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(SSP) GEOMETRY WITH DIRECTIONAL HOMEOMORPHISMS

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Dedicated to Professor David Trotman for his 60th birthday

Abstract. In a previous paper [6] we discussed several directional properties of sets satisfying the sequence selection property, denoted by (SSP) for short, and developed the (SSP) geometry via bi-Lipschitz transformations. In this paper we introduce the notion of directional homeomorphism and show that we can develop also the (SSP) geometry with directional transformations. For many important results proved in [6] for bi-Lipschitz homeomorphisms we describe the analogues for directional homeomorphisms as well.

1. Introduction.

In [4] we introduced the notion of sequence selection property, denoted by (SSP) for short, in order to show that the dimension of the common direction set of two subanalytic subsets is preserved by a bi-Lipschitz homeomorphism provided that their images are also subanalytic. The condition (SSP) is one of the three main ingredients in the given proof. Subsequently we generalised the result above to the case of a general real closed field in [5], where we also discussed several (SSP) properties.

Following the above works, we have started to work on condition (SSP) both on the field of real numbers and on the field of complex numbers. In fact, we proved essential directional properties of sets satisfying (SSP) with respect to bi-Lipschitz homeomorphisms in [6]. Amongst the main results in [6] are the following:

(1) Weak transversality theorem,

(2) (SSP) structure preserving theorem,

(3) Important property: $LD(h(LD(A))) = LD(h(A))$,

(4) Directional property of intersection sets.

Concerning (2), we proved two types of (SSP) structure preserving theorems in [6]. The main purpose in this paper is to introduce a new notion of homeomorphism, called *directional homeomorphism*, which enables us to show general results including those theorems mentioned above, using the new notion of homeomorphism without the assumption on the sequence selection property. We shall discuss several properties of the directional homeomorphism in §3.1, and give the main results in §4.

Throughout this paper we use the following notations:

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Let $\{a_m\}$, $\{b_m\}$ be sequences of points of $\mathbb{R}^n$ tending to the origin $0 \in \mathbb{R}^n$. If there are a natural number $N \in \mathbb{N}$ and a real number $K > 0$ such that

$$\|a_m\| \leq K\|b_m\|, \quad \forall m \geq N$$

then we write $\|a_m\| \precsim \|b_m\|$ (or $\|b_m\| \succsim \|a_m\|$). If $\|a_m\| \precsim \|b_m\|$ and $\|b_m\| \precsim \|a_m\|$, we write $\|a_m\| \approx \|b_m\|$.

2. Directional Properties of Sets

In this section we recall the notions of direction set and sequence selection property, and describe several elementary properties.

2.1. **Direction set.** Let us recall the notion of direction set.

Definition 2.1. Let A be a set-germ at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A}$. We define the *direction set* $D(A)$ of A at $0 \in \mathbb{R}^n$ by

$$D(A) := \{a \in S^{n-1} \mid \exists \{x_i\} \subset A \setminus \{0\},\ x_i \to 0 \in \mathbb{R}^n \text{ s.t. } \frac{x_i}{\|x_i\|} \to a,\ i \to \infty\}.$$

Here S^{n-1} denotes the unit sphere centred at $0 \in \mathbb{R}^n$.

For a subset $A \subset S^{n-1}$, we denote by $L(A)$ a half-cone of A with the origin $0 \in \mathbb{R}^n$ as the vertex:

$$L(A) := \{ta \in \mathbb{R}^n \mid a \in A,\ t \geq 0\}.$$

In the case where $A \subset S^{n-1}$ is a point we call $L(A)$ a *semiline*. Therefore a semiline $\ell \subset \mathbb{R}^n$ means a half line whose starting point is the origin $0 \in \mathbb{R}^n$. For a set-germ A at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A}$, we put $LD(A) := L(D(A))$, and call it the *real tangent cone* of A at $0 \in \mathbb{R}^n$.

