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SIMPLE CURVE SINGULARITIES

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Abstract. In this paper we classify simple parametrisations of complex curve singularities of arbitrary embedding dimension. Simple means that all neighbouring singularities fall in finitely many equivalence classes. We take the neighbouring singularities to be the ones occurring in the versal deformation of the parametrisation. This leads to a smaller list than that obtained by looking at the neighbours in the space of multi-germs with a fixed number of branches. Our simple parametrisations are the same as the complex version of the fully simple singularities of Zhitomirskii, who classified real plane and space curve singularities. The list of simple parametrisations of plane curves is the A-D-E list. Also for space curves the list coincides with the lists of simple curves of Giusti and Frühbis-Krüger, in the sense of deformations of the curve. For higher embedding dimension no classification of simple curves is available, but we conjecture that even there the list is exactly that of curves with simple parametrisations.

Introduction

Curve singularities can be described by parametrisations or by systems of equations. These two view points lead to different list of simple objects, with simple meaning that all neighbouring singularities fall in finitely many equivalence classes. This phenomenon was already observed by Bruce and Gaffney, who classified simple parametrisations of irreducible plane curve singularities [BrGa]. In this setting the neighbouring singularities are to be found among the maps $(\mathbb{C},0) \to (\mathbb{C}^2,0)$, with image given by an irreducible function, whereas in Arnold's A-D-E classification [Ar1] all functions are considered. The classifications were extended to irreducible space curves by Gibson and Hobbs [GiHo], irreducible curves of any embedding dimension by Arnold [Ar2] and finally to reducible curves by Kolgushkin and Sadykov [KoSa] on the one hand and to complete intersections by Giusti [Gi] and determinantal codimension 2 singularities by Frühbis-Krüger [F-K, FrNe] on the other hand.

A more restricted definition of simpleness for parametrisations was given by Zhitomirskii, who introduced fully simple singularities [Zh]. The idea is that the neighbouring singularities of multi-germs of maps should be all curves in the neigbourhood of the image, even those with more irreducible components. For plane curves he finds exactly the A-D-E singularities, and also his list of space curves (when corrected) coincides with the lists of Giusti and Frühbis-Krüger together. The definition is quite natural from the point of view of a somewhat different approach to simpleness and modality, explicitly formulated by Wall [Wa1]. Given a singularity, the neighbouring singularities are those occurring in its versal deformation. For contact equivalence this yields the same concept of simpleness as the one obtained by using the space of all functions. For a parametrisation $\varphi\colon (\overline{C},\overline{0}) \to (\mathbb{C}^n,0)$, where $(\overline{C},\overline{0})$ is a smooth multi-germ, we can consider deformations of the map φ (see [GLS, II.2.3], and [GrCo]). We call the parametrisation *simple*, if there are only finitely many isomorphism classes in the versal deformation of φ. A curve is fully simple in the sense of Zhitomirskii [Zh] if and only if its parametrisation is simple in our sense.

Actually, we consider the complex version of Zhitomirskii's notion. In contrast to most of the cited classifications Zhitomirskii [Zh] treats the real case. For curves in 3-space we refer to his paper. Starting from there it should not be difficult to extend our results to the reals. Only the relation between equations and parametrisations becomes more complicated. A curve, defined by real equations, is only the image of a real parametrisation, if it has no complex conjugate branches.

In this paper we classify simple parametrisations of any embedding dimension, for complex map germs. Rather than striking the non-simple ones from the long lists of [Ar2, KoSa] we start from scratch; it is however a good check to compare our list with theirs. Proving simpleness is more difficult in our context, whereas showing that a singularity is not simple is easier: in all cases we succeed by giving a deformation to a confining singularity. The list of these is very simple and contains only the L^n_{n+2}, the curves consisting of $n+2$ lines through the origin in $\mathbb{C}^n$. For $n = 1$ and $n = 2$ the definition has to be modified (Definition 2.3).

For a plane curve singularity every deformation of the parametrisation gives a deformation of the image curve, but not every deformation of the curve comes from a deformation of the parametrisation: a necessary and sufficient condition is that the δ-invariant is constant (see [GLS, II.2.6]). Without comparing lists we prove that a plane curve with simple parametrisation is itself simple by showing that a deformation to a confining singularity can always be realised by a deformation of the parametrisation. We use the characterisation of simple plane curves, given by Barth, Peters and Van de Ven, as curves without points of multiplicity four on the (reduced) total transform in each step of the embedded resolution [BPV, II.8].

For space curves the δ-invariant can go down in a deformation of the parametrisation. Then it is not a (flat) deformation of the image. The simplest example is that of two intersecting lines which are moved from each other, forming two skew lines. In this case we have only a partial explanation of the coincidence of the two classifications. The simple parametrisations come in infinite series, which all are deformations of $A_k \vee L^n_n$, $D_k \vee L^n_n$ or $E_k \vee L^n_n$ (the union of a plane germ with n smooth branches in independent directions), and a finite number of sporadic parametrisations. The sporadic curves have $\delta \leq 5$. As $\delta \geq 5$ for all confining singularities, all curves with $\delta \leq 4$ are simple, and non-simple curves with $\delta = 5$ have a δ-constant deformation to a confining singularity.

Beyond embedding dimension three not much is known about simpleness of curves, in the sense of deforming the image. The curves L^r_r, having $\delta = r - 1$ are simple [BuGr, 7.2.8], and also the curves with $\delta = r$ [Gr]. This follows because the genus $\delta - r + 1$ of the Milnor fibre is upper semi-continuous. Determining adjacencies by explicit computations with the versal deformation seems prohibiting difficult, as may be seen from our computations for partition curves [St1]. Any parametrisation of a curve of multiplicity m can be deformed to a parametrisation of L^m_m, but this is not true for deformations of the image. As shown by Mumford, there exist non-smoothable curves, who only deform to curves of the same type, cf. [Gr]. The argument is that the number of moduli is too large compared to the dimension of a smoothing component; such curves are therefore not simple. Our lack of knowledge is shown by the old unsolved question whether rigid reduced curve singularities exist. Such a singularity, having no nontrivial deformations at all, is certainly simple. But we expect them not to exist. In fact, we believe that our list is also the list of simple curves (for the problem of deforming the image).

Conjecture. *The simple reduced curve singularities are exactly those with simple parametrisation.*

The contents of this paper is as follows. After defining the basic concepts and fixing our notations we formulate our main results. We give the list of simple parametrisations in Section 4. The proof of the classification is in the next Section. In Section 6 we treat plane curves, while the final Section discusses our Conjecture about simple curves. There we give equations and parametrisations for the simple space curves, together with the names from [Gi] and [F-K].

1. Basic concepts

1.1. **Simple curves and parametrisations.** We consider germs of irreducible complex curves $(C,0)$, classified up to analytic isomorphism. Let $n\colon (\overline{C},\overline{0}) \to (C,0)$ be the normalisation. Here $(\overline{C},\overline{0})$ denotes a smooth multi-germ. The δ-invariant of the curve is $\delta(C) = \dim_{\mathbb{C}} \mathcal{O}_{\overline{C}}/\mathcal{O}_C$. Given an embedding $i\colon (C,0) \to (\mathbb{C}^n,0)$ the composed map $\varphi = i \circ n\colon (\overline{C},\overline{0}) \to (\mathbb{C}^n,0)$ is a *parametrisation* of the curve. Classifying curves is equivalent to classifying parametrisations.

