Journal *of* Singularities
Volume 12 (2015), 27-52

Proc. of Singularities in Geometry and Appl. III, Edinburgh, Scotland, 2013
DOI: 10.5427/jsing.2015.12c

GEOMETRY OF D_4 CONFORMAL TRIALITY AND SINGULARITIES OF TANGENT SURFACES

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Abstract. It is well known that projective duality can be understood in the context of geometry of A_n-type. In this paper, as D_4-geometry, we construct explicitly a flag manifold, its triple-fibration and differential systems which have D_4-symmetry and conformal triality. Then we give the generic classification for singularities of the tangent surfaces to associated integral curves, which exhibits the triality. The classification is performed in terms of the classical theory on root systems combined with the singularity theory of mappings. The relations of D_4-geometry with G_2-geometry and B_3-geometry are mentioned. The motivation of the tangent surface construction in D_4-geometry is provided.

1. Introduction

The projective structure and the conformal structure are the most important ones among various kinds of geometric structures. For the projective structures, we do have an important notion, the projective duality. Then we can ask the existence of any counterpart to the projective duality for the conformal structures. Let us try to find it from the view point of Dynkin diagrams. The projective duality can be understood in the context of geometry of A_n-type. In fact, Dynkin diagrams of A_n-type, which lay under the projective structures, enjoy the obvious $\mathbb{Z}_2$-symmetry. It induces the projective duality after all. On the other hand, the base of the conformal structures is provided by diagrams of type B_n and D_n. We observe that only the diagram of type D_4 possesses $\mathfrak{S}_3$-symmetry. In fact, among all simple Lie algebras, only D_4 has $\mathfrak{S}_3$ as the outer automorphism group.

The triality was first discussed by Cartan ([7], see also [19]). Then algebraic triality was studied via octonions by Chevelley, Freudenthal, Springer, Jacobson and so on ([21]). The real geometric triality was studied first by Study [22]. Porteous, in [20], gave a modern exposition on geometric triality. Note that in [20], the null Grassmannians in B_n- and D_n-geometry are called "quadric Grassmannians" and the D_4 triality is called "quadric triality". For relations to representation theory of $SO(4,4)$ and to mathematical physics, also see [10][18].

The triality has close relations with singularity theory, in particular, theory of simple singularities (see [3]). The D_4-singularities of function-germs, wavefronts, caustics, etc. have the natural $\mathfrak{S}_3$-symmetry and also the relations of D_4-singularities and G_2-singularities are found([2][9][19]).

In general, for each complex semi-simple Lie algebra, to construct geometric homogeneous models in terms of Borel subalgebras and parabolic subalgebras is known, for instance, in the classical Tits geometry ([23][24][1]). However it is another non-trivial problem to construct the explicit real model from an appropriate real form of the complex Lie algebra, with the detailed analysis on associated canonical geometric structures. Moreover singularities naturally arising from the geometric model provide new problems. We do treat in this paper both the realization problem of geometric models and the classification problem of singularities for D_4.

2010 *Mathematics Subject Classification.* Primary 58K40; Secondary 57R45, 53A20.
This work was supported by KAKENHI No. 22340030, No. 22540109, and No. 23740041.

We would like to call a "conformal triality" any phenomenon which arises from this $\mathfrak{S}_3$-symmetry of D_4. In this paper, we construct an explicit diagram of fibrations, which is called a *tree of fibrations*, or a *cascade of fibrations* or a *quiver of fibrations*, and associated geometric structures on it with D_4-symmetry. Moreover we show, as one of conformal trialities, the classification of singularities of surfaces arising from conformal geometry on the explicit tree of fibrations arising from the D_4-diagram. The appearance of singularities often depends on geometric structure behind. Thus the geometric triality becomes visible via the triality on the data of singularities.

We provide, as the real geometric model for D_4-diagram, the tree of fibrations on null flag manifolds on the 8-space with $(4,4)$-metric in §2. In §3, we recall the structure of $\mathfrak{so}(4,4) = \mathfrak{o}(4,4)$, the Lie algebra of the orthogonal group $O(4,4)$ on $\mathbf{R}^{4,4}$, as a basic structure of our constructions, and then we describe the canonical geometric structures. In §4, we give the statement of the main classification result (Theorem 4.3). We describe explicitly the tree of fibrations of D_4 in §5, and the canonical differential system on null flags in §6, where Theorem 4.3 is proved. In §7, we provide one of motivations for the tangent surface construction in D_4-geometry, introducing the notion of "null frontals", and a relation to "bi-Monge-Ampère equations".

The authors thank to the referees for valuable comments to improve the paper.

2. Null flag manifolds associated to D_4-diagram

Let $V = \mathbf{R}^{4,4}$ and $(\cdot \,|\, \cdot)$ be the inner product of signature $(4,4)$. A linear subspace $W \subset V$ is called *null* if $(u|v) = 0$ for any $u, v \in W$. We set

$$Q_0 := \{V_1 \mid V_1 \subset V, \ \dim(V_1) = 1, V_1 \text{ is null}\}.$$

Then Q_0 is a 6-dimensional quadric in the projective space $P^7 = P(V) = G_1(V)$. The set of 2-dimensional null subspaces,

$$M := \{V_2 \mid V_2 \subset V, \ \dim(V_2) = 2, V_2 \text{ is null}\},$$

is a 9-dimensional submanifold of the Grassmannian $G_2(V)$. The set of 3-dimensional null subspaces,

$$R := \{V_3 \mid V_3 \subset V, \ \dim(V_3) = 3, V_3 \text{ is null}\},$$

is a 9-dimensional submanifold of the Grassmannian $G_3(V)$.

The totality of maximal null subspaces, namely, 4-dimensional null subspaces, form disjoint two families $Q_+ = \{V_4^+\}$ and $Q_- = \{V_4^-\}$, which are both 6-dimensional submanifolds of the Grassmannian $G_4(V)$.

Remark 2.1. We have diffeomorphisms $Q_0 \cong Q_+ \cong Q_- \cong \mathrm{SO}(4) \cong S^3 \times_{\mathbb{Z}_2} S^3$, where $S^3 \times_{\mathbb{Z}_2} S^3$ means the quotient by the diagonal action of the $\mathbb{Z}_2$-action on S^3 by the antipodal map (see [20][18]).

For any $V_4^+ \in Q_+$ and $V_4^- \in Q_-$ from the two families, we have that

$$\dim(V_4^+ \cap V_4^-) = 1 \text{ or } 3.$$

We call V_4^+ and V_4^- *incident* if $\dim(V_4^+ \cap V_4^-) = 3$. For $W, W' \in Q_+$ (resp. $W, W' \in Q_-$) from one family, we have $\dim(W \cap W') = 0, 2$ or 4. For any $V_3 \in R$, there exists unique incident pair $V_4^+ \in Q_+, \ V_4^- \in Q_-$ with $V_3 = V_4^+ \cap V_4^-$. For null subspaces $V_i, V_j \subset V$ of dimensions i, j respectively with $i < j$, we call them *incident* if $V_i \subset V_j$.

Now we consider flags of mutually incident null subspaces in $\mathbf{R}^{4,4}$. We define the 11-dimensional flag manifold

$$\begin{aligned} N &:= \{(V_1, V_4^+, V_4^-) \in Q_0 \times Q_+ \times Q_- \mid V_1 \subset V_4^+ \cap V_4^-,\ \dim(V_4^+ \cap V_4^-) = 3.\} \\ &= \{(V_1, V_4^+, V_4^-) \in Q_0 \times Q_+ \times Q_- \mid V_1, V_4^+, V_4^- \text{ are mutually incident.}\}, \end{aligned}$$

which is diffeomorphic to

$$N' := \{(V_1, V_3) \in Q_0 \times R \mid V_1 \subset V_3\}.$$

In fact the map $\Phi : N \to N'$ defined by $\Phi(V_1, V_4^+, V_4^-) = (V_1, V_4^+ \cap V_4^-)$ is a diffeomorphism.

Moreover we define the 12-dimensional complete flag manifold

$$\begin{aligned} Z := \{(V_1, V_2, V_4^+, V_4^-) \in Q_0 \times M \times Q_+ \times Q_- \mid V_1 \subset V_2 \subset V_4^+ \cap V_4^-, \\ \dim(V_4^+ \cap V_4^-) = 3\}, \end{aligned}$$

which is diffeomorphic to

$$Z' := \{(V_1, V_2, V_3) \in Q_0 \times M \times R \mid V_1 \subset V_2 \subset V_3\},$$

by the diffeomorphism $(V_1, V_2, V_4^+, V_4^-) \mapsto (V_1, V_2, V_4^+ \cap V_4^-)$.

Thus we get the tree of fibrations for the D_4-diagram:

$$\begin{array}{ccccccc}
P^1 & & \longrightarrow & Z^{12}(\subset N \times M) & & \longleftarrow & P^1 \times P^1 \times P^1 \\
 & & \pi_N \swarrow & & & \searrow \pi_M & \\
 & & N^{11} & & & & \\
 & & & & & M^9 & \\
 & \pi'_0 \swarrow & \pi'_+ \downarrow & \pi'_- \searrow & & & \\
Q_0^6 & & Q_+^6 & & Q_-^6 & &
\end{array}$$

where $\pi_N, \pi_M, \pi'_0, \pi'_+$ and π'_- are natural projections.

Let $O(4,4)$ be the orthogonal group of $V = \mathbf{R}^{4,4}$, and $\mathfrak{g} = \mathfrak{o}(4,4)$ its Lie algebra. Note that $O(4,4)$ has 4 connected component. Let $O(4,4)_e$ be the identity component of $O(4,4)$, and G the universal covering of $O(4,4)_e$. Then G is a simply connected Lie group having $\mathfrak{g}$ as its Lie algebra. Here we consider the Lie group G in order to realize the triality not only in the level of Lie algebras but also in the level of Lie groups ([18]).

In the above diagram, each flag manifold is in fact G-homogeneous, as well as $O(4,4)$-homogeneous, and each projection is G-equivariant.

The lower left diagram indicates the conformal triality.

3. Gradations to $\mathfrak{o}(4,4)$ and geometric structures on null flag manifolds

We recall the structure of $\mathfrak{g} = \mathfrak{o}(4,4)$, the Lie algebra of the orthogonal group $O(4,4)$ on $\mathbf{R}^{4,4}$, that is the split real form of $\mathfrak{o}(8,\mathbf{C})$. See [11][6][25] for details and for other simple Lie algebras.

With respect to a basis $e_1, \dots, e_8$ of $\mathbf{R}^{4,4}$ with inner products

$$(e_i | e_{9-j}) = \frac{1}{2}\delta_{ij}, 1 \leq i, j \leq 8,$$

we have

$$\begin{aligned} \mathfrak{o}(4,4) &= \{A \in \mathfrak{gl}(8,\mathbf{R}) \mid {}^tAK + KA = O\}, \\ &= \{A = (a_{ij}) \in \mathfrak{gl}(8,\mathbf{R}) \mid a_{9-j,9-i} = -a_{ij}, 1 \leq i, j \leq 8\}, \end{aligned}$$

where $K = (k_{ij})$ is the 8×8-matrix defined by $k_{i,9-j} = \frac{1}{2}\delta_{ij}$. Let E_{ij} denote the 8×8-matrix whose (k,ℓ)-component is defined by $\delta_{ik}\delta_{j\ell}$. Then

$$\mathfrak{h} := \mathfrak{g}_0 = \langle E_{ii} - E_{9-i,9-i} \mid 1 \leq i \leq 4 \rangle_{\mathbf{R}}$$

is a Cartan subalgebra of $\mathfrak{g}$. Let $(\varepsilon_i \mid 1 \leq i \leq 4)$ denote the dual basis of $\mathfrak{h}^*$ to the basis $(E_{ii} - E_{9-i,9-i} \mid 1 \leq i \leq 4)$ of $\mathfrak{h}$. Then the root system is given by $\pm\varepsilon_i \pm \varepsilon_j, 1 \leq i < j \leq 4$, and $\mathfrak{g}$ is decomposed, over $\mathbf{R}$, into the direct sum of root spaces

$$\mathfrak{g}_{\varepsilon_i-\varepsilon_j} = \langle E_{i,j} - E_{9-j,9-i}\rangle_{\mathbf{R}},\ \mathfrak{g}_{\varepsilon_i+\varepsilon_j} = \langle E_{i,9-j} - E_{j,9-i}\rangle_{\mathbf{R}},$$
$$\mathfrak{g}_{-\varepsilon_i+\varepsilon_j} = \langle E_{j,i} - E_{9-i,9-j}\rangle_{\mathbf{R}},\ \mathfrak{g}_{-\varepsilon_i-\varepsilon_j} = \langle E_{9-j,i} - E_{9-i,j}\rangle_{\mathbf{R}},$$

$(1 \leq i < j \leq 4)$.

The simple roots are given by

$$\alpha_1 := \varepsilon_1 - \varepsilon_2, \quad \alpha_2 := \varepsilon_2 - \varepsilon_3, \quad \alpha_3 := \varepsilon_3 - \varepsilon_4, \quad \alpha_4 := \varepsilon_3 + \varepsilon_4.$$

(The numbering of simple roots is the same as in [5] and is slightly different from [18].)

By labeling the root just on the left-upper-half part, we illustrate the structure of $\mathfrak{g}$:

ε_1	α_1	$\alpha_1+\alpha_2$	$\alpha_1+\alpha_2$ $+\alpha_3$	$\alpha_1+\alpha_2$ $+\alpha_4$	$\alpha_1+\alpha_2$ $+\alpha_3+\alpha_4$	$\alpha_1+2\alpha_2$ $+\alpha_3+\alpha_4$	0
$-\alpha_1$	ε_2	α_2	$\alpha_2+\alpha_3$	$\alpha_2+\alpha_4$	$\alpha_2+\alpha_3$ $+\alpha_4$	0	
$-\alpha_1-\alpha_2$	$-\alpha_2$	ε_3	α_3	α_4	0		
$-\alpha_1-\alpha_2$ $-\alpha_3$	$-\alpha_2-\alpha_3$	$-\alpha_3$	ε_4	0			
$-\alpha_1-\alpha_2$ $-\alpha_4$	$-\alpha_2-\alpha_4$	$-\alpha_4$	0	$-\varepsilon_4$			
$-\alpha_1-\alpha_2$ $-\alpha_3-\alpha_4$	$-\alpha_2-\alpha_3$ $-\alpha_4$	0			$-\varepsilon_3$		
$-\alpha_1-2\alpha_2$ $-\alpha_3-\alpha_4$	0					$-\varepsilon_2$	
0							$-\varepsilon_1$

The Borel subalgebra is given by $\mathfrak{g}_{\geq 0} = \mathfrak{g}_0 \oplus \sum_{\alpha>0} \mathfrak{g}_\alpha$, the sum of Cartan subalgebra $\mathfrak{h} = \mathfrak{g}_0$ and positive root spaces $\mathfrak{g}_\alpha$ with respect to the simple root system $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$.