Let $U,\ V \subset \mathbb{R}^n$ such that $0 \in \overline{U} \cap \overline{V}$. The following properties hold:

(1) $D(\overline{U}) = D(U)$,
(2) $D(U \cup V) = D(U) \cup D(V)$,
(3) $\overline{\cup_i D(U_i)} \subseteq D(\cup U_i)$,
(4) If U_i are half-cones then $\overline{\cup_i D(U_i)} = D(\cup U_i)$,
(5) $D(U \cap V) \subseteq D(U) \cap D(V)$.

2.2. **Sequence selection property.** Let us recall the notion of condition (SSP). In fact here we give a generalised notion of (SSP) relatively to a subset of $\mathbb{R}^n$.

Definition 2.2. Let A, B be two set-germs at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A} \cap \overline{B}, D(A) \subseteq D(B)$. We say that A satisfies *condition* (SSP)-relative to B, if for any sequence of points $\{a_m\}$ of B tending to $0 \in \mathbb{R}^n$, such that $\lim_{m\to\infty} \frac{a_m}{\|a_m\|} \in D(A)$, there is a sequence of points $\{b_m\} \subset A$ such that,

$$\|a_m - b_m\| \ll \|a_m\|,\ \|b_m\|,$$

i.e., $\lim_{m\to\infty} \frac{\|a_m - b_m\|}{\|a_m\|} = 0$.

In the case $B = \mathbb{R}^n$ we will not mention B (it is the usual (SSP) condition).

Clearly the direction set and the sequence selection property are conditions in the spirit of Whitney [7], who consistently studied directional properties at singular points and their behaviour while approaching a singularity via sequences of points. We concentrate our study to sets for which their direction sets are essentially independent on the ambient space.

For the reader's convenience we give some remarks on the relative condition (SSP) ((2) and (3) follow from the transitivity of the relative condition (SSP)).

Remark 2.3. (1) A (resp. $\overline{A}$) satisfies *condition* (SSP)-relative to $\overline{A}$ (resp. A).
(2) A satisfies *condition* (SSP) if and only if A satisfies *condition* (SSP)-relative to $LD(A)$.
(3) A satisfies *condition* (SSP) if and only if $\overline{A}$ satisfies *condition* (SSP).
(4) A satisfies *condition* (SSP)-relative to $ST_d(A; C), d > 1$
(see [4] for $ST_d(A : C)$).

In this note we also consider the notion of weak sequence selection property, denoted by $(WSSP)$ for short; in fact they are equivalent notions.

Definition 2.4. Let A, B be two set-germs at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A} \cap \overline{B}, D(A) \subseteq D(B)$. We say that A satisfies *condition* $(WSSP)$-relative to B, if for any sequence of points $\{a_m\}$ of B tending to $0 \in \mathbb{R}^n$ such that $\lim_{m\to\infty} \frac{a_m}{\|a_m\|} \in D(A)$, there is a subsequence $\{m_j\}$ of $\{m\}$ and a sequence $\{b_{m_j}\} \subset A$ such that

$$\|a_{m_j} - b_{m_j}\| \ll \|a_{m_j}\|,\ \|b_{m_j}\|.$$

We have the following characterisation of condition (SSP). As mentioned in [6], the proof in the relative case is similar to the non-relative case, for which we gave a detailed proof in [5].

Lemma 2.5. *([6] Proposition 2.7) Let A, B be two set-germs at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A} \cap \overline{B}$. If A satisfies condition $(WSSP)$-relative to B, then it satisfies condition (SSP)-relative to B. Namely, the conditions relative (SSP) and relative $(WSSP)$ are equivalent.*

Below we give several examples of sets satisfying the condition (SSP). Consult [6] for more examples.