We can now consider two deformation problems, that of deforming the curve, and that of deforming the parametrisation. These are very different problems. By a result of Teissier a deformation of the parametrisation gives a deformation of the curve and vice versa if and only if the δ-invariant is constant (see [GLS, II.2.6]). In a deformation of the curve the number of components can go down: a simple example is the deformation of A_3 into A_2, given by $y^2 = x^4 + sx^3$. In a deformation of the parametrisation the number of components cannot decrease. The simplest example of deformation of the parametrisation which does not give a deformation of the image curve, is the deformation of $A_1 \subset \mathbb{C}^3$, which pulls apart the two lines. The first branch is parametrised by $(x,y,z) = (t_1,0,0)$, while the second is $(x,y,z) = (0,t_2,s)$. The ideal I of the image needs four generators:

$$I = \big(yx, zx, y(z-s), z(z-s)\big) .$$

For $s = 0$ the ideal defines the two intersecting lines together with an embedded component at the origin.

Given a deformation problem, suppose that every object X has a versal deformation $\mathcal{X} \to S$.

Definition 1.1. An object X is *simple* if there occur only finitely many isomorphism classes in the versal deformation $\mathcal{X} \to S$.

So an object is simple if it has no moduli and it also does not deform to objects with moduli.

Definition 1.2. A collection of objects forms a collection of *confining* objects, if no object of the collection is simple, and every other non-simple object deforms into one of the objects of the collection.

In particular, the two deformation problems for curve singularities give two notions of simpleness. We will refer to the simple objects as *simple parametrisations*, and *simple curves* respectively.

1.2. **$\mathcal{A}$-simple map germs.** The first results on simple curve singularities were obtained by Bruce–Gaffney [BrGa], for irreducible plane curve singularities, using a different concept of simpleness obtained by considering parametrisations in a fixed space of germs. In fact for any of Mather's groups $\mathcal{R}$, $\mathcal{K}$ and $\mathcal{A}$ (say $\mathcal{G}$) one can define the notion of a $\mathcal{G}$-simple map germ $(k^n,0) \to (k^p,0)$, where k is $\mathbb{R}$ or $\mathbb{C}$: a germ is *$\mathcal{G}$-simple*, if all neighbouring singularities in the space of map germs $(k^n,0) \to (k^p,0)$ fall into finitely many $\mathcal{G}$-equivalence classes.

A parametrisation of an irreducible complex plane curve singularity is a map germ

$$\varphi\colon (\mathbb{C},0) \to (\mathbb{C}^2,0).$$

Two such map germs φ_1 and φ_2 are $\mathcal{A}$-equivalent if and only defining equations f_1 and f_2 for their images are $\mathcal{K}$-equivalent, but $\mathcal{A}$-simpleness of φ is not equivalent to $\mathcal{K}$-simpleness of a defining equation f.

Example 1.3. The germ $\varphi(t) = (t^4, t^5)$ is $\mathcal{A}$-simple [BrGa, Theorem 3.8] but its defining equation $f = y^4 - x^5$ is unimodal; it is W_{12} in Arnold's notation. The function f has a deformation $F(x, y, s) = y^4 - x^5 + s(x^2y^2 + x^4)$ to X_9. This deformation can be parametrised as $\Phi(t, s) = (t^4 + s(t^2 + 1), t^5 + s(t^3 + t))$, but as germ at the origin $\varphi_s(t) = \Phi(t, s)$ is an immersion for $s \neq 0$.

Bruce and Gaffney call an irreducible function germ $f\colon (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ *irreducible $\mathcal{K}$-simple* if all neighbouring irreducible functions fall into finitely many $\mathcal{K}$-equivalence classes. The parametrisation of a curve, which is irreducible $\mathcal{K}$-simple, is $\mathcal{A}$-simple. The confining singularities for irreducible plane curve singularities are those with Puiseux pairs $(4, 9)$ and $(5, 6)$. All irreducible curves below these ones have only finitely many $\mathcal{K}$-orbits, so are therefore irreducible $\mathcal{K}$-simple. The complete list consists A_{2k}, E_{6k}, E_{6k+2}, W_{12}, W_{18} and $W^{\#}_{1,2q-1}$. In particular, the list of $\mathcal{A}$-simple parametrisations coincides with that of irreducible $\mathcal{K}$-simple functions.

The classification of $\mathcal{A}$-simple curves was extended to space curves by Gibson–Hobbs [GiHo] and by Arnol'd [Ar2] to irreducible curves of arbitrary embedding dimension and finally to reducible curves by Kolgushkin–Sadykov [KoSa]. The lists become rather long.

The other possibility in the situation of Example 1.3 is to change the concept of simpleness for parametrisatrions. This approach was taken by Zhitomirskii [Zh]. We recall his definition of fully simple singularities, for real parametrised curves.

Definition 1.4. An *arc* $F\colon [a, b] \to \mathbb{R}^n$ is said to *represent* a multi-germ

$$\gamma\colon \coprod_{i=1}^{r} (\mathbb{R}_{(i)}, 0) \to (\mathbb{R}^n, 0)$$

if the multi-germ $(F, F^{-1}(0))$ is $\mathcal{A}$-equivalent to γ. Here we assume that the image of F contains the origin, and that the endpoints $F(a)$ and $F(b)$ are different from the origin.

Definition 1.5. A multi-germ γ of a parameterized curve in $\mathbb{R}^n$ is *fully simple* is there exists an arc $F\colon [a, b] \to \mathbb{R}^n$ representing γ such that the singularities of all arcs in a neighbourhood of F at all points of their images sufficiently close to the origin belong to finitely many equivalence classes.

As Zhitomirskii remarks, this definition extends in a natural way to complex parametrisations. It is convenient to represent a reducible curve by a finite number of arcs. A nearby fibre in a good representative of the germ of the versal deformation of a parametrisation gives a finite collection of complex arcs. The versal deformation contains representatives for the isomorphism classes of all neighbouring arcs. Therefore the simple complex parametrisations, in the sense of Definition 1.1 are exactly the complex fully simple parametrised curves of Zhitomirskii [Zh].

1.3. **Stably equivalent parametrisations.** In a deformation of the parametrisation the embedding dimension can increase. Therefore the collection of confining singularities depends on the chosen target dimension for the parametrisation we start with. Two parametrisations which only differ in target dimension are called *stably equivalent* [Ar2]. A parametrisation is *stably simple* if all stably equivalent parametrisations are simple.

Lemma 1.6. *A simple parametrisation is stably simple.*

Proof. Suppose a simple parametrisation $\varphi\colon (\overline{C}, \overline{0}) \to (\mathbb{C}^n, 0)$ deforms with higher target dimension into a parametrisation with moduli, so there exist a family $\psi_s\colon (\overline{C}, \overline{0}) \to (\mathbb{C}^{n+k}, 0)$ with moduli. For a generic projection $\pi\colon (\mathbb{C}^{n+k}, 0) \to (\mathbb{C}^n, 0)$ the family $\pi \circ \psi_s$ is a deformation of φ. One expects a generic projection of a singularity to have more moduli than the singularity itself, so $\pi \circ \psi_s$ has moduli, contradicting that φ is simple. It suffices to prove this for the confining singularities for stable simpleness. By Theorem 3.2 they are the curves L^n_{n+2} of Definition 2.3, and for them the expectation is indeed true. $\square$

We classify stably simple parametrisations. The Lemma justifies that we speak only of simple parametrisations and drop the word 'stable'. We always consider a curve as embedded in $(\mathbb{C}^N, 0)$ for N large enough, except in the section on plane curves.

