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COMPLETE TRANSVERSALS OF SYMMETRIC VECTOR FIELDS

MIRIAM MANOEL AND IRIS DE OLIVEIRA ZELI

Abstract. We use group representation theory to obtain complete transversals of singularities of vector fields in nonsymmetric as well as reversible and equivariant contexts. The method is an algebraic alternative to compute complete transversals, producing normal forms to be applied systematically in the local analysis of symmetric dynamics.

1. Introduction

In singularity theory there are many results concerned with determining normal forms of map germs defined on different domains under different equivalence relations. Among a great number of papers in this direction, we cite for example the classical works by Bruce *et al.* [7], Gaffney and du Plessis [14], Gaffney [13] and Wall [23, 24]. On the classification of singularities applied to bifurcation theory we mention Golubitsky *et al.* [15, 16] and Melbourne [20, 21], these in the contexts with and without symmetries. In [8] the authors present the *complete transversal method*, an algebraic tool for the classification of finitely determined map germs. In [17] Kirk presents the programme *Transversal*, that implements this method.

In dynamical systems, normal forms of vector fields are obtained up to conjugacy and are extensively used in the study of local dynamics around a singularity. Some classical works are due to Poincaré [22], Birkhoff [6], Dulac [11], Belitskii [5] and Elphick *et al.* [12]. The method developed by Belitskii [5] consists of calculating the kernel of the homological operator associated with the adjoint L^t of the linearization L of the original vector field. This calculation in turn is associated with finding polynomial solutions of a PDE. Elphick *et al.* in [12] give an algebraic method for obtaining the normal form introducing an action of a group of symmetries $\mathbf{S}$, namely

$$\mathbf{S} = \overline{\{e^{sL^t}, s \in \mathbb{R}\}}, \tag{1}$$

so that the polynomial nonlinear terms are equivariant under this action. In [4] we treat formal normal forms of smooth vector fields in the simultaneous presence of symmetric and reversing symmetric transformations. The algebraic treatment shows advantage at once, since the set Γ formed by such transformations has a group structure. As a consequence, the vector field, called Γ-reversible-equivariant, has a well-determined general form that can be given explicitly in an algorithmic way (see [1] and [2]). Purely reversible systems have been studied for a long time, and in more recent years, reversible and equivariant systems have also become an object of great interest; for surveys see [10] and [18]. In particular, in [3] a relationship between purely equivariant systems (without reversing symmetries) and a class of reversible equivariant systems is established. The normal form of a Γ-reversible-equivariant system inherits the symmetries and reversing symmetries if the changes of coordinates are equivariant under the group Γ. Belitskii normal form has been used by many authors in different aspects; for example, in the analysis

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of occurrence of limit cycles or families of periodic orbits either in purely reversible vector fields or in reversible equivariant ones (see [19] and references therein). Motivated by these works, in [4] we have established an algebraic result related to those by Belitskii [5] and Elphick [12] in the reversible equivariant context using tools from invariant theory. In this process we have proved that the normal form comes from the description of the reversible equivariant theory of the semidirect product $\mathbf{S} \rtimes \Gamma$. After that recognition, we use results of [1, 2] to produce a formal normal form of a reversible equivariant vector field by means of an alternative algebraic method, without passing through a search for solutions of a PDE, which is the basis of Belitskii's method.

In the present work we put together the approaches from singularities and dynamical systems in the study of normal forms. We show how the complete transversal method is closely related to the normal form method developed in [4]. Let us stress that our intention here is not to apply the method for specific classifications. The goal is, instead, to explore this relation to recognize an algebraic alternative to compute complete transversals of singularities. Clearly the result is also valid without symmetries. The idea is to introduce Lie groups of changes of coordinates in both contexts. In the nonsymmetric case we recognize the complete transversal as being the space of polynomial map germs that commute with the group $\mathbf{S}$; in the reversible equivariant case, the space of polynomial map germs are reversible equivariant under the action of $\mathbf{S} \rtimes \Gamma$.