Remark 2.6. Let $A, B \subseteq \mathbb{R}^n$ be a set-germs at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A} \cap \overline{B}$, then the following hold:
(1) The cone $LD(A)$ satisfies condition (SSP),
(2) If A is subanalytic ([3]) or definable in some o-minimal structure ([5]), then it satisfies condition (SSP),
(3) If A is a finite union of sets, all of which satisfy condition (SSP), then A satisfies condition (SSP),
(4) If $0 \in A$, a C^1 manifold, then it satisfies condition (SSP) and $LD(A) = T_0(A)$ i.e., the tangent space of A at $0 \in \mathbb{R}^n$, (this is not necessarily true for C^0 manifolds or if $0 \notin A$),
(5)If $A \subseteq B, D(A) = D(B)$ and A satisfies condition (SSP), then B satisfies condition (SSP).

3. Directional Homeomorphism and some fundamental lemmas

In this section we introduce the notion of directional homeomorphism, and describe some fundamental properties of it.

3.1. **Directional homeomorphism.** In this subsection we describe the condition semiline-(SSP), and we use it to give some characterisations of the condition (SSP) and our definition of directional homeomorphism.

Definition 3.1. We say that a homeomorphism $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ satisfies condition *semiline*-(SSP), if $h(\ell)$ has a unique direction for all semilines ℓ.

Remark 3.2. Take a germ of a semiarc $\gamma : ([0, \epsilon), 0) \to (\mathbb{R}^n, 0)$ with a unique direction, say $\ell = LD(\gamma)$. (It is not difficult to see that in this case γ necessarily satisfies condition (SSP).) It follows that for a bi-Lipschitz homeomorphism $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ where h^{-1} satisfies condition semiline-(SSP), we do have that $h(\gamma)$ has also a unique direction, i.e., h also satisfies condition semiline-(SSP). Indeed, we can easily see that $LD(h(\gamma)) = LD(h(LD(\gamma))) = LD(h(\ell))$ is also a semiline. Let

$$\mathscr{SL} := \{\gamma : ([0,\epsilon),0) \to (\mathbb{R}^n,0) \mid LD(\gamma) \text{ is a semiline}\}.$$

The above argument implies that if h^{-1} satisfies condition semiline-(SSP), then the map

$$h : \mathscr{SL} \to \mathscr{SL}$$

induces a map $\overline{h} : S^{n-1} \to S^{n-1}$ defined by

$$\overline{h}(D(\gamma)) = D(h(\gamma)) \text{ for } \gamma \in \mathscr{SL}.$$

If both h, h^{-1} satisfy condition semiline-(SSP), then $\overline{h} : S^{n-1} \to S^{n-1}$ is a one-to-one correspondence, in other words, $\overline{h} : S^{n-1} \to S^{n-1}$ is bijective.

Note that in the case where $\gamma : ([0,\epsilon),0) \to (\mathbb{C}^n,0),\ \gamma \in \mathscr{SL}$, we have that the complex cone $LD^*(\gamma) := LD(S^1 D\gamma)$ is a complex line, and all complex lines can be obtained in this way.

Theorem 3.3. *([6] Theorem 2.25) Let $h : (\mathbb{R}^n,0) \to (\mathbb{R}^n,0)$ be a bi-Lipschitz homeomorphism such that h (so h^{-1}) satisfies condition semiline-(SSP). Then the induced map $\overline{h} : S^{n-1} \to S^{n-1}$ given in Remark 3.2 extends to a bi-Lipschitz homeomorphism $\overline{h} : \mathbb{R}^n \to \mathbb{R}^n$, and for any $A \subset \mathbb{R}^n$ such that $0 \in \overline{A}$, we have*

$$\overline{h}(D(A)) = D(h(LD(A))) = D(h(A)) = D(\overline{h}(A)).$$

In particular, we have $\dim D(A) = \dim D(h(A))$.

Conversely the radial extension of a self bi-Lipschitz homeomorphism of the sphere S^{n-1} satisfies the condition semiline-(SSP). As we shall see below, there is a clear correspondence between these radial bi-Lipschitz homeomorphisms and the bi-Lipschitz homeomorphisms which satisfy condition semiline-(SSP). It is not difficult to see that the bi-Lipschitz semiline-(SSP) homeomorphisms preserve condition (SSP). We shall discuss a more general result in §4.