2. Notations

2.1. Curves with smooth branches.

Definition 2.1. A curve singularity $C = C_1 \cup C_2$ is *decomposable* if the curves C_1 and C_2 lie in smooth spaces intersecting each other transversally in one point, the singular point of C. We write $C = C_1 \vee C_2$.

We write $C \vee L$ for the wedge of C with a smooth branch.

Definition 2.2. The curve $L^n_n = L \vee \cdots \vee L \subset \mathbb{C}^n$ is the curve isomorphic to the singularity consisting of the coordinate axes in $\mathbb{C}^n$. The curve L^n_{n+1}, $n \geq 2$ is the curve consisting of $n+1$ lines in $\mathbb{C}^n$ through the origin in general position, meaning that each subset consisting of n lines span $\mathbb{C}^n$.

Note that L^2_3 is the plane curve singularity D_4.

Points in projective space are in generic position if each subset imposes independent conditions on hypersurfaces of each degree [Gr]. The curve L^n_{n+2}, which is the cone over $n+2$ points in generic position in $\mathbb{P}^{n-1}$, has $\mu = \delta + 2$, if $n \geq 3$. But the singularity $\widetilde{E}_7$, four lines though the origin, has $\mu = \delta + 3$. There exists a curve with the same tangent cone, having $\mu = \delta + 2$; we lift one branch out of the plane. Let $\widetilde{E}_7$ be given by $xy(x-y)(x-\lambda y) = 0$. We take the same first three lines, but parametrise the last one as $(x, y, z) = (\lambda t, t, t^2)$. The equations are determinantal:

$$\text{Rank} \begin{pmatrix} z & \lambda(x-y) & y(x-y) \\ 0 & x-\lambda y & z-y^2 \end{pmatrix} \leq 1\,. \tag{1}$$

We will call this curve L^2_4. As it is not a complete intersection, there is no deformation from $\widetilde{E}_7$, but there is a deformation of the parametrisation.

The curve L^1_3 consists of three smooth branches with common tangent. The plane curve $\widetilde{E}_8\colon x(x-y^2)(x-\lambda y^2)$ has $\mu = \delta + 3$. We can again lift one branch out of the plane and parametrise $(x, y, z) = (\lambda t^2, t, t^3)$. Equations are

$$\text{Rank} \begin{pmatrix} z & \lambda(\lambda-1)y & \lambda x \\ 0 & x-\lambda y^2 & \lambda z - xy \end{pmatrix} \leq 1\,. \tag{2}$$

If we lift the line further out of the plane, as $(x, y, z) = (\lambda t^2, t, t^2)$, the coefficient of the first t^2 in x can be transformed into 1, and we get the simple curve denoted $J_{2,0}(2)$ by Frühbis-Krüger [F-K] and denoted S^t_3 in [St2]. The difference between the curves S^t_3 and L^1_3 can be seen from the 2-jet of the parametrisation. Following [Zh] we say that the 2-jet $j^2\varphi$ is *planar* if the image of φ lies modulo terms of third order on a smooth surface.

Definition 2.3. The curve L^n_{n+2} is for $n \geq 3$ the curve consisting of $n+2$ lines through the origin in generic position in $\mathbb{C}^n$, the curve L^2_4 is the curve with equations (1) and L^1_3 the curve with equations (2).

2.2. **Notation for singular curves.** We will denote monomial curves by their semigroup, so the curve $Z_{10}\colon z^2+yx^2=y^2+x^3=0$ of [Gi] is $(4,6,7)$. Plane curves $(2,2k{+}1)$ are mostly referred to by their name A_{2k}. Also for the monomial curve of minimal multiplicity $(k,k+1,\dots,2k-1)$ with $\delta=k-1$ we use a special name M_k. We extend this notation to quasi-homogeneous reducible curves by writing the exponents of the parameter. The union of curves is indicated by a plus sign. If for some coordinate function $z_i=\varphi_i(t)=0$, we write a dash. For example, the curve $S_3^t=J_{2,0}(2)$ described above is notated $(1,\_,\_)+(1,2,\_)+(1,\_,2)$.

2.3. **Notation for adjacencies.** The name or symbol denotes both a curve and its parametrisation. There are two types of adjacencies, for deformations of the parametrisation and for deformations of the image curve. We refrain from the most logical notation $\succ\!\!-$ for adjacency of parametrisations, $\rightarrow$ for adjacency of image curves and $\rightarrowtail$ for an adjacency, which can be obtained in both ways, as the latter is the most frequent. We will use only twice a symbol for adjacency of image curves, and we choose $\dashrightarrow$ for it. Adjacencies of parametrisations occur more frequently and we use $\succ\!\!-$ for them. This leaves the usual arrow $\rightarrow$ for adjacency in both ways.

3. Main results on parametrisations

With the notations introduced above we can formulate our classification result.

Theorem 3.1. *The infinite series of curves A_k, $A_k\vee L_n^n$ $(n\geq 1)$, D_k, $D_k\vee L_n^n$ and E_k, $E_k\vee L_n^n$ and the sporadic curves $(5,6,7,8)$, $(4,6,7)$, $(2,3,\_,\_)+(\_,4,5,3)$, and $(4,5,7)\vee L$ have simple parametrisations. Any other simple parametrisation occurs in the versal deformation of one of these parametrisations.*

A complete list of simple parametrisations is given in the next section. In the course of the classification we also determine the confining singularities, thereby proving (in the complex case) Conjecture A1 of Zhitomirskii [Zh].

Theorem 3.2. *The confining singularities for deformations of parametrisations are the curves L_n^{n-2} from Definition 2.3.*

The list of simple parametrisations shows that also Conjecture B1 of Zhitomirskii [Zh] is true:

Corollary 3.3. *The curve singularities with simple parametrisations are quasi-homogeneous.*

4. List of simple parametrisations

We list the curves together with some adjacencies. These are by no means all adjacencies, but we rather use them to organise the list. We start with the sporadic curves.

4.1. **Sporadic curves.** For all curves listed the δ-invariant satisfies $\delta\leq 5$. In each case the most singular curve has $\delta=5$ and an adjacency of parametrisations and image curves (given by an arrow $\leftarrow$ or $\downarrow$) is δ-constant, while the other adjacencies lower δ by one.

4.1.1. *Irreducible curves.* There are eight unibranch sporadic curves.

$$
\begin{array}{ccccccccc}
 & & & & (5,6,7,8,9) & -\!\!\prec & (5,6,7,8) & & \\
 & & & & \downarrow & & \Upsilon & & \\
(4,5,6,7) & -\!\!\prec & (4,5,6) & \leftarrow & (4,5,7) & \leftarrow & (4,6,7,9) & -\!\!\prec & (4,6,7) \\
 & & & & & & \downarrow & & \\
 & & & & & & (3,7,8) & &
\end{array}
$$

4.1.2. *One branch of multiplicity four and a line.*

$$
\begin{array}{ccccc}
 & & (4,5,6,7)+(\_,\_,1,\_) & \leftarrow & (4,5,7)\vee L \\
 & & \downarrow & & \downarrow \\
(4,5,6,7)\vee L & \mathrel{-\!\!\prec} & (4,5,6,7)+(\_,\_,\_,1) & \leftarrow & (4,5,6)\vee L
\end{array}
$$

4.1.3. *One branch of multiplicity three and a cusp.*

$$
A_2\vee M_3 \quad \mathrel{-\!\!\prec} \quad (2,3,\_,\_)+(\_,5,4,3) \quad \leftarrow \quad (2,3,\_,\_)+(\_,5,4,3)
$$

