanalytic curve (possibly empty). But this set is exactly $\Gamma^Y_{\theta,A} \setminus (\Sigma^Y_\theta \cup \{0\})$. □

Let $R \subset Y$ be a subanalytic set of dimension strictly less than $\dim Y$. We will need the following lemma.

Lemma 2.5. *For almost all A in $\mathrm{Sym}^{+,*}(\mathbb{R}^n)$, $\Gamma^Y_{\theta,A} \setminus (\Sigma^Y_\theta \cup \{0\}) \cap R$ is a subanalytic set of dimension at most 0 in a neighborhood of 0.*

Proof. Let us put $l = \dim Y$. Since R admits a locally finite subanalytic stratification, we can assume that R is a C^2-subanalytic manifold of dimension d with $d < l$. Let W be the following subanalytic set:

$$W = \Big\{(x,A) \in \mathbb{R}^n \times \mathrm{Sym}^{+,*}(\mathbb{R}^n) \mid x \in R \setminus (\Sigma^Y_\theta \cup \{0\}) \text{ and rank} \Big[\nabla \theta_{|Y}(x), \nabla \rho_{A|Y}(x)\Big] < 2\Big\}.$$

Using the same method as in the previous lemma, we can prove that W is a C^1-subanalytic manifold of dimension $\frac{n(n+1)}{2} + 1 + d - l$ and conclude, remarking that $d - l \leq -1$. □

Now we introduce a new C^2-subanalytic function $\beta : \mathbb{R}^n \to \mathbb{R}$ such that $\beta(0) = 0$. We denote by $\Gamma^Y_{\theta,\beta,A}$ the following subanalytic polar set:

$$\Gamma^Y_{\theta,\beta,A} = \Big\{x \in Y \mid \text{rank} \Big[\nabla \theta_{|Y}(x), \nabla \beta_{|Y}(x), \nabla \rho_{A|Y}(x)\Big] < 3\Big\},$$

and by $\Gamma^Y_{\theta,\beta}$ the following subanalytic polar set:

$$\Gamma^Y_{\theta,\beta} = \big\{x \in Y \mid \text{rank} \big[\nabla \theta_{|Y}(x), \nabla \beta_{|Y}(x)\big] < 2\big\}.$$

Lemma 2.6. *For almost all A in $\mathrm{Sym}^{+,*}(\mathbb{R}^n)$, $\Gamma^Y_{\theta,\beta,A} \setminus (\Gamma^Y_{\theta,\beta} \cup \{0\})$ is a C^1-subanalytic set of dimension at most 2 (possibly empty) in a neighborhood of 0.*

Proof. We can assume that $\dim Y > 2$. Let

$$Z = \Big\{(x,A) \in \mathbb{R}^n \times \mathrm{Sym}^{+,*}(\mathbb{R}^n) \mid x \in Y, \text{rank} \big[\nabla \theta_{|Y}(x), \nabla \beta_{|Y}(x)\big] = 2$$
$$\text{and rank} \Big[\nabla \theta_{|Y}(x), \nabla \beta_{|Y}(x), \nabla \rho_{A|Y}(x)\Big] < 3\Big\}.$$

Let (y,B) be a point in Z. We can suppose that around y, Y is defined by the vanishing of k subanalytic functions $f_1,\dots,f_k$ of class C^2. Hence in a neighborhood of (y,B), Z is defined by the vanishing of $f_1,\dots,f_k$ and the minors

$$\frac{\partial(f_1,\dots,f_k,\theta,\beta,\rho_A)}{\partial(x_{i_1},\dots,x_{i_{k+3}})}.$$

Since y does not belong to $\Gamma^Y_{\theta,\beta}$, we can assume that

$$\frac{\partial(f_1,\ldots,f_k,\theta,\beta)}{\partial(x_1,\ldots,x_k,x_{k+1},x_{k+2})} \neq 0,$$

in a neighborhood of y. Therefore Z is locally defined by $f_1,\ldots,f_k = 0$ and

$$\frac{\partial(f_1,\ldots,f_k,\theta,\beta,\rho_A)}{\partial(x_1,\ldots,x_{k+2},x_{k+3})} = \cdots = \frac{\partial(f_1,\ldots,f_k,\theta,\beta,\rho_A)}{\partial(x_1,\ldots,x_{k+2},x_n)} = 0.$$

It is clear that we can apply the same method as Lemma 2.4 to get the result. $\square$

3. Lê-Greuel type formula

In this section, we prove the Lê-Greuel type formula announced in the introduction.

Let $(X,0) \subset (\mathbb{R}^n,0)$ be the germ of a closed subanalytic set and let $f:(X,0)\to(\mathbb{R},0)$ be a subanalytic function. We assume that X is contained in a open set U of $\mathbb{R}^n$ and that f is the restriction to X of a C^2-subanalytic function $F:U\to\mathbb{R}$. We denote by X^f the set $f^{-1}(0)$ and by [4], we can equip X with a subanalytic Thom stratification $\mathcal{V}=\{V_\alpha\}_{\alpha\in A}$ adapted to X^f. This means that $\{V_\alpha\in\mathcal{V} \mid V_\alpha \nsubseteq X^f\}$ is a Whitney stratification of $X\setminus X^f$ and that for any pair of strata (V_α,V_β) with $V_\alpha\nsubseteq X^f$ and $V_\beta\subset X^f$, the Thom condition is satisfied.

Let us denote by $\Sigma_{\mathcal{V}}f$ the critical locus of f. It is the union of the critical loci of f restricted to each stratum, i.e. $\Sigma_{\mathcal{V}}f=\cup_\alpha\Sigma(f_{|V_\alpha})$, where $\Sigma(f_{|V_\alpha})$ is the critical set of $f_{|V_\alpha}:V_\alpha\to\mathbb{R}$. Since $\Sigma_{\mathcal{V}}f\subset f^{-1}(0)$ (see Lemma 2.1), the fibre $f^{-1}(\delta)$ intersects the strata V_α's, $V_\alpha\nsubseteq X^f$, transversally if δ is sufficiently small. Hence $f^{-1}(\delta)$ is Whitney stratified with the induced stratification $\{f^{-1}(\delta)\cap V_\alpha \mid V_\alpha\nsubseteq X^f\}$.

By Lemma 2.1, we know that if $\epsilon>0$ is sufficiently small then the sphere S_ϵ intersects X^f transversally. By the Thom condition, this implies that there exists $\delta(\epsilon)>0$ such that for each δ with $0<|\delta|\le\delta(\epsilon)$, the sphere S_ϵ intersects the fibre $f^{-1}(\delta)$ transversally as well. Hence the set $f^{-1}(\delta)\cap B_\epsilon$ is a Whitney stratified set equipped with the following stratification:

$$\{f^{-1}(\delta)\cap V_\alpha\cap \mathring{B}_\epsilon, f^{-1}(\delta)\cap V_\alpha\cap S_\epsilon \mid V_\alpha\nsubseteq X^f\}.$$

Definition 3.1. *We call the set $f^{-1}(\delta)\cap B_\epsilon$, where $0<|\delta|\ll\epsilon\ll1$, a real Milnor fibre of f.*

We will use the following notation: $M_f^{\delta,\epsilon}=f^{-1}(\delta)\cap B_\epsilon$.

Now we consider another subanalytic function $g:(X,0)\to(\mathbb{R},0)$ and we assume that it is the restriction to X of a C^2-subanalytic function $G:U\to\mathbb{R}$. We denote by X^g the set $g^{-1}(0)$. Under some restrictions on g, we will study the topological behaviour of $g_{|M_f^{\delta,\epsilon}}$.

First we assume that g satisfies the following Condition (A):

- Condition (A): $g:(X,0)\to(\mathbb{R},0)$ has an isolated stratified critical point at 0.

This means that for each strata V_α of $\mathcal{V}$, $g:V_\alpha\setminus\{0\}\to\mathbb{R}$ is a submersion in a neighborhood of the origin.

In order to give the second assumption on g, we need to introduce some polar sets. Let V_α be a stratum of $\mathcal{V}$ not contained in X^f. Let $\Gamma^{V_\alpha}_{f,g}$ be the following set:

$$\Gamma^{V_\alpha}_{f,g}=\left\{x\in V_\alpha \mid \operatorname{rank}[\nabla f_{|V_\alpha}(x),\nabla g_{|V_\alpha}(x)]<2\right\},$$

and let $\Gamma_{f,g}$ be the union $\cup\Gamma^{V_\alpha}_{f,g}$ where $V_\alpha\nsubseteq X^f$. We call $\Gamma_{f,g}$ the relative polar set of f and g with respect to the stratification $\mathcal{V}$. We will assume that g satifies the following Condition (B):

- Condition (B): the relative polar set $\Gamma_{f,g}$ is a 1-dimensional C^1-subanalytic set (possibly empty) in a neighborhood of the origin.

Note that Condition (B) implies that $\overline{\Gamma_{f,g}} \cap X^f \subset \{0\}$ in a neighborhood of the origin because the frontiers of the $\Gamma_{f,g}^{V_\alpha}$'s are 0-dimensional.

From Condition (A) and Condition (B), we can deduce the following result.

Lemma 3.2. *We have $\overline{\Gamma_{f,g}} \cap X^g \subset \{0\}$ in a neighborhood of the origin.*

Proof. If it is not the case then there is a C^1-subanalytic curve $\gamma : [0,\nu[\to \Gamma_{f,g} \cap X^g$ such that $\gamma(0) = 0$ and $\gamma(]0,\nu[) \subset X^g \setminus \{0\}$. We can also assume that $\gamma(]0,\nu[)$ is contained in a stratum V. For $t \in]0,\nu[$, we have

$$0 = (g \circ \gamma)'(t) = \langle \nabla g_{|V}(\gamma(t)), \gamma'(t) \rangle.$$

Since $\gamma(t)$ belongs to $\Gamma_{f,g}$ and $\nabla g_{|V}(\gamma(t))$ does not vanish for $g : (X,0) \to (\mathbb{R},0)$ has an isolated stratified critical point at 0, we can conclude that $\langle \nabla f_{|V}(\gamma(t)), \gamma'(t) \rangle = 0$ and that $(f \circ \gamma)'(t) = 0$ for all $t \in]0,\nu[$. Therefore $f \circ \gamma \equiv 0$ because $f(0) = 0$ and $\gamma([0,\nu[)$ is included in X^f. This is impossible by the above remark. $\square$

Let $\mathcal{B}_1, \ldots, \mathcal{B}_l$ be the connected components of $\Gamma_{f,g}$, i.e. $\Gamma_{f,g} = \sqcup_{i=1}^l \mathcal{B}_i$. Each $\mathcal{B}_i$ is a C^1-subanalytic curve along which f is strictly increasing or decreasing and the intersection points of the $\mathcal{B}_i$'s with the fibre $M_f^{\delta,\epsilon}$ are exactly the critical points (in the stratified sense) of g on $f^{-1}(\delta) \cap \mathring{B}_\epsilon$. Let us write

$$M_f^{\delta,\epsilon} \cap \sqcup_{i=1}^l \mathcal{B}_i = \{p_1^{\delta,\epsilon}, \ldots, p_r^{\delta,\epsilon}\}.$$

Note that $r \leq l$.

Let us recall now the definition of the index of an isolated stratified critical point.

Definition 3.3. *Let $Z \subset \mathbb{R}^n$ be a closed subanalytic set, equipped with a Whitney stratification. Let $p \in Z$ be an isolated critical point of a subanalytic function $\phi : Z \to \mathbb{R}$, which is the restriction to Z of a C^2-subanalytic function Φ. We define the index of ϕ at p as follows:*

$$\mathrm{ind}(\phi, Z, p) = 1 - \chi\big(Z \cap \{\phi = \phi(p) - \eta\} \cap B_\epsilon(p)\big),$$

where $0 < \eta \ll \epsilon \ll 1$ and $B_\epsilon(p)$ is the closed ball of radius ϵ centered at p.

Our aim is to give a topological interpretation to the following sum:

$$\sum_{i=1}^r \mathrm{ind}(g, f^{-1}(\delta), p_i^{\delta,\epsilon}) + \mathrm{ind}(-g, f^{-1}(\delta), p_i^{\delta,\epsilon}).$$

For this, we will apply stratified Morse theory to $g_{|M_f^{\delta,\epsilon}}$. Note that the points p_i's are not the only critical points of $g_{|M_f^{\delta,\epsilon}}$ and other critical points can occur on the "boundary" $M_f^{\delta,\epsilon} \cap S_\epsilon$.

The next step is to study the behaviour of these "boundary" critical points for a generic choice of the C^2-distance function to the origin. Let $\rho : \mathbb{R}^n \to \mathbb{R}$ be a subanalytic C^2-distance function to the origin. We denote by $\tilde{S}_\epsilon$ the level $\rho^{-1}(\epsilon)$ and by $\tilde{B}_\epsilon$ the set $\{\rho \leq \epsilon\}$. We will focus on the critical points of $g_{|X^f \cap \tilde{S}_\epsilon}$ and $g_{|f^{-1}(\delta) \cap \tilde{S}_\epsilon}$, with $0 < |\delta| \ll \epsilon \ll 1$.

For each stratum V of X^f, let

$$\Gamma_{g,\rho}^V = \left\{ x \in V \mid \mathrm{rank}[\nabla g_{|V}(x), \nabla \rho_{|V}(x)] < 2 \right\},$$

and let $\Gamma_{g,\rho}^{X^f} = \cup_{V \subset X^f} \Gamma_{g,\rho}^V$. By Lemma 2.4 and the fact that $g : (X^f,0) \to (\mathbb{R},0)$ has an isolated stratified critical point at 0, we can assume that $\Gamma_{g,\rho}^{X^f}$ is a C^1-subanalytic curve in a neighborhood of the origin.

Lemma 3.4. *We have $\Gamma_{g,\rho}^{X^f} \cap X^g \subset \{0\}$ in a neighborhood of the origin.*

Proof. Same proof as Lemma 3.2. $\square$

Therefore if $\epsilon > 0$ is small enough, $g_{|\tilde{S}_\epsilon \cap X^f}$ has a finite number of critical points. They do not lie in the level $\{g = 0\}$ so by Lemma 2.3, they are outwards-pointing for $g_{|X^f \cap \tilde{B}_\epsilon}$ if they lie in $\{g > 0\}$ and inwards-pointing if they lie in $\{g < 0\}$.

Let us study now the critical points of $g_{|f^{-1}(\delta)\cap\tilde{S}_\epsilon}$. We will need the following lemma.

Lemma 3.5. *For every $\epsilon > 0$ sufficiently small, there exists $\delta(\epsilon) > 0$ such that for $0 < |\delta| \leq \delta(\epsilon)$, the points $p_i^{\delta,\epsilon}$ lie in $\tilde{B}_{\epsilon/4}$.*

Proof. Let

$$W = \left\{(x, r, y) \in U \times \mathbb{R} \times \mathbb{R} \mid \rho(x) = r, y = f(x) \text{ and } x \in \overline{\Gamma_{f,g}}\right\}.$$

Then W is a subanalytic set of $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$ and since it is a graph over $\overline{\Gamma_{f,g}}$, its dimension is less or equal to 1. Let

$$\begin{array}{cccc} \pi & : & \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} & \to & \mathbb{R} \times \mathbb{R} \\ & & (x, r, y) & \mapsto & (r, y), \end{array}$$

be the projection on the last two factors. Then $\pi_{|W} : W \to \pi(W)$ is proper and $\pi(W)$ is a closed subanalytic set in a neighborhood of the origin.

Let us write $Y_1 = \mathbb{R} \times \{0\}$ and let Y_2 be the closure of $\pi(W) \setminus Y_1$. Since Y_2 is a curve for W is a curve, 0 is isolated in $Y_1 \cap Y_2$. By Lojasiewicz's inequality, there exists a constant $C > 0$ and an integer $N > 0$ such that $|y| \geq Cr^N$ for (r, y) in Y_2 sufficiently close to the origin. So if $x \in \Gamma_{f,g}$ then $|f(x)| \geq C\rho(x)^N$ if $\rho(x)$ is small enough.

Let us fix $\epsilon > 0$ small. If $0 < |\delta| \leq \frac{1}{C}(\frac{\epsilon}{4})^N$ and $x \in f^{-1}(\delta) \cap \Gamma_{f,g}$ then $\rho(x) \leq \frac{\epsilon}{4}$. ☐

For each stratum $V \nsubseteq X^f$, let

$$\Gamma^V_{f,g,\rho} = \left\{x \in V \mid \operatorname{rank}[\nabla f_{|V}(x), \nabla g_{|V}(x), \nabla\rho_{|V}(x)] < 3\right\},$$

and let $\Gamma_{f,g,\rho} = \cup_{V \nsubseteq X^f} \Gamma^V_{f,g,\rho}$. By Lemma 2.6, we can assume that $\Gamma_{f,g,\rho} \backslash \Gamma_{f,g}$ is a C^1-subanalytic manifold of dimension 2. Let us choose $\epsilon > 0$ small enough so that $\tilde{S}_\epsilon$ intersects $\Gamma_{f,g,\rho} \setminus \Gamma_{f,g}$ transversally. Therefore $(\Gamma_{f,g,\rho} \setminus \Gamma_{f,g}) \cap \tilde{S}_\epsilon$ is a subanalytic curve. By Lemma 3.4, we can find $\delta(\epsilon) > 0$ such that $f^{-1}\big([-\delta(\epsilon), \delta(\epsilon)]\big) \cap \tilde{S}_\epsilon \cap \Gamma_{f,g}$ is empty and so

$$f^{-1}\big([-\delta(\epsilon), \delta(\epsilon)]\big) \cap (\Gamma_{f,g,\rho} \setminus \Gamma_{f,g}) \cap \tilde{S}_\epsilon = f^{-1}\big([-\delta(\epsilon), \delta(\epsilon)]\big) \cap \Gamma_{f,g,\rho} \cap \tilde{S}_\epsilon.$$

Let $C_1, \ldots, C_t$ be the connected components of $f^{-1}\big([-\delta(\epsilon), \delta(\epsilon)]\big) \cap \Gamma_{f,g,\rho} \cap \tilde{S}_\epsilon$ whose closure intersects $X^f \cap \tilde{S}_\epsilon$. Note that by Thom's (a_f)-condition, for each $i \in \{1, \ldots, t\}$, $\overline{C_i} \cap X^f$ is a subset of $\Gamma^{X^f}_{g,\rho}$. Let z_i be a point in $\overline{C_i} \cap X^f$. Since $C_i \cap X^f = \emptyset$, there exists $0 < \delta'_i(\epsilon) \leq \delta(\epsilon)$ such that the fibre $f^{-1}(\delta)$, $0 < |\delta| \leq \delta'_i(\epsilon)$, intersects C_i transversally in a neighborhood of z_i.

Let us choose δ such that $0 < |\delta| \leq \operatorname{Min}\{\delta'_i(\epsilon) \mid i = 1, \ldots, t\}$. Then the fibre $f^{-1}(\delta)$ intersects the C_i's transversally and $f^{-1}(\delta) \cap (\cup_i C_i)$ is exactly the set of critical points of $g_{|f^{-1}(\delta)\cap\tilde{S}_\epsilon}$. We have proved:

Lemma 3.6. *For $0 < |\delta| \ll \epsilon \ll 1$, $g_{|f^{-1}(\delta)\cap\tilde{S}_\epsilon}$ has a finite number of critical points, which are exactly the points in $\Gamma_{f,g,\rho} \cap \tilde{S}_\epsilon \cap f^{-1}(\delta)$.*

☐

Let $\{s_1^{\delta,\epsilon}, \ldots, s_u^{\delta,\epsilon}\}$ be the set of critical points of $g_{|f^{-1}(\delta)\cap\tilde{S}_\epsilon}$.

Lemma 3.7. *For $i \in \{1, \ldots, u\}$, $g(s_i^{\delta,\epsilon}) \neq 0$ and $s_i^{\delta,\epsilon}$ is outwards-pointing (resp. inwards-pointing) if and only if $g(s_i^{\delta,\epsilon}) > 0$ (resp. $g(s_i^{\delta,\epsilon}) < 0$).*

Proof. Note that $s_i^{\delta,\epsilon}$ is necessarily outwards-pointing or inwards-pointing because $s_i^{\delta,\epsilon} \notin \Gamma_{f,g}$.

Assume that for each $\delta > 0$ small enough, there exists a point $s_i^{\delta,\epsilon}$ such that $g(s_i^{\delta,\epsilon}) = 0$. Then we can construct a sequence of points $(\sigma_n)_{n\in\mathbb{N}}$ such that $g(\sigma_n) = 0$ and σ_n is a critical point of $g_{|f^{-1}(\frac{1}{n})\cap X\cap\tilde{S}_\epsilon}$. We can also assume that the points σ_n's belong to the same stratum S and that they tend to $\sigma \in V$ where $V \subseteq X^f$ and $V \subset \partial\overline{S}$. Therefore we have a decomposition:

$$\frac{\nabla g_{|S}(\sigma_n)}{\|\nabla g_{|S}(\sigma_n)\|} = \lambda_n \frac{\nabla f_{|S}(\sigma_n)}{\|\nabla f_{|S}(\sigma_n)\|} + \mu_n \frac{\nabla \rho_{|S}(\sigma_n)}{\|\nabla \rho_{|S}(\sigma_n)\|}.$$

Now by Whitney's condition (a), $T_{\sigma_n}S$ tends to a linear space T such that $T_\sigma V \subset T$. So $\nabla g_{|S}(\sigma_n)$ tends to a vector u in T whose orthogonal projection on $T_\sigma V$ is exactly $\nabla g_{|V}(\sigma)$. Since $g_{|V\setminus\{0\}}$ is a submersion, $\nabla g_{|V}(\sigma) \neq 0$ and so $u \neq 0$ and u is not orthogonal to $T_\sigma V$. So $\frac{\nabla g_{|S}(\sigma_n)}{\|\nabla g_{|S}(\sigma_n)\|}$ tends to $\frac{u}{\|u\|}$. Similarly $\nabla \rho_{|S}(\sigma_n)$ tends to a vector $u' \neq 0$ in T, not orthogonal to $T_\sigma V$ and whose orthogonal projection on $T_\sigma V$ is exactly $\nabla \rho_{|V}(\sigma)$. So $\frac{\nabla \rho_{|S}(\sigma_n)}{\|\nabla \rho_{|S}(\sigma_n)\|}$ tends to $\frac{u'}{\|u'\|}$.

By Thom's condition, $\frac{\nabla f_{|S}(\sigma_n)}{\|\nabla f_{|S}(\sigma_n)\|}$ tends to a vector w in T which is orthogonal to $T_\sigma V$. Since $\left|\langle w, \frac{u'}{\|u'\|}\rangle\right| < 1$, there exist C, $0 \leq C < 1$, and n_0 such that for $n \geq n_0$, we have

$$\left|\langle \frac{\nabla f_{|S}(\sigma_n)}{\|\nabla f_{|S}(\sigma_n)\|}, \frac{\nabla \rho_{|S}(\sigma_n)}{\|\nabla \rho_{|S}(\sigma_n)\|}\rangle\right| \leq C.$$

Since $\langle \frac{\nabla g_{|S}(\sigma_n)}{\|\nabla g_{|S}(\sigma_n)\|}, \frac{\nabla g_{|S}(\sigma_n)}{\|\nabla g_{|S}(\sigma_n)\|}\rangle = 1$, this implies that for $n \geq n_0$, $\lambda_n^2 + \mu_n^2 + 2C\lambda_n\mu_n \leq 1$ or $\lambda_n^2 + \mu_n^2 - 2C\lambda_n\mu_n \leq 1$. Then it is not difficult to see that $(\lambda_n)_{n\geq n_0}$ and $(\mu_n)_{n\geq n_0}$ are bounded. Taking a subsequence if necessary, we can assume that λ_n tends to a real λ and μ_n tends to a real μ. Taking the limit in the above equality, we obtain

$$\frac{u}{\|u\|} = \lambda w + \mu \frac{u'}{\|u'\|},$$

and so

$$u = \lambda \|u\| w + \mu \frac{\|u\|}{\|u'\|} u'.$$

Projecting this equality on $T_\sigma V$, we see that $\nabla g_{|V}(\sigma)$ and $\nabla \rho_{|V}(\sigma)$ are colinear which means that σ is a critical point of $g_{|X^f\cap\tilde{S}_\epsilon}$. But since $g(\sigma_n) = 0$, we find that $g(\sigma) = 0$, which is impossible by Lemma 3.4. This proves the first assertion.

To prove the second one, we use the same method. Assume that for each $\delta > 0$ small enough, there exists a point $s_i^{\delta,\epsilon}$ such that $g(s_i^{\delta,\epsilon}) > 0$ and $s_i^{\delta,\epsilon}$ is an inwards-pointing critical point for $g_{|f^{-1}(\delta)\cap\tilde{S}_\epsilon}$. Then we can construct a sequence of points $(\tau_n)_{n\in\mathbb{N}}$ such that $g(\tau_n) > 0$ and τ_n is an inwards-pointing critical point for $g_{|f^{-1}(\frac{1}{n})\cap X\cap\tilde{S}_\epsilon}$. We can also assume that the points τ_n's belong to the same stratum S and that they tend to $\tau \in V$ where $V \subseteq X^f$ and $V \subset \partial\overline{S}$. Therefore, we have a decomposition:

$$\frac{\nabla g_{|S}(\tau_n)}{\|\nabla g_{|S}(\tau_n)\|} = \lambda_n \frac{\nabla f_{|S}(\tau_n)}{\|\nabla f_{|S}(\tau_n)\|} + \mu_n \frac{\nabla \rho_{|S}(\tau_n)}{\|\nabla \rho_{|S}(\tau_n)\|},$$

with $\mu_n < 0$. Using the same arguments as above, we find that $\nabla g_{|V}(\tau) = \mu \nabla \rho_{|S}(\tau)$ with $\mu \leq 0$ and $g(\tau) \geq 0$. This contradicts the remark after Lemma 3.4. Of course, this proof works for $\delta < 0$. ☐

Let $\Gamma_{g,\rho}$ be the following polar set:

$$\Gamma_{g,\rho} = \{x \in U \mid \operatorname{rank}[\nabla g(x), \nabla \rho(x)] < 2\}.$$

By Lemma 2.5 and Lemma 2.1, we can assume that $\Gamma_{g,\rho} \setminus \{g=0\}$ does not intersect $X^f \setminus \{0\}$ in a neighborhood of 0 and so $\Gamma_{g,\rho} \setminus \{g=0\}$ does not intersect $X^f \cap \tilde{S}_\epsilon$ for $\epsilon > 0$ sufficiently small. Since the critical points of $g_{|X^f \cap \tilde{S}_\epsilon}$ lie outside $\{g=0\}$, they do not belong to $\Gamma_{g,\rho} \cap \tilde{S}_\epsilon$ and so the critical points of $g_{|f^{-1}(\delta)\cap X \cap \tilde{S}_\epsilon}$ do not neither if δ is sufficiently small. Hence at each critical point of $g_{|f^{-1}(\delta)\cap X\cap \tilde{S}_\epsilon}$, $g_{|\tilde{S}_\epsilon}$ is a submersion. We are in position to apply Theorem 3.1 and Lemma 2.1 in [15]. For $0 < |\delta| \ll \epsilon \ll 1$, we set

$$I(\delta,\epsilon,g) = \sum_{i=1}^{r} \operatorname{ind}(g, f^{-1}(\delta), p_i^{\delta,\epsilon}),$$

$$I(\delta,\epsilon,-g) = \sum_{i=1}^{r} \operatorname{ind}(-g, f^{-1}(\delta), p_i^{\delta,\epsilon}).$$

Theorem 3.8. *We have*

$$I(\delta,\epsilon,g) + I(\delta,\epsilon,-g) = 2\chi\big(f^{-1}(\delta)\cap \tilde{B}_\epsilon\big) - \chi\big(f^{-1}(\delta)\cap\tilde{S}_\epsilon\big) - \chi\big(X^g \cap f^{-1}(\delta)\cap \tilde{S}_\epsilon\big).$$

Proof. Let us denote by $\{a_j^+\}_{j=1}^{\alpha^+}$ (resp. $\{a_j^-\}_{j=1}^{\alpha^-}$) the outwards-pointing (resp. inwards-pointing) critical points of $g : f^{-1}(\delta)\cap\tilde{S}_\epsilon \to \mathbb{R}$. Applying Morse theory type theorem ([15], Theorem 3.1) and using Lemma 2.1 in [15], we can write

$$I(\delta,\epsilon,g) + \sum_{j=1}^{\alpha^-} \operatorname{ind}(g, f^{-1}(\delta)\cap\tilde{S}_\epsilon, a_j^-) = \chi(f^{-1}(\delta)\cap\tilde{B}_\epsilon) \qquad (1),$$

$$I(\delta,\epsilon,-g) + \sum_{j=1}^{\alpha^+} \operatorname{ind}(-g, f^{-1}(\delta)\cap\tilde{S}_\epsilon, a_j^+) = \chi(f^{-1}(\delta)\cap\tilde{B}_\epsilon) \qquad (2).$$

Let us evaluate

$$\sum_{j=1}^{\alpha^-} \operatorname{ind}(g, f^{-1}(\delta)\cap\tilde{S}_\epsilon, a_j^-) + \sum_{j=1}^{\alpha^+} \operatorname{ind}(-g, f^{-1}(\delta)\cap\tilde{S}_\epsilon, a_j^+).$$

Since the outwards-pointing critical points of $g_{|f^{-1}(\delta)\cap\tilde{S}_\epsilon}$ lie in $\{g>0\}$ and the inwards-pointing critical points of $g_{|f^{-1}(\delta)\cap\tilde{S}_\epsilon}$ lie in $\{g<0\}$, we have

$$\chi(f^{-1}(\delta)\cap\tilde{S}_\epsilon\cap\{g\geq 0\}) - \chi(f^{-1}(\delta)\cap\tilde{S}_\epsilon\cap\{g=0\}) = \sum_{j=1}^{\alpha^+} \operatorname{ind}(g, f^{-1}(\delta)\cap\tilde{S}_\epsilon, a_j^+) \qquad (3),$$

and

$$\chi(f^{-1}(\delta)\cap\tilde{S}_\epsilon\cap\{g\leq 0\}) - \chi(f^{-1}(\delta)\cap\tilde{S}_\epsilon\cap\{g=0\}) = \sum_{j=1}^{\alpha^-} \operatorname{ind}(-g, f^{-1}(\delta)\cap\tilde{S}_\epsilon, a_j^-) \quad (4).$$

Therefore making (3) + (4) and using the Mayer-Vietoris sequence, we find

$$\chi(f^{-1}(\delta)\cap\tilde{S}_\epsilon) - \chi(f^{-1}(\delta)\cap\tilde{S}_\epsilon\cap\{g=0\}) = \\ \sum_{j=1}^{\alpha^+} \operatorname{ind}(g, f^{-1}(\delta)\cap\tilde{S}_\epsilon, a_j^+) + \sum_{j=1}^{\alpha^-} \operatorname{ind}(-g, f^{-1}(\delta)\cap\tilde{S}_\epsilon, a_j^-) \quad (5).$$

Moreover we have

$$\chi(f^{-1}(\delta)\cap\tilde{S}_\epsilon)=\sum_{j=1}^{\alpha^+}\operatorname{ind}(g,f^{-1}(\delta)\cap\tilde{S}_\epsilon,a_j^+)+\sum_{j=1}^{\alpha^-}\operatorname{ind}(g,f^{-1}(\delta)\cap\tilde{S}_\epsilon,a_j^-)\qquad(6),$$

$$\chi(f^{-1}(\delta)\cap\tilde{S}_\epsilon)=\sum_{j=1}^{\alpha^+}\operatorname{ind}(-g,f^{-1}(\delta)\cap\tilde{S}_\epsilon,a_j^+)+\sum_{j=1}^{\alpha^-}\operatorname{ind}(-g,f^{-1}(\delta)\cap\tilde{S}_\epsilon,a_j^-)\quad(7).$$

The combination $-(5)+(6)+(7)$ leads to

$$\chi(f^{-1}(\delta)\cap\tilde{S}_\epsilon)+\chi(f^{-1}(\delta)\cap\tilde{S}_\epsilon\cap\{g=0\})=$$
$$\sum_{j=1}^{\alpha^+}\operatorname{ind}(-g,f^{-1}(\delta)\cap\tilde{S}_\epsilon,a_j^+)+\sum_{j=1}^{\alpha^-}\operatorname{ind}(g,f^{-1}(\delta)\cap\tilde{S}_\epsilon,a_j^-).$$

□

Let us assume now that $(X,0)$ is equipped with a Whitney stratification $\mathcal{W}=\cup_{\alpha\in A}W_\alpha$ and $f:(X,0)\to(\mathbb{R},0)$ has an isolated critical point at 0. In this situation, our results apply taking for $\mathcal{V}$ the following stratification:

$$\left\{W_\alpha\setminus f^{-1}(0),W_\alpha\cap f^{-1}(0)\setminus\{0\},\{0\}\mid W_\alpha\in\mathcal{W}\right\}.$$

Corollary 3.9. *If $f:(X,0)\to(\mathbb{R},0)$ has an isolated stratified critical point at 0, then*

$$I(\delta,\epsilon,g)+I(\delta,\epsilon,-g)=2\chi\big(f^{-1}(\delta)\cap\tilde{B}_\epsilon\big)-\chi\big(X^f\cap\tilde{S}_\epsilon\big)-\chi\big(X^f\cap X^g\cap\tilde{S}_\epsilon\big).$$

Proof. For each stratum W of X, let

$$\Gamma_{f,\rho}^W=\left\{x\in W\mid \operatorname{rank}[\nabla f_{|W}(x),\nabla\rho_{|W}(x)]<2\right\},$$

and let $\Gamma_{f,\rho}=\cup_W\Gamma_{f,\rho}^W$. By Lemma 3.4 applied to X and f instead of X^f and g,

$$\Gamma_{f,\rho}\cap\{f=0\}\subset\{0\}$$

in a neighborhood of the origin and so 0 is a regular value of $f:X\cap\tilde{S}_\epsilon\to\mathbb{R}$ for ϵ sufficiently small. By Thom-Mather's second isotopy lemma, $f^{-1}(0)\cap\tilde{S}_\epsilon$ is homeomorphic to $f^{-1}(\delta)\cap\tilde{S}_\epsilon$ for δ sufficiently small.

Now let p be a stratified critical point of $f:X^g\to\mathbb{R}$. By Lemma 2.1, we know that p belongs to $f^{-1}(0)\cap X^g$ and so p is also a critical point of $g:X^f\to\mathbb{R}$. Hence $p=0$ by Condition (A) and $f:X^g\to\mathbb{R}$ has an isolated stratified critical point at 0. As above, we conclude that $X^f\cap X^g\cap\tilde{S}_\epsilon$ is homeomorphic to $X^g\cap f^{-1}(\delta)\cap\tilde{S}_\epsilon$. □

Let $\omega(x)=\sqrt{x_1^2+\cdots+x_n^2}$ be the euclidian distance to the origin. As explained by Durfee in [10], Lemma 1.8 and Lemma 3.6, there is a neighborhood Ω of 0 in $\mathbb{R}^n$ such that for every stratum V of X^f, $\nabla\omega_{|V}$ and $\nabla\rho_{|V}$ are non-zero and do not point in opposite direction in $\Omega\setminus\{0\}$. Applying Durfee's argument ([10], Proposition 1.7 and Proposition 3.5), we see that $X^f\cap\tilde{S}_\epsilon$ is homeomorphic to $X^f\cap S_{\epsilon'}$ for $\epsilon,\epsilon'>0$ sufficiently small. Similarly $X^f\cap X^g\cap\tilde{S}_\epsilon$ and $X^f\cap X^g\cap S_{\epsilon'}$ are homemorphic. Now let us compare $f^{-1}(\delta)\cap\tilde{B}_\epsilon$ and $f^{-1}(\delta)\cap B_{\epsilon'}$. Let us choose ϵ' and ϵ such that

$$f^{-1}(\delta)\cap B_{\epsilon'}\subset f^{-1}(\delta)\cap\tilde{B}_\epsilon\subset\Omega.$$

If δ is sufficiently small then, for every stratum $V\not\subseteq X^f$, $\nabla\omega_{|V\cap f^{-1}(\delta)}$ and $\nabla\rho_{|V\cap f^{-1}(\delta)}$ are non-zero and do not point in opposite direction in $\tilde{B}_\epsilon\setminus\mathring{B}_{\epsilon'}$. Otherwise, by Thom's (a_f)-condition, we would find a point p in $X^f\cap(\tilde{B}_\epsilon\setminus\mathring{B}_{\epsilon'})$ such that either $\nabla\omega_{|S}(p)$ or $\nabla\rho_{|S}(p)$ vanish or $\nabla\omega_{|S}(p)$ and $\nabla\rho_{|S}(p)$ point in opposite direction, where S is the stratum of X^f that contains p. This is impossible if we are sufficiently close to the origin. Now, applying the same arguments as Durfee [10], Proposition 1.7 and Proposition 3.5, we see that $f^{-1}(\delta) \cap \tilde{B}_\epsilon$ is homeomorphic to $f^{-1}(\delta) \cap B_{\epsilon'}$ and that $f^{-1}(\delta) \cap \tilde{S}_\epsilon$ is homeomorphic to $f^{-1}(\delta) \cap S_{\epsilon'}$.

Theorem 3.10. *We have*

$$I(\delta,\epsilon,g) + I(\delta,\epsilon,-g) = 2\chi(M_f^{\delta,\epsilon}) - \chi(f^{-1}(\delta) \cap S_\epsilon) - \chi(X^g \cap f^{-1}(\delta) \cap S_\epsilon).$$

□

Corollary 3.11. *If $f : (X,0) \to (\mathbb{R},0)$ has an isolated stratified critical point at 0, then*

$$I(\delta,\epsilon,g) + I(\delta,\epsilon,-g) = 2\chi(M_f^{\delta,\epsilon}) - \chi(\mathrm{Lk}(X^f)) - \chi(\mathrm{Lk}(X^f \cap X^g)).$$

□

Let us remark if $\dim X = 2$ then in Theorem 3.10 and in Corollary 3.11, the last term of the right-hand side of the equality vanishes. If $\dim X = 1$ then in Theorem 3.10 and in Corollary 3.11, the last two terms of the right-hand side of the equality vanish.

4. An infinitesimal Gauss-Bonnet formula

In this section, we apply the results of the previous section to the case of linear forms and we establish a Gauss-Bonnet type formula for the real Milnor fibre.

We will first show that generic linear forms satisfy Condition (A) and Condition (B). For $v \in S^{n-1}$, let us denote by v^* the function $v^*(x) = \langle v, x\rangle$.

Lemma 4.1. *There exists a subanalytic set $\Sigma_1 \subset S^{n-1}$ of positive codimension such that if $v \notin \Sigma_1$, $\{v^* = 0\}$ intersects $X \setminus \{0\}$ transversally (in the stratified sense) in a neighborhood of the origin.*

Proof. It is a particular case of Lemma 3.8 in [14]. □

Corollary 4.2. *If $v \notin \Sigma_1$ then $v^*_{|X} : (X,0) \to (\mathbb{R},0)$ has an isolated stratified critical point at 0.*

Proof. By Lemma 2.1, we know that the stratified critical points of $v^*_{|X}$ lie in $\{v^* = 0\}$. But since $\{v^* = 0\}$ intersects $X \setminus \{0\}$ transversally, the only possible critical point of $v^*_{|X} : (X,0) \to (\mathbb{R},0)$ is the origin. □

Lemma 4.3. *There exists a subanalytic set $\Sigma_2 \subset S^{n-1}$ of positive codimension such that if $v \notin \Sigma_2$, then Γ_{f,v^*} is a C^1-subanalytic curve (possibly empty) in a neighborhood of 0.*

Proof. Let V be stratum of dimension e such that $V \nsubseteq X^f$. We can assume that $e \geq 2$. Let

$$M_V = \left\{(x,y) \in V \times \mathbb{R}^n \mid \mathrm{rank}[\nabla f_{|V}(x), \nabla y^*_{|V}(x)] < 2\right\}.$$

It is a subanalytic manifold of class C^1 and of dimension $n+1$. To see this, let us pick a point (x,y) in M_V. In a neighborhood of x, V is defined by the vanishing of $k = n-e$ C^2-subanalytic functions $f_1, \ldots, f_k$. Since V is not included in X^f, $f : V \to \mathbb{R}$ is a submersion and we can assume that in a neighborhood of x, the following $(k+1)\times(k+1)$-minor:

$$\frac{\partial(f_1,\ldots,f_k,f)}{\partial(x_1,\ldots,x_k,x_{k+1})},$$

does not vanish. Therefore, in a neighborhood of (x,y), M_V is defined by the vanishing of the following $(k+2)\times(k+2)$-minors:

$$\frac{\partial(f_1,\ldots,f_k,f,y^*)}{\partial(x_1,\ldots,x_k,x_{k+1},x_{k+2})}, \ldots, \frac{\partial(f_1,\ldots,f_k,f,y^*)}{\partial(x_1,\ldots,x_k,x_{k+1},x_n)}.$$

A simple computation of determinants shows that the gradient vectors of these minors are linearly independent. As in previous lemmas, we show that Σ_{f,v^*} is one-dimensional considering the projection

$$\begin{array}{rccc} \pi_2 & : & M^V & \to & \mathbb{R}^n \\ & & (x,y) & \mapsto & y. \end{array}$$

Since $\Gamma_{f,v^*} = \cup_{V \not\subseteq X^f} \Gamma^V_{f,v^*}$, we get the result. □

Let $\Sigma = \Sigma_1 \cup \Sigma_2$, it is a subanalytic subset of S^{n-1} of positive codimension and if $v \notin \Sigma$ then v^* satisfies Conditions (A) and (B). In particular, $v^*_{|f^{-1}(\delta)\cap X \cap \mathring{B}_\epsilon}$ has a finite number of critical points $p_1^{\delta,\epsilon}, \dots, p_{r_v}^{\delta,\epsilon}$. We recall that

$$I(\delta,\epsilon,v^*) = \sum_{i=1}^{r_v} \mathrm{ind}(v^*, f^{-1}(\delta), p_i^{\delta,\epsilon}),$$

$$I(\delta,\epsilon,-v^*) = \sum_{i=1}^{r_v} \mathrm{ind}(-v^*, f^{-1}(\delta), p_i^{\delta,\epsilon}).$$

In this situation, Theorem 3.10 and Corollary 3.11 become

Corollary 4.4. *If $v \notin \Sigma$ then*

$$I(\delta,\epsilon,v^*) + I(\delta,\epsilon,-v^*) = 2\chi(M_f^{\delta,\epsilon}) - \chi(f^{-1}(\delta)\cap S_\epsilon) - \chi(X^{v^*} \cap f^{-1}(\delta) \cap S_\epsilon).$$

Furthermore, if $f : (X,0) \to (\mathbb{R},0)$ has an isolated stratified critical point at 0, then

$$I(\delta,\epsilon,v^*) + I(\delta,\epsilon,-v^*) = 2\chi(M_f^{\delta,\epsilon}) - \chi(\mathrm{Lk}(X^f)) - \chi(\mathrm{Lk}(X^f \cap X^{v^*})).$$

□

As an application, we give a Gauss-Bonnet formula for the Milnor fibre $M_f^{\delta,\epsilon}$.

Let $\Lambda_0(f^{-1}(\delta), -)$ be the Gauss-Bonnet measure on $f^{-1}(\delta)$ defined by

$$\Lambda_0(f^{-1}(\delta), U') = \frac{1}{s_{n-1}} \int_{S^{n-1}} \sum_{x \in U'} \mathrm{ind}(v^*, f^{-1}(\delta), x) dx,$$

where U' is a Borel set of $f^{-1}(\delta)$ (see [6], page 299) and s_{n-1} is the volume of the unit sphere S^{n-1}. Note that if x is not a critical point of $v^*_{|f^{-1}(\delta)}$ then $\mathrm{ind}(v^*, f^{-1}(\delta), x) = 0$. We are going to evaluate

$$\lim_{\epsilon\to 0}\lim_{\delta\to 0} \Lambda_0(f^{-1}(\delta), M_f^{\delta,\epsilon}).$$

Theorem 4.5. *We have*

$$\lim_{\epsilon\to 0}\lim_{\delta\to 0} \Lambda_0(f^{-1}(\delta), M_f^{\delta,\epsilon}) = \chi(M_f^{\delta,\epsilon}) - \frac{1}{2}\chi(f^{-1}(\delta)\cap S_\epsilon) - \frac{1}{2s_{n-1}} \int_{S^{n-1}} \chi(f^{-1}(\delta) \cap \{v^*=0\} \cap S_\epsilon) dv.$$

Furthermore, if $f : (X,0) \to (\mathbb{R},0)$ has an isolated stratified critical point at 0, then

$$\lim_{\epsilon\to 0}\lim_{\delta\to 0} \Lambda_0(f^{-1}(\delta), M_f^{\delta,\epsilon}) = \chi(M_f^{\delta,\epsilon}) - \frac{1}{2}\chi(\mathrm{Lk}(X^f)) - \frac{1}{2s_{n-1}} \int_{S^{n-1}} \chi(\mathrm{Lk}(X^f \cap X^{v^*})) dv.$$

Proof. By definition, we have

$$\Lambda_0(f^{-1}(\delta), M_f^{\delta,\epsilon}) = \frac{1}{s_{n-1}} \int_{S^{n-1}} \sum_{x \in M_f^{\delta,\epsilon}} \mathrm{ind}(v^*, f^{-1}(\delta), x) dv.$$

It is not difficult to see that

$$\Lambda_0(f^{-1}(\delta), M_f^{\delta,\epsilon}) = \frac{1}{2s_{n-1}} \int_{S^{n-1}} \Big[\sum_{x \in M_f^{\delta,\epsilon}} \mathrm{ind}(v^*, f^{-1}(\delta), x) + \mathrm{ind}(-v^*, f^{-1}(\delta), x) \Big] dv.$$

Note that if $v \notin \Sigma$ then

$$\sum_{x \in M_f^{\delta,\epsilon}} \mathrm{ind}(v^*, f^{-1}(\delta), x) + \mathrm{ind}(-v^*, f^{-1}(\delta), x)$$

is equal to $I(\delta, \epsilon, v^*) + I(\delta, \epsilon, -v^*)$ and is uniformly bounded by Hardt's theorem. By Lebesgue's theorem, we obtain

$$\lim_{\epsilon \to 0} \lim_{\delta \to 0} \Lambda_0(f^{-1}(\delta), M_f^{\delta,\epsilon}) = \frac{1}{2s_{n-1}} \int_{S^{n-1}} \lim_{\epsilon \to 0} \lim_{\delta \to 0} [I(\delta, \epsilon, v^*) + I(\delta, \epsilon, -v^*)] dv.$$

We just have to apply the previous corollary to conclude. □

5. Infinitesimal linear kinematic formulas

In this section, we apply the results of the previous section to the case of a linear function in order to obtain "infinitesimal" linear kinematic formulas for closed subanalytic germs.

We start recalling known facts on the geometry of subanalytic sets. We need some notations:

- for $k \in \{0, \ldots, n\}$, G_n^k is the Grassmann manifold of k-dimensional linear subspaces in $\mathbb{R}^n$ and g_n^k is its volume,
- for $k \in \mathbb{N}$, b_k is the volume of the k-dimensional unit ball and s_k is the volume of the k-dimensional unit sphere.

In [17], Fu developed integral geometry for compact subanalytic sets. Using the technology of the normal cycle, he associated with every compact subanalytic set $X \subset \mathbb{R}^n$ a sequence of curvature measures

$$\Lambda_0(X, -), \ldots, \Lambda_n(X, -),$$

called the Lipschitz-Killing measures. He proved several integral geometry formulas, among them a Gauss-Bonnet formula and a kinematic formula. Later another description of the measures using stratified Morse theory was given by Broecker and Kuppe [6] (see also [5]). The reader can refer to [14], Section 2, for a rather complete presentation of these two approaches and for the definition of the Lipschitz-Killing measures.

Let us give some comments on these Lipschitz-Killing curvatures. If dim $X = d$ then

$$\Lambda_{d+1}(X, U') = \cdots = \Lambda_n(X, U') = 0,$$

for any Borel set U' of X and $\Lambda_d(X, U') = \mathcal{L}_d(U')$, where $\mathcal{L}_d$ is the d-dimensional Lebesgue measure in $\mathbb{R}^n$. Furthemore if X is smooth then for any Borel set U' of X and for $k \in \{0, \ldots, d\}$, $\Lambda_k(X, U')$ is related to the classical Lipschitz-Killing-Weil curvature K_{d-k} through the following equality:

$$\Lambda_k(X, U') = \frac{1}{s_{n-k-1}} \int_{U'} K_{d-k}(x) dx.$$

In [14], Section 5, we studied the asymptotic behaviour of the Lipschitz-Killing measures in the neighborhood of a point of X. Namely we proved the following theorem ([14], Theorem 5.1).

Theorem 5.1. *Let $X \subset \mathbb{R}^n$ be a closed subanalytic set such that $0 \in X$. We have:*

$$\lim_{\epsilon\to 0} \Lambda_0(X, X\cap B_\epsilon) = 1 - \frac{1}{2}\chi(\mathrm{Lk}(X)) - \frac{1}{2g_n^{n-1}} \int_{G_n^{n-1}} \chi(\mathrm{Lk}(X\cap H))dH.$$

Furthermore for $k \in \{1, \ldots, n-2\}$, we have:

$$\lim_{\epsilon\to 0} \frac{\Lambda_k(X, X\cap B_\epsilon)}{b_k \epsilon^k} = -\frac{1}{2g_n^{n-k-1}} \int_{G_n^{n-k-1}} \chi(\mathrm{Lk}(X\cap H))dH + \frac{1}{2g_n^{n-k+1}} \int_{G_n^{n-k+1}} \chi(\mathrm{Lk}(X\cap L))dL,$$

and:

$$\lim_{\epsilon\to 0} \frac{\Lambda_{n-1}(X, X\cap B_\epsilon)}{b_{n-1} \epsilon^{n-1}} = \frac{1}{2g_n^2} \int_{G_n^2} \chi(\mathrm{Lk}(X\cap H))dH,$$

$$\lim_{\epsilon\to 0} \frac{\Lambda_n(X, X\cap B_\epsilon)}{b_n \epsilon^n} = \frac{1}{2g_n^1} \int_{G_n^1} \chi(\mathrm{Lk}(X\cap H))dH.$$

In the sequel, we will use these equalities and Theorem 4.5 to establish linear kinematic types formulas for the quantities $\lim_{\epsilon\to 0} \frac{\Lambda_k(X, X\cap B_\epsilon)}{b_k \epsilon^k}$, $k = 1, \ldots, n$. Let us start with some lemmas. We work with a closed subanalytic set X such that $0 \in X$, equipped with a Whitney stratification $\{W_\alpha\}_{\alpha\in A}$.

Lemma 5.2. *Let f be a C^2-subanalytic function such that $f_{|X} : X \to \mathbb{R}$ has an isolated stratified critical point at 0. Then for $0 < \delta \ll \epsilon \ll 1$, we have*

$$\chi(M_f^{\delta,\epsilon}) + \chi(M_f^{-\delta,\epsilon}) = \chi(\mathrm{Lk}(X)) + \chi(\mathrm{Lk}(X^f)).$$

Proof. With the same technics and arguments as the ones we used in order to establish Corollary 3.11, we can prove that

$$\mathrm{ind}(f, X, 0) + \mathrm{ind}(-f, X, 0) = 2\chi(X\cap B_\epsilon) - \chi(\mathrm{Lk}(X)) - \chi(\mathrm{Lk}(X^f)).$$

We conclude thanks to the following equalities

$$\mathrm{ind}(f, X, 0) = 1 - \chi(M_f^{-\delta,\epsilon}),\ \mathrm{ind}(-f, X, 0) = 1 - \chi(M_f^{\delta,\epsilon}),\ \text{ and }\ \chi(X\cap B_\epsilon) = 1.$$

□

Corollary 5.3. *There exist a subanalytic set $\Sigma_1 \subset S^{n-1}$ of positive codimension such that if $v \notin \Sigma$ then for $0 < \delta \ll \epsilon \ll 1$,*

$$\chi(M_{v^*}^{\delta,\epsilon}) + \chi(M_{v^*}^{-\delta,\epsilon}) = \chi(\mathrm{Lk}(X)) + \chi(\mathrm{Lk}(X\cap \{v^* = 0\})).$$

Proof. Apply Corollary 4.2 and Lemma 5.2. □

Lemma 5.4. *Let $S \subset \mathbb{R}^n$ be a C^2-subanalytic manifold. Let $H \in G_n^{n-k}$, $k \in \{1, \ldots, n\}$, such that H intersects $S \setminus \{0\}$ transversally and let $G_{H^\perp}^1$ be the Grassmann manifold of lines in the orthogonal complement $H^\perp$ of H. There exists a subanalytic set $\Sigma'_H \subset G_{H^\perp}^1$ of positive codimension such that if $\nu \notin \Sigma'_H$ then $H \oplus \nu$ intersects $S \setminus \{0\}$ transversally.*

Proof. Assume that S has dimension e and that H is given by the equations $x_1 = \ldots = x_k = 0$ so that $H^\perp = \mathbb{R}^k$ with coordinate system $(x_1, \ldots, x_k)$. Since H intersects $S \setminus \{0\}$ transversally, we just have to consider points outside H. Let W be defined by

$$W = \Big\{(x, v_1, \ldots, v_{k-1}) \in \mathbb{R}^n \times (\mathbb{R}^k)^{k-1} \mid x \in S \setminus H \text{ and } \langle x, v_1\rangle = \cdots = \langle x, v_{k-1}\rangle = 0\Big\},$$

where $v_i \in \mathbb{R}^k \times \{0\} \subset \mathbb{R}^n$. Let us show that W is a C^2-subanalytic manifold of dimension $e + (k-1)^2$. Let (y, w) be a point in W. We can assume that around y, S is defined by the vanishing of $n-e$ C^2-subanalytic functions $f_1, \ldots, f_{n-e}$. Hence in a neighborhood of (y, w), W is defined by the equations:

$$f_1(x) = \ldots = f_{n-e}(x) = 0 \text{ and } \langle x, v_1 \rangle = \cdots = \langle x, v_{k-1} \rangle = 0.$$

The gradient vectors of this $n-e+k-1$ functions are linearly independent in a neighborhood of (y, w). To see this, we observe that there exists $j \in \{1, \ldots, k\}$ such that $x_j \neq 0$ because y does not belong to H. Therefore, writing $v_i = (v_i^1, \ldots, v_i^k, 0, \ldots, 0)$ for $i \in \{1, \ldots, k-1\}$, we see that

$$\frac{\partial \langle x, v_i \rangle}{\partial v_i^j}(x) \neq 0,$$

for $i = 1, \ldots, k-1$. This enables us to conclude that W is a C^2-subanalytic manifold of dimension $e + (k-1)^2$. Let π_2 be the following projection:

$$\pi_2 : W \to (\mathbb{R}^n)^{n-k}, (x, v_1, \ldots, v_{n-k}) \mapsto (v_1, \ldots, v_{n-k}).$$

Bertini-Sard's theorem implies that the set of critical values of π_2 is a subanalytic set of positive codimension. If $(v_1, \ldots, v_{k-1})$ lies outside this subanalytic set then the $(n-k+1)$-plane

$$\{x \in \mathbb{R}^n \mid \langle x, v_1 \rangle = \cdots = \langle x, v_{k-1} \rangle = 0\}$$

contains H and intersects $S \setminus \{0\}$ transversally. $\square$

Now we can present our infinitesimal linear kinematic formulas.

Let $H \in G_n^{n-k}$, $k \in \{1, \ldots, n\}$, and let $S_{H^\perp}^{k-1}$ be the unit sphere of the orthogonal complement of H. Let v be an element in $S_{H^\perp}^{k-1}$. For $\delta > 0$, we denote by $H_{v,\delta}$ the $(n-k)$-dimensional affine space $H + \delta v$ and we set

$$\beta_0(H, v) = \lim_{\epsilon \to 0} \lim_{\delta \to 0} \Lambda_0(H_{\delta,v} \cap X, H_{\delta,v} \cap X \cap B_\epsilon).$$

Then we set

$$\beta_0(H) = \frac{1}{s_{k-1}} \int_{S_{H^\perp}^{k-1}} \beta_0(H, v) dv.$$

Theorem 5.5. *For $k \in \{1, \ldots, n\}$, we have*

$$\lim_{\epsilon \to 0} \frac{\Lambda_k(X, X \cap B_\epsilon)}{b_k \epsilon^k} = \frac{1}{g_n^{n-k}} \int_{G_n^{n-k}} \beta_0(H) dH.$$

Proof. We treat first the case $k \in \{1, \ldots, n-2\}$. By Theorem 5.1 , we know that

$$\lim_{\epsilon \to 0} \frac{\Lambda_k(X, X \cap B_\epsilon)}{b_k \epsilon^k} = -\frac{1}{2g_n^{n-k-1}} \int_{G_n^{n-k-1}} \chi(\mathrm{Lk}(X \cap H)) dH$$
$$+ \frac{1}{2g_n^{n-k+1}} \int_{G_n^{n-k+1}} \chi(\mathrm{Lk}(X \cap L)) dL.$$

By Lemma 3.8 in [14], we know that generically H intersects $X \setminus \{0\}$ transversally in a neighborhood of the origin. Let us fix H that satisfies this generic property. For any $v \in S_{H^\perp}^{k-1}$, let ν be the line generated by v and let L_v be the $(n-k+1)$-plane defined by $L_v = H \oplus \nu$. By Lemma 5.4, we know that for v generic in $S_{H^\perp}^{k-1}$, L_v intersects $X \setminus \{0\}$ transversally in a neighborhood of the origin. Therefore, $v^*_{|X \cap L_v}$ has an isolated singular point at 0 and we can apply Theorem 4.5. We have

$$\lim_{\epsilon\to 0}\lim_{\delta\to 0}\Lambda_0(X\cap L_v\cap\{v^*=\delta\}, X\cap L_v\cap\{v^*=\delta\}\cap B_\epsilon) =$$
$$\chi(X\cap L_v\cap\{v^*=\delta\}\cap B_\epsilon) - \tfrac{1}{2}\chi(\mathrm{Lk}(X\cap L_v\cap\{v^*=0\}))$$
$$-\tfrac{1}{2s_{n-k}}\int_{S^{n-k}_{L_v}}\chi(\mathrm{Lk}(X\cap L_v\cap\{v^*=0\}\cap\{w^*=0\}))dw,$$

where $S^{n-k}_{L_v}$ is the unit sphere of L_v. Let us remark that $L_v\cap\{v^*=\delta\}$ is exactly $H_{v,\delta}$ and that $L_v\cap\{v^*=0\}$ is H. We can also apply Lemma 5.2 to $v^*_{|X\cap L_v}$ to obtain the following relation:

$$\beta_0(H,v)+\beta_0(H,-v) = \chi(\mathrm{Lk}(X\cap L_v)) - \frac{1}{s_{n-k}}\int_{S^{n-k}_{L_v}}\chi(\mathrm{Lk}(X\cap H\cap\{w^*=0\}))dw.$$

Since $\beta_0(H)$ is equal to

$$\frac{1}{2s_{k-1}}\int_{S^{k-1}_{H^\perp}}[\beta_0(H,v)+\beta_0(H,-v)]\,dv,$$

we find that

$$\beta_0(H) = \frac{1}{2s_{k-1}}\int_{S^{k-1}_{H^\perp}}\chi(\mathrm{Lk}(X\cap L_v))dv$$
$$-\frac{1}{2s_{k-1}s_{n-k}}\int_{S^{k-1}_{H^\perp}}\int_{S^{n-k}_{L_v}}\chi(\mathrm{Lk}(X\cap H\cap\{w^*=0\}))dwdv.$$

Replacing spheres with Grassman manifolds in this equality, we obtain

$$\beta_0(H) = \frac{1}{2g^1_k}\int_{G^1_{H^\perp}}\chi(\mathrm{Lk}(X\cap H\oplus\nu))d\nu$$
$$-\frac{1}{2g^1_k g^{n-k}_{n-k+1}}\int_{G^1_{H^\perp}}\int_{G^{n-k}_{H\oplus\nu}}\chi(\mathrm{Lk}(X\cap H\cap K))dKd\nu.$$

Therefore, we have

$$\frac{1}{g^{n-k}_n}\int_{G^{n-k}_n}\beta_0(H)dH = \frac{1}{2g^1_k g^{n-k}_n}\int_{G^{n-k}_n}\int_{G^1_{H^\perp}}\chi(\mathrm{Lk}(X\cap H\oplus\nu))d\nu dH-$$
$$\frac{1}{2g^{n-k}_n g^1_k g^{n-k}_{n-k+1}}\int_{G^{n-k}_n}\int_{G^1_{H^\perp}}\int_{G^{n-k}_{H\oplus\nu}}\chi(\mathrm{Lk}(X\cap H\cap K))dKd\nu dH.$$

Let us compute

$$\mathcal{I} = \frac{1}{2g^{n-k}_n g^1_k}\int_{G^{n-k}_n}\int_{G^1_{H^\perp}}\chi(\mathrm{Lk}(X\cap H\oplus\nu))d\nu dH.$$

Let $\mathcal{H}$ be the flag variety of pairs (L,H), $L\in G^{n-k+1}_n$ and $H\in G^{n-k}_L$. This variety is a bundle over G^{n-k}_n, each fibre being a G^1_k. Hence we have

$$\int_{G^{n-k}_n}\int_{G^1_{H^\perp}}\chi(\mathrm{Lk}(X\cap H\oplus\nu))d\nu dH = \int_{G^{n-k+1}_n}\int_{G^{n-k}_L}\chi(\mathrm{Lk}(X\cap L))dHdL =$$
$$g^{n-k}_{n-k+1}\int_{G^{n-k+1}_n}\chi(\mathrm{Lk}(X\cap L))dL.$$

Finally, we get that

$$\mathcal{I} = \frac{g^{n-k}_{n-k+1}}{2g^{n-k}_n g^1_k}\int_{G^{n-k+1}_n}\chi(\mathrm{Lk}(X\cap L))dL = \frac{1}{2g^{n-k+1}_n}\int_{G^{n-k+1}_n}\chi(\mathrm{Lk}(X\cap L))dL.$$

Let us compute now

$$\mathcal{J} = \frac{1}{2g_n^{n-k} g_k^1 g_{n-k+1}^{n-k}} \int_{G_n^{n-k}} \int_{G^1_{H^\perp}} \int_{G^{n-k}_{H\oplus\nu}} \chi(\mathrm{Lk}(X\cap H\cap K)) dK d\nu dH.$$

First, as we have just done above, we can write

$$\mathcal{J} = \frac{1}{2g_n^{n-k} g_k^1 g_{n-k+1}^{n-k}} \int_{G_n^{n-k+1}} \int_{G_L^{n-k}} \int_{G_L^{n-k}} \chi(\mathrm{Lk}(X\cap H\cap K)) dK dH dL.$$

Then we remark (see [14], Corollary 3.11 for a similar argument) that

$$\frac{1}{g_{n-k+1}^{n-k}} \int_{G_L^{n-k}} \chi(\mathrm{Lk}(X\cap H\cap K)) dK = \frac{1}{g_{n-k}^{n-k-1}} \int_{G_H^{n-k-1}} \chi(\mathrm{Lk}(X\cap J)) dJ,$$

and so

$$\mathcal{J} = \frac{1}{2g_n^{n-k} g_k^1 g_{n-k}^{n-k-1}} \int_{G_n^{n-k+1}} \int_{G_L^{n-k}} \int_{G_H^{n-k-1}} \chi(\mathrm{Lk}(X\cap J)) dJ dH dL.$$

Considering the flag variety of pairs (H, J), $H \in G_L^{n-k}$ and $J \in G_H^{n-k-1}$, and proceeding as above, we find

$$\int_{G_L^{n-k}} \int_{G_H^{n-k-1}} \chi(\mathrm{Lk}(X\cap J)) dJ dH = g_2^1 \int_{G_L^{n-k-1}} \chi(\mathrm{Lk}(X\cap J)) dJ,$$

so

$$\mathcal{J} = \frac{g_2^1}{2g_n^{n-k} g_k^1 g_{n-k}^{n-k-1}} \int_{G_n^{n-k+1}} \int_{G_L^{n-k-1}} \chi(\mathrm{Lk}(X\cap J)) dJ.$$

To finish the computation, we consider the flag variety of pairs (L, J), $L \in G_n^{n-k+1}$ and $J \in G_L^{n-k-1}$. It is a bundle over G_n^{n-k-1}, each fibre being a G_{k+1}^2. Hence we have

$$\mathcal{J} = \frac{g_2^1}{2g_n^{n-k} g_k^1 g_{n-k}^{n-k-1}} \int_{G_n^{n-k-1}} \int_{G^2_{J^\perp}} \chi(\mathrm{Lk}(X\cap J)) dJ dM,$$

$$\mathcal{J} = \frac{g_2^1 g_{k+1}^2}{2g_n^{n-k} g_k^1 g_{n-k}^{n-k-1}} \int_{G_n^{n-k-1}} \chi(\mathrm{Lk}(X\cap J)) dJ = \frac{1}{2g_n^{n-k-1}} \int_{G_n^{n-k-1}} \chi(\mathrm{Lk}(X\cap J)) dJ.$$

This ends the proof for the case $k \in \{1, \ldots, n-2\}$. For $k = n-1$ or n, the proof is the same. We just have to remark that in these cases

$$\beta_0(H, v) + \beta_0(H, -v) = \chi(\mathrm{Lk}(X \cap L_v)),$$

and if $k = n-1$, $\dim L_v = 2$ and if $k = n$, $\dim L_v = 1$. $\square$

Let us end with some remarks on the limits $\lim_{\epsilon\to 0} \frac{\Lambda_k(X, X\cap B_\epsilon)}{b_k \epsilon^k}$. We already know that if $\dim X = d$ then $\lim_{\epsilon\to 0} \frac{\Lambda_k(X, X\cap B_\epsilon)}{b_k \epsilon^k} = 0$ for $k \geq d+1$. This is also the case if $l < d_0$, where d_0 is the dimension of the stratum that contains 0. To see this let us first relate the limits $\lim_{\epsilon\to 0} \frac{\Lambda_k(X, X\cap B_\epsilon)}{b_k \epsilon^k}$ to the polar invariants defined by Comte and Merle in [9]. They can be defined as follows. Let $H \in G_n^{n-k}$, $k \in \{1, \ldots, n\}$, and let v be an element in $S^{k-1}_{H^\perp}$. For $\delta > 0$, we set

$$\lambda_0(H, v) = \lim_{\epsilon\to 0} \lim_{\delta\to 0} \chi(H_{\delta,v} \cap X \cap B_\epsilon),$$

$$\lambda_0(H) = \frac{1}{s_{k-1}} \int_{S^{k-1}_{H^\perp}} \lambda_0(H, v) dv,$$

and then

$$\sigma_k(X, 0) = \frac{1}{g_n^{n-k}} \int_{G_n^{n-k}} \lambda_0(H) dH.$$

Moreover, we put $\sigma_0(X,0)=1$.

Theorem 5.6. *For $k \in \{0,\dots,n-1\}$, we have*

$$\lim_{\epsilon\to 0}\frac{\Lambda_k(X,X\cap B_\epsilon)}{b_k\epsilon^k}=\sigma_k(X,0)-\sigma_{k+1}(X,0).$$

Furthermore, we have

$$\lim_{\epsilon\to 0}\frac{\Lambda_n(X,X\cap B_\epsilon)}{b_n\epsilon^n}=\sigma_n(X,0).$$

Proof. It is the same proof as Theorem 5.5. For example if $k \in \{0,\dots,n-1\}$, we just have to remark that

$$\lambda_0(H,v)+\lambda_0(H,-v)=\chi(\mathrm{Lk}(X\cap L_v))+\chi(\mathrm{Lk}(X\cap H)),$$

by Lemma 5.2, which implies that

$$\sigma_k(X,0)=\frac{1}{2g_n^{n-k+1}}\int_{G_n^{n-k+1}}\chi(\mathrm{Lk}(X\cap L))dL+\frac{1}{2g_n^{n-k}}\int_{G_n^{n-k}}\chi(\mathrm{Lk}(X\cap H))dH.$$

□

It is explained in [9] that $\sigma_k(X,0)=1$ if $0\le k\le d_0$, so if $k<d_0$ then $\lim_{\epsilon\to 0}\frac{\Lambda_k(X,X\cap B_\epsilon)}{b_k\epsilon^k}=0$.

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STRATIFICATIONS OF INERTIA SPACES OF COMPACT LIE GROUP ACTIONS

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Dedicated to David Trotman on the occasion of his 60th birthday.

ABSTRACT. We study the topology of the inertia space of a smooth G-manifold M where G is a compact Lie group. We construct an explicit Whitney stratification of the inertia space, demonstrating that the inertia space is a triangulable differentiable stratified space. In addition, we demonstrate a de Rham theorem for differential forms defined on the inertia space with respect to this stratification.

1. INTRODUCTION

Let G be a compact Lie group acting (from the left) on a smooth manifold M. In the case where G acts locally freely on M, the orbit space $X := G\backslash M$ is an orbifold. Moreover, in this situation, the inertia space ΛX of the orbifold X can be defined as the quotient of the disjoint union $\bigsqcup_{g\in G} M^g$ of the fixed point manifolds M^g by the natural action of the Lie group G. It turns out that ΛX is an orbifold as well which in general has several connected components of varying dimension. The inertia space of an orbifold has originally been introduced by Kawasaki in [KAW78, p. 77] and subsequently used in [KAW79, KAW84]. In these papers, the inertia orbifold served as a bookkeeping device for the formulation of the topological index in an orbifold signature theorem resp. an orbifold index formula. Since Kawasaki's work, the inertia orbifold has played a major role for all formulations of index theorems on orbifolds; see e.g. [FAR92, FAR07, PFPOTA07, VERG]. In addition, the inertia orbifold has been widely studied in connection with the Chern character for orbifolds, which provides an isomorphism between the (rationalized) orbifold K-theory and the cohomology of the inertia orbifold, as well as with Chen–Ruan orbifold cohomology, which is additively isomorphic to the cohomology of the inertia orbifold; see e.g. [ADLERU, BACO, BABRMPH].

In the general case, where the action of G is no longer assumed to be locally free, the space $G\backslash M$ is not necessarily an orbifold but rather a differentiable stratified space; see [PFL, Chap. 4]. In this case, however, an analog of the inertia orbifold has appeared in connection with the study of the convolution algebra $\mathcal{C}^\infty(M)\rtimes G$ from the point of view of noncommutative geometry. More precisely, in connection with his study of the Hochschild cohomology of the convolution algebra $\mathcal{C}^\infty(M)\rtimes G$ in [BRY], Brylinsky considered the space of relative basic differential forms on an appropriately defined space which in this paper we will identify with the inertia space of the groupoid $G\ltimes M$. Similarly, Block and Getzler proved in [BLGE] that the periodic cyclic cohomology of the convolution algebra G is isomorphic to a sheaf of equivariant differential forms defined as a sheaf on G. When G is connected, this sheaf can also be understood as a sheaf of equivariant relative differential forms on the inertia space. In their paper [LUUR], Lupercio–Uribe defined for every topological groupoid G an inertia groupoid $\Lambda\mathsf{G}$. More precisely,

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the inertia groupoid $\Lambda\mathsf{G}$ is the transformation groupoid $\mathsf{G} \ltimes \mathsf{B}_0$, where B_0 is the space of loops of the groupoid G, i.e. the set of all arrows g such that the source $s(g)$ coincides with the target $t(g)$. The inertia space of G then is orbit space of the inertia groupoid $\Lambda\mathsf{G}$ or in other words the quotient space $\mathsf{G}\backslash\mathsf{B}_0$. If G is a proper étale Lie groupoid representing an orbifold X, the thus defined inertia space coincides with the inertia orbifold of X as defined originally by Kawasaki and subsequent authors. See also [AdGo] for recent results on the K-theory of inertia spaces of compact Lie group actions where the fundamental group of the Lie group is torsion-free and all isotropy groups have maximal rank.

With this paper, we aim at defining a general notion of the inertia space of a proper Lie groupoid and studying its fundamental properties in the basic situation where the Lie groupoid is a transformation groupoid $G \ltimes M$ with G a compact Lie group. Under this hypothesis, we give an explicit stratification of the inertia space in Theorem 4.1. Additionally, we demonstrate a de Rham theorem for differential forms on the inertia space in Theorem 5.1. Note that (locally), the inertia space is a subanalytic set, hence is known to admit a stratification by [MaSh]. However, the stratification constructed here is given explicitly in terms of local data on M and G, similar to the well-known stratification of $G\backslash M$ by orbit types.

In the case that G is a torus and M is stably almost complex, an inertia space is implicitly realized as a differentiable stratified space in [GoHoKn], where the Chen–Ruan orbifold cohomology is extended to this case. This construction differs from ours in that their inertia space is implicitly defined as a subquotient of the space $M \times \underline{G}$ where $\underline{G}$ denotes the group G with the *discrete* topology; the inertia space then appears as the disjoint union of an infinite family of quotients of G-invariant submanifolds of M. Our construction considers the inertia space as a subquotient of the manifold $M \times G$ where G is given its usual topology as a Lie group. Hence, while these inertia spaces are the same as sets, the topology of our inertia space does not coincide with that of [GoHoKn].

This paper is organized as follows. In Section 2, we review the notions of differentiable spaces and differentiable stratified spaces. In Section 3, we define the inertia space as well as the structure sheaf with which it is a differentiable space. In the same section, we also study the local properties of the inertia space and in particular demonstrate that it is locally contractible and triangulable. In Section 4, we explicitly describe the stratification of the inertia space; we give several examples of the main construction before proving the corresponding main result, Theorem 4.1. In Section 5, we prove Theorem 5.1, a de Rham Theorem for the inertia space.

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2. Preliminaries

In this section, we recall the definitions of differentiable spaces and differentiable stratified spaces. Hereby, we use the notion of a differentiable space as originally introduced by Spallek [Spa69, Spa70, Spa71, Spa72], and follow the exposition and notation of [GoSa]; see also [Bie75, Bie80, Bre, Pfl] for more details on stratified spaces.