Remark 3.4. In particular the above property holds for any definable bi-Lipschitz homeomorphism, and for any subanalytic bi-Lipschitz homeomorphism.

Corollary 3.5. *Let A be a set-germ at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A}$, and let $h : (\mathbb{R}^n,0) \to (\mathbb{R}^n,0)$ be a bi-Lipschitz homeomorphism such that h (h^{-1}) satisfies condition semiline-(SSP). Then $LD(A)$ and $LD(h(A))$ are bi-Lipschitz homeomorphic.*

Let $h : (\mathbb{R}^n,0) \to (\mathbb{R}^n,0)$ be a Lipschitz homeomorphism. Then it is not difficult to see the following property holds for h.

Proposition 3.6. *The following conditions are equivalent.*

(1) $\forall y,\ \|y\| = 1,\ \exists \lim_{t\to 0_+} \frac{\|h(ty)\|}{t} := \alpha(y)$,
(2) $\forall x,\ \frac{x}{\|x\|} \to y,\ \exists \lim_{|x|\to 0} \frac{\|h(x)\|}{\|x\|} := \alpha(y)$.

Let us assume that a Lipschitz homeomorphism h satisfies the equivalent conditions of the proposition above. Then α is Lipschitz (as h is) and we have

$$\lim_{x\to 0, \frac{x}{\|x\|}\to y} \frac{\alpha(\frac{x}{\|x\|})\|x\|}{\|h(x)\|} = 1,$$

provided that h is not vanishing outside the origin.

Also if h satisfies condition semiline-(SSP), h induces $\overline{h}$ (as we know) and we have the following:

$$\lim_{x\to 0,\frac{x}{\|x\|}\to y}\frac{h(x)}{\|h(x)\|}=\overline{h}(y),\forall y\in S^{n-1}.$$

From now on we will use the bar notation, as $\overline{h}(x)$, for the corresponding extension

$$\overline{h}(tx)=tx,t\geq 0,\|x\|=1.$$

In consequence, for a Lipschitz homeomorphism h which satisfies condition semiline-(SSP), we have the following structure, relating h and $\overline{h}$:

$$\lim_{x\to 0}\frac{\|\alpha(\frac{x}{\|x\|})\overline{h}(x)-h(x)\|}{\|x\|}=0.$$

More generally, one can show all the above properties, even for h of the form $h(x)=\tau(x)+\mathrm{o}(x)$, where τ is merely Lipschitz and satisfies the following:

$$\forall y,\|y\|=1,\exists\lim_{t\to 0_+}\frac{\|\tau(ty)\|}{t}:=\alpha(y),$$

and $\lim_{x\to 0}\frac{\|\mathrm{o}(x)\|}{\|x\|}=0$. After this, we use the notation $\mathrm{o}(x)$ as a mapping satisfying the limit condition. The above comments justify the next definition, inspired by the notion of weak diffeomorphism in [2] (see §4.2 for this notion).

Definition 3.7. A *directional homeomorphism* is a homeomorphism $h(x)=\tau(x)+\mathrm{o}(x)$, where τ is a bi-Lipschitz semiline-(SSP) homeomorphism.

Accordingly, for directional homeomorphisms we also have the remarkable decomposition

$$h(x)=\alpha(\frac{x}{\|x\|})\overline{\tau}(x)+\mathrm{o}(x),$$

with $\overline{\tau}$ defined as above. This kind of decomposition, in principle allows us to replace (when studying direction sets), a directional homeomorphism with a homeomorphism which is both positively homogeneous $\overline{\tau}(tx)=t\overline{\tau}(x),t\geq 0$, and norm preserving $\|\overline{\tau}(x)\|=\|x\|$. In a small neighbourhood of the origin, h and $\overline{h}=\overline{\tau}$ are homotopically equivalent.

3.2. **Fundamental lemmas.** In this subsection we describe some properties on homeomorphisms even weaker than directional homeomorphisms. Let $h:(\mathbb{R}^n,0)\to(\mathbb{R}^n,0)$ denote a homeomorphism which can be expressed as $h(x)=\tau(x)+\mathrm{o}(x)$, similar to a directional homeomorphism.