We take parabolic subalgebras $\mathfrak{g}^1, \mathfrak{g}^2, \mathfrak{g}^3, \mathfrak{g}^4$, where $\mathfrak{g}^i$ is the sum of $\mathfrak{g}_{\geq 0}$ and all $\mathfrak{g}_\alpha$ for a negative root α without α_i-term. For instance,

$$\mathfrak{g}^1 = \langle E_{ij} - E_{9-j,9-i} \mid 2 \leq j \leq 7, 1 \leq i \leq 8-j\ \rangle_{\mathbf{R}} + \langle E_{11} - E_{88}\rangle_{\mathbf{R}}.$$

Moreover we have a parabolic subalgebra

$$\mathfrak{g}^{134} := \mathfrak{g}^1 \cap \mathfrak{g}^3 \cap \mathfrak{g}^4 = \mathfrak{g}_{\geq 0} \oplus \mathfrak{g}_{-\alpha_2}.$$

Let $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ denote the adjoint representation, B (resp. G^i) the normalizer in G under Ad of the subalgebra $\mathfrak{g}_{\geq 0}$ (resp. the subalgebras $\mathfrak{g}^i$, $i = 1, 2, 3, 4$). Then B (resp. G^i) has $\mathfrak{g}_{\geq 0}$ (resp. $\mathfrak{g}^i$) as its Lie algebra. The subgroup

$$G^{134} := G^1 \cap G^3 \cap G^4$$

has $\mathfrak{g}^{134}$ as its Lie algebra. Then the flag manifolds Z, Q_0, M, Q_+, Q_- and N are G-homogeneous spaces with isotropy groups B, G^1, G^2, G^3, G^4 and G^{134} respectively. We have

$$Z = G/B,\ Q_0 = G/G^1,\ M = G/G^2,\ Q_+ = G/G^3,\ Q_- = G/G^4,\ N = G/G^{134}.$$

Define the linear isomorphisms $\sigma, \tau : \mathfrak{h}^* \to \mathfrak{h}^*$ on the dual space

$$\mathfrak{h}^* = \langle \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\rangle_{\mathbf{R}} = \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4\rangle_{\mathbf{R}}$$

of the Cartan subalgebra $\mathfrak{h}$ by

$$\sigma(\alpha_1) = \alpha_3, \sigma(\alpha_2) = \alpha_2, \sigma(\alpha_3) = \alpha_4, \sigma(\alpha_4) = \alpha_1,$$

and

$$\tau(\alpha_1) = \alpha_1, \tau(\alpha_2) = \alpha_2, \tau(\alpha_3) = \alpha_4, \tau(\alpha_4) = \alpha_3,$$

which induce Lie algebra isomorphisms $\sigma, \tau : \mathfrak{g} \to \mathfrak{g}$, expressed by the same letters, satisfying

$$\sigma(\mathfrak{g}_{\pm\alpha_1}) = \mathfrak{g}_{\pm\alpha_3}, \sigma(\mathfrak{g}_{\pm\alpha_2}) = \mathfrak{g}_{\pm\alpha_2}, \sigma(\mathfrak{g}_{\pm\alpha_3}) = \mathfrak{g}_{\pm\alpha_4}, \sigma(\mathfrak{g}_{\pm\alpha_4}) = \mathfrak{g}_{\pm\alpha_1},$$

and

$$\tau(\mathfrak{g}_{\pm\alpha_1}) = \mathfrak{g}_{\pm\alpha_1}, \tau(\mathfrak{g}_{\pm\alpha_2}) = \mathfrak{g}_{\pm\alpha_2}, \tau(\mathfrak{g}_{\pm\alpha_3}) = \mathfrak{g}_{\pm\alpha_4}, \tau(\mathfrak{g}_{\pm\alpha_4}) = \mathfrak{g}_{\pm\alpha_3}.$$

(See the related references [18] §1.8, [19] §7.1. For the general theory, see [11] Ch.III, Theorem 5.4.)

The isomorphisms σ, τ are of order $3, 2$ respectively. Thus $\mathfrak{g}$ has $\mathfrak{S}_3$-symmetry. Since G, the universal covering of $O(4,4)_e$, is simply connected, the $\mathfrak{S}_3$-symmetry on $\mathfrak{g}$ lifts to the $\mathfrak{S}_3$-symmetry of G. In particular the associated isomorphism $\sigma : G \to G$ satisfies

$$\sigma(B) = B, \sigma(G^1) = G^3, \sigma(G^2) = G^2, \sigma(G^3) = G^4, \sigma(G^4) = G^1, \sigma(G^{134}) = G^{134}.$$

Thus, in particular, we have induced diffeomorphisms $Q_0 \cong Q_+ \cong Q_-$.

The null quadric $Q_0 \subset P(V) = P(\mathbf{R}^{4,4})$ has the canonical conformal structure of type $(3,3)$. In fact, for each $V_1 \in Q_0$, consider $V_1^\perp \subset V = \mathbf{R}^{4,4}$. Then the tangent space $T_{V_1}Q_0$ is isomorphic to $V_1^\perp/V_1$, up to similarity transformation. Therefore the metric on V induces the canonical conformal structure on Q_0 of signature $(3,3)$. In other words, the conformal structure on Q_0 is defined by the quadric tangent cone C_x of the *Schubert variety*

$$S_x := \{W_1 \in Q_0 \mid W_1 \subset V_1^\perp\} = P(V_1^\perp) \cap Q_0 \subset Q_0,$$

for each $x = V_1 \in Q_0$. Note that $S_x = \pi_0\pi_M^{-1}\pi_M\pi_0^{-1}(x)$, in terms of the tree of fibrations.

Also Q_+ (resp. Q_-) has a conformal structure of type $(3,3)$. In fact, for each $y = V_4^\pm \in Q_\pm$, the *Schubert variety*

$$S_y^\pm := \{W_4 \in Q_\pm \mid W_4 \cap V_4^\pm \neq \{0\}\} \subset Q_\pm$$

induces invariant quadratic cone field (conformal structure) $C_y^\pm$ on $Q_\pm$ defined by the Pfaffian, respectively. Note that $S_y^\pm = \pi_\pm\pi_M^{-1}\pi_M\pi_\pm^{-1}(y)$. The triality

$$Q_0 \cong Q_+ \cong Q_-$$

preserves the conformal structures.

Now we turn to construct the invariant differential systems on null flag manifolds.

Let

$$\mathfrak{g}_{-1} := \mathfrak{g}_{-\alpha_1} \oplus \mathfrak{g}_{-\alpha_2} \oplus \mathfrak{g}_{-\alpha_3} \oplus \mathfrak{g}_{-\alpha_4}.$$

The subspace

$$\mathfrak{g}_{\geq -1} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{\geq 0} = \mathfrak{g}^{134} + \mathfrak{g}^2$$

in $\mathfrak{g}$ satisfies $\mathrm{Ad}(G)(\mathfrak{g}_{\geq -1}) = \mathfrak{g}_{\geq -1}$ and defines a left invariant distribution $\widetilde{E}$ on G, which induces the standard differential system $E \subset TZ$ with rank 4 and with growth $(4, 7, 10, 11, 12)$ (see [25]). In fact we can read the growth from the above table. We call E the D_4 *Engel distribution* on Z.

Remark 3.1. We would like to call the distribution E "Engel", simply because it lives on the top place (heaven) of our real spaces, referring the contributions of the mathematician Friedrich Engel on the theory of Lie algebras.

The flag manifold M^9 has the canonical contact structure D_M with growth $(8,9)$. In fact we define the subspace

$$\begin{aligned}\mathfrak{d}_M &:= (\mathfrak{g}_{-\varepsilon_1+\varepsilon_3} \oplus \mathfrak{g}_{-\varepsilon_2+\varepsilon_3} \oplus \mathfrak{g}_{-\varepsilon_1+\varepsilon_4} \oplus \mathfrak{g}_{-\varepsilon_2+\varepsilon_4} \\ &\qquad \oplus \mathfrak{g}_{-\varepsilon_1-\varepsilon_4} \oplus \mathfrak{g}_{-\varepsilon_2-\varepsilon_4} \oplus \mathfrak{g}_{-\varepsilon_1-\varepsilon_3} \oplus \mathfrak{g}_{-\varepsilon_2-\varepsilon_3}) \oplus \mathfrak{g}^2 \\ &= (\mathfrak{g}_{-\alpha_1-\alpha_2} \oplus \mathfrak{g}_{-\alpha_2} \oplus \mathfrak{g}_{-\alpha_1-\alpha_2-\alpha_3} \oplus \mathfrak{g}_{-\alpha_2-\alpha_3} \\ &\qquad \oplus \mathfrak{g}_{-\alpha_1-\alpha_2-\alpha_4} \oplus \mathfrak{g}_{-\alpha_2-\alpha_4} \oplus \mathfrak{g}_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4} \oplus \mathfrak{g}_{-\alpha_2-\alpha_3-\alpha_4}) \oplus \mathfrak{g}^2\end{aligned}$$

in $\mathfrak{g}$. Then we have that $\mathrm{Ad}(G^2)\mathfrak{d}_M = \mathfrak{d}_M$ and therefore $\mathfrak{d}_M$ defines the invariant distribution $D_M \subset TM = T(G/G^2)$ with rank 8, which is a contact structure. We call D_M the D_4 *contact structure* on M.

The contact structure D_M carries a structure of $2 \times 2 \times 2$-hyper-matrices and it possesses a Lagrange cone field defined by a decomposable cubic. In fact, define the subalgebra $\mathfrak{g}^0_M$ of $\mathfrak{g}$ by

$$\mathfrak{g}^0_M := \mathfrak{g}_0 \oplus \mathfrak{g}_{\pm\alpha_1} \oplus \mathfrak{g}_{\pm\alpha_3} \oplus \mathfrak{g}_{\pm\alpha_4}.$$

Then $\mathfrak{g}^0_M$ is isomorphic to $\mathfrak{sl}(2,\mathbf{R}) \oplus \mathfrak{sl}(2,\mathbf{R}) \oplus \mathfrak{sl}(2,\mathbf{R}) \oplus \mathbf{R}$ and it acts on $\mathfrak{d}_M$. Thus the group $SL(2,\mathbf{R}) \times SL(2,\mathbf{R}) \times SL(2,\mathbf{R}) \times \mathbf{R}^\times$ acts on the contact structure D_M. We set

$$\mathfrak{d}^1_M = \mathfrak{g}_{-\alpha_2} \oplus \mathfrak{g}_{-\alpha_1-\alpha_2} \oplus \mathfrak{g}^2,\ \mathfrak{d}^3_M = \mathfrak{g}_{-\alpha_2} \oplus \mathfrak{g}_{-\alpha_2-\alpha_3} \oplus \mathfrak{g}^2,\ \mathfrak{d}^4_M = \mathfrak{g}_{-\alpha_2} \oplus \mathfrak{g}_{-\alpha_2-\alpha_4} \oplus \mathfrak{g}^2.$$

Then they induce subbundles D^1_M, D^3_M, D^4_M of rank 2 on M respectively. Moreover an isomorphism

$$\mathfrak{d}_M/\mathfrak{g}^2 \cong (\mathfrak{d}^1_M/\mathfrak{g}^2) \otimes (\mathfrak{d}^3_M/\mathfrak{g}^2) \otimes (\mathfrak{d}^4_M/\mathfrak{g}^2)$$

between vector spaces of dimension 8 induces an isomorphism

$$D_M \cong D^1_M \otimes D^3_M \otimes D^4_M.$$

of vector bundles on M. This means that the distribution D_M has a structure of $2\times 2\times 2$-hyper-matrices. By the diagonal action of $SL(2,\mathbf{R})$ we have a Lagrange cone field in D_M, which we call the D_4 *Monge cone structure* on M.

The flag manifold N^{11} has a distribution D_N with growth $(6,9,11)$ with a direct sum decomposition into three subbundles of rank two. We define the subspace

$$\begin{aligned}\mathfrak{d}_N &:= (\mathfrak{g}_{-\varepsilon_1+\varepsilon_2} \oplus \mathfrak{g}_{-\varepsilon_1+\varepsilon_3}) \oplus (\mathfrak{g}_{-\varepsilon_2+\varepsilon_4} \oplus \mathfrak{g}_{-\varepsilon_3+\varepsilon_4}) \oplus (\mathfrak{g}_{-\varepsilon_2-\varepsilon_4} \oplus \mathfrak{g}_{-\varepsilon_3-\varepsilon_4}) \oplus \mathfrak{g}^{134} \\ &= (\mathfrak{g}_{-\alpha_1} \oplus \mathfrak{g}_{-\alpha_1-\alpha_2}) \oplus (\mathfrak{g}_{-\alpha_2-\alpha_3} \oplus \mathfrak{g}_{-\alpha_3}) \oplus (\mathfrak{g}_{-\alpha_2-\alpha_4} \oplus \mathfrak{g}_{-\alpha_4}) \oplus \mathfrak{g}^{134}\end{aligned}$$

of $\mathfrak{g}$. Then we have that $Ad(G^{134})\mathfrak{d}_N = \mathfrak{d}_N$, and therefore $\mathfrak{d}_N$ defines the invariant distribution $D_N \subset TN = T(G/G^{134})$ with rank 6. Define the subalgebra $\mathfrak{g}^0_N := \mathfrak{g}_0 \oplus \mathfrak{g}_{\pm\alpha_2}$ of $\mathfrak{g}$. Then $\mathfrak{g}^0_N$ is isomorphic to $\mathfrak{sl}(2,\mathbf{R}) \oplus \mathbf{R} \oplus \mathbf{R} \oplus \mathbf{R}$ and acts on $\mathfrak{d}_N$. We set

$$\mathfrak{d}^1_N = \mathfrak{g}_{-\alpha_1} \oplus \mathfrak{g}_{-\alpha_1-\alpha_2} \oplus \mathfrak{g}^{134},\ \mathfrak{d}^3_N = \mathfrak{g}_{-\alpha_2-\alpha_3} \oplus \mathfrak{g}_{-\alpha_3} \oplus \mathfrak{g}^{134},\ \mathfrak{d}^4_N = \mathfrak{g}_{-\alpha_2-\alpha_4} \oplus \mathfrak{g}_{-\alpha_4} \oplus \mathfrak{g}^{134}.$$

Then we have an invariant decomposition

$$D_N = D^1_N \oplus D^3_N \oplus D^4_N,$$

into subbundles D^1_N, D^3_N, D^4_N of rank 2. We call D_N the D_4 *Cartan distribution.*

Remark 3.2. The D_4 Engel distributions E on Z and the D_4 Cartan distribution D_N on N are related, via the projection $\pi_N : Z \to N$, as follows: The pull-back $(\pi_{N*})^{-1}(D_N)$ is equal to the square $E^2 := E + [E,E]$ of the distribution E, which is a distribution on Z of rank 7. The Cauchy characteristic of E^2 is equal to $\mathrm{Ker}(\pi_{N*} : TZ \to TN)$. Therefore the reduced space $Z/\mathrm{Ker}(\pi_{N*})$ is identified with N and the reduction of E^2 on N is identified with D_N. (See for instance, [25]).