Recall that a *locally* $\mathbb{R}$-*ringed space* $(X, \mathcal{O})$ consists of a topological space X equipped with a sheaf $\mathcal{O}$ of $\mathbb{R}$-algebras such that at each point $x \in X$ the stalk $\mathcal{O}_x$ is a local ring. Note that by definition there then exists for each point $x \in X$ and open neighborhood U of x an evaluation map $e_x : \mathcal{O}(U) \to \mathbb{R}$, $f \mapsto f(x)$. A *morphism* of locally $\mathbb{R}$-ringed spaces $(X,\mathcal{O}) \to (Y,\mathcal{Q})$ is a pair (f,F), where $f : X \to Y$ is a continuous map and $F : \mathcal{Q} \to f_*\mathcal{O}$ a morphism of sheaves over Y such that for each $x \in X$ the induced map on the stalks $F_x : \mathcal{Q}_{f(x)} \to \mathcal{O}_x$ is a local ring homomorphism. Obviously, locally $\mathbb{R}$-ringed spaces with their morphisms form a category.

We consider $\mathcal{C}^\infty(\mathbb{R}^n)$ with the unique topology with respect to which it is a Frechét algebra, see [GoSa, Section 2.3]. A locally $\mathbb{R}$-ringed space is called an *affine differentiable space*, if for some $n \in \mathbb{N}^*$ there is a closed ideal $\mathfrak{a} \subseteq \mathcal{C}^\infty(\mathbb{R}^n)$ such that $(X,\mathcal{O})$ is isomorphic as a locally $\mathbb{R}$-ringed space to $(\mathrm{Spec}_\mathrm{r}(A),\mathcal{A})$. Here, A denotes the differentiable algebra $\mathcal{C}^\infty(\mathbb{R}^n)/\mathfrak{a}$, $\mathrm{Spec}_\mathrm{r}(A)$ is the *real spectrum* of A, i.e. the collection of all continuous $\mathbb{R}$-algebra homomorphisms $A \to \mathbb{R}$ equipped with the Gelfand topology, and $\mathcal{A}$ is the structure sheaf on $\mathrm{Spec}_\mathrm{r}(A)$, i.e. the sheaf associated to the presheaf $U \mapsto A_U$, where U runs through the open sets of $\mathrm{Spec}_\mathrm{r}(A)$ and A_U is the localization of A over U. A locally ringed space $(X,\mathcal{O})$ is a *differentiable space*, if for each point $x \in X$ there is an open neighborhood U such that the restriction $(U,\mathcal{O}_{|U})$ is an affine differentiable space. If in addition the map

$$\mathcal{O}(U) \mapsto \mathcal{C}(U),\ f \mapsto \hat{f} := \big(U \ni x \mapsto f(x) \in \mathbb{R}\big)$$

is injective for each open $U \subseteq X$, one calls $(X,\mathcal{O})$ a *reduced differentiable space*. A reduced differentiable space is called a *smooth differentiable space*, if for each point $x \in X$ there is an open neighborhood U such that the restriction $(U,\mathcal{O}_{|U})$ is isomorphic as a locally $\mathbb{R}$-ringed space to some $(\mathbb{R}^n,\mathcal{C}^\infty_{\mathbb{R}^n})$, where $\mathcal{C}^\infty_{\mathbb{R}^n}$ denotes the sheaf of smooth functions on $\mathbb{R}^n$. Examples of reduced differentiable spaces include smooth manifolds, the orbit space of a proper smooth action of a Lie group on a manifold or more generally of a proper Lie groupoid [PfPoTa11], algebraic varieties, and symplectically reduced spaces.

Let now $Y \subset X$ be a locally closed subspace of a differentiable space $(X,\mathcal{O})$. Then, Y carries a natural structure sheaf $\mathcal{O}_{|Y}$ such that $(Y,\mathcal{O}_{|Y})$ becomes a reduced differentiable space. More precisely, if $V \subseteq Y$ is relatively open and $U \subseteq X$ open with $V = U \cap Y$, we define $\mathcal{O}_{|Y}(U \cap Y)$ to be the quotient of $\mathcal{O}_{|X}(U)$ by the closed ideal of functions that vanish on $U \cap Y$, see [GoSa, Example 3.21 and Section 5.1]. Note that in general, the restricted sheaf $\mathcal{O}_{|Y}$ coincides with the pullback sheaf $i^*\mathcal{O}$ for the embedding $i : Y \hookrightarrow X$ only if Y is open in X.

We now recall the definitions of a decomposition and stratification of a (paracompact, second-countable Hausdorff) topological space X in the sense of Mather [Mat73], see also [Pfl, Chap. 1] for further details. A *decomposition* of X is a locally finite partition of X into locally closed subspaces, the *pieces* of the decomposition, such that each piece is a smooth manifold in the induced topology and the pieces satisfy the *condition of frontier*: If R and S are pieces such that $R \cap \overline{S} \neq \emptyset$, then $R \subseteq \overline{S}$. A *stratification* $\mathcal{S}$ of X is an assignment to each $x \in X$ the germ $\mathcal{S}_x$ of a closed subset of X such that there is a neighborhood U of x and a decomposition $\mathcal{Z}$ of U such that for each $y \in U$, $\mathcal{S}_y$ coincides with the germ of the piece of $\mathcal{Z}$ containing y.

Suppose now that $(X,\mathcal{O})$ is a reduced differentiable space which in addition carries a stratification. Then every stratum S of X is locally closed, hence one obtains for every stratum S the restricted sheaf $\mathcal{O}_{|S}$. Denote by $\mathcal{C}^\infty_S$ the sheaf of smooth functions on the smooth manifold S. We say that $(X,\mathcal{O})$ is a *differentiable stratified space*, if for each stratum S of X, the sheaves $\mathcal{O}_{|S}$ and $\mathcal{C}^\infty_S$ coincide. Note that this notion of a differentiable stratified space is equivalent to the notion of a stratified space with $\mathcal{C}^\infty$-structure as defined in [Pfl, Sec. 1.3]. In particular, given an affine set U (i.e. an open subset $U \subseteq X$ such that $(U,\mathcal{O}_{|U})$ is an affine differentiable space), an isomorphism of $(U,\mathcal{O}_{|U})$ with $\mathrm{Spec}_r(\mathcal{C}^\infty(\mathbb{R}^n)/\mathfrak{a})$ defines a singular chart for U in the sense of [Pfl, Sec. 1.3]. Often, we denote the structure sheaf of a reduced differentiable space X or a differentiable stratified space X by $\mathcal{C}^\infty_X$.

To give an example of a differentiable stratified space, consider a Lie group G acting properly on a smooth manifold M. Let $\varrho : M \to G\backslash M$ be the quotient map. It is well known (cf. e.g. [Pfl, Sec. 4.3] or [DuKo, Sec. 2.7]) that the orbit space $G\backslash M$ is stratified by orbit types. Specifically, let $G_x \leq G$ denote the isotropy group of a point $x \in M$, let (G_x) denote the G-conjugacy class of G_x, and let $M_{(G_x)}$ denote the collection of $y \in M$ such that G_y is conjugate to G_x. Then the stratification of $G\backslash M$ by orbit types is given by assigning to each $x \in M$ the germ of the set $G\backslash M_{(G_x)}$. Moreover, by [Pfl, Thm. 4.4.6], the orbit space carries a canonical differentiable structure which is compatible with the stratification by orbit types. In other words, $G\backslash M$ thus becomes a differentiable stratified space. The structure sheaf $\mathcal{C}^\infty_{G\backslash M}$ is given by assigning to an open subset U of $G\backslash M$ the $\mathbb{R}$-algebra of continuous functions on U which pull back to smooth G-invariant functions on $\varrho^{-1}(U)$, i.e.

$$\mathcal{C}^\infty_{G\backslash M}(U) := \left\{ f \in \mathcal{C}(U) \mid f \circ \varrho_{|\varrho^{-1}(U)} \in \mathcal{C}^\infty(\varrho^{-1}(U))^G \right\}.$$

3. The Inertia Space of a Proper Lie Groupoid

Recall that by a *groupoid* one understands a small category G such that all arrows are invertible, cf. [MoMr]. Denote by G_0 the set of objects and by G_1 the set of arrows of a groupoid G. The source (resp. target) map will then be denoted by $s : \mathsf{G}_1 \to \mathsf{G}_0$ (resp. $t : \mathsf{G}_1 \to \mathsf{G}_0$), the unit map by $u : \mathsf{G}_0 \to \mathsf{G}_1$, the inversion by $i : \mathsf{G}_1 \to \mathsf{G}_1$, and finally the composition map by $m : \mathsf{G} \times_{\mathsf{G}_0} \mathsf{G}_1 \to \mathsf{G}_1$. If G_1 and G_0 are both topological spaces, and all structure maps continuous, the groupoid is called a *topological groupoid*. If in addition, G_1 and G_0 are smooth differentiable spaces, all structure maps are smooth maps, and s and t are both submersions, G is called a *Lie groupoid*. Note that the arrow set of a Lie groupoid in general need not be a Hausdorff topological space.

If G is a topological groupoid, and both s and t are local homeomorphisms, G is called an *étale groupoid*, in case the pair $(s,t) : \mathsf{G}_1 \to |sfG_0 \times \mathsf{G}_0$ is a proper map, one says that G is a *proper groupoid*.

Fundamental examples of proper Lie groupoids are given by transformation groupoids $G \ltimes M$, where G is a Lie group which acts properly on a smooth manifold M. The object space of such a transformation groupoid is given by $(G \ltimes M)_0 := M$, the arrow space by $(G \ltimes M)_1 := G \times M$, and the structure maps are defined as follows:

$$\begin{aligned}
s &: (G \ltimes M)_1 \to (G \ltimes M)_0,\ (g,p) \mapsto p,\\
t &: (G \ltimes M)_1 \to (G \ltimes M)_0,\ (g,p) \mapsto gp,\\
u &: (G \ltimes M)_0 \to (G \ltimes M)_1,\ p \mapsto (e,p),\\
i &: (G \ltimes M)_1 \to (G \ltimes M)_1,\ (g,p) \mapsto (g^{-1}, gp), \quad \text{and}\\
m &: (G \ltimes M)_1 \times_{(G \ltimes M)_0} (G \ltimes M)_1 \to (G \ltimes M)_1,\ \big((g,hp),(h,p)\big) \mapsto (gh,p).
\end{aligned}$$

Let now G be an arbitrary proper Lie groupoid. One then defines the *loop space* of G as the subspace

$$\mathsf{B}_0 := \left\{ k \in \mathsf{G}_1 \mid s(k) = t(k) \right\}. \tag{3.1}$$

Sometimes, we denote the loop space also by $\Lambda(\mathsf{G}_0)$. The groupoid G acts on the loop space in the following way:

$$\mathsf{G}_1 \times_{\mathsf{G}_0} \mathsf{B}_0 \to \mathsf{B}_0,\ (g,k) \mapsto g\,k\,g^{-1}.$$

We can now define:

Definition 3.1 (cf. [LuUr]). Let G be a proper Lie groupoid, and B_0 its loop space. The action groupoid $\mathsf{G} \ltimes \mathsf{B}_0$ then is called the *inertia groupoid* of G. It will be denoted by $\Lambda\mathsf{G}$. The quotient space $\mathsf{G}\backslash\Lambda\mathsf{G}$ will be called the *inertia space* of the groupoid G. If X denotes the orbit space $\mathsf{G}\backslash\mathsf{G}_0$, we sometimes write (by slight abuse of notation) ΛX for the inertia space of G.

Remark 3.2. The loop space B_0 is a closed subset of the smooth manifold G_1, hence inherits the structure of a differentiable space. Moreover, B_0 is locally semialgebraic, hence possesses a minimal Whitney B stratification, and a triangulation subordinate to it. The inertia space ΛX inherits these properties from the loop space as well. We will elaborate on this in a forthcoming publication.

Remark 3.3. The inertia space ΛX as a topological space depends in fact only on the Morita equivalence class of the proper Lie groupoid G. So if one thinks of X as a topological space together with a Morita equivalence class of Lie groupoids having X as orbit space, the notation ΛX is fully justified. The stratification defined by Theorem 4.1 and the corresponding de Rham complex are not in general Morita invariant. For example, in the case of $\mathsf{SO}(3)$ acting on $\mathbb{R}^3 \setminus \{0\}$ with the restriction of the standard action, the resulting translation groupoid is easily seen to be Morita equivalent to that associated to the trivial action of $\mathsf{SO}(2)$ on $\mathbb{R}$ (for a compact example, one may restrict the action to the unit sphere in $\mathbb{R}^3$). The stratifications of the inertia space associated to these two presentations do not coincide; see 4.2.6. Other relationships between the inertia spaces of Morita equivalent groupoids will be explored elsewhere.

Let us now describe the inertia space in the particular situation, where the underlying proper Lie groupoid is a transformation groupoid $G \ltimes M$ with G a compact Lie group and M a smooth G-manifold. The loop space B_0 then is given as the closed subspace

$$\Lambda M := \mathsf{B}_0 := \big\{(k,x) \in G \times M \mid kx = x\big\}$$

of $G \times M$. Moreover, G acts on $G \times M$ by

$$G \times (G \times M) \to (G \times M),\ (g,(k,x)) \mapsto g(k,x) := (gkg^{-1}, x).$$

This action leaves ΛM invariant. The inertia space of $G \ltimes M$ now coincides with the quotient space $\Lambda X := G\backslash \Lambda M$, where $X := G\backslash M$. Sometimes, we call ΛX the *inertia space of the G-manifold M*.

Proposition 3.4. *Let G be a compact Lie group. The inertia space ΛX of a G-manifold M carries a natural and uniquely determined structure of a differentiable space such that the embedding $\iota : \Lambda X \hookrightarrow G\backslash(G \times M)$ becomes a smooth map, where $G\backslash(G \times M)$ carries the unique differentiable structure such that the canonical projection $\varrho : G \times M \to G\backslash(G \times M)$ is smooth.*

Remark 3.5. In the following, we denote the canonical projection $M \to G\backslash M$ of a G-manifold M to its orbit space by ϱ^M. Instead of $\varrho^{G\times M}$ we often write ϱ, if no confusion can arise. The restriction of ϱ to ΛM will be denoted by $\widehat{\varrho}$, that means $\widehat{\varrho} : \Lambda M \to \Lambda X$ is the orbit map from the loop space to the inertia space.

Proof. Recall that by [GoSa, Thm. 11.17], the quotient $G\backslash(G \times M)$ is a differentiable space, and that the structure sheaf on $G\backslash(G \times M)$ is uniquely determined by the requirement that the quotient map ϱ is smooth. Since ΛM is a closed G-invariant subspace of $G \times M$, it follows from [GoSa, Lem. 11.15] that ΛX is a differentiable space. Again, the structure sheaf is uniquely determined by the requirement that the $\iota : \Lambda X \hookrightarrow G\backslash(G \times M)$ is smooth which according to [GoSa, Lem. 11.15] is the case indeed. $\square$

Let us briefly give a more explicit description of the structure sheaf on the inertia space. Let $U \subset \Lambda X$ be open. Then $\mathcal{C}^\infty_{\Lambda X}(U)$ is the space of all $f \in \mathcal{C}(U)$ such that there exists an open $W \subset G \times M$ and a function $F \in \mathcal{C}^\infty(W)$ which have the property that $W \cap \Lambda M = \varrho^{-1}(U)$ and that $F_{|W\cap\Lambda M} = f \circ \varrho_{|W\cap\Lambda M}$. In other words,

$$\mathcal{C}^\infty_{\Lambda X}(U) \cong \big(\mathcal{C}^\infty(\varrho^{-1}(U))\big)^G.$$

Hence, the structure sheaf of ΛX is given by the restriction of the smooth G-invariant functions on $G \times M$ to ΛM.

The inertia space ΛX of a G-manifold M carries even more structure. In the following considerations we will explain this in more detail.

3.1. **Reduction to Slices in M.** Fix a point $x \in M$, and let Y_x be a slice at x for the G-action on M. By a *slice at x* we hereby mean a submanifold $Y_x \subset M$ transverse to the orbit Gx such that the following conditions are satisfied (cf. [Bre, II. Theorem 4.4]):

(SL1) Y_x is closed in GY_x,
(SL2) GY_x is an open neighborhood of Gx,
(SL3) $G_xY_x = Y_x$, and
(SL4) $gY_x \cap Y_x \neq \emptyset$ implies $g \in G_x$.

After possibly shrinking Y_x, we can even assume that Y_x is a *linear slice*, which means that

(SL5) there exists a G_x-equivariant diffeomorphism $Y_x \to N_x$ of the slice Y_x onto the normal space $N_x := T_xM/T_xGx$.

Note that we implicitly have used here the fact that G_x acts linearly on the normal space N_x. After choosing a G-invariant riemannian metric on M, the image $\exp(B_x)$ of every sufficiently small open ball B_x around the origin of N_x under the exponential map is a linear slice at x. We will assume from now on that all slices are linear. By (SL5) this implies in particular that there is a G_x-equivariant contraction $[0,1] \times Y_x \to Y_x$ to the point x. Let us also recall at this point the slice theorem [Kos] which tells that the map

$$\theta : G \times_{G_x} Y_x \longrightarrow GY_x,\ [h,y] \mapsto hy \tag{3.2}$$

is a G-equivariant diffeomorphism between $G \times_{G_x} Y_x$ and the tube $GY_x \subseteq M$ about Gx.

Let us now examine the loop and the inertia space of the G-manifold $G \times_{G_x} Y_x$. By definition, the loop space is given by

$$\Lambda(G \times_{G_x} Y_x) = \big\{(g,[h,y]) \in G \times (G \times_{G_x} Y_x) \mid g[h,y] = [h,y]\big\}.$$

In addition, the map

$$\mathrm{id}_G \times \theta :\ G \times (G \times_{G_x} Y_x) \to G \times M$$

is a $G \times G$-equivariant diffeomorphism onto a $G \times G$-invariant open neighborhood of (e,x) in $G \times M$. Hence the restriction

$$\Lambda\theta := (\mathrm{id}_G \times \theta)_{|\Lambda(G \times_{G_x} Y_x)} :\ \Lambda(G \times_{G_x} Y_x) \to (G \times GY_x) \cap \Lambda M \subseteq \Lambda M$$

becomes a G-equivariant homeomorphism onto the G-invariant neighborhood $(G \times GY_x) \cap \Lambda M$ of (e,x) in ΛM. Moreover, it follows that $\Lambda\theta$ is an isomorphism between the differentiable spaces $\Lambda(G \times_{G_x} Y_x)$ and $(G \times GY_x) \cap \Lambda M$ by [GoSa, Lem. 11.15].

Next let us consider the loop space $\Lambda Y_x := \{(h,y) \in G_x \times Y_x \mid hy = y\}$ of the G_x-manifold Y_x. Then we have the following result, which provides a local picture of the inertia space of a G-manifold M.

Proposition 3.6. *Let Y_x be a slice at the point x of a G-manifold M for G a compact Lie group. Then the inertia space $\Lambda(G_x \backslash Y_x)$ of the G_x-manifold Y_x is isomorphic as a differentiable space to the open neighborhood $\Lambda(G \backslash GY_x)$ of the point $G(e,x)$ in the inertia space $\Lambda(G \backslash M)$.*

Proof. Consider the map $\phi :\ \Lambda Y_x \to \Lambda(G \times_{G_x} Y_x)$ defined as the restriction of the smooth map

$$G_x \times Y_x \to G \times (G \times_{G_x} Y_x),\ (h,y) \mapsto (h,[e,y]).$$

to ΛY_x. Obviously, by elementary considerations, ϕ is continuous and injective. Moreover, ϕ is a morphism of differentiable spaces since it is the restriction of a smooth map between manifolds.

Since ϕ is equivariant with respect to the canonical embedding $G_x \hookrightarrow G$, it also induces a continuous map between the quotients

$$\Phi : \Lambda(G_x\backslash Y_x) \to \Lambda(G\backslash (G \times_{G_x} Y_x)) \cong \Lambda(G\backslash GY_x),\ G_x(h,y) \mapsto G(h,[e,y]).$$

Let us show that Φ is bijective. This will prove the claim.

To show that Φ is injective, suppose that (h,y) and $(\overline{h},\overline{y})$ are elements of ΛY_x such that $\Phi(G_x(h,y)) = \Phi(G_x(\overline{h},\overline{y}))$. Then $G(h,[e,y]) = G(\overline{h},[e,\overline{y}])$ so that there is a $g \in G$ such that $(h,[e,y]) = g(\overline{h},[e,\overline{y}])$. Therefore, $[e,y] = [g,\overline{y}]$ and $h = g\overline{h}g^{-1}$, so that there is an $\widetilde{h} \in G_x$ such that $(\widetilde{h}^{-1}, \widetilde{h}y) = (g,\overline{y})$. But this implies that $\widetilde{h}^{-1} = g \in G_x$ and $y = g\overline{y}$, so that $g(\overline{h},\overline{y}) = (h,y)$ with $g \in G_x$. It follows that Φ is injective.

To show that Φ is surjective, let $(k,[g,y])$ be an arbitrary element of the loop space $\Lambda(G \times_{G_x} Y_x)$ which means that $k[g,y] = [g,y]$. Then $g^{-1}kg[e,y] = [e,y]$, so that by (SL4) $g^{-1}kg \in G_x$. Since $g^{-1}(k,[g,y]) = (g^{-1}kg,[e,y])$, it follows that $\Phi(G_x(g^{-1}kg,y)) = G(k,[g,y])$, and Φ is surjective.

Finally, we claim that Φ is even an isomorphism between differentiable spaces. To this end note first that for all $f \in \mathcal{C}^\infty\big(\Lambda(G\backslash (G \times_{G_x} Y_x))\big)$ the pullback $\Phi^*(f)$ is a smooth function on $\Lambda G\backslash Y_x$, since

$$\Phi^*(f) \circ \varrho^{Y_x}_{|\Lambda Y_x} = f \circ \varrho^{G\times_{G_x} Y_x}_{|\Lambda(G\times_{G_x} Y_x)} \circ \phi,$$

where we have used the notation as explained in Remark 3.5. By surjectivity of Φ, the pullback $\Phi^* : \mathcal{C}^\infty\big(\Lambda(G\backslash (G \times_{G_x} Y_x))\big) \to \mathcal{C}^\infty\big(\Lambda(G_x\backslash Y_x)\big)$ is injective.

To show that Φ^* is surjective, let $h \in \mathcal{C}^\infty\big(\Lambda(G_x\backslash Y_x)\big)$. Since ΛY_x is a closed differentiable subspace of $G \times Y_x$, there exists a smooth H on $G \times Y_x$, such that $H_{|\Lambda Y_x} = h \circ \varrho^{Y_x}_{|\Lambda Y_x}$. By possibly averaging over G_x one can even assume that H is G_x-invariant. With this, we define $\tilde{f} \colon G \times (G \times_{G_x} Y_x) \to \mathbb{R}$ by setting $\tilde{f}(k,[g,y]) = H(g^{-1}kg, y)$. By G_x-invariance, $\tilde{f}$ is well-defined and smooth. Moreover, $\tilde{f}$ is G-invariant, hence on has

$$\tilde{f}_{|\Lambda G\times_{G_x} Y_x} = f \circ \varrho^{G\times_{G_x} Y_x}_{|\Lambda(G\times_{G_x} Y_x)}$$

for some smooth $f : \Lambda(G \times_{G_x} Y_x) \to \mathbb{R}$. By construction it is clear that then $\Phi^*(f) = h$, hence Φ^* is surjective. This proves that Φ is even an isomorphism of differentiable spaces. $\square$

In the preceding proposition, one can choose for each $x \in M$ the slice Y_x to be equivariantly isomorphic to a ball B_x in some finite dimensional orthogonal G_x-representation space V_x. Moreover, since every compact Lie group has a linear faithful representation, we can assume that G_x is (represented as) a compact real algebraic group. The loop space

$$\Lambda B_x = \big\{(k,x) \in G_x \times B_x \mid kx = x\big\}$$

then is a semi-algebraic set in $G_x \times V_x$. Since by the Proposition the inertia space $\Lambda(G\backslash M)$ has a locally finite cover by open subsets such that each of the elements of the cover is isomorphic as a differentiable space to some inertia space $\Lambda(G_x\backslash B_x)$ the inertia space of the G-manifold M is locally semi-algebraic in the sense of Delfs–Knebusch [DeKn, Chap. I].

Corollary 3.7. *The inertia space $\Lambda(G\backslash M)$ of a G-manifold M with G compact is locally semi-algebraic in a way that is compatible with its structure as a differentiable space.*

A semi-algebraic set X has a minimal $\mathcal{C}^\infty$-Whitney stratification according to [Mat73, Thm. 4.9 & p. 210]. As the definition of a Whitney stratification as in [Mat73, Sec. I.2] is local, and because the $\mathcal{C}^\infty$ structure is compatible with the global differential structure of the inertia space, by the Corollary, the inertia space of a G-manifold M therefore possesses a minimal $\mathcal{C}^\infty$-Whitney stratification and becomes a differentiable stratified space. We will call this stratification the *canonical stratification* or the *minimal Whitney stratification* of the inertia space. Since the canonical stratification satisfies Whitney's condition B, there even exists a system of smooth control data for the canonical stratification (cf. [MAT70] and [PFL, Thm. 3.6.9]). According to [GOR, Sec. 5] [VERO, Cor. 3.7] there exists a triangulation of the inertia space subordinate to the canonical stratification. Hence, the inertia space is triangulable and in particular locally contractible. We thus obtain the following result.

Theorem 3.8. *Let G be a compact Lie group. The inertia space ΛX of a G-manifold M is in a canonical way a locally compact locally contractible differentiable stratified space. Its canonical stratification satisfies Whitney's condition B and is minimal among the stratifications of ΛX with this property. Moreover, there exists a triangulation of the inertia space subordinate to the canonical stratification.*

That the inertia space is locally contractible can be shown directly as well. The main point hereby lies in the following result which also will be needed later to prove a de Rham Theorem for inertia spaces.

Proposition 3.9. *Let M and G be as above, and consider a point (h,x) in the loop space ΛM. Then there is a linear slice $V_{(h,x)}$ at (h,x) such that the action of scalars $t \in [0,1]$ on $V_{(h,x)}$ leaves the set $V_{(h,x)} \cap \Lambda M$ invariant. The linearization $V_{(h,x)} \hookrightarrow B_{(h,x)} \subseteq N_{(h,x)}$ from $V_{(h,x)}$ to an open convex neighborhood of the origin in the normal space $N_{(h,x)} := T_{(h,x)}(G \times M)/T_{(h,x)}\big(G(h,x)\big)$ can hereby be chosen as the inverse of the restriction $\exp_{|B_{(h,x)}}$ of the exponential map corresponding to an appropriate G-invariant riemannian metric on $G \times M$.*

Proof. Choose a G-invariant riemannian metric on M, a bi-invariant riemannian metric on G (cf. [DUKO, Prop. 2.5.2 and Sec. 3.1]), and let $G \times M$ carry the product metric. Recall that under these assumptions, the exponential map $T_{(h,x)}(G \times M) \to G \times M$ decomposes into a product $\exp^G_h \times \exp^M_x$. Moreover, the riemannian exponential map $\exp^G$ on TG then coincides with the map

$$TG \cong G \times \mathfrak{g} \to G,\ (g,\xi) \mapsto g\, e^{\xi}.$$

Hereby, e^{ξ} denotes the exponential map on the Lie algebra, and the isomorphism between $G \times \mathfrak{g}$ and TG is given by $(g,\xi) \mapsto (L_g)_*\xi$, where $L_g : G \to G$ denotes the left action by g. Now let $B_{(h,x)}$ denote a sufficiently small open ball around the origin of the normal space $N_{(h,x)}$ so that the exponential map is injective on $B_{(h,x)}$, and let $V_{(h,x)} := \exp\big(B_{(h,x)}\big)$.

Note that every element g of the isotropy group $H := G_{(h,x)}$ commutes with H. Hence, for $(k,y) \in V_{(h,x)}$ one has $h = ghg^{-1} \in gH_{(k,y)}g^{-1}$ if and only if $h \in H_{(k,y)}$, and the same holds for each connected component of $H_{(k,y)}$. Therefore, if $(H_1), \ldots, (H_s)$ denotes the collection of isotropy types for the linear H-action on $V_{(h,x)}$, which is finite by [PFL, Lem. 4.3.6], we can assume they are ordered in such a way that $h \in gH_ig^{-1}$ for each $g \in H$ and $i = 1, \ldots, r$, and $h \notin gH_ig^{-1}$ for each $g \in H$ and $i = r+1, \ldots, s$. Moreover, for $i = 1, \ldots, r$, let H_i^h denote the connected component of H_i containing h, which implies that $h \in gH_i^hg^{-1}$ for each $g \in H$.

With this, we define

$$C = \left(\bigcup_{i=1}^{r} \ \bigcup_{g \in H} gH_ig^{-1} \smallsetminus gH_i^hg^{-1} \right) \cup \left(\bigcup_{i=r+1}^{s} \ \bigcup_{g \in H} gH_ig^{-1} \right).$$

That is, C is the union of all conjugates of isotropy groups not containing h as well as, for each isotropy group containing h, all conjugates of the connected components not containing h. Since the quotient map $H \to Ad_H\backslash H$ is closed by [TDIE, Prop. 3.6] it follows immediately that C is closed in H. By construction, C is also H-invariant. This implies that $G \smallsetminus C$ is an Ad_H-invariant open neighborhood of h in H. Hence there exists a connected open and Ad_H-invariant neighborhood O_h of h in $G \smallsetminus C$ small enough to be contained in a logarithmic chart around h in the Lie algebra $\mathfrak{h}$ of H, see [DuKo, Thm. 1.6.3]. Therefore, $O_h \times M$ is an H-invariant open neighborhood of (h,x) in $G \times M$. With this, we may shrink $V_{(h,x)}$ to assume that $V_{(h,x)} \subseteq O_h \times M$.

Now, suppose $(k,y) \in V_{(h,x)} \cap \Lambda M$ so that $ky = y$ and then clearly $k(k,y) = (k,y)$. By property (SL4) of the slice $V_{(h,x)}$, it follows that $k \in H$. Since $k \in O_h$ we have that k is not contained in any H-conjugate of H_i for $i = r+1, \dots, s$ so that $H_{(k,y)}$ is conjugate to H_i for some $i \le r$. Moreover, O_h is connected and does not intersect C so that k is contained in the same connected component of $H_{(k,y)}$ as h. Hence, by using logarithmic coordinates near h, we may express $k = he^{\xi}$ for some $\xi \in \mathfrak{h}_{(k,y)}$, the Lie algebra of $H_{(k,y)}$. Additionally, $he^{t\xi} \in H_{(k,y)}$ for $t \in [0,1]$, so that $he^{t\xi}(k,y) = (k,y)$. Next, let $w \in T_x M$ such that $\exp_x^M(w) = y$. Then we have $\exp_{(h,x)}(\xi, w) = (k,y)$, and $(\xi, w) \in N_{(h,x)}$. Moreover, we get

$$\exp_{(h,x)}\big(t(\xi,w)\big) = \exp_{(h,x)}\big((t\xi, tw)\big) = \big(he^{t\xi}, y(t)\big),$$

where $y(t) = \exp_x^M(tw)$. However, since the action of H on the normal space $N_{(h,x)}$ is linear, and since $he^{t\xi}(k,y) = (k,y)$, it follows that $he^{t\xi}\big(he^{t\xi}, y(t)\big) = \big(he^{t\xi}, y(t)\big)$ for $t \in [0,1]$. This of course implies that $he^{t\xi}y(t) = y(t)$ so that $\big(he^{t\xi}, y(t)\big) \in \Lambda M$, proving the claim. $\square$

Corollary 3.10. *The inertia space $\Lambda(G\backslash M)$ of a compact Lie group action is locally contractible.*

Proof. Let $(h,x) \in \Lambda M$, $H = G_{(x,h)}$, and choose a linear slice $V_{(h,x)}$ as in the preceding Proposition. Then $GV_{(h,x)}$ is an open G-invariant neighborhood of (h,x) in $G \times M$. Define the map $\mathcal{H}\colon G \times_H V_{(h,x)} \times [0,1] \to G \times_H V_{(h,x)}$ by $\mathcal{H}([g,(k,y)],t) = [g,(1-t)(k,y)]$. Then $\mathcal{H}$ is a G-equivariant deformation retraction of $G \times_H V_{(h,x)}$ onto $G \times_H \{(h,x)\}$ which induces a retraction in the quotient onto the single orbit $G(h,x)$. Moreover, by the preceding Proposition, the map $\mathcal{H}$ restricts to a G-invariant retraction of $(G \times_H V_{(h,x)}) \cap \Lambda M$ onto a single orbit. $\square$

4. The orbit Cartan type Stratification

In this section, we present the explicit stratification of the inertia space ΛX. We give the definition of this stratification in Subsection 4.1 and state our first main result, Theorem 4.1. Before turning to the proof of Theorem 4.1 in Subsection 4.4, we first give several examples of the stratifications in Subsection 4.2 and establish some useful results for actions of abelian groups in Subsection 4.3.

Let G° denote the connected component of the identity of G. Recall that a *Cartan subgroup* T of G is a closed topologically cyclic subgroup that has finite index in its normalizer $N_G(\mathsf{T})$ (cf. [BrDi, IV. Def. 4.1], see also [Seg]). If $g \in G$, then by [BrDi, IV. Prop. 4.2], there is a Cartan subgroup T of G such that $g \in \mathsf{T}$ and $\mathsf{T}/\mathsf{T}^\circ$ is generated by $g\mathsf{T}^\circ$. We will say that such a T is a *Cartan subgroup associated to g*. If $g \in G^\circ$, then T is a maximal torus of G° containing g; in general, T is isomorphic to the product of a torus and a finite cyclic group. We will make frequent use of [BrDi, IV. Prop. 4.6], which states that the homomorphism

$$\begin{array}{rcl} \mathsf{T}/\mathsf{T}^\circ & \longrightarrow & G/G^\circ \\ t\mathsf{T}^\circ & \longmapsto & tG^\circ \end{array} \tag{4.1}$$

defines a correspondence between Cartan subgroups of G and cyclic subgroups of G/G° that induces a bijection on conjugacy classes. That is, given $g,h \in G$ and Cartan subgroups T_g and T_h associated to g and h, respectively, T_g and T_h are conjugate in G if and only if $\langle gG^\circ\rangle \le G/G^\circ$ and $\langle hG^\circ\rangle \le G/G^\circ$ are conjugate in G/G°. For $g,h \in G^\circ$, this corresponds to the well-known fact that all maximal tori in a compact connected Lie group are conjugate; see e.g. [DuKo, Thm. 3.7.1].

4.1. **Definition of the Stratification.** Let $(h,x) \in \Lambda M$ and let H denote the isotropy group of (h,x) with respect to the G-action on $G \times M$. Then $H = G_x \cap Z_G(h) = Z_{G_x}(h)$ where $Z_G(h)$ denotes the centralizer of h in G and G_x denotes the isotropy group of x with respect to the G-action on M. Let $\mathsf{T}_{(h,x)}$ be a Cartan subgroup of H associated to h; note that if G_x is connected, we have by [DuKo, Thm. 3.3.1 (i)] that $h \in (Z_{G_x}(h))^\circ = H^\circ$, so that $\mathsf{T}_{(h,x)}$ is a maximal torus of H° containing h. Choose a slice $V_{(h,x)}$ at (h,x) for the G-action on $G \times M$, and define an equivalence relation $\sim$ on $\mathsf{T}_{(h,x)}$ by $s \sim t$ if $(GV_{(h,x)})^s = (GV_{(h,x)})^t$. Let $\mathsf{T}^*_{(h,x)}$ denote the connected component of the $\sim$ class $[h]$ containing h.

We define a stratification of ΛM by assigning to the point $(h,x) \in \Lambda M$ the germ

$$\mathcal{S}_{(h,x)} = \left[G\left(V^H_{(h,x)} \cap \left(\mathsf{T}^*_{(h,x)} \times M \right)\right)\right]_{(h,x)} . \tag{4.2}$$

After applying the quotient map $\widehat{\varrho} : \Lambda M \to \Lambda X$, we can similarly define a stratification of ΛX by assigning to the orbit $G(h,x)$ the germ

$$\mathcal{R}_{G(h,x)} = \left[\widehat{\varrho}\left(G\left(V^H_{(h,x)} \cap \left(\mathsf{T}^*_{(h,x)} \times G \right)\right)\right)\right]_{G(h,x)} . \tag{4.3}$$

We will see below that the germs defined by Equations (4.2) and (4.3) do not depend on the choice of slice $V_{(h,x)}$ and Cartan subgroup $\mathsf{T}_{(h,x)}$, see Lemmas 4.14 and 4.15.

The following result shows that $\mathcal{S}$ and $\mathcal{R}$ are stratifications of the loop space and inertia space, indeed. We call these the stratifications by *orbit Cartan type*.

Theorem 4.1. *Let G be a compact Lie group and let M be a smooth G-manifold. Then Equation (4.2) defines a Whitney stratification of ΛM with respect to which ΛM is a differentiable stratified space. Moreover, this stratification induces a stratification on ΛX by Equation (4.3) with respect to which ΛX is a differentiable stratified space fulfilling Whitney's condition B.*

An immediate consequence of this result and Theorem 3.8 is the following.

Corollary 4.2. *The orbit Cartan type stratification is in general finer than the canonical stratification of the inertia space ΛX. Moreover, there exists a triangulation of the inertia space subordinate to the orbit Cartan type stratification.*

Remark 4.3. Assume in addition that M itself is partitioned into a finite number of G- and T-isotropy types for any Cartan subgroup T of G. The definition of $\mathsf{T}^*_{(h,x)}$ above can be modified by saying that $s \sim t$ if $(G \times M)^s = (G \times M)^t$. The modified definitions of $\mathcal{S}_{(h,x)}$ and $\mathcal{R}_{G(h,x)}$ also result in Whitney stratifications of ΛM and ΛX, respectively. The proof of this fact is identical to the proof of Theorem 4.1 below with minor simplifications. The modified stratifications are generally finer and depend on global data in $G \times M$, though they can be easier to compute in examples.

Before we prove Theorem 4.1, we provide several examples which illustrate our definition.

4.2. **Examples of the Stratification.**

4.2.1. *Cases Where ΛM is Smooth.* Suppose G acts freely on M. Then

$$\Lambda M = \{e\} \times M \subseteq G \times M$$

is diffeomorphic to M. Each point (e,x) has trivial isotropy group, and it is easy to see that the stratifications of ΛM and ΛX given by Equations (4.2) and (4.3) are trivial. The result in both cases is a smooth manifold with a single stratum, and hence trivially a stratified differentiable space.

Similarly, suppose L is a (necessarily normal) subgroup of G that acts trivially on M, and suppose G/L acts freely on M. Then

$$\Lambda M = L \times M \subseteq G \times M$$

is a smooth manifold. The isotropy types of elements of ΛM correspond to the isotropy types of L with respect to the G-action by conjugation; that is, elements (h,x) and (k,y) of ΛM have the same isotropy type if and only if the centralizers $Z_G(h)$ and $Z_G(k)$ are conjugate. We claim that in this case, the stratifications of ΛM and ΛX given by Equations (4.2) and (4.3) coincide with the stratifications by G-isotropy types.

Choose a slice $V_{(h,x)}$ at $(h,x) \in \Lambda M$ for the G-action on $G \times M$. By construction, it is clear that $\mathcal{S}_{(h,x)}$ is a subgerm of the germ of the isotropy type of (h,x) at (h,x). Let

$$(\tilde{k},\tilde{y}) \in GV_{(h,x)} \cap (L \times M)$$

be a point in the orbit of this slice with the same G-isotropy type as (h,x). Then there is a $\tilde{g} \in G$ such that $(k,y) := \tilde{g}(\tilde{k},\tilde{y}) \in V_{(h,x)}$, and hence $G_{(k,y)} \leq H = G_{(h,x)}$. However, as $G_{(k,y)} = \tilde{g} G_{(\tilde{k},\tilde{y})} \tilde{g}^{-1}$ is conjugate to H, we have by [Pfl, Lem. 4.2.9] that $G_{(k,y)} = H$. Therefore, $(k,y) \in V_{(h,x)}^H$, which is connected, so that k is in the same connected component of H as h. It follows that kH° and hH° generate the same subgroup of H/H°, so that by [BrDi, IV. Prop. 4.6], Cartan subgroups $\mathsf{T}_{(h,x)}$ and $\mathsf{T}_{(k,y)}$ of H associated to h and k, respectively, are conjugate in H. Hence there is a $g \in H$ such that $gkg^{-1} \in \mathsf{T}_{(h,x)}$; however, as $g \in H = G_{(k,y)}$, it follows that $k = gkg^{-1} \in \mathsf{T}_{(h,x)}$. Moreover, as $Z_L(k) = G_{(k,y)} = G_{(h,x)} = Z_L(h)$, we have that $(G \times M)^k = (G \times M)^h$ so that $(GV_{(h,x)})^h = (GV_{(h,x)})^k$ and $k \in \mathsf{T}^*_{(h,x)}$. We conclude that $(\tilde{k},\tilde{y}) \in G\left(V_{(h,x)}^H \cap \left(\mathsf{T}^*_{(h,x)} \times M\right)\right)$, and hence that $\mathcal{S}_{(h,x)}$ is the germ of the G-isotropy type of (h,x) in $M \times G$.

More generally, we have the following. The proof is an elementary argument applied to slices for the G-action on M using a local section of the fiber bundle $G \to G/J$.

Proposition 4.4. *Suppose the stratification of M by G-orbit types has depth zero which means that there is a $K \leq G$ such that every point has orbit type (K). Then ΛM is a smooth submanifold of $G \times M$ that is locally diffeomorphic to $K \times M$.*

Proof. To show that ΛM is a differentiable manifold, let $(h,x) \in \Lambda M$, and let Y_x be a slice at x for the G-action on M. Without loss of generality, we can assume that $G_x = K$. Then for each $y \in Y_x$, as $G_y \leq K$ and G_y is conjugate to K, it must be that $G_y = K$. Therefore, $Y_x^K = Y_x$, and a neighborhood of G_x in M is diffeomorphic to

$$G \times_K Y_x = G/K \times Y_x$$

via the map

$$\tau : \; G/K \times Y_x \longrightarrow M, \; (gK, y) \longmapsto gy.$$

To prove that ΛM is a differentiable submanifold of $G \times M$, choose a neighborhood U of eK in G/K small enough so that the fiber bundle $G \to G/K$ admits a differentiable section on U. Let $\sigma : U \to G$ for $G \to G/K$ be a choice of such a section, and consider the map

$$\begin{aligned} \widetilde{\tau} : G \times U \times Y_x &\longrightarrow G \times \tau(U \times Y_x) \subseteq G \times M, \\ (\widehat{g}, gK, y) &\longmapsto \left(\sigma(gK)\, \widehat{g}\, \sigma(gK)^{-1}, \sigma(gK)y\right). \end{aligned}$$

Since U is an open neighborhood of eK in G/K, we have that $U \times Y_x$ is an open neighborhood of (eK, x) in $G/K \times Y_x$. Therefore, $\tau(U \times Y_x)$ is an open neighborhood of x in M. Simple computations demonstrate that $\widetilde{\tau}$ is a diffeomorphism from the neighborhood $G \times U \times Y_x$ of (h, eK, x) in $G \times G/K \times Y_x$ onto the neighborhood $G \times \tau(U \times Y_x)$ of (h, x) in $G \times M$. Moreover,

$$\widetilde{\tau}(K \times U \times Y_x) = (G \times \tau(U \times Y_x)) \cap \Lambda M,$$

so that $\widetilde{\tau}$ restricts to a diffeomorphism between a neighborhood of (h, x) in $K \times M$ to a neighborhood of (h, x) in ΛM. $\square$

In this case, however, it may happen that the stratifications of ΛM and ΛX given by Equations (4.2) and (4.3) are strictly finer than the respective stratifications by isotropy types. This is the case, for instance, when ΛX is the inertia space of $\mathbb{R}^3 \setminus \{0\}$ with its usual $\mathsf{SO}(3)$-action; see 4.2.6 below.

4.2.2. *Locally Free Actions.* If the action of G on M is locally free, i.e. the isotropy group of each $x \in M$ is finite, then the quotient $X = G\backslash M$ is an orbifold. The corresponding inertia space ΛX then is an orbifold as well and is called the *inertia orbifold* of X, see e.g. [AdLeRu] or [PfPoTa07]. Let us briefly sketch this within our framework and let us show that the above defined stratification of the inertia space ΛX coincides with the orbit type stratification, if the action of G is locally free.

To this end consider first the case, where G is a finite group. The loop space ΛM then is the disjoint union $\bigsqcup_{h \in G} \{h\} \times M^h$ of smooth manifolds of possibly different dimensions. Choose a G-invariant riemannian metric on M. Since G is finite, the linear slice $V_{(h,x)}$ at some point $(h, x) \in \Lambda M$ can be chosen to be of the form $\{h\} \times V_x$, where $V_x \subset M^h$ is an open ball around x in M^h; note that M^h is totally geodesic in M. Denote by H the isotropy group of (h, x), i.e. let $H := Z_{G_x}(h)$. Because under the assumptions made the Cartan subgroups are discrete, the set germ $\mathcal{S}_{(h,x)}$ at $(h, x) \in \Lambda M$ from Eq. (4.2) coincides with

$$[G(V^H_{(h,x)} \cap (\{h\} \times M))]_{(h,x)} = [G(\{h\} \times V^H_x)]_{(h,x)} = [\{h\} \times V^H_x]_{(h,x)} \, .$$

The second equality hereby follows from the fact that for every $g \in Z_G(h)$ and $y \in V^H_x$ with $gy \in V_x$ one has $gy \in V^H_x$, since $g \in H$ by (SL4). Observe that the orbit map

$$\varrho : G \times M \to G\backslash(G \times M)$$

is injective on $\{h\} \times V^H_x$ by the slice theorem, hence the set germ $\mathcal{R}_{G(h,x)}$ at $G(h, x) \in \Lambda M$ is given by

$$\mathcal{R}_{G(h,x)} = [\varrho(\{h\} \times V^H_x)]_{G(h,x)} \, .$$

In other words this means that the stratification by orbit Cartan type of the inertia space ΛX of a finite group action on M is given by the orbit type stratification.

Let us now consider the case where G is a compact Lie group acting locally freely on M. According to Theorem 3.8, the inertia space ΛX is a differentiable stratified space with stratification given by the canonical stratification. Recall that the canonical stratification is minimal among all Whitney stratifications of ΛX. Now observe that by Proposition 3.6 the neighborhood $\Lambda(G\backslash GY_x)$ of (e, x) in ΛX is isomorphic as a differentiable space to the inertia space $\Lambda(G_x\backslash Y_x)$, where Y_x is a slice of M at x. Since G_x is finite, it follows by the above considerations that the stratification $\mathcal{R}$ of $\Lambda(G_x\backslash Y_x)$ coincides with the stratification by orbit types. But the latter is known to be the minimal Whitney stratification, hence $\Lambda(G_x\backslash Y_x)$ with the orbit type stratification is even isomorphic as a differentiable stratification to $\Lambda(G\backslash GY_x)$ with the canonical stratification. Since ΛX is covered by the open sets $\Lambda(G\backslash GY_x)$, $x \in X$, it follows that both the canonical stratification of ΛX and the stratification by orbit Cartan type coincide with the stratification by orbit type, and that ΛX is an orbifold, indeed.

4.2.3. *Semifree Actions.* Suppose G acts semifreely on M so that there is a collection $N = M^G$ of submanifolds of M fixed by G, and G acts freely on $M \smallsetminus N$. Then

$$\Lambda M = [\{e\} \times (M \smallsetminus N)] \cup (G \times N).$$

The isotropy group of (h, x) is trivial if $x \notin N$ and is equal to the centralizer $Z_G(h)$ if $x \in N$. With respect to the adjoint action Ad_G of G on itself, let $(K_1), \dots, (K_m)$ denote the isotropy types of elements of G so that the centralizer of every element of G is conjugate to some K_j. We assume that $K_m = G$ is the isotropy group of the center of G. For $j = 1, \dots, m-1$, we let Ad_G^j denote the set of elements of G with centralizer exactly K_j and let $Ad_G^{(j)}$ denote the set of elements of G with centralizer conjugate to K_j. For $j = m$, we let $Ad_G^m = Ad_G^{(m)}$ denote the set of *nontrivial* central elements of G, which may be empty. The sets Ad_G^j and $Ad_G^{(j)}$ are disjoint unions of smooth submanifolds of G by [PFL, Cor. 4.2.8]. Moreover, we have for $h \in G$ that $(G \times M)^h = Z_G(h) \times N$, so that if (h, x) and (k, y) have the same isotropy group, then h and k have the same fixed point sets in neighborhoods of the orbits $G(h, x)$ and $G(k, y)$.

Let

$$\mathcal{S}_0 := \{e\} \times (M \smallsetminus N),$$

and for each $j \in \{1, \dots, m\}$, let

$$\mathcal{S}_j := Ad_G^{(j)} \times N$$

(which is empty for $j = m$ if G has trivial center), and

$$\mathcal{S}_{m+1} := \{e\} \times N.$$

Projection under the quotient map $\widehat{\varrho} \colon \Lambda M \to \Lambda X$ provides manifolds

$$\mathcal{R}_j := \widehat{\varrho}(\mathcal{S}_j).$$

The decompositions of ΛM and ΛX given by the connected components of the $\mathcal{S}_j$ and $\mathcal{R}_j$, respectively, coincide with the stratifications defined in Equations (4.2) and (4.3).

In particular, note that this stratification is strictly finer than the stratification by orbit types in the case where G has nontrivial center. In fact, the piece $Z(G) \times N$, which consists of points of the same isotropy type, must be split into $\{e\} \times N$ and $(Z(G) \smallsetminus \{e\}) \times N$ in order for the pieces to satisfy the condition of frontier. The reason for this is the occurrence of $\{e\}$ as the isotropy group for points of the form (e, x) with $x \in M \smallsetminus N$. Indeed, the closure of the stratum $\mathcal{S}_0 = \{e\} \times (M \smallsetminus N)$ is $\{e\} \times M$, and hence cannot contain the entire isotropy type of points (e, n) with $n \in N$.

As a simple, concrete example, consider the action of the circle $\mathsf{SO}(2)$ on the sphere $\mathbb{S}^2$ by rotations about the z-axis; this action is semifree with $N = (\mathbb{S}^2)^{\mathsf{SO}(2)}$ given by the north and south poles. It is easy to see that the isotropy types

$$A = \{(e, x) \mid x \in \mathbb{S}^2 \smallsetminus N\}$$

and

$$B = \{(t, x) \mid t \in S^1, x \in N\}$$

do not yield a decomposition of $\Lambda\mathbb{S}^2$, as $\overline{A} \cap B = \{e\} \times N$.

Remark 4.5. As illustrated by this example, the inertia space of a G-manifold need not have a top (i.e. open, dense, connected) stratum. In particular, it is clear that any decomposition of the loop space $\Lambda\mathbb{S}^2$ must have both 1- and 2-dimensional strata that are not contained in the closures of other strata. Moreover, it follows from Proposition 3.9 that the inertia space of a G-manifold is locally homeomorphic to a cone on a stratified topological space, called the *link* (see [PFL, 1.4.1]). However, the link of some points in the inertia space cannot be chosen to be connected. In particular, the link of the north pole in the above example is the union of a circle and two points in the loop space $\Lambda\mathbb{S}^2$ and link of the corresponding orbit is the union of three points in the inertia space. Hence, the inertia space does not share the nice topological properties of some other singular spaces, see e.g. [SjLe, Section 5].

4.2.4. *Actions of Abelian Groups.* Suppose G is abelian, and let $\{H_i \mid i \in I\}$ be the (possibly infinite) collection of isotropy groups for the G-action on M. Note that the isotropy group of $(h,x) \in \Lambda M$ is equal to G_x. For each $x \in M$, let $I_x \subseteq I$ be the finite subset consisting of all i such that every neighborhood of x contains points with isotropy group H_i.

Choose $(h,x) \in \Lambda M$, and note that the Cartan subgroup $\mathsf{T}_{(h,x)}$ is in this case unique. For $k \in \mathsf{T}_{(h,x)}$, we have $k \sim h$ if and only if h and k fix the same points in a neighborhood of x in M, or equivalently if and only if h and k are in exactly the same isotropy groups H_i for $i \in I_x$. Therefore,the equivalence class $\mathsf{T}^*_{(h,x)}$ is determined by the set $I_{(h,x)} = \{i \in I_x \mid h \in H_i\}$. Specifically,

$$\mathsf{T}^*_{(h,x)} = \left(\bigcup_{i \in I_{(h,x)}} H_i \right) \cap \left(\bigcup_{j \notin I_{(h,x)}} H_j \right)^c$$

where ^{c} denote the complement (cf. Subsection 4.3 below). The stratification of ΛM given by Equation (4.2), then, is given by sets of the form $\mathsf{T}^*_{(h,x)} \times M_{H_i}$ where $h \in H_i$ and $x \in M_{H_i}$. Intuitively, ΛM is partitioned by isotropy types, and then further decomposed to separate the closures of nearby strata with lower-dimensional fibers in the G–direction.

As a particularly elucidating example, consider $G = \mathbb{T}^2 = \{(s,t) \mid s,t \in \mathbb{T}^1\}$ and $M = \mathbb{CP}^2$ with action given by

$$(s,t)[z_0, z_1, z_2] = [sz_0, tz_1, stz_2].$$

Note that the points $[1,0,0]$, $[0,1,0]$, and $[0,0,1]$ are fixed by $\mathbb{T}^2$. Near these three points, respectively, using coordinates

$$(u_1,u_2) := \left(\frac{z_1}{z_0}, \frac{z_2}{z_0}\right), \quad (v_0,v_2) := \left(\frac{z_0}{z_1}, \frac{z_2}{z_1}\right), \quad \text{and} \quad (w_0,w_1) := \left(\frac{z_0}{z_2}, \frac{z_1}{z_2}\right),$$

the action is given by

$$\begin{array}{rcl} (s,t)(u_1,u_2) & = & (s^{-1}tu_1, tu_2), \\ (s,t)(v_0,v_2) & = & (st^{-1}v_0, sv_2), \text{ and} \\ (s,t)(w_0,w_1) & = & (t^{-1}w_0, s^{-1}w_1). \end{array}$$

Note in particular that the action near each fixed point is different, and hence the torus $\mathbb{T}^2$ is partitioned into $\sim$ classes in different ways at each. However, the strata in $\Lambda\mathbb{CP}^2$ whose closures contain two fixed points have torus fiber given by a subtorus of $\mathbb{T}^2$ whose partition into $\sim$ classes is compatible with both.

For instance, the torus fiber over $R = \{[z_0, z_1, 0] \mid z_0, z_1 \neq 0\}$ in $\Lambda\mathbb{CP}^2$ is the 1–dimensional subtorus T of $\mathbb{T}^2$ consisting of points of the form (s,s). Any open neighborhood of the orbit of a point in R contains points with trivial isotropy and points with isotropy T, so that $T \times R$ is partitioned into $\{e\} \times R$ and $(T \smallsetminus \{e\}) \times R$. It is easy to see that this partition is the restriction to T of the partitions of $\mathbb{T}^2$ at $[1,0,0]$ and $[0,1,0]$. Though it is not compatible with the partition at $[0,0,1]$, this causes no difficulty as $[0,0,1]$ is separated from the closure of R.

Remark 4.6. In the above example, the link in the loop space $\Lambda\mathbb{CP}^2$ of the point

$$(e, [1,0,0]) \in \mathbb{T}^2 \times \mathbb{CP}^2$$

can be described as follows. Let (x,y,u_1,u_2) denote coordinates for $\mathfrak{t} \times \mathbb{C}^2$ where $\mathfrak{t}$ denotes the Lie algebra of $\mathbb{T}^2$, and let $S = \{(x,y,u_1,u_2) : x^2 + y^2 + |u_1|^2 + |u_2|^2 = 1\}$ denote the unit sphere. Then the link is the set of points in S of the form $(x,y,0,0)$ (corresponding to the isotropy group of the origin), $(x, x, u_1, 0)$ (corresponding to the points fixed by the circle (s, s)), $(x, 0, 0, u_2)$ (corresponding to the points fixed by the circle (s, e)), and $(0, 0, u_1, u_2)$ (corresponding to the identity element of $\mathbb{T}^2$). In general, the link at a point in the abelian case can be described similarly.

4.2.5. *The Adjoint Action.* Let G act on itself by conjugation. Then

$$\Lambda G = \{(h,k) \in G^2 \mid kh = hk\}$$

is the set of commuting ordered pairs of elements of G with diagonal G-action by conjugation.

The isotropy group of (h, k) is given by $Z_G(h) \cap Z_G(k)$, and the set of points fixed by $Z_G(h) \cap Z_G(k)$ is given by points of the form (l, j) where l and j are elements of the center of $Z_G(h) \cap Z_G(k)$. Similarly, $\mathsf{T}^*_{(h,k)}$ consists of elements j of $\mathsf{T}_{(h,k)}$ such that $Z_G(h)$ and $Z_G(j)$ coincide on a neighborhood of the G-orbit $G(h, k)$.

4.2.6. *The Standard Action of* $\mathsf{SO}(3)$ *on* $\mathbb{R}^3$*.* Let $G = \mathsf{SO}(3)$ act on $M = \mathbb{R}^3$ in the usual way. For each point $x \in \mathbb{R}^3$ with $x \neq 0$, we let $R_{x,\theta}$ with $\theta \in [0, 2\pi)$ denote rotation through the angle θ about the line spanned by x where we assume θ is a positive rotation with respect to an oriented basis for $\mathbb{R}^3$ whose third element is x. In particular, $R_{x,0} = 1$ and $R_{x,\theta} = R_{-x,2\pi-\theta}$ for each $x \in \mathbb{R}^3 \setminus \{0\}$. See [DuKo, Sec. 1.2 and 3.4] for a careful description of this action, and note that our notation differs slightly to adapt to our situation.

There are three isotropy types that occur in $\Lambda\mathbb{R}^3$: the point $(e, 0)$ has isotropy group $\mathsf{SO}(3)$, points of the form $(R_{x,\pi}, 0)$ have conjugate isotropy groups isomorphic to $O(2)$, and all other points have conjugate isotropy groups isomorphic to $\mathsf{SO}(2)$. If $(h, x) \in \Lambda\mathbb{R}^3$ such that $x \neq 0$ and $\mathsf{T}_{(h,x)} = G_{(h,x)} \cong \mathsf{SO}(2)$, then any neighborhood of the orbit $G(h, x)$ small enough to not intersect $\{0\} \times \mathsf{SO}(3)$ contains only points with $\mathsf{T}_{(h,x)}$-isotropy type $\mathsf{T}_{(h,x)}$ and $\{e\}$. Hence there are only two $\sim$ classes, the identity and the nontrivial elements. Similarly, if $(h, x) = (R_{x,\theta}, 0)$ with $\theta \neq \pi$, it can be seen in a neighborhood of the orbit $G(h, x)$ small enough to contain no points of the form $(R_{y,\pi}, 0)$ that $\mathsf{T}_{(h,x)}$ is as well partitioned into the same two $\sim$ classes. If $(h, x) = (R_{x,\pi}, 0)$, then as $G(h, x)$ contains $(R_{y,\pi}, 0)$ for each $y \in \mathbb{R}^3$ and as $R_{x,\pi}$ fixes $(R_{y,\pi}, 0)$ when x and y are orthogonal, the torus $\mathsf{T}_{(h,x)}$ is partitioned into the $\sim$ classes $\{e\}$, $\{R_{x,\pi}\}$, and $\{R_{x,\theta} \mid \theta \in (0, \pi) \cup (\pi, 2\pi)\}$. It follows that the maximal decomposition of $\Lambda\mathbb{R}^3$ induced by the stratification $\mathcal{S}_{(h,x)}$ consists of four sets:

$$\begin{aligned}
S_1 &= \{(e, 0)\},\\
S_2 &= \{(R_{x,\pi}, 0) \mid x \in \mathbb{R}^3 \setminus \{0\}\},\\
S_3 &= \{(e, x) \mid x \in \mathbb{R}^3 \setminus \{0\}\}, \quad \text{and}\\
S_4 &= \{(R_{x,\theta}, x) \mid \theta \in (0, \pi) \cup (\pi, 2\pi), x \in \mathbb{R}^3\} \cup \{(R_{x,\pi}, x) \mid x \in \mathbb{R}^3 \setminus \{0\}\}.
\end{aligned}$$

Note in particular that the map $\mathsf{SO}(3)\backslash\Lambda\mathbb{R}^3 \to \mathsf{SO}(3)\backslash\mathbb{R}^3$ given by $\mathsf{SO}(3)(h, x) \mapsto \mathsf{SO}(3)x$ is not a stratified mapping; for $\theta \in (0, \pi) \cup (\pi, 2\pi)$, the points $\mathsf{SO}(3)(R_{x,\theta}, x)$ and $\mathsf{SO}(3)(R_{x,\theta}, 0)$ are mapped to points with different isotropy types.

Similarly, consider the restriction of the $\mathsf{SO}(3)$-action to $M = \mathbb{R}^3 \setminus \{0\}$. The maximal decomposition of ΛM given by Equation (4.2) has two pieces,

$$\{(e, x) \mid x \in \mathbb{R}^3 \setminus \{0\}\} \text{ and } \{(R_{x,\theta}, x) \mid \theta \in (0, 2\pi), x \in \mathbb{R}^3 \setminus \{0\}\}.$$

Note in particular that in this case, ΛM is a smooth manifold with a single isotropy type, and hence that this stratification is strictly finer than the stratification by isotropy types.

To understand this phenomenon, let H be the subgroup of $\mathsf{SO}(3)$ isomorphic to $\mathsf{SO}(2)$ given by rotations about the z-axis. Then considering the H-space $\mathbb{R}^3$ given by the restricted action, there are two isotropy types; points on the z-axis are fixed by all of H, while points off the z-axis are fixed only by the identity. It is easy to see, then, that the partition of $\Lambda\mathbb{R}^3$ given by the restriction of the isotropy type stratification of $\mathbb{R}^3 \times H$ does not yield a stratification of $\Lambda\mathbb{R}^3$. Hence, while the stratifications given by Equations (4.2) and (4.3) are in general not the coarsest stratifications of ΛM and ΛX, they have the benefit of giving a uniform, explicit stratification of the loop space ΛM and the inertia space for all smooth G-manifolds under consideration.

4.3. **A Partition of Cartan Subgroups in Isotropy Groups.** In this subsection, we prove a number of auxiliary results on topological properties of the equivalence classes of the relation $\sim$ which has been defined in Subsection 4.1. Throughout this section, let Q be a smooth, not necessarily connected G-manifold and fix a closed abelian subgroup $\mathsf{T} \leq G$ which need not be connected. Assume that Q is partitioned into a finite number of T-isotropy types. We have in mind the case $Q = GV$ where V is a slice for the G-action on $G \times M$ and T is a Cartan subgroup of the isotropy group of the origin in V, but we state the results of this subsection more generally.

As above, for $s, t \in \mathsf{T}$, we say that $s \sim t$ when $Q^s = Q^t$. Let $H_0, H_1, \dots H_r$ be the finite collection of isotropy groups for the action of T on Q. Then the $\sim$ class $[t]$ of $t \in \mathsf{T}$ is given by

$$[t] = \bigcap_{t \in H_i} H_i \cap \left(\bigcup_{t \notin H_j} H_j \right)^c .$$

That is, each $\sim$ class is determined by a subset of $\{1, 2, \dots, r\}$; note that a nonempty subset

$$I \subseteq \{1, 2, \dots, r\}$$

need not correspond to a nonempty $\sim$ class. Using this together with a dimension counting argument the following result is derived immediately.

Lemma 4.7. *The group T is partitioned into a finite number of $\sim$ classes. Each $\sim$ class $[t]$ is an open subset of the closed subgroup $t^\bullet$ of T defined by*

$$t^\bullet := \bigcap_{t \in H_i} H_i = \bigcap_{q \in Q^t} \mathsf{T}_q,$$

and $\overline{[t]}$ consists of a union of connected components of $t^\bullet$. Moreover, each $\sim$ class has a finite number of connected components.

Also note the following.

Lemma 4.8. *Suppose $s, t \in \mathsf{T}$ such that $[s] \cap \overline{[t]} \neq \emptyset$. Then for each connected component $[s]^\circ$ of $[s]$ and $[t]^\circ$ of $[t]$ such that $[s]^\circ \cap \overline{[t]^\circ} \neq \emptyset$ the relation $[s]^\circ \subseteq \overline{[t]^\circ}$ holds true.*

Proof. Let $u \in [s]^\circ \cap \overline{[t]^\circ}$. Then $Q^s = Q^u$, and by continuity of the action, $Q^t \subseteq Q^u$. It follows that $Q^t \subseteq Q^s$, and hence that $s^\bullet = \bigcap_{q \in Q^s} \mathsf{T}_q \leq \bigcap_{q \in Q^t} \mathsf{T}_q = t^\bullet$.

Note that $[s]^\circ$ is contained in a connected component $(s^\bullet)^\circ$ of $s^\bullet$ which is contained in a connected component $(t^\bullet)^\circ$ of $t^\bullet$. Similarly, $\overline{[t]}$ consists of entire connected components of $t^\bullet$, so $u \in \overline{[t]^\circ} = (t^\bullet)^\circ$. Then $[s]^\circ \subseteq (t^\bullet)^\circ = \overline{[t]^\circ}$, completing the proof. □

For each $g \in G$, we let $\sim_g$ denote the equivalence relation defined on $g\mathsf{T}g^{-1}$ in terms of its action on Q. In particular, if $g \in N_G(\mathsf{T})$, then $\sim_g$ coincides with $\sim$. It is easy to verify the following.

Lemma 4.9. *Let $s, t \in \mathsf{T}$. Then $s \sim t$ if and only if $gsg^{-1} \sim_g gtg^{-1}$, i.e.*

$$[gtg^{-1}]_g = g[t]g^{-1},$$

where $[-]_g$ denotes the equivalence class with respect to $\sim_g$.

Similarly, the following will be important when showing that certain $\sim$ classes are sufficiently separated.

Lemma 4.10. *Suppose $s, t \in \mathsf{T}$ such that $s \not\sim t$, and $[s]$ is diffeomorphic to $[t]$. Then $[s] \cap \overline{[t]} = \emptyset$.*

Proof. If $Q^s \subseteq Q^t$, then $\bigcap_{q \in Q^t} \mathsf{T}_q \leq \bigcap_{q \in Q^s} \mathsf{T}_q$, so that $t^\bullet \leq s^\bullet$. By Lemma 4.7, $[s]$ and $[t]$ are open subsets of $s^\bullet$ and $t^\bullet$, respectively, so that as $[s]$ and $[t]$ are diffeomorphic, $s^\bullet$ and $t^\bullet$ have the same dimension. Additionally, $[s]$ is open and dense in each connected component of $s^\bullet$ it intersects, and similarly $[t]$ in $t^\bullet$, so that $[s]$ and $[t]$ do not intersect the same connected components of the closed group $t^\bullet$. The claim follows, and the argument is identical if $Q^t \subseteq Q^s$.

So suppose $Q^s \not\subseteq Q^t$ and $Q^t \not\subseteq Q^s$. If $l \in \overline{[s]} \cap \overline{[t]}$, then by continuity of the action, l fixes $Q^s \cup Q^t$. Since Q^t is a proper subset of $Q^s \cup Q^t$, it follows that $l \not\sim s$. Therefore, $l \in \overline{[s]} \smallsetminus [s]$ and $[s] \cap \overline{[t]} = \emptyset$. $\square$

For each $n \in N_G(\mathsf{T})$, conjugation by n induces a diffeomorphism from T to itself which by Lemma 4.9 acts on the set of $\sim$ classes. More precisely:

Lemma 4.11. *The normalizer $N_G(\mathsf{T})$ acts on the finite set of $\sim$ classes in T in such a way that for each $n \in N_G(\mathsf{T})$ and $t \in \mathsf{T}$, the submanifold $n[t]n^{-1}$ is diffeomorphic to $[t]$. Moreover, either $n[t]n^{-1} = [t]$ or $\overline{n[t]n^{-1}} \cap [t] = \emptyset$.*

4.4. **Proof of Theorem 4.1.** In this subsection, we prove Theorem 4.1, establishing that for a G-manifold M the germs $\mathcal{S}_{(h,x)}$ and $\mathcal{R}_{G(h,x)}$ given by Equations (4.2) and (4.3) define a smooth Whitney stratification of the loop space ΛM and a smooth stratification of the inertia space ΛX, respectively.

The general strategy is to first decompose ΛM into its G-isotropy types. Roughly speaking, isotropy types consisting of smaller manifolds have larger G-fibers, so the fibers must further be decomposed as illustrated in the examples in Subsection 4.2.4. This is accomplished by first decomposing a Cartan subgroup in the G-fiber into $\sim$ classes using the results of Subsection 4.3 and then partitioning nearby by taking the G-orbits of these pieces. A brief outline of the proof follows.

We begin with Lemma 4.12 which essentially guarantees that we can apply the results of the preceding section on the G-saturation of a (linear) slice. Then, in Lemma 4.13 we confirm that the germ $\mathcal{S}_{(h,x)}$ is that of a subset of ΛM consisting of points with the same G-isotropy type. Afterwards, we prove Lemmas 4.14 and 4.15, demonstrating that the germs $\mathcal{S}_{(h,x)}$ and hence the $\mathcal{R}_{G(h,x)}$ do not depend on the choices of the slice, the Cartan subgroup associated to h, and the representative (h,x) of the orbit $G(h,x)$. With this, we prove Proposition 4.16, showing that $\mathcal{S}_{(h,x)}$ and $\mathcal{R}_{G(h,x)}$ are germs of smooth submanifolds of $G \times M$.

With this, we are required to define a decomposition $\mathcal{Z}$ of a neighborhood of U of each point $(h,x) \in \Lambda M$; indicating this decomposition and verifying its properties involve the main technical details of the proof. The definition of $\mathcal{Z}$ is given in Equation (4.7) in a manner similar to the stratification; the piece containing (k,y) is defined in terms of the isotropy type of (k,y) and the $\sim$ class $\mathsf{T}^*_{(k,y)}$ of k with respect to the action near $G(k,y)$. However, this definition is given in terms of a slice at (h,x) rather than (k,y), so that we must take into consideration the orbit of the $\sim$ class $\mathsf{T}^*_{(k,y)}$ under the action of the normalizer $N_{G_{(h,x)}}(\mathsf{T}_{(h,x)})$. In particular, the pieces of $\mathcal{Z}$ are defined to be connected components so that, though they are G°-invariant, they need not be G-invariant. However, the G-action simply permutes the pieces of $\mathcal{Z}$ that are connected components of the same G-invariant set.

As the definition of each piece of $\mathcal{Z}$ involves choosing a particular point in each orbit near that of (h,x) as well as a Cartan subgroup, Lemmas 4.18 and 4.19 demonstrate that the definition is independent of these choices and the resulting partition is well-defined. We then show in

Proposition 4.20 that the germs of the pieces of the decomposition $\mathcal{Z}$ coincide with the stratification. This in particular requires a careful description of a G-invariant neighborhood W of a (k, y) small enough not to intersect certain $\sim$ classes in the Cartan subgroup $\mathsf{T}_{(k,y)}$. Roughly speaking, W is formed by removing the closures of the finite collection of conjugates of $\mathsf{T}^*_{(k,y)}$ by $N_{G_{(h,x)}}(\mathsf{T}^*_{(k,y)})$ from the G-factor; it is on this neighborhood that the connected component of the stratum containing (k, y) coincides with the piece containing (k, y). As the stratum containing (k, y) has finitely many connected components in this neighborhood, it follows that the germs coincide. With this, we demonstrate that the partition of a neighborhood of (h, x) is finite in Lemma 4.21, that it satisfies the condition of frontier in Proposition 4.22, and that it satisfies Whitney's condition B in Proposition 4.23. This completes the outline, and we now proceed with the proof.

First we assume to have fixed a G-invariant riemannian metric on M, a bi-invariant metric on G, and that $G\times M$ carries the product metric. By (h, x) we will always denote a point of the loop space ΛM, and by $V_{(h,x)}$ a linear slice in $G \times M$ at (h, x). The isotropy group $G_{(h,x)} = Z_{G_x}(h)$ of (h, x) will be denoted by H, and the normal space $T_{(h,x)}(G \times M)/T_{(h,x)}(G(h, x))$ by $N_{(h,x)}$.

Lemma 4.12. *Let K be a closed subgroup of G, and $V_{(h,x)}$ a (linear) slice for the G-action on $G \times M$ as above. Then the K-manifold $Q := GV_{(h,x)}$ has a finite number of K-isotropy types.*

Proof. Let $\Psi : V_{(h,x)} \to N_{(h,x)}$ denote an H-invariant embedding of the slice $V_{(h,x)}$ into the normal space $N_{(h,x)}$ such that its image is an open convex neighborhood of the origin. Choose an H-invariant open convex neighborhood B of the origin of $N_{(h,x)}$ which is relatively compact in $\Psi\big(V_{(h,x)}\big)$. For each point $(k, y) \in GV_{(h,x)}$ choose a slice $Y_{(k,y)}$ for the K-action on $GV_{(h,x)}$. Then the family $\{KY_{(k,y)}\}_{(k,y)\in GV_{(h,x)}}$ is an open cover of $G\Psi^{-1}(\overline{B})$ which has to admit a finite subcover by compactness of $G\Psi^{-1}(\overline{B})$. Since each $KY_{(k,y)}$ has a finite number of K-isotropy types by [Pfl, Lem. 4.3.6], it follows that $G\Psi^{-1}(\overline{B})$, hence $G\Psi^{-1}(B)$ has a finite number of K-isotropy types. However, $GV_{(h,x)}$ contains the same isotropy types as $G\Psi^{-1}(B)$, since the action by $t \in (0, 1]$ on $V_{(h,x)}$ is G-equivariant, and for each $v \in V_{(h,x)}$ there is a $t \in (0, 1]$ with $tv \in \Psi^{-1}(B)$. Hence $GV_{(h,x)}$ itself has a finite number of K-isotropy types. □

The Lemma implies in particular that the results of Subsection 4.3 apply to $Q = GV_{(h,x)}$ for each abelian subgroup $\mathsf{T} = K$ of G.

Lemma 4.13. *The set germ $\mathcal{S}_{(h,x)}$ is contained in the set germ at (h, x) of points of ΛM having the same isotropy type as (h, x) with respect to the G-action on $G \times M$.*

Proof. Suppose

$$(k, y) \in V^H_{(h,x)} \cap (\mathsf{T}^*_{(h,x)} \times M).$$

Since H fixes (k, y) and $k \in \mathsf{T}_{(h,x)} \leq H$, one obtains $ky = y$ and $(k, y) \in \Lambda M$. By G-invariance of ΛM we get $G(k, y) \subseteq \Lambda M$, hence $\mathcal{S}_{(h,x)}$ is the germ of a subset of ΛM. Now observe that $V^H_{(h,x)} = (V_{(h,x)})_H \subseteq (G \times M)_H$, where $(V_{(h,x)})_H$ and $(G \times M)_H$ denote the subsets of points having isotropy group H. Hence the isotropy group of every point in the G-orbits defining $\mathcal{S}_{(h,x)}$ is conjugate to H, and $\mathcal{S}_{(h,x)}$ is a subgerm of $(G \times M)_{(H)}$. □

The following two lemmas demonstrate that the stratification $\mathcal{S}_{(h,x)}$ does not depend on the choice of a Cartan subgroup $\mathsf{T}_{(h,x)}$ nor on the particular choice of a slice $V_{(h,x)}$.