We have organized this paper as follows. In Section 2 we briefly present notation and collect basic concepts from reversible equivariant mappings and from normal form theory. In Section 3 we present the algebraic way to compute complete transversals. According to the action of the group of equivalences, we characterize the tangent space to the orbit of a map germ (Proposition 3.2), and recognize the complete transversal (Theorem 3.3). In Subection 3.2 we give the reversible equivariant versions, Proposition 3.5 and Theorem 3.4.

2. Preliminaries

Throughout we use the language of germs from singularity theory for the local study of C^∞ applications around a singularity, which we assume to be the origin.

2.1. **Reversible equivariant map germs.** Let Γ be a compact Lie group with a linear action on a finite-dimensional real vector space V: $\Gamma \times V \to V$, $(\gamma, x) \mapsto \gamma x$.

Consider a group homomorphism

$$\sigma : \Gamma \to \mathbf{Z}_2 = \{\pm 1\}, \tag{2}$$

defining elements of Γ as follows: if $\sigma(\gamma) = 1$ then γ is a symmetry, if $\sigma(\gamma) = -1$, then γ is a reversing symmetry. We denote by Γ_+ the subgroup of symmetries of Γ. If Γ_+ is nontrivial, then $\Gamma_+ = \ker \sigma$ is a proper normal subgroup of Γ of index 2.

We recall that to a linear action of Γ on V there corresponds a representation ρ of the group Γ on V. In other words, there is a linear group homomorphism $\rho : \Gamma \to \mathbf{GL}(V)$, $\rho(\gamma)x = \gamma x$, where $\mathbf{GL}(V)$ is the vector space of invertible linear maps $V \mapsto V$. The representation $\rho_\sigma : \Gamma \to \mathbf{GL}(V)$, $\rho_\sigma(\gamma) = \sigma(\gamma)\rho(\gamma)$ is called the dual of ρ.

Let us denote by $\mathcal{E}_V$ the ring of smooth function germs $f : V, 0 \to \mathbb{R}$, by $\vec{\mathcal{E}}_V$ the module of smooth map germs $g : V, 0 \to V$ and by $\vec{\mathcal{P}}_V$ the submodule of $\vec{\mathcal{E}}_V$ of polynomial map germs. A germ $f \in \mathcal{E}_V$ is called $\Gamma-$invariant if

$$f(\rho(\gamma)x) = f(x), \ \forall \gamma \in \Gamma, \ x \in V, 0. \tag{3}$$

We denote by $\mathcal{P}_V(\Gamma)$ the ring of $\Gamma-$invariant polynomial function germs and by $\mathcal{E}_V(\Gamma)$ the ring of $\Gamma-$invariant smooth function germs.

A map germ $g \in \vec{\mathcal{E}}_V$ is called (purely) $\Gamma-$equivariant if

$$g(\rho(\gamma)x) = \rho(\gamma)g(x), \ \forall \gamma \in \Gamma, \ x \in V, 0. \tag{4}$$

We denote by $\vec{\mathcal{P}}_V(\Gamma)$ the module of $\Gamma-$equivariant polynomial map germs and by $\vec{\mathcal{E}}_V(\Gamma)$ the module of $\Gamma-$equivariant smooth map germs.

A smooth map germ $g : V, 0 \to V$ is called Γ-reversible-equivariant if

$$g(\rho(\gamma)x) = \rho_\sigma(\gamma)g(x), \ \forall \gamma \in \Gamma, \ x \in V, 0. \tag{5}$$

We denote by $\vec{\mathcal{Q}}_V(\Gamma)$ the module of Γ-reversible-equivariant polynomial map germs and by $\vec{\mathcal{F}}_V(\Gamma)$ the module of Γ-reversible-equivariant smooth map germs.