Remark 3.3. We can compare the above mentioned facts with G_2-diagram: We consider the purely imaginary split octonions $\mathrm{Im}\mathbb{O}'$ with the inner product of type $(3,4)$ and consider the null projective space N^5 (resp. the null Grassmannian M^5, the flag manifold Z^6) which consists of 1-dimensional null subalgebras (resp. 2-dimensional null subalgebras, the incident pairs of 1-dimensional null subalgebras and 2-dimensional null subalgebras) for the multiplication on the split octonions $\mathbb{O}'$. The flag manifold Z has the Engel distribution with growth $(2,3,4,5,6)$, N^5 has a distribution with growth $(2,3,5)$, and the null projective space M^5 has a contact structure with growth $(4,5)$ with a cubic Lagrange cone field ([15])

4. D_4-triality and singularities of null tangent surfaces

We consider the canonical projections

$$\pi_0 = \pi_0' \circ \pi_N : Z \longrightarrow Q_0,\ \pi_+ = \pi_+' \circ \pi_N : Z \longrightarrow Q_+,\ \pi_- = \pi_-' \circ \pi_N : Z \longrightarrow Q_-,$$

and the diagram

$$\begin{array}{ccccc} & & Z^{12} & \xrightarrow{\pi_M} & M^9 \\ & \pi_0 \swarrow & \pi_+ \downarrow & \pi_- \searrow & \\ Q_0^6 & & Q_+^6 & & Q_-^6 \end{array}$$

induced by D_4 Dynkin diagram.

The D_4 *Engel distribution* E on Z is described from the tree of fibrations, by

$$E = (\ker \pi_{0*} \cap \ker \pi_{+*} \cap \ker \pi_{-*}) \oplus \ker \pi_{M*} \subset TZ,$$

which is of rank 4. We regard the definition of E as the *standard differential system* for $\mathfrak{o}(4,4)$ in §3.

A curve $f : I \to Z$ on Z is called *E-integral* if it is tangent to E, namely, if $f_*(TI) \subset E(\subset TZ)$.

Definition 4.1. For the given (indefinite) conformal structure $\{C_x\}_{x\in Q_0}$ on Q_0, we call a curve $\gamma : I \to Q_0$ a *null curve* if

$$\gamma'(t) \in C_{\gamma(t)}, (t \in I).$$

A geodesic on Q_0 is called a *null geodesic* if it is a null curve.

A surface $F : U \to Q_0$ is called a *null surface* if

$$F_*(T_u U) \subset C_{F(u)}, (u \in U).$$

The same definition is applied also to $Q_\pm$.

Proposition 4.2. (Guillemin-Sternberg [10]) *The null geodesics on Q_0 for the conformal structure on Q_0 are given by null lines, namely, projective lines on $Q_0 \subset P(V) = P(\mathbf{R}^{4,4})$.*

We will take null geodesics, namely, null lines as "tangent lines" for null curves in Q_0. Note that any null line in Q_0 is given by $\pi_0(\pi_M^{-1}(V_2))$ for some $V_2 \in M$. Then we are naturally led to consider tangent surfaces of null curves in Q_0, Q_+ and Q_-. For $Q_\pm$ we take, as the family of "lines" in $Q_\pm$,

$$\pi_\pm(\pi_M^{-1}(V_2)) = \{W_4 \in Q_\pm \mid V_2 \subset W_4\}, \quad V_2 \in M.$$

If we consider a special class of null curves which are projections of E-integral curves $f : I \to Z$ to Q_0, Q_+ or Q_-, then their tangent surfaces turn to be null surfaces in Q_0, Q_+ or Q_- in the above sense. In fact we show later more strict results (Proposition 7.4).

For M, we regard

$$\pi_M(\pi_0^{-1}(V_1) \cap \pi_+^{-1}(V_4^+) \cap \pi_-^{-1}(V_4^-)) = \{W_2 \mid V_1 \subset W_2 \subset V_4^+ \cap V_4^-\},\ (V_1, V_4^+, V_4^-) \in N,$$

as lines in M.

We will give the explicit classification of singularities of "tangent surfaces" in the viewpoint of geometry of D_4-triality:

Theorem 4.3. (*Triality of singularities.*) *For a generic E-integral curve $f : I \longrightarrow Z$, the singularities of tangent surfaces, to the curves $\gamma_0 = \pi_0 \circ f, \gamma_+ = \pi_+ \circ f, \gamma_- = \pi_- \circ f, \gamma_M = \pi_M \circ f$ on Q_0, Q_+, Q_-, M,*

$$\mathrm{Tan}(\gamma_0) = \pi_0 \pi_M^{-1} \pi_M f(I)(\subset Q_0),$$
$$\mathrm{Tan}(\gamma_+) = \pi_+ \pi_M^{-1} \pi_M f(I)(\subset Q_+), \quad \mathrm{Tan}(\gamma_-) = \pi_- \pi_M^{-1} \pi_M f(I)(\subset Q_-),$$
$$\mathrm{Tan}(\gamma_M) = \pi_M(\pi_0^{-1} \pi_0 f(I) \cap \pi_+^{-1} \pi_+ f(I) \cap \pi_-^{-1} \pi_- f(I))(\subset M),$$

at any point $t \in I$ is classified, up to local diffeomorphisms, as follows:

Tan(γ_0)	*Tan*(γ_+)	*Tan*(γ_-)	*Tan*(γ_M)
CE	*CE*	*CE*	*CE*
OSW	*CE*	*CE*	*CE*
CE	*OSW*	*CE*	*CE*
CE	*CE*	*OSW*	*CE*
OM	*OM*	*OM*	*OSW*

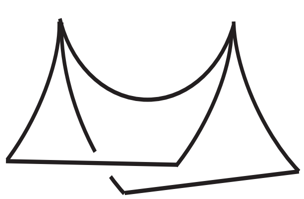
Here CE *(resp.* OSW*,* OM*) means the cuspidal edge (resp. open swallowtail, open Mond surface).*

The *cuspidal edge* (resp. *open swallowtail*, *open Mond surface*) is defined as a diffeomorphism class of the tangent surface-germ to a curve of type $(1, 2, 3, \cdots)$ (resp. $(2, 3, 4, 5, \cdots)$, $(1, 3, 4, 5, \cdots)$) in an affine space. The type of a curve is the strictly increasing sequence of orders (degrees of initial terms) of components in an appropriate system of linear coordinates. Their normal forms are given as follows:

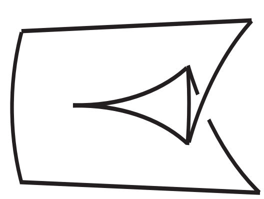
$$\begin{aligned} \mathrm{CE} \quad : \quad & (u,t) \mapsto (u, t^2 - 2ut, 2t^3 - 3ut^2, 0, 0, 0), (\mathbf{R}^2, 0) \to (\mathbf{R}^6, 0), \\ & (u,t) \mapsto (u, t^2 - 2ut, 2t^3 - 3ut^2, 0, 0, 0, 0, 0, 0), (\mathbf{R}^2, 0) \to (\mathbf{R}^9, 0), \end{aligned}$$

$$\begin{aligned} \mathrm{OSW} \quad : \quad & (u,t) \mapsto (u, t^3 - 3ut, t^4 - 2ut^2, 3t^5 - 5ut^3, 0, 0), (\mathbf{R}^2, 0) \to (\mathbf{R}^6, 0), \\ & (u,t) \mapsto (u, t^3 - 3ut, t^4 - 2ut^2, 3t^5 - 5ut^3, 0, 0, 0, 0, 0), (\mathbf{R}^2, 0) \to (\mathbf{R}^9, 0), \end{aligned}$$

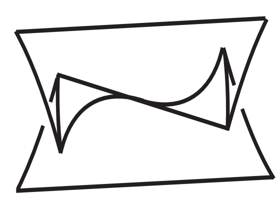
$$\mathrm{OM} \quad : \quad (u,t) \mapsto (u, 2t^3 - 3ut^2, 3t^4 - 4ut^3, 4t^5 - 5ut^4, 0, 0), (\mathbf{R}^2, 0) \to (\mathbf{R}^6, 0),$$



cuspidal edge



open swallowtail



open Mond surface

The classification is performed in terms of the classical theory on root systems combined with the singularity theory of mappings. From the root system which defines the flag manifolds, we have the type of an appropriate projection of the E-integral curve and we can determine the normal forms of tangent surfaces.

We have the following sequence of diagrams from the D_4-diagram by "foldings" and "removings":

$$\begin{array}{ccccccc} & & & & & D_4 & \\ & & & & \swarrow & \downarrow & \\ & & A_3 & = & D_3 & \leftarrow & B_3 \\ & \swarrow & & \downarrow & \swarrow & \downarrow & \\ A_2 & \leftarrow & C_2 & = & B_2 & \leftarrow & G_2. \end{array}$$

In fact for each Dynkin diagram P we can associate an explicit tree of fibrations T_P. See for the general theory [4]. A folding of Dynkin diagram $P \to Q$ corresponds to an embedding $T_Q \to T_P$ of tree of fibrations, and a removing $R \to S$ corresponds to a local projection $T_R \to T_S$. In fact, an embedding $\mathfrak{g}(P) \to \mathfrak{g}(Q)$ is induced, via the root decompositions, from a folding $P \to Q$ such that any parabolic subalgebra of $\mathfrak{g}(P)$ is the pull-back of a parabolic subalgebra of $\mathfrak{g}(Q)$. A projection $\mathfrak{g}(R) \to \mathfrak{g}(S)$ of Lie algebras is induced by a removing $R \to S$ such that any parabolic subalgebra of $\mathfrak{g}(R)$ projects to a parabolic subalgebra of $\mathfrak{g}(S)$.

From this perspective on Dynkin diagrams, we can observe relations between geometry, singularity and differential equations arising from diagrams of fibrations.

For example, in G_2-diagram, the singularities of tangent surfaces to projections of a generic E-integral curve on Z^6 to N^5, M^5 respectively has the duality

$$\begin{array}{ccc} \text{CE} & \longleftrightarrow & \text{CE}, \\ \text{OM} & \longleftrightarrow & \text{OSW}, \\ \text{OGFP} & \longleftrightarrow & \text{OS}. \end{array}$$

Here OGFP (resp. OS) means the *open generic folded pleat* (resp. *open Shcherbak surface*) which is the tangent surface to a generic curve of type $(2, 3, 5, 7, 8)$ (resp. a curve of type $(1, 3, 5, 7, 8)$) ([15]) For the cases $C_2 = B_2$ and A_2, see [14][15] and [16].

5. Fibrations via flag coordinates

Let $(V_1, V_2, V_3) \in Z' = Z'(D_4)$ or $(V_1, V_2, V_4^+, V_4^-) \in Z = Z(D_4)$ with $V_3 = V_4^+ \cap V_4^-$. Then the flag is completed into the multiple double flag:

$$V_1 \subset V_2 \subset V_3 \begin{array}{c} \subset \\ \subset \end{array} \begin{array}{c} V_4^+ \\ V_4^- \end{array} \begin{array}{c} \subset \\ \subset \end{array} V_3^\perp \subset V_2^\perp \subset V_1^\perp \subset V = \mathbf{R}^{4,4},$$

combined with the intermediate V_4^+, V_4^-, the unique pair of 4-null subspaces containing V_3, which are contained in $V_3^\perp$.

Fix any $(V_1^0, V_2^0, V_3^0) \in Z' = Z'(D_4)$ and set $V_3^0 = V_4^{0+} \cap V_4^{0-}$. Then there exists a basis $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8$ of $V = \mathbf{R}^{4,4}$ such that

$$V_1^0 = \langle e_1 \rangle_{\mathbf{R}}, \quad V_2^0 = \langle e_1, e_2 \rangle_{\mathbf{R}}, \quad V_3^0 = \langle e_1, e_2, e_3 \rangle_{\mathbf{R}},$$
$$V_4^{0+} = \langle e_1, e_2, e_3, e_4 \rangle_{\mathbf{R}}, \quad V_4^{0-} = \langle e_1, e_2, e_3, e_5 \rangle_{\mathbf{R}}, \quad V_3^{0\perp} = \langle e_1, e_2, e_3, e_4, e_5 \rangle_{\mathbf{R}},$$
$$V_2^{0\perp} = \langle e_1, e_2, e_3, e_4, e_5, e_6 \rangle_{\mathbf{R}}, \quad V_1^{0\perp} = \langle e_1, e_2, e_3, e_4, e_5, e_6, e_7 \rangle_{\mathbf{R}}$$

and with inner products

$$(e_1|e_8) = \tfrac{1}{2}, (e_2|e_7) = \tfrac{1}{2}, (e_3|e_6) = \tfrac{1}{2}, (e_4|e_5) = \tfrac{1}{2},$$

other pairings being null. Such a basis $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8$ of $V = \mathbf{R}^{4,4}$ is called an *adapted basis* for $(V_1, V_2, V_3) \in Z' = Z'(D_4)$ or $(V_1, V_2, V_4^+, V_4^-) \in Z = Z(D_4)$. Then the metric on V is expressed via the coordinates $x_1, \dots, x_8$ associated to the above basis by

$$ds^2 = dx_1dx_8 + dx_2dx_7 + dx_3dx_6 + dx_4dx_5.$$

For any curve $f : I \to Z$, we can take a moving frame $\boldsymbol{f} : I \to O(4,4)$ such that $\boldsymbol{f}(t)$ is an adapted basis for $f(t)$, which is called an *adapted frame* for f.

Remark 5.1. If we set

$$\widetilde{Z} := \left\{(V_1, V_2, V_3, V_4) \mid V_1 \subset V_2 \subset V_3 \subset V_4 \subset \mathbf{R}^{4,4},\ \dim(V_i) = i, V_i \text{ is null}, i = 1,2,3,4\right\},$$

then the projection $\pi : \widetilde{Z} \to Z'$, $\pi(V_1, V_2, V_3, V_4) = (V_1, V_2, V_3)$ is a trivial double covering. In fact, if we set

$$Z_\pm := \left\{(V_1, V_2, V_3, V_4) \in \widetilde{Z} \mid V_4 \in Q_\pm\right\},$$

then $\widetilde{Z} = Z_+ \cup Z_-$, disjoint union, and $\pi|_{Z_\pm} : Z_\pm \to Z'$ is a diffeomorphism. As is seen as above, we have an embedding $\widetilde{Z}$ into the complete flag manifold $\mathcal{F}_{1,2,3,4,5,6,7}(\mathbf{R}^{4,4})$.