Lemma 4.14. *Let $(h, x) \in \Lambda M$ with isotropy group $H = Z_{G_x}(h)$. The germ $\mathcal{S}_{(h,x)}$ does not depend on the choice of the Cartan subgroup $\mathsf{T}_{(h,x)}$ of H.*

Proof. Suppose $\mathsf{T}_{(h,x)}$ and $\mathsf{T}'_{(h,x)}$ are two Cartan subgroups of H associated to h. Then $\mathsf{T}_{(h,x)}/\mathsf{T}^{\circ}_{(h,x)}$ is generated by $h\mathsf{T}^{\circ}_{(h,x)}$ and $\mathsf{T}'_{(h,x)}/\mathsf{T}'^{\circ}_{(h,x)}$ is generated by $h\mathsf{T}'^{\circ}_{(h,x)}$, where here and in the rest of this section K° denotes the connected component of the neutral element in a Lie group K. Under the correspondence given in (4.1) both $\mathsf{T}_{(h,x)}/\mathsf{T}^{\circ}_{(h,x)}$ and $\mathsf{T}'_{(h,x)}/\mathsf{T}'^{\circ}_{(h,x)}$ then correspond to $\langle hH^{\circ}\rangle \leq H/H^{\circ}$. It follows by [BrDi, IV. Prop. 4.6] that $\mathsf{T}_{(h,x)}$ and $\mathsf{T}'_{(h,x)}$ are conjugate, so that there is a $g \in H$ such that $g\mathsf{T}_{(h,x)}g^{-1} = \mathsf{T}'_{(h,x)}$. Then, as $g \in H$, the space $V^{H}_{(h,x)}$ is left invariant by g, hence $ghg^{-1} = h$. Therefore, if $k \in \mathsf{T}_{(h,x)}$ with $k \sim h$ as elements of $\mathsf{T}_{(h,x)}$ acting on $GV_{(h,x)}$, Lemma 4.9 implies that $gkg^{-1} \sim ghg^{-1} = h$ as elements of $\mathsf{T}'_{(h,x)}$ acting on $GV_{(h,x)}$. It follows that conjugation by g induces a diffeomorphism of $\mathsf{T}^{*}_{(h,x)}$ onto $\mathsf{T}'^{*}_{(h,x)}$, so that

$$g\left(V^{H}_{(h,x)} \cap \left(\mathsf{T}^{*}_{(h,x)} \times M\right)\right) = \left(V^{H}_{(h,x)} \cap \left(\mathsf{T}'^{*}_{(h,x)} \times M\right)\right),$$

and

$$G\left(V^{H}_{(h,x)} \cap \left(\mathsf{T}^{*}_{(h,x)} \times M\right)\right) = G\left(V^{H}_{(h,x)} \cap \left(\mathsf{T}'^{*}_{(h,x)} \times M\right)\right).$$

□

Lemma 4.15. *The germ $\mathcal{S}_{(h,x)}$ is independent of the particular choice of the slice $V_{(h,x)}$ at (h,x).*

Proof. Suppose $V_{(h,x)}$ and $W_{(h,x)}$ are two choices of linear slices at (h,x) for the G-action on $G \times M$. Let $\mathsf{T}_{(h,x)}$ be Cartan subgroup of H associated to h. By Lemma 4.14, we may assume that the stratum containing (h,x) is defined with respect to each of the two slices using this Cartan subgroup. Note that by the slice theorem and the assumptions on $V_{(h,x)}$ and $W_{(h,x)}$ the open sets $GV_{(h,x)} \cong G \times_H V_{(h,x)}$ and $GW_{(h,x)} \cong G \times_H W_{(h,x)}$ are G-diffeomorphic and hence $\mathsf{T}_{(h,x)}$-diffeomorphic. Therefore, the $\sim$ classes in $\mathsf{T}_{(h,x)}$ do not depend on the choice of the slice.

Letting $\mathcal{N} = \{n \in N_G(\mathsf{T}_{(h,x)}) : n\mathsf{T}^{*}_{(h,x)}n^{-1} \neq \mathsf{T}^{*}_{(h,x)}\}$, we have by Lemma 4.11 that the set

$$C = (H \smallsetminus hH^{\circ}) \cup \bigcup_{n \in \mathcal{N}} n\left(\overline{\mathsf{T}^{*}_{(h,x)}}\right)n^{-1}$$

is a closed subset of G disjoint from $\mathsf{T}^{*}_{(h,x)}$. Hence $V_{(h,x)} \cap (C \times M)^{c}$ is an open neighborhood of (h,x) in $V_{(h,x)}$. We may therefore assume after possibly shrinking $V_{(h,x)}$ and $W_{(h,x)}$ that $V_{(h,x)} \cap (C \times M) = W_{(h,x)} \cap (C \times M) = \emptyset$. Clearly, shrinking the slice does not affect the germ of the stratum. With this, we let O denote the G-invariant open neighborhood $O := GV_{(h,x)} \cap GW_{(h,x)}$ of (h,x) and claim that

$$O \cap G\left(V^{H}_{(h,x)} \cap \left(\mathsf{T}^{*}_{(h,x)} \times M\right)\right) = O \cap G\left(W^{H}_{(h,x)} \cap \left(\mathsf{T}^{*}_{(h,x)} \times M\right)\right).$$

Any element of $O \cap G\left(V^{H}_{(h,x)} \cap \left(\mathsf{T}^{*}_{(h,x)} \times M\right)\right)$ is in the G-orbit of some

$$(k,y) \in O \cap V^{H}_{(h,x)} \cap \left(\mathsf{T}^{*}_{(h,x)} \times M\right).$$

As $(k,y) \in O$, there is a $g \in G$ such that $g(k,y) \in W_{(h,x)}$. Since $G_{(k,y)} = H$ and

$$G_{g(k,y)} = gG_{(k,y)}g^{-1} \leq H,$$

[Pfl, Lem. 4.2.9] implies that $G_{g(k,y)} = H$. In particular, $g \in N_G(H)$, and $g(k,y) \in W^{H}_{(h,x)}$.

Now, $k \in \mathsf{T}^{*}_{(h,x)} \subseteq \mathsf{T}_{(h,x)}$ by definition, so that $gkg^{-1} \in g\mathsf{T}_{(h,x)}g^{-1}$. As $g \in N_G(H)$, it follows that $g\mathsf{T}_{(h,x)}g^{-1} \leq H$. Noting that k is an element of the connected set $\mathsf{T}^{*}_{(h,x)} \ni h$, we have that k is in the same connected component of $\mathsf{T}_{(h,x)}$ as h and so $\mathsf{T}_{(h,x)}$ is a Cartan subgroup of H associated to k as well as h. It is then easy to see that $g\mathsf{T}_{(h,x)}g^{-1}$ is a Cartan subgroup of H

associated to gkg^{-1}. Moreover, because $W_{(h,x)}$ is disjoint from $(H \smallsetminus hH^\circ) \times M \subseteq C \times M$, it must be that $gkg^{-1} \in hH^\circ$. By [BrDi, IV. Prop. 4.6], there is a $\tilde{h} \in H$ such that

$$\tilde{h} g \mathsf{T}_{(h,x)} g^{-1} \tilde{h}^{-1} = \mathsf{T}_{(h,x)},$$

and hence $\tilde{h}g \in N_G(\mathsf{T}_{(h,x)})$. That $\tilde{h} \in H = G_{g(k,y)}$ implies $\tilde{h}gkg^{-1}\tilde{h}^{-1} = gkg^{-1}$, so that $gkg^{-1} \in \tilde{h}g\mathsf{T}_{(h,x)}g^{-1}\tilde{h}^{-1} = \mathsf{T}_{(h,x)}$. Moreover, as $k \in \mathsf{T}^*_{(h,x)}$, we have in addition that

$$gkg^{-1} = \tilde{h}gkg^{-1}\tilde{h}^{-1} \in \tilde{h}g\mathsf{T}^*_{(h,x)}g^{-1}\tilde{h}^{-1}.$$

Therefore,

$$g(k,y) \in O \cap W^H_{(h,x)} \cap \left(\tilde{h}g\mathsf{T}^*_{(h,x)}g^{-1}\tilde{h}^{-1} \times M\right).$$

However, as $W_{(h,x)} \cap (C \times M) = \emptyset$, as C contains all of the nontrivial conjugates of elements of $\mathsf{T}^*_{(h,x)}$ by elements of $N_G(\mathsf{T}_{(h,x)})$, and as $\tilde{h}g \in N_G(\mathsf{T}_{(h,x)})$, it must be that

$$\tilde{h}g\mathsf{T}^*_{(h,x)}g^{-1}\tilde{h}^{-1} = \mathsf{T}^*_{(h,x)}.$$

Hence,

$$g(k,y) \in O \cap W^H_{(h,x)} \cap \left(\mathsf{T}^*_{(h,x)} \times M\right).$$

Switching the roles of $W_{(h,x)}$ and $V_{(h,x)}$ completes the proof. □

Note that if $(h,x) \in \Lambda M$ and $g \in G$, then $gV_{(h,x)}$ is a slice at $g(h,x)$, $g(V^H_{(h,x)}) = (gV_{(h,x)})^{gHg^{-1}}$, and $g\mathsf{T}_{(h,x)}g^{-1}$ is a Cartan subgroup of gHg^{-1} associated to ghg^{-1}. Therefore, as the $\sim$ classes depend only on the action on $GV_{(h,x)}$, Lemmas 4.14 and 4.15 imply that $g\mathcal{S}_{(h,x)} = \mathcal{S}_{g(h,x)}$, so that in particular $\mathcal{R}_{G(h,x)}$ is well-defined.

Now we have the means to verify the following crucial result.

Proposition 4.16. *Each $\mathcal{S}_{(h,x)}$ is the germ of a smooth G-submanifold of $G \times M$, and each $\mathcal{R}_{(h,x)}$ is the germ of a smooth submanifold of $G\backslash(G \times M)$.*

Of course, $G\backslash(G \times M)$ is not itself a smooth manifold but rather a differentiable space. By a smooth submanifold of $G\backslash(G \times M)$, we mean a differentiable subspace of $G\backslash(G \times M)$ that is itself a smooth manifold.

Proof. Since the germ $\mathcal{S}_{(h,x)}$ does not depend on the choice of a particular slice by Lemma 4.15, we choose the slice $V_{(h,x)}$ at (h,x) to be the image under the exponential map of an open ball $B_{(h,x)}$ around the origin of the normal space $N_{(h,x)}$. Note that $N_{(h,x)}$ naturally carries an H-invariant inner product since we have initially fixed an invariant riemannian metric on M and a bi-invariant riemannian metric on G.

Since $(G \times_H V_{(h,x)})^H$ is a totally geodesic submanifold of $G \times_H V_{(h,x)}$ by [Mic, 6.1], the exponential map at (h,x) maps $B^H_{(h,x)} = N^H_{(h,x)} \cap B_{(h,x)}$ onto $V^H_{(h,x)}$. Similarly, as $\mathsf{T}^*_{(h,x)}$ is an open subset of the closed subgroup $h^\bullet$ of H by Lemma 4.7, the relation $T_h\mathsf{T}^*_{(h,x)} = T_h h^\bullet$ holds true. It follows that the exponential map associated to the product metric on $G \times M$ maps the subspace

$$N^H_{(h,x)} \cap (T_h h^\bullet \oplus T_x M) \cap B_{(h,x)}$$

onto $V^H_{(h,x)} \cap \left(\mathsf{T}^*_{(h,x)} \times M\right)$, which is then diffeomorphic to an open neighborhood of the origin in a linear space.

Noting that $G \times_H V^H_{(h,x)} \cong G/H \times V^H_{(h,x)}$, the G-diffeomorphism

$$\Psi\colon\ G \times_H V_{(h,x)} \longrightarrow GV_{(h,x)} \subseteq G \times M$$

induced by the exponential map restricts to a G-diffeomorphism

$$G/H \times \left(V^H_{(h,x)} \cap \left(\mathsf{T}^*_{(h,x)} \times M\right)\right) \longrightarrow G\left(V^H_{(h,x)} \cap \left(\mathsf{T}^*_{(h,x)} \times M\right)\right).$$

Moreover, Ψ induces a map on quotient spaces which is a homeomorphism

$$G\backslash \left(G/H \times \left(V^H_{(h,x)} \cap \left(\mathsf{T}^*_{(h,x)} \times M\right)\right)\right) \longrightarrow G\backslash \left(G\left(V^H_{(h,x)} \cap (\mathsf{T}^*_{(h,x)} \times M)\right)\right).$$

Hence,

$$V^H_{(h,x)} \cap \left(\mathsf{T}^*_{(h,x)} \times M\right) \cong G\backslash \left(G\left(V^H_{(h,x)} \cap (\mathsf{T}^*_{(h,x)} \times M)\right)\right)$$

is a topological submanifold of $G\backslash(G \times M)$. On the differentiable space $G\backslash(G \times M)$, the structure sheaf $\mathcal{O}^\infty_{G\backslash(G\times M)}$ is locally that of G-invariant functions on $G \times M$ (see Section 3). Similarly, the G-invariant $\mathcal{C}^\infty$ functions on $G/H \times \left(V^H_{(h,x)} \cap \left(\mathsf{T}^*_{(h,x)} \times M\right)\right)$ are exactly the $\mathcal{C}^\infty$ functions on $V^H_{(h,x)} \cap \left(\mathsf{T}^*_{(h,x)} \times M\right)$ by [tDie, Prop. 5.2] and [GoSa, Thm. 1.22 (5)]. Therefore, $G\backslash \left(G\left(V^H_{(h,x)} \cap (\mathsf{T}^*_{(h,x)} \times M)\right)\right)$, whose set germ at (h,x) coincides with $\mathcal{R}_{(h,x)}$, is a smooth submanifold of the differentiable space $G\backslash(G \times M)$. □

In order for the germs $\mathcal{S}_{(h,x)}$ to define a stratification, one must verify that for each $(h,x) \in \Lambda M$ there is a neighborhood U in ΛM and a decomposition $\mathcal{Z}$ of U such that for all $(k,y) \in \Lambda M$, the germ $\mathcal{S}_{(k,y)}$ coincides with the germ of the piece of $\mathcal{Z}$ containing (k,y). Set $U := GV_{(h,x)} \cap \Lambda M$. We now define the decomposition $\mathcal{Z}$ of U. Given $(\tilde{k},\tilde{y}) \in U$ there is a $\tilde{g} \in G$ such that $\tilde{g}(\tilde{k},\tilde{y}) \in V_{(h,x)}$. Put $(k,y) = \tilde{g}(\tilde{k},\tilde{y})$ and $K = G_{(k,y)} \leq H$, and let $\mathsf{T}_{(k,y)}$ be a Cartan subgroup in K associated to k. We define the piece of $\mathcal{Z}$ containing $(\tilde{k},\tilde{y})$ to be the connected component containing $(\tilde{k},\tilde{y})$ of the set $\mathcal{U}^{\mathsf{T}_{(k,y)}}_{\tilde{g}}(\tilde{k},\tilde{y})$ which is defined as the G-saturation of the set of points $(l,z) \in (V_{(h,x)})_K \cap (\mathsf{T}_{(k,y)} \times M)$ such that $\mathsf{T}_{(k,y)}$ is a Cartan subgroup of K associated to l and such that the $\sim$ class $\mathsf{T}^*_{(l,z)}$ is conjugate to $\mathsf{T}^*_{(k,y)}$ via an element of $N_H(\mathsf{T}_{(k,y)})$. Note that as above $\mathsf{T}^*_{(l,z)}$ is the $\sim$ class of l in $\mathsf{T}_{(l,z)} = \mathsf{T}_{(k,y)}$ with respect to its action on $GV_{(l,z)}$, where $V_{(l,z)}$ is a slice for the G-action of on $G \times M$ at (l,z). Observe that by [Sch, Proposition 1.3(2)]. as the action of K on $V_{(h,x)}$ is linear, the slice representation of each point in $V_{(h,x)}$ with isotropy group K is isomorphic. It follows that the set $\mathcal{U}^{\mathsf{T}_{(k,y)}}_{\tilde{g}}(\tilde{k},\tilde{y})$ can be written as

$$\mathcal{U}^{\mathsf{T}_{(k,y)}}_{\tilde{g}}(\tilde{k},\tilde{y}) = G\Big(\bigcup_{n \in N_H(\mathsf{T}_{(k,y)})} (V_{(h,x)})_K \cap \left(n\mathsf{T}^*_{(k,y)}n^{-1} \times M\right)\Big) \tag{4.4}$$

Observe that $(V_{(h,x)})_K$ is closed under the action of scalars $\in (0,1]$ and open in $V^K_{(h,x)}$ as a consequence of Lemma 4.12. Moreover,

$$(V_{(h,x)})_K = V^K_{(h,x)} \setminus \bigcup_{\{(H_v) \mid v \in V_{(h,x)} \& K < H_v\}} (V_{(h,x)})_{(H_v)},$$

where the union is over the (finitely many) isotropy classes that (properly) contain K. An analogous relation holds true for $(N_{(h,x)})_K$. Since all fixed point sets $N^{H_v}_{(h,x)}$ are algebraic, the set $(N_{(h,x)})_K \cap S_{(h,x)}$ is a semialgebraic subset of $N_{(h,x)}$, where $S_{(h,x)}$ is a sphere in $N_{(h,x)}$. Hence, $(N_{(h,x)})_K \cap S_{(h,x)}$ and $(N_{(h,x)})_K$ have finitely many connected components by [BoCoRo, Sec. 2.4]. Therefore, $(V_{(h,x)})_K$ has finitely many components, too.

Next, we want to show that $(V_{(h,x)})_K \cap \left(n\mathsf{T}^*_{(k,y)}n^{-1} \times M\right)$ for $n \in N_H(\mathsf{T}_{(k,y)})$ has finitely many components as well. To this end note first that for each $(\tilde{k},\tilde{y}) \in V_{(h,x)}$ the group element $\tilde{k}$ lies in the same connected component hH° of H as h, since $V_{(h,x)}$ is connected. Therefore, for any Cartan subgroup $\mathsf{T}_{(\tilde{k},\tilde{y})}$ of the isotropy group K of $(\tilde{k},\tilde{y})$ associated to $\tilde{k}$, $\mathsf{T}_{(\tilde{k},\tilde{y})}$ is conjugate to a subgroup of $\mathsf{T}_{(h,x)}$ in H. To see this, note that by [BrDi, IV. Prop. 4.2] and its proof, a Cartan subgroup of K associated to $\tilde{k}$ is generated by a maximal torus in $Z_K(\tilde{k})$ and $\tilde{k}$, while a Cartan subgroup of H associated to $\tilde{k}$ is generated by a maximal torus in $Z_H(\tilde{k})$ and $\tilde{k}$. Say $\tilde{h}\mathsf{T}_{(\tilde{k},\tilde{y})}\tilde{h}^{-1} \leq \mathsf{T}_{(h,x)}$ for some $\tilde{h} \in H$. Then $(k,y) := \tilde{h}(\tilde{k},\tilde{y}) \in V_{(h,x)}$ and $k \in \mathsf{T}_{(h,x)}$. It follows that we can always choose a representative $(k,y) \in V_{(h,x)}$ from an orbit such that $\mathsf{T}_{(k,y)} \leq \mathsf{T}_{(h,x)}$.

Since $(V_{(h,x)})_K$ is closed under multiplication by scalars $t \in (0,1]$ using the linear structure it inherits from $N_{(h,x)}$, each point $t(k,y)$ has isotropy group K, hence its G-coordinate lies in K. As K and hence $K \times M$ is closed, it then must be that $\lim_{t\to 0} t(k,y) = (h,x) \in K \times M$, which means $h \in K$. In particular, h and k are in the same connected component of K and hence that the Cartan subgroup $\mathsf{T}_{(k,y)}$ of K associated to k is conjugate in K to a Cartan subgroup of K associated to h. It follows in particular that there is a $\tilde{k} \in K$ such that $h \in \tilde{k}\mathsf{T}_{(k,y)}\tilde{k}^{-1}$. However, this implies that $\tilde{k}^{-1}h\tilde{k} \in \mathsf{T}_{(k,y)}$, so that as $\tilde{k} \in K \leq H$ fixes h, we have from the beginning that $h \in \mathsf{T}_{(k,y)}$.

Now, recall that the equivalence class $\mathsf{T}^*_{(h,x)}$ is an open subset of the closed subgroup $h^\bullet$ of $\mathsf{T}_{(h,x)}$. With respect to the action of $\mathsf{T}_{(h,x)}$ on $GV_{(h,x)}$, there are a finite number of $\sim$ classes in $\mathsf{T}_{(h,x)}$. Hence, there is a neighborhood O of h in G which only intersects $\sim$ classes in $\mathsf{T}_{(h,x)}$ whose closures contain h. Assume $V_{(h,x)}$ is a slice chosen small enough so that $V_{(h,x)} \subseteq O \times M$, and pick $(k,y) \in V_{(h,x)}$. Then, one can chose the Cartan subgroup $\mathsf{T}_{(k,y)} \leq G_{(k,y)}$ with $k \in \mathsf{T}_{(k,y)}$ such that $\mathsf{T}_{(k,y)} \leq \mathsf{T}_{(h,x)}$. Since the slice $V_{(k,y)}$ at (k,y) may be shrunk such that $GV_{(k,y)} \subseteq GV_{(h,x)}$, it follows from the definition of $\sim$ that $\mathsf{T}^*_{(k,y)}$ is the intersection of a union of $\sim$ classes in $\mathsf{T}_{(h,x)}$ with $\mathsf{T}_{(k,y)}$. In particular, as the closure of each such $\sim$ classes contains h, $\overline{\mathsf{T}^*_{(k,y)}}$ and $\mathsf{T}_{(k,y)}$ both contain h. By the proof of Proposition 4.16, the relation $T_h(n\overline{\mathsf{T}^*_{(k,y)}}n^{-1}) = T_h(nk^\bullet n^{-1})$ holds true for all $n \in N_H(\mathsf{T}_{(k,y)})$, where $k^\bullet$ is the intersection of all isotropy groups of the $\mathsf{T}_{(k,y)}$-action on $GV_{(k,y)}$ which contain k. It follows that the exponential map associated to the product metric on $G \times M$ maps the subspace

$$(N_{(h,x)})_K \cap \big(T_h(nk^\bullet n^{-1}) \oplus T_xM\big) \cap B_{(h,x)} \tag{4.5}$$

onto $(V_{(h,x)})_K \cap \left(n\overline{\mathsf{T}^*_{(k,y)}}n^{-1} \times M\right)$. By construction, (4.5) is a semialgebraic subset of $N_{(h,x)}$, and invariant under the action of $t \in (0,1]$.

Let us now describe the preimage of $(n\mathsf{T}^*_{(k,y)}n^{-1} \times M)$ under the exponential map. Since there are only finitely many $\sim$ classes in $\mathsf{T}_{(k,y)}$, one can find finitely many elements $l_1,\ldots,l_\alpha \in \mathsf{T}_{(k,y)}$ such that each group $l_\iota^\bullet$, $\iota = 1,\ldots,\alpha$ has dimension less than $\dim k^\bullet$, and such that

$$\mathsf{T}^*_{(k,y)} = \overline{\mathsf{T}^*_{(k,y)}} \setminus \bigcup_{\iota=1}^{\alpha} l_\iota^\bullet \,.$$

This implies that exp maps the set

$$(N_{(h,x)})_K \cap \Big(\big(T_h(nk^\bullet n^{-1}) \setminus \bigcup_{\iota=1}^{\alpha} T_h(nl_\iota^\bullet n^{-1})\big) \oplus T_xM\Big) \cap B_{(h,x)} \tag{4.6}$$

onto $(V_{(h,x)})_K \cap \big(n\mathsf{T}^*_{(k,y)}n^{-1} \times M\big)$. But (4.6) is semialgebraic by construction, and invariant under the action of $t \in (0,1]$. Hence, (4.6) and thus $(V_{(h,x)})_K \cap \big(n\mathsf{T}^*_{(k,y)}n^{-1} \times M\big)$ have both finitely many connected components, and are invariant under the action of $t \in (0,1]$, too. Since G is compact, and since there are only finitely many different sets $n\mathsf{T}^*_{(k,y)}n^{-1}$, when n runs through the elements of $N_H(\mathsf{T}_{(k,y)})$, Eq. (4.4) entails the following.

Lemma 4.17. *Suppose $V_{(h,x)}$ is given by the image under the exponential map of a sufficiently small ball $B_{(h,x)}$ in the normal space $N_{(h,x)}$, and $(k,y) \in V_{(h,x)}$. Then the set*

$$\exp^{-1}_{(h,x)}\left(\mathcal{U}_{\tilde{g}}^{\mathsf{T}_{(k,y)}}\big(G(k,y)\big)\right) \cap B_{(h,x)}$$

is invariant under multiplication by scalars in $(0,1]$. Moreover, each set $\mathcal{U}_{\tilde{g}}^{\mathsf{T}_{(k,y)}}\big(G(k,y)\big)$ has a finite number of connected components.

At the moment, the set $\mathcal{U}_{\tilde{g}}^{\mathsf{T}_{(k,y)}}(\tilde{k},\tilde{y})$ appears to depend both on the choice of $(k,y) \in V_{(h,x)}$ in the G-orbit of $(\tilde{k},\tilde{y})$ and the Cartan subgroup $\mathsf{T}_{(k,y)}$. With the following two lemmas, we will demonstrate that this is not the case, allowing us to simplify the notation.

Lemma 4.18. *The set $\mathcal{U}_{\tilde{g}}^{\mathsf{T}_{(k,y)}}(\tilde{k},\tilde{y})$ does not depend on the particular representative $(k,y) \in V_{(h,x)}$ of the G-orbit of $(\tilde{k},\tilde{y})$, hence does not depend on $\tilde{g}$.*

Proof. Suppose $g \in G$ also satisfies

$$g(\tilde{k},\tilde{y}) =: (k',y') \in V_{(h,x)}.$$

It follows that $(k',y') = g\tilde{g}^{-1}(k,y)$, so that we have $g\tilde{g}^{-1} \in H$ by (SL4), since

$$g\tilde{g}^{-1}V_{(h,x)} \cap V_{(h,x)} \neq \emptyset.$$

Let $\tilde{h} = g\tilde{g}^{-1}$. Then the isotropy group of (k',y') is $K' := \tilde{h}K\tilde{h}^{-1} \leq H$. Let $\mathsf{T}_{(k',y')}$ be a choice of a Cartan subgroup in K' associated to k'. Note that since $\tilde{h}\mathsf{T}_{(k,y)}\tilde{h}^{-1}$ is clearly a Cartan subgroup of K' associated to k' as well, there exists by [BrDi, IV. Prop. 4.6] a $\hat{k} \in K'$ such that $\mathsf{T}_{(k',y')} = \hat{k}\tilde{h}\mathsf{T}_{(k,y)}\tilde{h}^{-1}\hat{k}^{-1}$. Moreover, by Lemma 4.9, $\mathsf{T}^*_{(k',y')} = \hat{k}\tilde{h}\mathsf{T}^*_{(k,y)}\tilde{h}^{-1}\hat{k}^{-1}$.

Let $(\tilde{l},\tilde{z}) \in \mathcal{U}_{\tilde{g}}^{\mathsf{T}_{(k,y)}}(\tilde{k},\tilde{y})$. Then there exists a $\tilde{g}' \in G$ such that $(l,z) := \tilde{g}'(\tilde{l},\tilde{z}) \in (V_{(h,x)})_K$ and an $n \in N_H(\mathsf{T}_{(k,y)})$ such that $\mathsf{T}^*_{(l,z)} = n\mathsf{T}^*_{(k,y)}n^{-1}$. As (l,z) has isotropy group K, one has $\tilde{h}(l,z) \in (V_{(h,x)})_{K'}$. Since $\hat{k} \in K'$, we get $\hat{k}\tilde{h}(l,z) = \tilde{h}(l,z)$. Again by Lemma 4.9, we therefore obtain

$$\begin{aligned}
\mathsf{T}^*_{\tilde{h}(l,z)} &= \mathsf{T}^*_{\hat{k}\tilde{h}(l,z)} \\
&= \hat{k}\tilde{h}\mathsf{T}^*_{(l,z)}\tilde{h}^{-1}\hat{k}^{-1} \\
&= \hat{k}\tilde{h}\left(n\mathsf{T}^*_{(k,y)}n^{-1}\right)\tilde{h}^{-1}\hat{k}^{-1} \\
&= \hat{k}\tilde{h}n\left(\tilde{h}^{-1}\hat{k}^{-1}\mathsf{T}^*_{(k',y')}\hat{k}\tilde{h}\right)n^{-1}\tilde{h}^{-1}\hat{k}^{-1}.
\end{aligned}$$

Using the fact that $n \in N_H(\mathsf{T}_{(k,y)})$, a routine computation verifies that

$$m := \hat{k}\tilde{h}n\tilde{h}^{-1}\hat{k}^{-1} \in N_H(\mathsf{T}_{(k',y')}).$$

But then $\tilde{h}(l,z) \in (V_{(h,x)})_{K'}$, and $\mathsf{T}^*_{\tilde{h}(l,z)} = m\mathsf{T}^*_{(k',y')}m^{-1}$ with $m \in N_H(\mathsf{T}_{(k',y')})$. It follows that $\tilde{h}(l,z) \in \mathcal{U}_{\tilde{g}}^{\mathsf{T}_{(k',y')}}(k',y')$. Since $\mathcal{U}_{\tilde{g}}^{\mathsf{T}_{(k,y)}}(\tilde{k},\tilde{y})$ is G-invariant,

$$\mathcal{U}_{\tilde{g}}^{\mathsf{T}_{(k,y)}}(\tilde{k},\tilde{y}) \subseteq \mathcal{U}_{g}^{\mathsf{T}_{(k',y')}}(k',y').$$

Switching the roles of (k,y) and (k',y') completes the proof. □

Note that we may now denote $\mathcal{U}_{\tilde{g}}^{\mathsf{T}_{(k,y)}}(\tilde{k},\tilde{y})$ simply as $\mathcal{U}^{\mathsf{T}_{(k,y)}}(\tilde{k},\tilde{y})$.

Lemma 4.19. *If $(\tilde{l},\tilde{z}) \in \mathcal{U}^{\mathsf{T}_{(k,y)}}(k,y)$, $(l,z) \in V_{(h,x)}$ is in the same orbit as $(\tilde{l},\tilde{z})$, and $\mathsf{T}'_{(l,z)}$ a Cartan subgroup of $G_{(l,z)}$ associated to l, then $\mathcal{U}^{\mathsf{T}_{(k,y)}}(\tilde{k},\tilde{y}) = \mathcal{U}^{\mathsf{T}'_{(l,z)}}(\tilde{l},\tilde{z})$. In particular, $\mathcal{U}^{\mathsf{T}_{(k,y)}}(\tilde{k},\tilde{y})$ does not depend on the choice of a Cartan subgroup $\mathsf{T}_{(k,y)}$ of K associated to k.*

Proof. Let $K = Z_{G_y}(k)$ as above. By Eq. (4.4), we may assume that

$$(l,z) \in (V_{(h,x)})_K \cap (\mathsf{T}_{(k,y)} \times M),$$

and that $\mathsf{T}^*_{(l,z)} = n\mathsf{T}^*_{(k,y)}n^{-1}$ for some $n \in N_H(\mathsf{T}_{(k,y)})$. Let $\mathsf{T}'_{(l,z)}$ be a choice of a Cartan subgroup in K associated to l. Then, by [BrDi, IV. Prop. 4.6], there is an $i \in K$ such that $i\mathsf{T}'_{(l,z)}i^{-1} = \mathsf{T}_{(k,y)}$. Recall that by Lemma 4.9, $i\mathsf{T}'^*_{(l,z)}i^{-1} = \mathsf{T}^*_{(l,z)}$.

Given $(j,w) \in (V_{(h,x)})_K \cap (\mathsf{T}'_{(l,z)} \times M)$ such that $\mathsf{T}'^*_{(j,w)} = m\mathsf{T}'^*_{(l,z)}m^{-1}$ for some $m \in N_H(\mathsf{T}'_{(l,z)})$, it is now easy to see that

$$(j,w) = i(j,w) \in i\left((V_{(h,x)})_K \cap (\mathsf{T}'_{(l,z)} \times M)\right) = (V_{(h,x)})_K \cap (\mathsf{T}_{(k,y)} \times M).$$

In particular, $j \in \mathsf{T}_{(k,y)}$ so that by Lemma 4.9, $\mathsf{T}^*_{(j,w)} = i\mathsf{T}'^*_{(j,w)}i^{-1}$. Then

$$\mathsf{T}^*_{(j,w)} = i\mathsf{T}'^*_{(j,w)}i^{-1} = imi^{-1}n\mathsf{T}^*_{(k,y)}n^{-1}im^{-1}i^{-1}.$$

A routine computation verifies that $imi^{-1}n \in N_H(\mathsf{T}_{(k,y)})$. Therefore, $(j,w) \in \mathcal{U}^{\mathsf{T}_{(k,y)}}(\tilde{k},\tilde{y})$, so that $\mathcal{U}^{\mathsf{T}'_{(l,z)}}(\tilde{l},\tilde{z}) \subseteq \mathcal{U}^{\mathsf{T}_{(k,y)}}(\tilde{k},\tilde{y})$. Switching the roles of $(\tilde{k},\tilde{y})$ and $(\tilde{l},\tilde{z})$ completes the proof that $\mathcal{U}^{\mathsf{T}'_{(l,z)}}(\tilde{l},\tilde{z}) = \mathcal{U}^{\mathsf{T}_{(k,y)}}(\tilde{k},\tilde{y})$.

If $\mathsf{T}'_{(k,y)}$ is another choice of a Cartan subgroup of K associated to k, repeating the above argument with $(l,z) = (k,y)$ yields $\mathcal{U}^{\mathsf{T}_{(k,y)}}(k,y) = \mathcal{U}^{\mathsf{T}'_{(k,y)}}(k,y)$. □

Because of the preceding considerations, the set $\mathcal{U}^{\mathsf{T}_{(k,y)}}(\tilde{k},\tilde{y})$ depends only on the orbit $G(k,y)$, hence we will denote it simply as $\mathcal{U}\big(G(k,y)\big)$. For $(\tilde{k},\tilde{y})$ in the same orbit as (k,y), we denote by $\mathcal{U}\big(G(k,y)\big)^{\mathsf{c}}_{(\tilde{k},\tilde{y})}$ or even shorter by $\mathcal{U}_{(\tilde{k},\tilde{y})}$ the connected component of $(\tilde{k},\tilde{y})$ in $\mathcal{U}\big(G(k,y)\big)$. The partition $\mathcal{Z}$ of U then can be written as

$$\mathcal{Z} = \big\{\mathcal{U}_{(\tilde{k},\tilde{y})} \in \mathcal{P}(U) \mid (\tilde{k},\tilde{y}) \in U\big\}. \tag{4.7}$$

Having established that the sets $\mathcal{U}\big(G(k,y)\big)$ are well-defined, we now confirm that the set germs of the $\mathcal{U}\big(G(k,y)\big)$ coincide with the stratification given by Equation (4.2).

Proposition 4.20. *For each $(\tilde{k},\tilde{y}) \in U$, the germs $[\mathcal{U}\big(G(k,y)\big)]_{(\tilde{k},\tilde{y})}$, $[\mathcal{U}_{(\tilde{k},\tilde{y})}]_{(\tilde{k},\tilde{y})}$ and $\mathcal{S}_{(\tilde{k},\tilde{y})}$ coincide.*

Proof. As $\mathcal{S}_{(\tilde{k},\tilde{y})}$ and $\mathcal{U}\big(G(k,y)\big)$ depend only on the orbit of $(\tilde{k},\tilde{y})$, it is clearly sufficient to consider the case of $(\tilde{k},\tilde{y}) = (k,y) \in V_{(h,x)}$. Set $K = Z_{G_y}(k)$ and fix a linear slice $V_{(k,y)}$ at (k,y) for the G-action on $G \times M$. By [Bre, II. Corollary 4.6], we may assume that $V_{(k,y)} \subseteq V_{(h,x)}$, though it need not be the case that $V_{(k,y)}$ is the image under the exponential map of a subset of the normal space $V_{(k,y)}$. As in the proof of Lemma 4.15, we define a closed subset C of G consisting of the (finitely many) connected components of K not containing k as well as the (finitely many) nontrivial $N_H(\mathsf{T}_{(k,y)})$-conjugates of $\mathsf{T}^*_{(k,y)}$. Let $O = C^c$ be the complement of C in G. Then $O \times M$ is an open subset of $G \times M$ containing $\mathsf{T}^*_{(k,y)}$. Hence $V_{(k,y)} \cap (O \times M)$ is an open neighborhood of (k,y) in $V_{(k,y)}$, so we may shrink $V_{(k,y)}$ to assume that $V_{(k,y)} \subseteq O \times M$. Put $Q = GV_{(k,y)}$. We now show that the set germs $[\mathcal{U}\big(G(k,y)\big)]_{(k,y)}$ and $\mathcal{S}_{(k,y)}$ coincide by proving that

$$\mathcal{U}\big(G(k,y)\big) \cap Q = G\left(V^K_{(k,y)} \cap (\mathsf{T}^*_{(k,y)} \times M)\right).$$

Let $(\tilde{l},\tilde{z}) \in \mathcal{U}\big(G(k,y)\big) \cap Q$. Then there is a $\tilde{g}' \in G$ such that

$$(l,z) := \tilde{g}'(\tilde{l},\tilde{z}) \in (V_{(h,x)})_K \cap (\mathsf{T}_{(k,y)} \times M)$$

and an $n \in N_H(\mathsf{T}_{(k,y)})$ such that $\mathsf{T}^*_{(l,z)} = n\mathsf{T}^*_{(k,y)}n^{-1}$. In particular, $\mathsf{T}_{(k,y)}$ is a Cartan subgroup of K associated to l. As $(l,z) \in Q$, there is a $g \in G$ such that $g(l,z) \in V_{(k,y)}$. Moreover, as $G_{g(l,z)} \leq K$ and $G_{(l,z)} \leq K$, we have $G_{g(l,z)} = K$ by [PFL, Lem. 4.2.9] and hence $g \in N_H(K)$. Similarly, as $g(l,z) \in V_{(k,y)} \subseteq V_{(h,x)}$ and $(l,z) \in V_{(h,x)}$, $g \in H$ by (SL4).

As $k, glg^{-1} \in K$ and $(k,y), g(l,z) \in V_{(k,y)}$, which is disjoint from $(K \smallsetminus kK^\circ) \times M$, k and glg^{-1} are in the same connected component of K. By [BRDI, IV. Prop. 4.6], there is a $\tilde{k} \in K$ such that $\tilde{k}g\mathsf{T}_{(k,y)}g^{-1}\tilde{k}^{-1} = \mathsf{T}_{(k,y)}$, and hence $\tilde{k}g \in N_H(\mathsf{T}_{(k,y)})$. Recalling that $l \in \mathsf{T}^*_{(l,z)} = n\mathsf{T}^*_{(k,y)}n^{-1}$ for some $n \in N_H(\mathsf{T}_{(k,y)})$, we have

$$\tilde{k}glg^{-1}\tilde{k}^{-1} \in \tilde{k}g\mathsf{T}^*_{(l,z)}g^{-1}\tilde{k}^{-1} = \tilde{k}gn\mathsf{T}^*_{(k,y)}n^{-1}g^{-1}\tilde{k}^{-1}.$$

Recalling that $\tilde{k} \in K$, and K is the isotropy group of $g(l,z) = (glg^{-1}, gz)$, we have

$$glg^{-1} \in \tilde{k}gn\mathsf{T}^*_{(k,y)}n^{-1}g^{-1}\tilde{k}^{-1}.$$

However, as $g(l,z) \in V_{(k,y)} \subseteq O \times M$, which is disjoint from $C \times M$, and as $\tilde{k}gn \in N_H(\mathsf{T}_{(k,y)})$, it must be that $\tilde{k}gn\mathsf{T}^*_{(k,y)}n^{-1}g^{-1}\tilde{k}^{-1} = \mathsf{T}^*_{(k,y)}$. It follows that $g(l,z) \in V^K_{(k,y)} \cap (\mathsf{T}^*_{(k,y)} \times M)$, and hence $\mathcal{U}\big(G(k,y)\big) \cap Q \subseteq G\left(V^K_{(k,y)} \cap (\mathsf{T}^*_{(k,y)} \times M)\right)$.

Conversely, if $(\tilde{j}, \tilde{w}) \in G\left(V^K_{(k,y)} \cap (\mathsf{T}^*_{(k,y)} \times M)\right)$, then there is a $\hat{g} \in G$ such that

$$(j,w) := \hat{g}(\tilde{j},\tilde{w}) \in V^K_{(k,y)} \cap (\mathsf{T}^*_{(k,y)} \times M).$$

Then as $V_{(k,y)} \subseteq V_{(h,x)}$, we have

$$(j,w) \in (V_{(h,x)})_K \cap (\mathsf{T}^*_{(k,y)} \times M),$$

and so $(j,w) \in \mathcal{U}\big(G(k,y)\big)$ using the trivial element of the normalizer. Therefore,

$$\mathcal{U}\big(G(k,y)\big) \cap Q = G\left(V^K_{(k,y)} \cap (\mathsf{T}^*_{(k,y)} \times M)\right),$$

which shows the first part of the claim.

As the set $\left(V^K_{(k,y)} \cap (\mathsf{T}^*_{(k,y)} \times M)\right)$ in the right hand side of the preceding equation is connected, the set $\mathcal{U}\big(G(k,y)\big) \cap Q$ has a finite number of connected components. Since Q is a G-invariant open neighborhood of the orbit $G(k,y)$, this implies that $\mathcal{U}\big(G(k,y)\big)$ has locally only finitely many connected components and that the germ of $\mathcal{U}_{(k,y)}$ at (k,y) coincides with the germ of $\mathcal{S}_{(k,y)}$ at (k,y). $\square$

Since the $\mathcal{S}_{(h,x)}$ are germs of smooth G-submanifolds of $G \times M$, and the piece associated to a point $(\tilde{k}, \tilde{y}) \in U$ has the same set germ as $\mathcal{S}_{(l,z)}$ at $(l,z) \in \mathcal{U}_{(\tilde{k},\tilde{y})}$, it follows that the pieces of $\mathcal{Z}$ are smooth submanifolds of $G \times M$ invariant under the G-action.

Lemma 4.21. *The partition $\mathcal{Z}$ of $U = GV_{(h,x)}$ given by Equation (4.7) is finite.*

Proof. The set $\mathcal{U}\big(G(k,y)\big)$ is determined by the H-conjugacy class of $G_{(k,y)} = Z_{G_y}(k) \leq H$ as well as the connected component containing k of the $\sim$ class of k for the $\mathsf{T}_{(k,y)}$-action on $GV_{(k,y)}$. By Lemma 4.19, the set $\mathcal{U}\big(G(k,y)\big)$ does not depend on the choice of a Cartan subgroup associated to k. As H acts linearly on $N_{(h,x)}$, there is a finite number of H-conjugacy classes of isotropy groups with respect to the H-action on $N_{(h,x)}$, hence on $V_{(h,x)}$.

Choose a representative K of each H-isotropy type in $V_{(h,x)}$. Then as K/K° is finite, there are a finite number of conjugacy classes of cyclic subgroups of K/K° and hence by [BRDI, IV. Prop. 4.6] a finite number of K-conjugacy classes of Cartan subgroups of K. Given a Cartan subgroup $\mathsf{T}_{(k,y)}$ of K, there are a finite number of connected components of $\sim$ classes in $\mathsf{T}_{(k,y)}$

with respect to the action of $\mathsf{T}_{(k,y)}$ on U. Of course, $\mathsf{T}^*_{(k,y)}$ is defined with respect to the action of $\mathsf{T}_{(k,y)}$ on a subset of U, but this implies that $\mathsf{T}^*_{(k,y)}$ is given by a union of connected components of $\sim$ classes with respect to the action on U, of which there are finitely many. It follows that there are a finite number of $\mathcal{U}\big(G(k,y)\big)$. Finally, each $\mathcal{U}\big(G(k,y)\big)$ has a finite number of connected components, which completes the proof. ☐

We now verify that $\mathcal{Z}$ is a decomposition indeed, cf. [Pfl, Def. 1.1.1 (DS2)].

Proposition 4.22. *The pieces of $\mathcal{Z}$ satisfy the condition of frontier.*

Proof. Suppose $\mathcal{U}\big(G(k,y)\big) \cap \overline{\mathcal{U}\big(G(l,z)\big)} \neq \emptyset$ where the closure is taken in ΛM. As the pieces of $\mathcal{Z}$ are defined to be the connected components of the $\mathcal{U}\big(G(k,y)\big)$ and $\mathcal{U}\big(G(l,z)\big)$, it is sufficient to show that $\mathcal{U}\big(G(k,y)\big) \cap \overline{\mathcal{U}\big(G(l,z)\big)}$ is both open and closed in $\mathcal{U}\big(G(k,y)\big)$. It is obvious that $\mathcal{U}\big(G(k,y)\big) \cap \overline{\mathcal{U}\big(G(l,z)\big)}$ is closed in $\mathcal{U}\big(G(k,y)\big)$, so we need only establish that

$$\mathcal{U}\big(G(k,y)\big) \cap \overline{\mathcal{U}\big(G(l,z)\big)}$$

is open in $\mathcal{U}\big(G(k,y)\big)$.

By Lemma 4.19, the piece $\mathcal{U}\big(G(k,y)\big)$ may be defined in terms of any orbit it contains, so we may assume that some element of the G-orbit of (k,y) is contained in $\mathcal{U}\big(G(k,y)\big) \cap \overline{\mathcal{U}\big(G(l,z)\big)}$. Then the G-invariance of these two sets implies that $G(k,y) \subseteq \mathcal{U}\big(G(k,y)\big) \cap \overline{\mathcal{U}\big(G(l,z)\big)}$. By Proposition 4.20, an open neighborhood of (k,y) in $\mathcal{U}\big(G(k,y)\big)$ is given by

$$G\big(V_{(k,y)}^K \cap (\mathsf{T}^*_{(k,y)} \times M)\big)$$

for a sufficiently small slice $V_{(k,y)}$ at (k,y). As above, we may assume $V_{(k,y)} \subseteq V_{(h,x)}$ by [Bre, II. Corollary 4.6] so that while $V_{(k,y)}$ can then be taken to be linear, it need not be the image under the exponential map of a subset of the normal space $N_{(k,y)}$. We will show that

$$G\big(V_{(k,y)}^K \cap (\mathsf{T}^*_{(k,y)} \times M)\big)$$

is contained in $\overline{\mathcal{U}\big(G(l,z)\big)}$.

As $GV_{(k,y)}$ must contain some element of $\mathcal{U}\big(G(l,z)\big)$, we may assume again by Lemma 4.19 that $G(l,z) \subseteq GV_{(k,y)}$. Moreover, by the proof of Lemma 4.17, we may choose the representative (l,z) from the orbit $G(l,z)$ such that $(l,z) \in V_{(k,y)}$, $k \in \mathsf{T}_{(l,z)} \leq \mathsf{T}_{(k,y)}$, and $k \in \overline{\mathsf{T}^*_{(l,z)}}$. Let $K = G_{(k,y)}$ and $L = G_{(l,z)}$ so that $L \leq K$, and then $(k,y) \in V_{(k,y)}^K \subseteq (V_{(h,x)})_K \subseteq \overline{(V_{(h,x)})_L}$. Then we have

$$(k,y) \in \overline{(V_{(h,x)})_L} \cap \big(\overline{\mathsf{T}^*_{(l,z)}} \times M\big).$$

In particular, note that by our choice of $(l,z) \in V_{(k,y)}$ used to define the set $\mathcal{U}\big(G(l,z)\big)$, (k,y) is in the closure of the set corresponding to the trivial element of $N_H(\mathsf{T}_{(l,z)})$ in Equation (4.4).

For any $(j,w) \in V_{(k,y)}^K \cap (\mathsf{T}^*_{(k,y)} \times M)$, as $j \in \mathsf{T}^*_{(k,y)}$, it follows that $(GV_{(k,y)})^j = (GV_{(k,y)})^k$. In particular, $k \in \mathsf{T}_{(l,z)} \leq L$ implies that k fixes $(l,z) \in V_{(k,y)}$ so that $j \in L$ as well. Since $V_{(k,y)}^K \cap (\mathsf{T}^*_{(k,y)} \times M)$ is invariant under the action of scalar $\in [0,1]$, and k is in the same connected component of L as l, each such j is in the same connected component of L as l also. Fix a $(j,w) \in V_{(k,y)}^K \cap (\mathsf{T}^*_{(k,y)} \times M)$. Then there is a $\tilde{l} \in L$ such that $\tilde{l}\mathsf{T}_{(l,z)}\tilde{l}^{-1}$ is a Cartan subgroup of L associated to j. Hence $\tilde{l}^{-1}j\tilde{l} \in \mathsf{T}_{(l,z)}$, so that as $\tilde{l} \in L \leq K = G_{(j,w)}$, we have $\tilde{l}^{-1}j\tilde{l} = j \in \mathsf{T}_{(l,z)}$.

Finally, note that as $j \in \mathsf{T}^*_{(k,y)}$, it is clear that $(GV_{(l,z)})^j = (GV_{(l,z)})^k$ for a slice $V_{(l,z)}$ chosen small enough so that $GV_{(l,z)} \subseteq GV_{(k,y)}$. Therefore $j \sim k$ as elements of $\mathsf{T}_{(l,z)}$. Then as the connected component $[k]^\circ$ of the $\sim$ class of k as an element of $\mathsf{T}_{(l,z)}$ intersects $\overline{\mathsf{T}^*_{(l,z)}}$, we have by Lemma 4.8 that $[k]^\circ \subseteq \overline{\mathsf{T}^*_{(l,z)}}$. It follows that

$$(j,w) \in \overline{(V_{(h,x)})_L} \cap \big(\overline{\mathsf{T}^*_{(l,z)}} \times M\big),$$

so as $(j,w) \in V^K_{(k,y)} \cap (\mathsf{T}^*_{(k,y)} \times M)$ was arbitrary,

$$V^K_{(k,y)} \cap (\mathsf{T}^*_{(k,y)} \times M) \subseteq \overline{(V_{(h,x)})_L} \cap \big(\overline{\mathsf{T}^*_{(l,z)}} \times M\big).$$

Considering the G-saturations of both sides of this inclusion, it follows that each element of $\mathcal{U}\big(G(k,y)\big) \cap \overline{\mathcal{U}\big(G(l,z)\big)}$ is contained in a neighborhood that is both open and closed in $\mathcal{U}\big(G(k,y)\big)$, completing the proof. $\square$

Finally, we have the following.

Proposition 4.23. *The orbit Cartan type stratifications of ΛM and the inertia space ΛX both satisfy Whitney's condition B.*

The proof follows [Pfl, Thm. 4.3.7].

Proof. Let $(h,x) \in \Lambda M$, $H = Z_{G_x}(h)$, and $V_{(h,x)}$ a slice at (h,x) of the form $\exp(B_{(h,x)})$, where $B_{(h,x)}$ is a ball around the origin in the normal space $N_{(h,x)}$. We work in the neighborhood $U := GV_{(h,x)}$ of (h,x) in ΛM, and show that for any stratum $S \in \mathcal{Z}$ with $(h,x) \in \overline{S}$ Whitney's condition B is satisfied at (h,x) for the pair of strata (R,S), where R is the piece of $\mathcal{Z}$ containing (h,x). Recall that $\mathcal{Z}$ is the decomposition of U given by Eq. (4.7). Recall also, that R is the connected component of $G\left(V^H_{(h,x)} \cap (\mathsf{T}^*_{(h,x)} \times M)\right)$ containing (h,x). To describe the stratum S in some more detail, consider an orbit $G(k,y)$ for $(k,y) \in S$. As in the proof of Lemma 4.17, we may choose the representative (k,y) of the orbit $G(k,y)$ such that $(k,y) \in V_{(h,x)}$, $h \in \mathsf{T}_{(k,y)} \leq \mathsf{T}_{(h,x)}$, and $h \in \overline{\mathsf{T}^*_{(k,y)}}$. In particular, we then have the relation $K \leq H$ for the isotropy group $K := Z_{G_y}(k)$ of (k,y). As shown above, S coincides with the connected component of $\mathcal{U}\big(G(k,y)\big)$ containing (k,y).

Suppose now that $(h_i,x_i)_{i\in\mathbb{N}}$ is a sequence in R and $(k_i,y_i)_{i\in\mathbb{N}}$ a sequence in S, and that both sequences converge to (h,x). Assume in addition that in a smooth chart around (h,x) the secants $\ell_i = \overline{(h_i,x_i),(k_i,y_i)}$ converge to a straight line ℓ, and the tangent spaces $T_{(k_i,y_i)}S$ converge to a subspace τ. Then we must show that $\ell \subseteq \tau$.

Note that the hypotheses imply that $(h,x) \in \mathcal{U}\big(G(h,x)\big) \cap \overline{\mathcal{U}\big(G(k,y)\big)}$. By the proof of Proposition 4.22 and the choices of (k,y) and $\mathsf{T}_{(k,y)} \subseteq K$ we obtain the relation

$$V^H_{(h,x)} \cap (\mathsf{T}^*_{(h,x)} \times M) \subseteq \overline{(V_{(h,x)})_K} \cap \big(\overline{\mathsf{T}^*_{(k,y)}} \times M\big). \tag{4.8}$$

Moreover, since every element $n \in N_H(\mathsf{T}_{(k,y)})$ fixes $V^H_{(h,x)} \cap (\mathsf{T}^*_{(h,x)} \times M)$, it follows that

$$V^H_{(h,x)} \cap (\mathsf{T}^*_{(h,x)} \times M) \subseteq n\overline{\mathsf{T}^*_{(k,y)}}n^{-1} \times M$$

as well, hence

$$V^H_{(h,x)} \cap (\mathsf{T}^*_{(h,x)} \times M) \subseteq \overline{(V_{(h,x)})_K} \cap \big(n\overline{\mathsf{T}^*_{(k,y)}}n^{-1} \times M\big). \tag{4.9}$$

Denote by $\mathfrak{g}$ the Lie algebra of G, by $\mathfrak{h}$ the Lie algebra of H, and let $\mathfrak{m}$ denote the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$ with respect to the initially chosen bi-invariant metric on G. Then there is a neighborhood $U' \subseteq U \cong G \times_H V_{(h,x)}$ of (h,x) in $G \times M$ such that

$$\Psi : U' \longrightarrow \mathfrak{m} \times N_{(h,x)},\ [\exp_{|\mathfrak{m}} \xi, \exp_{(h,x)}(v)] \longmapsto (\xi, v)$$

is a smooth chart at (h,x), where $\exp_{|\mathfrak{m}}$ denotes the restriction of the exponential map of the Lie group G to $\mathfrak{m}$, and $\exp_{(h,x)}$ the exponential function restricted to the open ball $B_{(h,x)} \subseteq N_{(h,x)}$. By shrinking U' if necessary, we have that there is an open neighborhood O of H in G such that

$$\Psi\left(O\left(V^H_{(h,x)} \cap (\mathsf{T}^*_{(h,x)} \times M)\right)\right) \subset \mathfrak{m} \times \left(N^H_{(h,x)} \cap T_{(h,x)}(\mathsf{T}^*_{(h,x)} \times M)\right).$$

We may assume that the sequences $(h_i,x_i)_{i\in\mathbb{N}}$ and $(k_i,y_i)_{i\in\mathbb{N}}$ are contained in U'. Since $(k_i,y_i) \in \mathcal{U}\big(G(k,y)\big)$, one knows that

$$\Psi(k_i,y_i) \in \mathfrak{m} \times H\left((V_{(h,x)})_K \cap \left(\bigcup_{n\in N_H(\mathsf{T}_{(k,y)})} n\mathsf{T}^*_{(k,y)}n^{-1} \times M\right)\right).$$

Recall that there are only finitely many and pairwise disjoint sets $n\mathsf{T}^*_{(k,y)}n^{-1}$, where n runs through the elements of $N_H(\mathsf{T}_{(k,y)})$. Moreover, by Lemma 4.11, $n\mathsf{T}^*_{(k,y)}n^{-1}$ is disjoint from $\overline{m\mathsf{T}^*_{(k,y)}m^{-1}} = m\left(\overline{\mathsf{T}^*_{(k,y)}}\right)m^{-1}$ for every $m \in N_H(\mathsf{T}_{(k,y)})$ with $n\mathsf{T}^*_{(k,y)}n^{-1} \neq m\mathsf{T}^*_{(k,y)}m^{-1}$. Hence, we may assume without loss of generality that

$$(k_i,y_i) \in G\left((V_{(h,x)})_K \cap \left(m_0\mathsf{T}^*_{(k,y)}m_0^{-1} \times M\right)\right)$$

for all i and some $m_0 \in N_H(\mathsf{T}_{(k,y)})$.

Choose $\tilde{l}_i \in G$ such that $(\tilde{k}_i,\tilde{y}_i) := \tilde{l}_i(k_i,y_i) \in (V_{(h,x)})_K$ for all $i \in \mathbb{N}$. Put $(\tilde{h}_i,\tilde{x}_i) := l_i(h_i,x_i)$. After possibly passing to a subsequence, $(\tilde{l}_i)_{i\in\mathbb{N}}$ converges to some $\tilde{l} \in H$, the secant lines $\tilde{\ell}_i = \overline{(\tilde{h}_i,\tilde{x}_i),(\tilde{k}_i,\tilde{y}_i)}$ converge to a straight line $\tilde{\ell}$, and the tangent spaces $T_{(\tilde{k}_i,\tilde{y}_i)}S$ converge to a subspace $\tilde{\tau}$. By definition, and since $\tilde{l}_iT_{(k_i,y_i)}S = T_{(\tilde{k}_i,\tilde{y}_i)}S$ for all i, one obtains $\tilde{\ell} = \tilde{l}\ell$, and $\tilde{\tau} = \tilde{l}\tau$. Hence, the first claim is shown, if $\tilde{\ell} \subseteq \tilde{\tau}$. Without loss of generality we may therefore assume that for all $i \in \mathbb{N}$

$$(k_i,y_i) \in (V_{(h,x)})_K \cap \big(m_0\mathsf{T}^*_{(k,y)}m_0^{-1} \times M\big), \tag{4.10}$$

and then show $\ell \subseteq \tau$ for the sequences $(k_i,y_i)_{i\in\mathbb{N}}$ and $(h_i,x_i)_{i\in\mathbb{N}}$.

Eq. (4.10) now means in particular that

$$\Psi(k_i,y_i) \in \{0\} \times \left((N_{(h,x)})_K \cap \exp^{-1}_{(h,x)}\left(m_0\mathsf{T}^*_{(k,y)}m_0^{-1} \times M\right)\right).$$

Since by Lemma 4.7 and the above observations $m_0\overline{\mathsf{T}^*_{(k,y)}}m_0^{-1}$ is an open and closed subset of a closed subgroup of G and also contains h, the set

$$V := N_{(h,x)} \cap T_{(h,x)}\left(m_0\big(\overline{\mathsf{T}^*_{(k,y)}}\big)m_0^{-1} \times M\right)$$

is a subspace of $N_{(h,x)}$. Let W be the orthogonal complement of the invariant space V^H in V with respect to the H-invariant scalar product induced from $V_{(h,x)}$. Then the image under the chart Ψ of every element of $G\big(V^H_{(h,x)} \cap (\mathsf{T}^*_{(h,x)} \times M)\big) \cap U'$ and every (k_i,y_i) is contained in

$$\mathfrak{m} \times (W_K \cup \{0\}) \times V^H.$$

With respect to this decomposition, (h,x) has coordinates $(0,0,0)$, each element of

$$G\left(V^H_{(h,x)} \cap (\mathsf{T}^*_{(h,x)} \times M)\right)$$

has coordinates contained in $\mathfrak{m} \times 0 \times V^H$, and each sequence element (k_i,y_i) has coordinates contained in $\{0\} \times W_K \times V^H$. In particular, let

$$\Psi(k_i,y_i) = (0,w_i,v_i)$$

for every i. Then as W_K is invariant under multiplication by non-vanishing scalars, we have

$$(\xi, w, v) := \lim_{i\to\infty} \frac{\Psi(k_i, y_i) - \Psi(h_i, x_i)}{\|\Psi(k_i, y_i) - \Psi(h_i, x_i)\|} \in \mathfrak{m} \times \overline{W_K} \times V^H$$

Now, as the unit sphere in W is compact, the sequence $\frac{w_i}{\|w_i\|}$ converges to some $\hat{w} \in SW$ after possibly passing to a subsequence. Then $w = \|w\|\hat{w}$. Since W_K is invariant by non-vanishing scalars, we have

$$\mathfrak{m} \times \operatorname{span} \hat{w} \times V^H \subseteq \tau,$$

and

$$\ell = \operatorname{span}(\xi, \hat{w}, v) \subseteq \tau,$$

proving the first claim.

Now let us show that the orbit Cartan type stratification of ΛX satisfies Whitney's condition B as well. To this end let us first choose a Hilbert basis of H-invariant polynomials

$$p_1, \ldots, p_\kappa : \left(N^H_{(h,x)}\right)^\perp \to \mathbb{R}$$

of the orthogonal complement of the invariant space $N^H_{(h,x)}$ in $N_{(h,x)}$. Next let

$$p_{\kappa+1}, \ldots, p_N : N^H_{(h,x)} \to \mathbb{R}$$

with $N = \kappa + \dim N^H_{(h,x)}$ be a linear coordinate system of the invariant space. We can even choose these p_i in such a way that $p_{\kappa+1}, \ldots, p_{\kappa+\dim V^H}$ is a linear coordinate system of V^H. By construction, $p_1, \ldots, p_N$ then is a Hilbert basis of the normal space $N_{(h,x)}$. Denote by $p : N_{(h,x)} \to \mathbb{R}^N$ the corresponding Hilbert map. Recall that p induces a chart of ΛX over $G\backslash U$ by

$$\widehat{\Psi} : G\backslash U \to \mathbb{R}^N,\ G\exp_{(h,x)}(v) \mapsto p(v).$$

Note that by H-invariance of p and since for every orbit in U there is a representative in $V_{(h,x)}$, the chart $\widehat{\Psi}$ is well-defined indeed. A decomposition of $\widehat{U} := \widehat{\Psi}(G\backslash U)$ inducing the orbit Cartan type stratification on $G\backslash U$ is given by

$$\widehat{\mathcal{Z}} := \left\{\widehat{\Psi}(G\backslash GS) \mid S \in \mathcal{Z}\right\}.$$

Let $\widehat{S} \in \widehat{\mathcal{Z}}$ denote the stratum containing the orbit $G(h,x)$, and $\widehat{S} \in \mathcal{Z}$ a stratum $\neq \widehat{R}$ such that $G(h,x)$ lies in the closure of $\widehat{S}$. Now consider sequences of orbits $\left(G(h_i,x_i)\right)_{i\in\mathbb{N}}$ in $\widehat{R}$ and $\left(G(k_i,y_i)\right)_{i\in\mathbb{N}}$ in $\widehat{S}$ such that both sequences converge to $G(h,x)$. Moreover, assume that the sequence of secants $\overline{\widehat{\Psi}(G(h_i,x_i)), \widehat{\Psi}(G(k_i,y_i))}$ converges to a line $\widehat{\ell}$, and that the sequence of tangent spaces $T_{\widehat{\Psi}(G(k_i,y_i))}\widehat{S}$ converges to some subspace $\widehat{\tau} \subseteq \mathbb{R}^N$. Using notation from before, we can choose representatives (h_i,x_i) and (k_i,y_i) having coordinates in $\mathfrak{m}\times(W_K\cup\{0\})\times V^H \subseteq N_{(h,x)}$ such that

$$\begin{aligned} \Psi(h_i,x_i) &= (0,0,v_i') \in \{0\} \times \{0\} \times V^H \ \text{ and} \\ \Psi(k_i,y_i) &= (0,w_i,v_i) \in \{0\} \times W_K \times V^H. \end{aligned} \tag{4.11}$$

Next observe that by the Tarski–Seidenberg Theorem and the proof of Lemma 4.17, the stratum $\widehat{S}$ is semialgebraic as the image of the semialgebraic set $(W_K \times V^H) \cap B_{(h,x)}$ under the Hilbert map p. By the same argument, $p(W_K)$ is semialgebraic, too, and an analytic manifold, since $p(W_K) \cong N_H(K)\backslash W_K \cong H\backslash W_{(K)}$. Moreover, the equality

$$\widehat{S} = (p(W_K) \times V^H) \cap p(B_{(h,x)})$$

holds true, where we have canonically identified V^H with its image under the Hilbert map p. By Eq. (4.11), this entails that

$$\widehat{\tau} = \lim_{i\to\infty} T_{\widehat{\Psi}(G(k_i,y_i))}\widehat{S} = \lim_{i\to\infty} T_{p(w_i)} p(W_K) \times V^H. \tag{4.12}$$

Since $p(W_K)$ is semialgebraic and an analytic manifold, [Loj65, Prop. 3, p. 103] (see also [Loj70, Section 9]) by Łojasiewicz entails that $p(W_K)$ satisfies Whitney's condition B over the origin. This means after possibly passing to subsequences, that $\ell_{W_K} \subset \tau_{W_K}$, where ℓ_{W_K} is the limit line of the secants $\overline{p(w_i), 0}$, and τ_{W_K} the limit of the tangent spaces $T_{p(w_i)}p(W_K)$ for $i \to \infty$. By Eqs. (4.11) and (4.12) this entails that

$$\widehat{\ell} \subseteq \ell_{W_K} \times V^H \subseteq \tau_{W_K} \times V^H = \widehat{\tau}.$$

This finishes the proof. □

Recall that $\widehat{\rho} : \Lambda M \to \Lambda X$ denotes the quotient map, which is both open and closed by [tDie, Prop. 3.1 (iv) and Prop. 3.6 (i)]. Hence, as the sets defining the $\mathcal{S}_{(h,x)}$ consist of entire G-orbits, and as the pieces of $\mathcal{Z}$ consist of connected components of G-orbits, Proposition 4.22 extends to the local decomposition in ΛX given by the $\mathcal{R}_{G(h,x)}$. Therefore, combining Propositions 4.16, 4.20, 4.22, and 4.23, we have completed the proof of Theorem 4.1.

5. A De Rham Theorem for the Inertia Space

In this section, we prove a de Rham theorem for the inertia space ΛX analogous to that of [Sja] for singular symplectic reduced spaces.

5.1. Differential Forms on the Inertia Space.

Before constructing differential forms on the inertia space, let us briefly recall from [Pfl, Prop. 1.2.7] that a stratification (in the sense of Mather [Mat73]) of a locally compact topological space X induces a uniquely determined coarsest decomposition of X into strata. Applied to our situation, where we consider a compact Lie group G acting on a smooth manifold M, we thus obtain a coarsest decomposition $\mathscr{D}$ of ΛM which induces the stratification from Theorem 4.1. The elements of $\mathscr{D}$ are the strata of ΛM. It is easy to see that each stratum from $\mathscr{D}$ is G-invariant and that the family of quotients $\{G\backslash Z \mid Z \in \mathscr{D}\}$ forms a decomposition of ΛX which induces the natural stratification of the inertia space from Theorem 4.1. Let us introduce some notation: $\iota\colon \Lambda M \to G \times M$ denotes the natural embedding of ΛM as a subspace and $\rho\colon G\times M \to G\backslash(G\times M)$ the quotient map. Moreover, for each $Z \in \mathscr{D}$, we denote by $\iota_Z\colon Z \to G\times M$ the inclusion and by $\rho_Z\colon Z \to G\backslash Z$ the restricted quotient map.

Let us construct in the following the sheaf of differential forms on the inertia space. Given $k \in \mathbb{N}$ we denote by Ω^k_{inv} the sheaf of G-invariant differential k-forms on $G \times M$ treated as a sheaf on $G\backslash(G\times M)$. That is, if U is an open subset of $G\backslash(G\times M)$, then $\Omega^k_{\mathrm{inv}}(U)$ consists of the differential k-forms $\omega \in \Omega^k(\rho^{-1}(U))$ on $\rho^{-1}(U) \subseteq G\times M$ such that $L_g^*\omega = \omega$ for all $g \in G$, where $L_g : G\times M \to G\times M$ denotes the left action by g on $G\times M$. Similarly, we let Ω^k_{bas} denote the subsheaf of Ω^k_{inv} consisting of G-basic differential forms on $G\times M$ or any of the G-manifolds $Z \subseteq G\times M$. More precisely, $\Omega^k_{\mathrm{bas}}(U)$ consists of all G-invariant k-forms ω on $\rho^{-1}(U)$ such that the interior product $i_{\xi_{G\times M}}\omega$ of ω with the fundamental vector field $\xi_{G\times M}$ vanishes for every $\xi \in \mathfrak{g}$ (cf. [Pfl, Sec. 5.3.1]). Now let $W \subseteq \Lambda X$ be relatively open, and $U \subseteq G\backslash(G\times M)$ open such that $W = U \cap \Lambda X$. By a *differential k-form* $\widetilde{\omega}$ on W we now understand a collection of differential forms $\widetilde{\omega}_Z$ on $W \cap (G\backslash Z)$ for $Z \in \mathscr{D}$ with $W\cap(G\backslash Z) \neq \emptyset$ such that there is an $\omega \in \Omega^k_{\mathrm{inv}}(U)$ with $\rho_Z^*\widetilde{\omega}_Z = \iota_Z^*\omega$ on its domain $\rho^{-1}(W)\cap Z$. We denote the space of differential k-forms on W by $\Omega^k(W)$. One checks immediately that Ω^k then becomes a sheaf on ΛX. This sheaf is even fine, since by construction Ω^k is a $\mathcal{C}^\infty_X$-module sheaf, and $\mathcal{C}^\infty_X$ is fine as the structure sheaf of a differentiable space.

Note that the form ω on U which represents the differential form $\widetilde{\omega}$ on W need not be globally basic. We let Ω^k_{ibas} denote the subsheaf of Ω^k_{inv} consisting of k-forms ω such that for every $Z \in \mathscr{D}$ the pull-back $\iota_Z^*\omega$ is a basic form on Z. That is, for each $U \subseteq G\backslash(G \times M)$ open, we define

$$\Omega^k_{\mathrm{ibas}}(U) = \{\omega \in \Omega^k(\rho^{-1}(U))^G \mid i_{\xi_Z}\iota_Z^*\omega = 0 \text{ for all } \xi \in \mathfrak{g} \text{ and } Z \in \mathscr{D}\}.$$

We refer to sections of Ω^k_{ibas} as *inertia-basic* k-forms. Intuitively, these correspond to k-forms that are basic on each of the strata of ΛM. A form $\omega \in \Omega^k_{\mathrm{bas}}(G\backslash(G \times M))$ is inertia-basic, but an inertia-basic form need not be basic on all of $G \times M$.

By definition, it is clear that we have a surjective linear map

$$\Omega^k_{\mathrm{ibas}}(U) \longrightarrow \Omega^k(W)$$

and that this map has kernel

$$\mathcal{I}^k(U) = \{\omega \in \Omega^k(\rho^{-1}(U))^G \mid \iota_Z^*\omega = 0 \text{ for all } Z \in \mathscr{D}\}.$$

Hence we obtain isomorphisms

$$\Omega^k(W) \cong \Omega^k_{\mathrm{ibas}}(U)/\mathcal{I}^k(U).$$

In particular, when $k = 0$,

$$\Omega^0(W) \cong \Omega^0_{\mathrm{ibas}}(U)/\mathcal{I}^0(U) = \mathcal{C}^\infty(\rho^{-1}(U))^G/\mathcal{I}^0(U),$$

where $\mathcal{I}^0(U)$ is the ideal of G-invariant smooth functions on $\rho^{-1}(U)$ which vanish on ΛM. By its definition in Section 3 the structure sheaf $\mathcal{C}^\infty_{\Lambda X}$ of ΛX can be naturally identified with the sheaf Ω^0 on ΛX.

Next let us show that the exterior derivative maps inertia-basic forms to inertia-basic forms. Suppose ω is an inertia-basic k-form on $\rho^{-1}(U)$, i.e. that $\omega \in \Omega^k_{\mathrm{ibas}}(U)$. By Cartan's Magic Formula, we then conclude for each $Z \in \mathscr{D}$ which intersects $\rho^{-1}(U)$ and each $\xi \in \mathfrak{g}$ that

$$i_{\xi_Z}\iota_Z^* d\omega = \iota_Z^* i_{\xi_{G\times M}} d\omega = \iota_Z^*(-d i_{\xi_{G\times M}}\omega + \mathcal{L}_{\xi_{G\times M}}\omega) = -d\iota_Z^* i_{\xi_{G\times M}}\omega = -d i_{\xi_Z}\iota_Z^*\omega = 0.$$

Therefore, $d\omega$ is inertia-basic as well, and we obtain a complex of sheaves

$$0 \longrightarrow \mathbb{R}_{\Lambda X} \longrightarrow \mathcal{C}^\infty_{\Lambda X} = \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \cdots, \tag{5.1}$$

where $\mathbb{R}_{\Lambda X}$ denotes the sheaf of locally constant $\mathbb{R}$-valued functions on ΛX.

5.2. **The Poincaré Lemma for the Inertia Space.** Let us show that the complex of sheaves (5.1) is acyclic, or in other words that a Poincaré Lemma holds true for forms on the inertia space. So suppose that ω is a k-form on $\rho^{-1}(U)$ for some open $U \subseteq G\backslash(G \times M)$, and that $d\omega \in \mathcal{I}^{k+1}(U)$. Choose a slice $V_{(h,x)}$ at $(h,x) \in \rho^{-1}(U)$ according to Proposition 3.9 so that $GV_{(h,x)} \subseteq \rho^{-1}(U)$. By possibly shrinking $V_{(h,x)}$ if necessary, we may assume by the slice theorem that $Z \cap V_{(h,x)}$ is invariant under the action of $t \in (0,1]$ for every $Z \in \mathscr{D}$. Let $H = Z_{G_x}(h)$ denote the isotropy group of (h,x). Following [Pfl, Lemmas 5.2.1 and 5.3.2], we define

$$\mathcal{H}\colon\ G \times_H V_{(h,x)} \times [0,1] \longrightarrow G \times_H V_{(h,x)}$$

by setting

$$\mathcal{H}([g,(k,y)],t) = [g,(1-t)(k,y)].$$

Then $\mathcal{H}$ is a G-invariant retraction of $G \times_H V_{(h,x)}$ onto $G \times_H \{(h,x)\}$ which restricts to a G-invariant retraction of $(G \times_H V_{(h,x)}) \cap \Lambda M$ onto a single orbit by Proposition 3.9. Let us point out that by the slice theorem we can naturally identify $G \times_H V_{(h,x)}$ with the set $GV_{(h,x)} \subseteq \rho^{-1}(U)$.