Lemma 3.8. *Let $h:(\mathbb{R}^n,0)\to(\mathbb{R}^n,0)$ be a homeomorphism, $h(x)=\tau(x)+\mathrm{o}(x)$, where τ is a Lipschitz homeomorphism, such that $\|x\|\approx\|\tau(x)\|$, and let $\{a_m\}$, $\{b_m\}$ be sequences of points of $\mathbb{R}^n$ tending to $0\in\mathbb{R}^n$. Suppose that $\|a_m-b_m\|\ll\|a_m\|$. Then we have*

$$\|h(a_m)-h(b_m)\|\ll\|h(a_m)\|.$$

Proof. We first have

$$\|h(a_m)-h(b_m)\|\leq\|\tau(a_m)-\tau(b_m)\|+\|\mathrm{o}(a_m)-\mathrm{o}(b_m)\|.$$

Note that $\|a_m\|\approx\|\tau(a_m)\|\approx\|h(a_m)\|$. Therefore we see that

$$\frac{\|h(a_m)-h(b_m)\|}{\|h(a_m)\|}\precsim\frac{\|\tau(a_m)-\tau(b_m)\|}{\|\tau(a_m)\|}+\frac{\|\mathrm{o}(a_m)-\mathrm{o}(b_m)\|}{\|a_m\|}\to 0$$

as $m\to\infty$. It follows that

$$\|h(a_m)-h(b_m)\|\ll\|h(a_m)\|.$$

□

In the above lemma we do not assume that τ satisfies the condition semiline-(SSP).

We next give a useful lemma to show some of the main results on directional homeomorphisms.

Lemma 3.9. *Let $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a homeomorphism, $h(x) = \tau(x) + \mathrm{o}(x)$, such that $\|\tau(x)\| \geq C\|x\|$ in a neighbourhood of $0 \in \mathbb{R}^n$ for some $C > 0$ (thus so does h), and let $A \subset \mathbb{R}^n$ be a set-germ at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A}$. Then we have:*

(1) $D(h(A)) = D(\tau(A))$, and $h(A)$ satisfies condition (SSP) if and only if so does $\tau(A)$,

(2) $D(graph(h)) = D(graph(\tau))$, and $graph(h)$ satisfies condition (SSP) if and only if so does $graph(\tau)$.

Proof. We first show (1). Let $\{a_m\}$ be a sequence of points of A tending to $0 \in \mathbb{R}^n$ such that

$$a := \lim_{m\to\infty} \frac{h(a_m)}{\|h(a_m)\|} \in D(h(A)).$$

Then

$$\frac{h(a_m)}{\|h(a_m)\|} = \frac{\tau(a_m) + \mathrm{o}(a_m)}{\|\tau(a_m) + \mathrm{o}(a_m)\|}.$$

By assumption, we have $\|\tau(a_m)\| \gtrsim \|a_m\|$, therefore $\|\mathrm{o}(a_m)\| \ll \|\tau(a_m)\|$. This implies that

$$a = \lim_{m\to\infty} \frac{\tau(a_m)}{\|\tau(a_m)\|} \in D(\tau(A)).$$

It follows that $D(h(A)) \subset D(\tau(A))$. Since $\tau(x) = h(x) - \mathrm{o}(x)$, the opposite inclusion follows similarly and we have $D(h(A)) = D(\tau(A))$.

We can easily see the latter statement in (1) from the definition of (SSP) and the above arguments.