Let us give local charts on Z', Z and Q_0. Take another flag defined by

$$W_1^0 = \langle e_8\rangle_{\mathbf{R}}, \quad W_2^0 = \langle e_8, e_7\rangle_{\mathbf{R}}, \quad W_3^0 = \langle e_8, e_7, e_6\rangle_{\mathbf{R}},$$
$$W_4^{0+} = \langle e_8, e_7, e_6, e_5\rangle_{\mathbf{R}}, \quad W_4^{0-} = \langle e_8, e_7, e_6, e_4\rangle_{\mathbf{R}}, \quad W_3^{0\perp} = \langle e_8, e_7, e_6, e_5, e_4\rangle_{\mathbf{R}},$$
$$W_2^{0\perp} = \langle e_8, e_7, e_6, e_5, e_4, e_3\rangle_{\mathbf{R}}, \quad W_1^{0\perp} = \langle e_8, e_7, e_6, e_5, e_4, e_3, e_2\rangle_{\mathbf{R}},$$

and take the open neighborhood

$$U' = \{(V_1, V_2, V_3) \in Z' \mid V_1 \cap W_1^{0\perp} = \{0\}, V_2 \cap W_2^{0\perp} = \{0\}, V_3 \cap W_3^{0\perp} = \{0\}\}$$

of (V_1^0, V_2^0, V_3^0) in Z'. Then, for any $(V_1, V_2, V_3) \in U'$, there exist unique $f_1, f_2, f_3 \in V_3$ such that f_1 forms a basis of V_1, f_1, f_2 form a basis of V_2 and f_1, f_2, f_3 form a basis of V_3 respectively and they are of form

$$\left\{\begin{array}{lcl} f_1 &=& e_1 + x_{21}e_2 + x_{31}e_3 + x_{41}e_4 + x_{51}e_5 + x_{61}e_6 + x_{71}e_7 + x_{81}e_8, \\ f_2 &=& \qquad\qquad e_2 + x_{32}e_3 + x_{42}e_4 + x_{52}e_5 + x_{62}e_6 + x_{72}e_7 + x_{82}e_8, \\ f_3 &=& \qquad\qquad\qquad\qquad e_3 + x_{43}e_4 + x_{53}e_5 + x_{63}e_6 + x_{73}e_7 + x_{83}e_8, \end{array}\right.$$

for some $x_{ij} \in \mathbf{R}$. Then we have

$$\begin{array}{rcl} (f_1|f_1) &=& x_{81} + x_{21}x_{71} + x_{31}x_{61} + x_{41}x_{51} = 0, \\ 2(f_1|f_2) &=& x_{82} + x_{21}x_{72} + x_{31}x_{62} + x_{41}x_{52} + x_{51}x_{42} + x_{61}x_{32} + x_{71} = 0, \\ 2(f_1|f_3) &=& x_{83} + x_{21}x_{73} + x_{31}x_{63} + x_{41}x_{53} + x_{51}x_{43} + x_{61} = 0, \\ (f_2|f_2) &=& x_{72} + x_{32}x_{62} + x_{42}x_{52} = 0, \\ 2(f_2|f_3) &=& x_{73} + x_{32}x_{63} + x_{42}x_{53} + x_{52}x_{43} + x_{62} = 0, \\ (f_3|f_3) &=& x_{63} + x_{43}x_{53} = 0. \end{array}$$

Therefore we see that

$$(x_{21}, x_{31}, x_{41}, x_{51}, x_{61}, x_{71}, x_{32}, x_{42}, x_{52}, x_{62}, x_{43}, x_{53})$$

is a chart on $U' \subset Z'$.

Moreover we take

$$f_4 \;=\; e_4 + x_{54}e_5 + x_{64}e_6 + x_{74}e_7 + x_{84}e_8,$$

from V_4^+ so that f_1, f_2, f_3, f_4 form a basis of V_4^+, and take

$$f_5 \;=\; x_{45}e_4 + e_5 + x_{65}e_6 + x_{75}e_7 + x_{85}e_8,$$

from V_4^- so that f_1, f_2, f_3, f_5 form a basis of V_4^-. We have

$$\begin{array}{rcl} 2(f_1|f_4) &=& x_{84}+x_{21}x_{74}+x_{31}x_{64}+x_{41}x_{54}+x_{51}=0,\\ 2(f_2|f_4) &=& x_{74}+x_{32}x_{64}+x_{42}x_{54}+x_{52}=0,\\ 2(f_3|f_4) &=& x_{64}+x_{43}x_{54}+x_{53}=0,\\ (f_4|f_4) &=& x_{54}=0,\\ 2(f_1|f_5) &=& x_{85}+x_{21}x_{75}+x_{31}x_{65}+x_{41}+x_{51}x_{45}=0,\\ 2(f_2|f_5) &=& x_{75}+x_{32}x_{65}+x_{42}+x_{52}x_{45}=0,\\ 2(f_3|f_5) &=& x_{65}+x_{43}+x_{53}x_{45}=0,\\ (f_4|f_5) &=& x_{45}=0. \end{array}$$

We set

$$U := \{(V_1, V_2, V_4^+, V_4^-) \in Z \mid V_1 \cap W_1^{0\perp} = \{0\}, V_2 \cap W_2^{0\perp} = \{0\}, V_4^{\pm} \cap W_4^{0\pm} = \{0\}, \}$$

Consider the diffeomorphism $\Phi : Z \to Z'$ defined by

$$\Phi(V_1, V_2, V_4^+, V_4^-) = (V_1, V_2, V_4^+ \cap V_4^-)(= (V_1, V_2, V_3)).$$

Then $\Phi(U) = U'$. After replacing x_{43}, x_{53} by x_{64}, x_{65}, we have a chart

$$(x_{21}, x_{31}, x_{41}, x_{51}, x_{61}, x_{71}, x_{32}, x_{42}, x_{52}, x_{62}, x_{64}, x_{65})$$

on $U = \Phi^{-1}(U') \subset Z$ and the mapping Φ is locally given by just $x_{53} = -x_{64}, x_{43} = -x_{65}$. In fact other components are calculated as follows:

$$\begin{cases} x_{81} &= -x_{71}x_{21} - x_{61}x_{31} - x_{51}x_{41},\\ x_{72} &= -x_{62}x_{32} - x_{52}x_{42},\\ x_{82} &= x_{62}(x_{32}x_{21} - x_{31}) + x_{52}(x_{42}x_{21} - x_{41}) - x_{51}x_{42} - x_{61}x_{32} - x_{71},\\ x_{43} &= -x_{65},\\ x_{53} &= -x_{64},\\ x_{63} &= -x_{65}x_{64},\\ x_{73} &= x_{65}x_{64}x_{32} + x_{64}x_{42} + x_{65}x_{52} - x_{62},\\ x_{83} &= x_{65}x_{64}(x_{31} - x_{32}x_{21}) + x_{64}(x_{41} - x_{42}x_{21}) + x_{65}(x_{51} - x_{52}x_{21}) - x_{61} + x_{62}x_{21},\\ x_{74} &= -x_{64}x_{32} - x_{52},\\ x_{84} &= x_{64}(x_{32}x_{21} - x_{31}) + x_{52}x_{21} - x_{51},\\ x_{75} &= -x_{65}x_{32} - x_{42},\\ x_{85} &= x_{65}(x_{32}x_{21} - x_{31}) + x_{42}x_{21} - x_{41}. \end{cases}$$

Now we will explicitly describe π_0, π_+, π_- and π_M locally on $U \subset Z$.

It is easy to describe π_0 in terms of our charts: Consider the open neighborhood of $V_1^0 \in Q_0$:

$$U_0 := \{V_1 \in Q_0 \mid V_1 \cap W_1^{0\perp} = \{0\}\}.$$

Then, using the above notations, $(x_{21}, x_{31}, x_{41}, x_{51}, x_{61}, x_{71})$ provides a chart on $U_0 \subset Q_0$. Moreover

$$\pi_0 : U \to U_0$$

is given by

$$(x_{21}, x_{31}, x_{41}, x_{51}, x_{61}, x_{71}, x_{32}, x_{42}, x_{52}, x_{62}, x_{64}, x_{65}) \mapsto (x_{21}, x_{31}, x_{41}, x_{51}, x_{61}, x_{71}).$$

Remark 5.2. We have the description of the conformal structure on Q_0 using the local coordinates: The Schubert variety $S_x = P(V_1^{\perp}) \cap Q_0, x = V_1 \in Q_0$ (see §3) is given in U_0 by

$$\{X \in U_0 \mid (X_{21} - x_{21})(X_{71} - x_{71}) + (X_{31} - x_{31})(X_{61} - x_{61}) + (X_{41} - x_{41})(X_{51} - x_{51}) = 0\}.$$

Then the null cone filed $C \subset TQ_0$ of the conformal structure on Q_0 is given, in our local coordinates, by

$$dx_{21}dx_{71} + dx_{31}dx_{61} + dx_{41}dx_{51} = 0,$$

in terms of the symmetric two tensor.

Next we describe π_M. Set

$$U_M := \{V_2 \in M \mid V_2 \cap W_2^{0\perp} = \{0\}\},$$

and take a basis of $V_2 \in M$ of form

$$\left\{\begin{array}{lcllllllll} h_1 & = & e_1 & & +z_{31}e_3 + z_{41}e_4 + z_{51}e_5 + z_{61}e_6 + z_{71}e_7 + z_{81}e_8, \\ h_2 & = & & e_2 & +z_{32}e_3 + z_{42}e_4 + z_{52}e_5 + z_{62}e_6 + z_{72}e_7 + z_{82}e_8. \end{array}\right.$$

Then we have a chart on $U_M \subset M$ by

$$(z_{31}, z_{41}, z_{51}, z_{61}, z_{71}, z_{32}, z_{42}, z_{52}, z_{62}).$$

Using the modification $h_1 = f_1 - x_{21}f_2, h_2 = f_2$, we have that the projection

$$\pi_M : U \to U_M$$

is given by

$$\begin{array}{l} z_{31} = x_{31} - x_{32}x_{21}, \quad z_{41} = x_{41} - x_{42}x_{21}, \quad z_{51} = x_{51} - x_{52}x_{21}, \quad z_{61} = x_{61} - x_{62}x_{21}, \\ z_{71} = x_{71} + x_{62}x_{32}x_{21} + x_{52}x_{42}x_{21}, \quad z_{32} = x_{32}, \quad z_{42} = x_{42}, \quad z_{52} = x_{52}, \quad z_{62} = x_{62}. \end{array}$$

To describe π_+, we set

$$U_+ := \{V_4^+ \in Q_+ \mid V_4^+ \cap W_4^{0+} = \{0\}\}.$$

and take a basis of $V_4^+ \in U_+$ of form

$$\left\{\begin{array}{lcllllllll} g_1 & = & e_1 & & & & +y_{51}e_5 & +y_{61}e_6 & +y_{71}e_7, & \\ g_2 & = & & e_2 & & & +y_{52}e_5 & +y_{62}e_6 & & -y_{71}e_8, \\ g_3 & = & & & e_3 & & -y_{64}e_5 & & -y_{62}e_7 & -y_{61}e_8, \\ g_4 & = & & & & e_4 & & +y_{64}e_6 & -y_{52}e_7 & -y_{51}e_8. \end{array}\right.$$

Then we have a chart on U_+ by

$$(y_{51}, y_{61}, y_{71}, y_{52}, y_{62}, y_{64}).$$

We use the modifications

$$\left\{\begin{array}{lcl} g_1 & = & f_1 - x_{21}f_2 - (x_{31} - x_{32}x_{21})f_3 - (x_{41} - x_{42}x_{21} - x_{43}(x_{31} - x_{32}x_{21}))f_4, \\ g_2 & = & f_2 - x_{32}f_3 - (x_{42} - x_{43}x_{32})f_4, \\ g_3 & = & f_3 - x_{43}f_4. \end{array}\right.$$

Then the projection

$$\pi_+ : U \to U_+$$

is described in terms of our charts, by

$$\left\{\begin{array}{lcl} y_{51} & = & x_{51} - x_{52}x_{21} + x_{64}(x_{31} - x_{32}x_{21}), \\ y_{61} & = & x_{61} - x_{62}x_{21} - x_{64}(x_{41} - x_{42}x_{21}), \\ y_{71} & = & x_{71} + x_{62}x_{31} + x_{52}x_{41} - x_{64}(x_{42}x_{31} - x_{41}x_{32}), \\ y_{52} & = & x_{52} + x_{64}x_{32}, \\ y_{62} & = & x_{62} - x_{64}x_{42}, \\ y_{64} & = & x_{64}. \end{array}\right.$$

To describe π_-, similarly we set

$$U_- := \{V_4^- \in Q_- \mid V_4^- \cap W_4^{0-} = \{0\}\},$$

and take a basis of $V_4^- \in U_-$:

$$\left\{\begin{array}{lllllllll} g_1 & = & e_1 & & & +y_{41}e_4 & & +y_{61}e_6 & +y_{71}e_7, & \\ g_2 & = & & e_2 & & +y_{42}e_4 & & +y_{62}e_6 & & -y_{71}e_8, \\ g_3 & = & & & e_3 & -y_{65}e_4 & & & -y_{62}e_7 & -y_{61}e_8, \\ g_5 & = & & & & & +e_5 & +y_{65}e_6 & -y_{42}e_7 & -y_{41}e_8. \end{array}\right.$$

Then a chart on U_- is given by

$$(y_{41}, y_{61}, y_{71}, y_{42}, y_{62}, y_{65}).$$

Use the modifications

$$\left\{\begin{array}{lll} g_1 & = & f_1 - x_{21}f_2 - (x_{31} - x_{32}x_{21})f_3 - (x_{51} - x_{52}x_{21} - x_{53}(x_{31} - x_{32}x_{21}))f_5, \\ g_2 & = & f_2 - x_{32}f_3 - (x_{52} - x_{53}x_{32})f_5, \\ g_3 & = & f_3 - x_{53}f_5. \end{array}\right.$$

Then the projection

$$\pi_- : U \to U_-$$

is given by

$$\left\{\begin{array}{lll} y_{41} & = & x_{41} - x_{42}x_{21} + x_{65}(x_{31} - x_{32}x_{21}), \\ y_{61} & = & x_{61} - x_{62}x_{21} - x_{65}(x_{51} - x_{52}x_{21}), \\ y_{71} & = & x_{71} + x_{62}x_{31} + x_{51}x_{42} - x_{65}(x_{51}x_{32} - x_{52}x_{31}), \\ y_{42} & = & x_{42} + x_{65}x_{32}, \\ y_{62} & = & x_{62} - x_{65}x_{52}, \\ y_{65} & = & x_{65}. \end{array}\right.$$

Remark 5.3. We have also the description of the conformal structure on $Q_\pm$ using the local coordinates: The Schubert variety $S_y = \{W \in Q_\pm \mid W \cap V_4^\pm \neq \{0\}\}, y = V_4^\pm \in Q_\pm$ (see §3), is given in U_+ (resp. in U_-) by

$$\{Y \in U_+ \mid (Y_{51} - y_{51})(Y_{62} - y_{62}) - (Y_{61} - y_{61})(Y_{52} - y_{52}) - (Y_{71} - y_{71})(Y_{64} - y_{64}) = 0\},$$

(resp. $\{Y \in U_- \mid (Y_{41} - y_{41})(Y_{62} - y_{62}) - (Y_{61} - y_{61})(Y_{42} - y_{42}) - (Y_{71} - y_{71})(Y_{65} - y_{65}) = 0\}$).