Since Γ is compact, $\vec{\mathcal{P}}_V(\Gamma)$ and $\vec{\mathcal{Q}}_V(\Gamma)$ are finitely generated modules over $\mathcal{P}_V(\Gamma)$, which in turn is a finitely generated ring (see [16]). If σ is trivial, then $\vec{\mathcal{P}}_V(\Gamma)$ and $\vec{\mathcal{Q}}_V(\Gamma)$ coincide. In [1], the authors present an algorithm that produces a generating set of $\vec{\mathcal{Q}}_V(\Gamma)$ over $\mathcal{P}_V(\Gamma)$. A result in [2] provides a simple way to compute a set of generators of $\mathcal{P}_V(\Gamma)$ from the knowledge of generators of $\mathcal{P}_V(\Gamma_+)$.

Notice that $\vec{\mathcal{P}}_V(\Gamma)$ and $\vec{\mathcal{Q}}_V(\Gamma)$ are graded modules,

$$\vec{\mathcal{P}}_V(\Gamma) = \bigoplus_{k \geq 0} \vec{\mathcal{P}}^k_V(\Gamma) \quad \text{and} \quad \vec{\mathcal{Q}}_V(\Gamma) = \bigoplus_{k \geq 0} \vec{\mathcal{Q}}^k_V(\Gamma), \tag{6}$$

for $\vec{\mathcal{P}}^k_V(\Gamma) = \vec{\mathcal{P}}_V(\Gamma) \cap \vec{\mathcal{P}}^k_V$ and $\vec{\mathcal{Q}}^k_V(\Gamma) = \vec{\mathcal{Q}}_V(\Gamma) \cap \vec{\mathcal{P}}^k_V$, where $\vec{\mathcal{P}}^k_V$ is the subset of $\vec{\mathcal{P}}_V$ of homogeneous polynomial germs of degree k defined on V, $k \geq 0$.

2.2. **Belitskii-Elphick method.**

For $h \in \vec{\mathcal{E}}_V$, consider the ODE

$$\dot{x} = h(x), \ x \in V, 0. \tag{7}$$

The interest of the theory is local, around a singular point which we assume to be the origin, so $h(0) = 0$. The normal form method consists of successive changes of coordinates in the domain that are perturbations of the identity, $x = \xi(y) = y + \xi_k(y)$, for $\xi_k \in \vec{\mathcal{P}}^k_V$, $k \geq 2$. In the new variables, the system is

$$\dot{y} = g(y), \ y \in V, 0.$$

where

$$g(y) = (d\xi)^{-1}_x h(\xi(y)), \tag{8}$$

For each x we have

$$(d\xi)^{-1}_x = (I + (d\xi_k)_x)^{-1} = I - (d\xi_k)_x + \varphi\left((d\xi_k)_x\right), \ k \geq 2, \tag{9}$$

where $\varphi((d\xi_k)_x)$ contains no terms of degree strictly less than $2(k-1)$.

The aim is to annihilate as many terms of degree k as possible in the original vector field, obtaining a conjugate vector field written in a simpler and more convenient form. The method is based on the reduction of this problem to computing $\ker Ad^k_{L^t}$ where $Ad^k_L : \vec{\mathcal{P}}^k_V \to \vec{\mathcal{P}}^k_V$ is the homological operator defined by

$$Ad^k_L(p)(x) = (dp)_x Lx - Lp(x), \ x \in V, 0, \tag{10}$$

where L^t is the adjoint of the linearization L. We refer to [16] for the details.