Next, let $\mathcal{K} : \Omega^k(G \times_H V_{(h,x)} \times [0,1]) \to \Omega^{k-1}(G \times_H V_{(h,x)})$ denote the homotopy operator which maps ω to $\mathcal{K}(\omega)$, where

$$\mathcal{K}(\omega)([g,(k,y)]) = \int_0^1 \omega([g,(k,y)],s)\Big(\frac{\partial}{\partial s}, -, \ldots, -\Big)\, ds \text{ for } g \in G,\ (k,y) \in V_{(h,x)}.$$

One checks (see [PFL, Lemma 5.2.1]), that then

$$d\mathcal{K}\mathcal{H}^* + \mathcal{K}\mathcal{H}^* d = \mathcal{H}_1^* - \mathcal{H}_0^*,$$

where $\mathcal{H}_s = \mathcal{H}(-,s)$ for $s \in [0,1]$. Hence we obtain for the restriction of ω to $GV_{(h,x)}$ that

$$\omega_{|GV_{(h,x)}} - d\mathcal{K}\mathcal{H}^*\omega_{|GV_{(h,x)}} = \mathcal{K}\mathcal{H}^* d\omega_{|GV_{(h,x)}}. \tag{5.2}$$

To prove that the right hand side of this equation lies in $\mathcal{I}^k(U')$, where $U' := \rho(V_{(h,x)})$, we will show that $\mathcal{K}\mathcal{H}^*$ maps $\mathcal{I}^k(U')$ into $\mathcal{I}^{k-1}(U')$. So suppose that $\eta \in \mathcal{I}^k(U')$ which means that $\iota_Z^*\eta = 0$ on $\rho^{-1}(U') \cap Z$. Let $\mathcal{H}_Z$ denote the homotopy

$$\mathcal{H}_Z\colon\ G \times_H (Z \cap V_{(h,x)}) \times [0,1] \longrightarrow G \times_H (\overline{Z} \cap V_{(h,x)})$$

given by restricting $\mathcal{H}$. Similarly, let $\mathcal{K}_Z$ denote the restriction of the operator $\mathcal{K}$ to

$$\Omega^k(G \times_H (Z \cap V_{(h,x)})).$$

Then the diagram

$$\begin{tikzcd}
G \times_H (Z \cap V_{(h,x)}) \times [0,1] \arrow[r, "\iota_Z \times \mathrm{id}_{[0,1]}"] \arrow[d, "\mathcal{H}_Z"'] & (G \times_H V_{(h,x)}) \times [0,1] \arrow[d, "\mathcal{H}"] \\
G \times_H (\overline{Z} \cap V_{(h,x)}) \arrow[r, "\iota_Z"] & G \times_H V_{(h,x)}
\end{tikzcd}$$

commutes. Since the operator $\mathcal{K}$ clearly commutes with the restriction to Z, this entails

$$\iota_Z^*\mathcal{K}\mathcal{H}^*\eta = \mathcal{K}_Z\mathcal{H}_Z^*\iota_Z^*\eta = 0.$$

Moreover, since $\mathcal{K}$ and $\mathcal{H}$ commute with the G-action, we obtain for $\xi \in \mathfrak{g}$

$$i_{\xi_{GV_{(h,x)}}}\mathcal{K}\mathcal{H}^*\eta = \mathcal{K}\mathcal{H}^* i_{\xi_{GV_{(h,x)}}}\eta = 0.$$

It follows that $\mathcal{K}\mathcal{H}^*$ maps $\mathcal{I}^k(U')$ into $\mathcal{I}^{k-1}(U')$, so that the right hand side of Eq. (5.2) lies in $\mathcal{I}^k(U')$, since $d\omega \in \mathcal{I}^{k-1}(U)$ by hypothesis. But this means that the sheaf complex $\Omega^\bullet$ on ΛX is exact, or in other words that the Poincaré Lemma for forms on the inertia space holds true.

Theorem 5.1. *The cohomology of the complex $\Omega^*(\Lambda X)$ of differential forms on ΛX naturally coincides with the singular (or Čech) cohomology of ΛX. Moreover, if X is compact, the cohomology of the de Rham complex $\Omega^*(\Lambda X)$ on the inertia space is finite dimensional.*

Proof. By the Poincaré Lemma for forms on the inertia space, $\Omega^\bullet$ provides a fine resolution of the sheaf of $\mathbb{R}$-valued locally constant functions on ΛX. Since ΛX is locally compact and locally contractible, the cohomology of the complex $\Omega^\bullet(\Lambda X)$ of global sections then has to coincide naturally with the singular cohomology on ΛX. Since ΛX is even triangulable, the cohomology of $\Omega^*(\Lambda X)$ even coincides with the Čech cohomology. The triangulability of ΛX also implies that for every open covering of ΛX there exists a locally finite subordinate good covering (see [PFPOTA11, Sec. 7]). This implies that under the assumption that X, hence ΛX is compact, the Čech cohomology of ΛX has to be finite dimensional. This completes the proof. □

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ON STRATIFIED MORSE THEORY: FROM TOPOLOGY TO CONSTRUCTIBLE SHEAVES

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Abstract. Stratified Morse theory is the generalization of usual Morse theory to functions on stratified spaces. There are versions for the topological type, homotopy type or (co)homology. A standard reference is the book of Goresky-MacPherson which primarily treats the topological type. Corresponding results about the homotopy type or cohomology may be expected to be consequences but in fact usually one needs some extra information, in particular in the case of cohomology of constructible sheaves, as we will see in this paper.

Introduction

This paper is based on a talk given at the conference "Geometry and topology of singular spaces" (10/29 - 11/02, 2012) in Luminy/Marseille, France, on the occasion of David Trotman's 60th birthday.

We will study the relation between stratified Morse theory concerning the topological type and cohomology, including the cohomology of constructible sheaves. It is quite instructive to look at classical Morse theory first, because already here one has to pay attention - in this case the geometry is so clear that it may seem pedantic to emphasize this point but one sees where one should be careful in more general situations.

Stratified Morse theory is the generalization of usual Morse theory to functions on stratified spaces. A standard reference is the book of Goresky - MacPherson [GM2]. The transition from topology to constructible sheaves in full generality is indicated there in an appendix ([GM2] II 6.A, p. 222-224). Cf. [Ms], too.

The main purpose of the present paper is to make this step more explicit, showing that the setup in [GM2] is indeed strong enough to enable the transition, with some extra care.

In fact there are more direct ways to get the statements about cohomology of constructible sheaves: directly, see Kashiwara-Schapira [KS] or Schürmann [S], or using some weaker version of stratified Morse theory which is sufficient for this purpose [H2].

In special cases one can argue more simply, as we will see. This holds especially for singular cohomology, or for homotopy groups which are discussed in [GM2]. Even in this case, however, one has to be careful, too, and we take the opportunity for some corresponding comments.

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We also take the opportunity to adjust the technique of Moving the Wall which has been developed and used by Goresky - MacPherson, see [GM2] I 4.3, p. 71f.

1. Classical Morse theory

We start with usual Morse theory which is well-known, see e.g. [Ma]. We treat this case because we want to stress some point which we will encounter in the general case, too, it can be more easily discussed in this simple context.
In particular, we will see that it is not completely true that the usual statements about the topological type imply the ones about homotopy or cohomology groups.

Let M be a C^∞ manifold of dimension n and $f : M \to \mathbb{R}$ a C^∞ function. Let us assume that f has isolated critical points which are non-degenerate. Put $M_a := \{p \in M \mid f(p) \leq a\}$. Let $a < b$ be regular values, $f^{-1}([a, b])$ compact. We want to compare M_a with M_b.

First suppose that $f^{-1}([a, b])$ contains no critical point. Then we have that M_b is homeomorphic (and even diffeomorphic) to M_a.

As a consequence we have that M_a and M_b have the same homotopy type.
Furthermore, we obtain that $H^k(M_b; \mathbb{Z}) \simeq H^k(M_a; \mathbb{Z})$ for all k.
More precisely: if $h : M_a \to M_b$ is a homeomorphism we obtain that

$$h^* : H^k(M_b; \mathbb{Z}) \to H^k(M_a, \mathbb{Z})$$

is bijective for all k.

In fact we want that it is the inclusion $i : M_a \to M_b$ which induces bijective mappings for all k. This is needed e.g. if one wants to reformulate the cohomological result by saying that $H^k(M_b, M_a; \mathbb{Z}) = 0$ for all k; similarly for homotopy groups.

But this is not obvious, the best is to go back to the proof and show that $i \sim h$ (homotopic). This implies that i is a homotopy equivalence, i.e. M_a is a weak deformation retract of M_b (see [Sp] 1.4, p. 30), which is in turn sufficient to show that $H^k(M_b, M_a; \mathbb{Z}) = 0$ for all k.
Furthermore, we want to have that $H^k(M_b, \mathcal{S}) \simeq H^k(M_a, \mathcal{S})$ if $\mathcal{S}$ is a locally constant sheaf (of abelian groups) on M_b. Here the situation is even worse: h induces isomorphisms

$$H^k(M_b, \mathcal{S}) \to H^k(M_a, h^*\mathcal{S}),$$

and we cannot simply replace $h^*\mathcal{S}$ by $\mathcal{S}|M_a$. But if i is a homotopy equivalence we have that $i^* : H^k(M_b, \mathcal{S}) \to H^k(M_a, \mathcal{S})$ is an isomorphism for all k, see [H1] Theorem 2.6.

So let us recall how one can obtain the homeomorphism h: Choose a vector field v on M with compact support such that $df_x(v(x)) = b - a$ for $x \in f^{-1}([a, b])$. Let σ be the corresponding flow, $h_t(p) := \sigma(p, t)$, $0 \leq t \leq 1$. Then (h_t) defines a one-parameter family of homeomorphisms $M_a \to M_{a+t(b-a)}$ with $h_0 = id$. In particular, $h := h_1$ is a homeomorphism of M_a onto M_b. Since $M_{a+t(b-a)} \subset M_b, 0 \leq t \leq 1$, we have that i is homotopic to h.

So M_a is, in particular, a weak deformation retract of M_b. In fact, M_a is even a strong deformation retract of M_b (see [Sp] loc. cit.). This is not completely obvious: Note that h^{-1} cannot be a retraction (except for the trivial case $M_a = M_b$) because otherwise $h^{-1} \circ i = id$ which would imply that i is bijective.

But (M_b, M_a) is a polyhedral pair, cf. [Mu] Theorem 10.6, p. 103, so M_a is a strong deformation retract of M_b if and only if M_a is a weak deformation retract of M_b : This equivalence follows from a homotopy extension property, cf. [Sp] Cor. 1.4.10, Theorem 1.4.11, p. 31, which holds, in particular, in the case of polyhedral pairs, cf. [Sp] Cor. 3.2.5, p. 118.

That we have a strong deformation retract can in our case also be shown directly using the flow σ above, of course.

Now we pass to the case where $f^{-1}([a,b])$ contains exactly one non-degenerate critical point p and λ is defined to be the corresponding index. Then we have that M_b is homeomorphic to a space obtained from M_a by attaching a handle of index λ, i.e. $D^\lambda \times D^{n-\lambda}$ along $S^{\lambda-1} \times D^{n-\lambda}$. Here we have the same problem when passing to cohomology: We want that

$$H^k(M_b, M_a; \mathbb{Z}) \simeq H^k(D^\lambda \times D^{n-\lambda}, S^{\lambda-1} \times D^{n-\lambda}; \mathbb{Z}) \simeq H^k(D^\lambda, S^{\lambda-1}; \mathbb{Z}) \simeq \mathbb{Z}$$

if $k = \lambda$ and $= 0$ if $k \neq \lambda$.

So we look at the proof more closely. It is sufficient to show that there is a space X with $M_a \subset X \subset M_b$ such that there is a homeomorphism $h : X \to M_b$ which is homotopic to the inclusion i and such that X is obtained from M_a by attaching a handle of index λ:
Then $H^k(M_b, X; \mathbb{Z}) = 0$ for all k, hence $H^k(M_b, M_a; \mathbb{Z}) \simeq H^k(X, M_a; \mathbb{Z})$. By excision,

$$H^k(X, M_a; \mathbb{Z}) \simeq H^k(D^\lambda \times D^{n-\lambda}, S^{\lambda-1} \times D^{n-\lambda}; \mathbb{Z}) \simeq \mathbb{Z}$$

if $k = \lambda$ and $= 0$ if $k \neq \lambda$.

Such a space X can be found as follows: Choose a suitable closed neighbourhood U of p, a and b sufficiently close to the critical value. Put $X := M_a \cup (U \cap M_b)$. Then

$$(U \cap \{a \leq f \leq b\}, U \cap \{f = a\})$$

is homeomorphic to $(D^\lambda \times D^{n-\lambda}, S^{\lambda-1} \times D^{n-\lambda})$.

If we look at a locally constant sheaf $\mathcal{S}$ instead of $\mathbb{Z}$ we do not meet new difficulties: As before we can deduce $H^k(M_b, X; \mathcal{S}) = 0$ for all k. Then, by excision:

$$H^k(X, M_a; \mathcal{S}) \simeq H^k(U \cap \{a \leq f \leq b\}, U \cap \{f = a\}; \mathcal{S}),$$

and U is contractible, which implies that $\mathcal{S}|U$ is isomorphic to the constant sheaf $\mathcal{S}_p$ (as usual, $\mathcal{S}_p$ denotes the stalk of $\mathcal{S}$ at p). So $H^k(M_b, M_a; \mathcal{S}) \simeq \mathcal{S}_p$ for $k = \lambda$ and $= 0$ if $k \neq \lambda$.

2. Decomposed homotopy equivalence

Now let us prepare the case of stratified Morse theory.

Let I be a partially ordered set (denoted by $\mathcal{S}$ in [GM2] I 1.1, p. 36). Let X be an I-decomposed space, i.e. a topological space with a locally finite decomposition (= partition) into locally closed subsets $S_{(i)}, i \in I$, such that $S_{(i)} \cap \bar{S}_{(j)} \neq \emptyset \Leftrightarrow i \leq j$. Similarly let Y be an I-decomposed space with subsets $R_{(i)}$. An I-decomposed map $f : X \to Y$ is a continuous map such that $f(S_{(i)}) \subset R_{(i)}$ for all i. See [GM2] I 1.1, p. 36. A homotopy F between I-decomposed maps $f_0, f_1 : X \to Y$ is a homotopy such that $F(S_{(i)} \times [0,1]) \subset R_{(i)}$ for all i.

We will fix I and speak of decomposed instead of I-decomposed.

It is now straightforward to define a decomposed homotopy equivalence and a decomposed weak/strong deformation retract.

An important ingredient in [GM2] is the technique of Moving the Wall which is based on Thom's first isotopy lemma. In fact there are two versions of Moving the Wall in [GM2], here we will concentrate on the first one. The moving is parametrized by a parameter t. In the corresponding theorem ([GM2] I 4.3, p. 72) the parameter space is $\mathbb{R}$. However, in later applications obviously $[0,1]$ is taken as a parameter space. Therefore it is appropriate to modify Theorem I 4.3 of [GM2] as follows. Note that we weaken the properness hypothesis, too. In order to facilitate the comparison we use the notations of [GM2]:

Let M, N be smooth manifolds, $f : M \to N$ smooth, $Z \subset M$ a Whitney stratified closed subset, see [GM2] I 1.2, p. 37. Then Z is a space which is decomposed by the strata; so I is the corresponding index set. Subsets of Z are naturally decomposed, too. Let $-\infty \leq \alpha < 0$, $1 < \beta \leq \infty$, $Y \subset N \times]\alpha, \beta[$ a closed Whitney stratified subset such the projection on the second factor yields a stratified submersion $\pi : Y \to]\alpha, \beta[$, cf. [GM2] I 1.5, p. 41. Assume that for each $(p,t) \in Y$ with $p \in f(Z)$, $t \in [0,1]$, and each non-zero characteristic covector $\lambda \in T_p^*N$ of $f|Z : Z \to N$, we have $\lambda|T_pS_t \neq 0$, where S is the stratum of Y which contains (p,t) and $S_t = \pi^{-1}(t) \cap S$. Recall that a covector $\lambda \in T_p^*N$, $p \in N$, is characteristic if and only if for all $z \in Z \cap f^{-1}(p)$ we have that $f^*\lambda|T_zS = 0$, where S is the stratum of Z which contains z, see [GM2] I 1.9, p. 46, together with [GM2] I 1.8, p. 44.

Furthermore assume that the mapping $(Z \times]\alpha, \beta[) \cap (f \times id_{\mathbb{R}})^{-1}(Y) \to]\alpha, \beta[$ given by the projection onto the second factor is proper.
Put $Y_t := \{q \in N \mid (q,t) \in Y\}$.

Now we have the following modified version of Moving the Wall ([GM2] I Theorem 4.3), cf. [S] Lemma 4.3.5, p. 267, too:

Theorem 2.1: Under these hypotheses there is a decomposed homeomorphism

$$h : Z \cap f^{-1}(Y_0) \to Z \cap f^{-1}(Y_1)$$

which preserves the Whitney stratification of both sides and is smooth on each stratum.

Note that these spaces must be compact!

Proof. We may assume that α, β are arbitrarily near to 0 resp. 1. Then we may assume that the assumption about characteristic covectors holds for all $t \in]\alpha, \beta[$ instead of $t \in [0,1]$, by continuity. This means that we have the hypotheses of [GM2] loc. cit. with $]\alpha, \beta[$ instead of $\mathbb{R}$, except for a weaker properness assumption.

Since $]\alpha, \beta[$ is diffeomorphic to $\mathbb{R}$ we may reduce to $]\alpha, \beta[= \mathbb{R}$ by base change.

Now proceed similarly as in the proof loc. cit.:

Our hypothesis guarantees that $f \times id_{\mathbb{R}}|_{Z \times \mathbb{R}}$ is transverse to Y in the stratified sense (cf. [GM2] I 1.3.1, p. 38), hence $(Z \times \mathbb{R}) \cap (f \times id_{\mathbb{R}})^{-1}(Y)$ inherits an induced stratification, and that $\pi \circ (f \times id_{\mathbb{R}}) : (Z \times \mathbb{R}) \cap (f \times id_{\mathbb{R}})^{-1}(Y) \to \mathbb{R}$ (projection onto the second factor) is a proper stratified submersion. Then apply Thom's first isotopy lemma, see [GM2] I 1.5, p. 41, with $\mathbb{R}$ instead of $\mathbb{R}^n$, $f =$ canonical projection. ∎

In order to handle certain situations where we get difficulties with the compactness assumption involved above it is useful to have

Remark 2.2: Suppose moreover that there is a closed subset Y_+ of N such that $Y_+ \times]\alpha, \beta[$ is a union of strata of Y of the form $S \times]\alpha, \beta[$. Then we may achieve that $h|Z \cap f^{-1}(Y_+)$ is the identity.

In order to prove this we need the following complement to Thom's first isotopy lemma ([M], [GM2] I 1.5, p. 41):

Theorem 2.3 (see [M] if $X_+ = \emptyset$): Suppose that M is a smooth manifold and that $X \subset M \times \mathbb{R}$ is a Whitney stratified subset. Let $f : X \to \mathbb{R}$ be the restriction of the projection onto the second factor. Let X_+ be a closed subset of M such that $X_+ \times \mathbb{R}$ is a union of strata of X of the form $S \times \mathbb{R}$. Assume that f is a proper stratified submersion. Then there is a stratum preserving homeomorphism $H : f^{-1}(\{0\}) \times \mathbb{R} \to X$ such that:

(1) $f(H(p,t)) = t$ for $p \in f^{-1}(\{0\}), t \in \mathbb{R}$,
(2) $H((q,0),t) = (q,t)$ for $q \in X_+, t \in \mathbb{R}$.

Proof. The isotopy lemma is proved in [M] using a vector field which is constructed inductively with respect to the strata. On $X_+ \times \mathbb{R}$ choose the obvious one, using control data for $X_+ \times \mathbb{R}$ which come from control data for X_+. □

Because of the difficulty when passing from topological type to homotopy or cohomology groups mentioned in the first section a statement about the homotopy type is appropriate, too:

Theorem 2.4: Beyond the hypotheses of Theorem 2.1 suppose that $Y_t \subset Y_1$ for $0 \le t \le 1$. Then $Z \cap f^{-1}(Y_0)$ is a decomposed weak deformation retract of $Z \cap f^{-1}(Y_1)$. Cf. [S] loc. cit., too.

Proof. The proof of Theorem 2.1 shows that we may assume that $]\alpha, \beta[= \mathbb{R}$ and that the assumption about covectors holds with $t \in \mathbb{R}$ instead of $t \in [0,1]$. So we can apply Thom's isotopy lemma to $pr : (Z \times \mathbb{R}) \cap (f \times id_{\mathbb{R}})^{-1}(Y) \to \mathbb{R}$, where pr is the projection onto the second factor, and get a homeomorphism

$$H : pr^{-1}(\{0\}) \times \mathbb{R} \to (Z \times \mathbb{R}) \cap (f \times id_{\mathbb{R}})^{-1}(Y)$$

such that $f(H(p,t)) = t$ for $p \in pr^{-1}(\{0\}), t \in \mathbb{R}$.
We may achieve that $H(p,0) = p$ for all such p because H is obtained by integration of a vector field. Note that $pr^{-1}(\{0\}) = (Z \cap f^{-1}(Y_0)) \times \{0\}$, and H can be written as

$$H((q,0),t) = (H'(q,t),t)$$

with a continuous mapping $H' : (Z \cap f^{-1}(Y_0)) \times \mathbb{R} \to Z$. Then

$$H'(q,t) \in Z \cap f^{-1}(Y_t) \subset Z \cap f^{-1}(Y_1)$$

for $t \in [0,1]$. Put $h_t : Z \cap f^{-1}(Y_0) \to Z \cap f^{-1}(Y_1) : h_t(q) := H'(q,t)$. Then H' yields the desired homotopy between the the inclusion h_0 and a homeomorphism h_1. □

We have a remark similar to Remark 2.2:

Remark 2.5: Suppose moreover that Y_+ is a closed subset of N such that $Y_+ \times]\alpha, \beta[$ is a union of strata of Y. Then there is a decomposed homotopy H' between the inclusion and a homeomorphism $h : Z \cap f^{-1}(Y_0) \to Z \cap f^{-1}(Y_1)$ such that $H'(p,t) = p$ for all $p \in Z \cap f^{-1}(Y_+)$.

The proof is as before but apply Theorem 2.3 instead of the usual Thom's first isotopy lemma.

It is not clear whether one can get a decomposed strong deformation retract by this method.

We need some preparation for dealing with constructible sheaves.

A constructible sheaf on the decomposed space $X = \bigcup_i S_{(i)}$ is a sheaf which is locally constant on each $S_{(i)}$. A constructible sheaf complex is a nonnegative complex of sheaves whose cohomology sheaves are constructible on X. We do not impose any finiteness condition.

We have the following general fact: If $\mathcal{S}$ is a sheaf complex on a topological space Y and $f : X \to Y$ is continuous we get induced homomorphisms $\mathbb{H}^k(Y, \mathcal{S}) \to \mathbb{H}^k(X, f^*\mathcal{S})$.
In particular, if f is a homeomorphism it induces isomorphisms.
Here $\mathbb{H}^k$ denotes the k-th hypercohomology group.

Theorem 2.6: Let $\mathcal{S}$ be a constructible sheaf complex on the decomposed space Y.

a) Let $f_0, f_1 : X \to Y$ be decomposed maps which are decomposed homotopic. Then $f_0^*\mathcal{S}$ and $f_1^*\mathcal{S}$ are quasiisomorphic, and the mappings $f_i^* : \mathbb{H}^k(Y, \mathcal{S}) \to \mathbb{H}^k(X, f_i^*\mathcal{S})$, $i = 0, 1$, coincide (if we identify $\mathbb{H}^k(X, f_i^*\mathcal{S}), i = 0, 1$).
b) If $f : X \to Y$ is a decomposed homotopy equivalence we have that the mappings
$$f^* : \mathbb{H}^k(Y, \mathcal{S}) \to \mathbb{H}^k(X, f^*(\mathcal{S}))$$
are isomorphisms.
c) In particular, if $X \subset Y$ is a decomposed weak deformation retract we have that the mappings $\mathbb{H}^k(Y, \mathcal{S}) \to \mathbb{H}^k(X, \mathcal{S})$ are isomorphisms for all k.
d) If (X, X_1) and (Y, Y_1) are pairs of spaces and X resp. X_1 is a decomposed weak deformation retract of Y resp. Y_1 we have that $\mathbb{H}^k(Y, Y_1; \mathcal{S}) \simeq \mathbb{H}^k(X, X_1; \mathcal{S})$ for all k.

Proof. a) The case where $\mathcal{S}$ consists of a single sheaf can be attacked in an elementary way, cf. [H1] Theorem 2.2, 2.7. In general we argue as follows: Let $p : X \times [0, 1] \to X$ be the projection, and let $i_t : X \to X \times [0, 1]$ be defined by $i_t(x) := (x, t)$. Let $\mathcal{T}$ be a constructible sheaf complex on the I-decomposed space $X \times [0, 1]$. By [KS] Prop. 2.7.8, p. 122, we have $\mathcal{T} \sim p^*\mathcal{Q}$ with $\mathcal{Q} = Rp_*\mathcal{T}$, where $\sim$ denotes "quasiisomorphic". So $\mathbb{H}^k(X \times [0, 1], \mathcal{T}) \to \mathbb{H}^k(X, i_t^*\mathcal{T})$ can be rewritten as $\mathbb{H}^k(X, (Rp_*)p^*\mathcal{Q}) \to \mathbb{H}^k(X, i_t^*p^*\mathcal{Q})$. This mapping is induced by $(Rp_*)p^*\mathcal{Q} \to i_t^*p^*\mathcal{Q}$ which is independent of $t \in \{0, 1\}$ because $i_t^*p^*\mathcal{Q} \sim \mathcal{Q}$. So $i_0^*\mathcal{T} \sim i_1^*\mathcal{T}$, and
$$\mathbb{H}^k(X \times [0, 1], \mathcal{T}) \to \mathbb{H}^k(X, i_t^*\mathcal{T})$$
is independent of $t \in \{0, 1\}$ under the corresponding identification of cohomology.

Now let $F : X \times [0, 1] \to Y$ be a decomposed homotopy between f_0 and f_1. Then $f_t = F \circ i_t$, $t = 0, 1$, so $f_t^*\mathcal{S} = i_t^*F^*\mathcal{S}$, $t = 0, 1$, are quasiisomorphic: put $\mathcal{T} := F^*\mathcal{S}$ above. Furthermore look at the composition $\mathbb{H}^k(Y, \mathcal{S}) \to \mathbb{H}^k(X \times [0, 1], F^*\mathcal{S}) \to \mathbb{H}^k(X, i_t^*F^*\mathcal{S})$. Here the right arrow is independent of t, see above.
The rest (b - d) is easy. □

3. Stratified Morse theory

a) The Main Theorem of Goresky-MacPherson

Now pass to stratified Morse theory in the sense of Goresky-MacPherson [GM2] which constitutes a deep generalization of usual Morse theory. Let Z be a Whitney stratified subset of a manifold M, see [GM2] I 1.2, p. 37, $\hat{f} : M \to \mathbb{R}$ smooth, $f := \hat{f}|Z$. Let $Z_c := \{f \leq c\}$.

In [GM2] it is supposed that f is proper (see [GM2] I 3.1, p. 61). Note that this does not imply that Z_c is compact, for this we need an extra assumption:

$$f \text{ is bounded from below.} \tag{*}$$

However we will not assume that (*) is fulfilled and weaken the properness assumption: Let $a < b$ be fixed. Then we assume that there are a_1, b_1 such that $a_1 < a < b < b_1$ and that $f^{-1}([a_1, b_1])$ is compact.

Let us begin with the easiest case:

Theorem 3.1: Suppose that $[a, b]$ contains no critical value.
a) Z_a is homeomorphic to Z_b, the homeomorphism being decomposed, compatible with the stratifications.
b) Z_a is a decomposed strong deformation retract of Z_b.
Note that Z_a, Z_b are stratified in an obvious way.

As in the case of classical Morse theory we need b) if we want to show the vanishing of relative homotopy or cohomology groups.

Proof. We assume without loss of generality that $[a, b] = [0, 1]$.

a) Similar to [GM2] I 7.2, p. 90, we may use the technique of Moving the Wall as modified in Theorem 2.1.

We can choose $\alpha < 0, \beta > 1$ sufficiently near to 0 resp. 1 so that t is not a critical value, $t \in [\alpha, \beta]$.

First suppose that (*) is fulfilled.

Then $Y := \{(y,t) \,|\, y \in \mathbb{R}, \alpha < t < \beta, y \le t\}$. The hypothesis of Theorem 2.1 (Moving the Wall) is fulfilled, and we get the assertion. Note that the properness assumption is guaranteed because of (*), whereas the projection $Y \to]\alpha, \beta[$, $(y,t) \mapsto t$, is not proper.
Note that we cannot take $\mathbb{R}$ here instead of $[0, 1]$ and $]\alpha, \beta[$ because then the condition on covectors may not be satisfied because of critical points of f. Also, if we modify Y_t by taking $Y_t := Y_0$ for $t \le 0$, $Y_t := Y_1$ for $t \ge 1$ we have to introduce the strata $\{(0, 0)\}$ resp. $\{(1, 1)\}$ in Y which are not mapped submersively to $\mathbb{R}$. So we need our modified version of Moving the Wall (Theorem 2.1).

If assumption (*) does not hold we take a different Y: $Y := \{(y,t) \,|\, \alpha < t < \beta, \alpha \le y \le t\}$. Now the hypothesis of Remark 2.2 is fulfilled, and we obtain a decomposed homeomorphism $h : f^{-1}([\alpha, 0]) \to f^{-1}([\alpha, 1])$ such that $h|f^{-1}(\{\alpha\}) = id$. We glue with $f^{-1}(]\infty, \alpha])$ in order to obtain the desired decomposed homeomorphism $Z_0 \to Z_1$.
Alternative: Use Thom's first isotopy lemma ([GM2] I 1.5, p. 41) more directly. Choose $\alpha < 0$ close to 0. There is a decomposed homeomorphism $H : f^{-1}(\{\alpha\}) \times [\alpha, 1] \to f^{-1}([\alpha, 1])$ such that $f(H(p,t)) = t$ for all (p,t), $H(p, \alpha) = p$. Now the homeomorphism $h : Z_0 \to Z_1$ is defined as follows: $h(p) := p$ if $f(p) \le \alpha$, $h(p) := H(q, (1 - \frac{1}{\alpha})t + 1)$ if $f(p) > \alpha$, $p = H(q,t)$.

b) Use moreover Theorem 2.4 in order to obtain a weak decomposed deformation retract. In the case where (*) is not fulfilled use Remark 2.5, too.

In order to obtain a strong decomposed deformation retract we use again Thom's isotopy lemma directly. Let H' be, similarly as in the alternative above, a decomposed homeomorphism $f^{-1}(\{0\}) \times [0, 1] \to f^{-1}([0, 1])$ such that $f(H'(p,t)) = t$ for all (p,t), $H'(p, 0) = p$. It is sufficient to show that $f^{-1}(\{0\})$ is a strong decomposed deformation retract of $f^{-1}([0, 1])$. Using H'

this amounts to proving that $f^{-1}(\{0\}) \times \{0\}$ is a strong decomposed deformation retract of $f^{-1}(\{0\}) \times [0,1]$ which is obvious. $\square$

Now suppose that $f^{-1}([a,b])$ contains exactly one critical point p. Let S be the stratum which contains p. Assume that p is a nondepraved critical point of f, see [GM2] I 2.3, p. 55. This involves a condition on $f|S$ which holds automatically if the critical point of $f|S$ is non-degenerate or if S and $f|S$ are real analytic, see [GM2] I 2.3, 2.4. Moreover it is demanded that the critical point p of f is normally nondegenerate (called nondegenerate in [GM2]), i.e. $d\hat{f}_p|T \neq 0$ for every generalized tangent space to Z at p, $T \neq T_pS$. Furthermore we call p a nondegenerate point of index λ if p is a nondegenerate point of $f|S$ of index λ and p is normally nondegenerate, too.

Put $v := f(p)$. We may take a, b as close to v as we wish, namely $a = v - \epsilon, b = v + \epsilon$, where $\epsilon > 0$ can be taken arbitrarily small.

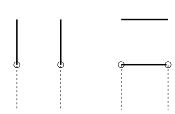
In order to express the main theorem use the following notations, see [GM2] I 3.3-3.6, pp. 62-65:

If (A, B) is a pair of decomposed topological spaces such that Z_b is decomposed homeomorphic to a space obtained from Z_a by attaching A along B we say that (A, B) is a Morse data for f at p.

Example: $(A, B) := (f^{-1}([a,b]), f^{-1}(a))$: "coarse" Morse data.

Morse data (A, B) are not well-defined (this even holds for the homotopy type of A/B):

Examples: a) $Z = \mathbb{Z}$, $f(x) = x$, $v = 0$. Then $(\emptyset, \emptyset)$ as well as $(\{0\}, \emptyset)$ are Morse data for f at 0.
b) $Z = \{0,1\} \times [-1,1], f(x,y) = y, v = 0$. Then not only $(\{0,1\} \times [0,1], \{0,1\} \times \{0\})$ but also $([0,1] \times \{0,1\}, \{0,1\} \times \{0\})$ is Morse data for f at $(0,0)$ (it is harmless to regard the regular point $(0,0)$ as a critical one, too).
In the following drawings A consists of the fat lines and B of the encircled points. On the left side the whole space is Z, on the right side the whole space is homeomorphic to Z.



Choose a Riemannian metric which is the canonical one with respect to some local coordinates near p, and let r be the square of the distance from p.

Let U be a suitable closed neighbourhood of p in Z: $U := Z \cap \{r \leq \delta\}$, $\delta > 0$ small. Choose ϵ above small compared with δ. Then the coarse Morse data of $f|U$ at p is called the local Morse data of f at p. The local Morse data of $f|S$ at p are called the tangential, the local Morse data of $f|N$ at p the normal Morse data at p, where N is a normal slice at p, see [GM2] I 1.4, p. 41. It is of the form $N = N^* \cap \{r \leq \delta\}$, N^* being the intersection of Z and some submanifold of M.

In a first step it is shown that local Morse data is Morse data. More precisely:

Theorem 3.2: a) $(Z_a \cup U) \cap Z_b$ is homeomorphic to Z_b, the homeomorphism being decomposed ([GM2] I 7.6, p. 95),

b) $(Z_a \cup U) \cap Z_b$ is a decomposed strong deformation retract of Z_b.

Again b) is needed, too, in order to pass to the vanishing of relative homotopy or cohomology groups.

Proof. a) Use Moving the Wall, see [GM2] I 7.6, i.e. use Theorem 2.1.

We encounter the same difficulties as in the proof of Theorem 3.1a), so we assume first (*).

Note that $Y_t, t \in [0,1]$, is depicted on [GM2] p. 96, it is obvious how to define Y_t for $t < 0$ close to 0 and $t > 1$ close to 1.

In general replace Y_t by its intersection with $\{(x,y) \,|\, y \geq c\}$ for a suitable c and proceed as in the proof of Theorem 3.1a).

Or: Apply the methods of [H2]. By [H2] Lemma 3.6 we have that (f,r) is submersive along $\{r = \epsilon, a \leq f \leq b\}$. By the Preparatory theorem (Theorem 1.2) of [H2] we get our statement.

b) If we apply Moving the Wall in the proof of a) we can use Theorem 2.4 in order to show that we have a decomposed weak deformation retract. If (*) is not fulfilled use Remark 2.5, too. Or apply the Preparatory Theorem of [H2] loc. cit. □

Now the Main Theorem says:

Theorem 3.3 ([GM2] I 3.7, p. 65): Local Morse data is homeomorphic to Tangential Morse data $\times$ Normal Morse data.

In particular, the product Tangential Morse data $\times$ Normal Morse data is a Morse data - a consequence which can be proved directly much more easily, as proved in [H2] (Theorem 1.9) (see also King [K] Theorem 5).

As we will see in the next section, the Main Theorem has corresponding consequences for singular cohomology groups and simple cases of constructible sheaves. For treating constructible sheaves in general one needs to look at the proof again, see section 5. Applications will be given in section 6.

Remark 3.4: In [GM2], stratified Morse theory is mainly applied to homotopy groups or homotopy type instead of cohomology. In particular, Lefschetz type theorems are proved. Here one needs the following argument: If the local Morse data is k-connected the same holds for the pair $(Z_{\leq b}, Z_{\leq a})$, too. But here one needs Theorem 3.2b), as in the case of singular cohomology which will be treated in section 4a.

b) Variants

There are variants of the Main Theorem of [GM2] developed in the same book.

Relative case: Suppose that $g : X \to Z$ is a proper stratified mapping, i.e. X is Whitney stratified, too, and each stratum of X is mapped submersively to a stratum of Z. We consider X as a decomposed space, the decomposition being given by the stratification. Let f be as before. Put $X_a := X \cap \{f \circ g \leq a\}$.
Relative local Morse data: inverse image of local Morse data of f under g. Relative normal Morse data: local relative Morse data of $f|N$, N being a normal slice, under g.

Theorem 3.5: Local relative Morse data is Morse data, more precisely, there is a decomposed homeomorphism $h : X_a \cup (X \cap \{f \circ g \leq b, r \circ g \leq \epsilon\}) \to X_b$ ([GM2] I 9.4, p. 115).
Moreover we can achieve that $h \sim i$, i inclusion, via a decomposed homotopy, so we have a decomposed weak deformation retract.

The proof is based on Moving the Wall again.

Theorem 3.6 (Main Theorem in relative case) ([GM2] I 9.5, p. 116): Local relative Morse data is homeomorphic to Tangential Morse data of $f \times$ Relative normal Morse data.

Nonproper case: Suppose that X is an open subset of Z which is a union of strata. We can define local nonproper Morse data similarly as before, using the inclusion of X in Z instead of g. Similarly: nonproper normal Morse data. See [GM2] I 10.3, p. 120.

Again we have that nonproper local Morse data are Morse data, see [GM2] I 10.4, p. 120. Moreover, $X_a \cup (X \cap \{f \leq b, r \leq \epsilon\})$ is a decomposed weak deformation retract of X_b.

Main Theorem in the nonproper case: the formulation is straightforward ([GM2] I Theorem 10.5, p. 121).

c) Additional remarks

Instead of Z_a we can also study $Z_{<a} := \{p \in Z \mid f(p) < a\}$. This will be useful when treating intersection cohomology.

Theorem 3.7: Suppose that $[a, b]$ contains no critical value.
a) $Z_{<a}$ is homeomorphic to $Z_{<b}$, the homeomorphism being decomposed and compatible with the stratifications.
b) $Z_{<a}$ is a weak decomposed deformation retract of $Z_{<b}$.
Of course, $Z_{<a}, Z_{<b}$ are stratified in an obvious way.

It is not true that $Z_{<a}$ is a retract of $Z_{<b}$ if f is surjective: if r is a retraction, we must have $r(z) = z$ for $z \in Z_{<a}$, hence for $z \in Z_a$ by continuity, which contradicts $r(Z_{<b}) \subset Z_{<a}$.

Proof. a) This follows from Theorem 3.1 a) because the homeomorphism there preserves strata. So the homeomorphism is obtained by the technique of Moving the Wall.
b) This follows by application of Theorem 2.4 resp. Remark 2.5. □

In fact we can compare the spaces $Z_{<a}$ and Z_a:

Theorem 3.8: Suppose that $[a, b]$ contains no critical value. Then Z_a is a strong decomposed deformation retract of $Z_{<b}$.

Proof. This is obvious by Thom's first isotopy lemma because $f^{-1}(\{a\}) \times \{a\}$ is a strong decomposed deformation retract of $f^{-1}(\{a\}) \times [a, b[$. But it does not follow from Theorem 2.4. □

4. Transition to cohomology

The assumptions are those of section 3.

a) Cohomology with integral coefficients

If $f^{-1}([a,b])$ contains no critical points, $H^k(Z_b;\mathbb{Z}) \simeq H^k(Z_a;\mathbb{Z})$ for all k. As in classical Morse theory, the isomorphism is induced by the inclusion but one needs Theorem 3.1b) rather than Theorem 3.1a) to see this: Z_a is a deformation retract of Z_b.

If $f^{-1}([a,b])$ contains exactly one critical point p which is non-degenerate of index λ,

$$H^k(Z_b, Z_a;\mathbb{Z}) \simeq H^{k-\lambda}(N \cap \{a \le f \le b\}, N \cap \{f = a\};\mathbb{Z})$$

Here one needs more information than that the product Tangential $\times$ Normal Morse data is Morse data. We need Theorem 3.2b), too:
$H^k(Z_b, Z_a \cup (U \cap \{a \le f \le b\});\mathbb{Z}) = 0$ for all k,
so the exact cohomology sequence of a triple gives

$$\begin{aligned} H^k(Z_b, Z_a;\mathbb{Z}) &\simeq H^k(Z_a \cup (U \cap \{a \le f \le b\}), Z_a;\mathbb{Z}) \simeq H^k(U \cap \{a \le f \le b\}, U \cap \{f = a\};\mathbb{Z}) \\ &\simeq H^k((D^\lambda \times D^{m-\lambda}, S^{\lambda-1} \times D^{m-\lambda}) \times (N \cap \{a \le f \le b\}, N \cap \{f = a\});\mathbb{Z}) \\ &\simeq H^{k-\lambda}(N \cap \{a \le f \le b\}, N \cap \{f = a\};\mathbb{Z}) \end{aligned}$$

where m denotes the dimension of the stratum which contains p.
Here we have used the Main Theorem (Theorem 3.3).

b) Relative case

Suppose first that $[a,b]$ contains no critical value. Then X_a is a weak deformation retract of X_b, so $H^k(X_b, X_a;\mathbb{Z}) = 0$. Here argue as in a) with $f \circ g$ instead of f.

If $f^{-1}([a,b])$ contains exactly one non-degenerate critical point of index λ,

$$X_a \cup \{f \circ g \le b, r \circ g \le \delta\}$$

is a decomposed weak deformation retract of X_b, hence $H^k(X_b, X_a \cup \{f \circ g \le b, r \circ g \le \delta\};\mathbb{Z}) = 0$.
Now use the Main Theorem in the relative case and apply Künneth. So

$$H^k(X_b, X_a;\mathbb{Z}) \simeq H^{k-\lambda}(g^{-1}(N \cap \{a \le f \le b\}), g^{-1}(N \cap \{f = a\});\mathbb{Z})$$

c) Nonproper case

Similarly as before we get:

If $[a,b]$ contains no critical value of f we have that $H^k(X_b, X_a;\mathbb{Z}) = 0$.
If $f^{-1}([a,b])$ contains exactly one non-degenerate critical point of index λ,

$$H^k(X_b, X_a;\mathbb{Z}) \simeq H^{k-\lambda}(N \cap X \cap \{a \le f \le b\}, N \cap X \cap \{f = a\});\mathbb{Z}).$$

d) Intersection cohomology

Let p be any perversity. Then the corresponding intersection cohomology can be defined on a purely n-dimensional pseudomanifold Z using the Deligne intersection complex

$$IC_p(Z) = IC_p(Z;\mathbb{Z})$$

which is constructible. Then look at $IH^k_p(Z;\mathbb{Z}) := \mathbb{H}^{k-n}(Z, IC_p(Z))$. See [GM1] p. 98.

Now let Z be as before, Z being purely n-dimensional. In order to have a pseudomanifold we need that there are no strata of codimension 1.

For the same reason we cannot take Z_a directly. So look at $Z_{<a} := Z \cap \{f < a\}$ instead.

By Theorem 2.6 and 3.7 we obtain $IH_p^k(Z_{<b};\mathbb{Z}) \simeq IH_p^k(Z_{<a};\mathbb{Z})$ for all k if $[a,b]$ contains no critical value.

Note that $IH_p^k(Z_{<a};\mathbb{Z}) \simeq \mathbb{H}^{k-n}(Z_a, IC_p(Z))$ if a is a regular value, by Theorem 5.2 below.

Now assume that $f^{-1}([a,b])$ contains exactly one non-degenerate critical point p of index λ. Let d be the dimension of the stratum S which contains p.

Let us look at

$$IH_p^k(Z_{<b}, Z_{<a};\mathbb{Z}) := \mathbb{H}^{k-n}(Z_{<b}, Z_{<a}; IC_p(Z)) \simeq \mathbb{H}^{k-n}(Z_b, Z_a; IC_p(Z)).$$

The Main Theorem of Goresky-MacPherson implies, using Theorem 3.2b) and 2.6c), that

$$\mathbb{H}^{k-n}(Z_b, Z_a; IC_p(Z)) \simeq$$

$$\mathbb{H}^{k-n}((D^\lambda \times D^{d-\lambda}, S^{\lambda-1} \times D^{d-\lambda}) \times (N \cap \{a \le f \le b\}, N \cap \{f = a\}), IC_p(S \times N^*)).$$

Here N^* is chosen as in the definition of a normal slice, it contains N.

Note that first we should take a pull-back of $IC_p(Z)$ on the right hand side but the Deligne intersection complex can be characterized axiomatically, see [GM1] §4, p. 107.

Let $i : N^* \to S \times N^*$ be defined by $q \mapsto (p,q)$. Then we have

$$\mathbb{H}^{k-n}((D^\lambda \times D^{d-\lambda}, S^{\lambda-1} \times D^{d-\lambda}) \times (N \cap \{a \le f \le b\}, N \cap \{f = a\}); IC_p(S \times N^*)$$

$$\simeq \mathbb{H}^{k-n-\lambda}(N \cap \{a \le f \le b\}, N \cap \{f = a\}); i^* IC_p(S \times N^*)).$$

Here we argue as in part (iv) of the proof of Theorem 5.4 below, replacing the commutative square there by

$$\begin{array}{ccc}
(\overset{o}{D}{}^\lambda \times D^{d-\lambda}) \times (N \cap \{a < f \le b\}) & \overset{p_1}{\to} & N \cap \{a < f \le b\} \\
\downarrow \pi_1 & & \downarrow \pi_0 \\
\overset{o}{D}{}^\lambda \times D^{d-\lambda} & \overset{p_0}{\to} & \{p\}
\end{array}$$

where p_1 and π_1 are canonical projections.

Then, $i^* IC_p(S \times N') \sim IC_p(N')[d]$, by [GM1] 5.4.1, p. 115.

Finally,

$$\mathbb{H}^{k-n+d-\lambda}(N \cap \{a \le f \le b\}, N \cap \{f = a\}); IC_p(N^*))$$

$$\simeq \mathbb{H}^{k-n+d-\lambda}(N \cap \{r < \delta, a < f < b\}, N \cap \{r < \delta, a < f < a'\}); IC_p(N^*))$$

$$\simeq IH_p^{k-\lambda}(N \cap \{r < \delta, a < f < b\}, N \cap \{r < \delta, a < f < a'\}; \mathbb{Z})$$

where $a' > a$ is sufficiently close to a.

In total,

$$IH_p^k(Z_{<b}, Z_{<a};\mathbb{Z}) \simeq IH_p^{k-\lambda}(N \cap \{r < \delta, a < f < b\}, N \cap \{r < \delta, a < f < a'\}; \mathbb{Z})$$

e) Locally constant coefficients

Let $\mathcal{L}$ be a locally constant sheaf on Z. Then $H^k(Z_b, Z_a; \mathcal{L}) = 0$ if $[a,b]$ contains no critical value: use Theorem 2.6 and Theorem 3.1b).

If there is just one critical point in $f^{-1}([a,b])$ which is non-degenerate of index λ we have $H^k(Z_b, Z_a; \mathcal{L}) \simeq H^k(U \cap Z \cap \{a \le f \le b\}, U \cap Z \cap \{f = a\}; \mathcal{L})$. Now $U \cap Z$ is contractible, so $\mathcal{L}|U \cap Z$ is constant, therefore

$$H^k(U \cap Z \cap \{a \le f \le b\}, U \cap Z \cap \{f = a\}; \mathcal{L}) \simeq H^k(U \cap Z \cap \{a \le f \le b\}, U \cap Z \cap \{f = a\}; \mathcal{L}_p).$$

Now we can continue as in the case of constant coefficients:

$$H^k(Z_b, Z_a; \mathcal{L}) \simeq H^{k-\lambda}(N \cap \{a \le f \le b\}, N \cap \{f = a\}; \mathcal{L}_p) \simeq$$
$$H^{k-\lambda}(N \cap \{a \le f \le b\}, N \cap \{f = a\}; \mathcal{L}).$$

5. Stratified Morse theory for constructible sheaves

Let $\mathcal{S}$ be a constructible sheaf complex on the decomposed space Z. So the cohomology groups of $\mathcal{S}$ are locally constant along the strata.

We take up the assumptions of the beginning of section 3.

By Theorem 3.1 b) and 2.6 we obtain immediately:

Theorem 5.1: $\mathbb{H}^k(Z_b, Z_a; \mathcal{S}) = 0$ for all k if $[a, b]$ contains no critical values.

We can also compare the cohomology of Z_a and $Z_{<a}$:

Theorem 5.2: If a is a regular value, the inclusion induces isomorphisms

$$\mathbb{H}^k(Z_a, \mathcal{S}) \simeq \mathbb{H}^k(Z_{<a}, \mathcal{S})$$

for all k.

Proof. It is an exercise to prove this using Theorem 5.1 and Theorem 3.7: Let $a' < a$ and $b > a$ sufficiently close to a so that $[a', b]$ contains no critical value. Then $\mathbb{H}^k(Z_{<b}, \mathcal{S}) \simeq \mathbb{H}^k(Z_{<a}, \mathcal{S})$, $\mathbb{H}^k(Z_a, \mathcal{S}) \simeq \mathbb{H}^k(Z_{a'}, \mathcal{S})$, which implies our statement.
Or: $Z_{a'}$ is a strong decomposed deformation retract of $Z_{<a}$, see Theorem 3.8. By Theorem 2.6 we have $\mathbb{H}^k(Z_{<a}, Z_{a'}; \mathcal{S}) = 0$ for all k. Finally use Theorem 5.1, too. □

Now suppose that there is just one critical point p in $f^{-1}([a, b])$ with $a < f(p) < b$ which is non-degenerate of index λ.
Then we can also pass to (co)homology, see e.g. [GM2] II Remark (2) after Theorem 6.4, p. 211: conclusion for $H_i(Z_b, Z_a; \mathbb{Z})$, but again one has to be more careful!

Let r be chosen as in section 3, $U := Z \cap \{r \le \delta\}$, where $\delta > 0$ is sufficiently small, $v := f(p)$, $\epsilon > 0$ small compared with δ, $a := v - \epsilon, b := v + \epsilon$.

Using Theorem 3.2 and 2.6 we obtain first:

Theorem 5.3: $\mathbb{H}^k(Z_b, Z_a \cup (U \cap Z_b); \mathcal{S}) = 0$ for all k.

By excision, $\mathbb{H}(Z_b, Z_a; \mathcal{S}) \simeq \mathbb{H}^k(Z_a \cup (U \cap Z_b), Z_a; \mathcal{S}) \simeq \mathbb{H}^k(U \cap \{a \le f \le b\}, U \cap \{f = a\}; \mathcal{S})$.

The final aim is to show that

$$\mathbb{H}^k(Z_b, Z_a; \mathcal{S}) \simeq \mathbb{H}^{k-\lambda}(N \cap \{a \le f \le b\}, N \cap \{f = a\}; \mathcal{S})$$

(i) By Theorem 3.3 (Main Theorem of Goresky-MacPherson) we have a homeomorphism

$$h : (U \cap S \cap \{a \le f \le b\}, U \cap S \cap \{f = a\}) \times (N \cap \{a \le f \le b\}, N \cap \{f = a\})$$
$$\to (U \cap \{a \le f \le b\}, U \cap \{f = a\})$$

This implies:

$$\begin{aligned} &\mathbb{H}^k(U \cap \{a \le f \le b\}, U \cap \{f = a\}; \mathcal{S}) \\ &\simeq \mathbb{H}^k((U \cap S \cap \{a \le f \le b\}, U \cap S \cap \{f = a\}) \times (N \cap \{a \le f \le b\}, N \cap \{f = a\}), h^*\mathcal{S}) \\ &\simeq \mathbb{H}^{k-\lambda}(N \cap \{a \le f \le b\}, N \cap \{f = a\}), i^*h^*\mathcal{S}) \end{aligned}$$

where

$$i : N \cap \{a \le f \le b\} \to (U \cap S \cap \{a \le f \le b\}) \times (N \cap \{a \le f \le b\})$$

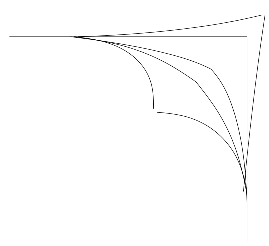
is defined by $x \mapsto (p, x)$.

There are cases where we can replace $i^*h^*\mathcal{S}$ by $\mathcal{S}$ without difficulty: if $\mathcal{S}$ is constant or merely locally constant (because we are dealing with a small neighbourhood). Similarly for the intersection cohomology complex which extends a constant sheaf on the union of the maximal strata of Z. See Section 4. But in other situations - e.g. if we look at an open subspace X of Z and a locally constant sheaf on this space or at the intersection cohomology complex extending a locally constant sheaf, see Section 6 - we must be more careful and look at the proof of Goresky-MacPherson's Main Theorem:

(ii) One considers a pair (A, B) of subspaces of Z which is more easily seen to be homeomorphic to the product of normal and tangential Morse data. The main difficulty is to construct a homeomorphism of the local Morse data onto (A, B). This is obtained as a composition of homeomorphisms each of which is obtained by the technique of "moving the wall".

For technical reasons, 2δ will be taken instead of δ, and let us assume $v = 0$.

More precisely: one considers a sequence (A_i, B_i) of subspaces and shows that two subsequent pairs are homeomorphic via a decomposed homeomorphism. In fact one applies the technique of Moving the Wall. This is indicated in [GM2] I 8.4, 8.5, pp. 103-113. In particular one has to describe walls depending on a parameter t which varies not only in $[0, 1]$ but in a slightly larger interval. But it is straightforward in most cases how to do this, except maybe for the stage of "rounding the corner" (I 8.5.1, p. 107) where the family of walls can be extended like follows:



Note that each A_i is defined as the "realization" of a diagram which is a pair of stratified regions in $\mathbb{R}^2$, together with functions to $\mathbb{R}$. In [GM2] pp. 103-106 these diagrams are depicted, with the two regions on the left and right respectively, the functions are written along the coordinate axes. Each time a subspace is indicated which is a union of strata, the realization of which yields B_i. With Moving the Wall one obtains a homeomorphism $A_i \to A_{i+1}$. Since it is stratum preserving it maps B_i homeomorphically onto B_{i+1}.

Note that we can ignore the transition $D_0 \to D_1$ and $D_5 \to D_6$ because nothing happens there. Let us be more specific about Moving the Wall in the other cases. On [GM2] p. 71 it is said that the wall space is taken to be 4-dimensional. We prefer $\mathbb{R}^2$ instead, because in each case only one of the two "pictures" P_a, P_b (left/right) is varied.

Example: $D_6 \to D_7$ (cf. [GM2] pp. 105, 111). Then we have a variation of P_a and get a corresponding subset $Y = \bigcup_{t\in]\alpha,\beta[}(P_a(t) \times \{t\}) \subset \mathbb{R}^2\times]\alpha,\beta[$. Here $]\alpha,\beta[$ is a small neighbourhood of $[0,1]$. Furthermore, replace Z in "Moving the Wall" by the inverse image of P_b under the mapping on the right hand side, i.e. by $Z \cap \{r \leq 2\delta\}$. The mapping f is replaced by the mapping $(f\circ\pi, f)$ on the left. In the case of other pictures proceed similarly but intersect also by $\{r < 2\delta'\}$, $\delta' > \delta$ near δ, in order to stay in a neighbourhood of p.

There is a technical problem because π is not defined everywhere but extend $f\circ\pi, r\circ\pi, \rho$ outside $\{r \leq 2\delta\}$ arbitrarily: this is harmless because the relevant considerations concern subsets of $\{r \leq 2\delta\}$.

Furthermore, in most cases we can apply Theorem 2.4 to the transition from A_i to A_{i+1} as well as from B_i to B_{i+1} or vice versa. However we cannot proceed in this way for B_i in all cases: it may happen that neither $B_i \subset B_{i+1}$ nor $B_{i+1} \subset B_i$. Therefore we modify the diagrams D_2, D_3, D_4 in order to pass from D_2 to D_3, D_3 to D_4: On the left we have to consider a "region" P_a together with a subregion Q_a. Replace the region P_a by $P'_a := \{(x,y)\,|\,y \geq -\epsilon'\}$ instead, where $\epsilon' > \epsilon$ is sufficiently near to ϵ. Also Q_a is replaced by $Q'_a :=$ closure of the complement of Q_a in P'_a, i.e. by

$$\{(x,y)\,|\,-\epsilon' \leq y \leq -\epsilon\} \text{ in the case of } D_2$$

$$\{(x,y)\,|\,y \geq -\epsilon', x \leq -\frac{3\epsilon}{4} \,\text{or}\, y \leq -\epsilon\} \text{ in the case of } D_3$$

$$\{(x,y)\,|\,y \geq -\epsilon', x \leq -\frac{3\epsilon}{4} \,\text{or}\, y - x \leq -\frac{\epsilon}{4}\} \text{ in the case of } D_4$$

We take obvious stratifications so that the subspaces are unions of strata. Instead of pairs (A_i, B_i), we thus obtain (A'_i, B'_i). Now we have $A'_i = A'_{i+1}$, $B'_i \subset B'_{i+1}$, $i = 2,3$. By Theorem 2.1, we obtain a homeomorphism $A'_i \to A'_{i+1}$ such that the restriction gives homeomorphisms $B'_i \to B'_{i+1}$, $A_i \to A_{i+1}$, $B_i \to B_{i+1}$. Furthermore, by Theorem 2.4 and 2.6 the inclusion defines isomorphisms $\mathbb{H}^k(A'_{i+1}, B'_{i+1}; \mathcal{S}) \to \mathbb{H}^k(A'_i, B'_i; \mathcal{S})$. By excision, we obtain

$$\mathbb{H}^k(A_{i+1}, B_{i+1}, \mathcal{S}) \simeq \mathbb{H}^k(A_i, B_i; \mathcal{S})$$

for all k. This shows that we obtain isomorphisms for the cohomology groups with the same constructible sheaf $\mathcal{S}$.

So we have $\mathbb{H}^k(U \cap f^{-1}([a,b]), U \cap \{f = a\}, \mathcal{S}) \simeq \mathbb{H}^k(A, B; \mathcal{S})$.

The precise description of (A, B) will be recalled in (iv) below.

In total we now have: $\mathbb{H}^k(Z_b, Z_a, \mathcal{S}) \simeq \mathbb{H}^k(A, B; \mathcal{S})$.

(iii) This result can be obtained more easily using different techniques, as in [H2]. Then we get:
The space Z_a is a decomposed strong deformation retract of $Z'_a := \{f \leq -\epsilon\} \cup E$, with

$$E := \{f\circ\pi \leq -\frac{3\epsilon}{4}, f - f\circ\pi \leq \frac{\epsilon}{4}, \rho \leq \delta, r\circ\pi \leq \delta\} \cup \{f - f\circ\pi \leq -\frac{\epsilon}{4}, f\circ\pi \leq \frac{3\epsilon}{4}, \rho \leq \delta, r\circ\pi \leq \delta\}$$

and

$$Z'_b := \{f \leq -\epsilon\} \cup \{f\circ\pi \leq \frac{3\epsilon}{4}, f - f\circ\pi \leq \frac{\epsilon}{4}, \rho \leq \delta, r\circ\pi \leq \delta\}$$

is a decomposed strong deformation retract of Z_b. See [H2] Prop. 4.4.
Therefore, $\mathbb{H}^k(Z'_a, \mathcal{S}) \simeq \mathbb{H}^k(Z_a, \mathcal{S})$, and $\mathbb{H}^k(Z_b, \mathcal{S}) \simeq \mathbb{H}^k(Z'_b, \mathcal{S})$.

Finally, $Z'_b = Z'_a \cup A$, and $B = Z'_a \cap A$, so $H^k(Z'_b, Z'_a, \mathcal{S}) \simeq H^k(A, B, \mathcal{S})$, which shows again that $H^k(Z_b, Z_a, \mathcal{S}) \simeq H^k(A, B, \mathcal{S})$.

(iv) Now

$$A = Z \cap \{|f - f \circ \pi| \le \frac{\epsilon}{4}, |f \circ \pi| \le \frac{3\epsilon}{4}, r \circ \pi \le \delta, \rho \le \delta\}$$

$$B = Z \cap \{r \circ \pi \le \delta, \rho \le \delta\} \cap (\{|f - f \circ \pi| \le \frac{\epsilon}{4}, f \circ \pi = -\frac{3\epsilon}{4}\} \cup \{f - f \circ \pi = -\frac{\epsilon}{4}, |f \circ \pi| \le \frac{3\epsilon}{4}\})$$

cf. [GM2] I Prop. 8.2. p. 101.
Furthermore,

$$\pi : (Z \cap \{r \circ \pi \le \delta, \rho \le \delta, |f - f \circ \pi| \le \frac{\epsilon}{4}\}, Z \cap \{r \circ \pi \le \delta, \rho \le \delta, f - f \circ \pi = -\frac{\epsilon}{4}\})$$
$$\to S \cap \{r \le \delta\}$$

is a fibre bundle pair with contractible base, hence trivial. The fibre pair over p is

$$(N \cap \{|f| \le \frac{\epsilon}{4}, r \le \delta\}, N \cap \{f = -\frac{\epsilon}{4}, r \le \delta\})$$

A trivialization yields a mapping pair

$$pr : (Z \cap \{r \circ \pi \le \delta, \rho \le \delta, |f - f \circ \pi| \le \frac{\epsilon}{4}\}, Z \cap \{r \circ \pi \le \delta, \rho \le \delta, f - f \circ \pi = -\frac{\epsilon}{4}\})$$
$$\to (N \cap \{|f| \le \frac{\epsilon}{4}, r \le \delta\}, N \cap \{f = -\frac{\epsilon}{4}, r \le \delta\})$$

The fibres are contractible, and $\mathcal{S}$ is cohomologically locally constant along the fibres. By [KS] Prop. 2.7.8, p. 122, we can conclude that $\mathcal{S}$ is quasiisomorphic to $pr^*\mathcal{T}$ with

$$\mathcal{T} := \mathcal{S}|N \cap \{|f| \le \frac{\epsilon}{4}, r \le \delta\} = i_0^*\mathcal{S},$$

where $i_0 : N \cap \{|f| \le \frac{\epsilon}{4}, r \le \delta\} \to (Z \cap \{r \circ \pi \le \delta, \rho \le \delta, |f - f \circ \pi| \le \frac{\epsilon}{4}\}$ is the inclusion:

Indeed, $\mathcal{S} \sim pr^*(R\,pr_*\mathcal{S})$, by [KS] loc. cit., so $i_0^*\mathcal{S} \sim i_0^* pr^*(R\,pr_*\mathcal{S}) \sim R\,pr_*\mathcal{S}$ because $pr \circ i_0 = id$.

From now on it is easier to work with cohomology with compact support instead of relative cohomology. Note that

$$A \setminus B = Z \cap \{-\frac{\epsilon}{4} < f - f \circ \pi \le \frac{\epsilon}{4}, -\frac{3\epsilon}{4} < f \circ \pi \le \frac{3\epsilon}{4}, r \circ \pi \le \delta, \rho \le \delta\}$$

Put $C := S \cap \{-\frac{3\epsilon}{4} < f \le \frac{3\epsilon}{4}, r \le \delta\}$, $D := N \cap \{-\frac{\epsilon}{4} < f \le \frac{\epsilon}{4}, r \le \delta\}$. Let $\pi_1 : A \setminus B \to C$ and $\pi_0 : D \to \{p\}$ be the restrictions of π, and let $p_1 : A \setminus B \to D$ and $p_0 : C \to \{p\}$ be the restrictions of pr, so that we have a commutative diagram:

$$\begin{array}{ccc} A \setminus B & \xrightarrow{p_1} & D \\ \downarrow \pi_1 & & \downarrow \pi_0 \\ C & \xrightarrow{p_0} & \{p\} \end{array}$$

Then we have:

$$\mathbb{H}^k(A, B, \mathcal{S}) \simeq \mathbb{H}^k(A, B, pr^*\mathcal{T}) \simeq \mathbb{H}^k_c(A \setminus B, p_1^*\mathcal{T}') \simeq H^k(R(p_0)_! R(\pi_1)_! p_1^*\mathcal{T}')$$

where $\mathcal{T}' := \mathcal{T}|D$.

Now

$$R(p_0)_! R(\pi_1)_! p_1^*\mathcal{T}' \sim R(p_0)_!(\mathbb{Z}_C \otimes^L R(\pi_1)_! p_1^*\mathcal{T}')$$
$$\sim R(p_0)_!(\mathbb{Z}_C \otimes^L p_0^* R(\pi_0)_! \mathcal{T}') \sim (R(p_0)_! \mathbb{Z}_C) \otimes^L R(\pi_0)_! \mathcal{T}'$$

The second quasiisomorphism follows by base change, the third one by some kind of projection formula, see [KS] Prop. 2.6.6, p. 113.

Hence

$$\mathbb{H}^k(A, B, \mathcal{S}) \simeq H^k((R(p_0)_! \mathbb{Z}_C) \otimes^L R(\pi_0)_! \mathcal{T}') \simeq \mathbb{H}_c^{k-\lambda}(D, \mathcal{T}')$$
$$\simeq \mathbb{H}^{k-\lambda}(N \cap \{|f| \leq \frac{\epsilon}{4}, r \leq \delta\}, N \cap \{f = -\frac{\epsilon}{4}, r \leq \delta\}, \mathcal{S})$$

Altogether we obtain as final result, with $N = N^* \cap \{r \leq \delta\}$:

Theorem 5.4: $\mathbb{H}^k(Z_b, Z_a; \mathcal{S}) \simeq \mathbb{H}^{k-\lambda}(N \cap \{a \leq f \leq b\}, N \cap \{f = a\}; \mathcal{S})$.

The final result has been shown by J. Schürmann [S] directly, too, with milder conditions on the critical point. See [S] Theorem 5.3.3.

6. Applications of stratified Morse theory for constructible sheaves

a) Cohomology with locally constant coefficients in the relative case

Let us look at the relative case as in section 4. Let $\mathcal{L}$ be a locally constant sheaf on X. Then we obtain:

If $[a, b]$ contains no critical value, $H^k(X_b, X_a; \mathcal{L}) = 0$ for all k.

If $f^{-1}([a, b])$ contains exactly one non-degenerate critical point of index λ,

$$H^k(X_b, X_a; \mathcal{L}) = H^{k-\lambda}(g^{-1}(N \cap \{a \leq f \leq b\}), g^{-1}(N \cap \{f = a\}), \mathcal{L})$$

In order to prove this, proceed as in the last section (Theorem 5.4) with $f \circ g$ instead of f, $r \circ g$ instead of r etc.

Or apply our theorem above to $Rg_*\mathcal{L}$, similarly as in [S] p. 275.

Note that Rg_* commutes with restriction to closed subsets because $Rg_* = Rg_!$, g being proper.

b) Cohomology with locally constant coefficients in the nonproper case

Suppose that X is an open subset of Z which is a union of strata and $\mathcal{L}$ a locally constant sheaf on X. Put $X_a := X \cap Z_a$. Then:

If $[a, b]$ contains no critical value, $H^k(X_b, X_a; \mathcal{L}) = 0$ for all k.

If $f^{-1}[a, b]$ contains exactly one non-degenerate critical point of index λ,

$$H^k(X_b, X_a; \mathcal{L}) = H^{k-\lambda}(N \cap X \cap \{a \leq f \leq b\}, N \cap X \cap \{f = a\}, \mathcal{L})$$

In order to prove this, proceed as in the last section finding decomposed weak deformation retracts.

Or apply Theorem 5.4 to $Rj_*\mathcal{L}$, where $j : X \to Z$ is the inclusion, similarly as in [S] p. 275. But the conclusion is not evident. Note that Rj_* commutes in general with $i^!$, hence with i^* if i is the inclusion of an open but not of a closed subset.

One needs a base change property which is proved in [S] Prop. 4.3.1, p. 261.

We argue in the same way for the normal slice and obtain

$$\mathbb{H}^k(N \cap \{a \leq f \leq b\}, Rj_*\mathcal{L}) \simeq H^k(N \cap X \cap \{a \leq f \leq b\}; \mathcal{L})$$

Similarly with $f = a$ instead of $a \leq f \leq b$.

c) Intersection cohomology with coefficients in a locally constant sheaf

Note that the locally constant sheaf has to be given outside codimension 2. We assume that Z is pure-dimensional. Again the reduction to the local case does not allow to assume that the locally constant sheaf is constant when applying the Main Theorem.

Similarly as in section 4 we obtain, using the complex $IC_p(Z;\mathcal{L})$:

$IH^k_p(Z_{<b}, Z_{<a};\mathcal{L}) = 0$ if $[a,b]$ does not contain critical values.

If there is exactly one non-degenerate critical point of index λ in $f^{-1}([a,b])$:

$$IH^k_p(Z_{<b}, Z_{<a};\mathcal{L}) = IH^{k-\lambda}_p(N \cap \{a < f < b, r < \delta\}, N \cap \{a < f < a', r < \delta\};\mathcal{L})$$

where $a' > a$ is sufficiently close to a.

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CROSS-RATIOS OF QUADRILATERAL LINKAGES

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ABSTRACT. We discuss the cross-ratio map of planar quadrilateral linkages, also in the case when one of the links is telescopic. Most of our results are valid for a planar quadrilateral linkage with generic lengths of the sides. In particular, we describe the image of cross-ratio map for quadrilateral linkage and planar robot 3-arm.

1. INTRODUCTION

We deal with quadrilaterals in Euclidean plane $\mathbb{R}^2$ with coordinates (x, y) identified with the complex plane $\mathbb{C}$ with coordinate $z = x + \imath y$. Given such a quadrilateral Q we define *cross-ratio of* Q as the cross-ratio of the four complex numbers representing its vertices in the prescribed order. Using complex numbers in the study of polygons has a long tradition (see, e.g., [1], [2]). We present several new developments concerned with the above notion of cross-ratio of quadrilateral.

The main aim of this paper is to investigate the values of cross-ratio in certain families of planar quadrilaterals. Two types of such families are discussed: (1) the 1-dimensional moduli spaces of quadrilateral linkage [4] and (2) the 2-dimensional moduli spaces of planar robot arms.

In the first part of this paper we deal with quadrilateral linkages (or 4-*bar mechanisms* [7]). In spite of apparent simplicity of these objects their study is related to several deep results of algebraic geometry and function theory, in particular, to the theory of elliptic functions and Poncelet Porism [5]. Comprehensive results on the geometry of planar 4-bar mechanisms are presented in [7]. Some recent results may be found in [5], [11],[13].

We complement results of [7] and [11] by discussing several new aspects which emerged in course of our study of extremal problems on moduli spaces of polygonal linkages (cf. [10],[11], [12], [13], [14]). In this context it is natural to consider polygonal linkage as a purely mathematical object defined by a collection of positive numbers and investigate its moduli spaces [4]. In this paper we deal with quadrilateral linkages and planar moduli spaces.

Two types of quadrilateral linkages are considered: (1) *conventional* quadrilateral linkages with the fixed lengths of the sides, and (2) quadrilateral linkages with one *telescopic link* [4]. Obviously, the latter concept is equivalent to the so-called *planar robot* 3-*arm* (or planar triple pendulum [14]). To unify and simplify terminology it is convenient to refer to these two cases by speaking of *closed* and *open* 4-*vertex linkages*.

The necessary background for our considerations is presented in Section 2. We begin with recalling the definition and basic geometric properties of planar moduli spaces of 4-vertex linkages (Proposition 2.1). With a planar 4-vertex linkage Q one can associate the *cross-ratio map* Cr_Q from its planar moduli space $M(Q)$ into the extended complex plane $\overline{\mathbb{C}}$ (Riemann sphere).

Our first main result gives a precise description of the image of cross-ratio map for a generic quadrilateral linkage (Theorem 3.1). It turns out that cross-ratio is a stable mapping in the sense of singularity theory and that its image is an arc of a circle or a full circle, depending on the type of the moduli space. This eventually enables us to obtain an analogous result for

Key words and phrases. quadrilateral linkage, robot 3-arm, moduli space, cross-ratio.

a planar robot 3-arm (Theorem 4.4). Here again cross-ratio is a stable map, having only folds (and no cusps) and the image is an annulus. Moreover the Jacobian of cross-ratio is a non-zero multiple of the signed area and the critical points correspond to quadrilaterals and arms with signed area zero.

In conclusion we mention several possible generalizations of and research perspectives suggested by our results.

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2. Moduli spaces of planar 4-vertex linkages

We freely use some notions and constructions from the mathematical theory of linkages, in particular, the concept of *planar moduli space* of a polygonal linkage [4]. Recall that *closed n-lateral linkage* $L(l)$ is defined by a n-tuple l of positive real numbers l_j called its side-lengths such that the biggest of side-lengths does not exceed the sum of remaining ones. The latter condition guarantees the existence of a n-gon in Euclidean plane $\mathbb{R}^2$ with the lengths of the sides equal to numbers l_j. Each such polygon is called a *planar realization* of linkage $L(l)$.

Linkage with a *telescopic side* is defined similarly but now the last side-length l_n is allowed to take any positive value. For brevity we will distinguish these two cases by speaking of closed and open linkages.

For a closed or open linkage L, its planar configuration space $M(L) = M_2(L)$ is defined as the set of its planar realizations (configurations) taken modulo the group of orientation preserving isometries of $\mathbb{R}^2$ [4]. It is easy to see that moduli spaces $M(L)$ have natural structures of compact real algebraic varieties. For an open n-linkage its planar moduli space is diffeomorphic to the $(n-2)$-dimensional torus T^{n-2}. For a closed n-linkage with a *generic* side-length vector l, its planar moduli space is a smooth compact $(n-3)$-dimensional manifold. As usual, here and below the term "generic" means "for an open dense subset of parameter space" (in our setting, this is the space $\mathbb{R}^n_+$ of side-lengths).