Then the null cone field $C \subset TQ_+$ (resp. TQ_-) of the conformal structure on Q_+ (resp. Q_-) is given locally by

$$dy_{51}dy_{62} - dy_{61}dy_{52} - dy_{71}dy_{64} = 0, \quad (\text{resp.} \quad dy_{41}dy_{62} - dy_{61}dy_{42} - dy_{71}dy_{65} = 0 \ ),$$

in terms of two tensors.

6. The Engel system via flag coordinates

Recall that

$$E = (\ker \pi_{0*} \cap \ker \pi_{+*} \cap \ker \pi_{-*}) \oplus \ker \pi_{M*} \subset TZ.$$

First we show

Lemma 6.1. *Let $f = (V_1, V_2, V_4^+, V_4^-) \in Z$ and $e = (e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8)$ be an adapted basis for f (see §5). For each tangent vector $v \in T_f Z$, the following conditions are equivalent to each other:*
(1) *The tangent vector v belongs to E_f.*
(2) *There exists a representative $c : (\mathbf{R}, 0) \to (Z, f), c(t) = (V_1(t), V_2(t), V_4^+(t), V_4^-(t))$ of the tangent vector v, with a framing*

$$V_1(t) = \langle f_1(t)\rangle_{\mathbf{R}}, \ V_2(t) = \langle f_1(t), f_2(t)\rangle_{\mathbf{R}},$$
$$V_4^+(t) = \langle f_1(t), f_2(t), f_3(t), f_4(t)\rangle_{\mathbf{R}}, \ V_4^-(t) = \langle f_1(t), f_2(t), f_3(t), f_5(t)\rangle_{\mathbf{R}},$$

by a curve-germ $\boldsymbol{f} : (\mathbf{R}, 0) \to \mathrm{GL}(\mathbf{R}^{4,4})$,

$$\boldsymbol{f}(t) = (f_1(t), f_2(t), f_3(t), f_4(t), f_5(t), f_6(t), f_7(t), f_8(t)),$$

with $\boldsymbol{f}(0) = e$*, which satisfies that* $f_1'(0) \in V_2, f_2'(0) \in V_4^+ \cap V_4^-$.

(3) *The tangent vector* v *satisfies that*

$$\pi_{0*}v \in T_{V_1}(G_1(V_2)) \text{ and } \pi_{M*}v \in T_{V_2}(G_2(V_4^+ \cap V_4^-)).$$

Proof. (1) $\Rightarrow$ (2): Let $v = w + u, w \in \ker \pi_{0*} \cap \ker \pi_{+*} \cap \ker \pi_{-*}, u \in \ker \pi_{M*}$. Take a frame

$$\boldsymbol{g}(t) = (g_1(t), g_2(t), g_3(t), g_4(t), g_5(t), g_6(t), g_7(t), g_8(t))$$

of V such that $\boldsymbol{g}(t)$ defines the tangent vector u at $t = 0$ and that $\langle g_1(t), g_2(t)\rangle_{\mathbf{R}} = V_2$. Take a frame

$$\boldsymbol{h}(t) = (h_1(t), h_2(t), h_3(t), h_4(t), h_5(t), h_6(t), h_7(t), h_8(t)))$$

such that $\boldsymbol{h}(t)$ defines the tangent vector w at $t = 0$ and that

$$\langle h_1(t)\rangle_{\mathbf{R}} = V_1, \langle h_1(t), h_2(t), h_3(t), h_4(t)\rangle_{\mathbf{R}} = V_4^+, \langle h_1(t), h_2(t), h_3(t), h_5(t)\rangle_{\mathbf{R}} = V_4^-$$

with $\boldsymbol{g}(0) = \boldsymbol{h}(0) = e$. Then the curve $\boldsymbol{f}(t) := \boldsymbol{g}(t) + \boldsymbol{h}(t) - \boldsymbol{g}(0)$ represents v. Moreover $f_1'(0) = g_1'(0) + h_1'(0) \in V_2, f_2'(0) = g_2'(0) + h_2'(0) \in V_4^+ \cap V_4^-$.
The assertion (2) $\Rightarrow$ (3) is clear.
(3) $\Rightarrow$ (1): We take a frame $\boldsymbol{f}(t) = (f_1(t), f_2(t), f_3(t), f_4(t), f_5(t))$ for v such that $f_1(t) \in V_2, f_2(t) \in V_3 = V_4^+ \cap V_4^-$. Write

$$\begin{cases} f_1 = e_1 + x_{21}e_2, \\ f_2 = e_2 + x_{32}e_3, \\ f_3 = e_3 - x_{65}e_4 - x_{64}e_5 + x_{63}e_6 + x_{73}e_7 + x_{83}e_8, \\ f_4 = e_4 + x_{64}e_6 + x_{74}e_7 + x_{84}e_8, \\ f_5 = e_5 + x_{65}e_6 + x_{75}e_7 + x_{85}e_8, \end{cases}$$

with functions $x_{ij} = x_{ij}(t)$ with $x_{ij}(0) = 0$. Then we have

$$x_{83} = -x_{21}x_{73}, x_{84} = -x_{21}x_{74}, x_{85} = -x_{21}x_{75}, x_{73} = -x_{32}x_{63}, x_{74} = -x_{32}x_{64}, x_{75} = -x_{32}x_{65}.$$

Therefore $x_{83}'(0) = 0, x_{84}'(0) = 0, x_{85}'(0) = 0, x_{73}'(0) = 0, x_{74}'(0) = 0, x_{75}'(0) = 0$. We define $\boldsymbol{g}(t)$ and $\boldsymbol{h}(t)$ by

$$\begin{cases} g_1 = e_1, \\ g_2 = e_2 + x_{32}e_3, \\ g_3 = e_3, \\ g_4 = e_4, \\ g_5 = e_5, \end{cases}$$

and

$$\begin{cases} h_1 = e_1 + x_{21}e_2, \\ h_2 = e_2, \\ h_3 = e_3 - x_{65}e_4 - x_{64}e_5 + x_{63}e_6, \\ h_4 = e_4 + x_{64}e_6, \\ h_5 = e_5 + x_{65}e_6. \end{cases}$$

Let $w \in T_fZ$ (resp. $u \in T_fZ$) be tangent vectors defined by the curve $\boldsymbol{g}(t)$ (resp. $\boldsymbol{h}(t)$) at $t = 0$. Then w (resp. u) belongs to $\ker \pi_{0*} \cap \ker \pi_{+*} \cap \ker \pi_{-*}$ (resp. to $\ker \pi_{M*}$). Set $\boldsymbol{k}(t) = \boldsymbol{g}(t) + \boldsymbol{h}(t) - \boldsymbol{g}(0)$. Then we see that $\boldsymbol{f}'(0) = \boldsymbol{k}'(0) = \boldsymbol{g}'(0) + \boldsymbol{h}'(0)$. Thus we have that $v = w + u \in (\ker \pi_{0*} \cap \ker \pi_{+*} \cap \ker \pi_{-*}) \oplus \ker \pi_{M*}$. □

Regarding Lemma 6.1, the differential system $E \subset TZ$ is given by the condition

$$f_1' \in \langle f_1, f_2 \rangle_{\mathbf{R}}, f_2' \in \langle f_1, f_2, f_3 \rangle_{\mathbf{R}}.$$

In terms of component functions x_{ij} introduced in §5, the condition $f_1' \in \langle f_1, f_2 \rangle_{\mathbf{R}}$ is equivalent to that for any t, there exists $p_1, p_2 \in \mathbf{R}$ satisfying

$$\begin{aligned}(0, x_{21}'(t), x_{31}'(t), x_{41}'(t), x_{51}'(t), x_{61}'(t), x_{71}'(t), x_{81}'(t))& \\ = p_1(1, x_{21}(t), x_{31}(t), x_{41}(t), x_{51}(t), x_{61}(t), x_{71}(t), x_{81}(t))& \\ +p_2(0, 0, x_{32}(t), x_{42}(t), x_{52}(t), x_{62}(t), x_{72}(t), x_{82}(t)).&\end{aligned}$$

Then $p_1 = 0, p_2 = x_{21}'(t)$ and

$$(x_{21}', x_{31}', x_{41}', x_{51}', x_{61}', x_{71}', x_{81}') = p_2(1, x_{32}, x_{42}, x_{52}, x_{62}, x_{72}, x_{82}).$$

Similarly, the condition $f_2' \in \langle f_1, f_2, f_3 \rangle_{\mathbf{R}}$ is equivalent to that, for each t, there exists $q \in \mathbf{R}$ satisfying

$$(x_{32}'(t), x_{42}'(t), x_{52}'(t), x_{62}'(t), x_{72}'(t), x_{82}'(t)) = q(1, x_{43}(t), x_{53}(t), x_{63}(t), x_{73}(t), x_{83}(t)).$$

Then $q = x_{32}'(t)$. Therefore we have that the differential system $E \subset TZ$ on our coordinate neighborhood U is given by

$$dx_{i1} = x_{i2}dx_{21}(3 \le i \le 8), \quad dx_{j2} = x_{j3}dx_{32}(4 \le j \le 8).$$

We introduce a weight $w_{ij} \in \mathbf{R}$ on each component x_{ij}. From the above equations for E, we impose the relations

$$w_{i1} = w_{i2} + w_{21}(3 \le i \le 8), \quad w_{j2} = w_{j3} + w_{32}(4 \le j \le 8).$$

Then the weights of all components x_{ij} are well-defined and they are explicitly expressed by w_{21}, w_{32}, w_{65} and w_{64}. Moreover we have

Lemma 6.2. (*Triality of weights.*) *The projections π_0, π_+, π_- and π_M are equivariant under the action generated by the Cartan subalgebra. Each component of projections for the flag coordinates is weighted homogeneous. The weights of components of the projections π_0, π_+, π_- to Q_0, Q_+, Q_- are given by the following table:*

Q_0	Q_+	Q_-
w_{21}	w_{65}	w_{64}
$w_{32} + w_{21}$	$w_{65} + w_{32}$	$w_{64} + w_{32}$
$w_{64} + w_{32} + w_{21}$	$w_{65} + w_{32} + w_{21}$	$w_{64} + w_{32} + w_{21}$
$w_{65} + w_{32} + w_{21}$	$w_{65} + w_{64} + w_{32}$	$w_{65} + w_{64} + w_{32}$
$w_{65} + w_{64} + w_{32} + w_{21}$	$w_{65} + w_{64} + w_{32} + w_{21}$	$w_{65} + w_{64} + w_{32} + w_{21}$
$w_{65} + w_{64} + 2w_{32} + w_{21}$	$w_{65} + w_{64} + 2w_{32} + w_{21}$	$w_{65} + w_{64} + 2w_{32} + w_{21}$

The weights of components of the projection π_M to M are given by

$$\begin{gathered} w_{32},\ w_{32} + w_{21},\ w_{65} + w_{32},\ w_{64} + w_{32}, \\ w_{65} + w_{32} + w_{21},\ w_{64} + w_{32} + w_{21},\ w_{65} + w_{64} + w_{32}, \\ w_{65} + w_{64} + w_{32} + w_{21},\ w_{65} + w_{64} + 2w_{32} + w_{21}. \end{gathered}$$

Remark 6.3. We observe that the formula of weights coincides with the formula of negative (or positive) roots of D_4 (see [5] for example). In fact, given a simple root system $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, we identify $-\alpha_1, -\alpha_2, -\alpha_3, -\alpha_4$ with $w_{21}, w_{32}, w_{65}, w_{64}$. Then the weight w of a component for a negative root α is given by $w = m_1w_{21} + m_2w_{32} + m_3w_{65} + m_4w_{64}$ if

$$\alpha = -m_1\alpha_1 - m_2\alpha_2 - m_3\alpha_3 - m_4\alpha_4.$$

See the following D_4 diagram with weights $w_{21}, w_{32}, w_{65}, w_{64}$ at appropriate positions:

$$\begin{tikzcd} & & w_{65} \\ w_{21} \arrow[r, dash] & w_{32} \arrow[ur, dash] \arrow[dr, dash] & \\ & & w_{64} \end{tikzcd}$$

Then we have the orders of flag coordinates for generic E-integral curves, and normal forms of singularities appeared in tangent surfaces.

Lemma 6.4. *Let $f : I \to Z$ be a generic E-integral curve. Then, for any $t_0 \in I$ and for any flag chart (x_{ij}) on Z centered at $f(t_0)$, the sets of orders on components for the projections $\pi_0 f, \pi_+ f, \pi_- f, \pi_M f$ are given as in the following table:*

$(w_{21}, w_{65}, w_{64}, w_{32})$	$\pi_0 f$	$\pi_+ f$	$\pi_- f$	$\pi_M f$
$(1,1,1,1)$	$1,2,3,3,4,5^*$	$1,2,3,3,4,5^*$	$1,2,3,3,4,5^*$	$1,2,2,2,3,3,3,4,5^*$
$(2,1,1,1)$	$2,3,4,4,5,6$	$1,2,4,3,5,6$	$1,2,4,3,5,6$	$1,3,2,2,4,4,3,5,6$
$(1,2,1,1)$	$1,2,3,4,5,6$	$2,3,4,4,5,6$	$1,2,3,4,5,6$	$1,2,3,2,4,3,4,5,6$
$(1,1,2,1)$	$1,2,4,3,5,6$	$1,2,3,4,5,6$	$2,3,4,4,5,6$	$1,2,2,3,3,4,4,5,6$
$(1,1,1,2)$	$1,3,4,4,5,7$	$1,3,4,4,5,7$	$1,3,4,4,5,7$	$2,3,3,3,4,4,4,5,7$

where 5^* means 5 or 6 on an isolated points.