In [12], Elphick *et al.* give an alternative algebraic method to obtain the normal form developed by Belitskii, which consists of computing nonlinear terms that are equivariant under the action of the group

$$\mathbf{S} = \overline{\{e^{sL^t}, s \in \mathbb{R}\}}. \tag{11}$$

The authors show that for each $k \geq 2$, $\ker Ad^k_{L^t} = \vec{\mathcal{P}}^k_V(\mathbf{S})$ and, since $Ad^k_{L^t} = (Ad^k_L)^t$, it follows that

$$\vec{\mathcal{P}}^k_V = \vec{\mathcal{P}}^k_V(\mathbf{S}) \oplus Ad^k_L(\vec{\mathcal{P}}^k_V). \tag{12}$$

From that, we show in [4, Theorem 4.1], that if the vector field h is Γ-reversible-equivariant, with $L = (dh)_0$, then for each $k \geq 2$ we have

$$\vec{\mathcal{Q}}^k_V(\Gamma) = \vec{\mathcal{Q}}^k_V(\mathbf{S} \rtimes \Gamma) \oplus Ad^k_L(\vec{\mathcal{P}}^k_V(\Gamma)), \tag{13}$$

where the semidirect product is induced from the homomorphism $\mu : \Gamma \to Aut(\mathbf{S})$ given by

$$\mu(\gamma)(e^{sL^t}) = e^{\sigma(\gamma)L^t}.$$

Hence, the normal form deduction reduces to the computation of a basis for the vector space $\vec{\mathcal{Q}}^k_V(\mathbf{S} \rtimes \Gamma)$ for each $k \geq 2$. In practice, via algorithmic methods we can obtain the general form of elements in $\vec{\mathcal{Q}}_V(\mathbf{S} \rtimes \Gamma)$ and, once this module is graded, we easily extract from this gradution a basis for $\vec{\mathcal{Q}}^k_V(\mathbf{S} \rtimes \Gamma)$. The main tools we use to obtain this general form are [1, Algorithm 3.7] and [2, Theorem 3.2] which hold in particular if the group is compact. There are many cases for which the group $\mathbf{S}$ fails to be compact; nevertheless, these tools can still be used as long as the ring $\mathcal{P}_V(\mathbf{S})$ and the module $\vec{\mathcal{P}}_V(\mathbf{S})$ are finitely generated.

3. The algebraic alternative for complete transversals

3.1. **Nonsymmetric case.** Let $\mathcal{G}$ be the group of formal changes of coordinates $\xi : V, 0 \to V$, $\xi = I + \tilde{\xi}$, where I is the germ of the identity and $\tilde{\xi} \in \bigoplus_{l \geq 2} \vec{\mathcal{P}}^l_V$. For $\mathcal{M}$ denoting the maximal ideal of $\mathcal{E}_V$, we consider the action of $\mathcal{G}$ on $\mathcal{M}\vec{\mathcal{E}}_V$ given as follows: for $\xi \in \mathcal{G}$ and $h \in \mathcal{M}\vec{\mathcal{E}}_V$,

$$(\xi \cdot h)(x) = (d\xi)^{-1}_{\xi(x)} h(\xi(x)), \quad x \in V, 0. \tag{14}$$

For each $k \geq 2$, consider now the vector space J^k formed by all $k-$jets $j^k h$ of elements $h \in \mathcal{M}\vec{\mathcal{E}}_V$. We introduce the group $J^k\mathcal{G} = \{j^k\xi,\ \xi \in \mathcal{G}\}$, which is a Lie group with an action on J^k induced by (14), namely

$$j^k\xi \cdot (j^k h)(x) = j^k(\xi \cdot h)(x),\ \xi \in \mathcal{G},\ h \in \mathcal{M}\vec{\mathcal{E}}_V.$$

For this action, we define the tangent space $T\mathcal{G} \cdot h$ to the orbit of h by the set of elements of the form

$$\frac{d}{dt}\phi(x,t)_{|t=0}, \tag{15}$$

for the one-parameter family $\phi(\cdot, t)$, where $\phi(x,t) = (d\xi)^{-1}_{(x,t)} h(\xi(x,t))$ and $\xi(x, 0) = x$.