In particular, a closed 4-linkage $Q = Q(l)$ is defined by a quadruple of positive numbers $l = (a, b, c, d) \in \mathbb{R}^4_+$. An open planar 4-linkage (or planar robot 3-arm $A = A(l)$) is analogously defined by a triple of positive numbers $l = (a, b, c) \in \mathbb{R}^3_+$ and its planar moduli space is diffeomorphic to the two-torus T^2. The complete list of possible topological types of planar moduli spaces of closed 4-linkages is also well known (see, e.g., [9]).

Proposition 2.1. *The complete list of homeomorphism types of planar moduli spaces of a 4-bar linkages is as follows: circle, disjoint union of two circles, bouquet of two circles, two circles with two common points, three circles with pairwise intersections equal to one point.*

Closed linkages with smooth moduli spaces are called non-degenerate. It is well-known that non-degeneracy is equivalent to the generic condition $a \pm b \pm c \pm d \neq 0$. It excludes aligned configurations. In the sequel we mainly focus on non-degenerate quadrilateral linkages, but in our study of robot 3-arms we will meet also the degenerate quadrilateral linkages. See the first row of Figure 2 for pictures of the moduli spaces.

3. Cross-ratio map of quadrilateral linkage

In this section we use some basic properties of cross-ratio which can be found in [2]. Recall that the complex cross-ratio of four points (where no three of them coincide) $p, q, z, w \in \mathbb{C}$ is defined as

$$[p,q;z,w] = \frac{z-p}{z-q} : \frac{w-p}{w-q} = \frac{p-z}{p-w} \cdot \frac{q-z}{q-w}. \tag{1}$$

and takes values in $\mathbb{C} \cup \infty = \mathbb{P}^1(\mathbb{C})$. Coinciding pairs correspond to the values $0, 1, \infty$.
Group S_4 acts by permuting points so one can obtain up to six values of the cross-ratio for a given unordered quadruple of points which are related by well-known relations [2]. For further use notice also that the value of cross-ratio is real if and only if the four points lie on the same circle of straight line [2].

Consider now a quadrilateral linkage $Q = Q(a,b,c,d)$. Note that no three vertices can coincide. Then, for each planar configuration $V = (v_1, v_2, v_3, v_4) \in \mathbb{C}^4$ of Q, put

$$Cr(V) = \mathrm{Cr}((v_1,v_2,v_3,v_4)) = [v_1,v_2;v_3,v_4] = \frac{v_3-v_1}{v_3-v_2} : \frac{v_4-v_1}{v_4-v_2}. \tag{2}$$

This obviously defines a continuous (in the non-degenerate case actually a real-analytic) mapping $Cr_Q : M(Q) \to \mathbb{P}^1(\mathbb{C})$. Our main aim in this section is to describe its image $\Gamma_Q = \operatorname{Im} Cr_Q$ which is obviously a continuous curve in $\mathbb{P}^1(\mathbb{C})$. Taking into account some well-known properties of cross-ratio and moduli space, one immediately obtains a few geometric properties of Γ_Q.

In particular, its image should be symmetric with respect to real axis. If the linkage Q does not have aligned configurations the points of intersection Γ_Q with real axis correspond to cyclic configurations of Q. It is known that Q can have no more then four distinct cyclic configurations which come into complex conjugate pairs [10]. Hence Γ_Q can intersect the real axis in no more than two points. In case $M(Q)$ has two components then they are complex-conjugate and the image of Cr is equal to the image of each component, which implies that Γ_Q is connected even though $M(Q)$ may have two components.

In further considerations it is technically more convenient to work with another map $R : M(Q) \to \mathbb{P}^1(\mathbb{C})$ defined by the formula

$$R(v_1,v_2,v_3,v_4) = Cr(v_1,v_3,v_2,v_4) = [v_1,v_3;v_2,v_4] = \frac{v_2-v_1}{v_2-v_3} : \frac{v_4-v_1}{v_4-v_3} \tag{3}$$

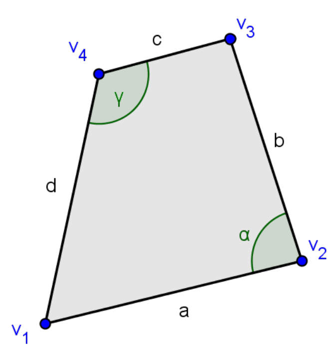

FIGURE 1. Quadrilateral.

From the transformation properties of cross-ratio follows that

$$Cr(V) = 1 - R(V).$$

So the properties of Cr can be immediately derived from the properties of R. For brevity we will call R the *uniformizer* of Q.

The main advantage of R is that, for any configuration V of closed linkage Q, the moduli of numbers $v_{i+1} - v_i$ are constant by its very definition. Consequently, for any $V \in M(Q)$, one has

$$|R(V)| = \frac{ac}{bd}. \tag{4}$$

In other words, R maps $M(Q)$ into the circle of radius $\frac{ac}{bd}$ with the center at point $0 \in \mathbb{C}$. Later (in the robot arm case) it is more convenient to consider the chart around ∞ and we get a circle with radius $\frac{bd}{ac}$. Let

$$\alpha = \arg \frac{v_3 - v_2}{v_1 - v_2} \ \ , \ \ \gamma = \arg \frac{v_1 - v_4}{v_3 - v_4} \tag{5}$$

be the angles at points v_2 and v_4 in the configuration V.

It follows that

$$\arg R(V) = -(\alpha + \gamma), \tag{6}$$

These observations enable us to get a very precise description of the image $\operatorname{Im} R$ given in the proposition below. Notice that since a non-singular moduli space is homeomorphic to a circle or the disjoint union of two circles, one may use the natural orientations of $M(Q)$ and $0 \in \mathbb{C}$ to define the mapping degree of uniformizer map.

Theorem 3.1. *For a non-degenerate quadrilateral linkage Q, the following statements hold:*

(1) the image $\operatorname{Im} R$ is a subset of the circle of radius ac/bd centered at the point $0 \in \mathbb{C}$;

(2) the image $\operatorname{Im} R$ is connected and symmetric about the real axis containing the point $\frac{ac}{bd}$;

(3) R is surjective if and only if $(a + b - c - d)(a - b + c - d)(a - b - c + d) \le 0$.

(4) the mapping degree of R equals zero and multiplicity at each point does not exceed two.

Proof. The first two statements follow from the preceding discussion. The third property can be proved as follows. Take a point $e^{\imath\tau} \in S^1$. We wish to solve the equation $\operatorname{Arg} R(V) = \tau$ with $V \in M(Q)$. Using the above notation this is equivalent to solving the system

$$\{a^2 + b^2 - 2ab\cos\alpha = c^2 + d^2 - 2cd\cos\gamma \ , \ \alpha + \gamma = -\tau\}.$$

Substituting $\cos\alpha = \cos(\tau + \gamma)$ we get

$$a^2 + b^2 - 2ab\cos\tau\cos\gamma + 2ab\sin\tau\sin\gamma \ - c^2 - d^2 + 2cd\cos\gamma = 0.$$

From this one easily obtains equation of the form

$$A\sin\gamma + B\cos\gamma = C, \tag{7}$$

where $A = 2ab\sin\tau,\ B = -2ab\cos\tau + 2cd,\ C = a^2 + b^2 - c^2 - d^2$. Now it is easy to see that this equation may have 0, 1 or 2 solutions in $[0, 2\pi]$ depending on the sign of expression

$$F_\tau = A^2 + B^2 - C^2 = 4a^2b^2 + 4c^2d^2 - (a^2 + b^2 - c^2 - d^2)^2 - 8abcd\cos\tau.$$

Namely, there are no solutions if $F_\tau < 0$, one solution if $F_\tau = 0$, and two solutions if $F_\tau > 0$.

It is now easy to conclude that if solution exists for certain $\tau \in [0, \pi]$ then it exists for any $\sigma \in [0, \pi], \sigma > \tau$ because in this case

$$F_\sigma(a, b, c, d) \ge F_\tau(a, b, c, d) \ge 0.$$

Hence surjectivity takes place if and only if the point with argument 0 is in the image of R. Notice that

$$F_0(a, b, c, d) = -(a + b - c - d)(a - b + c - d)(a - b - c + d)(a + b + c + d).$$

Thus surjectivity is equivalent to $F_0(a, b, c, d) \ge 0$ which differs from the criterion of (3) only by a negative factor $-(a + b + c + d)$. So property (3) is proved. Property (4) follows form the symmetry of R with respect to the real axis, which completes the proof of proposition. □

In the non-degenerate case we have:

Corollary 3.2. *The image of Cr is a conjugation-invariant arc of the circle of radius ac/bd centered at the point $1 \in \mathbb{C}$.*

This is immediate in view of the relation between Cr and R.

Corollary 3.3. *Cross-ratio map of Q is surjective if and only if Q has a self-intersecting cyclic configuration.*

This follows from the above proof since the argument of R of self-intersecting cyclic configuration is equal to 0.

Corollary 3.4. *Cross-ratio map of Q is surjective if and only if its planar moduli space has two components. In other words, surjectivity of cross-ratio map is a topological property.*

Indeed, it was shown in [10] that a self-intersecting cyclic configuration exists if and only if the moduli space has two components. Notice that these observations yield a simple criterion of connectedness of the moduli space.

Corollary 3.5. *The moduli space is connected if and only if*

$$(a+b-c-d)(a-b+c-d)(a-b-c+d) \geq 0.$$

Notice also that, in non-degenerate case, $R(Q)$ is a smooth mapping between two compact one-dimensional manifolds. We compute its differential with respect to an angular parameter on $M_2(Q)$ and identify its critical points as quadrilaterals V with signed area equal to zero. Signed area was defined and it properties were studied in [10]. Moreover we describe the global behaviour as follows:

Theorem 3.6. *For a non-degenerate quadrilateral linkage Q, the following statements hold:*
(5) If $M(Q)$ consists of one component then cross-ratio is a stable mapping with exactly 2 fold points. The image is an arc of a circle,
(6) If $M(Q)$ consists of two components then cross-ratio is a stable mapping, has no singularities and maps each circle bijectively to the image circle.

Proof. We use Lagrange multipliers for the function $\arg R(V) = -(\alpha+\gamma)$ with respect to

$$g(\alpha,\gamma) = a^2 + b^2 - 2ab\cos\alpha - c^2 - d^2 + 2cd\cos\gamma = 0. \tag{8}$$

The critical points of $\arg R(V)$ are given by:

$$2ab\sin\alpha + 2cd\sin\gamma = 0. \tag{9}$$

This is the condition that the signed Area (sA) of the quadrilateral is zero! In [10] it is shown, that sA has exactly two critical points on each component and that in the 2-component case it never takes the value 0. It follows that in the 1-component case there are precisely 2 quadrilaterals V with $sA(V) = 0$.

Also the second derivative can be computed by the Lagrange multipliers method, following [8].

The main ingredient for a function $f(x_1, x_2)$ and an equation $g(x_1, x_2) = 0$ is the Hessian matrix $H = (\frac{\partial^2 f}{\partial x_i \partial x_j} - \lambda \frac{\partial^2 g}{\partial x_i \partial x_j})$, where λ is defined by $\mathrm{grad} f = \lambda\, \mathrm{grad} g$. Evaluate this only on vectors in the tangent space to $g(x_1, x_2) = 0$ at a critical point. We have in our case (taking into account condition (9)):

$$\lambda^{-1} = 2ab\sin\alpha \;\; ; \;\; H = \lambda \begin{pmatrix} 2ab\cos\alpha & 0 \\ 0 & 2cd\cos\gamma \end{pmatrix}.$$

The tangent space is generated by $w^t = -2ab\sin\alpha\ (1,-1)$. The final result for the second derivative is: $w^tHw = -2ab\sin\alpha(2ab\cos\alpha - 2cd\cos\gamma)$. This is zero as soon as

$$2ab\cos\alpha = 2cd\cos\gamma.$$

Combining this with condition (9) it follows that $(\alpha,\gamma) \in \{(0,0),(0,\pi),(\pi,0),(\pi,\pi)\}$. These are the aligned configurations, which are degenerate.

So the second derivative is non-zero in the critical points of arg$R(V)$. By Morse lemma this is enough to conclude that each critical point is a fold.

□

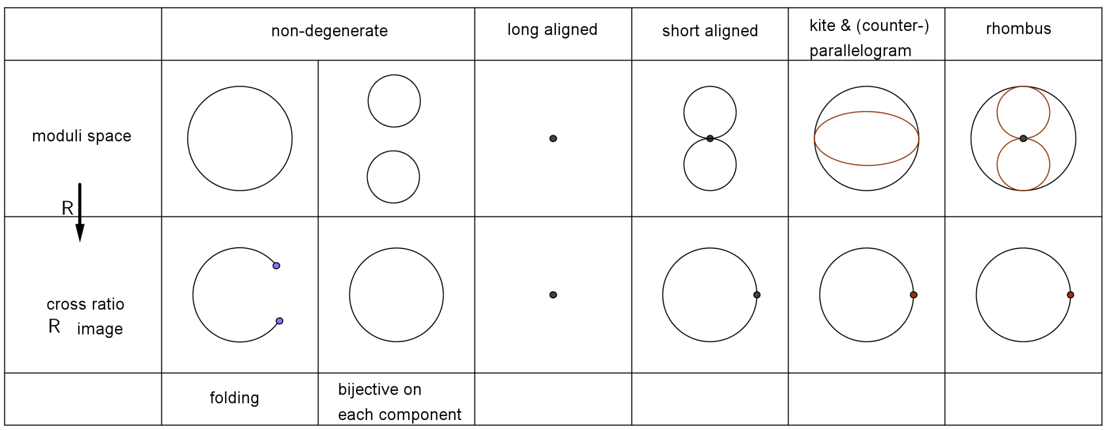

FIGURE 2. Moduli spaces of quadrilateral linkages and the cross-ratio images

Next we give the description of the image of R if linkage $Q(a,b,c,d)$ is not generic. We will use this in the section about robot arms.

3.1. **Long aligned.** In this case the length of one edge is equal to the sum of the three others. We have

$$F_\pi = (a-b-c-d)(a+b+c-d)(a-b+c+d)(a-c+b+d) = 0.$$

$\tau = 0$ is the only possibility, the moduli space is a point and the image is also one point.

3.2. **Short aligned.** In this case the sum of the lengths of two sides is equal to the sum of the two others. We have

$$F_0 = -(a+b-c-d)(a-b+c-d)(a-b-c+d)(a+c+b+d) = 0.$$

Consequently R is surjective. When when we are not in the cases 3.3, 3.4 or 3.5 the moduli space is a bouquet of two circles and the uniformizer R maps each of the two circles onto a full circle. The wedge point is mapped to the intersection of the circle with the positive real axis.

3.3. **Kite.** When $a = b$ and $c = d$ we have a moduli space, which consists of two circles having two points in common. R maps one circle $2:1$ (with degree 0) onto the image circle and the other circle collapses to the point on the positive real axis.

3.4. **Parallelogram and counter-parallelogram.** When $a = c$ and $b = d$ we have a moduli space, which consists of two circles having two points in common. R maps one circle (corresponding to the parallelograms) $2 : 1$ (with degree 0) onto the image circle and the other circle (corresponding to the counter-parallelograms) collapses to the point on the positive real axis.

3.5. **Rhomboid.** When $a = b = c = d$ the moduli space consists of three circles having pairwise a point in common [4]. Note that R maps one circle $2 : 1$ (with degree 0) onto the image circle and the other two circles collapse to the point on the positive real axis.

4. Cross-ratio map of robot 3-arm

The cross-ratio is defined as a map $Cr : M(A) \to \mathbb{P}^1(\mathbb{C})$ to the Riemann sphere. Only if $M(A(l))$ does contain configurations with coinciding vertices Cr attains the value ∞. This happens besides non-generic cases only if the arm forms a triangle. Since $M(A)$ is diffeomorphic to T^2 we may ask a number of natural questions about the behavior of Cr as a mapping between 2-dimensional manifolds. In particular, in the spirit of Whitney's results on stable mappings (see, e.g. [2]) one can wonder if Cr is stable in the sense of singularity theory: having only folds and cusps as singularities.

As before we work below with the uniformizer $R(Z) = 1 - Cr(Z)$.

Theorem 4.1. *The cross-ratio map for open linkages is a stable mapping with folds only.*

Proof. We show this in several steps. First we consider the 4-bar linkage Q_t obtained by adding to the arm Z a fourth side of length t and take this t as one of the local coordinates on open subsets of the torus $M(A)$. This is possible as long as Q_t is non-degenerate. Avoid $a \pm b \pm c \pm t = 0$, where aligned cases occur. We can take as other local coordinate α or γ, which are implicitly related by:

$$a^2 + b^2 - 2ab\cos\alpha = c^2 + t^2 - 2ct\cos\gamma.$$

Assume we can use (t, α) as local coordinates then $\gamma = \gamma(t, \alpha)$. We use polar coordinates $(|R^{-1}|, \arg R^{-1})$ on the chart at ∞ . Now

$$|R^{-1}(Z)| = \frac{bt}{ac}\ ,\ \arg R^{-1} = \alpha + \gamma$$

and therefore the critical points of R are just the union of the critical (=fold) points of each of the closed linkages. We next relate this to the criterium for a mapping $F(t, \alpha) = (t, f(t, \alpha))$ to have a fold singularity (cf.,[3], p. 74): $\frac{\partial f}{\partial \alpha} = 0$ and $\frac{\partial^2 f}{\partial^2 \alpha} \neq 0$ both taken in a point (t_0, α_0). Take for this point the fold point of the quadrilateral Q_{t_0} and it follows that we have indeed a fold for our open linkage. We treat the remaining cases in step 2 and 3.

The second step is to consider the aligned positions. We choose a complex coordinate on the torus, such that the vertices of the arm are given by $0, a, a + be^{i\phi}, a + be^{i\phi} + ce^{i\eta}$. This gives

$$R^{-1} = \tfrac{-b}{ac}(a + be^{i\phi} + ce^{i\eta})e^{i(\phi-\eta)}.$$

For each of the aligned positions we can compute its 2-jet. We take as example $(\phi, \eta) = (0, 0)$. The other cases behave in the same way. Up to a constant we have for the 2-jet:

$$(\phi, \eta) \to (a + b + c - \tfrac{1}{2}[a(\phi - \eta)^2 + b(2\phi - \eta)^2 + c\phi^2]\ ,\ a(\phi - \eta) + b(2\phi - \eta) + c\phi)$$

It's singular set is the line $\phi = 0$ and a coordinate transformation brings it in the standard equation of the fold. Since fold singularities are determined by its 2-jet it follows that also R^{-1} has a fold at the point $(0, 0)$.

The third step: In the special case of a closed telescopic arm ($t = 0$) (where polar coordinates are not well-defined) we compute in proposition 4.2 the Jacobian in general and show that for

$t = 0$ it equals to $bc \sin[\eta - \phi]$. This expression is non-zero as soon as the arm is not aligned (non-generic case).

This finishes the proof. □

Remark: Note that the other type of stable singularity, the cusp, does not occur!

Proposition 4.2. *The Jacobian of R^{-1} is equal to the signed area of the quadrangle defined by the arm Z. The critical points of R and of cross-ratio correspond to the arms with signed area equal to zero.*

Proof. A straightforward computation shows, that the Jacobian is (modulo a non-zero constant) given by:

$$ab \sin[\phi] + ac \sin[\eta] + bc \sin[\eta - \phi].$$

This is precisely twice the signed area of arm Z. □

Corollary 4.3. *The point $\infty \in \mathbb{P}^1(\mathbb{C})$ is a regular value of uniformizer R.*

Next we investigate the shape of the image. We slice with circles. Fix a number $t \in [0, a+b+c]$ and consider the 4-bar linkage Q_t obtained by adding to A a fourth side of length t. Then the image of Cr_A is simply the union of the the images $Cr(M(Q_t))$. These are arcs of the circles described in Theorem 3.6.

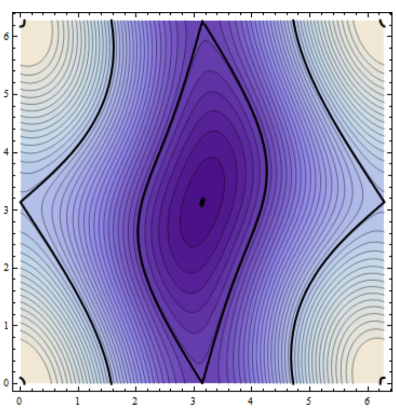

FIGURE 3. t-levels; case $a > b + c$

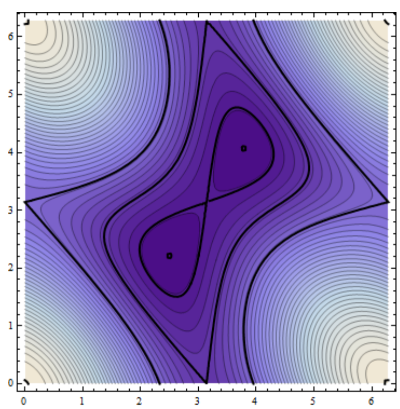

FIGURE 4. t-levels; case $a < b + c$

We consider the following two cases:

i. $M(A)$ contains no closed configurations,
ii. $M(A)$ contains a closed configuration (triangle).

We exclude non-generic arms. For sake of presentation we assume $a > b > c$. Other cases behave similar. So we distinguish now between:

i. $a > b + c$
ii. $a < b + c$

The topology of a slice changes at critical values of t (seen as function $M(A) \to \mathbb{R}$). Level curves are shown in Figures 3 and 4. According to [9] these are exactly the aligned positions (where Morse indices follow from the combinatorics) and in the second case also the value 0.

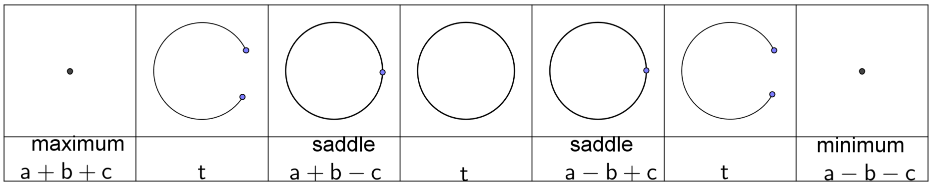

FIGURE 5. Movie of images of R in case $a > b + c$

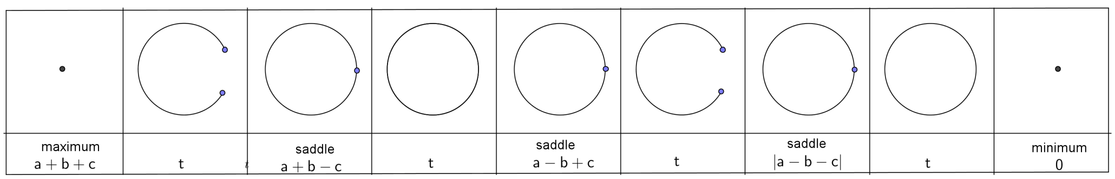

FIGURE 6. Movie of images of R in case $a < b + c$

In case i. we have a Morse function with one maximum, two saddles and a minimum; see the "movie" of R-images in figure 5. In case ii. (see figure 6) there appears an extra saddle and we end up with two minima, which correspond to $t = 0$ (two conjugate triangles). Pictures of the image of R are shown in figures 7 and 8.

The considerations from section 3 give the following analog of Theorems 3.1 and 3.6:

Theorem 4.4. *For a generic planar robot* 3*-arm* $A(a, b, c)$*, the cross-ratio map has degree zero, its image is a conjugation-invariant differentiable annulus and belongs to a disc with radius* $a + b + c$*. The cross-ratio map is 2-1 except on the critical set, with image the fold curves.*

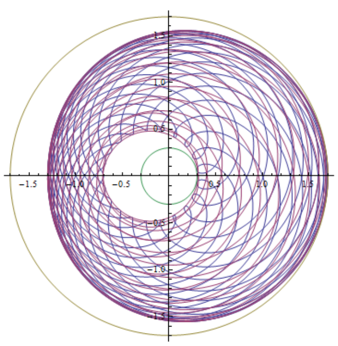

FIGURE 7. Image of R; case $a > b + c$

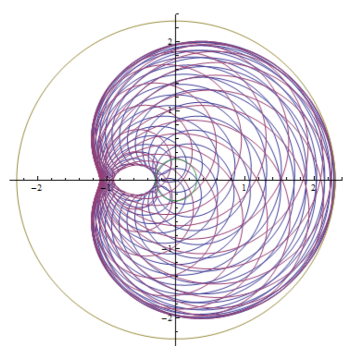

FIGURE 8. Image of R; case $a < b + c$

5. Concluding remarks

First of all, we wish to add that using stereographic projection one may introduce cross-ratio map for spherical quadrilaterals. The analogs of Theorems 3.1 and 4.4 follow in a straightforward way.

It is also interesting to describe the change of cross-ratio under the action of the so-called Darboux transformation of quadrilateral linkage [5]. Taking into account a version of Poncelet Porism for quadrilateral linkages obtained in [5] one might hope to get certain insights concerning the arising discrete dynamical system in the image of cross-ratio map.

In a future paper, by a way of analogy we investigate cross-ratios of one-dimensional families of the so-called *poristic* quadrilaterals arising from Poncelet Porism [6]. Analogs of our main results are available for bicentric poristic quadrilaterals and poristic quadrilaterals associated with confocal ellipses.

Next, one can also consider cross-ratios of families of quadrilaterals arising as the centers of circles of Steiner 4-chains [2] and try to describe the image of the corresponding cross-ratio map.

Finally, an analogous line of development arises in connection with the notion of *conformal modulus of a quadrilateral* [1]. In particular, one can try to describe the image and behavior of conformal modulus for families of poristic bicentric polygons and confocal ellipses. Developments in this direction will be published elsewhere.

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(SSP) GEOMETRY WITH DIRECTIONAL HOMEOMORPHISMS

SATOSHI KOIKE AND LAURENTIU PAUNESCU

Dedicated to Professor David Trotman for his 60th birthday

Abstract. In a previous paper [6] we discussed several directional properties of sets satisfying the sequence selection property, denoted by (SSP) for short, and developed the (SSP) geometry via bi-Lipschitz transformations. In this paper we introduce the notion of directional homeomorphism and show that we can develop also the (SSP) geometry with directional transformations. For many important results proved in [6] for bi-Lipschitz homeomorphisms we describe the analogues for directional homeomorphisms as well.

1. Introduction.

In [4] we introduced the notion of sequence selection property, denoted by (SSP) for short, in order to show that the dimension of the common direction set of two subanalytic subsets is preserved by a bi-Lipschitz homeomorphism provided that their images are also subanalytic. The condition (SSP) is one of the three main ingredients in the given proof. Subsequently we generalised the result above to the case of a general real closed field in [5], where we also discussed several (SSP) properties.

Following the above works, we have started to work on condition (SSP) both on the field of real numbers and on the field of complex numbers. In fact, we proved essential directional properties of sets satisfying (SSP) with respect to bi-Lipschitz homeomorphisms in [6]. Amongst the main results in [6] are the following:

(1) Weak transversality theorem,

(2) (SSP) structure preserving theorem,

(3) Important property: $LD(h(LD(A))) = LD(h(A))$,

(4) Directional property of intersection sets.

Concerning (2), we proved two types of (SSP) structure preserving theorems in [6]. The main purpose in this paper is to introduce a new notion of homeomorphism, called *directional homeomorphism*, which enables us to show general results including those theorems mentioned above, using the new notion of homeomorphism without the assumption on the sequence selection property. We shall discuss several properties of the directional homeomorphism in §3.1, and give the main results in §4.

Throughout this paper we use the following notations:

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Key words and phrases. direction set, sequence selection property, transversality, bi-Lipschitz homeomorphism.

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Let $\{a_m\}$, $\{b_m\}$ be sequences of points of $\mathbb{R}^n$ tending to the origin $0 \in \mathbb{R}^n$. If there are a natural number $N \in \mathbb{N}$ and a real number $K > 0$ such that

$$\|a_m\| \leq K\|b_m\|, \quad \forall m \geq N$$

then we write $\|a_m\| \precsim \|b_m\|$ (or $\|b_m\| \succsim \|a_m\|$). If $\|a_m\| \precsim \|b_m\|$ and $\|b_m\| \precsim \|a_m\|$, we write $\|a_m\| \approx \|b_m\|$.

2. Directional Properties of Sets

In this section we recall the notions of direction set and sequence selection property, and describe several elementary properties.

2.1. **Direction set.** Let us recall the notion of direction set.

Definition 2.1. Let A be a set-germ at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A}$. We define the *direction set* $D(A)$ of A at $0 \in \mathbb{R}^n$ by

$$D(A) := \{a \in S^{n-1} \mid \exists \{x_i\} \subset A \setminus \{0\},\ x_i \to 0 \in \mathbb{R}^n \text{ s.t. } \frac{x_i}{\|x_i\|} \to a,\ i \to \infty\}.$$

Here S^{n-1} denotes the unit sphere centred at $0 \in \mathbb{R}^n$.

For a subset $A \subset S^{n-1}$, we denote by $L(A)$ a half-cone of A with the origin $0 \in \mathbb{R}^n$ as the vertex:

$$L(A) := \{ta \in \mathbb{R}^n \mid a \in A,\ t \geq 0\}.$$

In the case where $A \subset S^{n-1}$ is a point we call $L(A)$ a *semiline*. Therefore a semiline $\ell \subset \mathbb{R}^n$ means a half line whose starting point is the origin $0 \in \mathbb{R}^n$. For a set-germ A at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A}$, we put $LD(A) := L(D(A))$, and call it the *real tangent cone* of A at $0 \in \mathbb{R}^n$.

Let $U,\ V \subset \mathbb{R}^n$ such that $0 \in \overline{U} \cap \overline{V}$. The following properties hold:

(1) $D(\overline{U}) = D(U)$,
(2) $D(U \cup V) = D(U) \cup D(V)$,
(3) $\overline{\cup_i D(U_i)} \subseteq D(\cup U_i)$,
(4) If U_i are half-cones then $\overline{\cup_i D(U_i)} = D(\cup U_i)$,
(5) $D(U \cap V) \subseteq D(U) \cap D(V)$.

2.2. **Sequence selection property.** Let us recall the notion of condition (SSP). In fact here we give a generalised notion of (SSP) relatively to a subset of $\mathbb{R}^n$.

Definition 2.2. Let A, B be two set-germs at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A} \cap \overline{B}, D(A) \subseteq D(B)$. We say that A satisfies *condition* (SSP)-relative to B, if for any sequence of points $\{a_m\}$ of B tending to $0 \in \mathbb{R}^n$, such that $\lim_{m\to\infty} \frac{a_m}{\|a_m\|} \in D(A)$, there is a sequence of points $\{b_m\} \subset A$ such that,

$$\|a_m - b_m\| \ll \|a_m\|,\ \|b_m\|,$$

i.e., $\lim_{m\to\infty} \frac{\|a_m - b_m\|}{\|a_m\|} = 0$.

In the case $B = \mathbb{R}^n$ we will not mention B (it is the usual (SSP) condition).

Clearly the direction set and the sequence selection property are conditions in the spirit of Whitney [7], who consistently studied directional properties at singular points and their behaviour while approaching a singularity via sequences of points. We concentrate our study to sets for which their direction sets are essentially independent on the ambient space.

For the reader's convenience we give some remarks on the relative condition (SSP) ((2) and (3) follow from the transitivity of the relative condition (SSP)).

Remark 2.3. (1) A (resp. $\overline{A}$) satisfies *condition* (SSP)-relative to $\overline{A}$ (resp. A).
(2) A satisfies *condition* (SSP) if and only if A satisfies *condition* (SSP)-relative to $LD(A)$.
(3) A satisfies *condition* (SSP) if and only if $\overline{A}$ satisfies *condition* (SSP).
(4) A satisfies *condition* (SSP)-relative to $ST_d(A;C), d > 1$
(see [4] for $ST_d(A : C)$).

In this note we also consider the notion of weak sequence selection property, denoted by $(WSSP)$ for short; in fact they are equivalent notions.

Definition 2.4. Let A, B be two set-germs at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A} \cap \overline{B}, D(A) \subseteq D(B)$. We say that A satisfies *condition* $(WSSP)$-relative to B, if for any sequence of points $\{a_m\}$ of B tending to $0 \in \mathbb{R}^n$ such that $\lim_{m\to\infty} \frac{a_m}{\|a_m\|} \in D(A)$, there is a subsequence $\{m_j\}$ of $\{m\}$ and a sequence $\{b_{m_j}\} \subset A$ such that

$$\|a_{m_j} - b_{m_j}\| \ll \|a_{m_j}\|,\ \|b_{m_j}\|.$$

We have the following characterisation of condition (SSP). As mentioned in [6], the proof in the relative case is similar to the non-relative case, for which we gave a detailed proof in [5].

Lemma 2.5. *([6] Proposition 2.7) Let A, B be two set-germs at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A} \cap \overline{B}$. If A satisfies condition $(WSSP)$-relative to B, then it satisfies condition (SSP)-relative to B. Namely, the conditions relative (SSP) and relative $(WSSP)$ are equivalent.*

Below we give several examples of sets satisfying the condition (SSP). Consult [6] for more examples.

Remark 2.6. Let $A, B \subseteq \mathbb{R}^n$ be a set-germs at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A} \cap \overline{B}$, then the following hold:
(1) The cone $LD(A)$ satisfies condition (SSP),
(2) If A is subanalytic ([3]) or definable in some o-minimal structure ([5]), then it satisfies condition (SSP),
(3) If A is a finite union of sets, all of which satisfy condition (SSP), then A satisfies condition (SSP),
(4) If $0 \in A$, a C^1 manifold, then it satisfies condition (SSP) and $LD(A) = T_0(A)$ i.e., the tangent space of A at $0 \in \mathbb{R}^n$, (this is not necessarily true for C^0 manifolds or if $0 \notin A$),
(5)If $A \subseteq B, D(A) = D(B)$ and A satisfies condition (SSP), then B satisfies condition (SSP).

3. Directional Homeomorphism and some fundamental lemmas

In this section we introduce the notion of directional homeomorphism, and describe some fundamental properties of it.

3.1. **Directional homeomorphism.** In this subsection we describe the condition semiline-(SSP), and we use it to give some characterisations of the condition (SSP) and our definition of directional homeomorphism.

Definition 3.1. We say that a homeomorphism $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ satisfies condition *semiline*-(SSP), if $h(\ell)$ has a unique direction for all semilines ℓ.

Remark 3.2. Take a germ of a semiarc $\gamma : ([0, \epsilon), 0) \to (\mathbb{R}^n, 0)$ with a unique direction, say $\ell = LD(\gamma)$. (It is not difficult to see that in this case γ necessarily satisfies condition (SSP).) It follows that for a bi-Lipschitz homeomorphism $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ where h^{-1} satisfies condition semiline-(SSP), we do have that $h(\gamma)$ has also a unique direction, i.e., h also satisfies condition semiline-(SSP). Indeed, we can easily see that $LD(h(\gamma)) = LD(h(LD(\gamma))) = LD(h(\ell))$ is also a semiline. Let

$$\mathscr{SL} := \{\gamma : ([0,\epsilon),0) \to (\mathbb{R}^n,0) \mid LD(\gamma) \text{ is a semiline}\}.$$

The above argument implies that if h^{-1} satisfies condition semiline-(SSP), then the map

$$h : \mathscr{SL} \to \mathscr{SL}$$

induces a map $\overline{h} : S^{n-1} \to S^{n-1}$ defined by

$$\overline{h}(D(\gamma)) = D(h(\gamma)) \quad \text{for} \quad \gamma \in \mathscr{SL}.$$

If both h, h^{-1} satisfy condition semiline-(SSP), then $\overline{h} : S^{n-1} \to S^{n-1}$ is a one-to-one correspondence, in other words, $\overline{h} : S^{n-1} \to S^{n-1}$ is bijective.

Note that in the case where $\gamma : ([0,\epsilon),0) \to (\mathbb{C}^n,0),\ \gamma \in \mathscr{SL}$, we have that the complex cone $LD^*(\gamma) := LD(S^1 D\gamma)$ is a complex line, and all complex lines can be obtained in this way.

Theorem 3.3. *([6] Theorem 2.25) Let $h : (\mathbb{R}^n,0) \to (\mathbb{R}^n,0)$ be a bi-Lipschitz homeomorphism such that h (so h^{-1}) satisfies condition semiline-(SSP). Then the induced map $\overline{h} : S^{n-1} \to S^{n-1}$ given in Remark 3.2 extends to a bi-Lipschitz homeomorphism $\overline{h} : \mathbb{R}^n \to \mathbb{R}^n$, and for any $A \subset \mathbb{R}^n$ such that $0 \in \overline{A}$, we have*

$$\overline{h}(D(A)) = D(h(LD(A))) = D(h(A)) = D(\overline{h}(A)).$$

In particular, we have $\dim D(A) = \dim D(h(A))$.

Conversely the radial extension of a self bi-Lipschitz homeomorphism of the sphere S^{n-1} satisfies the condition semiline-(SSP). As we shall see below, there is a clear correspondence between these radial bi-Lipschitz homeomorphisms and the bi-Lipschitz homeomorphisms which satisfy condition semiline-(SSP). It is not difficult to see that the bi-Lipschitz semiline-(SSP) homeomorphisms preserve condition (SSP). We shall discuss a more general result in §4.

Remark 3.4. In particular the above property holds for any definable bi-Lipschitz homeomorphism, and for any subanalytic bi-Lipschitz homeomorphism.

Corollary 3.5. *Let A be a set-germ at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A}$, and let $h : (\mathbb{R}^n,0) \to (\mathbb{R}^n,0)$ be a bi-Lipschitz homeomorphism such that h (h^{-1}) satisfies condition semiline-(SSP). Then $LD(A)$ and $LD(h(A))$ are bi-Lipschitz homeomorphic.*

Let $h : (\mathbb{R}^n,0) \to (\mathbb{R}^n,0)$ be a Lipschitz homeomorphism. Then it is not difficult to see the following property holds for h.

Proposition 3.6. *The following conditions are equivalent.*

(1) $\forall y,\ \|y\| = 1,\ \exists \lim_{t\to 0_+} \frac{\|h(ty)\|}{t} := \alpha(y)$,
(2) $\forall x,\ \frac{x}{\|x\|} \to y,\ \exists \lim_{|x|\to 0} \frac{\|h(x)\|}{\|x\|} := \alpha(y)$.

Let us assume that a Lipschitz homeomorphism h satisfies the equivalent conditions of the proposition above. Then α is Lipschitz (as h is) and we have

$$\lim_{x\to 0, \frac{x}{\|x\|}\to y} \frac{\alpha(\frac{x}{\|x\|})\|x\|}{\|h(x)\|} = 1,$$

provided that h is not vanishing outside the origin.

Also if h satisfies condition semiline-(SSP), h induces $\overline{h}$ (as we know) and we have the following:

$$\lim_{x\to 0, \frac{x}{\|x\|}\to y} \frac{h(x)}{\|h(x)\|} = \overline{h}(y), \forall y \in S^{n-1}.$$

From now on we will use the bar notation, as $\overline{h}(x)$, for the corresponding extension

$$\overline{h}(tx) = tx, t \geq 0, \|x\| = 1.$$

In consequence, for a Lipschitz homeomorphism h which satisfies condition semiline-(SSP), we have the following structure, relating h and $\overline{h}$:

$$\lim_{x\to 0} \frac{\|\alpha(\frac{x}{\|x\|})\overline{h}(x) - h(x)\|}{\|x\|} = 0.$$

More generally, one can show all the above properties, even for h of the form $h(x) = \tau(x) + \mathrm{o}(x)$, where τ is merely Lipschitz and satisfies the following:

$$\forall y, \|y\| = 1, \exists \lim_{t\to 0_+} \frac{\|\tau(ty)\|}{t} := \alpha(y),$$

and $\lim_{x\to 0} \frac{\|\mathrm{o}(x)\|}{\|x\|} = 0$. After this, we use the notation $\mathrm{o}(x)$ as a mapping satisfying the limit condition. The above comments justify the next definition, inspired by the notion of weak diffeomorphism in [2] (see §4.2 for this notion).

Definition 3.7. A *directional homeomorphism* is a homeomorphism $h(x) = \tau(x) + \mathrm{o}(x)$, where τ is a bi-Lipschitz semiline-(SSP) homeomorphism.

Accordingly, for directional homeomorphisms we also have the remarkable decomposition

$$h(x) = \alpha(\frac{x}{\|x\|})\overline{\tau}(x) + \mathrm{o}(x),$$

with $\overline{\tau}$ defined as above. This kind of decomposition, in principle allows us to replace (when studying direction sets), a directional homeomorphism with a homeomorphism which is both positively homogeneous $\overline{\tau}(tx) = t\overline{\tau}(x), t \geq 0$, and norm preserving $\|\overline{\tau}(x)\| = \|x\|$. In a small neighbourhood of the origin, h and $\overline{h} = \overline{\tau}$ are homotopically equivalent.

3.2. **Fundamental lemmas.** In this subsection we describe some properties on homeomorphisms even weaker than directional homeomorphisms. Let $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ denote a homeomorphism which can be expressed as $h(x) = \tau(x) + \mathrm{o}(x)$, similar to a directional homeomorphism.

Lemma 3.8. *Let $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a homeomorphism, $h(x) = \tau(x) + \mathrm{o}(x)$, where τ is a Lipschitz homeomorphism, such that $\|x\| \approx \|\tau(x)\|$, and let $\{a_m\}$, $\{b_m\}$ be sequences of points of $\mathbb{R}^n$ tending to $0 \in \mathbb{R}^n$. Suppose that $\|a_m - b_m\| \ll \|a_m\|$. Then we have*

$$\|h(a_m) - h(b_m)\| \ll \|h(a_m)\|.$$

Proof. We first have

$$\|h(a_m) - h(b_m)\| \leq \|\tau(a_m) - \tau(b_m)\| + \|\mathrm{o}(a_m) - \mathrm{o}(b_m)\|.$$

Note that $\|a_m\| \approx \|\tau(a_m)\| \approx \|h(a_m)\|$. Therefore we see that

$$\frac{\|h(a_m) - h(b_m)\|}{\|h(a_m)\|} \precsim \frac{\|\tau(a_m) - \tau(b_m)\|}{\|\tau(a_m)\|} + \frac{\|\mathrm{o}(a_m) - \mathrm{o}(b_m)\|}{\|a_m\|} \to 0$$

as $m \to \infty$. It follows that

$$\|h(a_m) - h(b_m)\| \ll \|h(a_m)\|. \qquad \square$$

In the above lemma we do not assume that τ satisfies the condition semiline-(SSP).

We next give a useful lemma to show some of the main results on directional homeomorphisms.

Lemma 3.9. *Let $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a homeomorphism, $h(x) = \tau(x) + \mathrm{o}(x)$, such that $\|\tau(x)\| \geq C\|x\|$ in a neighbourhood of $0 \in \mathbb{R}^n$ for some $C > 0$ (thus so does h), and let $A \subset \mathbb{R}^n$ be a set-germ at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A}$. Then we have:*

(1) $D(h(A)) = D(\tau(A))$, and $h(A)$ satisfies condition (SSP) if and only if so does $\tau(A)$,

(2) $D(graph(h)) = D(graph(\tau))$, and $graph(h)$ satisfies condition (SSP) if and only if so does $graph(\tau)$.

Proof. We first show (1). Let $\{a_m\}$ be a sequence of points of A tending to $0 \in \mathbb{R}^n$ such that

$$a := \lim_{m\to\infty} \frac{h(a_m)}{\|h(a_m)\|} \in D(h(A)).$$

Then

$$\frac{h(a_m)}{\|h(a_m)\|} = \frac{\tau(a_m) + \mathrm{o}(a_m)}{\|\tau(a_m) + \mathrm{o}(a_m)\|}.$$

By assumption, we have $\|\tau(a_m)\| \gtrsim \|a_m\|$, therefore $\|\,\mathrm{o}(a_m)\| \ll \|\tau(a_m)\|$. This implies that

$$a = \lim_{m\to\infty} \frac{\tau(a_m)}{\|\tau(a_m)\|} \in D(\tau(A)).$$

It follows that $D(h(A)) \subset D(\tau(A))$. Since $\tau(x) = h(x) - \mathrm{o}(x)$, the opposite inclusion follows similarly and we have $D(h(A)) = D(\tau(A))$.

We can easily see the latter statement in (1) from the definition of (SSP) and the above arguments.

We next show (2). Let $\{a_m\}$ be a sequence of points of $\mathbb{R}^n$ tending to $0 \in \mathbb{R}^n$ such that

$$(a, b) := \lim_{m\to\infty} \frac{(a_m, h(a_m))}{\|(a_m, h(a_m))\|} \in D(graph(h)).$$

Since $\|\,\mathrm{o}(a_m)\| \ll \|\tau(a_m)\|$ as above, we have

$$(a, b) = \lim_{m\to\infty} \frac{(a_m, \tau(a_m) + \mathrm{o}(a_m))}{\|(a_m, \tau(a_m) + \mathrm{o}(a_m))\|} = \lim_{m\to\infty} \frac{(a_m, \tau(a_m))}{\|(a_m, \tau(a_m))\|} \in D(graph(\tau)).$$

It follows that $D(graph(h)) \subset D(graph(\tau))$. The opposite inclusion similarly follows as above, thus we have $D(graph(h)) = D(graph(\tau))$.

In order to show the latter statement of (2), let us assume that $graph(h)$ satisfies condition (SSP) at $(0, 0) \in \mathbb{R}^n \times \mathbb{R}^n$. Let $\{(a_m, b_m)\}$ be a sequence of points of $\mathbb{R}^n \times \mathbb{R}^n$ tending to $(0, 0) \in \mathbb{R}^n \times \mathbb{R}^n$ such that

$$\lim_{m\to\infty} \frac{(a_m, b_m)}{\|(a_m, b_m)\|} \in D(graph(\tau)) = D(graph(h)).$$

By assumption, there is a sequence of points $\{c_m\}$ of $\mathbb{R}^n$ tending to $0 \in \mathbb{R}^n$ such that

$$\|(c_m, h(c_m)) - (a_m, b_m)\| \ll \|c_m\| + \|\tau(c_m) + \mathrm{o}(c_m)\| \approx \|c_m\| + \|\tau(c_m)\|.$$

Therefore we have

$$\|(c_m, \tau(c_m) + \mathrm{o}(c_m)) - (a_m, b_m)\| \ll \|c_m\| + \|\tau(c_m)\|.$$

It follows that

$$\|(c_m, \tau(c_m)) - (a_m, b_m)\| \ll \|c_m\| + \|\tau(c_m)\|.$$

This means that $graph(\tau)$ satisfies condition (SSP). Since we can similarly show the converse, the latter statement of (2) follows. $\square$

4. Main Results

In this section we give the new results for directional homeomorphisms concerning the properties mentioned in our Introduction. We first recall, in each subsection, the corresponding results for bi-Lipschitz homeomorphisms shown in [6]. The results for bi-Lipschitz homeomorphisms, except the (SSP) structure preserving theorem, assume the (SSP) condition, in contrast to the results for directional homeomorphisms. Concerning the structure preserving theorem, we give a generalisation of the results in [6].

4.1. **Weak transversality theorem.**

This is an important notion with potential important applications in Algebraic Geometry, where the tangent cones are important invariants. Let us recall the notion.

Definition 4.1. Let A, $B \subset \mathbb{R}^n$ be set-germs at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A} \cap \overline{B}$. We say that A and B are *weakly transverse at* $0 \in \mathbb{R}^n$ if $D(A) \cap D(B) = \emptyset$ (if and only if $LD(A)$ and B are weakly transverse).

For a bi-Lipschitz homeomorphism, we have the following weak transversality theorem.

Theorem 4.2. *([6] Theorem 3.5) Let A, B be two set-germs at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A} \cap \overline{B}$, and let $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a bi-Lipschitz homeomorphism. Suppose that A or B satisfies condition (SSP), and $h(A)$ or $h(B)$ satisfies condition (SSP). Then A and B are weakly transverse at $0 \in \mathbb{R}^n$ if and only if $h(A)$ and $h(B)$ are weakly transverse at $0 \in \mathbb{R}^n$.*

By Remark 3.2 we have the following weak transversality theorem for a directional homeomorphism.

Theorem 4.3. *Let A, B be two set-germs at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A} \cap \overline{B}$, and let $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a directional homeomorphism. Then A and B are weakly transverse at $0 \in \mathbb{R}^n$ if and only if $h(A)$ and $h(B)$ are weakly transverse at $0 \in \mathbb{R}^n$.*

Note that we do not assume the condition (SSP) of "A or B" or of "$h(A)$ or $h(B)$" in the case of directional homeomorphism.

4.2. **(SSP) structure preserving theorem.**

As mentioned in the Introduction, we proved two types of (SSP) structure preserving theorems in [6]. Let us first recall those theorems.

Definition 4.4. Let $A \subset \mathbb{R}^m$ be a set-germ at $0 \in \mathbb{R}^m$ such that $0 \in \overline{A}$ and $B \subset \mathbb{R}^n$ a set-germ at $0 \in \mathbb{R}^n$ such that $0 \in \overline{B}$. Let $h : (A, 0) \to (B, 0)$ be a map-germ. We say that h is an (SSP) *map* if the graph of h satisfies condition (SSP) at $(0, 0) \in \mathbb{R}^m \times \mathbb{R}^n$.

We have an (SSP) structure preserving theorem for this (SSP) bi-Lipshitz homeomorphism.

Theorem 4.5. *([6] Theorem 4.7) Let $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be an (SSP) bi-Lipschitz homeomorphism, and let $A \subset \mathbb{R}^n$ be a set-germ at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A}$. Then A satisfies condition (SSP) if and only if $h(A)$ satisfies condition (SSP).*

In [6] we give a characterisation of (SSP) bi-Lipschitz homeomorphisms.

Proposition 4.6. *([6] Proposition 4.13(3)) Let $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a bi-Lipschitz homeomorphism. Then h is an (SSP) map if and only if $I_n \times h : (\mathbb{R}^n \times \mathbb{R}^n, 0 \times 0) \to (\mathbb{R}^n \times \mathbb{R}^n, 0 \times 0)$ (or $I_n \times h^{-1}$) satisfies condition semiline-(SSP). Here $I_n : \mathbb{R}^n \to \mathbb{R}^n$ is the identity map.*

Applying the above proposition to any semiline $\ell \subset \mathbb{R}^n$ as $\ell = \{0\} \times \ell$ $(\subset \mathbb{R}^n \times \mathbb{R}^n)$, we can see that an (SSP) bi-Lipschitz homeomorphism satisfies condition semiline-(SSP). Therefore an (SSP) bi-Lipschitz homeomorphism is a directional homeomorphism.

We call a homeomorphism $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ a *weak diffeomorphism*, if h and h^{-1} admit derivative (= linear approximation) at $0 \in \mathbb{R}^n$. Y.-N. Gau and J. Lipman [2] have proved the Zariski conjecture on hypersuface multiplicity even in the non-hypersurface case under the assumption that the homeomorphism is a weak diffeomorphism. The hypersurface case was implicitly shown in [1].

A weak diffeomorphism h can be expressed in a neighbourhood of $0 \in \mathbb{R}^n$ as follows:

$$h(x) = M_h(x) + \mathrm{o}(x),$$

where M_h is a regular linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$, and $\lim_{x\to 0} \frac{\|\mathrm{o}(x)\|}{\|x\|} = 0$ as in the previous section. Therefore a weak diffeomorphism is clearly a directional homeomorphism by definition.

We also have an (SSP) structure preserving theorem for weak diffeomorphisms.

Theorem 4.7. *([6] Corollary 4.20) Let $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a weak diffeomorphism, and let $A \subset \mathbb{R}^n$ be a set-germ at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A}$. Then A satisfies condition (SSP) if and only if $h(A)$ satisfies condition (SSP).*

We next show the following (SSP) structure preserving theorem for directional homeomorphisms, generalising Theorems 4.5 and 4.7.

Theorem 4.8. *Let $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a directional homeomorphism, and let $A \subset \mathbb{R}^n$ be a set-germ at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A}$. Then A satisfies condition (SSP) if and only if $h(A)$ satisfies condition (SSP).*

Proof. We assume that A satisfies condition (SSP). Since $h(x) = \tau(x) + \mathrm{o}(x)$ is a directional homeomorphism, let us apply the following lemma to this τ.

Lemma 4.9. *([6] Corollary 2.22) Let $\tau : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a bi-Lipschitz homeomorphism, and let $A \subset \mathbb{R}^n$ be a set-germ at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A}$. Suppose that A satisfies condition (SSP), and τ satisfies condition semiline-(SSP). Then $\tau(A)$ satisfies condition (SSP).*

By this lemma $\tau(A)$ satisfies condition (SSP). Then it follows from Lemma 3.9 that $h(A)$ also satisfies condition (SSP). The converse can be shown similarly. $\square$

4.3. **Important property of (SSP).** We first recall an important property concerning the direction set of the image of a set by a bi-Lipschitz homeomorphism.

Theorem 4.10. *([4] Lemma 5.6) Let A be a set-germ at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A}$, and let $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a bi-Lipschitz homeomorphism. Suppose that A satisfies condition (SSP). Then we have $D(h(LD(A))) = D(h(A))$.*

This result takes a very important role in the proof of the main theorem in [4]. On the other hand, this result does not always hold on a real closed field which is not a complete metric space ([5]).

By Theorem 3.3 we have the following property for directional homeomorphisms.

Theorem 4.11. *Let A be a set-germ at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A}$, and let $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a directional homeomorphism. Then we have $D(h(LD(A))) = D(h(A))$.*

Note that we do not assume the condition (SSP) of A in the case of directional homeomorphisms.

Using Lemma 3.9, we have a corollary of the above theorem.

Corollary 4.12. *Let A be a set-germ at $0 \in \mathbb{R}^n$ with $0 \in \overline{A}$ such that $LD(A) = \ell$ is a semiline, and let $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a directional homeomorphism. Then we have*

$$LD(h(A)) = LD(h(\ell))$$

is a semiline.

4.4. **Image of intersection sets.** For bi-Lipschitz homeomorphisms, we have the following directional property of intersection sets.

Theorem 4.13. *([6] Theorem 2.30) Let $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a bi-Lipschitz homeomorphism, and let U, $V \subset \mathbb{R}^n$ be set-germs at $0 \in \mathbb{R}^n$ such that $0 \in \overline{U} \cap \overline{V}$. Suppose that*

$$D(U \cap V) = D(U) \cap D(V),$$

and $U \cap V$ and $h(U)$ satisfy condition (SSP). Then $D(h(U \cap V)) = D(h(U)) \cap D(h(V))$.

This result has an application to a local classification of spirals (see [6] §5. Appendix). On the other hand, we have the following property for directional homeomorphisms.

Theorem 4.14. *Let $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a directional homeomorphism, and let U, $V \subset \mathbb{R}^n$ be set-germs at $0 \in \mathbb{R}^n$ such that $0 \in \overline{U} \cap \overline{V}$. Suppose that $D(U \cap V) = D(U) \cap D(V)$. Then $D(h(U \cap V)) = D(h(U)) \cap D(h(V))$.*

Proof. It suffices to show $D(h(U)) \cap D(h(V)) \subset D(h(U \cap V))$. Therefore we show the following equivalent condition

$$LD(h(U)) \cap LD(h(V)) \subset LD(h(U \cap V)).$$

Let $\ell \subset LD(h(U)) \cap LD(h(V))$. By Lemma 3.9 we have

$$\ell \subset LD(h(U)) = LD(\tau(U)), \quad \ell \subset LD(h(V)) = LD(\tau(V)).$$

Applying τ^{-1} to the above, we have

$$LD(\tau^{-1}(\ell)) \subset LD(U), \quad LD(\tau^{-1}(\ell)) \subset LD(V).$$

By assumption we have

$$LD(\tau^{-1}(\ell)) \subset LD(U) \cap LD(V) = LD(U \cap V).$$

Applying τ to the above, it follows from Lemma 3.9 that

$$\ell \subset LD(\tau(U \cap V)) = LD(h(U \cap V)).$$

Therefore we have $LD(h(U)) \cap LD(h(V)) \subset LD(h(U \cap V))$. □

Note that we do not assume the condition (SSP) of $U \cap V$ or of $h(U)$ in the case of directional homeomorphisms.

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STRATIFIED SUBMERSIONS AND CONDITION (D)

CLAUDIO MUROLO

To my friend David Trotman for his 60th birthday

Abstract. In this paper we investigate Goresky's Condition *(D)* for a stratified submersion between two Whitney stratifications. After revisiting the main results on Condition *(D)* of 1976 and 1981 due to Goresky, we give new equivalent properties[1] and two sufficient analytic conditions and their geometric meaning.

1. Introduction.

Let $f : M \to M'$ be a C^1 map between C^1 manifolds and $\mathcal{W} \subseteq M$ and $\mathcal{W}' \subseteq M'$ Whitney stratified sets such that the restriction $f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ is a stratified surjective submersion. Condition (D) for $f : M \to M'$ with respect to $\mathcal{W}$ and $\mathcal{W}'$ was originally introduced by M. Goresky in his Ph.D. Thesis (1976) as a convenient technical condition to define the singular substratified objects $\mathcal{W}$ allowed to represent the geometric chains and cochains of a Thom-Mather abstract stratified space $\mathcal{X}$ (**[5]** 2.3 and 4.1) in the aim of introducing nice geometric homology and cohomology theories.

Condition (D) for $f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ at $x \in X \subseteq \overline{Y}$ (where $X < Y$ are strata of $\mathcal{W}$, see §2.2 for the definition) roughly speaking means that for every stratum Y of $\mathcal{W}$, the surjective differential map $f_{Y*} : TY \to TY'$ extends to a surjective map (see Remark 3.7) $f_{*x|C_xY} : C_xY \to C_{x'}Y'$ between the *Nash* tangent cones C_xY and $C_{x'}Y'$ (where $C_xY = \sqcup_{\{y_i\}_i \to x} \lim_i T_{y_i}Y$ is analogous in the real case to the Whitney tangent cone $C_4(Y, x)$ **[21]**).

1.1. ***Historical motivations.*** Using an appropriate definition of stratified cycles (Definition 2.4) Goresky proves that every abstract stratified cycle in a manifold is cobordant to one which is radial on M and that, thanks to the condition (D), this last admits a Whitney cellularisation (**[5]** 3.7).

This result is the main step in proving his important theorems on the bijective representability of the homology of a C^1 *manifold* M by its geometric stratified cycles and of the cohomology of an arbitrary Thom-Mather abstract stratified set (**[5]** 2.4 and 4.5).

For a Whitney stratification $\mathcal{X} = (A, \Sigma)$, in 1981 **[6]** Goresky redefines his geometric homology and cohomology theories using only Whitney (that is (b)-regular) substratified cycles and cocycles of $\mathcal{X}$, denoting them in this case $WH_k(\mathcal{X})$ and $WH^k(\mathcal{X})$, without assuming this time the condition (D) in their definition. With these new definitions and replacing the terminology (but essentially not the meaning) *"radial"* by *"with conical singularities"* (**[6]**, Appendices 1, 2, 3) Goresky again proves the bijectivity of his homology and cohomology representation maps:

Theorem 1.1. *If $\mathcal{X} = (M, \{M\})$ is the trivial stratification of a compact C^1 manifold, the homology representation map $R_k : WH_k(\mathcal{X}) \to H_k(M)$ is a bijection.*

Proof. **[6]** Theorem 3.4. ☐

Key words and phrases. Stratified sets and maps, Whitney Conditions, regular cellularisations.

[1] Used in **[16]** to give a new proof of the (b)-regularity of stratified mapping cylinders needed to Goresky in 1978 to prove a theorem of Whitney cellularisation of Whitney stratifications with conical singularities.

Theorem 1.2. *If $\mathcal{X} = (A, \Sigma)$ is a compact Whitney stratified space, the cohomology representation map $R^k : WH^k(\mathcal{X}) \to H^k(A)$ is a bijection.*

Proof. **[6]** Theorem 4.7. □

Later such geometric theories were improved by the author of the present paper by introducing a sum operation in $WH_k(M)$ and $WH^k(\mathcal{X})$ geometrically meaning transverse union of stratified cycles **[14, 15]**.

1.2. ***Problems related to condition*** (D)***.*** Although in the revised theory of 1981 **[6]**, condition (D) was not assumed in the definitions of the Whitney cycles and cocycles, it was once again the main tool to obtain the two important representation theorems, through a strategy of *using Condition* (D) *in order to construct Whitney cellularisations of Whitney stratifications with conical singularities* using stratified mapping cylinders whose (b)-regularity is obtained through the condition (D) (**[6]**, App. 1,2,3). We give a short survey of this in §2.2.

We underline here that in the homology case the main result, that $R_k : WH_k(\mathcal{X}) \to H_k(M)$ is a bijection, was established *only* when $\mathcal{X} = (M, \{M\})$ is a trivial stratification of a compact manifold M and that the complete homology statement for $\mathcal{X}$ an arbitrary compact (b)-regular stratification remains a famous problem of Goresky which is still unsolved (**[5]** p. 52, **[6]** p. 178):

Conjecture 1.1. *If $\mathcal{X} = (A, \Sigma)$ is a compact Whitney stratified space the homology, representation map $R_k : WH_k(\mathcal{X}) \to H_k(A)$ is a bijection.*

The proof of this conjecture would follow as a corollary if one could prove the following:

Conjecture 1.2. *Every compact Whitney stratified space $\mathcal{X}$ admits a Whitney cellularisation.*

This would be also a first important step of a possible proof of the celebrated conjecture:

Conjecture 1.3. *Every compact Whitney stratified space $\mathcal{X}$ admits a Whitney triangulation.*

Let us recall that in 2005 M. Shiota proved that semi-algebraic sets admit a Whitney triangulation **[18]** and in 2012 M. Czapla gave new proof of this result **[2]** as a corollary of a more general triangulation theorem for definable sets. On the other hand, our motivation being the applications to Goresky's geometric homology theory, we are interested in the stronger Conjectures 1.2 and 1.3 for stratifications having C^1 strata.

In 1978 Goresky also proved an important triangulation theorem for compact Thom-Mather stratified sets **[7]** whose proof (based on a double inductive step) can be used to obtain a Whitney cellularisation of a Whitney stratification provided that one knows how to obtain <u>Whitney</u> stratified mapping cylinders. Goresky used this idea *based on Condition* (D) for Whitney stratifications having *only conical singularities* (see Proposition 2.4) for which he gave a solution of Conjecture 1.2 and deduced as applications the proof of Theorems 1.1 and 1.2.

The strategy of Goresky could be used for an approach to a more general solution of Conjecture 1.2. In this context it is clear that Goresky's condition (D) might play an important role in answering affirmatively Conjecture 1.4 and in solving the famous conjectures 1.1 and 1.3.

1.3. ***Content of the paper.*** In §2.1 we review quickly some basic notions about the most important regular stratifications concerned by this paper: the Whitney (b)-regular stratifications **[21]** and the abstract stratified sets of Thom-Mather **[9, 10, 19]**. Then in §2.2 we introduce the definition of condition (D) for a stratified submersions $f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ as a technical tool to obtain (b)-regularity of stratified mapping cylinders and we recall all results of Goresky of 1976-81 **[5, 7]** necessary to prove that: *"Every Whitney stratification with conical singularities* *and conical control data admits a Whitney cellularisation"* (Proposition 2.4) which is a partial solution of Conjecture 1.2.

In §3.1, we analyze what condition (D) means for a C^1 submersion $f : M \to M'$ between C^1 manifolds at a regular point $y_0 \in M$. First we remark that submersivity can be interpreted as the $C^{0,1}$-regularity of the foliation defined by the fibres of f (from Proposition 3.5 to Corollary 3.2).

When $Y \subseteq M$ are riemannian manifolds, we show that the submersivity at $y_0 \in Y$ of the restriction $f_Y : Y \to Y'$ is equivalent to the continuity at y_0 of the *canonical distribution* $\mathcal{D}(y) = \perp(\ker f_{Y*y}, T_y Y)$ (Proposition 3.6).

Then we introduce two test functions h_Y and H_Y (Definition 3.5) given by the minimum and the maximum norm of the isomorphism $f_{Y*y|\mathcal{D}(y)} : \mathcal{D}(y) \to T_{y'}Y'$ and its inverse isomorphism $f^{-1}_{Y*y|\mathcal{D}(y)} : T_{y'}Y' \to \mathcal{D}(y)$, such that $\lim_{y\to y_0} h_Y(y)$ and $\lim_{y\to y_0} H_Y(y)$ characterize the submersivity of f_Y at y_0 (Proposition 3.7).

Finally in §2.2, thanks to this, we prove that submersivity at y_0 is also equivalent to the property "$f_{*y_0}(\lim_{y_i\to y_0} \mathcal{D}(y_i)) \supseteq \lim_i f_{*y_i}(\mathcal{D}(y_i))$" and to Condition (D) for f_Y at y_0, interpreted as stratified map defined on the stratification $Y - \{y_0\} \sqcup \{y_0\}$ (Proposition 3.8).

This preliminary analysis of §3 is necessary in introducing the results of §4.

In §4 we give the main results of this paper.

First in §4.1 we investigate the technical, geometric and analytic content of condition (D) at a point $x \in X < Y$ (X, Y being two strata of $\mathcal{W}$) for a general stratified submersion $f : \mathcal{W} \to \mathcal{W}'$ between two Whitney stratifications.

In Theorem 4.3 we prove that, in the context of stratified spaces, condition (D) at $x \in X < Y$ is equivalent to the key property (which is the most important technical content of Condition (D)):

"For every $\{y_i\}_i \subseteq Y$ *such that* $\lim_i y_i = x \in X$, *every* $v' \in \lim_i T_{y_i}Y$ *can be written as a limit* $\lim_i v'_i = v'$ *of a sequence* $\{v'_i \in T_{f(y_i)}f(Y)\}_i$ *having a bounded sequence of preimages* $\{w_i \in f^{-1}_{*y_i}(v'_i) \subseteq T_{y_i}Y\}_i$*"*

and it is again equivalent to the property of transforming "continuously" the limits of the canonical distributions: $f_{*x}(\lim_{y_i\to x} \mathcal{D}(y_i)) \supseteq \lim_{y_i\to x} f_{*y_i}(\mathcal{D}(y_i))$.

The author of the present paper used this properties in **[16]**, when $f_{\mathcal{W}} = \pi_{XY|\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ is the restriction of a projection $\pi_{XY} : S^{\epsilon}_{XY} \to X$, to give a different proof of the *essential* result of Goresky (Proposition 2.2) that *"Stratified mapping cillynders with conical singularities admit a (b)-regular natural stratification"*; the property which allow to prove the important Whitney Cellularisation Theorem (Proposition 2.4) recalled above.

In Theorem 4.4 and Corollary 4.3 we prove that the analytic conditions $\liminf_{y\to x} h_Y(y) > 0$ and $\liminf_{y\to x} H_Y(y) < +\infty$ are sufficient for condition (D) at $x \in X < Y$.

In §4.2 for U, V two vector subspaces of an Euclidian vector space E, we use the usual "distance" functions $\delta(u, V)$ and $\delta(U, V)$ ($u \in E$) to define the *essential minimal distance* $\delta'(U, V)$ between U and V, as the sinus of the *minimum essential angle* $\alpha(U, V)$ between two *essential mutual subspaces* U', V' of U and V and we prove some useful properties of $\delta(u, V)$, $\delta(U, V)$ and $\delta'(U, V)$.

In §4.3 using this new "distance" function $\delta'(U, V)$ we introduce two new *geometric* test functions δ_Y (intrinsic by x) and $\delta_{Y,x}$ (depending on x) for Condition (D) at $x \in X < Y$.

In Theorem 4.5 and Corollary 4.4 we prove, when $f : M \to M'$ is a submersion at x, equivalence between the more geometric condition $\liminf_{y\to x} \delta_Y(y) > 0$ and the analytic condition

$\liminf_{y\to x} h_Y(y) > 0$ (or $\limsup_{y\to x} H_Y(y) < +\infty$) and thanks to this that $\liminf_{y\to x} \delta_Y(y) > 0$ becomes a sufficient condition for Condition (D) at $x \in X < Y$ (Corollary 4.5).

After making precise relations between δ_Y and $\delta_{Y,x}$ (Propositions 4.9 and 4.10) we find that the analogous results of Theorem 4.5 and Corollary 4.4 hold by considering the function $\delta_{Y,x}$ instead of δ_Y (Theorem 4.6 and Corollary 4.6).

We conclude the section by explaining (by two examples) the geometric meaning of the sufficient conditions $\liminf_{y\to x} \delta_Y(y) > 0$ and $\liminf_{y\to x} \delta_{Y,x}(y) > 0$.

2. Stratified Spaces and Maps and Condition (D).

A *stratification* of a topological space A is a locally finite partition Σ of A into C^1 connected manifolds (called the *strata* of Σ) satisfying the *frontier condition*: if X and Y are disjoint strata such that X intersects the closure of Y, then X is contained in the closure of Y. We write then $X < Y$ and $\partial Y = \sqcup_{X<Y} X$ so that $\overline{Y} = Y \sqcup \left(\sqcup_{X<Y} X\right) = Y \sqcup \partial Y$ and $\partial Y = \overline{Y} - Y$ ($\sqcup$ = disjoint union). The pair $\mathcal{X} = (A, \Sigma)$ is called a *stratified space* with *support* A and *stratification* Σ.

A *stratified map* $f : \mathcal{X} \to \mathcal{X}'$ between stratified spaces $\mathcal{X} = (A, \Sigma)$ and $\mathcal{X}' = (B, \Sigma')$ is a continuous map $f : A \to B$ which sends each stratum X of $\mathcal{X}$ into a unique stratum X' of $\mathcal{X}'$, such that the restriction $f_X : X \to X'$ is C^1.

A *stratified submersion* is a stratified map f such that each $f_X : X \to X'$ is a C^1 submersion.

2.1. ***Regular Stratified Spaces and Maps.*** Extra regularity conditions may be imposed on the stratification Σ, such as to be an *abstract stratified set* in the sense of Thom-Mather **[9, 10, 19]** or, when A is a subset of a C^1 manifold, to satisfy conditions (a) or (b) of Whitney **[21]**, or (c) of K. Bekka **[1]** or, when A is a subset of a C^2 manifold, to satisfy conditions (w) of Kuo-Verdier **[22]**, or (L) of Mostowski **[17]**.

In this paper we will consider essentially Whitney ((b)-regular) stratifications so called because they satisfy Condition (b) of Whitney (1965, **[21]**).

Definition 2.1. Let Σ be a stratification of a subset $A \subseteq \mathbb{R}^N$, $X < Y$ strata of Σ and $x \in X$.

One says that $X < Y$ *is (b)-regular* (or that it satisfies *Condition (b) of Whitney*) at x if for every pair of sequences $\{y_i\}_i \subseteq Y$ and $\{x_i\}_i \subseteq X$ such that $\lim_i y_i = x \in X$ and $\lim_i x_i = x$ and moreover $\lim_i T_{y_i} Y = \tau$ and $\lim_i [y_i - x_i] = L$ in the appropriate Grassmann manifolds (here $[v]$ denotes the vector space spanned by v) then $L \subseteq \tau$.

The pair $X < Y$ *is called (b)-regular* if it is (b)-regular at every $x \in X$.

Σ is called a *(b)-regular* (or a *Whitney)* stratification if all $X < Y$ in Σ are (b)-regular.

Most important properties of Whitney stratifications follow because they are in particular abstract stratified sets **[9, 10]**.

Definition 2.2. (Thom-Mather 1970) Let $\mathcal{X} = (A, \Sigma)$ be a stratified space.

A family $\mathcal{F} = \{(\pi_X, \rho_X) : T_X \to X \times [0, \infty[)\}_{X\in\Sigma}$ is called *a system of control data* of $\mathcal{X}$ if for each stratum $X \in \Sigma$ we have that:

(1) T_X is a neighbourhood of X in A (called *tubular neighbourhood of* X);
(2) $\pi_X : T_X \to X$ is a continuous retraction of T_X onto X (called *projection on* X);
(3) $\rho_X : T_X \to [0, \infty[$ is a continuous function such that $X = \rho_X^{-1}(0)$ (called the *distance from* X);

and, furthermore, for every pair of adjacent strata $X < Y$, by considering the restriction maps $\pi_{XY} := \pi_{X|T_{XY}}$ and $\rho_{XY} := \rho_{X|T_{XY}}$, on the subset $T_{XY} := T_X \cap Y$, we have that:

5) the map $(\pi_{XY}, \rho_{XY}) : T_{XY} \to X \times]0, \infty[$ is a C^1 submersion (then $\dim X < \dim Y$);
6) for every stratum Z of $\mathcal{X}$ such that $Z > Y > X$ and for every $z \in T_{YZ} \cap T_{XZ}$ the following *control conditions* are satisfied:
i) $\pi_{XY}\pi_{YZ}(z) = \pi_{XZ}(z)$ (called the *π-control condition*)
ii) $\rho_{XY}\pi_{YZ}(z) = \rho_{XZ}(z)$ (called the *ρ-control condition*).

In what follows for every $\epsilon > 0$ we will pose $T^{\epsilon}_X := T_X(\epsilon) = \rho^{-1}_X([0, \epsilon[)$, $S^{\epsilon}_X := S_X(\epsilon) = \rho^{-1}_X(\epsilon)$, and $T^{\epsilon}_{XY} := T^{\epsilon}_X \cap Y$, $S^{\epsilon}_{XY} := S^{\epsilon}_X \cap Y$ and without loss of generality will assume $T_X = T_X(1)$ **[9,10]**.

The pair $(\mathcal{X}, \mathcal{F})$ is called an *abstract stratified set (ASS)* if A is Hausdorff, locally compact and admits a countable basis for its topology. Since one usually works with a unique system of control data $\mathcal{F}$ of $\mathcal{X}$, in what follows we will omit $\mathcal{F}$.

If $\mathcal{X}$ is an abstract stratified set, then A is metrizable and the tubular neighbourhoods $\{T_X\}_{X\in\Sigma}$ may (and will always) be chosen such that: "$T_{XY} \neq \emptyset \Leftrightarrow X \leq Y$" and

$$\text{“}T_X \cap T_Y \neq \emptyset \Leftrightarrow X \leq Y \text{ or } X \geq Y\text{”}$$

(where both implications $\Leftarrow$ automatically hold for each $\{T_X\}_X$) as in **[9, 10]**, pp. 41-46.