Remark 6.5. *From the formula on weights of components, we can estimate the orders of component functions of E-integral curves. However it is possible that the orders of some components become higher than expected by accidental cancelings of leading terms. Therefore, in order to determine the exact order of each component of generic curves, we need the explicit local expressions of the projections $\pi_0, \pi_+, \pi_-, \pi_M$ and the differential system $E \subset TZ$.*

Proof of Lemma 6.4. As we have seen in the above arguments, all components of $\pi_0 \circ f$ (resp. $\pi_+ \circ f, \pi_- \circ f, \pi_M \circ f$) are obtained just from the four components $x_{21} \circ f, x_{65} \circ f, x_{64} \circ f, x_{32} \circ f$ by differentiations, multiplications, summations and integrations. We can spell out, from the explicit expression of components obtained in §5, which component may have higher order than expected. For example, since $(x_{52} \circ f)' = (x_{53} \circ f)(x_{32} \circ f)'$, we see $x_{52} \circ f = \int (x_{53} \circ f)(x_{32} \circ f)' dt$. Therefore $\mathrm{ord}(x_{52} \circ f) = \mathrm{ord}(x_{53} \circ f) + \mathrm{ord}(x_{32} \circ f)$. As another example, for the component $z_{31} \circ f = (x_{31} - x_{32}x_{21}) \circ f$ of π_M, we have $(z_{31} \circ f)' = \{(x_{31} - x_{32}x_{21}) \circ f\}' = -(x_{32} \circ f)'(x_{21} \circ f)$. Therefore $z_{31} \circ f = -\int (x_{32} \circ f)'(x_{21} \circ f) dt$ and $\mathrm{ord}(z_{31} \circ f) = \mathrm{ord}((x_{32} \circ f) + \mathrm{ord}(x_{21} \circ f)$.

By the ordinary transversality theorem, we have, generically, just four cases where

$$\big(\mathrm{ord}(x_{21} \circ f), \mathrm{ord}(x_{65} \circ f), \mathrm{ord}(x_{64} \circ f), \mathrm{ord}(x_{32} \circ f)\big)$$

is equal to

$$(1,1,1,1), (2,1,1,1), (1,2,1,1), (1,1,2,1), (1,1,1,2),$$

respectively. The last four cases occur just on isolated points, where the orders of all components are equal to the weights of components. In the first case, the order of one component may increase by one from the weight of the component accidentally on an isolated points. Thus we have the above table. □

Proof of Theorem 4.3: We use several results proved in [12]. If the set of orders contains $1, 2, 3$ (resp. $2, 3, 4, 5$, $1, 3, 4, 5$), then the tangent surface to the projection of the Engel integral curve is locally diffeomorphic to the cuspidal edge (resp. the open swallowtail, the open Mond surface) in $(\mathbf{R}^6, 0)$ or $(\mathbf{R}^9, 0)$. This is proved essentially by the versality of the cuspidal edge (resp. the open swallowtail, the open Mond surface) as an "opening" of the fold map (resp. the Whitney's cusp, the beak-to beak map) $(\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$. For example, we show one case where the set of orders of components is given by $\{1, 2, 3, 3, 4, 5\}$. Then the projection of the Engel integral curve is locally expressed by $c : (\mathbf{R}, 0) \to (\mathbf{R}^6, 0)$ with components

$$\begin{cases} x_1(t) &= a_1 t + \cdots, \\ x_2(t) &= a_2 t^2 + \cdots, \\ x_3(t) &= a_3 t^3 + \cdots, \\ x_4(t) &= a_4 t^3 + \cdots, \\ x_5(t) &= a_5 t^4 + \cdots, \\ x_6(t) &= a_6 t^5 + \cdots. \end{cases}$$

where $a_i \neq 0, 1 \leq i \leq 6$ and $\cdots$ means higher order terms.

Then, by a local diffeomorphism on $(\mathbf{R}, 0)$ and a linear transformation on $(\mathbf{R}^6, 0)$ the curve is transformed into a curve $\tilde{c} : (\mathbf{R}, 0) \to (\mathbf{R}^6, 0)$ with components

$$x_1(t) = t,\ x_2(t) = t^2 + \varphi_2(t),\ x_3(t) = t^3 + \varphi_3(t),$$
$$x_4(t) = t^3 + \varphi_4(t),\ x_5(t) = t^4 + \varphi_5(t),\ x_6(t) = t^5 + \varphi_6(t),$$

where $\mathrm{ord}(\varphi_2) \geq 3, \mathrm{ord}(\varphi_3) \geq 4, \mathrm{ord}(\varphi_4) \geq 4, \mathrm{ord}(\varphi_5) \geq 5, \mathrm{ord}(\varphi_6) \geq 6$. The tangent surface of $\tilde{c}$ is parametrized by $F(t, s) = \tilde{c}(t) + s\tilde{c}'(t)$, namely,

$$\begin{array}{ll} x_1(t, s) = t + s, & x_2(t, s) = t^2 + 2st + \varphi_2(t) + s\varphi_2'(t), \\ x_3(t, s) = t^3 + 3st^2 + \varphi_3(t) + s\varphi_3'(t), & x_4(t, s) = t^3 + 3st^2 + \varphi_4(t) + s\varphi_4'(t), \\ x_5(t, s) = t^4 + 4st^3 + \varphi_5(t) + s\varphi_5'(t), & x_6(t, s) = t^5 + 5st^4 + \varphi_6(t) + s\varphi_6'(t). \end{array}$$

If we put $u = t + s$, then we have that F is diffeomorphic to a map-germ $G : (\mathbf{R}^2, 0) \to (\mathbf{R}^6, 0)$ with components

$$\begin{array}{ll} x_1(t, u) = u, & x_2(t, u) = -t^2 + 2ut + \psi_2(t, u), \\ x_3(t, u) = -2t^3 + 3ut^2 + \psi_3(t, u), & x_4(t, u) = -2t^3 + 3ut^2 + \psi_4(t, u), \\ x_5(t, u) = -3t^4 + 4ut^3 + \psi_5(t, u), & x_6(t, u) = -4t^5 + 5ut^4 + \psi_3(t, u), \end{array}$$

where $\psi_i(t, u) = \varphi_i(t) + (u - t)\varphi_i'(t)$. Now consider the set $\mathcal{R}$ of functions $h(t, u)$ such that $\frac{\partial h}{\partial t}$ is a functional multiple of $u - t$. All components of G belong to $\mathcal{R}$. We define $g, \tilde{g} : (\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$, by $g(t, u) = (u, -t^2 + 2ut + \psi_2(t, u))$ and $\tilde{g}(t, u) = (u, -t^2 + 2ut)$, both of which are diffeomorphic to the fold map. Then $\mathcal{R}$ coincides with $\mathcal{R}_g$, the totality of $h : (\mathbf{R}^2, 0) \to \mathbf{R}$ such that dh is a functional linear combination of du and $d(-t^2 + 2ut + \psi_2(t, u))$, and with $\mathcal{R}_{\tilde{g}}$ which is similarly defined. In this situation, we say that G is an *opening* of g. We can show that any $h \in \mathcal{R}$ is a function on

$$\widetilde{G} = (u, -t^2 + 2ut, -2t^3 + 3ut^2),$$

which is a *versal opening* of $\tilde{g}$. Thus we see, in fact, that there exist functions

$$\Phi_2, \Phi_3, \Phi_4, \Phi_5, \Phi_6 : (\mathbf{R}^3, 0) \to (\mathbf{R}, 0)$$

on $(\mathbf{R}^3, 0)$ with coordinates y_1, y_2, y_3 such that

$$\begin{array}{ll} x_1(t, u) = u, & x_2(t, u) = -t^2 + 2ut + \Phi_2 \circ \widetilde{G}, \\ x_3(t, u) = -2t^3 + 3ut^2 + \Phi_3 \circ \widetilde{G}, & x_4(t, u) = -2t^3 + 3ut^2 + \Phi_4 \circ \widetilde{G}, \\ x_5(t, u) = \Phi_5 \circ \widetilde{G}, & x_6(t, u) = \Phi_6 \circ \widetilde{G}. \end{array}$$

Then we see necessarily that $\frac{\partial \Phi_2}{\partial y_2}(0) = 0, \frac{\partial \Phi_3}{\partial y_3}(0) = 0$. Define a map-germ $\tau : (\mathbf{R}^6, 0) \to (\mathbf{R}^6, 0)$ by

$$\begin{aligned}\tau(y_1, y_2, y_3, y_4, y_5, y_6) \quad = \quad & (y_1, y_2 + \Phi_2(y_1, y_2, y_3), y_3 + \Phi_3(y_1, y_2, y_3), \\ & y_3 + y_4 + \Phi_4(y_1, y_2, y_3), y_5 + \Phi_5(y_1, y_2, y_3), y_6 + \Phi_6(y_1, y_2, y_3)).\end{aligned}$$

Then we have that τ is a diffeomorphism-germ of $(\mathbf{R}^6, 0)$ and $G = \tau \circ (\widetilde{G}, 0, 0, 0)$. Thus F is diffeomorphic to $(\widetilde{G}, 0, 0, 0)$, which is diffeomorphic to

$$(u, v) \mapsto (u, v^2, v^3, 0, 0, 0),$$

the cuspidal edge in $\mathbf{R}^6$. Note that $(\widetilde{G}, 0, 0, 0)$ provides a normal form among tangent mappings.

On the notions of openings and versal openings, and related results, see [12]. We can treat other cases similarly using Lemma 6.4. Thus we have Theorem 4.3. $\square$

7. D_4 Cartan distributions and null frontals

In the previous sections we have studied tangent surfaces to null curves of special kind, that is, null curves which are projections of E-integral curves (see Theorem 4.3). In general the tangent surface to a null curve is a ruled surface by null lines, which is not necessarily a null surface. However, for a projection of an E-integral curve, its null tangent lines do form a null surface, which we have called the *null tangent surface*. In this section we provide the geometric characterization (Proposition 7.4, Remark 7.5) of our main objects in this paper, the null tangent surfaces, by introducing the new notion of "null frontals" and by using the triality. Moreover we characterize the null tangent surfaces as geometric solutions to "bi-Monge-Ampère system". Thus we will make clear the significance of our constructions.

We have defined in §3 the distribution $D_N \subset TN$ on the flag manifold N.

Definition 7.1. A mapping $F : U \to Q_0$ (resp. $F : U \to Q_+, F : U \to Q_-$) from a 2-dimensional manifold U is called a *null frontal* if there exists a D_N-integral lift $\widetilde{F} : U \to N$ of F, i.e. which satisfies $\widetilde{F}_*(T_xU) \subset (D_N)_{\widetilde{F}(x)}$ and $\pi'_0(\widetilde{F}(x)) = F(x)$ (resp. $\pi'_+(\widetilde{F}(x)) = F(x)$, $\pi'_-(\widetilde{F}(x)) = F(x)$), for any $x \in U$.

Remark 7.2. In the above definition, if we can take $\widetilde{F}$ an immersion, then we call F a *null front*.

Recall that Q_0, Q_+, Q_- are endowed with conformal structures of type $(3, 3)$ and we have defined the notion of null surfaces (Definition 4.1).

Proposition 7.3. (1) *If $F : U \to Q_0$ (resp. $F : U \to Q_+, F : U \to Q_-$) is a regular (immersive) null surface, then F is a null frontal.*
(2) *If $F : U \to Q_0$ (resp. $F : U \to Q_+, F : U \to Q_-$) is a null frontal, then F is a null surface.*

As is mentioned in §4 (after Proposition 4.2), we have the following:

Proposition 7.4. *Let $f : I \to Z$ be an E-integral curve. Consider the projections $\gamma_0 = \pi_0 \circ f : I \to Q_0, \gamma_+ = \pi_+ \circ f : I \to Q_+$ and $\gamma_- = \pi_- \circ f : I \to Q_-$. Then the tangent surfaces $F_0 = \mathrm{Tan}(\gamma_0), F_+ = \mathrm{Tan}(\gamma_+)$ and $F_- = \mathrm{Tan}(\gamma_-)$ are null frontals. In fact, there exists a D_N-integral lifting $\widetilde{F_0}$ of F_0 (resp. $\widetilde{F_+}$ of F_+, $\widetilde{F_-}$ of F_-) such that $\pi_+ \circ \widetilde{F_0}$ and $\pi_- \circ \widetilde{F_0}$ (resp. $\pi_- \circ \widetilde{F_+}$ and $\pi_0 \circ \widetilde{F_+}$, $\pi_0 \circ \widetilde{F_-}$ and $\pi_+ \circ \widetilde{F_-}$) are constant along tangent lines.*

Remark 7.5. The converse of Proposition 7.4 holds in the following sense: Let $\gamma_0 : I \to Q_0$ be an immersion. Suppose that γ_0 is a null immersion, its tangent surface $F_0 = \mathrm{Tan}(\gamma_0)$ is a null frontal, and, for a D_N-integral lifting $\widetilde{F_0}$ of F_0, $\pi_+ \circ \widetilde{F_0}$ and $\pi_- \circ \widetilde{F_0}$ are constant along tangent lines. Then there exists an E-integral curve $f : I \to Z$ such that $\gamma_0 = \pi_0 \circ f$ and $\pi_+ \circ \widetilde{F_0}$ (resp. $\pi_- \circ \widetilde{F_0}$) is parametrized by $\pi_+ \circ f$ (resp. $\pi_- \circ f$). In fact, we set $f(t) = (V_1(t), V_2(t), V_4^+(t), V_4^-(t))$, where $V_1(t) = \gamma_0(t)$ regarded as a null line in $V = \mathbf{R}^{4,4}$, $V_2(t)$ is the tangent line to γ_0 at t regarded as a null plane containing $V_1(t)$. Moreover the null 4-space $V_4^+(t)$ (resp. $V_4^-(t)$) is given by the value of $\pi_+ \circ \widetilde{F_0}$ (resp. $\pi_- \circ \widetilde{F_0}$) along the tangent line corresponding to $V_2(t)$.

Note that D_N is described, in terms of tree of fibrations, by

$$(\ker \pi'_{+*} \cap \ker \pi'_{-*}) \oplus (\ker \pi'_{0*} \cap \ker \pi'_{-*}) \oplus (\ker \pi'_{0*} \cap \ker \pi'_{+*}) \subset TN.$$

To show Propositions 7.3 and 7.4, we need the following Lemma 7.6 which gives the equivalent descriptions of D_N in different forms.