4.1.4. *Two cusps and a line.*

$$
\begin{array}{ccccc}
 & & \begin{array}{c}(2,3,\_,\_)+(\_,3,2,\_)\\ +(\_,\_,\_,1)\end{array} & \leftarrow & \begin{array}{c}(2,3,\_,\_)+(3,\_,2,\_)\\ +(\_,\_,\_,1)\end{array} \\
 & & \downarrow & & \downarrow \\
A_2\vee A_2\vee L & \mathrel{-\!\!\prec} & \begin{array}{c}(2,3,\_,\_)+(\_,\_,3,2)\\ +(\_,1,1,\_)\end{array} & \leftarrow & \begin{array}{c}(2,3,\_,\_)+(\_,\_,3,2)\\ +(1,\_,1,\_)\end{array}
\end{array}
$$

4.1.5. *The union of two A_k-singularities.*

$$
\begin{array}{ccccc}
A_2\vee A_4 & \mathrel{-\!\!\prec} & (2,3,\_)+(\_,5,2) & & (2,3,\_)+(2,\_,3) \\
\downarrow & & \downarrow & & \Yup \\
A_2\vee A_3 & \mathrel{-\!\!\prec} & (2,3,\_)+(\_,\_,1)+(\_,2,1) & & (2,3,\_,\_)+(2,\_,3,4) \\
\downarrow & & \downarrow & & \swarrow \\
A_2\vee A_2 & \mathrel{-\!\!\prec} & (2,3,\_)+(\_,3,2) & \leftarrow & (2,3,\_)+(\_,2,3)
\end{array}
$$

4.1.6. *Other sporadic curves.*

$$
\begin{array}{ccc}
(3,4,5,\_)+(1,\_,\_,2) & \mathrel{-\!\!\prec} & (3,4,5)+(1,\_,\_) \\
\downarrow & & \Yup \\
(1,\_,\_)+(1,2,\_)+(1,\_,2) & \leftarrow & (2,5,\_)+(1,\_,2)
\end{array}
$$

4.2. **Infinite series, of the form $C\vee L_k^k$.** All singularities in this part of the list are related to A_k, D_k or E_k. We have therefore series of series and individual series. A series is of the form $C\vee L_k^k$ with C indecomposable. Here we allow $k=0$ and interpret L_0^0 as point, so $C\vee L_0^0$ is just the curve C itself. We list below only the indecomposable curves C. The only curve not of this form is the simplest of all, the totally decomposable curve L_n^n. This curve is singular if $n\geq 2$, with $L_2^2=A_1$. We include L_n^n by including A_1 in the list, even though it is decomposable.

4.2.1. *Indecomposable curves of type E and deformations.*

$$
\begin{array}{ccccccc}
(3,4,5) & \mathrel{-\!\!\prec} & E_6\colon(3,4) & \leftarrow & (3,5,7) & \mathrel{-\!\!\prec} & E_8\colon(3,5) \\
 & & & & \downarrow & & \downarrow \\
 & & & & (2,3,\_)+(1,\_,2) & \mathrel{-\!\!\prec} & E_7\colon(2,3)+(1,\_)
\end{array}
$$

4.2.2. *Deformations of $E_k\vee L_{n-2}^{n-2}$.* Here n is the embedding dimension, which has to satisfy $n\geq 3$. From E_8 and E_6 we get

$$
\begin{array}{c}
(3,5,7)\vee L_{n-3}^{n-3}+(\_,\_,1,\dots,1) \\
\Yup \\
(3,4,5)\vee L_{n-3}^{n-3}+(\_,1,\_,1\dots,1) \\
\downarrow \\
(3,4,5)\vee L_{n-3}^{n-3}+(\_,\_,1,\dots,1)
\end{array}
$$

and from E_7

$$A_2 \vee L_{n-2}^{n-2} + (1, \_, 1, \dots, 1) \prec A_2 \vee L_{n-2}^{n-2} + (1, \_, 2, \dots, 2)$$

4.2.3. *Indecomposable curves of type A.*

$$A_1 \colon (1, -) + (-, 1)$$
$$A_{2k-1} \colon (1, -) + (1, k) \leftarrow A_{2k} \colon (2, 2k+1)$$

4.2.4. *Deformations of $D_k \vee L_{n-2}^{n-2}$.*

$$L_n^n + (1, \dots, 1)$$
$$A_{2k} \vee L_{n-2}^{n-2} + (\_, 1, 1 \dots, 1) \leftarrow A_{2k-1} \vee L_{n-2}^{n-2} + (\_, 1, 1 \dots, 1)$$

Here $n \geq 2$ is again the embedding dimension. For $n = 2$ the curves are the plane curves D_4, D_{2k+3} and D_{2k+2}.

5. Classification

The proof of Theorems 3.1 and 3.2 proceeds by classifying all parametrisations which do not deform into a parametrisation of a curve L_n^{n-2}. The result is that these do not have moduli. Furthermore we show that all other parametrisations do deform into L_n^{n-2}. Therefore singularities of the list can only deform into other singularities of the list, implying simpleness.

We start by describing large classes of parametrisations, which are not simple. From them we derive restrictions on the multiplicities of the irreducible components of curves with simple parametrisation.

5.1. **Some adjacencies.**

5.1.1. *Every parametrisation of a curve $C = C_1 \cup C_2$ deforms into $C_1 \vee C_2$.* Parametrise C_1 with $\varphi^{(1)} \colon \overline{C}_1 \to \mathbb{C}^n$ and C_2 with $\varphi^{(2)} \colon \overline{C}_2 \to \mathbb{C}^n$, and consider the curve as lying in $\mathbb{C}^{2n}$. The parametrisation, given by $(\varphi^{(1)}, 0)$ and $(\varphi^{(2)}, s\varphi^{(2)})$ has for $s \neq 0$ image $C_1 \vee C_2$.

If a curve C with simple parametrisation is reducible, and can be written as union $C' \cup C''$, then both C' and C'' have a simple parametrisation.

5.1.2. *A parametrisation of an irreducible curve of multiplicity m deforms into the monomial curve M_m.* We may assume that we have a parametrisation $\varphi \colon \overline{C} \to \mathbb{C}^m$ with first component $z_1 = \varphi_1(t) = t^m$. Now deform $z_1 = t^m$, $z_i = \varphi_i(t) + st^{m+i-1}$ for $i \geq 2$.

5.1.3. *M_m deforms into $M_{m_1} \vee \dots \vee M_{m_k}$ for any partition $(m_1, \dots, m_k)$ of m.* A description in terms of equations is given in [St1, p. 199]. A simple argument in terms of the parametrisation is the following. The curve M_m is a special hyperplane section of the cone over the rational curve of multiplicity m and is resolved by one blow-up. Now deform the smooth strict transform such that it intersects the exceptional divisor in k points with multiplicities given by the partition $(m_1, \dots, m_k)$ and blow down again.

5.1.4. *$A_2 \vee L \to A_3$.* This is a special case of the adjacency $A_k \vee L \to D_{k+1}$ (in fact $D_3 = A_3$), which can be inferred from the formulas of [F-K, p. 1040], but is missing in [FrNe, Diagram 4]. Consider the deformation

$$\operatorname{Rank} \begin{pmatrix} x^k & y & z \\ y & x & s \end{pmatrix} \leq 1 \,.$$

One branch is $(0, 0, t_1)$ and for even k the second branch is $(t_2^2, t_2^{k+1}, st_2^{k-1})$.