The notion of system of control data of $\mathcal{X}$, introduced by Mather, is very important because it allows one to obtain good extensions of (stratified) vector fields **[9, 10]** which are the fundamental tool in showing that a stratified (controlled) submersion $f : \mathcal{X} \to M$ into a manifold, satisfies Thom's First Isotopy Theorem: the stratified version of Ehresmann's fibration theorem **[3,9,10,19]**.

Moreover by applying it to the projections $\pi_X : T_X \to X$ it follows in particular that $\mathcal{X}$ has a *locally trivial structure* and also a locally trivial topologically conical structure.

This fundamental property allows moreover to prove that ASS are triangulable spaces **[7]**.

Since Whitney (b)-regular) stratifications are ASS, they are locally trivial and triangulable.

2.2. ***Condition*** (D) ***and Goresky's results.*** The following definition was introduced by Goresky first in his Ph.D. Thesis **[5]** (1976) and later in **[6]** (1981).

Definition 2.3. Let $f : M \to M'$ be a C^1 map between C^1 manifolds and $\mathcal{W} \subseteq M$ and $\mathcal{W}' \subseteq M'$ Whitney stratifications such that the restriction $f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ is a <u>surjective</u> stratified submersion (so f takes each stratum Y of $\mathcal{W}$ to only one stratum $Y' = f(Y)$ of $\mathcal{W}' = f(\mathcal{W})$).

One says that $f : M \to M'$ *satisfies condition* (D) *with respect to* $\mathcal{W}$ *and* $\mathcal{W}'$ and we will say for short that the restriction $f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ *satisfies the condition* (D) if the following holds:

for every pair of adjacent strata $X < Y$ of $\mathcal{W}$ and every point $x \in X$ and every sequence $\{y_i\}_i \subseteq Y$ such that $\lim_i y_i = x \in X$, $\lim_i T_{y_i}Y = \tau$ and $\lim_i T_{f(y_i)}Y' = \tau'$ in the appropriate Grassmann manifolds, then $f_{*x}(\tau) \supseteq \tau'$. Starting from now we will write this for short by:

$$f_{*x}(\lim_i T_{y_i}Y) \;\supseteq\; \lim_i T_{f(y_i)}Y' \,.$$

and we will extend this notation also to some other limits of subspaces of the $\{T_{y_i}Y\}_i$.

Later on we will also consider given, with the obvious restricted meaning of the definition 2.3, what one intends by: *"$f : M \to M'$ satisfies condition (D) with respect to $X < Y$"* and *"$f : M \to M'$ satisfies condition (D) with respect to $X < Y$ at $x \in X$"* (*"at $x \in X < Y$"*).

In the whole of the paper we will denote $Y' = f(Y)$ and $y' = f(y)$, for every $y \in Y$.

Example 2.1. Let M be the horizontal plane $M = \{z = 1\} \subseteq \mathbb{R}^3$, $M' = L(0, 1, 0) = y$-axis in $\mathbb{R}^3$ and $f : M \to M'$ the standard projection $f(x, y, z) = y$.

Let $\mathcal{W} = (W, \Sigma_{\mathcal{W}})$ be the stratified space with support $W = \{y = tan(x) \; : \; x \geq 0\} \cap M$ the half graph of the tangent map in M and stratification $\Sigma_{\mathcal{W}} = \{R, S\}$ where $R = \{(0,0,1)\}$ and $S = W \cap \{x > 0\}$. Then $R < S$.

Let $\mathcal{W}'$ be the stratified space with support the half y-axis, $W' = M' \cap \{y \geq 0\}$ in M' and stratification $\Sigma_{\mathcal{W}'} = \{R', S'\}$ where $R' = \{(0,0,0)\}$ and $S' = M' \cap \{y > 0\}$. Then $R' < S'$.

Then $f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ satisfies condition (D) at $(0,0,1) \in R < S$.

If $\mathcal{W} = (W, \Sigma_{\mathcal{W}})$ is as above but taking now for W the half parabola $W = \{y = x^2, x \geq 0\} \cap M$ in M, then $f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ does not satisfy condition (D) at $(0,0,1) \in R < S$. $\square$

Figures 1 and 2 below represents both cases of Example 2.1. In figure 1, $f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ satisfies condition (D) at $(0,0,1) \in R < S$ while in figure 2 it does not.

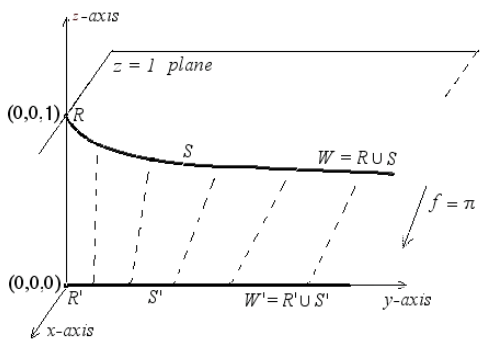

Figure 1

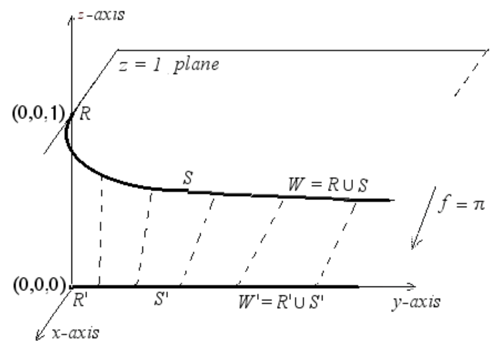

Figure 2

An important example in which condition (D) holds is the case of cellular maps **[5]**, **[16]**:

Proposition 2.1. *Let $f : M \to M'$ be a surjective C^1 submersion and h and h' two smooth cellularisations of two subsets $\mathcal{K} \subseteq M$ and $\mathcal{K}' \subseteq M'$ making the following diagram*

$$\begin{array}{ccc} \mathcal{H} & \stackrel{h}{\to} & \mathcal{K} \subseteq M \\ & & \\ g \downarrow & & \downarrow f \\ & & \\ \mathcal{H}' & \stackrel{h'}{\to} & \mathcal{K}' \subseteq M'. \end{array}$$

commutative where $g : \mathcal{H} \to \mathcal{H}'$ is a cellular map of cellular complexes.

Then $f_{\mathcal{K}} : \mathcal{K} \to \mathcal{K}'$ satisfies condition (D). $\square$

In 1976 Goresky used condition (D) to define a convenient class of stratified subspaces $\mathcal{W} \subseteq \mathcal{X}$ of a Thom-Mather ASS $\mathcal{X} = (A, \Sigma)$ equipped with a system of control data

$$\mathcal{F} = \{(\pi_X, \rho_X) : T_X^1 \to X \times [0, \infty[\}_{X \in \Sigma}$$

[9, 10] and a *family of lines* of $\mathcal{X}$, $\mathcal{R} = \{r_X^\epsilon : T_X^1 - X \to S_X^\epsilon\}_{X \in \Sigma, \epsilon \in]0,\delta[}$ **[7]** retracting every tubular neighbourhood $T_X^1 - X$ on its ϵ-sphere S_X^ϵ.

Definition 2.4. (**[5]** 2.3.2). Let $\mathcal{X}$ be a Thom-Mather ASS, equipped with a fixed system of control data $\mathcal{F}$ and a family of lines $\mathcal{R}$ and denote, for every stratum X of $\mathcal{X}$, by C_X^o the open cone operator associated to $\mathcal{R}$, that is: $C_X^o(Q) = {r_X^\epsilon}^{-1}(Q)$ for every $Q \subseteq S_X^\epsilon$.

A Thom-Mather ASS $\mathcal{W} \subseteq \mathcal{X}$ is called a *substratified object of* $\mathcal{X}$ and one says that $\mathcal{W}$ *follows the lines of* $\mathcal{X}$ if the following hold:

(1) Each stratum R of $\mathcal{W}$ is a submanifold of a stratum X of $\mathcal{X}$.
(2) For each stratum X of $\mathcal{X}$, $\mathcal{W} \cap X$ satisfies Whitney's condition (b).
(3) For each stratum X of $\mathcal{X}$, there exists $\epsilon > 0$ such that $\mathcal{W} \cap (T_X^\epsilon - X) = C_X^o(\mathcal{W} \cap S_X^\epsilon)$.
(4) If X is a stratum of $\mathcal{X}$, there exists $\epsilon > 0$ such that $\pi_{\mathcal{W}\cap S_X^\epsilon} : \mathcal{W} \cap S_X^\epsilon \to \mathcal{W} \cap X$ is a stratified submersion which satisfies condition (D).

Goresky commented on property 4) above as follows: *"Condition (D) is used in section 6.4 to guarantee that certain intersections of substratified objects will be substratified objects. It can be weakened considerably and perhaps omitted completely although this would necessitate considerably more technical analysis when intersections of substratified objects are considered".*

Later in 1981 Goresky redefined his geometric homology $WH_k(\mathcal{X})$ and cohomology $WH^k(\mathcal{X})$ (this time only) for a Whitney stratification $\mathcal{X}$ without asking that the substratified objects representing cycles and cocycles of $\mathcal{X}$ satisfy condition (D) above (**[6]** §3 and §4).

The main reason for which Goresky introduced Condition (D) in 1981 was that it allows one to obtain Condition (b) for the natural stratifications on the mapping cylinder of the stratified submersion:

Proposition 2.2. *Let $\pi : E \to M'$ be a C^1 riemannian vector bundle and $M = S_{M'}^\epsilon$ the ϵ-sphere bundle of E. If $\mathcal{W} \subseteq M$, $\mathcal{W}' = \pi(\mathcal{W}) \subseteq M'$ are two Whitney stratifications such that $\pi_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ is a stratified submersion which satisfies condition (D), then the closed stratified mapping cylinder*

$$C_{\mathcal{W}'}(\mathcal{W}) = \bigsqcup_{Y \subseteq \mathcal{W}} \left[(C_{\pi_{\mathcal{W}}(Y)}(Y) - \pi_{\mathcal{W}}(Y)) \sqcup \pi_{\mathcal{W}}(Y) \sqcup Y \right]$$

is a Whitney (i.e. (b)-regular) stratified space.

Proof. **[6]** Appendix A.1 or **[16]** for a different proof. □

Then, in order to use it together with Proposition 2.3 below:

Proposition 2.3. *Every Whitney stratification $\mathcal{W}$ in a manifold M can be deformed to a Whitney stratification $\mathcal{W}'$ having conical singularities.*

Proof. **[6]** Appendix A.3. Proposition. □

Goresky proved that:

Proposition 2.4. *Every Whitney stratified space $\mathcal{X}$ with conical singularities and conical control data admits a Whitney cellularisation.*

Proof. **[7]** Appendix A.2. Proposition. □

Proposition 2.4 gives hence a partial solution of Conjecture 1.2 in the introduction and suggests moreover new ideas for an approach to his general solution.

Proposition 2.4 was thus also the main tool which allowed Goresky to prove his two homology representation theorems, Theorem 1.1 and Theorem 1.2, recalled in the introduction.

A detailed account of condition (D), containing a finer analysis, new proofs and equivalent properties of Goresky's results is given in **[16]**.

3. $C^{0,1}$-Regular foliations and condition (D) for C^1 maps.

3.1. ***Regular foliations from C^1 maps.*** In this section we clarify some simple properties of C^1 maps that will be useful in §4.

Remark 3.1. *Let $f : M \to M'$ be a C^1 map between C^1 manifolds, $y_0 \in M$ and $\{y_i\}_i \subseteq M$ a sequence such that $\lim_i y_i = y_0$.*

1) For every sequence of vectors $\{v_i \in \ker f_{*y_i}\}_i$ such that $\lim_i v_i = v_0$ one has $v_0 \in \ker f_{*y_0}$.

2) If, in an appropriate Grassmann manifold, there exists

$$\lim_i \ker f_{*y_i} = \tau,$$

then $\tau \subseteq \ker f_{*y_0}$ (starting from now we will write this for short by: "$\lim_i \ker f_{*y_i} \subseteq \ker f_{*y_0}$").

Proof. Since f is C^1 one obviously has: $f_{*y_0}(v_0) = f_{*y_0}(\lim_i v_i) = \lim_i f_{*y_i}(v_i) = 0$. □

The opposite inclusion $\lim_i \ker f_{*y_i} \supseteq \ker f_{*y_0}$ would follow immediately when two such vector spaces have the same dimension. This happens when f is a submersion:

Proposition 3.5. *Let $f : M \to M'$ be a C^1 submersion on $M - \{y_0\}$ for a point $y_0 \in M$.*

Then the following conditions are equivalent:

1) $f : M \to M'$ is a submersion at y_0;

2) For all $\{y_i\}_i \subseteq M - \{y_0\}$ converging to y_0 there exists $\lim_i \ker f_{*y_i}$ and

$$\lim_i \ker f_{*y_i} = \ker f_{*y_0}.$$

This means that the map $\mathcal{K} : M \longrightarrow \mathbb{G}_k(TM)$, $\mathcal{K}(y) := \ker f_{*y}$ is continuous.

3) For all $\{y_i\}_i \subseteq M - \{y_0\}$ converging to y_0 there exists $\lim_i \ker f_{*y_i}$ and

$$\lim_i \ker f_{*y_i} \supseteq \ker f_{*y_0}.$$

Proof. Since f is a C^1 submersion at $M - \{y_0\}$, for every $y_i \in M - \{y_0\}$, if $y'_i = f(y_i)$, the fibre $f^{-1}(y'_i)$ is a C^1 manifold of dimension $k = \dim M - \dim M'$ such that $T_{y_i} f^{-1}(y'_i) = \ker f_{*y_i}$.

In particular, for every $i \in \mathbb{N}$, $\dim \ker f_{*y_i} = k$.

$(1 \Rightarrow 2)$. Let $\{\ker f_{*y_{i_h}}\}_h$ an arbitrary converging subsequence of the sequence $\{\ker f_{*y_i}\}_i$.

If f is a submersion at y_0, then $f^{-1}(y'_0)$ is a C^1 k-manifold too with tangent spaces

$$T_{y_0} f^{-1}(y'_0) = \ker f_{y_0 *}$$

and $\dim \ker f_{*y_0} = k = \dim \lim_h \ker f_{*y_{i_h}}$.

Since f is a C^1 map, $\lim_h \ker f_{*y_{i_h}} \subseteq \ker f_{*y_0}$ (Remark 3.1) and having both the same dimension k they coincide: $\lim_h \ker f_{*y_{i_h}} = \ker f_{*y_0}$.

All converging subsequences of the sequence $\{\ker f_{*y_i}\}_i$ have then the same limit $\ker f_{*y_0}$ in the Grassmann compact manifold and hence there exists $\lim_i \ker f_{*y_i}$ and

$$\lim_i \ker f_{*y_i} = \ker f_{*y_0}.$$

$(2 \Rightarrow 3)$. Obvious.

$(3 \Rightarrow 1)$. If $\lim_i \ker f_{*y_i} \supseteq \ker f_{*y_0}$, then, for every i, $\dim \ker f_{*y_0} \leq \dim \ker f_{*y_i}$ and by codimension $\dim Im\, f_{*y_0} \geq \dim Im\, f_{*y_i}$. Thus again f being a submersion at y_i one has:

$$\dim Im\, f_{*y_0} \ \geq\ \dim Im\, f_{*y_i} \ =\ \dim T_{y'_i} M' \ =\ \dim T_{y'_0} M'$$

and, since $Im\, f_{*y_0} \subseteq T_{y'_0} M'$, then necessarily $Im\, f_{*y_0} = T_{y'_0} M'$ and f is a submersion at y_0. □

With the same hypotheses and proof of the proposition 3.5 one has:

Remark 3.2. *The following conditions are equivalent:*

1) $f : M \to M'$ *is a submersion at* y_0*;*

2) For all $\{y_i\}_i \subseteq M - \{y_0\}$ *converging to* y_0 *there exists* $\lim_i \ker f_{*y_i}$ *and*

$$\dim \lim_i \ker\, f_{*y_i} = \dim \ker\, f_{*y_0};$$

3) For all $\{y_i\}_i \subseteq M - \{y_0\}$ *converging to* y_0 *there exists* $\lim_i \ker f_{*y_i}$ *and*

$$\dim \lim_i \ker\, f_{*y_i} \geq \dim \ker\, f_{*y_0}. \ \square$$

Corollary 3.1. *If* $f : M \to M'$ *is a* C^1*-submersion, the foliation of* M *defined by* $\mathcal{F} = \{M_y = f^{-1}(y')\}_{y \in M}$*, where* $y' = f(y)$*, is* $C^{0,1}$*-regular. I.e. for every sequence* $\{y_i\}_i \subseteq M$

$$\lim_i y_i = y_0 \Longrightarrow \lim_i T_{y_i} M_{y_i} = T_{y_0} M_{y_0}.$$

Proof. Since f is a C^1 submersion on M, for every $y_i \in M$, $f^{-1}(y_i')$ is a C^1 manifold of dimension $k = \dim M - \dim M'$ and $T_{y_i} f^{-1}(y') = \ker\, f_{*y_i}$. Then, by Proposition 3.5:

$$\lim_i T_{y_i} M_{y_i} \ = \ \lim_i \ker\, f_{*y_i} \ = \ \ker\, f_{y_0 *} \ = \ T_{y_0} M_{y_0}. \quad \square$$

Corollary 3.2. *Let* $f : M \to M'$ *be a* C^1 *map and* $\mathcal{F}' = \{M_i'\}_i$ *an* $C^{0,1}$*-regular foliation of* M' *whose leaves are transverse to* f *and such that there exists a submanifold* V *of* M' *of dimension* $h = \dim M' - \dim \mathcal{F}'$ *transverse to each leaf of* $\mathcal{F}'$ *and intersecting it in a singleton* $V \cap M_i' = \{y_i'\}$*.*

Then the foliation of M *defined by* $\mathcal{F} = \{M_i = f^{-1}(M_{i'}')\}_i$ *is* $C^{0,1}$*-regular.*

Proof. Let us consider the submersion $g : M' \to V$ defined for every $y' \in M'$, by

$$g_{|M_i'} = \text{ constant } = y_i'.$$

Thus g defines the foliation $\mathcal{F}' = \{M_{y'}'\}_{y' \in M'}$ via preimage.

Then the foliation $\mathcal{F} = \{M_i\}_i$ of M is defined by the C^1 submersion $g \circ f : M \to V$. $\square$

Starting from now we will suppose $M = M^n$ to be a riemannian manifold of dimension n.

For a C^1 map $f : M \to M'$ let us consider the distribution of vector subspaces $\mathcal{D}(y) := \mathcal{D}_f(y)$ obtained by splitting every $T_y M$ as the direct orthogonal sum:

$$T_y M \ = \ \mathcal{D}(y) \ \oplus \ \ker\, f_{*y} \qquad \text{where} \qquad \mathcal{D}(y) := \perp (\ker\, f_{*y}, T_y M)\,.$$

We call $\mathcal{D} : M \to \mathbb{G}_{n-k}(TM)$, $\{\mathcal{D}(y) = \perp (\ker\, f_{*y}, T_y M)\}_y$ the *canonical distributions* of f.

We will see that the study of the condition (D) for a submersive restriction $f_Y : Y \to Y'$ ($Y \subseteq M$ and $Y' \subseteq M'$) at a point x in the adherence $\overline{Y}$ of Y is strongly related to good properties of limits of the distribution

$$\mathcal{D}(y) \ = \ \mathcal{D}_{f_Y}(y) \ := \perp (\ker\, f_{Y*y}, T_y Y).$$

When $f_Y = \pi_{XY|} : S_{XY}^{\epsilon} \to X$ is the restriction of a projection $\pi_{XY} : T_{XY} \to X$ on a stratum $X < Y$, of a system of control data $\{(T_X, \pi_X, \rho_X)\}_X$ of a regular stratification, then $\mathcal{D}_f(y)$ is defined in the same way as the *canonical distribution* $\mathcal{D}_X(y)$ *relative to the stratum* X introduced in **[11, 12, 13]**. In this case, if $\mathcal{W}$ and $\mathcal{W}'$ are Whitney refinements of S_{XY}^{ϵ} and X, Condition (D) implies the (a)*-regularity* (see **[13]**) of a "horizontal" foliation related to $\mathcal{D}_X$ in a particular stratified mapping cylinder $C_{\mathcal{W}'}(\mathcal{W})$ **[16]** (from Lemma 3.1 to Theorem 3.4).

Lemma 3.1. *Let $V \subseteq U$ be two vector subspaces of $\mathbb{R}^n$.*

If $\{V_i\}_i$ and $\{U_i\}_i$ are two sequences of vector subspaces of $\mathbb{R}^n$ with $V_i \subseteq U_i$, $l = \dim V_i$, $k = \dim U_i$ for every i and such that $\lim_i U_i = U$ in $\mathbb{G}_k^n$, then

$$\lim_i V_i \;=\; V \;\text{in}\; \mathbb{G}_l^n \quad \Longleftrightarrow \quad \lim_i \perp(V_i, U_i) \;=\perp(V,U) \;\text{ in }\; \mathbb{G}_{k-l}^n\,.$$

Proof. ($\Rightarrow$). Let us denote $\mathcal{D}_i \;=\perp(V_i, U_i)$ and $\mathcal{D} \;=\perp(V,U)$ and show that $\lim_i \mathcal{D}_i \;=\; \mathcal{D}$.
Since $\dim V_i = l$ and $\dim U_i = k$ then $\dim \mathcal{D}_i = k-l$ for every i.
Since $U \;=\; \lim_i U_i \in \mathbb{G}_k^n$ and $V \;=\; \lim_i V_i$, then $\dim U = k$, $\dim V = l$ and $\dim \mathcal{D} \;=\; k-l$.

Let $\{\mathcal{D}_{i_h}\}_h$ be an arbitrary convergent subsequence of $\{\mathcal{D}_i\}_i$ and $\mathcal{D}' \;=\; \lim_h \mathcal{D}_{i_h}$.
Every vector $w \in \mathcal{D}' = \lim_h \mathcal{D}_{i_h}$ is a limit $w = \lim_h w_{i_h}$ of a sequence of vectors $\{w_{i_h} \in \mathcal{D}_{i_h}\}_h$ so that $< w_{i_h}, v_{i_h} > = \;0$ for every vector $v_{i_h} \in V_{i_h}$.

On the other hand $V = \lim_i V_i = \lim_h V_{i_h}$, so every vector $v \in V$ is also a limit $v = \lim_h v_{i_h}$ of a sequence of vectors $\{v_{i_h} \in V_{i_h}\}_h$ and we have $< w, v > = \;\lim_h < w_{i_h}, v_{i_h} > = \;0$ so that $w \in \perp(V,U) = \mathcal{D}'$. Hence $\mathcal{D}' \subseteq \mathcal{D}$ and, since they have the same dimension, $\mathcal{D}' = \mathcal{D}$.

Therefore every convergent subsequence $\{\mathcal{D}_{i_h}\}_h$ of $\{\mathcal{D}_i\}_i$ has limit $\mathcal{D}$ and so $\lim_i \mathcal{D}_i = \mathcal{D}$.

The proof of ($\Leftarrow$) follows from ($\Rightarrow$) because $V_i = \perp(\mathcal{D}_i, U_i)$ and $V = \perp(\mathcal{D}, U)$. $\square$

Proposition 3.6 below anticipates some arguments that will appear in §4.

Proposition 3.6. *Let M^n be a riemannian manifold and $f : M \to M'$ a C^1 submersion on $M - \{y_0\}$ with $y_0 \in M$. Then the following conditions are equivalent:*

1) $f : M \to M'$ is a submersion at y_0;

2) For all $\{y_i\}_i \subseteq M - \{y_0\}$ converging to y_0 there exists $\lim_i \mathcal{D}(y_i)$ and

$$\lim_i \mathcal{D}(y_i) = \perp(\ker f_{*y_0}, T_{y_0}M).$$

I. e.: the map $\mathcal{D} : M \to \mathbb{G}_{n-k}(TM)$, $\mathcal{D}(y) := \perp(\ker f_{*y}, T_y M)$ is continuous;

3) For all $\{y_i\}_i \subseteq M - \{y_0\}$ converging to y_0 there exists $\lim_i \mathcal{D}(y_i)$ and

$$\lim_i \mathcal{D}(y_i) \subseteq \perp(\ker f_{*y_0}, T_{y_0}M).$$

Proof. It follows immediately from Proposition 3.5 and the previous Lemma 3.1. $\square$

Definition 3.5. below will play an important role in the next section.

Definition 3.5. Let $f : M \to M'$ be a C^1 map of riemannian manifolds, $Y \subseteq M$, $Y' \subseteq M'$ two C^1-submanifolds whose restriction $f_Y : Y \to Y'$ is a C^1 surjective submersion; so $Y' = f(Y)$, $T_{y'}Y' = T_{f(y)}f(Y)$, $y' = f(y)$ for all y, and we will assume such notations in the whole of the paper.

Let $x \in \overline{Y} \subseteq M$ (*a priori* x could lie or not in Y) and $x' = f(x)$.

For every point $y \in Y$, let $\mathcal{D}(y) = \perp(\ker f_{Y*y}, T_y Y)$ be the canonical distribution of f_Y.
The restricted differential map:

$$f_{Y*y|\mathcal{D}(y)} \;:\; \mathcal{D}(y) \;\longrightarrow\; T_{y'}Y'$$

is then an isomorphism and for every unit vector $u \in \mathcal{D}(y)$, one has $f_{Y*y}(u) \neq 0$, so that by compactness of each unit sphere of $\mathcal{D}(y)$ one can define the continuous map h_Y:

$$h_Y : Y - \{x\} \to \,]0, +\infty[\quad , \quad h_Y(y) = \min\big\{ ||\, f_{Y*y|\mathcal{D}(y)}(u)\, || \;\;:\;\; ||\, u\, || \;= 1 \big\}.$$

Similarly, by considering the inverse map $f^{-1}_{Y*y|\mathcal{D}(y)} \;:\; T_{y'}Y' \to \mathcal{D}(y)$, every vector $v' \in T_y Y'$ has a unique (pre)image $v = f^{-1}_{Y*y|\mathcal{D}(y)}(v')$ such that $v \in \mathcal{D}(y)$ and $f_{Y*y}(v) = v'$.

We call such a vector $v = f^{-1}_{Y*y|\mathcal{D}(y)}(v')$ *the canonical lifting of* v'*:*
it is the unique vector $v \in T_y Y$ such that $f_{Y*y}(v) = v'$ and having no component along $\ker\, f_{Y*y}$.

Of course $v' \neq 0$ if and only if its lift $v \neq 0$.

So, starting from now, every vector that we will lift, will always be supposed $\neq 0$.
We will understand this also in many statements of §4 without say it explicitely every time.

We can then define the *dual* continuous map H_Y:

$$H_Y : Y - \{x\} \to \,]0, +\infty[\quad , \quad H_Y(y) = \max\left\{ || \ f^{-1}_{Y*y|\mathcal{D}(y)}(v') \ || \ : \ || \ v' \ || \ = 1 \right\}.$$

I.e. $H_Y(y)$ is the classical norm of the linear isomorphism $f^{-1}_{Y*y|\mathcal{D}(y)} \ : \ T_{y'}Y' \to \mathcal{D}(y)$.

Remark 3.3. *For every $y \in Y$ and every vector $v' \in T_{y'}Y' - \{0\}$ we have:*

1) The unit vector $u = \frac{v}{||\ v\ ||}$ of the canonical lifting $v := f^{-1}_{Y*y|\mathcal{D}(y)}(v') \in \mathcal{D}(y)$ of $v' \in T_{y'}Y'$ satisfies:

$$|| \ v \ || \ = \ \frac{|| \ v' \ ||}{|| \ f_{Y*y|\mathcal{D}(y)}(u)||}.$$

2) If $|| \ v' \ || = 1$ *then:* $\quad || \ v \ || \ = \ \frac{1}{|| \ f_{Y*y|\mathcal{D}(y)}(u)||}.$

3) $H_Y(y) = \dfrac{1}{h_Y(y)}$.

Proof. For 1) one easily finds:

$$|| \ v' \ || \ = || \ f_{Y*y}(v) \ || \ = || \ f_{Y*y}(\tfrac{v}{||\ v\ ||}) \ || \ \cdot || \ v \ || \ = || \ f_{Y*y|\mathcal{D}(y)}(u) \ || \ \cdot || \ v \ ||$$

which also obviously implies 2), while 3) follows by 2) thanks to:

$$H_Y(y) = \sup_{||\ v'\ ||=1} \left\{ \ || \ v \ || \ : \ v' \in T_{y'}Y' \ \right\} = \sup_{||\ u\ ||=1} \left\{ \ \frac{1}{|| \ f_{Y*y|\mathcal{D}(y)}(u)||} \ : \ u \in \mathcal{D}(y) \ \right\} =$$

$$\frac{1}{\inf_{||\ u\ ||=1} \left\{ || \ f_{Y*y|\mathcal{D}(y)}(u)|| \ : \ u \in \mathcal{D}(y) \right\}} = \frac{1}{h_Y(y)}. \qquad \square$$

Being interested in the properties of the maps h_Y and H_Y at a regular point we will suppose in Proposition 3.7 below that $Y \cup \{x\} = M$, and we will denote $y_0 = x$, $h = h_Y$ and $H = H_Y$.

Proposition 3.7. *Let $f : M \to M'$ be a C^1 map, submersion on $M - \{y_0\}$ with $y_0 \in M$.*
The following conditions are equivalent:

1) $f : M \to M'$ *is a submersion at y_0;*

2) There exists $\lim_{y \to y_0} h(y) > 0$*;*

3) There exists $\lim_{y \to y_0} H(y) < +\infty$.

Proof. 1) $\Rightarrow$ 2). If y_0 is a regular point of M, and f is a submersion at y_0 then Definition 3.5 of the continuous map h extends naturally to y_0 giving $\lim_{y \to y_0} h(y) = h(y_0) \in \,]0, +\infty[$.
2) $\Rightarrow$ 3). It follows obviously by Remark 3.3.
3) $\Rightarrow$ 1). Let us fix a unit vector $v' \in T_{y'_0}M'$.
By hypothesis for every sequence $\{y_i\}_i \subseteq M$ such that $\lim_i y_i = y_0$ one has $\lim_i H(y_i) < +\infty$.
Given then a sequence of *unit* vectors $\{v'_i \in T_{y'_i}M'\}_i$ such that $\lim_i v'_i = v'$, the sequence of canonical lifts $\{v_i := f^{-1}_{*y_i|\mathcal{D}(y_i)}(v'_i) \in \mathcal{D}(y_i)\}_i$, is bounded: $\sup_i \ ||v_i|| \ \leq \ \sup_i \ H(y_i) < +\infty$.

There exists thus a subsequence $\{v_{ih}\}_h$ converging to a vector $v = \lim_h v_{ih} \in T_{y_0}M$ and $f : M \to M'$ being C^1 at y_0 one finds:

$$f_{*y_0}(v) = f_{*y_0}(\lim_h v_{ih}) = \lim_h f_{*y_{i_h}}(v_{i_h}) = \lim_h v'_{ih} = v' .$$

Therefore $f_{*y_0} : T_{y_0}M \to T_{y'_0}M'$ is surjective and f is a submersion at y_0. $\square$

3.2. ***Condition* (D) *at a regular point.*** Let us recall now the definition of the condition (D) for $f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ at $x \in X < Y$.

Let $f : M \to M'$ be a C^1 map between C^1 manifolds, $\mathcal{W} \subseteq M$ and $\mathcal{W}' \subseteq M'$ Whitney stratifications and suppose that the restriction $f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ is a stratified (stratum for stratum) surjective submersion satisfying condition (D) at $x \in X < Y$.

This means that for every sequence $\{y_i\}_i \subseteq Y$ such that $\lim_i y_i = x \in X$ one has:

$$\exists \lim_i T_{y_i}Y = \tau \quad \text{and} \quad \exists \lim_i T_{y'_i}Y' = \tau' \quad \Longrightarrow \quad f_{*x}(\tau) \supseteq \tau'$$

where $Y' = f(Y)$ and $y' = f(y)$ for every $y \in Y$.

Remark 3.4. *The C^1 smoothness of f on M does not suffice to imply the inclusion $f_{*x}(\tau) \supseteq \tau'$ which as one sees with easy examples is false in general (see Example 2.1).* $\square$

We will show in the next section (Theorem 4.3) that it depends on the possibility of extracting a bounded sequence of vector preimages v_i, one in each fibre $f^{-1}_{*y_i}(v'_i)$ with $\lim_i v'_i \in \tau'$.

We will see moreover that the whole complexity of the condition (D) at x is contained in the behaviour near x of the maps h_Y and/or H_Y.

Remark 3.5. *Condition (D) for $f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ at $x \in X < Y$ does not depend on the stratum X containing x: to formulate it, one must consider a map f defined on a C^1 manifold M containing Y and $x \in \overline{Y}$ and which is C^1 on M.* $\square$

Remark 3.6. *With the same hypotheses and notations as above we have:*

i) Since $f : M \to M'$ is C^1 the opposite inclusion $f_{*x}(\tau) \subseteq \tau'$ is always satisfied.

ii) $f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ being a stratified submersion, $T_{y'_i}Y' = f_{*y_i}(T_{y_i}Y)$ for every i.

Proof i). If $v \in \tau$ we can write $v = \lim_i v_i$ for a sequence $\{v_i \in T_{y_i}Y\}_i$, hence:

$$f_{*x}(v) = f_{*x}(\lim_i v_i) = \lim_i f_{*y_i}(v_i) \in \lim_i f_{*y_i}(T_{y_i}Y) = \tau' \quad \text{and so:} \quad f_{*x}(\tau) \subseteq \tau' . \quad \square$$

Since $f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ is the restriction of a C^1 map $f : M \to M'$ between two manifolds, there exists a differential map $f_{*x} : T_xM \to T_{x'}M'$ and a unique possible way to define the restriction $f_{*x|C_xY}$ to the tangent cone (the Nash fiber) $C_xY := \bigsqcup_{\tau = \lim_i T_{y_i}Y} \tau$ of Y at x.

Condition (D) implies moreover that the "restriction" $f_{*x|C_xY} : C_xY \to C_{x'}Y'$ *must be surjective.* This is the most natural generalisation at a singular point of the submersivity:

Remark 3.7. *If $f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ satisfies condition (D) at $x \in X < Y$, then*

i) $f_{*x}(\tau) = \tau'$;

ii) The surjective differential map $f_{Y*} : TY \to TY'$ of the restriction $f_Y : Y \to Y'$ extends surjectively to the union of linear maps:

$$f_{Y*x|C_xY} = \bigsqcup_{\tau = \lim_i T_{y_i}Y} f_{*x|\tau} \ : \ C_xY = \bigsqcup_{\tau = \lim_i T_{y_i}Y} \tau \ \longrightarrow \ C_{x'}Y' = \bigsqcup_{\tau' = \lim_i T_{y'_i}Y'} \tau'$$

between the tangent cones C_xY and $C_{x'}Y'$. $\square$

Condition (D) for $f_{\mathcal{W}}$ also morally means that the differential maps $f_{Y*y} : T_yY \to T_{y'}Y'$ have to be surjective including all possible limit maps $\lim_{y_i \to x} f_{Y*y_i} : T_{y_i}Y \to T_{y'_i}Y'$: a kind of *"super-submersivity"* defined in the same spirit as Goresky's *super-transversality* **[5]**.

Look now at what condition (D) "means" at a regular point $y_0 \in Y$.

Let $f : M \to M'$ a C^1 map on a riemannian C^1 manifold M and $Y \subseteq M$ a submanifold.

If the restriction $f_Y : Y \to Y'$ is a surjective submersion out of a point $y_0 \in Y$, then condition (D) for f_Y at y_0 can be naturally defined as condition (D) for $f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ by considering for $\mathcal{W}$ and $\mathcal{W}'$ the Whitney stratifications $\mathcal{W} = (Y - \{y_0\}) \sqcup \{y_0\}$ and $\mathcal{W}' = (Y' - \{y'_0\}) \sqcup \{y'_0\}$ with $y'_0 = f(y_0)$ (we also include the possibility $Y = M$).

With such an extended meaning we have :

Proposition 3.8. *Let $f_Y : Y \to Y' = f(Y)$ be a surjective C^1 map and $y_0 \in Y$ such that f_Y is a submersion at every point of $Y - \{y_0\}$. Then the following conditions are equivalent:*

1) $f_Y : Y \to Y'$ is a submersion at y_0;

2) $\lim_i y_i = y_0$ and $\exists \lim_i \mathcal{D}(y_i) \Longrightarrow f_{Y*y_0}\big(\lim_i \mathcal{D}(y_i)\big) \supseteq \lim_i f_{Y*y_i}\big(\mathcal{D}(y_i)\big)$;

3) f_Y satisfies the condition (D) at y_0.

Proof. Since Y and Y' are C^1 manifolds, for every sequence $\{y_i\}_i \subseteq Y - \{y_0\}$ such that $\lim_i y_i = y_0$, we automatically have that both limits exist:

$$\tau = \lim_i T_{y_i}Y_0 = \lim_i T_{y_i}Y = T_{y_0}Y \quad \text{and} \quad \tau' = \lim_i T_{y'_i}Y'_0 = \lim_i T_{y'_i}Y' = T_{y'_0}Y' .$$

Moreover, f_Y being a submersion at every $y_i \in Y - \{y_0\}$, by decomposing $T_{y_i}Y$ in the orthogonal direct sum: $T_{y_i}Y = \mathcal{D}(y_i) \oplus \ker f_{Y*y_i}$, with $\mathcal{D}(y_i) = \perp(\ker f_{Y*y_i}, T_{y_i}Y)$, then $f_{Y*y_i|\mathcal{D}(y_i)} : \mathcal{D}(y_i) \to T_{y'_i}Y'$ is an isomorphism of vector spaces, and hence $\tau' = \lim_i f_{Y*y_i}\big(\mathcal{D}(y_i)\big)$.

$(1 \Rightarrow 2)$. Let us suppose that $f_Y : Y \to Y'$ is a submersion at y_0.
We fix a unit vector $v' \in \lim_i f_{Y*y_i}\big(\mathcal{D}(y_i)\big)$ and we will show that $v' \in f_{Y*y_0}\big(\lim_i \mathcal{D}(y_i)\big)$.

There exists then a sequence of unit vectors $\{v'_i \in f_{*y_i}\big(\mathcal{D}(y_i)\big)\}_i$ such that $v' = \lim_i v'_i$.
For every $v'_i \in f_{Y*y_i}(\mathcal{D}(y_i))$ the canonical lifting v_i satisfies $v_i \in \mathcal{D}(y_i)$ and $f_{Y*y}(v_i) = v'_i$.
Now f_Y being a submersion at y_0, by Proposition 3.7 $(1 \Rightarrow 3)$, we find that $\limsup_{y \to y_0} H_Y(y) < +\infty$ and that the sequence $\{v_i = f^{-1}_{*y_i|\mathcal{D}(y_i)}(v'_i)\}_i$ is bounded and admits a subsequence $\{v_{ih}\}_h$ converging to a vector $v = \lim_h v_{i_h} \in \lim_h \mathcal{D}(y_{ih}) = \lim_i \mathcal{D}(y_i)$ for which

$$f_{Y*y_0}(v) = f_{Y*y_0}(\lim_h v_{ih}) = \lim_h f_{Y*y_0}(v_{ih}) = \lim_h v'_{ih} = v' .$$

Therefore $v' \in f_{Y*y_0}\big(\lim_i \mathcal{D}(y_i)\big)$.

$(2 \Rightarrow 3)$. Chosen a subsequences such that there exists $\lim_h \mathcal{D}(y_{i_h})$ we immediately have :

$$f_{Y*y_0}(\tau) = f_{Y*y_0}\big(\lim_h T_{i_h}Y\big) \supseteq f_{Y*y_0}\big(\lim_h \mathcal{D}(y_{i_h})\big) \supseteq \lim_h f_{Y*y_{i_h}}\big(\mathcal{D}(y_{i_h})\big) = \lim_h T_{y'_{i_h}}Y' = \tau' .$$

Hence Condition (D) holds at y_0 for f_Y.

$(3 \Rightarrow 1)$. If f_Y satisfies condition (D) at y_0, we have $f_{Y*y_0}(\tau) \supseteq \tau'$ and since y_0 is a regular point of the manifold Y, $\tau = \lim_i T_{y_i}Y = T_{y_0}Y$ and $\tau' = \lim_i T_{y'_i}Y' = T_{y'_0}Y'$. Thus $f_{Y*y_0}(T_{y_0}Y) \supseteq T_{y'_0}Y'$.

Hence $f_{Y*y_0} : T_{y_0}Y \to T_{y'_0}Y'$ is surjective, and $f_Y : Y \to Y'$ is a submersion at y_0. $\square$

4. Sufficient conditions, analytic and geometric meanings for condition (D).

In this section we prove the main results of the paper given in Theorems 4.3, 4.4, 4.5, 4.6 and their Corollaries 4.3, 4.4, 4.5, 4.6.

Starting from the analysis of the technical content of condition (D), (Theorem 4.3) we find various equivalent analytic and geometric properties (Theorems 4.4, 4.5, 4.6), which are all sufficient conditions for Condition (D) (Corollaries 4.3, 4.5 and 4.6).

4.1. *Technical content and sufficient analytic conditions for Condition* (D).

Theorem 4.3 below explains the essential technical content of the condition (D).

The equivalence $(1 \Leftrightarrow 4)$ has been used by the author of the present paper in **[16]** (Theorem 3.3) when $f_{\mathcal{W}} = \pi_{XY|\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ is the restriction of a projection $\pi_{XY} : S^{\epsilon}_{XY} \to X$, to prove that certain stratified mapping cones $C_{\mathcal{W}'}(\mathcal{W})$ are (b)-regular, to obtain an equivalent version of Goresky's essential Proposition 2.2 and 2.4 (Theorem 3.4 and Corollary 3.2, **[16]**) .

Proposition 2.2 is really the key property in proving Proposition 2.4 which gives a partial solution of Conjecture 1.2, suggests new ideas for a general approach to it and is fundamental for the proof of Theorems 1.1 and 1.2 in the theories WH_*, WH^* of Goresky (see §2).

Theorem 4.3. *Let $f : M \to M'$ be a C^1 map between C^1 manifolds, $\mathcal{W} \subseteq M$ and $\mathcal{W}' \subseteq M'$ Whitney stratifications such that the restriction $f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ is a stratified surjective submersion.*

Let $X < Y$ be strata of $\mathcal{W}$, $x \in X$. By denoting $f_Y : Y \to Y' = f(Y)$ the restriction of f, and for all $y \in Y$, $y' = f(y)$ and $\mathcal{D}(y) = \perp (\ker f_{Y*y}, T_yY)$, the following conditions are equivalent:

1. *The map $f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ satisfies the condition (D) at $x \in X < Y$.*
2. *For all $\{y_i\}_i \subseteq Y$ such that $\lim_i y_i = x$ and both limits $\tau = \lim_i T_{y_i}Y$ and $\tau' = \lim_i T_{y'_i}Y'$ exist, for every $v' \in \tau' - \{0\}$ <u>there exists a sequence</u> $\{v'_i \in T_{y'_i}Y' - \{0\}\}_i$ such that $\lim_i v'_i = v'$ and having a bounded sequence of preimages $\{w_i \in f^{-1}_{Y*y_i}(v'_i) \in T_{y_i}Y\}_i$.*
3. *For all $\{y_i\}_i \subseteq Y$ such that $\lim_i y_i = x$ and both limits $\tau = \lim_i T_{y_i}Y$ and $\tau' = \lim_i T_{y'_i}Y'$ exist, for every $v' \in \tau' - \{0\}$ there exists a sequence $\{v'_i \in T_{y'_i}Y' - \{0\}\}_i$ such that $\lim_i v'_i = v'$ and having the sequence by canonical lifting $\{v_i \in f^{-1}_{Y*y_i|\mathcal{D}(y_i)}(v'_i) \in \mathcal{D}(y_i)\}_i$ bounded.*
4. *For all $\{y_i\}_i \subseteq Y$ such that $\lim_i y_i = x$ and both limits $\sigma = \lim_i \mathcal{D}(y_i)$ and $\tau' = \lim_i T_{y'_i}Y'$ exist, one has: $f_{*x}(\lim_i \mathcal{D}(y_i)) \supseteq \lim_i f_{Y*y_i}(\mathcal{D}(y_i))$.*

Proof. Let us consider a sequence $\{y_i\}_i \subseteq Y$ such that $\lim_i y_i = x$ and both limits $\tau = \lim_i T_{y_i}Y$ and $\tau' = \lim_i T_{y'_i}Y'$ exist in the appropriate Grassmann manifold.

Remark also that, $f_Y : Y \to Y'$ being submersive, $T_{y'_i}Y' = f_{Y*y_i}(T_{y_i}Y) = f_{*y_i}(T_{y_i}Y)$ for each i.

$(1 \Rightarrow 2)$. If $f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ satisfies the condition (D) at $x \in X < Y$, $f_{*x}(\tau) \supseteq \tau'$ then for every vector $v' \in \tau'$ there exists a vector $v \in \tau$ such that $v' = f_{*x}(v)$.

Since $v \in \tau = \lim_i T_{y_i}Y$, there exists a sequence $\{w_i \in T_{y_i}Y\}_i$ such that $v = \lim_i w_i$ and $\{w_i\}_i$ is in particular obviously bounded. The sequence of the images $\{v'_i := f_{*y_i}(w_i)\}_i$ satisfies then:

i) $\lim_i v'_i = \lim_i f_{*y_i}(w_i) = f_{*x}(\lim_i w_i) = f_{*x}(v) = v'$;

ii) $\{v'_i = f_{*y_i}(w_i)\}_i$ admits the bounded sequence of lifting $\{w_i \in f^{-1}_{*y_i}(v'_i)\}_i$.

$(2 \Rightarrow 3)$. Under the hypothesis 2), by decomposing every vector w_i in the orthogonal sum $w_i = v_i + u_i \in \mathcal{D}(y_i) \oplus \ker f_{Y*y_i}$ one immediately has $||v_i|| \leq ||w_i||$ so that if $\{w_i\}_i$ is bounded then $\{v_i\}_i$ is bounded too and moreover: $v_i \in \mathcal{D}(y_i)$ and $f_{*y_i}(v_i) = v'_i$.

$(3 \Rightarrow 4)$. Let $v' \in \lim_i f_{*y_i}(\mathcal{D}(y_i)) \subseteq \tau'$ and let us suppose that $\lim_i \mathcal{D}(y_i) = \sigma$ exists.

By hypothesis 3) for every $v' \in \tau'$ there exists a sequence $\{v'_i \in T_{y'_i}Y'\}_i$ such that $\lim_i v'_i = v'$ whose sequence of canonical lifting $\{v_i \in f^{-1}_{Y*y_i}(v'_i) \cap \mathcal{D}(y_i) \subseteq T_{y_i}Y\}_i$ is bounded.

Thus for a convenient subsequence of indexes $\{i_h\}_h$ there exist $v = \lim_h v_{i_h}$, $\tau = \lim_h T_{y_{i_h}}Y$ and (obviously) $\lim_h \mathcal{D}(y_{i_h})$ so that

$$v = \lim_h v_{i_h} \in \lim_h \mathcal{D}(y_{i_h}) = \lim_i \mathcal{D}(y_i)$$

and

$$v' = \lim_h v'_{i_h} = \lim_h f_{Y*y_{i_h}}(v_{i_h}) = f_{*x}(v) \in f_{*x}(\lim_i \mathcal{D}(y_i))$$

and in conclusion:

$$f_{*x}(\lim_i \mathcal{D}(y_i)) \supseteq \lim_i f_{Y*y_i}(\mathcal{D}(y_i))\,.$$

$(4 \Rightarrow 1)$. Let $\{y_i\}_i \subseteq Y$ be a sequence such that $\lim_i y_i = x$ and both limits $\tau = \lim_i T_{y_i}Y$ and $\tau' = \lim_i T_{y'_i}Y'$ exist in the appropriate Grassmann manifold.

The Grassmann manifold being compact, there exists a subsequence of indices $(i_h)_h$, such that there exists also $\lim_h \mathcal{D}(y_{i_h}) =: \sigma$.

Thus $f_Y : Y \to Y'$ being a submersion, $T_{y'_{i_h}}Y' = f_{Y*y_{i_h}}(T_{y_{i_h}}Y) = f_{*y_{i_h}}(T_{y_{i_h}}Y)$ and hence:

$$\tau' = \lim_i T_{y'_i}Y' = \lim_h T_{y'_{i_h}}Y' = \lim_h f_{Y*y_{i_h}}(\mathcal{D}(y_{i_h})) = \lim_h f_{*y_{i_h}}(\mathcal{D}(y_{i_h})) \subseteq$$

by the hypothesis 4)

$$\subseteq\ f_{*x}(\lim_h \mathcal{D}(y_{i_h}))\ \subseteq\ f_{*x}(\lim_h T_{y_{i_h}}Y)\ =\ f_{*x}(\lim_i T_{y_i}Y)\ =\ f_{*x}(\tau)\ .$$

Then in conclusion $f : \mathcal{W} \to \mathcal{W}'$ satisfies the condition (D) at $x \in X < Y$. $\square$

Theorem below extends to the stratiffied case the previous Propostion 3.7 and allows to give in Corollary 4.3 a sufficient analytic condition for Condition (D).

Theorem 4.4. *Let $f : M \to M'$ be a C^1 map between C^1 manifolds, $\mathcal{W} \subseteq M$ and $\mathcal{W}' \subseteq M'$ Whitney stratifications such that the restriction $f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ is a stratified surjective submersion.*

Let $X < Y$ be adjacent strata of $\mathcal{W}$, $x \in X$, $Y' = f(Y)$ and $y' = f(y)$ for all $y \in Y$.

Let us consider for $f_Y : Y \to Y'$ the distribution $\mathcal{D}(y) = \perp(\ker f_{Y*y}, T_yY)$ and the maps

$$h_Y : Y \to]0,\infty[\quad , \quad h_Y(y) = \min\{||f_{Y*y|\mathcal{D}(y)}(u)|| \ :\ ||u|| = 1\}\,,$$

$$H_Y : Y \to\ ]0,+\infty[\quad , \quad H_Y(y) = \max\{||\ f^{-1}_{Y*y|\mathcal{D}(y)}(v')\ ||\ :\ ||v'||\ = 1\}\,.$$

The following conditions are equivalent:

1) For all $\{y_i\}_i \subseteq Y$ such that $\lim_i y_i = x$ and both limits $\tau = \lim_i T_{y_i}Y$ and $\tau' = \lim_i T_{y'_i}Y'$ exist, for every vector $v' \in \tau' - \{0\}$, every sequence of vectors $\{v'_i \in T_{y'_i}Y' - \{0\}\}_i$ such that $\lim_i v'_i = v'$ has a bounded subsequence of canonical liftings $\{v_{i_h} = f^{-1}_{Y*y_{i_h}|\mathcal{D}(y_{i_h})}(v'_{i_h})\}_h$.

2) For all $\{y_i\}_i \subseteq Y$ such that $\lim_i y_i = x$ and both limits $\tau = \lim_i T_{y_i}Y$ and $\tau' = \lim_i T_{y'_i}Y'$ exist, for every unit vector $u' \in \tau'$, every sequence of unit vectors $\{u'_i \in T_{y'_i}Y'\}_i$ such that $\lim_i u'_i = u'$ has a bounded subsequence of canonical liftings $\{u_{i_h} = f^{-1}_{Y*y_{i_h}|\mathcal{D}(y_{i_h})}(u'_{i_h})\}_h$.

3) $\liminf_{y\to x} h_Y(y) > 0$.

4) $\limsup_{y\to x} H_Y(y) < +\infty$.

Proof 1) $\Rightarrow$ 2). Obvious.

Proof 2) $\Rightarrow$ 1). If $v' \in \tau' - \{0\}$ and $\{v'_i \in T_{y'_i}Y' - \{0\}\}_i$ is a sequence such that $\lim_i v'_i = v'$, then $u' := \frac{v'}{||v'||} \in \tau'$ and $u'_i := \frac{v'_i}{||v'_i||} \in T_{y'_i}Y'$ are unit vectors such that $\lim_i u'_i = u'$.

By the hypothesis 2) the sequence of canonical liftings $\{u_i := f^{-1}_{Y*y_i|\mathcal{D}(y_i)}(u'_i)\}$ admits a bounded subsequence $\{u_{i_h}\}_h$. So there exists $K > 0$ such that

$$||f^{-1}_{Y*y_{i_h}|\mathcal{D}(y_{i_h})}(\frac{v'_{i_h}}{||v'_{i_h}||})|| \leq K \quad \text{and hence:} \quad ||f^{-1}_{Y*y_{i_h}|\mathcal{D}(y_{i_h})}(v'_{i_h})|| \leq K \cdot ||v'_{i_h}|| \,.$$

The canonical liftings $\{v_{i_h} := f^{-1}_{Y*y_{i_h}|\mathcal{D}(y_{i_h})}(v'_{i_h})\}_h$ of the $\{v'_{i_h}\}_h$ are then bounded by:

$$||v_{i_h}|| = ||f^{-1}_{Y*y_{i_h}|\mathcal{D}(y_{i_h})}(v'_{i_h})|| \leq K \cdot ||v'_{i_h}|| \leq K \cdot \sup_h ||v'_{i_h}|| = K' < +\infty \,.$$

Proof 2) $\Rightarrow$ 3). Let $l = \liminf_{y\to x} h_Y(y)$ the minimum value of adherence of h_Y.

There exists then a sequence $\{y_i\}_i \subseteq Y$ such that $\lim_i y_i = x \in X$ and $\lim_i h_Y(y_i) = l \in \mathbb{R}$.

By definition of each $h_Y(y_i)$, there exists a sequence of unit vectors $\{u_i \in \mathcal{D}(y_i) \subseteq T_{y_i}Y\}_i$ such that each $h_Y(y_i) = ||f_{Y*y|\mathcal{D}(y_i)}(u_i)||$ realizes the minimum norm defining $h_Y(y_i)$ (Definition 3.5).

There exists a subsequence $\{y_{i_h}\}_h$, such that both limits exist:

$$\lim_i T_{y_{i_h}}Y =: \tau \quad \text{and} \quad \lim_i T_{y'_{i_h}}Y' =: \tau'.$$

Every u_{i_h} being a unit vector $\in \mathcal{D}(y_{i_h}) - \{0\}$, its image $u'_{i_h} := f_{Y*y_{i_h}|\mathcal{D}(y_{i_h})}(u_{i_h}) \in T_{y'_{i_h}}Y' - \{0\}$ is not zero (as well as for all images of vectors in $\mathcal{D}(y_{i_h}) - \{0\}$) and we can write:

$$u_{i_h} = f^{-1}_{Y*y_{i_h}|\mathcal{D}(y_{i_h})}({u'}_{i_h}) \in \mathcal{D}(y_{i_h}) \quad \text{and} \quad \frac{u_{i_h}}{||u'_{i_h}||} = f^{-1}_{Y*y_{i_h}|\mathcal{D}(y_{i_h})}(\frac{{u'}_{i_h}}{||{u'}_{i_h}||}) \in \mathcal{D}(y_{i_h}).$$

For a suitable further subsequence (note it again $\{i_h\}_h$), there exists then the limit :

$$u' := \lim_h \frac{{u'}_{i_h}}{||{u'}_{i_h}||} \in \lim_h T_{y'_{i_h}}Y' - \{0\}\,.$$

It follows that:

i) The unit vector $u' = \lim_h \frac{{u'}_{i_h}}{||{u'}_{i_h}||} \in \tau' - \{0\}$.

ii) Every vector $\frac{u_{i_h}}{||u'_{i_h}||} = f^{-1}_{Y*y_{i_h}|\mathcal{D}(y_{i_h})}(\frac{u'_{i_h}}{||u'_{i_h}||})$ is the canonical lifting of the unit vectors $\frac{u'_{i_h}}{||u'_{i_h}||}$.

Hence, by the hypothesis 2), there exists a bounded subsequence (let us denote it again) $\{\frac{u_{i_h}}{||u'_{i_h}||}\}_h$. That is there exists $K > 0$ such that $||f^{-1}_{Y*y_{i_h}|\mathcal{D}(y_{i_h})}(\frac{u'_{i_h}}{||u'_{i_h}||})|| \leq K$.

Therefore,

$$1 = ||u_{i_h}|| = ||f^{-1}_{Y*y_{i_h}|\mathcal{D}(y_{i_h})}(u'_{i_h})|| \leq K \cdot ||u'_{i_h}|| = K \cdot h_Y(y_{i_h})$$

and in conclusion:

$$l = \liminf_{y\to x} h_Y(y) = \lim_i h_Y(y_i) = \lim_h h_Y(y_{i_h}) \geq \frac{1}{K} > 0\,.$$

Proof 3) $\Rightarrow$ 4). It follows immediately because by Remark 3.3.3 one has: $H_Y(y) = \frac{1}{h_Y(y)}$.

Proof 4) $\Rightarrow$ 2). Let $\{y_i\}_i \subseteq Y$ be a sequence of points such that $\lim_i y_i = x$, $\lim_i T_{y_i}Y = \tau$, $\lim_i T_{y'_i}Y' = \tau'$ and let us fix $u' \in \tau'$ a unit vector and a sequence of unit vectors $\{u'_i \in T_{y'_i}Y'\}_i$ such that $\lim_i u'_i = u'$.

Since $L := \limsup_{y\to x} H_Y(y) < +\infty$, then $\limsup_i H_Y(y_i) \leq L$ is finite and so, by Definition 3.5 of each $H_Y(y_i)$, the sequence

$$||f^{-1}_{Y*y_i|\mathcal{D}(y_i)}(u'_i)|| \leq H_Y(y_i) \leq L \quad \text{is bounded.} \qquad \square$$

We deduce then, as corollary, a sufficient condition for Goresky's Condition (D):

Corollary 4.3. *Let $f : M \to M'$ be a C^1 map between C^1 manifolds, $\mathcal{W} \subseteq M$ and $\mathcal{W}' \subseteq M'$ Whitney stratifications such that the restriction $f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ is a stratified surjective submersion.*

Let $X < Y$ be adjacent strata of $\mathcal{W}$ and x a point of X.

If $\liminf_{y\to x} h_Y(y) > 0$ or equivalently $\limsup_{y\to x} H_Y(y) < +\infty$ then:

$$f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}' \qquad \textit{satisfies the condition } (D) \textit{ at } \ x \in X < Y\,.$$

Proof. It follows immediately by 3) $\Rightarrow$ 1) of Theorem 4.4 and 3) $\Rightarrow$ 1) of Theorem 4.3. $\square$

4.2. ***Distance functions between vector subspaces of an Euclidian space.*** We will give a sufficient condition for Condition (D) in terms of all possible limits of the sequences of *essential angles* $\{\alpha'(T_{y_i}Y, \ker f_{*y_i})\}_i$ between the vector subspaces $T_{y_i}Y$ and $\ker f_{*y_i}$ of $T_{y_i}M$. We introduce then the essential minimal distance between two vector subspaces.

Definition 4.6. Let V be a vector subspace of a Euclidian space E.

For every vector $u \in E$ let us define *the distance of u from V* as usual **[22]** by:

$$\delta(u,V) \;=\; \inf_{v\in V} \;||\,u - v\,||\;.$$

Such a minimum value $\inf_{v\in V} \;||\,u-v\,||$ is realized when $u-v$ is orthogonal to V, so precisely when $v = p_V(u)$ is the orthogonal projection of u on V. In particular:

$$\delta(u,V) \;=\; \inf_{v\in V} \;||\,u - v\,|| \;= ||\;u - p_V(u)||$$

and if $u \neq 0$ we let $\alpha(u,V) := \alpha(u, p_V(u))$ denote the *unoriented* angle $\in [0, \frac{\pi}{2}]$ between u and $p_V(u)$.

Let us recall now some simple properties of the fonction δ:

Remark 4.8. *Under the above hypotheses we have:*

1) $||\;u - p_V(u)\;|| \;= ||\;u\;||\; \sin\alpha(u,V)$ *and* $||\;u\;|| \;= 1 \;\Rightarrow\; ||\;u - p_V(u)\;|| \;= \sin\alpha(u,V)$;

2) $||\;p_V(u)\;|| \;=\; ||\;u\;||\; \cos\alpha(u,V)$ *and* $||\;u\;|| = 1 \;\Rightarrow\; ||\;p_V(u)\;|| \;= \cos\alpha(u,V)$;

3) $u \in V \iff \delta(u,V) \;=\; 0$;

4) $\delta(a,V) \;=\; ||a\,|| \cdot \delta(\frac{a}{||a\,||}, V)$, *for all* $a \in E - \{0\}$;

5) $\lim_i u_i \;=\; u \implies \lim_i \delta(u_i,V) \;=\; \delta(u,V)$;

6) $\lim_i V_i \;=\; V \implies \lim_i \delta(u,V_i) \;=\; \delta(u,V)$;

7) $\lim_i u_i \;=\; u$ *and* $\lim_i V_i \;=\; V \implies \lim_i \delta(u_i,V_i) \;=\; \delta(u,V)$.

Proof. 1), …, 4) are immediate, while 5) follows thanks to: $\lim_i p_V(u_i) = p_V(u)$ and 6) by: $\lim_i p_{V_i}(u) = p_V(u)$. The proof of 7) holds since the inequalities:

$$\delta(u,V) \;=\; ||u - p_V(u)|| \;\;\leq\;\; ||u - u_i|| + \delta(u_i,V_i) + ||p_{V_i}(u_i) - p_V(u)||$$

$$\delta(u_i, V_i) \;=\; ||u_i - p_{V_i}(u_i)|| \;\;\leq\;\; ||u - u_i|| \;+\; \delta(u, V) \;+\; ||p_V(u) - p_{V_i}(u_i)||$$

imply

$$|\delta(u, V) - \delta(u_i, V_i)| \;\leq\; ||u - u_i|| \;+\; ||p_{V_i}(u_i) - p_V(u)||$$

and since the hypotheses $\lim_i u_i = u$ and $\lim_i V_i = V$ imply $\lim_i p_{V_i}(u_i) = p_V(u)$. □

One usually considers as "distance" function between two vector subspaces $U, V \subseteq E$, not necessarily of the same dimension, the following :

$$\delta(U, V) \;:= \sup_{u \in U\;,\; ||u||=1} \delta(u, V) = \sup_{u \in U\;,\; ||u||=1} \inf_{v \in V} \;||\, u - v \,||\;.$$

Thanks to the equality (true since every $||\; u\; || \;= 1$):

$$\delta(U, V) = \sup_{u \in U\;,\; ||u||=1} ||\, u - p_V(u)\,|| \;= \sup_{u \in U\;,\; ||u||=1} sin\, \alpha(u, V) \;\in\; [0, 1],$$

by denoting $\alpha(U, V)$ the maximum angle $\in [0, \frac{\pi}{2}]$ between a vector of U and its projection on V, one can write:

$$\delta(U, V) \;=\; \sup_{u \in U} \, sin\, \alpha(u, V) \;\;=\;\; sin\; \alpha(U, V)\,.$$

One finds then:

Remark 4.9. *The function $\delta(U, V)$ satisfies the following properties:*

1) $\delta(U, V) = 0 \quad \Longleftrightarrow \quad U \subseteq V$;

2) $\delta(V, U) = 1 \quad \Longleftrightarrow \quad \exists\, v \in V - U \;:\; v \perp U$ (this holds if $U \subset V$ is strictly contained);

3) $\delta(U, V) \;\neq\; \delta(V, U)$ is not symmetric in general;

4) $||u|| = 1 \quad \Longrightarrow \quad \delta(L(u), V) = \delta(u, V)$ where $L(u)$ is the vector subspace spanned by u;

5) $\delta(a, V) \;\leq\; 2||a - b|| + \delta(b, V)$ for every unit vectors $a, b \in E$;

6) $\delta(a, U) \;\leq\; 2\delta(a, V) + \delta(V, U)$ for every unit vector $a \in E$;

7) $\lim_i U_i = U$, and $\lim_i V_i = V \quad \Longrightarrow \quad \lim_i \delta(U_i, V_i) \;=\; \delta(U, V)\,.$

Proof. $1), \ldots, 4)$ are immediate.

The proof of 5) follows easily by $\delta(a, V) = ||a - p_V(a)||$ and

$$||a - p_V(a)|| \leq ||a - b|| + ||b - p_V(b)|| + ||p_V(b) - p_V(a)|| \;\leq\; ||a - b|| + \delta(b, V) + ||b - a||\,.$$

The proof of 6) follows similarly, since:

$$\delta(a, U) = ||a - p_U(a)|| \leq ||a - p_V(a)|| + ||p_V(a) - p_U(p_V(a))|| + ||p_U(p_V(a)) - p_U(a)|| =$$
$$\delta(a, V) + \delta(p_V(a), U) + ||p_U(a - p_V(a))|| \;\leq\; \delta(a, V) + \delta(V, U) + ||a - p_V(a)|| =$$
$$2\delta(a, V) + \delta(V, U)\,.$$

To prove 7), let u be the unit vectors $\in U$ such that $\delta(U, V) = ||u - p_V(u)|| = \delta(u, V)$

Since $\lim_i U_i = U$ then $\lim_i p_{U_i}(u) = u$, so by Remark 4.8.7 and since every $p_{U_i}(u) \in U_i$ one has:

$$\delta(U, V) \;=\; \delta(u, V) \;=\; \lim_i \delta(p_{U_i}(u), V_i) \;\leq\; \lim_i \delta(U_i, V_i)\,.$$

Simalrly if u_i is the unit vector $\in U_i$ such that $\delta(U_i, V_i) = ||u_i - p_{V_i}(u_i)|| = \delta(u_i, V_i)$ (taking a subsequence if necessary), there exists $\lim_i u_i = a \in U$ and by 5) one finds:

$$\delta(U_i, V_i) \;=\; \delta(u_i, V_i) \;\leq\; 2||u_i - a|| \;+\; \delta(a, V_i) \;\leq\; 2||u_i - a|| \;+\; \delta(U, V_i)$$

hence also that :

$$\lim_i \delta(U_i, V_i) \;\leq\; 2\; \lim_i ||u_i - a|| \;+\; \lim_i \delta(U, V_i) \;=\; \delta(U, V)\,. \qquad \square$$

In order to define a finer "distance" $\delta'(U,V)$ between U and V, we will be interested in the "minimum essential angle", $\alpha'(U,V)$, between U and V, a notions which needs the following more detailed definition.

Definition 4.7. Let $U, V \subseteq E$ two vector subspaces not necessarily of the same dimension.

If $U = \{0\}$ or $V = \{0\}$ let us define $\delta'(U,V) = 0$. Suppose then $U \neq \{0\}$ and $V \neq \{0\}$.

If $U \cap V = \{0\}$, every unit vector $u \in U$ does not lie in V so $||\, u - p_V(u)\,|| > 0$ and using the previous Remark 4.8.1) one can simply define:

$$\delta'(U,V) \;=\; \min_{u\in U\;,\;||u||=1} \;||\, u - p_V(u)\,|| \;=\; \min_{u\in U\;,\;||u||=1} \; sin\, \alpha(u, p_V(u)) \;\in\;]0,1],$$

and denoting $\alpha'(U,V)$ the minimum positive angle between a vector of U and its projection on V, one can write

$$\delta'(U,V) = sin\ \alpha'(U,V)\,.$$

Thus using that $\alpha'(U,V) = \alpha'(V,U)$, one has:

Remark 4.10. *If $U, V \neq \{0\}$, then:*

$$U \cap V = \{0\} \quad \Longrightarrow \quad U \not\subseteq V \;\; and \;\; V \not\subseteq U \quad \Longrightarrow \quad \delta'(U,V) \;=\; \delta'(V,U) > 0. \qquad \square$$

Our definition 4.7 of $\delta'(U,V)$, in the case $U \neq \{0\}$ and $V \neq \{0\}$ and $U \cap V = \{0\}$, coincides with the definition given in **[8]** (p. 534, where it is denoted by $\delta(U,V)$).

On the other hand the definition in **[8]** in the case $U \cap V \neq \{0\}$ satisfies $\delta(U,V) = 0$.

This is not convenient enough for our aims, so we have to extend it in a finer way:

Definition 4.8. If $U \cap V \neq \{0\}$, we consider their *essential mutual subspaces*:

$$U' \;:= \perp (U \cap V; U) \quad \text{and} \quad V' \;:= \perp (U \cap V; V)\,,$$

that easily satisfy $U' \cap V' = \{0\}$ and define

$$\delta'(U,V) \;:=\; \delta'(U',V') \;=\; \min_{u'\in U'\;,\;||u'||=1} \;||\, u' - p_{V'}(u')\,|| \;=\; sin\ \alpha'(U',V')$$

and call $\alpha'(U,V) := \alpha'(U',V')$ the *minimum essential angle between U and V* and similarly we call $\delta'(U,V) := \delta'(U',V')$ the *minimum essential distance between U and V.*

Definition 4.8 and Remark 4.9, obviously imply:

Remark 4.11. *For every two arbitrary vector subspaces U, V of E :*
1) $U \cap V = \{0\} \iff U' = U$ and $V' = V \iff U' = U$ <u>or</u> $V' = V$.
2) $\delta'(U,V) := \delta'(U',V') = \delta'(V',U') = \delta'(V,U)$. $\square$

Thus Definition 4.8 extends Definition 4.7 and allows us to obtain that the fonction:

$$\delta' \;:\; \mathbb{G}(E) \times \mathbb{G}(E) \longrightarrow [0,1] \quad , \quad \delta'(U,V) := \delta'(U',V')$$

is a symmetric function, where $\mathbb{G}(E)$ denotes the Grassmann manifold of all vector subspaces of E. Moreover we have:

Remark 4.12. *For every pair of vector subspaces U, V of E:*

1) $\delta'(U,V) = 0 \iff U \subseteq V$ *or* $U \supseteq V$.

2) *If* $\dim U = \dim V$; $\delta'(U,V) = 0 \iff U = V$.

3) $\delta'(U,V) := \delta'(U',V') = \delta'(U',V) = \delta'(U,V')$.

Proof 1), 2). It follows easily since: $U \subseteq V$ if and only if $U' = \{0\}$ and then $\delta'(U,V) = 0$.

Proof 3). Since $V = (U \cap V) \oplus V'$ is an orthogonal sum, for every $u' \in U'$ its projection $p_V(u')$ on V decomposes into the orthogonal sum $p_V(u') = p_{U\cap V}(u') + p_{V'}(u')$.

Moreover, since u', lying in U', is orthogonal to $U \cap V$, one has $p_{U\cap V}(u') = 0$ and $p_V(u) = p_{V'}(u')$.

By definition 4.8,

$$\delta'(U,V) \quad = \quad \delta'(U',V') \quad = \quad \min_{u' \in U'\,,\, ||u'\,||=1} \quad ||\, u' - p_{V'}(u')\,||\,.$$

Since $U' \cap V \subseteq U \cap U' \cap V = U' \cap (U \cap V) = \{0\}$, then $U' \cap V = \{0\}$ and

$$\delta'(U',V) \quad = \quad \min_{u' \in U'\,,\, ||u'||=1} \quad ||\, u' - p_V(u')\,|| \;.$$

Since $p_V(u') = p_{V'}(u')$ for every $u' \in U'$ one finds: $\delta'(U,V) := \delta'(U',V') = \delta'(U',V)$.

Finally, δ' being a symmetric function (Remark 4.11.2), this last equality also implies:

$$\delta'(U,V) \;:=\; \delta'(U',V') \;=\; \delta'(V',U') \;=\; \delta'(V',U) \;=\; \delta'(U,V')\,. \qquad \square$$

One sees moreover easily that δ' is *a decreasing* function with respect to both variables U, V.

As one can see with simple examples, δ' *is not a metric* also when restricted to a family of subspaces of the same dimension, except for the 1-dimensional case.

4.3. ***Sufficient conditions and geometric meaning.*** With the same hypotheses and notations as in §4.1 and §4.2, if U, V are the two vector subspaces $U := T_yY$ and $V := \ker f_{*y}$ of $E := T_yM$, the essential mutual subspace U' is:

$$U' \;:=\; [T_yY]' \;= \perp (T_yY \cap \ker f_{*y}; T_yY) \;= \perp\, (\ker f_{Y*y}; T_yY) = \mathcal{D}(y)\,.$$

We can then define (using also Remark 4.12.3) the function

$$\delta_Y : Y \to [0,\infty[\quad , \quad \delta_Y(y) := \delta'(T_yY, \ker f_{*y}) = \delta'(\mathcal{D}(y), \ker f_{*y})$$

and we have:

Theorem 4.5. *Let $f : M \to M'$ be a C^1 map between C^1 manifolds, $\mathcal{W} \subseteq M$ and $\mathcal{W}' \subseteq M'$ Whitney stratifications such that the restriction $f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ is a stratified surjective submersion.*

Let $X < Y$ be strata of $\mathcal{W}$ and $x \in X$ and consider the function δ_Y defined by

$$\delta_Y : Y \to [0,\infty[\quad , \quad \delta_Y(y) := \delta'(T_yY, \ker f_{*y}) = \delta'(\mathcal{D}(y), \ker f_{*y})\,.$$

If $f : M \to M'$ is a submersion at x, the following conditions are equivalent:

1) $\liminf_{y\to x} \delta_Y(y) > 0$.

2) For every sequence $\{y_i\}_i \subseteq Y$ such that $\lim_i y_i = x \in X$ and $\lim_i \mathcal{D}(y_i) = \sigma$ exists, for every unit vector $u \in \lim_i \mathcal{D}(y_i)$ and every sequence $\{u_i \in \mathcal{D}(y_i)\}_i$, of unit vectors converging to $u = \lim_i u_i$, there exists a subsequence of images $\{u'_{i_h} = f_{Y*y_{i_h}}(u_{i_h})\}_h$ such that $\inf_h ||u'_{i_h}|| > 0$.

3) For every sequence $\{y_i\}_i \subseteq Y$ such that $\lim_i y_i = x \in X$ and $\lim_i f_{Y*y_i}(T_{y_i}Y) = \tau'$ exists, for every $v' \in \lim_i f_{Y*y_i}(T_{y_i}Y) - \{0\}$, every sequence $\{v'_i \in f_{Y*y_i}(T_{y_i}Y) - \{0\}\}_i$ converging to $v' = \lim_i v'_i$, has an upper bounded subsequence of canonical liftings $\{v_{i_h} = f^{-1}_{Y*y_{i_h}|\mathcal{D}(y_{i_h})}(v'_{i_h})\}_h$.

Proof $(1 \Rightarrow 2)$. Let suppose that 2) does not hold.

Then, for a sequence $\{y_i\}_i \subseteq Y$, $\lim_i y_i = x \in X$, $\lim_i \mathcal{D}(y_i) = \sigma$ and there exists a unit vector $u \in \lim_i \mathcal{D}(y_i)$ which is a limit of a sequence of unit vectors $\{u_i \in \mathcal{D}(y_i)\}_i$ such that $\lim_i \; ||f_{Y*y_i}(u_i)|| = 0$ and hence necessarily $\lim_i \; f_{Y*y_i}(u_i) \; = 0$.