We next show (2). Let $\{a_m\}$ be a sequence of points of $\mathbb{R}^n$ tending to $0 \in \mathbb{R}^n$ such that

$$(a, b) := \lim_{m\to\infty} \frac{(a_m, h(a_m))}{\|(a_m, h(a_m))\|} \in D(graph(h)).$$

Since $\|\mathrm{o}(a_m)\| \ll \|\tau(a_m)\|$ as above, we have

$$(a, b) = \lim_{m\to\infty} \frac{(a_m, \tau(a_m) + \mathrm{o}(a_m))}{\|(a_m, \tau(a_m) + \mathrm{o}(a_m))\|} = \lim_{m\to\infty} \frac{(a_m, \tau(a_m))}{\|(a_m, \tau(a_m))\|} \in D(graph(\tau)).$$

It follows that $D(graph(h)) \subset D(graph(\tau))$. The opposite inclusion similarly follows as above, thus we have $D(graph(h)) = D(graph(\tau))$.

In order to show the latter statement of (2), let us assume that $graph(h)$ satisfies condition (SSP) at $(0, 0) \in \mathbb{R}^n \times \mathbb{R}^n$. Let $\{(a_m, b_m)\}$ be a sequence of points of $\mathbb{R}^n \times \mathbb{R}^n$ tending to $(0, 0) \in \mathbb{R}^n \times \mathbb{R}^n$ such that

$$\lim_{m\to\infty} \frac{(a_m, b_m)}{\|(a_m, b_m)\|} \in D(graph(\tau)) = D(graph(h)).$$

By assumption, there is a sequence of points $\{c_m\}$ of $\mathbb{R}^n$ tending to $0 \in \mathbb{R}^n$ such that

$$\|(c_m, h(c_m)) - (a_m, b_m)\| \ll \|c_m\| + \|\tau(c_m) + \mathrm{o}(c_m)\| \approx \|c_m\| + \|\tau(c_m)\|.$$

Therefore we have

$$\|(c_m, \tau(c_m) + \mathrm{o}(c_m)) - (a_m, b_m)\| \ll \|c_m\| + \|\tau(c_m)\|.$$

It follows that

$$\|(c_m, \tau(c_m)) - (a_m, b_m)\| \ll \|c_m\| + \|\tau(c_m)\|.$$

This means that $graph(\tau)$ satisfies condition (SSP). Since we can similarly show the converse, the latter statement of (2) follows. □

4. Main Results

In this section we give the new results for directional homeomorphisms concerning the properties mentioned in our Introduction. We first recall, in each subsection, the corresponding results for bi-Lipschitz homeomorphisms shown in [6]. The results for bi-Lipschitz homeomorphisms, except the (SSP) structure preserving theorem, assume the (SSP) condition, in contrast to the results for directional homeomorphisms. Concerning the structure preserving theorem, we give a generalisation of the results in [6].

4.1. Weak transversality theorem.

This is an important notion with potential important applications in Algebraic Geometry, where the tangent cones are important invariants. Let us recall the notion.

Definition 4.1. Let A, $B \subset \mathbb{R}^n$ be set-germs at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A} \cap \overline{B}$. We say that A and B are *weakly transverse at* $0 \in \mathbb{R}^n$ if $D(A) \cap D(B) = \emptyset$ (if and only if $LD(A)$ and B are weakly transverse).

For a bi-Lipschitz homeomorphism, we have the following weak transversality theorem.

Theorem 4.2. *([6] Theorem 3.5) Let A, B be two set-germs at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A} \cap \overline{B}$, and let $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a bi-Lipschitz homeomorphism. Suppose that A or B satisfies condition (SSP), and $h(A)$ or $h(B)$ satisfies condition (SSP). Then A and B are weakly transverse at $0 \in \mathbb{R}^n$ if and only if $h(A)$ and $h(B)$ are weakly transverse at $0 \in \mathbb{R}^n$.*

By Remark 3.2 we have the following weak transversality theorem for a directional homeomorphism.

Theorem 4.3. *Let A, B be two set-germs at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A} \cap \overline{B}$, and let $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a directional homeomorphism. Then A and B are weakly transverse at $0 \in \mathbb{R}^n$ if and only if $h(A)$ and $h(B)$ are weakly transverse at $0 \in \mathbb{R}^n$.*

Note that we do not assume the condition (SSP) of "A or B" or of "$h(A)$ or $h(B)$" in the case of directional homeomorphism.