Lemma 7.6. *Let $f = (V_1, V_4^+, V_4^-) \in N$. For each tangent vector $\boldsymbol{v} \in T_f N$, the following conditions are equivalent to each other:*
(1) *The tangent vector $\boldsymbol{v}$ belongs to $(D_N)_f$.*
(2) *There exists a representative $c : (\mathbf{R}, 0) \to (N, f), c(t) = (V_1(t), V_4^+(t), V_4^-(t))$ of the tangent vector $\boldsymbol{v}$, with a framing*

$$V_1(t) = \langle f_1(t) \rangle_{\mathbf{R}},\ V_4^+(t) \cap V_4^-(t) = \langle f_1(t), f_2(t), f_3(t) \rangle_{\mathbf{R}},$$
$$V_4^+(t) = \langle f_1(t), f_2(t), f_3(t), f_4(t) \rangle_{\mathbf{R}},\ V_4^-(t) = \langle f_1(t), f_2(t), f_3(t), f_5(t) \rangle_{\mathbf{R}},$$

by a curve-germ $\boldsymbol{f} : (\mathbf{R}, 0) \to \mathrm{GL}(\mathbf{R}^{4,4})$,

$$\boldsymbol{f}(t) = (f_1(t), f_2(t), f_3(t), f_4(t), f_5(t), f_6(t), f_7(t), f_8(t)),$$

which satisfies that $\boldsymbol{f}(0)$ is an adapted basis for some flag in $\pi_N^{-1}(f) \subset Z$, and that

$$f_1'(0) \in V_4^+ \cap V_4^-, f_2'(0), f_3'(0) \in (V_4^+ \cap V_4^-)^\perp.$$

To show Lemma 7.6, we give local coordinates of N' and of N. First fix a complete flag as before

$$W_1^0 \subset W_2^0 \subset W_3^0 \begin{array}{ccc} \subset & W_4^{0+} & \subset \\ \subset & W_4^{0-} & \subset \end{array} W_3^{0\perp} \subset W_2^{0\perp} \subset W_1^{0\perp} \subset V = \mathbf{R}^{4,4},$$

and take the open neighborhood

$$\Omega' = \{(V_1, V_3) \in N' \mid V_1 \cap W_1^{0\perp} = \{0\}, V_3 \cap W_3^{0\perp} = \{0\}\}$$

of (V_1^0, V_3^0) in N'. Then, for any $(V_1, V_3) \in \Omega'$, there exist unique $f_1, f_2, f_3 \in V_3$ such that f_1 forms a basis of V_1, and f_1, f_2, f_3 form a basis of V_3 respectively and they are of form

$$\left\{ \begin{array}{lcllll} f_1 & = & e_1 + & x_{21}e_2 & +x_{31}e_3 & +x_{41}e_4 + x_{51}e_5 + x_{61}e_6 + x_{71}e_7 + x_{81}e_8, \\ f_2 & = & & e_2 & & +x_{42}e_4 + x_{52}e_5 + x_{62}e_6 + x_{72}e_7 + x_{82}e_8, \\ f_3 & = & & & e_3 & +x_{43}e_4 + x_{53}e_5 + x_{63}e_6 + x_{73}e_7 + x_{83}e_8, \end{array} \right.$$

for some $x_{ij} \in \mathbf{R}$. Then we have

$$\begin{array}{rcl}(f_1|f_1) & = & x_{81} + x_{21}x_{71} + x_{31}x_{61} + x_{41}x_{51} = 0, \\ 2(f_1|f_2) & = & x_{82} + x_{21}x_{72} + x_{31}x_{62} + x_{41}x_{52} + x_{51}x_{42} + x_{71} = 0, \\ 2(f_1|f_3) & = & x_{83} + x_{21}x_{73} + x_{31}x_{63} + x_{41}x_{53} + x_{51}x_{43} + x_{61} = 0, \\ (f_2|f_2) & = & x_{72} + x_{42}x_{52} = 0, \\ 2(f_2|f_3) & = & x_{73} + x_{32}x_{63} + x_{42}x_{53} + x_{52}x_{43} + x_{62} = 0, \\ (f_3|f_3) & = & x_{63} + x_{43}x_{53} = 0.\end{array}$$

Therefore we see that

$$(x_{21}, x_{31}, x_{41}, x_{51}, x_{61}, x_{71}, x_{42}, x_{52}, x_{62}, x_{43}, x_{53})$$

is a chart on $\Omega' \subset N'$. We take

$$f_4 \ = \ e_4 + x_{54}e_5 + x_{64}e_6 + x_{74}e_7 + x_{84}e_8,$$

from V_4^+ so that f_1, f_2, f_3, f_4 form a basis of V_4^+, and take

$$f_5 \ = \ x_{45}e_4 + e_5 + x_{65}e_6 + x_{75}e_7 + x_{85}e_8,$$

from V_4^- so that f_1, f_2, f_3, f_5 form a basis of V_4^-. Then we have a local chart for N:

$$(x_{21}, x_{31}, x_{41}, x_{51}, x_{61}, x_{71}, x_{42}, x_{52}, x_{62}, x_{64}, x_{65}).$$

Note that the calculations of coordinates for N' and N go similarly to that for Z' and Z, and we obtain the local forms of π'_0, π'_+, π'_- from those for π_0, π_+, π_- in §5, by just putting $x_{32} = 0$. In fact, we have the coordinate expressions for the projection

$$\pi'_0 : N \to Q_0$$

by

$$(x_{21}, x_{31}, x_{41}, x_{51}, x_{61}, x_{71}, x_{42}, x_{52}, x_{62}, x_{64}, x_{65}) \mapsto (x_{21}, x_{31}, x_{41}, x_{51}, x_{61}, x_{71}),$$

for

$$\pi'_+ : N \to Q_+$$

by

$$\left\{\begin{array}{rcl} y_{51} & = & x_{51} - x_{52}x_{21} + x_{64}x_{31}, \\ y_{61} & = & x_{61} - x_{62}x_{21} - x_{64}(x_{41} - x_{42}x_{21}), \\ y_{71} & = & x_{71} + x_{62}x_{31} + x_{52}x_{41} - x_{64}x_{42}x_{31}, \\ y_{52} & = & x_{52}, \\ y_{62} & = & x_{62} - x_{64}x_{42}, \\ y_{64} & = & x_{64}, \end{array}\right.$$

and for

$$\pi'_- : N \to Q_-$$

by

$$\left\{\begin{array}{rcl} y_{41} & = & x_{41} - x_{42}x_{21} + x_{65}x_{31}, \\ y_{61} & = & x_{61} - x_{62}x_{21} - x_{65}(x_{51} - x_{52}x_{21}), \\ y_{71} & = & x_{71} + x_{62}x_{31} + x_{51}x_{42} + x_{65}x_{52}x_{31}, \\ y_{42} & = & x_{42}, \\ y_{62} & = & x_{62} - x_{65}x_{52}, \\ y_{65} & = & x_{65}. \end{array}\right.$$

Proof of Lemma 7.6:
(1) $\Rightarrow$ (2) : Let $\boldsymbol{v} \in (D_N)_f$. Decompose $\boldsymbol{v} = v_1 + v_3 + v_4$ into

$$v_1 \in \ker \pi'_{+*} \cap \ker \pi'_{-*}, v_3 \in \ker \pi'_{0*} \cap \ker \pi'_{-*}$$

and $v_4 \in \ker \pi'_{0*} \cap \ker \pi'_{+*}$. We take representatives $\boldsymbol{g}(t), \boldsymbol{h}(t), \boldsymbol{k}(t)$ of v_1, v_3, v_4 at 0 respectively, such that $\boldsymbol{g}(0) = \boldsymbol{h}(0) = \boldsymbol{k}(0)$ is an adapted frame for f, and

$$\begin{gathered}\langle g_1(t), g_2(t), g_3(t), g_4(t)\rangle_{\mathbf{R}} = V_4^+, \langle g_1(t), g_2(t), g_3(t), g_5(t)\rangle_{\mathbf{R}} = V_4^-,\\ \langle h_1(t)\rangle_{\mathbf{R}} = V_1, \langle h_1(t), h_2(t), h_3(t), h_5(t)\rangle_{\mathbf{R}} = V_4^-,\\ \langle k_1(t)\rangle_{\mathbf{R}} = V_1, \langle k_1(t), k_2(t), k_3(t), k_4(t)\rangle_{\mathbf{R}} = V_4^+,\end{gathered}$$

for any t near 0. Set $\boldsymbol{f}(t) = \boldsymbol{g}(t) + \boldsymbol{h}(t) + \boldsymbol{k}(t) - 2\boldsymbol{g}(0)$. Then we have

$$f_1'(0) = g_1'(0) + h_1'(0) + k_1'(0) = g_1'(0) \in V_4^+ \cap V_4^-,$$

and

$$f_2'(0) = g_2'(0) + h_2'(0) + k_2'(0) \in V_4^+ + V_4^- = (V_4^+ \cap V_4^-)^{\perp}.$$

(2) $\Rightarrow$ (1) : Write down the first five components of $\boldsymbol{f}(t)$ as

$$\left\{\begin{array}{lllllllll} f_1 & = & e_1 & +x_{21}e_2 & +x_{31}e_3 & +x_{41}e_4 & +x_{51}e_5 & +x_{61}e_6 & +x_{71}e_7 & +x_{81}e_8,\\ f_2 & = & & e_2 & & +x_{42}e_4 & +x_{52}e_5 & +x_{62}e_6 & +x_{72}e_7 & +x_{82}e_8,\\ f_3 & = & & & e_3 & -x_{65}e_4 & -x_{64}e_5 & +x_{63}e_6 & +x_{73}e_7 & +x_{83}e_8,\\ f_4 & = & & & & e_4 & & +x_{64}e_6 & +x_{74}e_7 & +x_{84}e_8,\\ f_5 & = & & & & & e_5 & +x_{65}e_6 & +x_{75}e_7 & +x_{85}e_8,\end{array}\right.$$

where $x_{ij} = x_{ij}(t)$ with $x_{ij}(0) = 0$. Then, by the condition (2), we have $x'_{ij}(0) = 0$, except for the components $x_{21}, x_{31}, x_{42}, x_{52}, x_{64}, x_{65}, x_{74}, x_{75}$, and $x'_{74}(0) = -x'_{52}(0), x'_{75}(0) = -x'_{42}(0)$. Then we take curves $\boldsymbol{g}(t), \boldsymbol{h}(t), \boldsymbol{k}(t)$ satisfying

$$\left\{\begin{array}{llllllll} g_1 & = & e_1 & +x_{21}e_2 & +x_{31}e_3, & & & \\ g_2 & = & & e_2, & & & & \\ g_3 & = & & & e_3, & & & \\ g_4 & = & & & & e_4, & & \\ g_5 & = & & & & & e_5, & \end{array}\right.$$

$$\left\{\begin{array}{llllllll} h_1 & = & e_1, & & & & & \\ h_2 & = & & e_2 & & +x_{42}e_4, & & \\ h_3 & = & & & e_3 & -x_{65}e_4, & & \\ h_4 & = & & & & e_4, & & \\ h_5 & = & & & & e_5 & +x_{65}e_6 & -x_{42}e_7, \end{array}\right.$$

$$\left\{\begin{array}{llllllll} k_1 & = & e_1, & & & & & \\ k_2 & = & & e_2 & & +x_{52}e_5, & & \\ k_3 & = & & & e_3 & -x_{64}e_5, & & \\ k_4 & = & & & e_4 & & +x_{64}e_6 & -x_{52}e_7, \\ k_5 & = & & & & e_5. & & \end{array}\right.$$

Let $g : I \to N, h : I \to N, k : I \to N$ be curves with the frame $\boldsymbol{g}(t), \boldsymbol{h}(t), \boldsymbol{k}(t)$ respectively. Let $v_1, v_3, v_4 \in T_f N$ be tangent vectors defined by g, h, k respectively. Then $\boldsymbol{v} = v_1 + v_2 + v_3$. Since $\pi'_+ \circ g$ and $\pi'_- \circ g$ are constant (resp. $\pi'_0 \circ h$ and $\pi'_- \circ h$ are constant, $\pi'_0 \circ k$ and $\pi'_+ \circ k$ are constant), we have $v_1 \in \ker \pi'_{+*} \cap \ker \pi'_{-*}, v_3 \in \ker \pi'_{0*} \cap \ker \pi'_{-*}, v_4 \in \ker \pi'_{0*} \cap \ker \pi'_{+*}$.

□

Proof of Proposition 7.3:

(1) Regarding $F(u, v)$ as a 1-dimensional subspace in V, we take a frame $f(u, v)$ of $F(u, v)$. Since F is regular,

$$f(u, v), \frac{\partial f}{\partial u}(u, v), \frac{\partial f}{\partial v}(u, v)$$

are linearly independent and

$$V_3(u,v) := \langle f, \frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}\rangle_{\mathbf{R}}$$

is a null 3 space in $V = \mathbf{R}^{4,4}$, for any $(u,v) \in U$. Then by the partial differentiations with respect to u, v of the equalities

$$(f|\frac{\partial f}{\partial u}) = 0,\ (f|\frac{\partial f}{\partial v}) = 0,\ (\frac{\partial f}{\partial u}|\frac{\partial f}{\partial u}) = 0,\ (\frac{\partial f}{\partial u}|\frac{\partial f}{\partial v}) = 0,\ (\frac{\partial f}{\partial v}|\frac{\partial f}{\partial v}) = 0,$$

we have that

$$\frac{\partial^2 f}{\partial u^2}, \frac{\partial^2 f}{\partial u \partial v}, \frac{\partial^2 f}{\partial v^2} \in V_3(u,v)^{\perp}.$$

We set $V_1(u,v) = \langle f(u,v)\rangle_{\mathbf{R}} \subset V$, and take the unique null 4-spaces $V_4^+(u,v), V_4^-(u,v)$ such that $V_3(u,v) = V_4^+(u,v) \cap V_4^-(u,v)$. Then define $\widetilde{F} : U \to N$ by

$$\widetilde{F}(u,v) = (V_1(u,v), V_4^+(u,v), V_4^-(u,v)).$$

Then $\pi_0' \circ \widetilde{F} = F$. Moreover $\widetilde{F}$ is a D_N-integral map.