The complete transversal method by Bruce *et al.* [8] is a tool for the classification of singularities that is performed on each degree level in the Taylor expansion of the germ to be studied. The main idea is to classify, at each step, $k-$jets on J^k, since J^k is isomorphic to a quotient of $\mathcal{E}_V-$modules $\mathcal{M}\vec{\mathcal{E}}_V / \mathcal{M}^{k+1}\vec{\mathcal{E}}_V$. The result is transcribed below:

Proposition 3.1. *([8, Proposition 2.2]) For $k \geq 1$, let h be a $k-$jet in the jet space J^k. If W is a vector subspace of $\vec{\mathcal{P}}^{k+1}_V$ such that*

$$\mathcal{M}^{k+1}\vec{\mathcal{E}}_V \subset W + T\mathcal{G} \cdot h + \mathcal{M}^{k+2}\vec{\mathcal{E}}_V, \tag{16}$$

then every $k+1-$jet g with $j^k g = h$ is in the same $J^{k+1}\mathcal{G}-$orbit as some $(k+1)-$jet of the form $h + \omega$, for some $\omega \in W$.

The vector subspace W is the so-called complete transversal. In principle, the computation of W requires the knowledge of $T\mathcal{G}\cdot h$ modulo $\mathcal{M}^{k+2}\vec{\mathcal{E}}_V$. Now, in an investigation of this result, we have noticed the presence of a linear operator resembling the homological operator given in (10). This has led us to obtain an alternative way to compute complete transversals through an algebraic approach. The rest of the present work is devoted to developing the approach.

We start with the linear operator $Ad_h : \vec{\mathcal{E}}_V \to \vec{\mathcal{E}}_V$,

$$Ad_h(\xi)(x) = (d\xi)_x h(x) - (dh)_x \xi(x), \tag{17}$$

and consider the restriction $Ad_h^k = Ad_h|_{\vec{\mathcal{P}}_V^k}$. Write $h = L + \tilde{h}$ with $L = (dh)_0$ and $\tilde{h} \in \mathcal{M}^2\vec{\mathcal{E}}_V$. By linearity it follows that

$$Ad_h^k(\xi_k) = Ad_L^k(\xi_k) + Ad_{\tilde{h}}^k(\xi_k),\ \xi_k \in \vec{\mathcal{P}}_V^k. \tag{18}$$

We can now characterize the tangent space $T\mathcal{G}\cdot h$:

Proposition 3.2. *The tangent space to the orbit of* $h \in \mathcal{M}\vec{\mathcal{E}}_V$ *is given by*

$$T\mathcal{G}\cdot h = \left\{ Ad_h(\tilde{\xi}) + \varphi(-(d\tilde{\xi})_x)h,\ \tilde{\xi} \in \bigoplus^{l\geq k} \vec{\mathcal{P}}_V^l,\ \varphi((d\tilde{\xi})_x) \text{ as in } (9),\ k \geq 2 \right\}.$$

Proof: Let $\xi(\cdot,t)$ be a family on $\mathcal{G}$, $\xi(x,t) = x + \tilde{\xi}(x,t)$, with $\xi(x,0) = x$, and let

$$\phi(x,t) = (d\xi)^{-1}_{\xi(x,t)} h\left(\xi(x,t)\right).$$

We have

$$\frac{d}{dt}\phi(x,0) = \left(-\frac{d}{dt}(d\tilde{\xi})_x + \varphi\left(\frac{d}{dt}(d\tilde{\xi})_x\right)\right) h(x) + (dh)_x \frac{d}{dt}\tilde{\xi}(x,0),$$

with φ given by

$$(d\xi)^{-1}_{\xi(x,t)} = I - (d\tilde{\xi})_{\xi(x,t)} + \varphi((d\tilde{\xi})_{(x,t)}). \tag{19}$$

Rewriting

$$\frac{d}{dt}(d\tilde{\xi})_x \equiv (d\tilde{\xi})_x\ , \quad \varphi\left(\frac{d}{dt}(d\tilde{\xi})_x\right) \equiv \varphi((d\tilde{\xi})_x) \quad \text{and} \quad \frac{d}{dt}\tilde{\xi}(x,0) \equiv \tilde{\xi}(x), \tag{20}$$

the result follows immediately. ▯

The theorem below is now a direct consequence of Proposition 3.2:

Theorem 3.3. *For* $k \geq 1$ *let* $h \in J^k$*. Consider the vector subspace* $\vec{\mathcal{P}}_V^{k+1}(\mathbf{S})$ *of* $\vec{\mathcal{P}}_V^{k+1}$*, with* $\mathbf{S}$ *defined in* (11) *associated with* $L = (dh)_0$*. Then,*

$$\mathcal{M}^{k+1}\vec{\mathcal{E}}_V \subset \vec{\mathcal{P}}_V^{k+1}(\mathbf{S}) + T\mathcal{G}\cdot h + \mathcal{M}^{k+2}\vec{\mathcal{E}}_V.$$

Proof: Let $g \in \mathcal{M}^{k+1}\vec{\mathcal{E}}_V$. From the decomposition (12), for each degree-k term g_{k+1} in the Taylor expansion of g we have

$$g_{k+1} = q_{k+1}\ +\ p_{k+1},$$

with $q_{k+1} \in \vec{\mathcal{P}}_V^{k+1}(\mathbf{S})$ and $p_{k+1} \in \operatorname{Im} Ad_L^{k+1}$. Then, $p_{k+1} = Ad_L^{k+1}(\xi_{k+1})$ for some $\xi_{k+1} \in \vec{\mathcal{P}}_V^{k+1}$. Consider $\varphi(-(d\xi_{k+1})_x)$ as in (9). We write $h = L + \tilde{h}$, with $L = (dh)_0$ and $\tilde{h} \in \mathcal{M}^2\vec{\mathcal{E}}_V$, to obtain

$$g_{k+1} = q_{k+1} + Ad_h(\xi_{k+1}) + \varphi(-(d\xi_{k+1})_x)h - \left(Ad_{\tilde{h}}(\xi_{k+1}) + \varphi(-(d\xi_{k+1})_x)h\right).$$

By Proposition 3.2, $Ad_h(\xi_{k+1}) + \varphi\left(-(d\xi_{k+1})_x\right)h \in T\mathcal{G}\cdot h$. Furthermore, from the definition of the linear operator and $\tilde{h}$ it follows that

$$Ad_{\tilde{h}}(\xi_{k+1}) + \varphi(-(d\xi_{k+1})_x)h \in \mathcal{M}^{k+2}\vec{\mathcal{E}}_V.$$

□

We remark that the choice of a vector subspace W satisfying (16) is obviously not unique; however, from the decomposition (12) it follows that $\vec{\mathcal{P}}_V^k(\mathbf{S})$ is among those with the smallest dimension.

3.2. **Reversible equivariant case.** Let Γ be a compact Lie group and consider the homomorphism σ defined in (2). We extend the results of the previous subsection to the Γ-reversible-equivariant context. In particular, if σ is trivial then the result reduces to the (purely) Γ-equivariant context.

Let us denote by $\tilde{\mathcal{G}}$ the subgroup of $\mathcal{G}$ of formal changes of coordinates $\xi : V, 0 \to V$, $\xi = I + \tilde{\xi}$, where $\tilde{\xi} \in \bigoplus_{l \geq 2} \vec{\mathcal{P}}_V^l(\Gamma)$, with its action on $\vec{\mathcal{F}}_V(\Gamma)$ defined as in (14).