4.2. (SSP) structure preserving theorem.

As mentioned in the Introduction, we proved two types of (SSP) structure preserving theorems in [6]. Let us first recall those theorems.

Definition 4.4. Let $A \subset \mathbb{R}^m$ be a set-germ at $0 \in \mathbb{R}^m$ such that $0 \in \overline{A}$ and $B \subset \mathbb{R}^n$ a set-germ at $0 \in \mathbb{R}^n$ such that $0 \in \overline{B}$. Let $h : (A, 0) \to (B, 0)$ be a map-germ. We say that h is an (SSP) *map* if the graph of h satisfies condition (SSP) at $(0, 0) \in \mathbb{R}^m \times \mathbb{R}^n$.

We have an (SSP) structure preserving theorem for this (SSP) bi-Lipshitz homeomorphism.

Theorem 4.5. *([6] Theorem 4.7) Let $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be an (SSP) bi-Lipschitz homeomorphism, and let $A \subset \mathbb{R}^n$ be a set-germ at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A}$. Then A satisfies condition (SSP) if and only if $h(A)$ satisfies condition (SSP).*

In [6] we give a characterisation of (SSP) bi-Lipschitz homeomorphisms.

Proposition 4.6. *([6] Proposition 4.13(3)) Let $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a bi-Lipschitz homeomorphism. Then h is an (SSP) map if and only if $I_n \times h : (\mathbb{R}^n \times \mathbb{R}^n, 0 \times 0) \to (\mathbb{R}^n \times \mathbb{R}^n, 0 \times 0)$ (or $I_n \times h^{-1}$) satisfies condition semiline-(SSP). Here $I_n : \mathbb{R}^n \to \mathbb{R}^n$ is the identity map.*

Applying the above proposition to any semiline $\ell \subset \mathbb{R}^n$ as $\ell = \{0\} \times \ell$ $(\subset \mathbb{R}^n \times \mathbb{R}^n)$, we can see that an (SSP) bi-Lipschitz homeomorphism satisfies condition semiline-(SSP). Therefore an (SSP) bi-Lipschitz homeomorphism is a directional homeomorphism.

We call a homeomorphism $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ a *weak diffeomorphism*, if h and h^{-1} admit derivative (= linear approximation) at $0 \in \mathbb{R}^n$. Y.-N. Gau and J. Lipman [2] have proved the Zariski conjecture on hypersuface multiplicity even in the non-hypersurface case under the assumption that the homeomorphism is a weak diffeomorphism. The hypersurface case was implicitly shown in [1].

A weak diffeomorphism h can be expressed in a neighbourhood of $0 \in \mathbb{R}^n$ as follows:

$$h(x) = M_h(x) + \mathrm{o}(x),$$

where M_h is a regular linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$, and $\lim_{x \to 0} \frac{\| \mathrm{o}(x) \|}{\|x\|} = 0$ as in the previous section. Therefore a weak diffeomorphism is clearly a directional homeomorphism by definition.

We also have an (SSP) structure preserving theorem for weak diffeomorphisms.

Theorem 4.7. *([6] Corollary 4.20) Let $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a weak diffeomorphism, and let $A \subset \mathbb{R}^n$ be a set-germ at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A}$. Then A satisfies condition (SSP) if and only if $h(A)$ satisfies condition (SSP).*

We next show the following (SSP) structure preserving theorem for directional homeomorphisms, generalising Theorems 4.5 and 4.7.

Theorem 4.8. *Let $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a directional homeomorphism, and let $A \subset \mathbb{R}^n$ be a set-germ at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A}$. Then A satisfies condition (SSP) if and only if $h(A)$ satisfies condition (SSP).*

Proof. We assume that A satisfies condition (SSP). Since $h(x) = \tau(x) + \mathrm{o}(x)$ is a directional homeomorphism, let us apply the following lemma to this τ.