5.2. **First consequences.** The curve $A_3 \vee A_3$ is not simple, as it deforms to L_4^2 (with constant $\delta = 5$); just rotate some lines in the plane spanned by the tangent lines of both A_3-singularities. As $A_2 \vee L \to A_3$, the curve $A_2 \vee A_2 \vee L_2^2$ is not simple. Using the adjacencies for monomial curves (5.1.2) we obtain the following chain of adjacent, non-simple curves:

$$M_6 \to M_5 \vee L \to M_4 \vee L_2^2 \to A_2 \vee A_2 \vee L_2^2.$$

We conclude that the parametrisation of an irreducible curve of multiplicity at least 6 is not simple. A simple parametrisation with at least four branches has at most one singular component, of multiplicity at most three. A (sporadic) simple curve has at most two singular components ($A_2 \vee A_2 \vee A_2$ is not simple), and the multiplicity is at most 5.

5.3. **Irreducible curves.** We may assume that the parametrisation has the form $x_i = \varphi_i(t)$, $i = 1, \dots, k$, with $v(\varphi_i) < v(\varphi_j)$ for $i < j$, $v(\varphi_i)$ being the order in t of φ_i. We can also achieve that $v(\varphi_j)$ does not lie in the semigroup generated by the $v(\varphi_i)$ with $i < j$.

A parametrisation of a curve of multiplicity at least 5 (that is, $v(\varphi_1) \geq 5$) is not simple if $v(\varphi_4) \geq 10$: deform into L_5^3 by perturbing φ_1, φ_2 and φ_3 such that they are divisible by $t^5 - s$ and making φ_j, $j \geq 4$, divisible by $(t^5 - s)^2$. A parametrisation of a curve of multiplicity at least 4 is not simple if $v(\varphi_3) \geq 8$: deform into L_4^2 by perturbing φ_1 and φ_2 such that they are divisible by $t^4 - s$ and making φ_j, $j \geq 3$, divisible by $(t^4 - s)^2$. For example, the curve $(t^5, t^6 u_2(t), t^8 u_3(t))$ has the deformation $(t(t^4 - s), t^2(t^4 - s)u_2(t), (t^4 - s)^2 u_3(t))$. A multiplicity 3 curve with $v(\varphi_2) > 6$ deforms into L_3^1, a curve with planar 2-jet, if $v(\varphi_3) > 9$ (recall that by assumption $v(\varphi_3)$ is not divisible by 3). Irreducible double points are simple.

This leaves only a few possibilities for simple parametrisations. Their normal forms can be computed with standard methods; they can be found in the paper by Ebey [Eb].

Lemma 5.1. *The curve* $(5, 6, 7, 9)$ *is not simple, as* $(5, 6, 7, 9) \to L_3^1$.

Proof. Consider the deformation

$$\varphi_s(t) = ((t^3 - s)t^2, (t^3 - s)^2, (t^3 - s)^2 t, (t^3 - s)^3)\ .$$

The parametrisation satisfies the equations $w = sz - x^2 \equiv 0 \mod (t^3 - s)^3$, so for $s \neq 0$ the 2-jet of $\varphi_s(t)$ is planar. □

Proposition 5.2. *The parametrisations of the curves* $(5, 6, 7, 8)$ *and* $(4, 6, 7)$ *are simple. They deforms into the other unibranch sporadic curves of* 4.1.1 *and the irreducible triple points of* 4.2.1.

Proof. As explained above, we now only show that there is no deformation to L_n^{n-2}. It suffices to consider the ones with $\delta = 5$. A deformation of the parametrisation of $(5, 6, 7, 8)$ or $(4, 6, 7)$ to L_4^2 or L_3^1 is δ-constant, so also a deformation of the curve. The curves $(5, 6, 7, 8)$ and $(4, 6, 7)$ are Gorenstein, but L_4^2 and L_3^1 not. Therefore such a deformation does not exist.

The adjacencies of 4.1.1 and 4.2.1 are easily established. □

5.4. **Curves with one singular component of multiplicity three or four.**

5.4.1. *Multiplicity four.* The curve $(4, 6, 7, 9)$ deforms into the (simple) curve $J_{2,0}(2) = S_3^t$ consisting of three tangent lines with non-planar 2-jet and therefore $(4, 6, 7, 9) \vee L$ deforms into L_4^2 and is not simple. If the line in the curve $(4, 5, 6) \cup L$ is not transverse to the Zariski tangent space of $(4, 5, 6)$, then the curve deforms into L_5^3. This leaves $(4, 5, 7) \vee L$, $(4, 5, 6) \vee L$ and curves of the type $(4, 5, 6, 7) \cup L$. The classification of the latter curves follows from the general results of [St2, 2.2]: the isomorphism type depends on the osculating space of M_4, to which the line is tangent, and the line can be taken to be a coordinate axis, except in the most degenerate case, that the line is tangent to the tangent line of the curve. The curve M_4 deforms into D_4, with tangent plane the (x_1, x_2)-plane, so if the line is tangent to this plane, there is a deformation to L_4^2.

Proposition 5.3. *The parametrisation of the curve* $(4,5,7) \vee L$ *is simple. It deforms into the other sporadic curves of* 4.1.2.

Proof. If $(4,5,7) \vee L \to L_4^2$, then $(4,5,7)$ deforms into three smooth branches, tangent to the plane containing the line L. Projection along L onto $\mathbb{C}^3$ gives a δ-constant deformation to the space curve $J_{2,0}(2)$ consisting of three smooth branches with common tangent. According to the tables in [F-K] such a deformation does not exist. To be self-contained we give a proof along the lines of the proofs in [Zh].

So suppose $(4,5,7) \to J_{2,0}(2)$. The parametrisation has the form

$$\varphi_i(t,s) = (t-a_s)(t-b_s)(t-c_s)\psi_i(t,s)$$

for $i = 1,2,3$. The images of the germs (t, a_s), (t, b_s) and (t, c_s) are tangent to a line L_s, which has a limiting position for $s \to 0$. By a coordinate transformation we may suppose that the line L_s is constant. It is given by two linearly independent equations of the form $Az_1 + Bz_2 + Cz_3 = 0$. This implies that

$$A\varphi_1(t,s) + B\varphi_2(t,s) + C\varphi_3(t,s) \equiv 0 \mod (t-a_s)^2(t-b_s)^2(t-c_s)^2$$

for all s. Specialising to $s = 0$ leads to the equation $At^4 + Bt^5 + Ct^7 \equiv 0 \mod t^6$, from which we conclude that $A = B = 0$. But then there is only one linear equation.

If $(4,5,7) \vee L \to L_3^1$, then $(4,5,7)$ deforms into two smooth branches with the line L as common tangent. Projection onto $\mathbb{C}^3$ gives a deformation to the space curve consisting of two cusps with common tangent, as L_3^1 has planar 2-jet. But this curve is at least Z_9 with $\delta = 5 > 4 = \delta(4,5,7)$. □

5.4.2. *Multiplicity three.* Let C_3 be an irreducible curve of multiplicity 3. A parametrisation-simple union of C_3 and n smooth branches has embedding dimension at least $n+2$, for otherwise it deforms into L_{n+3}^{n+1}. The n smooth branches form an L_n^n: for $n = 1$ this is trivial; if $n = 2$ and the branches are A_{2k-1} with $k > 1$, then the parametrisation deforms into the non-simple $A_{2k-1} \vee A_4$, as M_3 deforms into A_4; finally, if $n > 2$ and the branches deform into L_n^{n-1}, then the parametrisation deforms into L_{n+1}^{n-1}, as the parametrisation of C_3 deforms into a smooth branch with arbitrary tangent.

The curve $E_{12}(2) = (3,7,8)$ deforms into $J_{2,0}(2) = S_3^t$, so $(3,7,8) \vee L$ is not simple.