As f is C^1 at x, one has:

$$f_{*x}(u) \;=\; f_{*x}(\lim_i u_i) \;=\; \lim_i f_{*y_i}(u_i) \;=\; 0 \qquad \text{that is:} \qquad u \in \ker f_{*x}\,.$$

Since, for every i, $\mathcal{D}(y_i) \cap \ker f_{*y_i} = \{0\}$ and $\delta_Y(y_i)$ is the essential *minimal* distance

$$\delta_Y(y_i) \;=\; \delta'(\mathcal{D}(y_i), \ker f_{*y_i}) \;=\; \min_{u'_i \in \mathcal{D}(y_i)\;,\;||u'_i\;||=1} \delta(u'_i,\;\ker\, f_{*y_i})\,,$$

and as $u_i \in \mathcal{D}(y_i)$ by Remark 4.9.6, we can write:

$$0 \le \delta_Y(y_i) = \delta'(\mathcal{D}(y_i),\;\ker\, f_{*y_i}) \le \delta(u_i, \ker\, f_{*y_i}) \le 2\delta(u_i, \ker\, f_{*x}) + \delta(\ker\, f_{*x}, \ker\, f_{*y_i})\,.$$

Since $\lim_i u_i = u$, and $u \in \ker\, f_{*x}$ (by Remark 4.8.5) we have: $\lim_i \delta(u_i, \ker\, f_{*x}) = 0$.

By hypothesis $f : M \to M'$ is a submersion at x[1] so by Proposition 3.5 and Remark 4.9.7:

$$\lim_i \ker\, f_{*y_i} = \ker\, f_{*x} \qquad \text{and} \qquad \lim_i \delta(\ker\, f_{*x}, \ker\, f_{*y_i}) = 0\,.$$

These two limits being 0, one concludes that $\lim_i \delta_Y(y_i) = 0$ which implies

$$\liminf_{y \to x} \delta_Y(y) = 0$$

in opposition to the hypothesis 1).

Proof $(2 \Rightarrow 1)$. Let us suppose in opposite that $\liminf_{y\to x} \delta_Y(y) = 0$.
There exists then a sequence $\{y_i\} \subseteq Y$ such that

$$\lim_i y_i = x \qquad\quad \text{and} \qquad\quad \lim_i \delta'(\mathcal{D}(y_i), \ker f_{*y_i}) \;=\; \lim_i \delta_Y(y_i) \;=\; 0\,.$$

Being δ' the essential *minimal* distance and $\mathcal{D}(y_i) \cap \ker f_{*y_i} = \{0\}$ for everi i, there exists then a sequence of unit vectors $\{u_i \in \mathcal{D}(y_i)\}_i$ realizing such a minimal essential distances, i.e. such that:

$$\lim_i \delta(u_i, \ker\, f_{*y_i}) = 0\,.$$

By Remark 4.9.6) one has:

$$(*): \qquad \delta(u_i, \ker\, f_{*x}) \;\le\; 2\delta(u_i, \ker\, f_{*y_i}) \;+\; \delta(\ker\, f_{*y_i}, \ker\, f_{*x})\,.$$

Now since f is C^1 at x, $\lim_i \ker\, f_{*y_i} \subseteq \ker\, f_{*x}$ (Remark 3.1) so by Remarks 4.9.7 and 4.9.1 one has[2]:

$$\lim_i \delta(\ker\, f_{*y_i}, \ker\, f_{*x}) \;=\; \delta(\lim_i \ker\, f_{*y_i}, \ker\, f_{*x}) \;=\; 0\,.$$

Then since one also has $\lim_i \delta(u_i, \ker\, f_{*y_i}) = 0$ by the $(*)$ above using Remark 4.8.5.($\Leftarrow$) one finds:

$$\lim_i \delta(u_i, \ker\, f_{*x}) = 0\,.$$

Every $u_i \in \mathcal{D}(y_i)$ being a unit vector, there exists a subsequence of indexes $\{i_k\}_k$ such that both limits $\lim_k \mathcal{D}(y_{i_k}) = \sigma$ and $u = \lim_k u_{i_k} \in \lim_k \mathcal{D}(y_{i_k})$ exist.

Then by Remark 4.8.3 one has:

$$\delta(u, \ker\, f_{*x}) = \lim_k \delta(u_{i_k}, \ker\, f_{*x}) = 0 \qquad \text{and hence} \qquad u \in \ker\, f_{*x}\,.$$

In conclusion, the sequence of images $u'_{i_k} := f_{*y_{i_k}}(u_{i_k})$ of the unit vectors $\{u_{i_k} \in \mathcal{D}(y_{i_k})\}_k$ satisfies:

$$\lim_k f_{*y_{i_k}}(u_{i_k}) \;=\; f_{*x}(\lim_k u_{i_k}) \;=\; f_{*x}(u) \;=\; 0$$

[1]If f is not a submersion at x, $\ker f_{*x} \supset \lim_i \ker f_{*y_i}$ strictly and by Remark 4.9.2: $\delta(\ker\, f_{*x}, \lim_i \ker f_{*y_i}) = 1$.

[2]Here we did not need the hypothesis: $f : M \to M'$ is a submersion at x.

and cannot have a subsequence such that $\inf_h ||u'_{i_{k_h}}|| > 0$.

Proof. $(3 \Leftrightarrow 2)$. If $v' \in \lim_i f_{Y*y_i}(T_{y_i}Y) - \{0\}$ and $\{v'_i \in f_{Y*y_i}(T_{y_i}Y) - \{0\}\}_i$ is a sequence such that $\lim_i v'_i = v'$, by Remark 3.3.1) the unit vectors $u_i := \frac{v_i}{||\ v_i\ ||}$ of the canonical liftings $v_i := f^{-1}_{Y*y|\mathcal{D}(y_i)}(v'_i) \in \mathcal{D}(y_i) - \{0\}$ of the v'_i satisfy:

$$|| \ v_i \ || \ = \ \frac{||\ v'_i\ ||}{||\ f_{Y*y_i|D(y_i)}(u_i)||} = \frac{||\ v'_i\ ||}{||\ f_{Y*y_i}(u_i)||} .$$

Hence, being $\{v'_i\}_i$ converging to v', the sequence of canonical liftings $\{v_i\}_i$ has an upper bounded subsequence $\{v_{i_h}\}_h$ if and only if the sequence of images $\{u'_i := f_{Y*y_i}(u_i)\}_i$ admits a subsequence $\{u'_{i_h} := f_{Y*y_{i_h}}(u_{i_h})\}_h$ such that $\inf_h ||u'_{i_h}|| > 0$. $\square$

By recalling the definition 3.5 of the fonctions h_Y and H_Y with the same proof as above, Theorem 4.5 can be simply and analytically stated as follows:

Corollary 4.4. *Let $f : M \to M'$ be a C^1 map between C^1 manifolds, $\mathcal{W} \subseteq M$ and $\mathcal{W}' \subseteq M'$ Whitney stratifications such that the restriction $f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ is a stratified surjective submersion.*

Let $X < Y$ be strata of $\mathcal{W}$, $x \in X$ and δ_Y the function:

$$\delta_Y : Y \to [0, \infty[\qquad , \qquad \delta_Y(y) = \delta'(T_yY, \ker f_{*y}) = \delta'(\mathcal{D}(y), \ker f_{*y}) .$$

If $f : M \to M'$ is a submersion at x, the following conditions are equivalent:

1) $\liminf_{y\to x} \delta_Y(y) > 0$ *;*

2) $\liminf_{y\to x} h_Y(y) > 0$ *;*

3) $\limsup_{y\to x} H_Y(y) < +\infty$ *.* $\square$

We deduce then the following analytic sufficient condition for $f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ to satisfy condition (D) at $x \in X < Y$:

Corollary 4.5. *Let $f : M \to M'$ be a C^1 map between C^1 manifolds, $\mathcal{W} \subseteq M$ and $\mathcal{W}' \subseteq M'$ Whitney stratifications such that the restriction $f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ is a stratified surjective submersion.*

Let $X < Y$ be strata of $\mathcal{W}$ and $x \in X$. If $f : M \to M'$ is a submersion at x, we have:

$$\liminf_{y\to x}\ \delta_Y(y) > 0 \qquad \Longrightarrow \qquad f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}' \textit{ satisfies condition } (D) \textit{ at } x \in X < Y.$$

Proof. The proof follows easily by Theorem 4.5 (or Corollary 4.4) and Corollary 4.3. $\square$

In Theorem 4.5 and its Corollaries 4.4 and 4.5, we gave sufficient conditions to obtain condition (D) at a point $x \in X < Y$ using a function $\delta_Y(y) = \delta'(T_yY, \ker f_{*y}) = \delta'(\mathcal{D}(y), \ker f_{*y})$ depending on the stratum Y and *intrinsically* defined with respect to the point $x \in X \subseteq \overline{Y}$.

We can also obtain a similar result using a function depending on Y and x, by setting this time $U := T_yY$ and $V := \ker f_{*x}$. In this case the essential mutual subspace U' is:

$$U' \ := \ [T_yY]' \ = \perp (T_yY \cap \ker f_{*x} \ ; \ T_yY)$$

and we can define the function:

$$\delta_{Y,x} : Y \to [0, \infty[\qquad , \qquad \delta_{Y,x}(y) := \delta'(T_yY, \ker f_{*x}) .$$

A priori, $[T_yY]'$ is not equal to $\mathcal{D}(y)$ and $\delta_{Y,x}(y)$ is not equal to $\delta'(\mathcal{D}(y), \ker f_{*x})$.

Later on we will denote $\mathcal{D}'(y)$ for $[T_yY]'$.

Proposition 4.9. *Let $f : M \to M'$ be a C^1 map, $\mathcal{W} \subseteq M$ and $\mathcal{W}' \subseteq M'$ Whitney stratifications such that the restriction $f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ is a stratified surjective submersion.*

Let $X < Y$ be strata of $\mathcal{W}$, $x \in X$ and $\{y_i\}_i \subseteq Y$ a sequence such that $\lim_i y_i = x$ and both limit below exist. If $f : M \to M'$ is a submersion at x, then:

$$\liminf_i \delta_{Y,x}(y_i) = 0 \qquad \Longleftrightarrow \qquad \liminf_i \delta_Y(y_i) = 0 \,.$$

Proof. For every $i \in \mathbb{N}$, let $\mathcal{D}'(y_i) := [T_{y_i}Y]'$ and $\mathcal{D}(y_i)$ be the vectors subspaces of $T_{y_i}Y$:

$$\mathcal{D}'(y_i) := \perp (T_{y_i}Y \cap \ker f_{*x} \; ; \; T_{y_i}Y) \qquad \text{then} \quad \mathcal{D}'(y_i) \cap \ker f_{*x} = \{0\}$$

$$\mathcal{D}(y_i) := \perp (T_{y_i}Y \cap \ker f_{*y_i} \; ; \; T_{y_i}Y) \qquad \text{then} \quad \mathcal{D}(y_i) \cap \ker f_{*y_i} = \{0\} \,.$$

By considering possibly subsequences we can suppose that both the limits exist:

$$\sigma' := \lim_i \mathcal{D}'(y_i) \qquad \text{and} \qquad \sigma := \lim_i \mathcal{D}(y_i) \,.$$

and since $f : M \to M'$ is a submersion at x, $\lim_i \ker f_{*y_i} = \ker f_{*x}$ (Proposition 3.5) and $\sigma' = \sigma$.

By Remark 4.12.3 and being every $\delta_{Y,x}(y_i) = \delta'(\mathcal{D}'(y_i), \ker f_{*x})$ a minimal essential distance, there exists, for every i, a unit vector $v_i \in \mathcal{D}'(y_i) \subseteq T_{y_i}Y$ such that:

$$\delta_{Y,x}(y_i) = \delta'(\mathcal{D}'(y_i), \ker f_{*x}) = \min_{u'_i \in \mathcal{D}'(y_i)\,,\, ||u'_i||=1} \delta(u'_i, \; \ker \, f_{*x}) = \delta(v_i, \; \ker \, f_{*x})$$

and (by taking possibly a subsequence) we can also suppose that there exists $\lim_i v_i = v \in \sigma'$.

Similarly there exists a unit vector $w_i \in \mathcal{D}(y_i) \subseteq T_{y_i}Y$ such that:

$$\delta_Y(y_i) = \delta'(\mathcal{D}(y_i), \ker f_{*y_i}) = \min_{u_i \in \mathcal{D}(y_i)\,,\, ||u_i||=1} \delta(u_i, \; \ker \, f_{*y_i}) = \delta(w_i, \; \ker \, f_{*y_i})$$

and such that there exists $\lim_i w_i = w \in \sigma$.

Proof ($\Rightarrow$). If $\liminf_i \delta_{Y,x}(y_i) = 0$, by extracting possibly a subsequence, one can write:

$$0 \; = \; \lim_i \delta_{Y,x}(y_i) \; = \; \lim_i \delta(v_i, \; \ker \, f_{*x}) \; = \; \delta(v, \; \ker \, f_{*x}) \qquad \text{and so:} \quad v \; \in \; \ker \, f_{*x}.$$

Let $p_i : T_{y_i}Y \to \mathcal{D}(y_i)$ be the orthogonal projection on $\mathcal{D}(y_i)$ and $\omega_i := \; p_i(v_i) \in \mathcal{D}(y_i)$. Then:

$$\lim_i \omega_i \; = \; \lim_i p_i(v_i) \; = \; p_\sigma(v) \; = \; v \qquad \text{as} \qquad v \in \sigma' = \sigma.$$

Since $\omega_i \in \mathcal{D}(y_i)$ and by Remark 4.9.6) we find:

$$\delta_Y(y_i) \; = \; \delta(w_i, \ker f_{*y_i}) \; \leq \; \delta(\omega_i, \ker f_{*y_i}) \; \leq 2\delta(\omega_i, \ker f_{*x}) \; + \; \delta(\ker f_{*x}, \ker f_{*y_i})$$

and being $\lim_i \omega_i = v \in \ker f_{*x}$ and $\lim_i \ker f_{*y_i} = \ker f_{*x}$ we conclude:

$$0 \; \leq \; \lim_i \delta_Y(y_i) \; \leq \; 2\delta(v, \ker f_{*x}) \; + \; \delta(\ker f_{*x}, \lim_i \ker f_{*y_i}) \; = 0 + 0 = 0.$$

Proof ($\Leftarrow$). It is completely dual to the proof ($\Rightarrow$) and it could be omitted.

If $\liminf_i \delta_Y(y_i) = 0$, by extracting possibly a subsequence, one can write:

$$0 = \lim_i \delta_Y(y_i) = \lim_i \delta(w_i, \; \ker \, f_{*y_i}) = \delta(w, \; \lim_i \ker \, f_{*y_i}) \qquad \text{and so:} \quad w \in \lim_i \ker \, f_{*y_i} \subseteq \ker \, f_{*x}.$$

Let $p'_i : T_{y_i}Y \to \mathcal{D}'(y_i)$ be the orthogonal projection on $\mathcal{D}'(y_i)$ and $\theta_i := p'_i(w_i) \in \mathcal{D}'(y_i)$. Then:

$$\lim_i \theta_i = \lim_i p'_i(w_i) = p_{\sigma'}(w) = w \qquad \text{as} \qquad w \in \sigma = \sigma'.$$

Since $\theta_i \in \mathcal{D}'(y_i)$ and by Remark 4.9.6) we find:

$$\delta_{Y,x}(y_i) \; = \; \delta(w_i, \ker f_{*y_i}) \; \leq \; \delta(\theta_i, \ker f_{*y_i}) \; \leq 2\delta(\theta_i, \ker f_{*y_i}) \; + \; \delta(\ker f_{*y_i}, \ker f_{*x})$$

and being $\lim_i \theta_i = w \in \lim_i \ker f_{*y_i} = \ker f_{*x}$ we conclude:

$$0 \leq \lim_i \delta_{Y,x}(y_i) \leq 2\delta(w, \lim_i \ker f_{*y_i}) + \delta(\lim_i \ker f_{*y_i}, \ker f_{*x}) = 0 + 0 = 0. \qquad \square$$

Proposition 4.10. *With the same notations as in Theorem 4.5 and Proposition 4.9:*

$$\liminf_{y\to x} \delta_{Y,x}(y) > 0 \qquad \Longleftrightarrow \qquad \liminf_{y\to x} \delta_Y(y) > 0\,.$$

Proof. Both implications follow by Proposition 4.9 using that $\liminf_{y\to x} \delta(y)$ is the minimum value of adherence of any function δ. $\square$

Using the specific (to x) function $\delta_{Y,x}$, instead of the intrinsic (by x) δ_Y, Corollary 4.4 gives:

Theorem 4.6. *Let $f : M \to M'$ be a C^1 map between C^1 manifolds, $\mathcal{W} \subseteq M$ and $\mathcal{W}' \subseteq M'$ Whitney stratifications such that the restriction $f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ is a stratified surjective submersion.*

Let $X < Y$ be strata of $\mathcal{W}$, $x \in X$ and $\delta_{Y,x}$ the function defined by

$$\delta_{Y,x} : Y \to [0, \infty[\quad , \quad \delta_{Y,x}(y) = \delta'(T_yY, \ker f_{*x}) = \delta'(\mathcal{D}'(y_i), \ker f_{*x})\,.$$

If $f : M \to M'$ is a submersion at x, the following conditions are equivalent:

1) $\liminf_{y\to x} \delta_{Y,x}(y) > 0$ *;*

2) $\liminf_{y\to x} h_Y(y) > 0$ *;*

3) $\limsup_{y\to x} H_Y(y) < +\infty$ *.*

Proof. ($1 \Leftrightarrow 2$). It follow by Proposition 4.10 and Corollary 4.4.

Proof. ($2 \Leftrightarrow 3$). It is formally the same of the proof of Theorem, 4.5. $\square$

By Theorem 4.6 and Theorem 4.4 (or Corollary 4.3) one has:

Corollary 4.6. *Let $f : M \to M'$ be a C^1 map between C^1 manifolds, $\mathcal{W} \subseteq M$ and $\mathcal{W}' \subseteq M'$ Whitney stratifications such that the restriction $f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ is a stratified surjective submersion.*

For every strata $X < Y$ of $\mathcal{W}$ and $x \in X$ we have:

$$\liminf_{y\to x} \delta_{Y,x}(y) > 0 \quad \Longrightarrow \quad f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}' \textit{ satisfies condition } (D) \textit{ at } x \in X < Y\,. \qquad \square$$

Geometric meanings. The analytic conditions $\liminf_{y\to x} \delta_Y(y) > 0$ (in Theorem 4.5 and Corollary 4.4), and $\liminf_{y\to x} \delta_{Y,x}(y) > 0$ (in Theorem 4.6 and Corollary 4.6) for $f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ at $x \in X < Y$, have respectively the following geometric meanings:

"No limit of essential subspaces $\lim_{y_i\to x} \mathcal{D}(y_i)$ has a common direction with $\lim_i \ker f_{*y_i}$".

"No limit of essential subspaces $\lim_{y_i\to x} \mathcal{D}'(y_i)$ has a common direction with $\ker f_{*x}$".

So, in Exemple 2.1 for $f : \mathbb{R}^2 \times \{1\} \to \{0\} \times \mathbb{R} \times \{0\}$, $f(a,b,1) = (0,b,0)$ and $x = (0,0,1)$ one has:

$$\lim_{y\to x} \ker f_{*y} = \ker f_{*x} = L(1,0,0) \qquad \text{and for both choices of } Y \qquad \mathcal{D}(y) = \mathcal{D}'(y) = T_yY.$$

Hence the limits of the essential subspaces $\mathcal{D}(y)$ and the limits of the test function $\delta_Y(y)$ are:

1) For $\mathcal{W} = Y \cup \{x\} = \{y = (a, \tan(a), 1) : a > 0\} \cup \{x\}$, when Condition (D) holds (Fig. 1):

$$\begin{cases} \lim\limits_{y\to x} \mathcal{D}(y) \; = \; \lim\limits_{a\to 0} L\Big(1, \dfrac{1}{\cos^2(a)}, 0\Big) \; = \; L(1,1,0) \; \not\subseteq \; L(1,0,0) \\ \\ \text{and} \\ \\ \lim\limits_{y\to x} \delta_Y(y) \; = \; \lim\limits_{a\to 0} \; \sin\arctan \; \dfrac{1}{\cos^2(a)} \; = \; \dfrac{\sqrt{2}}{2} \; > \; 0\,. \end{cases}$$

2) For $\mathcal{W} = Y \cup \{x\} = \{y = (a, a^2, 1) : a > 0\} \cup \{x\}$ when Condition (D) does not hold (Fig. 2):

$$\begin{cases} \lim\limits_{y\to x} \mathcal{D}(y) \; = \; \lim\limits_{a\to 0} L(1, 2a, 0) \; = \; L(1,0,0) \subseteq \; L(1,0,0) \\ \\ \text{and} \\ \\ \lim\limits_{y\to x} \delta_Y(y) \; = \; \lim\limits_{a\to 0} \; \sin\arctan\,(2a) \; = \; 0\,. \qquad \square \end{cases}$$

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DU TUMULUS AU GRADIENT HORIZONTAL

PATRICE ORRO

À David, pour son soixantième anniversaire

1. Résumé

Je voudrais, dans cet article, montrer l'inspiration et l'orientation que les échanges et la collaboration avec David ont eu sur une partie de mes travaux. En commençant par le tout début, via les tumulus, pour arriver à quelques réesultats sur le gradient sous-riemannien.

Après un rapide rappel historique dans la section 2, la section 3 regroupe quelques résultats obtenus à travers l'étude des espaces conormaux et du cône normal sur des stratifications : notamment sur l'existence et la densité des fonctions de Morse sur un espace stratifié, la relation avec les espaces conormaux et la condition (b^*), ainsi que les relations entre la dimension du cône normal et la pseudo-platitude normale dans le cadre de stratifications $(a + r^e)$-régulières.

La section 4 porte sur le gradient en géométrie sous-riemannienne, et contient en particulier deux résultats sur le gradient horizontal : le premier est que, pour un polynôme générique f, l'ensemble V_f des points critiques horizontaux de f est un ensemble algébrique lisse de dimension 1, ou est vide, et la restriction $f|_{V_f}$ est une fonction de Morse. Le second, toujours pour un polynôme générique f, indique que chaque trajectoire du gradient horizontal approchant V_f possède une limite.

2. Introduction

Les tumulus ou Barrow sont des petites surfaces utilisées à plusieurs occasions dans les travaux de David Trotman - ils sont présents dans sa thèse d'état, et sont apparus pour la première fois (pour moi) dans un article A. Kambouchner - D. Trotman [KT], "Whitney (a)-faults which are hard to detect" paru aux annales de l'ENS en 1979.

R. Thom dans son rapport sur la thèse d'état de D. Trotman indique que « Trotman a démontré [...] que cette condition (la condition (t)) était suffisante pour assurer (a) dans le cas des ensembles semi- et sous-analytiques, mais non pour les ensembles stratifiés C^∞ pour lesquels Trotman a construit des contre-exemples [...]. Dans ce but il utilise la notion de tumulus, notion qu'il a inventée. Ce sont des objets géométriques en forme de rides locales qu'on peut construire algébriquement ».

Mes recherches ont débutées, en thèse à Orsay avec David, par une étude des liens entre diverses conditions de régularités que l'on peut mettre sur une stratification et les fonctions de Morse stratifiées. Motivé en cela

- par les travaux de D. T. Lê et B. Teissier [LT],[Tei] en analytique complexe - qui montrent que (b) implique (b^*) - et ceux de V. Navarro et D. Trotman [NT] en réel - (w) implique (w^*) en sous-analytique et (b) implique (b^*) si la dimension de la petite strate est un,

- par les résultats de R. Pignoni [P] et les travaux de M. Goresky et R. Mac Pherson [GMP] sur la théorie de Morse stratifiée.

Nous avions de nombreuses discussions, et j'ai beaucoup profité de la connaissance impressionnante de David des publications dans des domaines variés allant de la théorie des stratifications (mais c'est une évidence) à la robotique en passant par la théorie du contrôle, les diverses théorie géométriques, la physique mathématique, etc... partout où la théorie des singularités pouvait interagir.

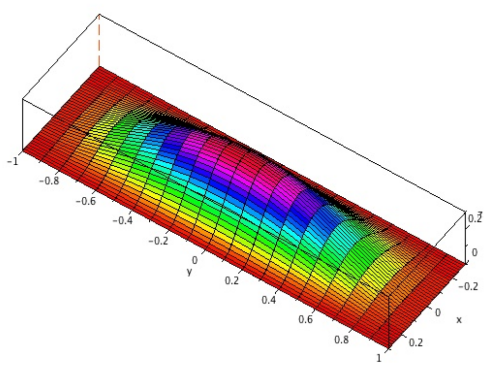
Trois articles que David m'avait indiqué à Orsay, puis à son arrivée à Marseille, me reviennent plus particulièrement à l'esprit, et ont beaucoup orienté une partie de mes travaux et collaborations futures : Subanalytic sets in the calculus of variations de M. Tamm [Tam] ; Subanalytic sets and feedback control de H. J. Sussman [Sus1] ; A new algebraic method for robot-motion-planning and real geometry de J. Canny [Can].

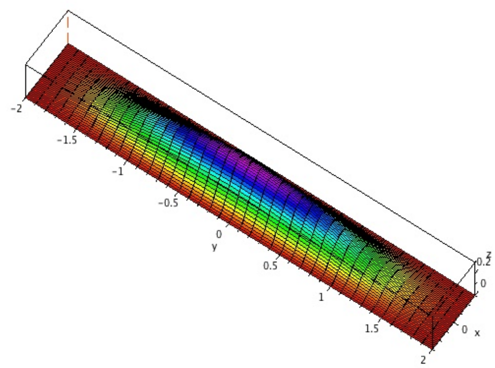
Ceux-ci mélangeaient la théorie de Morse, la propriété de Sard, l'utilisation de la théorie des ensembles semi- et sous-analytiques, des problèmes variés dans lesquels la théorie des espaces stratifiés pourrait apporter de nouvelles méthodes, la distance géodésique, les champs stratifiés, ... Leur influence se retrouve par exemple dans les articles [AOP1], [KO], [KOS], [O5], [J1] et dans [DKO] avec l'étude du gradient horizontal de fonctions polynomiales.

3. Tumulus

Soient m et r deux réels strictement positifs, un tumulus de paramètres m, r est l'ensemble $T_{m,r} = \{m^7r^3x_3 = (m^2 - x_2^2)^2(m^2r^2 - x_1^2)^2 : \quad |x_2| \leq m, |x_1| \leq mr\}$.

En voici deux illustrations :





Notant $f(x, y, z)$ la fonction $m^7r^3x_3 - (m^2 - x_2^2)^2(m^2r^2 - x_1^2)^2$ un petit calcul montre que

$$\nabla f = m^7r^3(4(1 - v^2)(1 - w^2)^2v, 4(1 - v^2)^2(1 - w^2)wr, 1)$$

où l'on a posé $v = \frac{x_1}{mr}$ et $w = \frac{x_2}{m}$.

Lorsque r tend vers 0 la direction $< \nabla f >$ tend vers un élément de l'ensemble

$$\{(4(1 - \lambda^2)(1 - \mu^2)^2\lambda, 0, 1) : \lambda, \mu \in [-1, 1]\}.$$

Rappelons qu'une fonction sur une stratification est de Morse, si sa restriction aux strates est de Morse, et si en un point y d'une strate Y le noyau de sa différentielle est transverse à toute limite d'espaces tangents à une strate X en une suite de points de X tendant vers y. A partir de cette notion de fonction de Morse sur un ensemble stratifié de Whitney, M. Goresky et R. MacPherson [GMP] ont donné les fondements d'une théorie de Morse stratifiée.

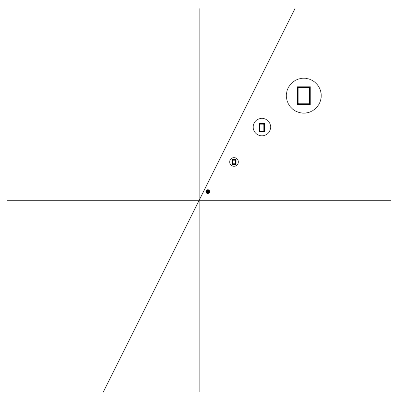
Il est très facile de voir que les fonctions de Morse ne sont pas denses en général, même sur un espace stratifié de Whitney - par exemple sur la spirale rapide $\{(r, \theta) : r = e^{-t^2}, \theta = t(2\pi), t \geq 0\}$ il n'y a pas de fonctions de Morse.

L'utilisation des tumulus a permis de construire dans [O1] une surface stratifiée vérifiant une condition de Whitney forte, tout en ayant un espace de tangents limites à l'origine de dimension topologique 2. Ce qui donne un exemple non trivial de stratification sur laquelle les fonctions de Morse ne sont pas denses, voir aussi [P] ; exemple généralisé à la classe C^q dans le second chapitre de ma thèse [O2].

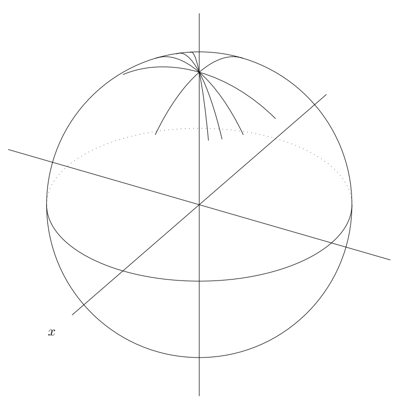
Théorème 3.1. [O1] *Il existe un espace stratifié fermé $\mathcal{Z}$, strictement whitney régulier, sur lequel les fonctions de Morse ne sont pas denses.*

Idée de démonstration. La structure normale des tumulus est reproduite à l'origine en y faisant arriver des suites de tumulus, tangentiellement à une famille dense de droites.

Le dessin ci-dessous montre une suite de cercles du plan Ox_1x_2 convergeant vers O le long d'une de ces droites, et à l'intérieur desquels des tumulus sont positionnés.



Chaque droite avec sa suite de tumulus associée donne un arc de cercle sur la sphère en structure limite



voir [O1] pour la justification complète et le détail des calculs.

L'espace des limites de normales étant fermé, on obtient ainsi $\mathcal{Z} = (S - \{0\}, \{0\})$ stratification telle que la fibre de Nash à l'origine $\tau(\mathcal{Z}, 0)$ soit de dimension 2. $\square$

Dans la catégorie sous-analytique les fonctions de Morse sont denses. Voir [P] dans le cas analytique, et [O2] et [O3] dans le cas sous-analytique et plus généralement dans le cas différentiel si l'espace conormal est "raisonnable".

La question de savoir quel type d'espaces de tangents limites on pouvait obtenir le long des strates d'une stratification s'était posée naturellement à la suite de [O1], [O2]. L'article [OT1] montre que les exemples de [O1] et [O2] peuvent être généralisés à l'aide des tumulus encore une fois :

Théorème 3.2. [OT1] *Pour tout compact étoilé K de $P^2(\mathbb{R})$, il existe une surface X de classe C^∞, telle que $X \cup \{0\}$ soit une stratification de Whitney homéomorphe à un disque, et dont la fibre de Nash à l'origine soit K.*

Idée de démonstration. La preuve se fait en prenant une suite ν_i dense dans le bord de K, puis en faisant converger des tumulus légèrement modifiés de manière à ce que la structure limite corresponde à l'arc de cercle allant de ν_i au pôle nord de la sphère. Encore un fois la fermeture de l'espace des limites de normales assure que K soit l'espace des limites de normales.

□

Ce théorème implique en particulier l'existence de stratifications à fibre de Nash de dimension topologique 1 et de dimension de Hausdorff égale à 2 (ou $1 + a$, avec a compris entre 0 et 1) - et donne aussi un exemple de stratification ayant des espaces de tangents limites fractaux et par la même un exemple de front d'onde fractal. Ces résultats ont été étendus par M. Kwiecinski et D. Trotman dans [KwT] qui ont montré que tout espace compact connexe de la grassmannienne peut être obtenu comme fibre de Nash d'une stratification de Whitney à singularité isolée.

Théorème 3.3. [KwT] *Let $n \geq 2$ and $d \leq n-1$ be positive integers. For any $K \in G^n_{n-d}$, compact and connected, there exists $\mathcal{Z} = (S - \{0\}, \{0\})$ smooth and Whitney stratified set such that* $\dim(S - \{0\}) = n - d$ *and* $\tau(\mathcal{Z}, \{0\}) = K$.

Il est immédiat à partir des définitions que l'existence des fonctions de Morse stratifiées est liée au comportement des espaces tangents limites, de même que pour la condition (b^*).

La condition (b^*) pour une stratification s'exprime comme suit. Définissons tout d'abord la notion d'aile : soit (X, Y) un couple de strates tel que $Y \subset \bar{X} \subset \mathbb{R}^N$, q un entier compris entre 1 et $\mathrm{cod}(Y)$, $1 \leq q < N - p$, et y un point de Y.

Une **aile de codimension q** en y est une sous-variété de R^N de codimension q contenant un voisinage de y dans Y. La topologie sur l'ensemble des ailes est induite par l'application $f : W \to T_yW \in G_{N-p-q}(N-p)$

Nous dirons alors que (X, Y) est b_{codq}**-régulier** en y s'il existe un ouvert dense U de l'espace des ailes de codimension q en y tel que : $\forall W \in U \;\; W \pitchfork X$ [1] **et** $(W \cap X, Y)$ est (b)-régulier en y. Nous dirons que (X, Y) est b^***-régulier** en y si b_{codq} est vérifiée en y pour tous les q entre 1 et $N - p$.

Aussi le lien entre ces deux problèmes se fait-il par les limites d'espaces tangents le long des strates. Les résultats de [O2], [O3], montrent les relations entre la densité des fonctions de Morse stratifiées et la condition (b_{cod1}).

A la suite de [O2] est apparu l'intérêt d'utiliser non pas les limites d'espaces tangents mais les limites d'hyperplans tangents, d'utilisation courante en géométrie complexe et qui était apparu à la fin de ma thèse en liaison avec les fonctions de Morse. L'introduction dans [OT2] d'un espace $B^1_Y X$ qui s'identifie au cône de Whitney dans le cas (b) régulier permet de donner une nouvelle preuve de l'existence de stratifications (b^*) - une autre était donnée dans [O2] - et ouvre la voie à une caractérisation complète de la condition (b^*) en terme de dimension du cône de Whitney, ce qui est réalisé dans [O4]. Quand au lien entre densité des fonctions de Morse, espaces conormaux et b_{cod1} il est donné par le théorème qui suit.

1. Si $\dim X + \dim Y < N, W \pitchfork X$ signifie $W \cap X = \emptyset$

Rappelons tout d'abord ce qu'est l'espace conormal : soit (X, Y) un couple de strates tel que $Y \subset \bar{X}$, l'espace conormal de X le long de Y - noté $W_Y(X)$ - est l'espace des limites d'hyperplans tangents à X le long de Y c'est à dire, en notant T_yX l'ensemble des limites en y d'espaces tangents à X :

$$W_Y(X) = \{(y, H) \in Y \times G^N_{N-1} : \exists T \in T_yX \quad T \subset H\}.$$

La fibre en y de la projection $W_Y(X) \to Y$ est notée $W_y(X)$.

Théorème 3.4. [O3] *Soient Σ une stratification (b)-régulière à espaces conormaux non fractal, alors on a équivalence de :*

(i) les fonctions de Morse sont denses dans $C^k(|\Sigma|)$

(ii) $\forall X, Y \in \Sigma$ tel que $Y \subset \bar{X}$ l'ensemble $\{y \in Y : \dim_h W_yX < N - \dim Y - 1\}$ est dense dans Y.

(iii) $\forall X, Y \in \Sigma$ tel que $Y \subset \bar{X}$ l'ensemble $\{y \in Y : (X, Y)$ est $b_{cod1}-$ régulier en $y\}$ est dense dans Y.

Les tumulus refont un apparition dans un article de 2002 [OT4] sur les cônes normaux. Si A est un sous-ensemble et Y une sous-variété de $\mathbb{R}^n$, le cône normal de A le long de Y peut être vu comme le diviseur exceptionnel de l'éclatement sphérique de A le long de Y.

La condition b^* est aussi liée au comportement de la multiplicité le long des strates ou plutôt, travaillant essentiellement en réel, avec la fonction de densité (voir [KR] ou la thèse de G. Comte [Com]). Ceci est une autre motivation pour l'étude du cône normal à une stratification, l'article [OT3] donne en particulier un contrôle de la dimension du cône normal - espace des limites de directions normales - à une stratification sous-analytique ou $(w + \delta)$-régulière.

Tout d'abord rappelons les définitions des conditions (n) et (ppn) :

Soit $\mathcal{Z}$ un fermé stratifié de $\mathbb{R}^n$. Pour chaque strate Y de $\mathcal{Z}$ on note $C_Y\mathcal{Z}$ le cône normal de $\mathcal{Z}$ le long de Y, c'est à dire la restriction au-dessus de Y de l'adhérence de l'ensemble

$$\{(x, \mu(x\pi(x))) : x \in \mathcal{Z} - Y\} \subset \mathbb{R}^n \times S^{n-1},$$

où π est la projection canonique locale sur Y, et $\mu(x)$ le vecteur unitaire $\dfrac{x}{\|x\|}$. Notons p_1 la projection $C_Y(\mathcal{Z}) \to Y$.

Condition (n) : La fibre $(C_Y\mathcal{Z})_y$ de p_1 en un point y de Y est le cône tangent $C_y(\mathcal{Z}_y)$ à la fibre $\mathcal{Z}_y = \mathcal{Z} \cap \pi^{-1}(y)$ de $\mathcal{Z}$ en y .

Condition de pseudo-platitude normale (ppn) : La projection $p : C_Y\mathcal{Z} \to Y$ est ouverte pour toute strate Y de $\mathcal{Z}$.

Les stratifications sous-analytiques vérifiant les conditions $(a+n)$ ou (ppn) ont un cône normal ayant un bon comportement du point de vue de la dimension des fibres. En effet elles vérifient la condition

$$\dim(C_Y\mathcal{Z})_y \leq \dim \mathcal{Z} - \dim Y - 1.$$

voir [OT3] et [OT4].

Le cône tangent $C_y(\mathcal{Z}_y)$ à la fibre $\mathcal{Z}_y = \mathcal{Z} \cap \pi^{-1}(y)$ (et donc la fibre $(C_Y\mathcal{Z})_y$ du cône normal, supposant (n)) peut être assez arbitraire : des travaux de Ferrarotti, Fortuna et Wilson montrent que tout cône semi-algébrique fermé de codimension ≥ 1 est réalisé comme le cône tangent en un point d'une certaine variété algébrique réelle [FFW], et comme déjà indiqué *tout* cône fermé est réalisé comme le cône tangent en une singularité isolée d'un certain espace stratifié $C^\infty(b)$-régulier [KwT].

Les premiers résultats dans la direction de l'étude de (n) et (ppn) ont été obtenus par H. Hironaka, qui montre dans [Hir] qu'une stratification de Whitney d'un ensemble analytique (réel ou complexe) est normalement pseudo-plate le long de chaque strate. J. P. Henry et M. Merle, dans [HM2], ont montré l'ouverture de la projection du cône normal d'une strate X le long d'une strate Y, où X et Y sont des strates adjacentes d'une stratification de Whitney sous-analytique.

Le théorème suivant extrait de [OT4] montre que (n) est vérifiée par toute stratification différentiable (a)-régulière ayant en plus une régularité (r^e) de type Kuo-Verdier : pour $0 \leq e < 1$, A vérifie la condition r^e, en y relativement à Y, si pour x dans A la quantité $R_e(x) = ||\pi(x)||^e d(T_xA, T_{\pi(x)}Y)/||x\pi(x)||$ est bornée près de y - π est la projection locale sur Y. Cette condition n'est autre que (w) pour $e = 0$, et est une variation de la condition de Kuo [Kuo].

Théorème 3.5. [OT4] *Soit A un fermé, stratifié par des variétés de classe $C^{k\geq 2}$ de manière $(a+r^e)$-régulière relativement à une strate Y. Alors $C_y(A_y) = (C_YA)_y$, pour tout point y de Y, c'est-à-dire que (n) est vérifiée.*

Toute stratification C^2 (w)-régulière vérifie automatiquement (a) et (r^e), c'est-à-dire $(a+r^e)$. Pour des strates sous-analytiques la combinaison $(a+r^e)$ est équivalente au critère (r) introduit par T.-C. Kuo en 1971, ce qui entraîne la condition (b) de Whitney [Kuo] ; on sait depuis [T1] que (r) est strictement plus faible que (w) dans le cas semi-algébrique, et il existe même des exemples algébriques réels. L'équivalence de (b), (r) et (w) pour les stratifications analytiques complexes est connue depuis 1982 ([Tei], [HM2]).

Les stratifications $(a+r^e)$-régulière vérifient aussi la pseudo-platitude normale [OT4]

Théorème 3.6. [OT4] *Sous les hypothèses du théorème précédent, la projection de $C_Y\mathcal{Z}$ dans Y est ouverte, i.e. $\mathcal{Z}$ est normalement pseudo-plate le long de Y.*

L'exemple d'un "*escargot de Kuo*", déjà utilisé dans [OT3], montre qu'une stratification différentiable (b)-régulière ne vérifie pas forcément (n) ou (ppn). Les deux exemples qui suivent sont construit à l'aide des tumulus.

Exemple 1. *(w_β), (n) et tumulus.*

Si l'on affaiblit (w) en $(w_\beta), \beta < 1$, c'est-à-dire si on suppose que le rapport $\dfrac{d(T_xX, T_zY)}{\|x-\pi(x)\|^\beta}$ est borné près de y pour x dans X et z dans Y, alors la condition (n) n'est pas vérifiée.

Considérons pour cela le demi-plan $x_3 = 0, x_1 > 0$ dans $\mathbb{R}^3$, et notons C_α le morceau de courbe $\{x_1 = x_2^{\frac{2+\alpha}{\alpha}}, x_1 > 0\}$, qui est tangent à $(0x_2)$. Centrons aux points $(x_1^i, x_2^i, 0) = (r_i^{1+\alpha}, r_i^{\frac{\alpha(1+\alpha)}{2+\alpha}}, 0)$ des tumulus $T_{r_i^\alpha, r_i}$, avec une suite r_i qui tend vers 0 de sorte que les tumulus soient disjoints.

Alors, si l'on note X le demi-plan perturbé le long de C_α et $Y = (0x_2)$, on obtient une stratification $(w_{\frac{1}{1+\alpha}})$-régulière, pour laquelle le cône normal n'est pas obtenu dans la fibre. En effet, en notant π la projection sur Y, et en notant

$$\xi = \frac{x_1 - x_1^i}{r_i^{1+\alpha}} \quad \text{et} \quad \chi = \frac{x_2 - x_2^i}{r_i^\alpha}$$

sur les tumulus, nous avons que

$$\|x - \pi(x)\| \equiv \frac{3}{2} r_i^{1+\alpha}, \quad \text{et} \quad d(T_xX, Y) \equiv -4\chi(\chi^2-1)(\xi^2-1)^2 r_i,$$

de sorte que

$$\frac{d(T_xX, Y)}{\|x-\pi(x)\|^\beta} \leq \text{Cte}, \quad \text{avec} \quad \beta = \frac{1}{1+\alpha},$$

c'est-à-dire que la stratification obtenue est (w_β)-régulière. De plus les fibres du cône tangent le long de Y sont des points, sauf en 0 où l'on a une courbe, étant donné que l'angle des sécantes passant par le sommet des tumulus a une ouverture constante (la tangente de cet angle est $\frac{2}{3}$). Il est clair par la construction que les limites des sécantes en 0 ne sont pas obtenues dans la fibre de π, c'est-à-dire que la condition (n) n'est pas vérifiée.

Les strates telles qu'elles sont données sont de classe C^1, mais elles peuvent être lissées sans difficulté de manière à obtenir des stratifications C^2 ayant les mêmes propriétés.

Exemple 2. *$(a+n)$ n'implique pas (ppn) avec tumulus.*

L'exemple précédent peut être modifié de sorte que la stratification obtenue soit (a)-régulière et que le cône normal soit obtenu dans la fibre en 0 de la projection sur Y.

En effet, centrons une suite de tumulus T_{m_i,m_i} aux points $(m_i^2, 0)$ de l'axe $(0x)$, où $m_i \to 0$ et les m_i soient tels que les tumulus ne se rencontrent pas. Notons encore X la surface obtenue, et $Y = (0y)$. Les tumulus donnent naissance à un cône tangent limite en 0 de dimension 1, provenant de suites de points situés sur l'axe $(0x)$. Les fibres du cône tangent le long de $Y = (0y)$ sont encore des points sauf en 0, où la fibre est de dimension 1, et la projection n'est donc pas ouverte. La condition (a) est vérifiée - il suffit de constater que les normales limites en 0 sont dans le plan $(0xz)$.

G. Valette et David Trotman ont observé par ailleurs comment construire un exemple algébrique de stratification $(a+n)$-régulière ne vérifiant pas (ppn) : considérer la surface donnée en coordonnées cylindriques par $\{r = (z^2 + \sin^2\theta)\cos\theta\}$.

4. Vers le gradient horizontal

Dans [Tam], M. Tamm donne des conditions pour que - Ξ étant une variété hilbertienne, π et E deux applications à valeurs respectivement dans une variété riemannienne M et dans $\mathbb{R}$ - la fonction $d(x) = \inf\{E(\xi) : \xi \in \pi^{-1}(x)\}$ soit sous-analytique.

$$\begin{tikzcd} \Xi \arrow[r, "\pi"] \arrow[d, "E"'] & M \arrow[dl, dashed, "d"] \\ \mathbb{R} & \end{tikzcd}$$

Ce résultat a été un des éléments déclencheurs de mes travaux en géométrie sous-riemannienne : appliqué dans le cas riemannien il donne une démonstration de la sous-analyticité de la distance géodésique riemannienne, mais il ne s'applique pas dans le cas des espaces stratifiés ou dans le cas sous-riemannien. Une extension de ce théorème nécessite une étude de l'espace des chemins horizontaux et des singularités des applications extrémités et énergie.

Dans ces directions j'ai codirigé plusieurs thèses dont celle de M. Alcheik soutenue en 1995, avait pour objectif principal une étude de l'espace des chemins horizontaux et une première approche des singularités de ces applications ; celle de S. Jacquet soutenue en 1997, une généralisation du théorème de Tamm et la sous-analyticité de la distance sous-riemannienne ; et celle de S. T. Dinh soutenue en 2007 sur le gradient horizontal de fonctions polynomiales. Je terminerai ce voyage dans le passé de la collaboration avec David en donnant quelques éléments de ce dernier travail, qui montre bien l'utilisation conjointe de la théorie des singularités, stratifications, sous-analytique et sous-riemannien.

Rappelons tout d'abord quelques éléments de géométrie sous-riemannienne : Δ désigne une distribution de rang constant sur une variété riemannienne M de dimension n. Un chemin horizontal est une courbe tangente à Δ de classe de Sobolev $W^{1,2}$ de $I = [0,1]$ dans M, c'est à dire absolument continue et d'énergie $\int_0^1 |\dot\gamma(t)|^2 dt$ finie. L'espace de ces chemins sera noté $\Omega(\Delta)$, si $\Delta = TM$ l'espace $\Omega(\Delta)$ est l'espace $\Omega(M)$ des chemins sur M.

Soient σ l'application origine $\gamma \in \Omega(\Delta) \to \gamma(0) \in M$ et π l'application extrémité, qui à $\gamma \in \Omega(\Delta)$ associe $\gamma(1) \in M$. L'espace $\Omega(\Delta)$ est une sous-variété hilbertienne de $\Omega(M)$, si a est un point de M l'image réciproque $\sigma^{-1}(a) = \Omega_a(\Delta)$ est une sous-variété de $\Omega(\Delta)$, l'application σ étant partout une submersion. L'application $\pi : \Omega_a(\Delta) \to M$ quant à elle n'est pas toujours une submersion, ses points critiques sont les chemins anormaux. Différentes caractérisations de ces chemins peuvent être trouvées dans la littérature, les chemins anormaux ont aussi beaucoup été étudiés en dimension 2 en liaison avec le problème de rigidité.

Considérons donc une variété analytique M de dimension n, et une distribution analytique Δ de dimension $p < n$ sur M, c'est-à-dire un sous-fibré analytique de dimension p du fibré tangent TM, muni d'une métrique analytique g_{SR} sur Δ, appelée métrique sous-riemannienne où de Carnot - Carathéodory. On suppose aussi que Δ vérifie la condition d'Hörmander, ce qui implique que la distribution est non-intégrable. Soient $X_1, \cdots, X_p$, des champs de vecteurs analytique orthonormés qui engendrent (localement) Δ, on définit le gradient horizontal d'une fonction $f \in C^\infty(M, \mathbb{R})$ par

$$\nabla^h f = \sum_{i=1}^{p} (X_i f) X_i.$$

On désigne par $V_f = \{\nabla^h f = 0\}$ l'ensemble des points critiques horizontaux de f.

Théorème 4.1. *Pour un polynôme f générique, V_f est un ensemble algébrique lisse de dimension 1 ou est vide, c-à-d qu'il existe un ensemble $L_d \subset \mathbb{R}_d[x]$, semi-algébrique ouvert et dense, tel que V_f est lisse de dimension 1 ou est vide, pour tout $f \in L_d$. De plus, pour tout f de L_d la fonction $f|V_f$ est de Morse.*

Idée de démonstration. Considérons $\bar{j}f(x) = (x, d_x f)$. Si $\Delta^0 \subset T^*M$ désigne l'orthogonal de la distribution, on a que $x \in V_f$ si et seulement si $\bar{j}f(x) \in \Delta^0$.

Posons $L(x, f) = \bar{j}f(x) : \mathbb{R}^n \times \mathbb{R}_d[x] \to \mathbb{R}^{2n}$. Une application du théorème de transversalité montre que $\{f \in \mathbb{R}_d[x] : L_f = L(\cdot, f) \pitchfork \Delta^0\}$ est dense et que $V_f = (\bar{j}f)^{-1}(\Delta^0)$ est de dimension 1. Montrer que $f_{|V_f}$ est de Morse est plus complexe et nécessite d'imposer d'autres conditions, suffisamment fines pour conserver la généricité. Voir [DKO] pour des précisions sur le type de distribution notamment et pour une démonstration complète. $\square$

L'inégalité de Łojasiewicz n'est pas valide pour le gradient horizontal, comme le montre l'exemple suivant : le gradient horizontal du polynôme $f = 2x_3$ pour la distribution d'Heisenberg, engendrée par les deux champs de vecteurs orthonormés pour la métrique sous-riemannienne

$$\left\{ \begin{aligned} X_1 &= \frac{\partial}{\partial x_1} - \frac{1}{2} x_2 \frac{\partial}{\partial x_3} \\ X_2 &= \frac{\partial}{\partial x_2} + \frac{1}{2} x_1 \frac{\partial}{\partial x_3}. \end{aligned} \right. \tag{1}$$

est $\nabla^h f = -x_2 X_1 + x_1 X_2$. Ainsi V_f est l'axe x_3, et, puisque la restriction de f à V_f n'est pas constante, en effet $f(V_f) = \mathbb{R}$, l'inégalité de Łojasiewicz n'est pas vérifiée.

Une deuxième observation est qu'une trajectoire du gradient horizontal peut être de longueur infinie, et peut même s'accumuler sur une courbe fermée.

Toutefois ces comportements sont exceptionnels dans un certain sens, et pour une fonction générique f, les trajectoires de son gradient horizontal ont des propriétés similaires au cas du gradient riemannien voir [DKO] pour plus de détails.

Donnons un autre résultat significatif extrait du même article :

Théorème 4.2. *Soit un $f \in \mathbb{R}_d[x]$ un polynôme générique, précisément supposons que $f \in L_d$, ouvert dense semi-algébrique donné par le théorème 4.1. Soient $B \subset \mathbb{R}^n$ un ensemble compact et $x(t) \subset B$ une trajectoire de $\nabla^h f$. Supposons que $x(t)$ s'approche de V_f, c-à-d qu'il existe une suite $t_m \to t_0 \in \mathbb{R} \cup \{+\infty\}$ telle que $d_{V_f}(x(t_m)) \to 0$. Alors $x(t)$ a une limite appartenant à $V_f \cap B$: il existe $x_0 \in V_f \cap B$ tel que*

$$dist_R(x(t), x_0) \to 0 \text{ quand } t \to t_0$$

où $dist_R$ est une distance riemannienne.

Les deux exemples finaux illustrent la différence de comportement du gradient horizontal (sous-riemannien) par rapport au gradient riemannien.

Exemple 3. *Plaçons nous dans $\mathbb{R}^3$ où l'on considère la distribution d'Heisenberg, et le polynôme $f(x_1, x_2, x_3) = 2x_3$. Le gradient horizontal de f s'écrit*

$$\nabla^h f = X_1 f X_1 + X_2 f X_2 = -x_2 X_1 + x_1 X_2.$$

L'ensemble des points critiques horizontaux V_f de f est l'axe x_3. Donc la généricité de dimension 1 de V_f, et la généricité sur la finitude de l'intersection de V_f avec une surface de niveau de f sont satisfaites.

Les trajectoires de $\nabla^h f$ sont déterminées par le système

$$(2) \qquad \begin{cases} \dot{x}_1 = -x_2 \\ \dot{x}_2 = x_1 \\ \dot{x}_3 = -\frac{1}{2}x_2(-x_2) + \frac{1}{2}x_1 x_1 = \frac{1}{2}(x_1^2 + x_2^2). \end{cases}$$

En résolvant ce système, nous obtenons

$$x_1 = r\cos(t) + c\sin(t)$$

où r, c sont des constantes.

Prenons $c = 0$, on a $x_1 = r\cos(t)$, $x_2 = r\sin(t)$ et $x_3 = \frac{1}{2}r^2 t + C$ où C est une constante, ainsi avec $C = 0$ nous avons $x_3 = \frac{1}{2}r^2 t$

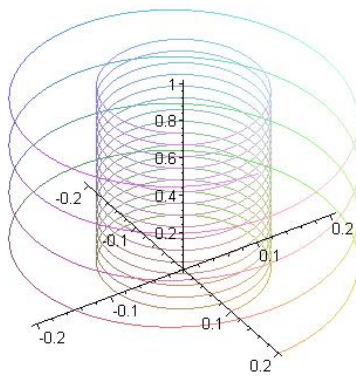
Calculons la longueur des trajectoires contenues dans la boîte

$$B = \{-\frac{1}{2} \leq x_1 \leq \frac{1}{2}, -\frac{1}{2} \leq x_2 \leq \frac{1}{2}, 0 \leq x_3 \leq 1\}$$

Soit t un temps pour lequel une trajectoire reste encore dans la boîte, on a $0 \leq x_3 = \frac{1}{2}r^2 t \leq 1$, donc $0 \leq t \leq \frac{2}{r^2}$, donc la longueur de toute trajectoire de $\nabla^h f$ dans B contenue dans le cylindre $\{x_1^2 + x_2^2 = r^2\}$ est

$$L(r) = \int_0^{\frac{2}{r^2}} ||\nabla^h f(x(t))||dt = \int_0^{\frac{2}{r^2}} \sqrt{\dot{x}_1^2(t) + \dot{x}_2^2(t)}dt = \int_0^{\frac{2}{r^2}} r dt = \frac{2}{r}.$$

Puisque r est la distance à l'axe x_3, la longueur des trajectoires du gradient n'est pas bornée uniformément dans cette boîte. En effet, la longueur devient de plus en plus grande quand la trajectoire s'approche de l'axe x_3. La vitesse de montée est plus petite que celle de rotation.



Dans cet exemple, les trajectoires de $\nabla^h f$ n'ont pas de points limites sur l'axe x_3.

Exemple 4. *On considère la distribution d'Heisenberg dans $\mathbb{R}^3$ et le polynôme*

$$f(x) = 2x_3 + \frac{1}{2}(x_1^2 + x_2^2).$$

Le gradient horizontal de f est égale à $\nabla^h f = X_1 f X_1 + X_2 f X_2$ où $X_1 f = x_1 - x_2$, $X_2 f = x_1 + x_2$. L'ensemble des points critiques horizontaux de f est l'axe x_3. Les trajectoires de $\nabla^h f$ satisfont le système d'équations différentielles suivant :

$$\begin{cases} \dot{x}_1 = x_1 - x_2 \\ \dot{x}_2 = x_1 + x_2 \\ \dot{x}_3 = \dfrac{1}{2}(x_1^2 + x_2^2). \end{cases}$$

En résolvant les deux premières équations, on obtient

$$\begin{cases} x_1(t) = e^t(a\sin(t) + b\cos(t)) \\ x_2(t) = e^t(-a\cos(t) + b\sin(t)) \end{cases}$$

où a, b sont des constantes. Alors $\dot{x}_3 = \dfrac{(a^2+b^2)}{2}e^{2t}$, donc

$$x_3 = \frac{(a^2+b^2)}{4}e^{2t} + c$$

où c est une constante. On remarque que toutes les trajectoires de $\nabla^h f$ possèdent une limite (quand $t \to -\infty$) sur l'axe x_3, qui est l'ensemble des points critiques horizontaux de f. Quand $t \to +\infty$, les trajectoires de $\nabla^h f$ s'éloignent de l'axe x_3 de manière exponentielle, donc, dans un compact, les trajectoires de $\nabla^h f$ possèdent au plus un point limite sur l'axe x_3.

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QUANTIZATION OF WHITNEY FUNCTIONS AND REDUCTION

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The paper is dedicated to David Trotman on the occasion of his 60th birthday.

Abstract. For a possibly singular subset of a regular Poisson manifold we construct a deformation quantization of its algebra of Whitney functions. We then extend the construction of a deformation quantization to the case where the underlying set is a subset of a not necessarily regular Poisson manifold which can be written as the quotient of a regular Poisson manifold on which a compact Lie group acts freely by Poisson maps. Finally, if the quotient Poisson manifold is regular as well, we show a "quantization commutes with reduction" type result. For the proofs, we use methods stemming from both singularity theory and Poisson geometry.

Introduction

In this paper we consider the synthesis of two, seemingly different, branches of mathematics, namely that of singularity theory and Poisson geometry and deformation quantization. There are motivations from both sides to consider such a blend: from the point of view of Poisson geometry and mathematical physics, singularities naturally appear when one considers Poisson manifolds with symmetries of which one wants to take the quotient. From the point of view of singularity theory, the general idea that a quantization can act as a kind of "noncommutative desingularization" has had quite a few striking applications. To make proper sense of this idea one needs to combine this with techniques coming from noncommutative geometry.

In this paper we use the notion of Whitney functions to describe the deformation quantization of a (singular) set inside a Poisson manifold. More specifically, we describe how the Fedosov method applies to construct such deformation quantizations inside a regular Poisson manifold, and prove a "quantization commutes with reduction" type of result for the quantized Whitney functions invariant under a free action of a compact Lie group that preserves the Poisson structure.

1. Formal deformation quantizations of Whitney functions

Recall that for a closed subset $X \subset M$ of a smooth manifold M the algebra of Whitney functions on X is defined as the quotient $\mathcal{E}^\infty(X; M) := \mathcal{C}^\infty(M)/\mathcal{J}^\infty(X, M)$, where

$$\mathcal{J}^\infty(X, M) := \left\{ f \in \mathcal{C}^\infty(M) \mid (Df)_{|X} = 0 \text{ for every differential operator } D \text{ on } M \right\}$$

denotes the ideal of smooth functions on M which are flat on X. If no confusion about the ambient space can arise, we briefly write $\mathcal{E}^\infty(X)$ instead of $\mathcal{E}^\infty(X; M)$. Moreover, we denote the canonical quotient map from $\mathcal{C}^\infty(M)$ to $\mathcal{E}^\infty(X; M)$, sometimes called the *jet map*, by $\mathsf{J}_{X;M}$ or J_X, if no confusion can ariese. Finally observe that if $\Phi : M \to N$ is a smooth map between

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manifolds M and N which maps the closed subset $X \subset M$ into a closed subset $Y \subset N$, then there is a canonical *pull-back map* for Whitney functions

$$\Phi^* : \mathcal{E}^\infty(Y;N) \to \mathcal{E}^\infty(X;M),$$

which maps the Whitney function $F = \mathsf{J}_{Y;N}(f)$ represented by $f \in \mathcal{C}^\infty(N)$ to the Whitney function $\mathsf{J}_{X;M}\big(f \circ \Phi\big)$. The reader will easily check that the pull-back is well-defined.

Recall further that by a *Whitney–Poisson* structure on X one understands a bilinear map $\{-,-\}$ on $\mathcal{E}^\infty(X)$ which satisfies for all $F, G, H \in \mathcal{E}^\infty(X)$ the following relations

(WP1) $\{F,G\} = -\{G,F\}$,
(WP2) $\{F,GH\} = \{F,G\}H + G\{F,H\}$, and
(WP3) $\{\{F,G\},H\} + \{\{H,F\},G\} + \{\{G,H\},F\} = 0$.

In other words, (WP1) tells that $\{-,-\}$ is an antisymmetric bilinear form, (WP2) says that $\{-,-\}$ is a derivation in each of its arguments, and (WP3) is the Jacobi identity. Hence there exists a smooth antisymmetric bivector field $\Lambda : X \to TM \otimes TM$ such that

$$\{F,G\} = \Lambda \lrcorner (dF \otimes dG) \quad \text{for all } F,G \in \mathcal{E}^\infty(X).$$

Note that we have used here the fact that $\mathcal{J}^\infty(X,M)\Omega^\bullet(M)$ is a graded ideal in $\Omega^\bullet(M)$ preserved by the exterior derivative d which gives rise to the differential graded quotient algebra

$$\Omega^\bullet_{\mathcal{E}^\infty}(X) := \Omega^\bullet(M)/\mathcal{J}^\infty(X;M)\Omega^\bullet(M).$$

Its differential will be denoted again by d. We call $\Omega^\bullet_{\mathcal{E}^\infty}(X)$ the complex of Whitney–de Rham forms on X. According to [BrPf], the cohomology of $\Omega^\bullet_{\mathcal{E}^\infty}(X)$ coincides with the singular cohomology (with values in $\mathbb{R}$), if M is an analytic manifold, and $X \subset M$ a subanalytic subset. Now we have the means to define what one understands by a formal deformation quantization of the algebra of Whitney functions.

Definition 1.1. Assume that $X \subset M$ is a closed subset of the smooth manifold M, and that $\mathcal{E}^\infty(X)$ carries a Whitney–Poisson structure. By a *formal deformation quantization* of the algebra $\mathcal{E}^\infty(X)$ or in other words by a *star product* on $\mathcal{E}^\infty(X)$ one then understands an associative product

$$\star : \mathcal{E}^\infty(X)[[\hbar]] \times \mathcal{E}^\infty(X)[[\hbar]] \to \mathcal{E}^\infty(X)[[\hbar]]$$

on the space $\mathcal{E}^\infty(X)[[\hbar]]$ of formal power series in the variable $\hbar$ with coefficients in $\mathcal{E}^\infty(X)$ such that the following is satisfied:

(DQ0) The product $\star$ is $\mathbb{R}[[\hbar]]$-linear and $\hbar$-adically continuous in each argument.
(DQ1) There exist $\mathbb{R}$-bilinear operators $c_k : \mathcal{E}^\infty(X) \times \mathcal{E}^\infty(X) \to \mathcal{E}^\infty(X)$, $k \in \mathbb{N}$ such that c_0 is the standard commutative product on $\mathcal{E}^\infty(X)$ and such that for all $F, G \in \mathcal{E}^\infty(X)$ there is an expansion of the product $F \star G$ of the form

$$F \star G = \sum_{k\in\mathbb{N}} c_k(F,G)\hbar^k. \tag{1.1}$$

(DQ2) The constant function $1 \in \mathcal{E}^\infty$ satisfies $1 \star F = F \star 1 = F$ for all $F \in \mathcal{E}^\infty(X)$.
(DQ3) The star commutator $[F,G]_\star := F \star G - G \star F$ of two Whitney functions $F, G \in \mathcal{E}^\infty(X)$ satisfies the commutation relation

$$[F,G]_\star = -i\hbar\{F,G\} + o(\hbar^2).$$

If in addition the condition

(DQ4) $\operatorname{supp}(F \star G) \subset \operatorname{supp}(F) \cap \operatorname{supp}(G)$ for all $F, G \in \mathcal{E}^\infty(X)$,

is satisfied, then the star product is called *local*.

Remark 1.2. If Π is a Poisson bivector on the smooth manifold M, then the ideal $\mathcal{J}^\infty(X;M)$ is even a Poisson ideal in $\mathcal{C}^\infty(M)$. This implies that the Poisson bracket on $\mathcal{C}^\infty(M)$ factors to the quotient $\mathcal{E}^\infty(X)$. We denote the inherited Poisson bracket on $\mathcal{E}^\infty(X)$ also by $\{-,-\}$, and call it a *global Whitney–Poisson structure.*

Now let us describe a method for constructing a formal deformation quantization of the algebra $\mathcal{E}^\infty(X)$ in case (M,Π) is a regular Poisson manifold and $\mathcal{E}^\infty(X)$ carries the corresponding global Whitney–Poisson structure. This method generalizes the original construction by Fedosov [Fed] to the Whitney function case, and has been explained in detail by the authors in [PPT12] for the particular case where the Poisson bivector comes from a symplectic structure. Recall that (M,Π) being a regular Poisson manifold means that the Poisson tensor field $\Pi : M \to TM \otimes TM$ has constant rank; see [Fed, Vai] for more details on regular Poisson manifolds. Moreover, regularity of Π implies that M is foliated in a natural way by symplectic manifolds. Denote by $\mathcal{S}$ the foliation of M by symplectic leaves which is induced by the regular Poisson tensor Π, and by $T\mathcal{S} \to M$ the subbundle of TM of all tangent vectors tangent to the symplectic leaves of the foliation. The following result then holds true. For its original proof we refer to Fedosov [Fed]; here we present a proof which also covers the later needed case of a regular Poisson manifold with a compatible G-action.

Proposition 1.3 (cf. [Fed, Sec. 5.7])**.** *For every regular Poisson manifold* (M,Π)*, there exists a* Poisson connection *which means a connection*

$$\nabla : \Gamma^\infty(T\mathcal{S}) \to \Gamma^\infty(T\mathcal{S} \otimes T^*\mathcal{S})$$

which leaves the Poisson bivector Π *invariant in the sense that*

$$\nabla \Pi = 0.$$

Moreover, if a compact Lie group G acts on M by Poisson maps, the Poisson connection ∇ can be chosen to be invariant.

Proof. Choose a riemannian metric η on M which is required to be G-invariant, if M carries a G-action compatible with the Poisson structure. Denote by

$$\nabla^{\mathrm{LC}} : \Gamma^\infty(T\mathcal{S}) \times \Gamma^\infty(T\mathcal{S}) \to \Gamma^\infty(T\mathcal{S})$$

the leafwise Levi–Civita connection of the riemannian metric restricted to $\mathcal{S}$. Moreover, let $\omega : T\mathcal{S} \otimes T\mathcal{S} \to \mathbb{R}$ be the leafwise symplectic structure induced by the Poisson bivector. Now we define a tensor field $\Delta' \in \Gamma^\infty(T^*\mathcal{S} \otimes T^*\mathcal{S} \otimes T^*\mathcal{S})$ by

$$\Delta'(X,Y,Z) = \nabla^{\mathrm{LC}}_Z \omega(X,Y) - \nabla^{\mathrm{LC}}_Y \omega(X,Z) \quad \text{for all } X,Y,Z \in \Gamma^\infty(T\mathcal{S}).$$

We then let $\Delta \in \Gamma^\infty(T^*\mathcal{S} \otimes T^*\mathcal{S} \otimes T\mathcal{S})$ be the tensor field such that

$$\omega\bigl(X, \Delta(Y,Z)\bigr) = \Delta'(X,Y,Z) \quad \text{for all } X,Y,Z \in \Gamma^\infty(T\mathcal{S}).$$

By construction it is clear that Δ' and Δ are both G-invariant, if Π and η (and hence ω) are. Now we put

$$\nabla_X Y := \nabla^{\mathrm{LC}}_X Y + \Delta(X,Y) \quad \text{for } X,Y \in \Gamma^\infty(T\mathcal{S}).$$

One readily checks that ∇ is a Poisson connection, and G-invariant, if Π and η are. □

Next, we consider the Weyl algebra bundle $\mathbb{W}_\mathcal{S} M \to M$ over M along the symplectic foliation $\mathcal{S}$. Its typical fiber over $p \in M$ is given by

$$\mathbb{W}_{\mathcal{S},p} M := \mathbb{W}(T_p\mathcal{S}) := \widehat{\mathrm{Sym}}(T^*_p\mathcal{S})[[\hbar]],$$

the space of formal power series in $\hbar$ with coefficients in the space of Taylor expansions at the origin of smooth functions on the fiber S_p of S over p. In other words, $\widehat{\mathrm{Sym}}(T_p^*\mathcal{S})$ coincides with the $\mathfrak{m}$-adic completion of the space $\mathrm{Sym}(T_p^*\mathcal{S})$ of polynomial functions on T_pS, where $\mathfrak{m}$ denotes the maximal ideal in $\mathrm{Sym}(T_p^*\mathcal{S})$. Hence, every element a of $\mathbb{W}(T_p\mathcal{S})$ can be uniquely expressed in the form

$$a = \sum_{s\in\mathbb{N},\, k\in\mathbb{N}} a_{s,k}\hbar^k, \tag{1.2}$$

where each $a_{s,k}$ is an element of $\mathrm{Sym}^s(T_p^*\mathcal{S})$, which can be naturally identified with the space of s-homogeneous polynomial functions on $T_p\mathcal{S}$. A section $a \in \mathcal{W}_{\mathcal{S}}(M) := \Gamma^\infty(\mathbb{W}_{\mathcal{S}}M)$ can be uniquely written in the form (1.2), where the $a_{s,k}$ with $s,k \in \mathbb{N}$ now are smooth sections of the symmetric powers $\mathrm{Sym}^s(T^*\mathcal{S})$. This representation allows us to define the *symbol map* $\sigma : \mathcal{W}_{\mathcal{S}} \to \mathcal{C}^\infty(M)[[\hbar]]$ by

$$\sigma(a) = \sum_{k\in\mathbb{N}} a_{0,k}\hbar^k \quad \text{for } a \in \mathcal{W}. \tag{1.3}$$

The space $\mathbb{W}(T_p\mathcal{S})$ is filtered by the *Fedosov-degree*

$$\deg_{\mathrm{F}}(a) := \min\{s + 2k \mid a_{s,k} \neq 0\}, \quad a \in \mathbb{W}(T_p\mathcal{S}).$$

The Fedosov-degree induces a filtration of the space of sections $\mathcal{W}_{\mathcal{S}}(M)$ of the Weyl algebra bundle along $\mathcal{S}$ by putting

$$\mathcal{F}^k\mathcal{W}_{\mathcal{S}}(M) := \{a \in \mathcal{W}(M) \mid \deg_{\mathrm{F}}(a(p)) \geq k \text{ for all } p \in M\}.$$

Now consider $\Omega^\bullet\mathbb{W}_{\mathcal{S}}$, the sheaf of leafwise smooth differential forms with values in the bundle $\mathbb{W}_{\mathcal{S}}M$, or in other words the sheaf of smooth sections of the (profinite dimensional) vector bundle $\mathbb{W}_{\mathcal{S}}M \otimes \Lambda^\bullet T^*\mathcal{S}$. Like $\mathcal{W}_{\mathcal{S}}(M)$, the space $\Omega^\bullet\mathcal{W}_{\mathcal{S}}(M)$ is also filtered by the Fedosov-degree.

Next, we define a non-commutative algebra structure on $\mathcal{W}_{\mathcal{S}}(M)$ and $\Omega^\bullet\mathcal{W}_{\mathcal{S}}(M)$. To this end observe first that the Poisson bivector $\Pi(p)$ on T_pM is linear and can be written in the form

$$\Pi(p) = \sum_{i=1}^{\frac{\dim S_p}{2}} \Pi_{i1}(p) \otimes \Pi_{i2}(p), \tag{1.4}$$

where $\Pi_{i1}(p), \Pi_{i2}(p) \in T_p\mathcal{S}$ for $i = 1, \cdots, \mathrm{rk}(\Pi)$. Since each of the tangent vectors $\Pi_{i1}(p), \Pi_{i2}(p)$ acts as a derivation on $\mathrm{Sym}(T_p^*\mathcal{S})$, this gives rise to the operator

$$\begin{aligned} \widehat{\Pi}(p) :\, &\mathrm{Sym}(T_p^*\mathcal{S}) \otimes \mathrm{Sym}(T_p^*\mathcal{S}) \to \mathrm{Sym}(T_p^*\mathcal{S}) \otimes \mathrm{Sym}(T_p^*\mathcal{S}), \\ &a \otimes b \mapsto \sum_{i=1}^{\frac{\mathrm{rk}(\Pi)}{2}} \Pi_{i1}(p)\cdot a \otimes \Pi_{i2}(p)\cdot b. \end{aligned} \tag{1.5}$$

Th operator $\widehat{\Pi}(p)$ does not depend on the particular representation (1.4). Note that by $\mathbb{C}[[\hbar]]$-linearity and $\mathfrak{m}$-adic continuity, $\widehat{\Pi}$ uniquely extends to an operator

$$\widehat{\Pi}(p) : \widehat{\mathrm{Sym}}(T_p^*\mathcal{S})[[\hbar]] \otimes \widehat{\mathrm{Sym}}(T_p^*\mathcal{S})[[\hbar]] \to \widehat{\mathrm{Sym}}(T_p^*\mathcal{S})[[\hbar]] \otimes \widehat{\mathrm{Sym}}(T_p^*\mathcal{S})[[\hbar]].$$

The so-called *Moyal–Weyl product* (see [BFFLS]) of two elements $a, b \in \mathbb{W}(S_p)$ is given by

$$a \circ_p b := \sum \frac{(-i\hbar)^k}{k!}\mu\big(\widehat{\Pi}(p)(a \otimes b)\big). \tag{1.6}$$

One checks easily that $\circ_p$ is a star product on $\mathbb{W}(S_p)$. Moreover, this fiberwise star product extends naturally to a noncommutative product $\star$ on $\mathcal{W}_{\mathcal{S}}(M)$, called the *Moyal–Weyl product* on the Weyl algebra bundle. For $a, b \in \mathcal{W}_{\mathcal{S}}(M)$ it is given by

$$a \circ b(p) := a(p) \circ_p b(p) \quad \text{for } p \in M. \tag{1.7}$$

Note that the Moyal–Weyl product on $\mathcal{W}_{\mathcal{S}}(M)$ satisfies by construction

$$[a,b]_\circ := a \circ b - b \circ a = -i\hbar\{a,b\} + o(\hbar^2)\,. \tag{1.8}$$

This indicates that $\mathcal{W}_{\mathcal{S}}(M)$ is already a kind of "formal deformation quantization", but just too big. It was Fedosov's fundamental idea to construct an appropriate flat connection D on $\mathcal{W}_{\mathcal{S}}(M)$ such that the subalgebra of flat sections, i.e., of sections a such that $Da = 0$, is isomorphically mapped by the symbol map onto $\mathcal{C}^\infty(M)[[\hbar]]$ and thus induces a star product on $\mathcal{C}^\infty(M)[[\hbar]]$. Let us explain Fedosov's construction of D.