Lemma 4.9. *([6] Corollary 2.22) Let $\tau : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a bi-Lipschitz homeomorphism, and let $A \subset \mathbb{R}^n$ be a set-germ at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A}$. Suppose that A satisfies condition (SSP), and τ satisfies condition semiline-(SSP). Then $\tau(A)$ satisfies condition (SSP).*

By this lemma $\tau(A)$ satisfies condition (SSP). Then it follows from Lemma 3.9 that $h(A)$ also satisfies condition (SSP). The converse can be shown similarly. □

4.3. **Important property of (SSP).** We first recall an important property concerning the direction set of the image of a set by a bi-Lipschitz homeomorphism.

Theorem 4.10. *([4] Lemma 5.6) Let A be a set-germ at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A}$, and let $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a bi-Lipschitz homeomorphism. Suppose that A satisfies condition (SSP). Then we have $D(h(LD(A))) = D(h(A))$.*

This result takes a very important role in the proof of the main theorem in [4]. On the other hand, this result does not always hold on a real closed field which is not a complete metric space ([5]).

By Theorem 3.3 we have the following property for directional homeomorphisms.

Theorem 4.11. *Let A be a set-germ at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A}$, and let $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a directional homeomorphism. Then we have $D(h(LD(A))) = D(h(A))$.*

Note that we do not assume the condition (SSP) of A in the case of directional homeomorphisms.

Using Lemma 3.9, we have a corollary of the above theorem.

Corollary 4.12. *Let A be a set-germ at $0 \in \mathbb{R}^n$ with $0 \in \overline{A}$ such that $LD(A) = \ell$ is a semiline, and let $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a directional homeomorphism. Then we have*

$$LD(h(A)) = LD(h(\ell))$$

is a semiline.

4.4. **Image of intersection sets.** For bi-Lipschitz homeomorphisms, we have the following directional property of intersection sets.

Theorem 4.13. *([6] Theorem 2.30) Let $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a bi-Lipschitz homeomorphism, and let U, $V \subset \mathbb{R}^n$ be set-germs at $0 \in \mathbb{R}^n$ such that $0 \in \overline{U} \cap \overline{V}$. Suppose that*

$$D(U \cap V) = D(U) \cap D(V),$$

and $U \cap V$ and $h(U)$ satisfy condition (SSP). Then $D(h(U \cap V)) = D(h(U)) \cap D(h(V))$.

This result has an application to a local classification of spirals (see [6] §5. Appendix). On the other hand, we have the following property for directional homeomorphisms.

Theorem 4.14. *Let $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a directional homeomorphism, and let U, $V \subset \mathbb{R}^n$ be set-germs at $0 \in \mathbb{R}^n$ such that $0 \in \overline{U} \cap \overline{V}$. Suppose that $D(U \cap V) = D(U) \cap D(V)$. Then $D(h(U \cap V)) = D(h(U)) \cap D(h(V))$.*

Proof. It suffices to show $D(h(U)) \cap D(h(V)) \subset D(h(U \cap V))$. Therefore we show the following equivalent condition

$$LD(h(U)) \cap LD(h(V)) \subset LD(h(U \cap V)).$$

Let $\ell \subset LD(h(U)) \cap LD(h(V))$. By Lemma 3.9 we have

$$\ell \subset LD(h(U)) = LD(\tau(U)), \quad \ell \subset LD(h(V)) = LD(\tau(V)).$$

Applying τ^{-1} to the above, we have

$$LD(\tau^{-1}(\ell)) \subset LD(U), \quad LD(\tau^{-1}(\ell)) \subset LD(V).$$

By assumption we have

$$LD(\tau^{-1}(\ell)) \subset LD(U) \cap LD(V) = LD(U \cap V).$$

Applying τ to the above, it follows from Lemma 3.9 that

$$\ell \subset LD(\tau(U \cap V)) = LD(h(U \cap V)).$$

Therefore we have $LD(h(U)) \cap LD(h(V)) \subset LD(h(U \cap V))$. □

Note that we do not assume the condition (SSP) of $U \cap V$ or of $h(U)$ in the case of directional homeomorphisms.

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