Proposition 5.4. *The curves* $E_6 \vee L_n^n$ *and* $E_8 \vee L_n^n$ *have simple parametrisations.*

Proof. As $E_8 \to E_6 + A_1$ it suffices to show simpleness for $E_8 \vee L_n^n$. We have to exclude deformations of the parametrisation into an L_{k+2}^k. If nm of the n deformed lines do not pass through the singular point of L_{k+2}^k, then there is also a deformation $E_8 \vee L_{n-m}^{n-m} \rightarrowtail L_{k+2}^k$. So we may assume that $m = 0$, and that the n lines are not deformed at all. The only possibilities for k are therefore $k = n$ and $k = n+1$.

If $E_8 \vee L_n^n \rightarrowtail L_{n+2}^n$, then E_8 is deformed into two smooth branches tangent to the space spanned by L_n^n. Projection onto the plane of the E_8 gives a deformation of the parametrisation into the union of two plane curves of multiplicity two, which is impossible.

If $E_8 \vee L_n^n \rightarrowtail L_{n+3}^{n+1}$, then projection onto the plane of the E_8 gives a deformation of E_8 into three tangent smooth branches, which is also impossible, as this would increase δ. □

For the curves C_3 of type $E_8(1) = (3,5,7)$ and $E_6(1) = M_3 = (3,4,5)$ with $(C_3 \cdot L_n^n) > 1$ we look at the 1-dimensional intersection T of the Zariski tangent spaces of the singular curve and L_n^n. If for $(3,5,7)$ the line T lies in the (x_1, x_2)-plane, then the curve deforms into L_{n+3}^{n+1}, as $(3,5,7) \to D_4$. Otherwise there is a transformation bringing T to the x_3-axis. In L_n^n the line T is in the direction $(1,\dots,1,0,\dots,0)$. The curve is indecomposable if and only if there are no zeroes. We transform to a different normal form, where the line T is a coordinate axis. The resulting curve is a deformation of $E_8 \vee L_n^n$.

The curve $(3,4,5)$ deforms into A_3 with the x_1-axis as tangent line, so if T is this axis, then the curve is not simple if $n>1$: it deforms to L_{n+2}^n. For $n=1$ the curve is simple: if the 2-jet of the parametrisation of L has image T, then it is the curve W_9, which is a δ-constant deformation of $Z_{10}=(4,6,7)$. There is also a curve with $\delta=4$, with $L=(t,0,0,t^2)$. For the other cases the intersection multiplicity $(M_3 \cdot L_n^n)$ is equal to 2, and [St2, 2.3] applies. If T does not lie in the (x_1,x_2)-plane, then the curve is a deformation of $E_6 \vee L$, otherwise of $E_8 \vee L$, under the deformation of the parametrisation $(t^3, st^4, t^5, 0, \dots, 0)$.

The classification of simple parametrisations with one singular branch of multiplicity three is now complete.

5.5. **Two singular components.** As every parametrisation of an irreducible curve other than A_2 deforms into A_3, one component has to be A_2. The curve $A_2 \vee A_5$ is not simple, as it deforms into L_3^1. This implies that the other singular component is M_3, A_4 or A_2.

5.5.1. *$A_2 \cup M_3$.* The embedding dimension is at least 4. Unless the curve is $A_2 \vee M_3$ we let T be the intersection line of the Zariski tangent spaces of the components. As A_2 deforms into $A_1 = L_2^2$ the curve is not simple, if T is the tangent line of M_3, by what was said above for $M_3 \cup L_2^2$. Otherwise $(A_2 \cdot M_3)=2$ and T may be taken as coordinate axis. There are four curves to consider.

Lemma 5.5. *The curve $(2,3,\_,\_)+(4,\_,5,3)$ and $(2,3,\_,\_)+(5,\_,4,3)$ are not simple, as they deform to L_3^1.*

Proof. The first curve deforms into the second. For that case we deform the cusp into a smooth branch by $(t^2,t^3,0,2st)$ and M_3 into A_3 by

$$((t^2-s^2)^2 t, 0, (t^2-s^2)^2, (t^2-s^2)(t+2s)) .$$

The A_3 lies on the smooth surface

$$12xs^6 - 3w^2s^4 - xw + z^2 + 2zws^2 + 12zs^8 = 0 .$$

The parametrisation of the smooth branch satisfies this equation modulo terms of degree 3. The intersection number of the branch with A_3 is 3, so we have three smooth tangent branches with $\delta=5$, which is L_3^1. □

Proposition 5.6. *The curves $(2,3,\_,\_)+(\_,4,5,3)$ and $(2,3,\_,\_)+(\_,5,4,3)$ are simple.*

Proof. Suppose first that such a curve deforms to L_4^2. The component M_3 deforms only to three smooth branches spanning 3-space, so both components have to deform to two smooth branches, and the two smooth branches, into which M_3 deforms, are tangent to the plane of the cusp. Then the last component of the parametrisation has the form $\varphi_4(t,s)=(t-a_s)^2(t-b_s)^2\psi_i(t,s)$, but $\varphi_4(t,0)=t^3$.

If the curve deforms to L_3^1, then the cusp deforms into a smooth branch and $M_3 \to A_3$ is a δ-constant deformation. The equation $z_1=0$ of M_3 deforms into $z_1+sf(z_1,\dots,z_4)=0$ and the first component of the parametrisation of A_2 is $\varphi_1(t,s)=t^2+s\psi_i(t,s)$. The intersection multiplicity of the smooth branch and A_3 is at most the order in t of $\varphi_1+sf(\varphi_1,\dots,\varphi_4)$, so at most 2. Therefore the three smooth branches form the simple singularity $J_{2,0}(2)$, with $\delta=4$. □

5.5.2. $A_2 \cup A_4$. Such a curve is not simple if the tangent line of A_4 lies in the plane of the cusp A_2, as it then deforms to L_4^2. If the tangent line of the cusp lies in the plane of the A_4, then there is a deformation to L_3^1. The curve $T_9 = (2,3,\_) + (\_,5,2)$ is a deformation of $Z_{10} = (4,6,7)$. It deforms into $A_2 \vee A_4$. The parametrisation of $A_2 \vee A_4 \vee L$ is not simple.

5.5.3. $A_2 \cup A_2$. All possibilities for the intersection line T yield simple curves. The curve Z_{10} deforms into $Z_9 = (2,3,\_) + (2,\_,3)$. A deformation of the parametrisation gives the curve $Z_9(1) = (2,3,\_,\_) + (2,\_,3,4)$ with $\delta = 4$. It deforms δ-constant into $T_7^* = (2,3,\_) + (3,\_,2)$ and then into $T_7 = (3,2,\_)+(3,\_,2)$. By a deformation of the parametrisation we obtain $A_2 \vee A_2$. The curves here and of the previous paragraph are listed in 4.1.5.

5.5.4. $A_2 \cup A_2 \cup L$. The curve $Z_9(1) \vee L$ deforms into L_3^1. The curve consisting of $A_2 \cup A_2$ and a smooth branch is not simple if the smooth branch is tangent to the plane spanned by the tangent lines of the cusps, for then there is again a deformation to L_3^1. The branch is also not tangent to the plane of one of the cusps, as $A_2 \vee D_4$ is not simple, deforming into L_4^2.

The singularity $T_7^* \vee L$ is a deformation of $W_8^* \vee L = (4,5,7) \vee L$, as $W_8^* \to T_7^*$ [F-K]: use the parametrisation $(t^2(t-s)^2, t^3(t-s)^2, t^4(t-s)^3)$.