In fact, for the differential map $\widetilde{F}_* : T_{(u,v)}U \to T_{\widetilde{F}(u,v)}N$ at any $(u,v) \in U$, we have that

$$\widetilde{F}_*(\frac{\partial}{\partial u}) \in (D_N)_{\widetilde{F}(u,v)}, \quad \widetilde{F}_*(\frac{\partial}{\partial v}) \in (D_N)_{\widetilde{F}(u,v)}.$$

To show the first assertion using Lemma 7.6, we set

$$f_1(t) := f(u+t, v) \in V_1(u+t, v),$$

and

$$f_2(t) := \frac{\partial f}{\partial u}(u+t,v) \in V_3(u+t,v), \quad f_3(t) := \frac{\partial f}{\partial v}(u+t,v) V_3(u+t,v).$$

Take $f_4(t)$ and $f_5(t)$ such that

$$V_4^+(u+t,v) = \langle f_1(t), f_2(t), f_3(t), f_4(t)\rangle_{\mathbf{R}},\ V_4^-(u+t,v) = \langle f_1(t), f_2(t), f_3(t), f_5(t)\rangle_{\mathbf{R}},$$

for any sufficiently small t. Note that $\widetilde{F}(u+t,v) = (V_1(u+t,v), V_4^+(u+t,v), V_4^-(u+t,v))$ regarded as a curve on N with parameter t represents the tangent vector $\widetilde{F}_*(\frac{\partial}{\partial u}) \in T_{\widetilde{F}(u,v)}N$. We can extend $(f_1(t), f_2(t), f_3(t), f_4(t), f_5(t))$ to a curve-germ $\boldsymbol{f} : (\mathbf{R}, 0) \to \mathrm{GL}(\mathbf{R}^{4,4})$,

$$\boldsymbol{f}(t) = (f_1(t), f_2(t), f_3(t), f_4(t), f_5(t), f_6(t), f_7(t), f_8(t)),$$

such that $\boldsymbol{f}(0)$ is an adapted basis for a flag in $\pi_N^{-1}(\widetilde{F}(u,v))) \subset Z$. Moreover, as is shown in above,

$$f_1'(0) \in V_4^+(u,v) \cap V_4^-(u,v), \quad f_2'(0), f_3'(0) \in (V_4^+(u,v) \cap V_4^-(u,v))^{\perp}.$$

Therefore, applying Lemma 7.6 to $\boldsymbol{v} = \widetilde{F}_*(\frac{\partial}{\partial u})$, we have that $\widetilde{F}_*(\frac{\partial}{\partial u})$ belongs to $(D_N)_{\widetilde{F}(u,v)}$. The assertion that $\widetilde{F}_*(\frac{\partial}{\partial v})$ belongs to $(D_N)_{\widetilde{F}(u,v)}$ is proved similarly. Thus we have that

$$\widetilde{F}_*(T_{(u,v)}U) \subset (D_N)_{\widetilde{F}(u,v)},$$

for any $(u,v) \in U$.

Therefore F is a null frontal. By triality we have the same result also for regular null surfaces in $Q_{\pm}$.

(2) Let $x = (u,v) \in U$. Let $\boldsymbol{v} \in T_xU$. Suppose $F_*(\boldsymbol{v}) \neq 0$. Then we have $\widetilde{F}_*(\boldsymbol{v}) \in (D_N)_{\widetilde{F}(x)}$. Take a curve $(V_1(t), V_4^+(t), V_4^-(t))$ on N which represents, at $t = 0$, the tangent vector $\widetilde{F}_*(\boldsymbol{v})$ at $\widetilde{F}(x)$. Then $f_1'(0) \in V_4^+(0) \cap V_4^-(0)$. The vector $f_1'(0)$ corresponds to $F_*(v)$. Therefore

$$F_*(\boldsymbol{v}) \in T_{F(x)}(P(V_4^+(0) \cap V_4^-(0))) \subset T_{F(x)}(P(V_1(0))^{\perp} \cap Q_0) = C_{F(x)},$$

and F is a null surface. By triality we have the same result also null frontals in $Q_\pm$. $\square$

Proof of Proposition 7.4:
Let $f : I \to Z$, $f(t) = (V_1(t), V_2(t), V_4^+(t), V_4^-(t))$ be an E-integral curve. Take a frame $f_1(t)$ of $V_1(t)$, $f_1(t), f_2(t)$ of $V_2(t)$, $f_1(t), f_2(t), f_3(t), f_4(t)$ of $V_4^+(t)$ and $f_1(t), f_2(t), f_3(t), f_5(t)$ of $V_4^-(t)$. Then the curve $\gamma_0(t)$ is defined by the family $V_1(t)$.

Consider, for each $t \in I$, $V_1(t,s) = f_1(t) + sf_2(t)$, which can be regarded a projective line. By the condition $f_1'(t) \in V_2(t)$, $V_1(t,s)$ gives the tangent line to γ at t, even when $f_1(t), f_1'(t)$ are linearly dependent. Then $F_0 = \mathrm{Tan}(\gamma_0(t))$ is given by $F_0(t,s) = V_1(t,s)$ and s is the parameter of tangent lines. We define the lift $\widetilde{F_0}$ of F_0 to N by

$$\widetilde{F_0}(t,s) := (V_1(t,s), V_4^+(t), V_4^-(t)).$$

We have that

$$\frac{\partial}{\partial t}(f_1(t) + sf_2(t)) = f_1'(t) + sf_2'(t) \in V_4^+(t) \cap V_4^-(t),$$
$$\frac{\partial}{\partial s}(f_1(t) + sf_2(t)) = f_2(t) \in V_2(t) \subset V_4^+(t) \cap V_4^-(t),$$

and that $\dfrac{\partial}{\partial t}f_3(t) \in (V_4^+(t) \cap V_4^-(t))^\perp$, $\dfrac{\partial}{\partial s}f_3(t) = 0$. Thus we have that $\widetilde{F_0}$ is D_N-integral by Lemma 7.6. Therefore we have that F_0 is a null frontal. Moreover $(\pi_+ \circ \widetilde{F_0})(t,s) = V_4^+(t)$ and $(\pi_- \circ \widetilde{F_0})(t,s) = V_4^-(t)$ do not depend on s.

By the triality, we have the results also for $F_+ = \mathrm{Tan}(\gamma_+(t))$ and $F_- = \mathrm{Tan}(\gamma_-(t))$.

In fact, under the diffeomorphism

$$\Phi : N \to N', \Phi(V_1, V_4^+, V_4^-) = (V_1, V_4^+ \cap V_4^-),$$

$\Phi \circ \widetilde{F_+} : I \to N'$ is given by

$$\Phi \circ \widetilde{F_+}(t) = (V_1(t), V_3(t,s)),\ V_3(t,s) := \langle f_1(t), f_2(t), f_3(t) + sf_5(t)\rangle_{\mathbf{R}},\ (t,s) \in I \times \mathbf{R},$$

and $\Phi \circ \widetilde{F_-} : I \to N'$ is given by

$$\Phi \circ \widetilde{F_-}(t) = (V_1(t), V_3(t,s)),\ V_3(t,s) := \langle f_1(t), f_2(t), f_3(t) + sf_4(t)\rangle_{\mathbf{R}},\ (t,s) \in I \times \mathbf{R}.$$

If we arrange to take an adapted frame $\boldsymbol{f} : I \to O(4,4)$,

$$\boldsymbol{f}(t) = (f_1(t), f_2(t), f_3(t), f_4(t), f_5(t), f_6(t), f_7(t), f_8(t)),$$

for the Engel integral curve $f : I \to Z$ (see §5), then we may write

$$\widetilde{F_+}(t,s) = (V_1(t), V_4^+(t,s), V_4^-(t)), V_4^+(t,s) := \langle f_1(t), f_2(t), f_3(t) + sf_5(t), f_3(t) - sf_6(t)\rangle_{\mathbf{R}},$$

and

$$\widetilde{F_-}(t,s) = (V_1(t), V_4^+(t), V_4^-(t,s)), V_4^-(t,s) := \langle f_1(t), f_2(t), f_3(t) + sf_4(t), f_3(t) - sf_6(t)\rangle_{\mathbf{R}},$$

for any $(t,s) \in I \times \mathbf{R}$. Therefore F_+ (resp. F_-) has a D_N-integral lift $\widetilde{F_+}$ (resp. $\widetilde{F_-}$) such that $\pi_- \circ \widetilde{F_+}$ and $\pi_0 \circ \widetilde{F_+}$ (resp. $\pi_0 \circ \widetilde{F_-}$ and $\pi_+ \circ \widetilde{F_-}$) do not depend on s $\square$

Let us describe D_N in coordinates. By Lemma 7.6, we pose the condition on a frame

$$\boldsymbol{f}(t) = (f_1(t), f_2(t), f_3(t), f_4(t), f_5(t), f_6(t), f_7(t), f_8(t))$$

such that

$$f_1'(0) \in \langle f_1(0), f_2(0), f_3(0)\rangle_{\mathbf{R}}, f_2'(0) \in \langle f_1(0), f_2(0), f_3(0), f_4(0), f_5(0)\rangle_{\mathbf{R}},$$
$$f_3'(0) \in \langle f_1(0), f_2(0), f_3(0), f_4(0), f_5(0)\rangle_{\mathbf{R}}.$$

Then there exist $p_i, q_i \in \mathbf{R}, i = 1, 2, 3$ such that

$$f_1'(0) = p_1 f_2(0) + q_1 f_3(0), \quad f_2'(0) = p_2 f_4(0) + q_2 f_5(0), \quad f_3'(0) = p_3 f_4(0) + q_3 f_5(0).$$

Then we have the differential system $D_{N'}$ on N' of rank 6:

$$\begin{cases} dx_{41} - x_{42}dx_{21} - x_{43}dx_{31} = 0, \\ dx_{51} - x_{52}dx_{21} - x_{53}dx_{31} = 0, \\ dx_{61} - x_{62}dx_{21} + x_{43}x_{53}dx_{31} = 0, \\ dx_{71} + x_{42}x_{52}dx_{21} + (x_{42}x_{53} + x_{43}x_{52} + x_{62})dx_{31} = 0, \\ dx_{62} + x_{53}dx_{42} + x_{43}dx_{52} = 0. \end{cases}$$

The integrability condition is given by

$$\begin{cases} dx_{42} \wedge dx_{21} + dx_{43} \wedge dx_{31} = 0, \\ dx_{52} \wedge dx_{21} + dx_{53} \wedge dx_{31} = 0, \\ dx_{53} \wedge dx_{42} + dx_{43} \wedge dx_{52} = 0. \end{cases}$$

By replacing x_{43}, x_{53} by $-x_{65}, -x_{64}$, we have the integrability condition for D_N:

$$\begin{cases} dx_{42} \wedge dx_{21} - dx_{65} \wedge dx_{31} = 0, \\ dx_{52} \wedge dx_{21} - dx_{64} \wedge dx_{31} = 0, \\ dx_{64} \wedge dx_{42} + dx_{65} \wedge dx_{52} = 0. \end{cases}$$

Thus we observe that the problem on the local construction of D_N-integral surfaces and null frontals is reduced to the construction of isotropic surface-germs for a kind of "tri-symplectic" structure on $\mathbf{R}^6$ as above.

Moreover we observe that, by Proposition 7.4, the tangent surfaces of π_0-projections of E-integral curves satisfy, in addition to the above system,

$$dx_{42} \wedge dx_{65} = 0, \quad dx_{52} \wedge dx_{64} = 0.$$

To make the situation clear, we consider $\mathbf{R}^6$ with coordinates $x_1, x_2, x_3, x_4, x_5, x_6$ with three 2-forms:

$$\begin{cases} \omega_1 = dx_3 \wedge dx_1 + dx_4 \wedge dx_2, \\ \omega_2 = dx_5 \wedge dx_1 + dx_6 \wedge dx_2, \\ \omega_3 = dx_6 \wedge dx_3 + dx_4 \wedge dx_5. \end{cases}$$

Let us consider an integral surface of the differential system $\omega_1 = \omega_2 = \omega_3 = 0$ which projects to (x_1, x_2) regularly. Then, from $\omega_1 = \omega_2 = 0$, it is written locally

$$x_3 = \frac{\partial f}{\partial x_1}, \; x_4 = \frac{\partial f}{\partial x_2}, \; x_5 = \frac{\partial g}{\partial x_1}, \; x_6 = \frac{\partial g}{\partial x_2}$$

for some functions $f = f(x_1, x_2), g = g(x_1, x_2)$. Then from $\omega_3 = 0$, we have the second order bilinear partial differential equation on $f = f(x_1, x_2), g = g(x_1, x_2)$,

$$\frac{\partial^2 f}{\partial x_1^2}\frac{\partial^2 g}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_2^2}\frac{\partial^2 g}{\partial x_1^2} - 2\frac{\partial^2 f}{\partial x_1 \partial x_2}\frac{\partial^2 g}{\partial x_1 \partial x_2} = 0.$$

This equation is regarded as an orthogonality condition of Lagrange-Gauss mapping of two Lagrange immersions defined by f and g.

Remark 7.7. Similarly to above, the calculations in B_3 geometry, namely geometry of $O(3,4)$, lead us to the differential system

$$\omega_1 = dx_3 \wedge dx_1 + dx_4 \wedge dx_2 = 0, \quad \omega_2 = dx_3 \wedge dx_4 = 0,$$

on $\mathbf{R}^4$ with coordinates x_1, x_2, x_3, x_4, which is expressed as the Monge-Ampère equation

$$\frac{\partial^2 f}{\partial x_1^2}\frac{\partial^2 f}{\partial x_2^2} - \left(\frac{\partial^2 f}{\partial x_1 \partial x_2}\right)^2 = 0$$

on "developable surfaces" (see [17][13]). We observe that the Monge-Ampère equation is obtained by the reduction $g = f$ or $x_5 = x_3, x_6 = x_4$ from the D_4 case to the B_3 case. See also [16] for relations of D_4-geometry and B_3-geometry.

Returning to D_4 case, consider the differential system on $\mathbf{R}^6$,

$$\omega_1 = 0,\ \omega_2 = 0,\ \omega_3 = 0,\ \Omega_1 := dx_3 \wedge dx_4 = 0,\ \Omega_2 := dx_5 \wedge dx_6 = 0,$$

which we call a "bi-Monge-Ampère system". Then the differential system is expressed by the system of equations

$$\frac{\partial^2 f}{\partial x_1^2}\frac{\partial^2 g}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_2^2}\frac{\partial^2 g}{\partial x_1^2} - 2\frac{\partial^2 f}{\partial x_1 \partial x_2}\frac{\partial^2 g}{\partial x_1 \partial x_2} = 0,$$

$$\frac{\partial^2 f}{\partial x_1^2}\frac{\partial^2 f}{\partial x_2^2} - \left(\frac{\partial^2 f}{\partial x_1 \partial x_2}\right)^2 = 0, \quad \frac{\partial^2 g}{\partial x_1^2}\frac{\partial^2 g}{\partial x_2^2} - \left(\frac{\partial^2 g}{\partial x_1 \partial x_2}\right)^2 = 0.$$

We conclude that the tangent surface construction in D_4-geometry offers geometric solutions with singularities of the above bi-Monge-Ampère system of equations.

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