Our space of germs is now $\vec{\mathcal{F}}_V(\Gamma)$. Let us denote by $J^k(\Gamma_\sigma)$ the space of Γ-reversible-equivariant $k-$jets and, for each $k \geq 1$, we denote by $\vec{\mathcal{F}}_{V\,k+1}(\Gamma)$ the space $\mathcal{M}^{k+1}\vec{\mathcal{E}}_V \cap \vec{\mathcal{F}}_V(\Gamma)$. Also, for each $k \geq 1$, let $J^k\tilde{\mathcal{G}}$ denote the group of $k-$jets $j^k\xi$ of elements $\xi \in \tilde{\mathcal{G}}$. Consider now the action of $J^k\tilde{\mathcal{G}}$ on $J^k(\Gamma_\sigma)$ induced by (14): for $\xi \in \tilde{\mathcal{G}},\ h \in \vec{\mathcal{F}}_V(\Gamma),\ h(0) = 0$,

$$j^k\xi \cdot \left(j^k h\right)(x) = j^k(\xi \cdot h)(x).$$

Castro and du Plessis have stated in [9] the equivariant version of Proposition 3.1. The reversible equivariant version adapts directly, just consider the group $\tilde{\mathcal{G}}$:

Theorem 3.4. *For $k \geq 1$ let h be a $k-$jet in the jet space $J^k(\Gamma_\sigma)$. If W is a vector subspace of $\vec{\mathcal{Q}}_V^{k+1}(\Gamma)$ such that*

$$\vec{\mathcal{F}}_{V\,k+1}(\Gamma) \subset W + T\tilde{\mathcal{G}} \cdot h + \vec{\mathcal{F}}_{V\,k+2}(\Gamma), \tag{21}$$

then every Γ-reversible-equivariant $k+1-$jet g with $j^k g = h$ is in the same $J^{k+1}\tilde{\mathcal{G}}-$orbit as some $(k+1)-$jet of the form $h + \omega$, for some $\omega \in W$.

As in the previous subsection, our aim here is to determine a subspace W satisfying (21). For that, we first characterize the tangent space $T\tilde{\mathcal{G}} \cdot h$ for $h \in \vec{\mathcal{F}}_V(\Gamma)$, $h(0) = 0$ through the linear operator defined in (17):

Proposition 3.5. *For $h \in \vec{\mathcal{F}}_V(\Gamma)$ with $h(0) = 0$, the tangent space to the orbit of h is given by*

$$T\tilde{\mathcal{G}} \cdot h = \left\{ Ad_h(\tilde{\xi}) + \varphi((d\tilde{\xi})_x)h,\ \tilde{\xi} \in \bigoplus^{l \geq k} \vec{\mathcal{P}}_V^l(\Gamma),\ \varphi((d\tilde{\xi})_x) \text{ as in (9)},\ k \geq 2 \right\}.$$

The proof of this proposition follows the steps of the proof of Proposition 3.2, accompanied with the Γ-equivariance.

The result below provides the complete transversal for the reversible equivariants:

Theorem 3.6. *For $k \geq 1$, let $h \in J^k(\Gamma_\sigma)$, $L = (dh)_0$. Consider the group* $\mathbf{S}$ *given in* (11) *associated with L. Then,*

$$\vec{\mathcal{F}}_{V\,k+1}(\Gamma) \subset \vec{\mathcal{Q}}_V^{k+1}(\mathbf{S} \rtimes \Gamma) + T\tilde{\mathcal{G}} \cdot h + \vec{\mathcal{F}}_{V\,k+2}(\Gamma).$$

Proof: Use the decomposition (13) and follow the steps of the proof of Theorem 3.3. □

As in the context without nontrivial symmetries, $\vec{\mathcal{Q}}_V^k(\mathbf{S} \rtimes \Gamma)$ is a complete transversal of smallest dimension that satisfies (21).

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Miriam Manoel, corresponding author. University of São Paulo, Mathematics Department, P.O. Box 668 13560-970, São Carlos - SP, Brazil

E-mail address: miriam@icmc.usp.br

Iris de Oliveira Zeli, University of Campinas, IMECC - Mathematics Department, P. O. Box 6065 13083-859, Campinas - SP, Brazil

E-mail address: irisfalkoliv@ime.unicamp.br