The curves $A_2 \cup A_2 \cup L$ in to which the parametrisation of $T_7^* \vee L$ deforms are listed in 4.1.4.

5.6. **At most one component of multiplicity two.** If there are only smooth branches it can happen that some branches have the same tangent line. As $A_3 \vee A_3$ is not simple, this can happen only for one direction. The curve $J_{2,0}(2)$ consisting of three smooth branches is a deformation of $J_{2,1}(2) = (2,5,\_) + (1,\_,2)$. The curve $J_{2,0}(2) \vee L$ deforms into L_4^2. So if the curve has at least four branches, only two of them can be tangent.

5.6.1. *Curves containing an A_k, $k \geq 3$.* As $A_3 \vee L_n^{n-1} \to L_{n+2}^n$, the n smooth branches in a curve, consisting of an A_k ($k \geq 3$) and these n branches, form an L_n^n. The intersection of the space spanned by this L_n^n with the tangent plane of the A_k is at most 1-dimensional. If it is a line, this line is not tangent to the A_k, for otherwise there is a deformation of the curve into L_{n+2}^n. So we can take the line to be a coordinate axis, and get the normal form listed above (4.2.4), see also [St2, Example 2-14]. Note that for $n = 1$ we have D_{k+3}. Any curve of this type is a deformation of $D_{k+3} \vee L_{n-1}^{n-1}$.

Proposition 5.7. *The curves $D_{k+3} \vee L_n^n$ have simple parametrisations.*

Proof. It suffices to prove the statement for $D_{2m+3} \vee L_n^n$. Again we have to exclude a deformation to L_{n+2}^n or L_{n+3}^{n+1}. In the first case the deformed line of D_{2m+3} does not pass through the singular point, and in the second case we can assume that this line and L_n^n are unchanged. In both cases the A_{2k} in D_{2k+3} deforms into two smooth branches, whose projection onto the plane is singular or always tangent to the line in D_{2k+3}, again impossible. □

5.6.2. *Curves containing an A_2.* If two smooth branches have the same tangent, then there are no more smooth branches ($A_3 \vee A_2 \vee L$ is not simple). The curve $A_2 \vee A_5$ is not simple, as it deforms to L_3^1. For $A_2 \vee A_3$ the smooth curves cannot be tangent to the plane of the cusp: there would be a deformation to L_4^2. The tangent line of the cusp cannot lie in the plane of A_3, otherwise there is a deformation to L_3^1. The curve $T_8 = (2,3,\_) + (\_,\_,1) + (\_,2,1)$ is a deformation of T_9.

As A_2 deforms by deforming the parametrisation into a smooth branch with arbitrary tangent, the n smooth branches in a curve containing A_2 form a L_n^n. Let T be the intersection of the tangent plane of the A_2 with the space spanned by the L_n^n. If the curve is indecomposable, then T is a line. If T is transverse to the cusp, then we get the same type of normal form as for higher A_k. But T can also be tangent to the cusp. For $n = 1$ we have E_7 and, by bending the line out of the plane, also $E_7(1)$. If there are more lines, and the cusp is tangent to one of the lines of L^n_n, then we have $E_7 \vee L^{n-1}_{n-1}$, $E_7(1) \vee L^{n-1}_{n-1}$ and also curves obtained by bending the line out of the plane in the direction of L^{n-1}_{n-1}. If the cusp is not tangent to one of the lines, we take a normal form where T is a coordinate axis.

All curves considered here are deformations of $E_7 \vee L^{n-1}_{n-1}$. This curve is simple, as it occurs in the versal deformation of the parametrisation of the curve $E_8 \vee L^{n-1}_{n-1}$, which is simple by Proposition 5.4.

6. Plane curve singularities

In this section we show that in the case of plane curves a parametrisation is simple if and only if its image is a simple curve. This fact was already observed by Zhitomirsky [Zh] as result of the classification. Here we give a direct argument. It is based on the characterisation of simple plane curve singularities given by Barth, Peters and Van de Ven [BPV, Section II.8].

Theorem 6.1. *A plane curve singularity is simple if and only if its multiplicity is at most three and in each step of the embedded resolution the multiplicity of the (reduced) total transform is at most three.*

Proof. If there is a point on the total transform of multiplicity at least four, then by a deformation of the parametrisation of the curve we can achieve that it is an ordinary multiple point. Then the blown-down deformed curve has moduli, as a trivialising coordinate transformation downstairs would lift to one of the ordinary multiple point on the blow-up.

For the converse we use a formula of Wall for the modality (for right equivalence) in terms of the multiplicity sequence of plane curve singularities [Wa2, Theorem 8.1]:

$$\operatorname{Mod}(C) = \sum_P \tfrac{1}{2}(m_P - 1)(m_P - 2) - r - s + 2\ ,$$

where the sum runs over all infinitely near points in a large enough embedded resolution, r is the number of branches and s the total number of satellite points. If the multiplicity of the singularity is two, then $\operatorname{Mod}(C) = 0$: if $r = 1$ there is at least one satellite point. For multiplicity three the strict transform has no point of multiplicity three. If $r = 2$ there is again at least one satellite point. In the case of one branch, if the strict transform on the first blow-up is smooth, there are two satellite points. The remaining possible multiplicity sequence is $(3, 2, 1, 1, \dots)$ with two satellite points. So again $\operatorname{Mod}(C) = 0$. □

Corollary 6.2. *The parametrisation of a plane curve is simple if and only if the curve is simple.*

Proof. For plane curves any deformation of the parametrisation gives a deformation of the image, so simpleness of the image implies simpleness of the parametrisation. Conversely, if the curve is not simple, then by the above proof the adjacency to a singularity with moduli can be realised by a deformation of the parametrisation. □

We classify the possible multiplicity sequences. They are given in Table 1. As the singularities in question have no moduli for right equivalence, it suffices to find one parametrisation for each sequence. This can be done using an explicit description of the charts of the blow-up.

Using deformations on the blow up we can also easily establish that the confining singularities are $\widetilde{E}_7\colon x^4 + ax^2y^2 + y^4 = 0$ and $\widetilde{E}_8\colon x^3 + axy^4 + y^6 = 0$. For instance, if the strict transform has a point of multiplicity three lying on an exceptional curve, then we deform it into an ordinary triple point. Blowing down the exceptional curve gives a singularity of type $\widetilde{E}_7$, which we can move off the exceptional curve, resulting in a deformation of the original singularity into $\widetilde{E}_7$.