We chooses a Poisson connection ∇ according to Prop. 1.3, which canonically lifts to a connection

$$\nabla : \Omega^\bullet \mathbb{W}_{\mathcal{S}}(M) \to \Omega^{\bullet+1}\mathbb{W}_{\mathcal{S}}(M).$$

Fedosov [Fed, Sec. 5.2] proved that there exists a section $A \in \Omega^1\mathcal{W}_{\mathcal{S}}(M)$ such that the connection

$$D := \nabla + \frac{i}{\hbar}[A, -]_\circ \tag{1.9}$$

is abelian, i.e., satisfies $D \circ D = 0$. Such an abelian connection D defined by a 1-form A will be called a *Fedosov connection*.

We briefly explain the uniqueness of the star product. Let $\{x^1, \cdots, x^{\mathrm{rk}(\Pi)}\}$ be leafwise coordinates along $\mathcal{S}$, and $\{y^1, \cdots, y^{\mathrm{rk}(\Pi)}\}$ be the dual elements in $T^*\mathcal{S}$. Define

$$\delta : \Omega^\bullet \mathbb{W}_{\mathcal{S}}(M) \to \Omega^{\bullet+1}\mathbb{W}_{\mathcal{S}}(M) \ \text{ and } \ \delta^* : \Omega^\bullet \mathbb{W}_{\mathcal{S}}(M) \to \Omega^{\bullet-1}\mathbb{W}_{\mathcal{S}}(M)$$

by

$$\delta a = \sum_{k=1}^{\mathrm{rk}(\Pi)} dx^k \wedge \frac{\partial a}{\partial y^k}, \qquad \delta^* a = \sum_{k=1}^{\mathrm{rk}(\Pi)} y^k \, \iota_{\frac{\partial}{\partial x^k}} a.$$

Given an abelian connection D of the form (1.9), direct computation shows that there is a canonical element $\Omega_D \in \mathcal{W}_{\mathcal{S}}(M)$, called the *curvature of* D, associated to the Poisson connection ∇ (cf. Prop. 1.3) and A such that

$$D^2 = \frac{i}{\hbar}[\Omega_D, -]_\circ.$$

Let $\Omega^\bullet_{\mathcal{S}}(M, \mathbb{R}[[\hbar]])$ be the space of leafwise differential forms along S with coefficients in $\mathbb{R}[[\hbar]]$. As $D^2 = 0$, Ω_D is in the center of $\mathcal{W}_{\mathcal{S}}(M)$, and therefore an element in $\Omega^2_{\mathcal{S}}(M, \mathbb{R}[[\hbar]])$ closed under the de Rham differential. Fedosov [Fed, Thm. 5.2.2] proved that under the requirements

(1) $\deg_{\mathrm{F}}(A) \geq 2$,
(2) $\delta^* A = 0$,

there is a unique Fedosov connection D associated to a given Poisson connection ∇ which has the given curvature form Ω. In what follows, we will always assume to work with Fedosov connections with the above assumptions.

Let us fix a Fedosov connection D and consider the space

$$\mathcal{W}_D(M) := \{a \in \mathcal{W}_{\mathcal{S}}(M) \mid Da = 0\}$$

of flat sections of the Weyl algebra bundle. $\mathcal{W}_D(M)$ is a subalgebra of $\mathcal{W}|_{\mathcal{S}}M$, as D is a derivation. Fedosov [Fed] observed that the restriction of the symbol map (1.3)

$$\sigma_{|\mathcal{W}_D(M)} : \mathcal{W}_D(M) \to \mathcal{C}^\infty(M)[[\hbar]]$$

is a linear isomorphism. Let

$$\mathfrak{q} : \mathcal{C}^\infty(M)[[\hbar]] \to \mathcal{W}_D(M)$$

be its inverse, the so-called *quantization map.* Then there exist uniquely determined differential operators $\mathfrak{q}_k : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ such that

$$\mathfrak{q}(f) = \sum_{k \in \mathbb{N}} \mathfrak{q}_k(f)\hbar^k \quad \text{for all } f \in \mathcal{C}^\infty(M), \tag{1.10}$$

and

$$\star : \mathcal{C}^\infty(M)[[\hbar]] \times \mathcal{C}^\infty(M)[[\hbar]], \quad (f,g) \mapsto \sigma\big(\mathfrak{q}(f) \circ \mathfrak{q}(g)\big)$$

is a star product on $\mathcal{C}^\infty(M)$.

Now observe that the Fedosov connection D leaves the module $\mathcal{J}^\infty(X;M) \cdot \Omega^\bullet(M;\mathbb{W}_\mathcal{S}M)$ invariant. This implies that D factors to the quotient

$$\Omega^\bullet_{\mathcal{E}^\infty}(X;\mathbb{W}_\mathcal{S}M) := \Omega^\bullet(M;\mathbb{W}_\mathcal{S}M)/\mathcal{J}^\infty(X;M) \cdot \Omega^\bullet(M;\mathbb{W}_\mathcal{S}M),$$

and acts on $\mathcal{E}^\infty(X;\mathbb{W}_\mathcal{S}M) := \mathcal{W}_\mathcal{S}(M)/\mathcal{J}^\infty(X;M)\cdot\mathcal{W}_\mathcal{S}(M)$. Moreover, the symbol map σ maps $\mathcal{J}^\infty(X;M)\cdot\mathcal{W}(M)$ to $\mathcal{J}^\infty(X;M)[[\hbar]]$, and $\mathfrak{q}\big(\mathcal{J}^\infty(X;M)[[\hbar]]\big)$ is contained in $\mathcal{J}^\infty(X;M)\cdot\mathcal{W}(M)$, since in the expansion (1.10) the operators $\mathfrak{q}_k$ are all differential operators. Hence σ and $\mathfrak{q}$ factor to $\mathcal{E}^\infty(X;\mathbb{W}M)$ respectively $\mathcal{E}^\infty(X)[[\hbar]]$. This entails the following result, which generalizes [PPT12, Thm. 1.5] to the regular Poisson case.

Theorem 1.4. *Let (M,Π) be a regular Poisson manifold, and ∇ a Poisson connection. Let $D = \nabla + A$ be the corresponding Fedosov connection on $\Omega^\bullet\mathbb{W}_\mathcal{S}$, and $X \subset M$ a closed subset. Then the space of flat sections*

$$\mathcal{W}_D(X) := \{a \in \mathcal{E}^\infty(X;\mathbb{W}_\mathcal{S}M) \mid Da = 0\}$$

is a subalgebra of $\mathcal{E}^\infty(X;\mathbb{W}_\mathcal{S}M)$, and the symbol map induces an isomorphism of linear spaces $\sigma_X : \mathcal{W}_D(X) \to \mathcal{E}^\infty(X)[[\hbar]]$. Moreover, the unique product $\star_X$ on $\mathcal{E}^\infty(X)[[\hbar]]$ with respect to which σ_X becomes an isomorphism of algebras is a formal deformation quantization of $\mathcal{E}^\infty(X)$.

By the uniqueness property of the Fedosov connection with respect to the curvature form Ω_D, we have the following functoriality property of the star products constructed in Thm. 1.4.

Proposition 1.5. *The Fedosov quantization of Whitney functions on closed subspaces of regular Poisson manifolds is functorial in the following sense. Let $\Phi : (N,\Lambda) \to (M,\Pi)$ be a Poisson map between regular Poisson manifolds which maps the closed subset $Y \subset N$ to the closed subset $X \subset M$. Assume that the restriction of Φ to each symplectic leaf of Λ is a (local) diffeomorphism, and further that ∇^N and ∇^M are Poisson connections on N respectively M such that $\nabla^N = \Phi^*\big(\nabla^M\big)$. Denote by $\mathcal{S}$ the symplectic foliation on M, by $\mathcal{R}$ the symplectic foliation on N. Let D^N resp. D^M be the corresponding Fedosov connection with the curvature form Ω_{D^N} resp. Ω_{D^M} and the induced star product $\star_Y$ resp. $\star_X$. Assume that $\Omega_{D^N} = \Phi^*\big(\Omega_{D^M}\big)$. Then the pullback $\Phi^* : \mathbb{W}_\mathcal{S}(M) \to \mathbb{W}_\mathcal{R}(N)$ is an algebra morphism*

$$\Phi^* : \big(\mathcal{E}^\infty(X;M),\star_X\big) \to \big(\mathcal{E}^\infty(Y;N),\star_Y\big)$$

which is functorial and contravariant in Φ with the above mentioned properties.

Proof. Since Φ restricts to a (local) symplectic diffeomorphism between symplectic leaves, it is straightforward to check that the pullback map $\Phi^* : T^*M \to T^*N$ lifts to a morphism of the corresponding Weyl algebra bundles,

$$\Phi^* : \mathbb{W}_\mathcal{S}(M) \to \mathbb{W}_\mathcal{R}(N).$$

As Φ is assumed to be compatible with the Poisson connections, i.e., $\nabla^N = \Phi^*\big(\nabla^M\big)$, and also with the curvature forms, i.e., $\Omega_{D^N} = \Phi^*\big(\Omega_{D^M}\big)$, the uniqueness property of the Fedosov connection with respect to the curvature form and Poisson connection implies that

$$D^N \circ \Phi^* = \Phi^* \circ D^M.$$

Hence, Φ^* restricts to an algebra morphism

$$\Phi^* : \mathcal{W}_{D^M}(M) \to \mathcal{W}_{D^N}(N),$$

and therefore a morphism

$$\Phi^* : \big(\mathcal{E}^\infty(X; M), \star_X\big) \to \big(\mathcal{E}^\infty(Y; N), \star_Y\big).$$

□

2. Whitney functions on an orbit space and their quantization

Assume that G is a compact Lie group acting freely on the smooth manifold M, and denote by $\pi : M \to N$ the canonical projection onto the orbit space $N := M/G$ which under our assumption is a smooth manifold as well. Let $X \subset M$ be a closed G-invariant subset, and $Y := X/G$. Then Y is a closed subset of N. Under these assumptions, the following result holds true.

Proposition 2.1. *The canonical projection induces a natural identification*

$$\pi^* : \mathcal{E}^\infty(Y; N) \cong \mathcal{E}^\infty(X; M)^G.$$

Here, $\mathcal{E}^\infty(X; M)^G$ denotes the set of Whitney functions represented by G-invariant smooth functions, i.e., the image of the space $\big(\mathcal{C}^\infty(M)\big)^G$ of G-invariant smooth functions on M under the jet map $\mathsf{J}_{X;M}$.

Proof. Observe first that the image of π^* lies in $\mathcal{E}^\infty(X; M)^G$ indeed by definition of the pull-back of Whitney functions and since $f \circ \pi$ is G-invariant for every $f \in \mathcal{C}^\infty(N)$. Since π is a surjective, the pull-back $\mathcal{C}^\infty(N) \to \big(\mathcal{C}^\infty(M)\big)^G$, $f \mapsto f \circ \pi$ is injective. Hence $\pi^* : \mathcal{E}^\infty(Y; N) \to \mathcal{E}^\infty(X; M)^G$ is injective as well, if we can yet show that $f \circ \pi \in \mathcal{J}^\infty(X, M)$ for $f \in \mathcal{J}^\infty(Y, N)$. But this follows from the multidimensional Faà di Bruno formula, cf. [Mic, Thm. 3.6]. More precisely, this formula says that for $x \in X$, a coordinate system $(x_1, \ldots, x_n)$ around x, a coordinate system $(y_1, \ldots, x_m)$ around $\pi(x)$, and a multiindex $\gamma \in \mathbb{N}^n$ the following equality holds true:

$$\partial^\gamma (f \circ \pi) = \sum_{\substack{\lambda = (\lambda_{i,\alpha}) \in \mathbb{N}^m \times \mathbb{N}^n \setminus \{0\} \\ \sum \lambda_{i,\alpha} \alpha = \gamma}} \frac{\gamma!}{\lambda!} \prod_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| > 0}} \Big(\frac{1}{\alpha!}\Big)^{\sum_i \lambda_{i,\alpha}} \Big(\partial^{\sum_\alpha (\lambda_{1,\alpha}, \ldots, \lambda_{m,\alpha})} f\Big) \circ \pi \prod_{i,\alpha} (\partial^\alpha \pi_i)^{\lambda_{i,\alpha}} ,$$

where π_i denotes the i-th component function of π (in a neighborhood x) with respect to the coordinate system y around $\pi(x)$. This implies that if all $\partial^{\sum_\alpha (\lambda_{1,\alpha}, \ldots, \lambda_{m,\alpha})} f$ vanish on Y then $\partial^\gamma (f \circ \pi)$ vanishes on X. Hence $f \in \mathcal{J}^\infty(Y, N)$ implies $f \circ \pi \in \mathcal{J}^\infty(X, M)$, and π^* is injective. Surjectivity of π^* follows from the Theorem by Schwarz–Mather [Schwa, Mat] which in particularly says that the map

$$\mathcal{C}^\infty(N) \to \big(\mathcal{C}^\infty(M)\big)^G, \; f \mapsto f \circ \pi$$

is split-surjective. □

Remark 2.2. This result has been proven in the general case without the restriction of the G-action to be free in [HerPfl].

Next we choose a G-invariant Poisson connection ∇ on M according to Thm. 1.3. Let us also fix the G-invariant curvature form $\Omega = -\omega$, where ω denotes the fiberwise symplectic form on TS. Then, by the preceeding section, there exists a uniquely determined Fedosov connection D having the given curvature form Ω. By construction, the connection D is G-invariant as well. Let $\star$ denote the corresponding star product on $\mathcal{C}^\infty(M)[[\hbar]]$. By invariance of D, the star product $\star$ is invariant as well, which means that for two G-invariant functions $f, g \in \mathcal{C}^\infty(M)^G$ their star product $f \star g$ is also G-invariant. This observation together with the previous proposition entails the first two claims of the following result.

Theorem 2.3. *The Fedosov star product $\star$ associated to a G-invariant Poisson connection ∇ on M (and to the G-invariant curvature form $\Omega = -\omega$) is G-invariant, hence*

$$\Big(\big(\mathcal{E}^\infty(X)[[\hbar]]\big)^G, \star\Big) \tag{2.1}$$

is a subalgebra of $\Big(\mathcal{E}^\infty(X)[[\hbar]], \star\Big)$. Moreover, under the isomorphism

$$\pi^* : \mathcal{E}^\infty(Y; N) \cong \mathcal{E}^\infty(X; M)^G$$

one obtains a star product algebra

$$\Big(\mathcal{E}^\infty(Y)[[\hbar]], \overline{\star}\Big),$$

where $F \,\overline{\star}\, G$ for $F, G \in \mathcal{E}^\infty(Y)$ is defined by $(\pi^*)^{-1}\big(\pi^*(F) \star \pi^*(G)\big)$. Finally, if (N, λ) is a regular Poisson manifold, then $\Big(\mathcal{E}^\infty(Y)[[\hbar]], \overline{\star}\Big)$ is isomorphic to the Fedosov deformation quantization $(\mathcal{E}^\infty(Y)[[\hbar]], \star_{\nabla^N})$ corresponding to a Poisson connection ∇^N on N and to the curvature form $-\omega^{\mathcal{R}}$, where $\omega^{\mathcal{R}}$ denotes the leafwise symplectic form on the symplectic foliation $\mathcal{R}$ of N.

Remark 2.4. The last statement of the theorem is a "quantization commutes with reduction" result for quantized Whitney functions.

Note that in general, the Poisson manifold N needs not be regular, hence the above theorem provides a quantization method for Whitney functions on subsets of not necessarily regular Poisson manifolds which can be written as the quotient of a regular Poisson manifold by a compact Lie group action.

Before proving the theorem, let us state some results needed in the proof.

Proposition 2.5. *Let (V, ω) be a presymplectic vector space and $W \subset V$ a linear subspace. Then the following equality holds true:*

$$\dim W + \dim W^\omega = \dim V + \dim(W \cap V^\omega).$$

Furthermore, if ω is non-degenerate and W is symplectic, then W^ω is symplectic as well.

Proof. This is a straightforward argument in linear symplectic geometry. □

Lemma 2.6. *Any element $g \in G$ maps symplectic leaves of M to symplectic leaves.*

Proof. Let $L \subset M$ be a symplectic leaf with symplectic form ω. Consider the connected submanifold $gL \subset M$, and two points $x, y \in gL$. Since Π is g-invariant, the restriction $\Pi_{|gL}$ is a Poisson bivector on gL of maximal rank, and its corresponding symplectic form coincides with $g_*\omega$. It remains to show that x and y can be connected by a piecewise smooth curve whose smooth parts are integral curves of Hamiltonian vector fields. But this is clear, since $g^{-1}x$ and $g^{-1}y$ are both elements of the symplectic leaf S, hence can be connected within L by a piecewise smooth curve γ whose smooth parts are integral curves of Hamiltonian vector fields. The curve $g\gamma$ then connects x and y and has the desired properties by G-invariance of Π. □

Proposition 2.7. *For every symplectic leaf $L \subset M$ there exists a closed subgroup $H_L \subset G$ called the* isotropy group *of L which leaves L invariant and which has the property that for each point $x \in L$ the fiber $\pi^{-1}\big(\pi(x)\big)$ coincides with the orbit $H_L x$. In other words, one has the natural isomorphism $\pi(L) \cong L/H$.*

Proof. By the preceeding lemma, the group G acts on the space Z of symplectic leaves of M. Let H_L be the isotropy group of the point $L \in Z$. Clearly, H_L then is a closed subgroup of G and has the desired properties. □

Proof of Thm. 2.3. It only remains to prove the last claim which says that the star product algebras $\big(\mathcal{E}^\infty(Y)[[\hbar]], \star_{\nabla^N}\big)$ and $\big(\mathcal{E}^\infty(Y)[[\hbar]], \overline{\star}\big)$ are isomorphic when N is regular Poisson. For this we use the well-known result [Del, Fed, Neu, BuDoWa] that on the regular Poisson manifold N, two deformation quantizations $\star$ and $\star'$ are isomorphic if and only if they have the same characteristic class in the formal cohomology $\omega/\hbar + H^2_{\mathcal{S}}(N, \mathbb{C}[[\hbar]])$, where $\mathcal{S}$ denotes the symplectic foliation. Precisely, this means that there exists a formal power series $G = 1+\hbar D_1+\ldots$ of differential operators tangent to the leaves of $\mathcal{S}$ such that

$$G^{-1}\left(G(f_1) \star G(f_2)\right) = f_1 \star' f_2.$$

Obviously, G preserves the ideal $\mathcal{J}^\infty(Y;N)[[\hbar]]$, so it induces an isomorphism between $(\mathcal{E}^\infty(Y), \star)$ and $(\mathcal{E}^\infty(Y), \star')$. Therefore, the claim follows from the fact that both $\big(\mathcal{E}^\infty(Y)[[\hbar]], \star_{\nabla^N}\big)$ and $\big(\mathcal{E}^\infty(Y)[[\hbar]], \overline{\star}\big)$ have the same characteristic class, namely $\omega^{\mathcal{R}}/\hbar$.

So finally it remains to prove that the characteristic class of $\overline{\star}$ is $\omega^{\mathcal{R}}/\hbar$, indeed (for every initially chosen G-invariant Poisson connection ∇^M and every Poisson connection ∇^N). To this end, it suffices to prove this claim for a particular choice of ∇^M and ∇^N. Fix a Poisson connection ∇^M. We first want to construct a "compatible" Poisson connection ∇^N.

Since M is foliated into symplectic leaves and the connections act leafwise, it suffices to prove the claim for each leaf separately. Due to Prop. 2.7 we can therefore assume without loss of generality, that M is symplectic, and G acts by symplectomorphisms on M. To prove the claim, we will decompose the tangent bundle TM in appropriate G-invariant subbundles which then will allow a unique lift of vector fields on N tangent to the symplectic foliation $\mathcal{R}$ of N to invariant vector fields on M having values in a certain subbundle.

To this end let G' be the standard polar pseudogroup associated to G as defined in [OrtRat, Sec. 5.5.1]. In other words, G' is the pseudogroup of local diffeomorphisms of M generated by the flows of Hamiltonian vector fields of the form $X_f := \Pi \lrcorner df : U \to TM$, where $f \in \big(\mathcal{C}^\infty(U)\big)^G$ and U is a G-invariant open subset of M. According to [OrtRat, Sec. 5.5.1 & Thm. 11.4.4], the actions by G and G' commute, and the symplectic leaves of M/G are given by the (piecewise) orbits of the induced G'-action on M/G. Let E be the vector bundle generated by such (invariant) Hamiltonian vector fields X_f. Then E together with the restriction of the symplectic form ω to E is a pre-symplectic bundle over M. By construction, the bundle E is mapped under $T\pi$ onto the tangent bundle $T\mathcal{R}$ of the symplectic foliation of N. Moreover,

$$E \subset T\mathcal{O}^\omega, \tag{2.2}$$

since one has for every $w \in E_p$, $p \in M$ and every fundamental X_ξ of an element $\xi \in \mathfrak{g}$ the relation

$$\omega\big(w, X_\xi(p)\big) = \omega\big(X_f(p), X_\xi(p)\big) = (X_\xi f)(p) = 0,$$

where the G-invariant smooth function f on M has been chosen such that $w = X_f(p)$. Now choose a G-invariant riemannian metric η on M, and let W be the orthogonal complement of $T\mathcal{O} \cap E$ in E, where $\mathcal{O}$ denotes the foliation of M by the G-orbits. By the regularity assumption on the induced Poisson structure on N it is clear that W is a vector bundle indeed. By construction, W is a G-invariant subbundle of E complementary to $E \cap T\mathcal{O}$. Since $T\mathcal{O}$ is the kernel bundle of the tangent map of the projection, $T\pi$, it follows that $T\pi$ maps W onto the tangent bundle $T\mathcal{R}$ of the symplectic foliation in such a way that fiberwise, $T\pi_{|W} : W \to T\mathcal{R}$ is a linear symplectic isomorphism. This observation allows us to construct for every vector field X on N which is tangent to $\mathcal{R}$ a unique lift $X^* : M \to W$ such that

$$T\pi\, X^*(p) = X(\pi(p)) \quad \text{for all } p \in M.$$

Now we can define a connection ∇^N on $T\mathcal{R}$ by putting, for any two vector fields X, Y on N tangent to the symplectic foliation $\mathcal{R}$,

$$\nabla^N_X Y := T\pi\, \nabla^M_{X^*} Y^* \,.$$

Clearly, ∇^N is torsion-free, so we only need to check that ∇^N is a Poisson connection. For X, Y as before let $A_{X,Y} : M \to TM$ be the vector field

$$A_{X,Y} := \big(T\pi\, \nabla^M_{X^*} Y^*\big)^* - \nabla^M_{X^*} Y^* \,.$$

By construction, $A_{X,Y}(p) \in T_p\mathcal{O}$ for all $p \in M$. This gives for the leafwise symplectic form $\omega^{\mathcal{R}}$ on $T\mathcal{R}$ and smooth vector fields X, Y, Z on N tangent to $\mathcal{R}$:

$$\begin{aligned}
& Z\big(\omega^{\mathcal{R}}(X,Y)\big)(\pi(p)) \\
&\quad = Z^*\big(\omega(X^*,Y^*)\big)(p) = \omega\big(\nabla^M_{Z^*}X^*, Y^*\big)(p) + \omega\big(X^*, \nabla^M_{Z^*}Y^*\big)(p) = \\
&\quad = \omega\big((\nabla^N_Z X)^*, Y^*\big)(p) + \omega\big(X^*, (\nabla^N_{Z^*} Y)^*\big)(p) + \\
&\qquad + \omega\big(A_{Z,X}, Y^*\big)(p) + \omega\big(X^*, A_{Z,Y}\big)(p) = \\
&\quad = \omega^{\mathcal{R}}\big(\nabla^N_Z X, Y\big)(\pi(p)) + \omega^{\mathcal{R}}\big(X, \nabla^N_{Z^*} Y\big)(\pi(p)),
\end{aligned}$$

where the last equality follows from the fact that the vector fields $A_{Z,X}$ and $A_{Z,Y}$ are tangent to the orbit direction, and that the lifted vector fields Y^* and X^* lie in the symplectic orthogonal complement of the orbit direction by Eq. 2.2. Hence, ∇^N is a Poisson connection.

Finally, observe that the leafwise symplectic form $\omega^{\mathcal{R}}$ on N and the symplectic form ω on M are related by

$$\omega(X^*,Y^*)(p) = \omega^{\mathcal{R}}(X,Y)(\pi(p)),$$

which implies that the induced Fedosov connections on N and M are related in an analogous fashion. This implies in particular that the characteristic classes of the star products $\overline{\star}$ and $\star_{\nabla^N}$ coincide in both cases with $\omega^{\mathcal{R}}/\hbar$. The proof is finished. $\square$

Remark 2.8. The proof of the theorem shows even more, namely that for the Poisson connection ∇^N constructed in the proof, the star products $\star_{\nabla^N}$ and $\overline{\star}$ on $(\mathcal{E}^\infty(Y)[[\hbar]]$ even coincide.

Example 2.9. Let (M, ω, G, J) be a Hamiltonian system with free G-action, and consider the stratification of $\mathfrak{g}^*$ with the coadjoint action by orbit types. Let $S_\circ \subset \mathfrak{g}^*$ be the open dense stratum, and put $U := J^{-1}(S_\circ)$. Then the quotient $V := U/G$ is a regular Poisson manifold, and the above "quantization commutes with reduction" result applies to any G-invariant closed $X \subset U$.

3. OUTLOOK

The results from the previous section indicate that methods of real algebraic geometry and singularity theory might be helpful in solving problems in Poisson geometry. In the following list we describe some of the problems, where we expect that combining methods from singularity theory with Poisson geometry could eventually lead to the solution of the outstanding questions.

- Even though one can construct deformation quantizations of Whitney functions over singular sets as explained above, a full (deformation) quantization theory of algebras of smooth functions over singular symplectic spaces is still lacking. Partial results exist, though, as the papers [BoHePf, HeIyPf] show, where deformation quantizations of a particular class of singular symplectically reduced spaces are constructed by homological perturbation theory. More precisely, the algebra of smooth functions on the zero level set of a G-hamiltonian system is resolved there by a Koszul complex defined by the moment map (again, under certain assumptions on the G-Hamiltonian system). The symmetry group G acts in a natural way on the Koszul complex which allows to represent the algebra of smooth functions on the symplectically reduced space as the cohomology group in degree 0 of the so-called (classical) BRST complex. Appropriately deforming the BRST complex eventually then gives rise to a deformation quantization on the symplectically reduced space. Generalizing this idea, one expects that the Koszul resolution appearing in this construction needs to be replaced by a Koszul–Tate resolution having infinite length. Sophisticated methods from commutative algebra and singularity theory then might eventually lead to the construction of star products on any symplectically reduced space.

- There are certain no go theorems on the existence of embeddings of a given symplectic (or Poisson) stratified space into a Poisson manifold, see [Egi, Dav]. It appears that methods from commutative algebra and singularity theory could share more light on this phenomenon and possibly will lead to a more precise characterization of the obstructions to such embeddings.
- Hochschild and cyclic homology theory of function algebras over singular spaces provide useful information on the existence of deformations of these algebras, and are the essential ingrediants in the study of the underlying singular spaces by methods of noncommutative geometry invented by A. Connes [Con]. Again, a synthesis of methods from singularity theory with those from differential geometry, and possibly even noncommutative geometry has already led to interesting results, see for example [NePfPoTa, PPT10, PPT12, PPT13], and might lead to further new observations in either of these areas. Work on this is in progress, see [HerPfl].

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THE NASH PROBLEM AND ITS SOLUTION: A SURVEY

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Abstract. The goal of this survey is to give a historical overview of the Nash Problem of arcs in arbitrary dimension, as well as its solution. This problem was stated by J. Nash around 1963 and has been an important subject of research in singularity theory. In dimension two the problem has been solved affirmatively by J. Fernández de Bobadilla and M. Pe Pereira in 2011. In 2002 S. Ishii and J. Kollár gave a counterexample in dimension four and higher, and in May 2012 T. de Fernex settled (negatively) the last remaining case — that of dimension three. After some history, we give an outline of the solution of the Nash problem for surfaces by Fernández de Bobadilla and Pe Pereira. We end this survey with the latest series of counterexamples, as well as the Revised Nash problem, both due to J. Johnson and J. Kollár.

1. Introduction

In this paper, $\Bbbk$ is an algebraically closed field of characteristic 0 (see Remark 1.7 below for the case of positive characteristic).

1.1. **The statement of the problem.** Let X be a singular algebraic variety over $\Bbbk$ and $\pi : \tilde{X} \longrightarrow X$ a *divisorial* resolution of singularities of X (this means that $\tilde{X}$ is a smooth variety and the **exceptional set** $E =: \pi^{-1}(Sing\ X)$ is a **divisor**, that is, is of pure codimension one). Let

$$E = \bigcup_{i \in \Delta} E_i \tag{1}$$

be the decomposition of E into its irreducible components. The set E has two kinds of irreducible components: essential and inessential. For each i let μ_i denote the divisorial valuation determined by E_i.

Definition 1.1. *We say that E_i is an* **essential divisor** *if for any other resolution $\pi' : (X', E') \to (X, Sing\ X)$ the center of μ_i on X' is an irreducible component of E'. The divisor E_i is* **inessential** *if it is not essential.*

Remark 1.2. *Intuitively, an irreducible divisor is essential if it appears, as an irreducible component, on every resolution of X.*

In general (that is, when $\dim\ X \geqslant 3$) it is quite difficult to show that a given component is essential (see [32] for a discussion of this question as well as some sufficient conditions for essentiality and [3] and [17] for new criteria of essentiality). In dimension two there exists a unique minimal resolution $\tilde{X}$ of X and each irreducible exceptional divisor of $\tilde{X}$ is essential.

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In order to study the resolution $\tilde{X}$ of X, J. Nash (around 1963, published in 1995 [26]) introduced the space X_∞^{sing} of *arcs* meeting the singular locus *Sing* X.

Definition 1.3. *An* **arc** *is a* $\Bbbk$*-morphism from Spec* $\Bbbk[[t]]$ *to* X*.*

Let X_∞^{sing} *be the set of arcs whose origin (that is, the image of the closed point) belongs to the singular locus of* X*.*

Intuitively, such an arc should be thought of as a parametrized formal curve, contained in X and meeting the singular locus of X. The analogue of an arc in complex analysis is a test map from a small disk around the origin on the complex plane to X. We will also need to consider more general arcs, which are morphisms from *Spec* $K[[t]]$ to X, where K is a field extension of $\Bbbk$; they are called K**-arcs**.

Let us denote the closed point (the origin) of *Spec* $\Bbbk[[t]]$ by 0 and the generic point by η.

An arc can be lifted to any resolution:

Lemma 1.4. *Let* $f : \tilde{X} \to X$ *be a resolution of the singularities. Every arc* $\alpha : Spec\ K[[t]] \to X$ *such that* $\alpha(\eta) \notin Sing(X)$ *can be lifted to an arc* $\tilde{\alpha} : Spec\ \Bbbk[[t]] \to \tilde{X}$*.*

The proof comes from the fact the resolution map π is proper (it is a special case of the valuative criterion of properness).

Nash showed that X_∞^{sing} has finitely many irreducible components, F_i, called *families of arcs*, and defined the following map:

Definition 1.5 (Nash [26])**.** *Let*

$$\mathcal{N} : \{\textit{irreducible components of } X_\infty^{sing}\} \to \{\textit{essential divisors of } \tilde{X}\}$$

be the map sending a family F_i *to the exceptional divisor* E_i *such that the generic arc of* F_i *has lifting to the resolution, passing through a general point of the component* E_i*.*

(see §2.2 for more details).

He showed that this map, now called the **Nash map**, is injective. The celebrated **Nash problem**, posed in [26], is the question whether the Nash map is surjective.

Let us fix a divisorial resolution of singularities $\tilde{X} \to X$ and let $E = \pi^{-1}(Sing\ X)$. Consider the decomposition (1) of E into irreducible components, as above. Let $\Delta' \subset \Delta$ denote the set which indexes the essential divisors.

M. Lejeune-Jalabert [20], inspired by Nash's original paper [26], proposed the following decomposition of the space X_∞^{sing}: for $i \in \Delta'$, let C_i be the set of arcs whose strict transform in $\tilde{X}$ intersects the essential divisor E_i transversally but does not intersect any other exceptional divisor E_j. M. Lejeune-Jalabert shows that $X_\infty^{sing} = \bigcup\limits_{i\in\Delta'} \overline{C_i}$ and the set $\overline{C_i}$ is an irreducible algebraic subvariety of the space of arcs; therefore the families of arcs are among the $\overline{C_i}$'s. Moreover there are as many $\overline{C_i}$ as essential divisors E_i. Then the Nash problem reduces to showing that the $\overline{C_i}$, $i \in \Delta'$, are precisely the irreducible components of X_∞^{sing}, that is, to proving $card(\Delta')(card(\Delta') - 1)$ non-inclusions:

Problem 1.6. *Is it true that* $\overline{C_i} \not\subset \overline{C_j}$ *for all* $i \neq j$*?*

Remark 1.7. *All of the above definitions make sense also when char* $\Bbbk > 0$*, with the following modification. An arc family is said to be* **good** *if its general element is not entirely contained in Sing* X*. When char* $\Bbbk = 0$ *it can be shown that all the arc families are good (*[16]*, Lemma 2.12). If the singularities of* X *are isolated (say,* $Sing(X) = \{\xi_1, \dots, \xi_l\}$*) then the only arcs* *contained in $Sing(X)$ are the trivial ones which map Spec $K[[t]]$ identically to one of the ξ_i. Every such arc is in the closure of every other arc passing through ξ_i. Hence the arcs contained in ξ_i belong to the closure of every irreducible component of X_∞^{sing} lying over ξ_i and cannot form an irreducible component by themselves. This proves that for X with isolated singularities X_∞^{sing} has no bad components, regardless of char $\Bbbk$. If char $\Bbbk > 0$ and $\dim Sing(X) \geqslant 1$, there may exist some bad families, and the Nash map is only defined on the set of good families. With this in mind, the Nash problem remains the same: is the Nash map, defined on the set of good families, surjective? See [37] for some recent work on the Nash problem in positive characteristic.*

1.2. **Some partial answers in dimension 2.** Before the work of Fernández de Bobadilla — Pe Pereira, the Nash problem for surfaces has been answered affirmatively in the following special cases: for A_n singularities by Nash, for minimal surface singularities by A. Reguera [34] (with other proofs by J. Fernandez-Sanchez [7] and C. Plénat [29]), for sandwiched singularities by M. Lejeune-Jalabert and A. Reguera (cf. [21] and [35]), for toric vareties in all dimensions by S. Ishii and J. Kollar [16] (using earlier work of C. Bouvier and G. Gonzalez-Sprinberg [1] and [2]), for a family of non-rational surface singularities by P. Popescu-Pampu and C. Plénat ([31]), for quotients of $\mathbb{C}^2$ by an action of finite group [27] by M. Pe Pereira in 2010 based on the work [5] of J. Fernández de Bobadilla (other proofs for $\mathbf{D_n}$ in 2004 by Plénat [30], for E_6 in 2010 by C. Plénat and M. Spivakovsky [33], (with a method that works for some normal hypersurface singularities), and by M. Leyton-Alvarez (2011) for E_6 and E_7, by applying the method for the following classes of normal hypersurfaces in $\mathbb{C}^3$: hypersurfaces $S(p, h_q)$ given by the equation $z^p + h_q(x, y) = 0$, where h_q is a homogeneous polynomial of degree q without multiple factors, and $p \geqslant 2$, $q \geqslant 2$ are two relatively prime integers [23]). A. Reguera [37] gave an affirmative answer to the Nash problem for rational surface singularities simultaneously and independently from the work [6].

See the bibliography for a (hopefully) complete list of references on the subject.

In 2011, J. Fernández de Bobadilla and M. Pe Pereira [6] showed that the answer is positive for any surface singularity. The main aim of this paper is to give an outline of their proof. Before going further into the details, we need to recall some earlier results that lead to the final proof.

The rest of the paper is organized as follows: §2 is dedicated to the work preceding the paper [6]; in §3 an outline of the proof is given. §4 contains a brief discussion of the Nash problem in dimension three and higher.

2. Previous results on the Nash problem

2.1. **The Wedge problem [18].** In 1980, M.Lejeune-Jalabert proposed to look at the Nash problem from a new point of view. She formulated in [18] what is now called “the wedge problem”, which is related to a “Curve Selection Lemma” in the space of arcs.

Let X be a singular algebraic variety over $\Bbbk$.

Let us first define **wedge**:

Definition 2.1. *Let K be a field extension of $\Bbbk$. A K-wedge on X is a $\Bbbk$-morphism*

$$\omega : Spec(K[[t, s]]) \to X$$

which maps the set $\{t = 0\}$ to Sing X.

The wedge ω induces two arcs on X as follows: a K-arc obtained by restricting ω to the set $\{s = 0\}$ (this arc is called the **special arc** of ω), and a $K((s))$-arc, obtained by restricting ω to the set $Spec(K[[t, s]]) \setminus \{s = 0\}$ (this arc is called the **general arc** of ω). We regard ω as a deformation of its special arc to its general arc or, alternatively, as an arc in the space of arcs X_∞^{sing}.

The wedge is said to be **centered** at an arc γ_0 if its special arc is γ_0.

Let $(X, 0)$ be a germ of a normal surface singularity, and let $\pi : (\tilde{X}, E) \to (X, 0)$ be its minimal (and so divisorial) resolution, with $E = \bigcup E_j = \pi^{-1}(0)$. Let E_i, E_j be irreducible components of E (they are essential as X is a surface). Let C_i and C_j be as above. Then if $C_j \subset \overline{C_i}$, E_j is not in the image of Nash map. If one had Curve Selection lemma in the space of arcs X_∞^{sing}, the inclusion above would just mean that one has a $\mathbb{k}$-wedge with special arc in C_j and generic arc in C_i. Then the morphism ω would not lift to the resolution $\tilde{X}$ as it has an indeterminacy at 0.

M. Lejeune-Jalabert proposed the following problem:

Problem 2.2. *For all irreducible essential divisors of the minimal resolution, any $\mathbb{k}$-wedge centered at $\gamma_i \in C_i$ can be lifted to $\tilde{X}$.*

It is not trivial to generalize the classical Curve Selection Lemma to the case of infinite-dimensional varieties such as X_∞^{sing}. A. Reguera proved a Curve Selection Lemma for X_∞^{sing} thus establishing the equivalence between the the Nash and the wedge problems. The wedges appearing in A. Reguera's theorem are K-wedges rather than $\mathbb{k}$-wedges, where K is an extension of $\mathbb{k}$ of infinite transcendence degree. This work of A. Reguera and its corollaries are discussed in §2.3. §2.2 is dedicated to an interpretation of the space of arcs in terms of representable functors. This interpretation is due to S. Ishii and J. Kollár [16]. It has been a great step in the resolution of the problem.

2.2. **Arc spaces as representable functors** [16]**.** Let X be a reduced scheme of finite type over $\mathbb{k}$.

Definition 2.3. *Let $\mathbb{k} \subset K$ be a field extension. A morphism $Spec(K[[t]]/(t^{n+1})) \to X$ is called an n-jet of X over K and a morphism Spec $(K[[t]]) \to X$ is called a K-arc of X. Let us denote the closed point (the origin) of Spec $K[[t]]$ by 0 and the generic point by η.*

Let $\mathcal{S}ch/\mathbb{k}$ be the category of $\mathbb{k}$-schemes and *Set* the category of sets. Define a contravariant functor

$$F_m : \mathcal{S}ch/\mathbb{k} \to Set$$

by

$$F_m(Y) = Hom_{\mathbb{k}}(Y \times_{Spec\ k} Spec(\mathbb{k}[[t]]/(t^m)), X)$$

Then, F_m is *representable* by a scheme X_m of finite type over $\mathbb{k}$. This means, by definition, that

$$Hom_{\mathbb{k}}(Y, X_m) = Hom_{\mathbb{k}}(Y \times_{Spec\ k} Spec(\mathbb{k}[[t]]/(t^m)), X)$$

for a $\mathbb{k}$-scheme Y.

This X_m is called the scheme of n-jets of X. The canonical surjection

$$\mathbb{k}[[t]]/(t^m) \to \mathbb{k}[[t]]/(t^{m-1})$$

induces a morphism $\phi_m : X_m \to X_{m-1}$. Define $\rho_m = \phi_1 \circ \cdots \circ \phi_m : X_m \to X$. A point $x \in X_m$ gives an m-jet $\alpha_x : Spec\ K[[t]]/(t^m) \to X$ and $\rho_m(x) = \alpha_x(0)$, where K is the residue field at x.

Let $X_\infty = \varprojlim X_m$ and call it the space of arcs of X. X_∞ is not of finite type over $\Bbbk$ but it is a $\Bbbk$-scheme. Denote the canonical projection $X_\infty \to X_m$ by η_m and the natural map $X_\infty \to X$ by ρ; it is the composition $\rho_m \circ \eta_m\ \forall m$. A point $x \in X_\infty$ with residue field K gives an arc $\alpha_x : Spec\ K[[t]] \to X$ with $\rho(x) = \alpha_x(0)$.

The scheme X_∞^{sing} defined earlier is nothing but the subscheme of X_∞ consisting of those arcs α for which $\alpha(0) \in Sing(X)$.

Lemma 1.4 applies equally well to K-arcs: any K-arc not contained in $Sing(X)$ has a unique lifting to any resolution of singularities $\tilde{X}$.

Let $\overline{C_i}$ be the closure of the set of arcs α that lift to a general point of a component E_i and such that $\alpha(\eta) \notin Sing(X)$ and $\alpha(0) \in Sing(X)$. Let γ_i denote the generic point of the closed irreducible set $\overline{C_i}$ and $\Bbbk_i$ the residue field of the local ring $\mathcal{O}_{X_\infty^{sing},\gamma_i}$.

Theorem 2.4 (Nash [26])**.** *The* **Nash map**

$$\mathcal{N} : \{C_i\} \to \{\textit{essential components of } \tilde{X}\}$$

given by $C_i \to E_i$ is injective.

In [16], after the reformulation of Nash problem (in any dimension), two beautiful results are shown: a positive answer to Nash problem for toric varieties in any dimension and a counter-exemple in dimension 4 and higher.

2.3. **A Curve Selection Lemma in X_∞^{sing} [36].** In the paper [36], A. Reguera has shown that a positive answer to the wedge problem is equivalent to the surjectivity of the Nash map. She has also extended the wedge problem to all dimensions. Note that she does not assume the singular varieties to be normal. More precisely, she proves the following:

Theorem 2.5. *Let X be a singular variety.*
Let E_i be an essential divisor over X. Let γ_i be the generic point of $\overline{C_i}$ (the closure of the set of arcs lifting transversally to E_i), $\Bbbk_i$ its residue field. The following are equivalent:

(1) *E_i belongs to the image of the Nash map.*
(2) *For any resolution of singularities $p : \tilde{X} \to X$ and for any field extension K of $\Bbbk_i$, any K-wedge whose special arc maps to γ_i, and whose generic arc maps to X_∞^{sing}, lifts to $\tilde{X}$.*
(3) *There exists a resolution of singularities $p : \tilde{X} \to X$ satisfying the conclusion of* (2)*.*

To prove this she needed a Curve Selection lemma for X_∞^{sing} for curves defined over K. This field is of infinite transcendence degree over $\Bbbk$, so it is quite difficult to work with. J. Fernández de Bobadilla [5] and M. Lejeune-Jalabert with A. Reguera [22] have shown, independently, that one may replace K by $\Bbbk$ in A. Reguera's theorem, provided that $\Bbbk$ is uncountable.

2.4. **The Nash Problem is a topological problem** [5]**.** In this paper, J. Fernández de Bobadilla looks at normal surface singularities, and the hypotheses of normality and dimension 2 are essential. He first gives the definition of wedges that realize an adjacency between two essential divisors.

Definition 2.6. *Let E_u and E_v be two essential divisors, and C_u and C_v the families of arcs associated to these divisors.*
A K-wedge realizes an adjacency from E_u to E_v if its generic arc belongs to C_u and its special arc belongs to C_v^o (i.e. it is transverse to E_v in a general point of E_v).

Note that if such a wedge exists, then C_v is not in the image of Nash map. This statement can be interpreted as the easy part of the Theorem of the previous section ($2 \implies 1$): a wedge realizing the adjacency cannot be lifted to any resolution.

J. Fernández de Bobadilla proves the following theorem:

Theorem 2.7. *Let $(X, 0)$ be a normal surface singularity defined over an uncountable algebraically closed field $\Bbbk$ of characteristic 0. Let E_v be an essential irreducible component of the exceptional divisor of a resolution. Then the following are equivalent:*

(1) *The set C_v is in the Zariski closure of C_u, where E_u is another component of the exceptional divisor.*
(2) *Given any proper closed subset $\mathcal{Z} \subset \overline{C_u}$, there exists an algebraic $\Bbbk$-wedge realizing an adjacency from E_u to E_v and avoiding $\mathcal{Z}$.*
(3) *There exists a formal $\Bbbk$-wedge realizing an adjacency from E_u to E_v.*
(4) *Given any proper closed subset $\mathcal{Z} \subset \overline{C_u}$, there exists a finite morphism realizing an adjacency from E_u to E_v and avoiding $\mathcal{Z}$.*

If the base field is $\mathbb{C}$ the following further conditions are equivalent to those above:

(5) *Given any convergent arc $\gamma \in C_u^o$ there exists a convergent $\mathbb{C}$-wedge realizing an adjacency from E_u to γ and avoiding the set Δ_u of arcs lifting to singular points of E_u or not transversal to E_u.*
(6) *Given any convergent arc $\gamma \in C_u^o$ there exists a convergent $\mathbb{C}$-wedge realizing an adjacency from E_u to γ.*
(7) *Given any convergent arc $\gamma \in C_u^o$ there exists a finite morphism realizing an adjacency from E_u to γ and avoiding Δ_u.*

See [5] for the definition of finite morphism realizing an adjacency from E_u to γ.

Sketch of the proof:
For 1) $\Rightarrow$ 2) J. Fernández de Bobadilla uses A. Reguera's results to obtain a K-wedge realizing an adjacency from E_u to E_v, with $\Bbbk \subset K$ an extension of $\Bbbk$. Then he uses a specialization process to obtain a $\Bbbk$-wedge realizing an adjacency from E_u to E_v and avoiding $\mathcal{Z}$. One can find a similar specialization process in [22] in which the authors characterize essential components that belong to the image of the Nash map and deduce that an irreducible exceptional divisor which is not uniruled is in the image of the Nash map (for uncountable fields).
For 4) $\Rightarrow$ 1), he needs to introduce some technical tools. First, he gets an algebraic $\Bbbk$-wedge using Popescu's theorem and Artin type approximation to replace the first formal $\Bbbk$-wedge. Then by Stein Factorization he obtains a finite morphism realizing an adjacency from E_u to E_v and avoiding $\mathcal{Z}$. He finally reduces to the case of $\Bbbk = \mathbb{C}$, and shows 1) in that case. For this, he proves a property that he calls "moving wedges":

Lemma 2.8. *Given two convergent arcs $\gamma, \gamma' \in C_v^o$, there exists a finite morphism realizing an adjacency from E_u to γ if and only if there exists a finite morphism realizing an adjacency from E_u to γ'.*

He uses the Lemma to prove the following theorem:

Theorem 2.9. *The set of adjacencies between exceptional divisors of a normal surface singularity is a combinatorial property of the singularity: it only depends on the dual weighted graph of* *the minimal good resolution. In the complex analytic case this means that the set of adjacencies only depends on the topological type of the singularity and not on the complex structure.*

The last important paper needed to understand the proof in dimension two is due to M. Pe Pereira [27], which gives an affirmative solution to the Nash problem for quotient singularities of surfaces. In that paper she has, in particular, introduced some useful tools needed in [6]. We will discuss them in the following section.

3. Solution of the Nash problem for surfaces

Theorem 3.1. *Let $\Bbbk$ be an algebraically closed field of characteristic 0 and $(X, 0)$ a normal singular surface over $\Bbbk$.*
The Nash map associated to $(X, 0)$ is bijective.

In [5] (7.2 p. 163), J. Fernández de Bobadilla shows that the families of arcs are stable under base change and so is the bijectivity of Nash map. Thus it remains to prove the theorem for complex normal surface singularities.

Let $(X, 0)$ be a normal surface singularity over $\mathbb{C}$.

The proof proceeds by contradiction.

Let $E = \bigcup_{i=0}^{n} E_i$ be the decomposition of E into irreducible components. Suppose there are two families $\overline{C_0}$ and $\overline{C_i}$ associated with two essential divisors E_0 and E_i of the minimal resolution such that $\overline{C_0} \subset \overline{C_i}$.

3.1. Definition of representatives of arcs and wedges.

The first ingredient is the definition of Milnor representative of arcs and wedges.

From now on, replace X by its underlying complex-analytic space. By abuse of notation, we will continue to denote this space by X. Let $\pi : \tilde{X} \to X$ be the minimal resolution of X.

Let us recall Milnor's work on isolated singularities.
Let B_ε denote the closed ball in $\mathbb{C}^N$ centered at the origin of radius ε and let S_ε be its boundary sphere. There exists for X a Milnor radius ε_0 such that all the spheres S_ε are transverse to X and $X \cap S_\varepsilon$ is a closed subset of S_ε for all $0 < \varepsilon \leqslant \varepsilon_0$. Let us call $X_{\varepsilon_0} = X \cap B_{\varepsilon_0}$ the Milnor representative of X. Let $\tilde{X}_{\varepsilon_0}$ be the minimal resolution of singularities of X_{ε_0}; $\tilde{X}_{\varepsilon_0}$ is nothing but the preimage of X_{ε_0} under π. Under these conditions X_{ε_0} has a conical structure and $\tilde{X}_{\varepsilon_0}$ admits E as a deformation retract.

Consider an arc $\gamma : (\mathbb{C}, 0) \to X_{\varepsilon_0}$. It is proved in [27] and [6] that there exists $\varepsilon \leqslant \varepsilon_0$ such that, restricted to X_ε, γ becomes a **Milnor arc**:

Definition 3.2. Milnor arc

A Milnor representative of γ is a map of the form

$$\gamma|_U : U \to X_\varepsilon$$

such that $\gamma|_U$ is a proper morphism, U is diffeomorphic to a closed disk, $\gamma^{-1}(\partial X_\varepsilon) = \partial U$ and the mapping $\gamma_{|U}$ is transverse to any sphere $S_{\varepsilon'}$ for $\varepsilon' \leqslant \varepsilon$. The radius ε is called a Milnor radius for γ.

Let $\alpha : (\mathbb{C}^2, 0) \to X_\varepsilon$ be an analytic wedge such that $\alpha(t, s) = \alpha_s(t)$ is the generic arc and $\alpha_0(t) = \gamma(t)$ is the special arc.
Let $\gamma\mid_U : U \to X_\varepsilon$ be a Milnor Representative of γ.

For the disk D_δ of radius δ around the origin in the complex plane we will use the notation ${D^o}_\delta = D_\delta \setminus \{0\}$.

Proposition 3.3. Milnor wedge
There exist $\delta > 0$ small enough, an open set $\mathcal{U} \subset U \times D_\delta$ and a map

$$\beta : U \times D_\delta \to X_\varepsilon \times D_\delta$$

$$(t, s) \quad \to (\alpha_s(t), s)$$

such that

- $\alpha_0(t) = \gamma\mid_U$ *is a Milnor representative of* α_0.
- *the restriction* $\beta\mid_{\mathcal{U}^o} : \mathcal{U}^o \to X^o_\varepsilon \times D^o_\delta$ *is a proper and finite morphism of analytic spaces and its image is a closed 2-dimensional closed analytic subset of* $X^o_\varepsilon \times D^o_\delta$.
- *the set* $U_s = \mathcal{U} \cap \mathbb{C} \times \{s\}$ *is diffeomorphic to a disk for all* s.
- *for any* $s \in D_\delta$, $\beta_{U\times\{s\}}$ *is transverse to* $S_\varepsilon \times D^o_\delta$ *(this means that every* $x \in \partial U_s$ *is a regular point of* $\beta_{U\times\{s\}}$ *and the vector space* $d\beta_{U\times\{s\}}$ *is transverse to the tangent space of* $S_\varepsilon \times D^o_\delta$ *at* $\beta_{U\times\{s\}}(x)$*).*
- $\mathcal{U}$ *is a smooth manifold with boundary* $\beta^{-1}(\partial X_\varepsilon \times D^o_\delta)$

Definition 3.4. *The map β restricted to $\mathcal{U}$ is a* **Milnor representative** *of the wedge α, whose special arc is $\gamma\mid_U$.*

Remark 3.5. *One has to prove that such a representative does exist, in particular that the set U can be taken to be differomorphic to a disk. See* [27] *or* [6].

Aiming for contradiction, we now consider a Milnor representative $\alpha : \mathcal{U} \to X_\varepsilon$ of an analytic wedge, realizing the adjacency from E_i to E_0, that is, a wedge such that the generic arc $\alpha_s(t)$ belongs to C_i and the special arc $\gamma(t)$ belongs to C_0.

Remark 3.6. *These definitions of representatives are a key point in the proof of the theorem. Let $\alpha_s : U_s \to X_\varepsilon$ be a generic arc of the wedge. By construction, U_s is a disk and thus has Euler characteristic equal to one. The aim of the rest of the proof is to show that the Euler Characteristic of U_s is bounded above by an expression less or equal to 0, and thus get the contradiction.*

We have the following lemma:

Lemma 3.7. *The mapping $\alpha_s : U_s \to X_\varepsilon$ is injective.*

Proof. The map α_s is a smooth deformation of $\alpha_0 : U_0 \to X_\varepsilon$. But the map $\alpha_0 : U_0 \to X_\varepsilon$ is injective since by construction it is transversal to every S_μ for $\mu \leqslant \varepsilon$, so α_0 is an injective and smooth mapping.
Moreover, for all $s \in D^o_\delta$ we have $\beta^{-1}(\partial B_\varepsilon \times \{s\}) \cong S^1$. The degree of a map of S^1 to itself is upper semi-continuous under smooth deformation, thus the map

$$\alpha_s\mid_{\partial U_s} : S^1 \to S^1$$

is of degree one. By Definition 3.4 and Proposition 3.3, α_s has no critical points on ∂U_s; this implies that $\alpha_s\mid_{\partial U_s}$ is one–to–one.
Hence α_s is a local homeomorphism and so is generically one–to–one. □

3.2. **Eliminating the indeterminacy of $\tilde{\alpha}$.** Let $\tilde{\beta}$ be the meromorphic map defined as the composition of $\sigma^{-1} \circ \beta$ with $\sigma = (\pi, id\,|_{D_\delta})$:

$$\begin{tikzcd} & \tilde{X}_\varepsilon \times D_\delta \arrow[d, "\sigma"] \\ \mathcal{U} \arrow[ur, dashed, "\tilde{\beta}"] \arrow[r, "\beta"] & X_\varepsilon \times D_\delta \end{tikzcd}$$

The indeterminacy locus of $\sigma^{-1} \circ \beta$ is of codimension 2. Thus we may assume that, shrinking the radius δ, if necessary, $(0,0)$ is the only indeterminacy point of $\tilde{\beta}$.

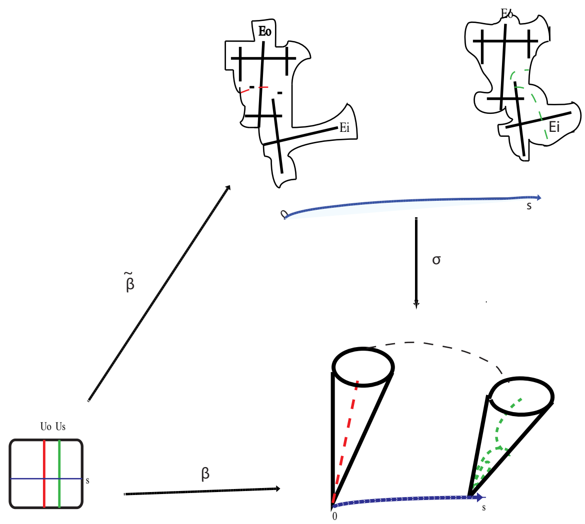

FIGURE 1 . Wedge representative

Moreover there exists a unique meromorphic lifting $\tilde{\alpha}$ of α such that:

$$\begin{tikzcd} Y \arrow[r, hook] \arrow[d] & \tilde{X}_\varepsilon \arrow[d, "\pi"] \\ \mathcal{U} \arrow[ur, dashed, "\tilde{\alpha}"] \arrow[r, "\alpha"] & X_\varepsilon \end{tikzcd}$$

Let $H = \beta(\mathcal{U})$ the image of $\mathcal{U}$ by β; it is an analytic subvariety of dimension two (as β is finite and proper). Let Y be the analytic Zariski closure of $\sigma^{-1}(H \backslash (\{0\} \times D_\delta))$ and let $Y_s = Y \cap (\tilde{X}_\varepsilon \times \{s\})$. The surface Y is reduced and is a Cartier divisor in the smooth threefold $\tilde{X} \times D_\delta$. One can prove the following ([5], p. 7):

(1) for all $s \in D^o_\delta$ one has $\tilde{\beta}(t,s) = (\tilde{\alpha}_s(t), s)$
(2) $Y \cap (\tilde{X}_\varepsilon \times D^o_\delta) = \tilde{\beta}(\mathcal{U} \backslash U_0)$.

Thus

$$Y_s = \tilde{\alpha}_s(U_s).$$

Lemma 3.8. *The mapping $\tilde{\alpha}_s : U_s \to Y_s$ is the morphism of normalization of Y_s.*

Proof. First, by the previous Lemma, α_s is generically one–to–one and proper. Hence so is $\tilde{\alpha_s}$. As U_s is a smooth disk, the mapping $\tilde{\alpha}_s : U_s \to Y_s$ is thus the morphism of normalization of Y_s. □

Definition 3.9. Returns
Elements of the set $\alpha_s^{-1}(0) \setminus \{(0,s)\}$ are called **returns**. *Their images by α_s are* 0 *and by $\tilde{\alpha_s}$ points of the exceptional set E.*

The curve $Y_0 = Y \cap (\tilde{X}_\varepsilon \times \{0\})$ does not need to be reduced. It contains $Z_0 := \tilde{\alpha_0}(U_0)$ and a sum of the exceptional components E_i with suitable multiplicities, which can be explicitly described as follows. For any point $\xi \in X$, let f_ξ denote the local defining equation of Y_0 near ξ.
We have a unique factorization

$$f_\xi = g_{n+1} \prod_{i=0}^{n} g_i^{a_i}$$

where $g_{n+1} = 0$ is the local defining equation of Z_0 near ξ and $g_i = 0$ the local defining equation of E_i near ξ (if $\xi \notin E_i$, we take $g_i = 1$, and similarly for g_{n+1}). It is very easy to see that, given two points $\xi, \xi' \in E_i$, one obtains the same exponent a_i from the local equations at ξ and ξ'; in other words, a_i is determined by E_i and not by the choice of the point ξ. We express this situation by the equation $Y_0 = Z_0 + \sum a_i E_i$; the analytic space Y_0 is reduced along $Z_0 \backslash E$.

Since Y_s is a deformation of Y_0, we have $b_i := Y_s.E_i = Y_0.E_i$; that is

$$M.(a_0, ..., a_n)^t = (1 + b_0, b_1, ..., b_n),$$

where M is the self-intersection matrix of E (the curve E_0 plays a special role in this equality because $Z_0.E_0 = 1$ and $Z_0.E_i = 0$ for $i \neq 0$). Note that the b_i's correspond to the number of returns that lift to E_i. By linear algebra, one obtains that $a_0 \neq 0$ (i.e. E_0 belongs to Y_0) and $b_0 = 0$, that is, Y_s must not intersect E_0.

3.3. **End of the proof.** As explained before, to obtain a contradiction we want to show that U_s has non-positive Euler characteristic. To do this, Fernández de Bobadilla and Pe Pereira give an upper bound on $\chi(U_s)$ in terms of $\chi(Y_s)$, $\chi(Y_0)$ and the possible returns.

Recall that $Y_0 = Z_0 + \sum_i a_i E_i$. We construct a tubular neighborhood of E in the following way.

Define $E_i^o = E_i \backslash Sing(Y_0^{red})$. Let $Sing(Y_0^{red}) = \{p_0, p_1, ..., p_m\}$, where $p_0 = Z_0 \cap E$. Let B_k be a small ball in $\tilde{X}$ centered at p_k. For $j \in \{0, \dots, n\}$, let T_j be a tubular neighborhood of E_j, small enough so that its intersection with each B_k is transverse. Let T_{n+1} be a tubular neighborhood of Z_0, small enough so that its intersection with B_0 is transverse. Let

$$W_j = T_j \backslash \left(\bigcup_{k=0}^{m} B_k \right).$$

All the neighborhoods are chosen so that

$$\chi(U_s) = \sum_{j=0}^{n+1} \chi(\tilde{\alpha_s}^{-1}(Y_s \cap W_j)) + \sum_{k=0}^{m} \chi(\tilde{\alpha_s}^{-1}(Y_s \cap B_k)). \tag{2}$$

We do not need to count $\chi(Y_s \cap T_j \cap B_k)$ since by the assumed transversality each of these intersections is a finite union of circles and thus

$$\chi(Y_s \cap T_j \cap B_k) = 0. \tag{3}$$

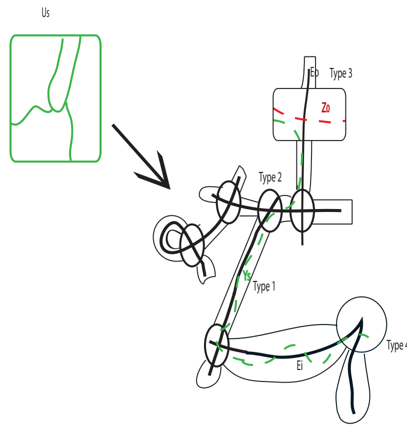

FIGURE 2 . Normalization map

It remains to bound above each summand on the right hand side of (2). To do this, we first consider the special case when $\tilde{X}_\varepsilon$ is a *good resolution* of X_ε, that is, when the exceptional set E has *normal crossings*. We divide the summands appearing in (2) into three types and deal separately with each type.

- Type 1: Terms of the form $\chi(\tilde{\alpha_s}^{-1}(Y_s \cap W_j))$. If $j \leqslant n$, by Hurwitz formula, we have

$$\chi(\tilde{\alpha_s}^{-1}(Y_s \cap W_j)) \leqslant a_i \chi(E_j^o) \tag{4}$$

  as the maps $\tilde{\alpha_s}^{-1}(Y_s \cap W_j) \to E_j \setminus \left(\bigcup_{k=0}^{m} B_k\right)$ are branched covers of degree a_i. For $j = n+1$, $Z_0 \setminus p_0$ is homeomorphic to a punctured disk, so Hurwitz formula gives

$$\chi(\tilde{\alpha_s}^{-1}(Y_s \cap W_{n+1})) \leqslant \chi(Z_0 \setminus p_0) = 0. \tag{5}$$

- Type 2: Terms of the form $\chi(\tilde{\alpha_s}^{-1}(Y_s \cap B_k))$ such that $k > 0$, $p_k \notin Y_s$ and $B_k \cap E$ has only normal crossings. Let (x, y) be local coordinates at p_k such that $f(x, y) = xy$ is a local defining equation of the set $E \cap B_k$. Let $Y_s^1, \ldots, Y_s^q$ be the connected components of the set $Y_s \setminus B_k$. Since the only connected orientable surfaces with boundary having positive Euler characteristic are disks, and in view of (3), we only have to be careful about those Y_s^l which are homeomorphic to disks.

  As Y_s is a deformation of Y_0, the boundary of such a component Y_s^l either deforms to $V(x) \cap S_\varepsilon$ or to $V(y) \cap S_\varepsilon$. This implies that Y_s^l must intersect either $V(x)$ or $V(y)$. In this case one has,

$$\chi(\tilde{\alpha_s}^{-1}(Y_s \cap B_k)) \leqslant \sum_{p \in Y_s \cap Y_0 \cap B_k} I_p(Y_s, Y_0^{red}). \tag{6}$$

- Type 3: Finally, we will show that $\chi(\tilde{\alpha_s}^{-1}(Y_s \cap B_0)) \leqslant a_0 - 1$. Indeed, as Y_0 is reduced locally at Z_0 let us suppose that the local equation at $Z_0 \cup E_0$ is of the form $f_0 = xy^{a_0} = 0$. Let Y_s^l be an irreducible component of $Y_s \cap B_0$ whose normalization is a disk. Then as Y_s is a deformation of Y_0, the boundary of that component Y_s^l either deforms to $V(x) \cap \partial B$ or to $V(y) \cap \partial B_0$. This implies that Y_s^l must intersect either $V(x)$ or $V(y)$. Therefore, as in the case of Type 2, we have

$$\chi(\tilde{\alpha_s}^{-1}(Y_s \cap B_0)) \leqslant \sum_{p \in Y_s \cap Y_0 \cap B_0} I_p(Y_s, Y_0^{red}) \leqslant a_0 + 1. \tag{7}$$

  As Y_s is a deformation of Y_0, there exists a connected component F of $Y_s \cap B_0$ whose boundary contains a circle K_s deforming to $V(x) \cap \partial B_0$. If $\partial F = K_s$ then, by the connectedness of Y_s, $Y_s \cap \partial B_0$ does not contain a circle deforming to $V(y) \cap \partial B_0$, which is impossible. Thus $K_s \subsetneq \partial F$, so ∂F must be a union of at least two circles. In particular, the normalization of F cannot be a disk. Since there are at least two circles in $Y_s \cap \partial B_0$ which bound a connected component of $Y_s \cap B_0$ having non-positive Euler characteristic, the inequality (7) remains true after we subtract 2 from the right hand side:

$$\chi(\tilde{\alpha_s}^{-1}(Y_s \cap B_0)) \leqslant a_0 - 1. \tag{8}$$

Combining (2), (4), (5), (6) and (8), we obtain

$$\chi(U_s) \leqslant a_0 - 1 + \sum_{i=0}^{n} a_i(\chi(E_i) - E_i.(Y_0^{red} - E_i)) + \sum_{p \in Y_s \cap (Y_0 \setminus B_0)} I_p(Y_s, Y_0^{red})$$

Rearranging the sum one has

$$\chi(U_s) \leqslant \sum_i a_i(2 - 2g_i + E_i.E_i).$$

This last sum is less or equal to 0 as each member is less than or equal to 0. We have proved that the disk U_s has non-positive Euler characteristic, which gives the desired contradiction. This completes the proof in the case when the minimal resolution $\tilde{X}$ is a good resolution.

We now briefly sketch the proof in the general case, that is, when E is not necessarily normal crossings.

The main difference with the normal crossings case is that now we must take more care to bound the terms in (2) of the form $\chi(\tilde{\alpha_s}^{-1}(Y_s \cap B_k))$ such that $B_k \cap E$ does not have normal crossings (in particular, $k > 0$). Assume that $Y_s \cap B_k \neq \emptyset$. Suppose, too, that Y_s does not pass through p_k (if not, one can reduce the problem to this case by suitably deforming Y_s).

To study the inequality (2), we use the following numerical characters of the singularities of the reduced exceptional set E. For each E_i consider the set of irreducible components of the germ of E_i at each point of $Sing(Y_0^{red})$. We denote by ν_i the total number of local branches of E_i at all the singular points of $Sing(Y_0^{red})$, by μ_i the sum of Milnor numbers of all these local branches, and by η_i the sum of all the pairwise intersection number between the branches.

Fix a sequence of point blowings up of $\tilde{X}_\varepsilon$ under which the total preimage of $E \cap B_k$ is normal crossings, and replace $\tilde{X}_\varepsilon$ by the resulting manifold. The Euler characteristic of the preimage of Y_s is equal to that of Y_s.

Analyzing the blown up surface by techniques similar to the ones used in the case of good resolution, we obtain the inequality

$$\chi(U_s) \leqslant a_0 - 1 + \sum_i a_i(\chi(E_i) + \nu_i - \mu_i - \eta_i - E_i.(Y_0^{red} - E_i)) + \sum_{p \in Y_s \cap (Y_0 \setminus B_0)} I_p(Y_s, Y_0^{red}).$$

Rearranging the sum one has

$$\chi(U_s) \leqslant \sum_i a_i(2 - 2g_i - \mu_i - \eta_i + E_i.E_i)$$

This last sum is less or equal to 0 as each member is less than 0. Contradiction.

3.4. **The non-normal case.** As before, thanks to Lefschetz Principle, Fernández de Bobadilla and Pe Pereira reduce the problem to the complex case.

Let X be a complex algebraic surface, not necessarily normal, and let $\pi : \tilde{X} \to X$ be its minimal resolution of the singularities of X. Let $E := \pi^{-1}(Sing(X))$. Let $E = \bigcup_{i=0}^{n} E_i$ be the decomposition of E into irreducible components.

Definition 3.10. *We say that E_i is of the* **first kind** *if* $\dim \pi(E_i) = 0$ *and of the* **second kind** *if* $\dim \pi(E_i) = 1$.

A priori, we have four types of possible adjacencies: an arc family of the first kind could be adjacent to one of the first or second kind and an arc family of the second kind could be adjacent to one of the first or second kind. The fact that a family of the second kind cannot be adjacent to another one of either first or second kind follows easily from the continuity of the wedge which realizes this hypothetical adjacency. The fact that an arc family C_i of the first kind cannot be adjacent to another family C_j of the first kind follows from the normal case: such an adjacency would induce an adjacency of the preimage of C_i to the preimage of C_j in the normalization of X, which is impossible by the normal case. To settle the last remaining case, that of an arc family C_i of the first kind adjacent to an arc family C_j of the second kind, J. Fernández de Bobadilla and M. Pe Pereira use plumbing to construct an auxiliary normal surface singularity (X', ξ') and two distinct Nash families C'_i and C'_j on X' such that C'_i is adjacent to C'_j, again contradicting the normal case.

4. Higher dimensions

For singularities of higher dimensions, the Nash Problem enunciated as above is false, though a few positive results have been proved: in [16], S. Ishii and J. Kollar give an affirmative answer for toric varieties in all dimensions. Affirmative answers for a family of singularities in dimension higher than 2 by P. Popescu-Pampu and C. Plénat ([32]) and another family by M. Leyton-Alvarez [23] (2011).

In [16], S. Ishii and J. Kollár give a counterexample to the Nash problem in dimension greater than or equal to 4: the hypersurface

$$x^3 + y^3 + z^3 + u^3 + w^6 = 0$$

which has a resolution with two irreducible exceptional components. These are essential, as one is the projectivization of the tangent cone at the singular point (hence it clearly corresponds to a Nash family), and the other one is not uniruled. Then the authors construct geometrically a wedge whose generic arc is in the Nash family, and whose special arc is in the second family.

In May 2012, T. de Fernex gave a counterexample in dimension 3 ([3], 2012). The equation is

$$(x^2 + y^2 + z^2)w + x^3 + y^3 + z^3 + w^5 + w^6 = 0 \tag{9}$$

In the algebraic setting, he can prove that the two exceptional components obtained after two blowing-ups are essential. But as an analytic variety, the hypersurface obtained from (9) by blowing up the origin is locally isomorphic to the non-degenerate quadratic cone, hence it admits a small resolution; this implies that the second exceptional component is not essential, so the counterexample does not apply in the analytic category. Deforming the equation (9), de Fernex obtains a counterexample to the Nash problem in dimension 3, valid in both the algebraic and the analytic setting:

$$(x^2+y^2)w + x^3 + y^3 + z^3 + w^5 + w^6 = 0.$$

An even more recent paper on the Nash problem is due J. Johnson and J. Kollár [17]. In that paper, J. Johnson and J. Kollár gives a new family of counterexamples to the Nash problem in dimension 3, called $c\mathcal{A}_1$-type singularities:

$$x^2+y^2+z^2+t^m = 0$$

with m odd, $m > 3$. These singularities are isolated and have only one Nash family, but two of the exceptional components in the resolution are essential.
Moreover, J. Johnson and J.Kollár formulates the Revised Nash problem, which we now explain.

Definition 4.1. *Let X be a variety over a field k, $k \subset K$ a field extension of k and ϕ : Spec $K[[t]] \to X$ an arc such that Supp $\phi^{-1}(Sing(X)) = \{0\}$. A* **sideways deformation** *of ϕ is an extension of ϕ to a morphism Φ : Spec $K[[t,s]] \to X$ such that*

$$Supp\ \Phi^{-1}(Sing(X)) = \{(0,0)\}.$$

Definition 4.2. *We say that X is* **arcwise Nash-trivial** *if every general arc in X_∞^{sing} has a sideways deformation.*

Definition 4.3. *Let X be a variety over k. A divisor over X is called* **very essential** *if the following holds. Let $p: Y \to X$ be a proper birational morphism such that Y is $\mathbb{Q}$-factorial and has only arcwise Nash-trivial singularities. Then center$_Y E$ is an irreducible component of $p^{-1}(Sing(X))$.*

In fact in the three counterexamples above, the components corresponding to Nash families are given precisely by the very essential divisor. Imitating and conceptualizing the proofs of non-essentiality appearing in the above counterexamples, one can show that divisors appearing in the image of the Nash map are very essential. We are lead to the following problem:

Problem 4.4. *Is the Nash map surjective onto the set of very essential divisors?*

In April 2014, when the present paper was well into the refereeing process, Tommaso de Fernex and Roi Docampo [4] made further significant progress on the Nash problem. They defined the notion of **terminal valuations** over X (where X is a variety of any dimension) and showed that any divisor associated to a terminal valuation is in the image of the Nash map. Restricting this result to the case dim $X = 2$ provides a new and completely algebraic proof of the Theorem of Fernández de Bobadilla – Pe Pereira. We acknowledge this very important paper even though we did not have a chance to discuss it in detail.

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