Table 1. Multiplicity sequences for simple plane curve

$$A_{2k-1}\colon \quad \boxed{\begin{matrix}1\\1\end{matrix}}\ \begin{matrix}\text{—}\cdots\text{—}\\\text{—}\cdots\text{—}\end{matrix}\ \boxed{\begin{matrix}1\\1\end{matrix}}\ \begin{matrix}\text{—}\,1\,\text{—}\cdots\\\text{—}\,1\,\text{—}\cdots\end{matrix}$$

$$A_{2k}\colon \quad 2\ \text{—}\cdots\text{—}\ 2\ \text{—}\,1\,\text{—}\cdots$$

$$D_{2k}\colon \quad \boxed{\begin{matrix}1\\1\\1\end{matrix}}\ \begin{matrix}\text{—}\\\text{—}\\\text{—}\end{matrix}\ \begin{array}{l}1\quad\text{—}\cdots\\ \boxed{\begin{matrix}1\\1\end{matrix}}\ \begin{matrix}\text{—}\cdots\text{—}\\\text{—}\cdots\text{—}\end{matrix}\ \boxed{\begin{matrix}1\\1\end{matrix}}\ \begin{matrix}\text{—}\,1\,\text{—}\cdots\\\text{—}\,1\,\text{—}\cdots\end{matrix}\end{array}$$

$$D_{2k+1}\colon \quad \boxed{\begin{matrix}1\\2\end{matrix}}\ \begin{matrix}\text{—}\\\text{—}\end{matrix}\ \begin{array}{l}1\quad\text{—}\cdots\\ 2\quad\text{—}\cdots\text{—}\quad 2\quad\text{—}\,1\,\text{—}\cdots\end{array}$$

$$E_6\colon \quad 3\ \text{—}\ 1\ \text{—}\,1\,\text{—}\cdots$$

$$E_7\colon \quad \boxed{\begin{matrix}1\\2\end{matrix}}\ \begin{matrix}\text{—}\\\text{—}\end{matrix}\ \boxed{\begin{matrix}1\\1\end{matrix}}\ \begin{matrix}\text{—}\,1\,\text{—}\cdots\\\text{—}\,1\,\text{—}\cdots\end{matrix}$$

$$E_8\colon \quad 3\ \text{—}\ 2\ \text{—}\,1\,\text{—}\cdots$$

7. Simple curves

For plane curves we showed without using the classification that the curves with simple parametrisation are exactly the simple curves for contact equivalence and even right equivalence of the defining equations.

Also for space curve singularities (in $\mathbb{C}^3$) both concepts of simpleness coincide, as a comparison of the lists of Giusti [Gi] and Frühbis-Krüger [F-K] with the space curves in our list shows; in fact, the comparison of the lists of simple curves with the list of Zhitomirskii [Zh] exposes some inaccuracies there, like the inclusion of the confining singularity $T^*_{10}\colon xy = x^3 + y^6 + z^2 = 0$ [AGV, I §9.8], which deforms into $\widetilde{E}_8$; it is $((I,I)^2, A_2)$ in [Zh, Table 4]. In Table 2 we list the indecomposable simple curves together with their names in the classifications by Giusti [Gi] and Frühbis-Krüger [F-K]. The equations are computed to agree with the parametrisations. The decomposable simple space curves are $A_k \vee L$, $D_k \vee L$ and $E_k \vee L$. The list of confining singularities for flat deformations of the curve is longer than for parametrisations, for complete intersections see [AGV, I §9.8] and for determinantal curves [F-K, Table 1].

The minimal δ-invariant for a confining singularity for parametrisations is $\delta = 5$. Therefore the list of all simple parametrisations contains all curves with $\delta \leq 4$. By the semi-continuity of δ we find the following corollaries of the classification.

Corollary 7.1. *Every curve singularity with $\delta \leq 4$ has a simple parametrisation and it is also simple as curve.*

Corollary 7.2. *A parametrisation of a curve singularity with $\delta = 5$ is simple if and only if the curve is simple.*

Proposition 7.3. *The sporadic curves with simple parametrisations are also simple as curve.*

Proof. A sporadic curve has $\delta \leq 5$. □

TABLE 2. Indecomposable simple space curves

type	parametrisation	equations
Z_{10}	$(4,6,7)$	$y^2-x^3,\ z^2-yx^2$
Z_9	$(2,3,\_)+(2,\_,3)$	$y^2-x^3,\ z^2-x^3$
W_9	$(3,4,5)+(1,\_,\_)$	$y^2-xz,\ z^2-yx^2$
W_8^*	$(4,5,7)$	$\begin{pmatrix} x & y & z \\ z & x^2 & y^2 \end{pmatrix}$
W_8	$(4,5,6)$	$y^2-xz,\ z^2-x^3$
U_9	$(3,5,7)+(\_,\_,1)$	$y^2-xz,\ yz-x^4$
U_8	$(2,3,\_)+(1,\_,2)+(\_,\_,1)$	$zy,\ y^2-x^3+zx$
U_7^*	$(3,4,5)+(\_,1,\_)$	$\begin{pmatrix} x & y & z \\ z & x^2 & xy \end{pmatrix}$
U_7	$(3,4,5)+(\_,\_,1)$	$y^2-xz,\ yz-x^3$
T_9	$(2,3,\_)+(\_,5,2)$	$xz,\ y^2-z^5-x^3$
T_8	$(2,3,\_)+(\_,\_,1)+(\_,2,1)$	$xz,\ y^2-yz^2-x^3$
T_7^*	$(2,3,\_)+(\_,2,3)$	$\begin{pmatrix} x & y & z \\ 0 & z & y^2-x^3 \end{pmatrix}$
T_7	$(2,3,\_)+(\_,3,2)$	$xz,\ y^2-z^3-x^3$
$E_{12}(2)$	$(3,7,8)$	$\begin{pmatrix} x^2 & y & z \\ y & z & x^3 \end{pmatrix}$
$J_{2,1}(2)$	$(2,5,\_)+(1,\_,2)$	$\begin{pmatrix} z & y & x^3 \\ 0 & x^2-z & y \end{pmatrix}$
$J_{2,0}(2)$	$(1,\_,\_)+(1,2,\_)+(1,\_,2)$	$\begin{pmatrix} z & y-x^2 & 0 \\ 0 & x^2-z & y \end{pmatrix}$
$E_8(1)$	$(3,5,7)$	$\begin{pmatrix} x & y & z \\ y & z & x^3 \end{pmatrix}$
$E_7(1)$	$(2,3,\_)+(1,\_,2)$	$\begin{pmatrix} z & x & y \\ 0 & y & x^2-z \end{pmatrix}$
$E_6(1)$	$(3,4,5)$	$\begin{pmatrix} x & y & z \\ y & z & x^2 \end{pmatrix}$
S_{2k+3}	$(1,\_,\_)+(1,k,\_)+(\_,\_,1)+(\_,1,1)$	$xz,\ y^2-yx^k-yz$
S_{2k+4}	$(2,2k+1,\_)+(\_,\_,1)+(\_,1,1)$	$xz,\ y^2-x^{2k+1}-yz$
S_6^*	$(2,3,\_)+(\_,\_,1)+(1,\_,1)$	$\begin{pmatrix} z & x & y \\ 0 & y & x^2-xz \end{pmatrix}$

This partly explains the coincidence of lists. The series of simple parametrisations are closely related to $A_k \vee L$, $D_k \vee L$ and $E_k \vee L$. In fact, this holds in any embedding dimension. We expect that our list gives the simple singularities.

Conjecture 7.4. *The simple reduced curve singularities are exactly those with simple parametrisation.*

This implies in particular a negative answer to the old unsolved problem whether rigid reduced curve singularities exist. The deformation theory of curve singularities of large codimension is complicated. There exist non-smoothable curves. They are not simple: the argument that they are not smoothable, is that the number of moduli is larger than the (computable) dimension of a smoothing component, cf. [Gr].

Proposition 7.5. *The curves L_n^n, $A_2 \vee L_k^k$, $A_3 \vee L_k^k$ and $L_{n+1}^n \vee L_k^k$ are simple.*

Proof. These are the curves with $\delta - r + 1 \leq 1$ [Gr], and $\delta - r + 1$ is upper semi-continuous [BuGr]. $\square$

Also the curves with $\delta - r + 1 = 2$ are classified, see [St2]. The ones with moduli are not Gorenstein, so the Gorenstein curves $A_2 \vee L_{n-2}^{n-2} + (\_, 1, 1 \ldots, 1)$ and $A_3 \vee L_{n-2}^{n-2} + (\_, 1, 1 \ldots, 1)$ are also simple.

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