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FREE DIVISORS IN A PENCIL OF CURVES

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Abstract. A plane curve $D \subset \mathbb{P}^2(\boldsymbol{k})$, where $\boldsymbol{k}$ is a field of characteristic zero, is free if its associated sheaf $\mathcal{T}_D$ of vector fields tangent to D is a free $\mathscr{O}_{\mathbb{P}^2(\boldsymbol{k})}$-module (see [6] or [5] for a definition in a more general context). Relatively few free curves are known. Here we prove that the union of all singular members of a pencil of plane projective curves with the same degree and with a smooth base locus is a free divisor.

1. Introduction

Let $\boldsymbol{k}$ be a field of characteristic zero and let $S = \boldsymbol{k}[x, y, z]$ be the graded ring such that $\mathbb{P}^2 = \mathrm{Proj}(S)$. We write $\partial_x := \frac{\partial}{\partial x}$, $\partial_y := \frac{\partial}{\partial y}$, $\partial_z := \frac{\partial}{\partial z}$ and $\nabla F = (\partial_x F, \partial_y F, \partial_z F)$ for a homogenous polynomial $F \in S$.

Let $D = \{F = 0\}$ be a reduced curve of degree n. The kernel $\mathcal{T}_D$ of the map ∇F is a rank two reflexive sheaf, hence a vector bundle on $\mathbb{P}^2$. It is the rank two vector bundle of vector fields tangent along D, defined by the following exact sequence:

$$0 \longrightarrow \mathcal{T}_D \longrightarrow \mathscr{O}^3_{\mathbb{P}^2} \xrightarrow{\nabla F} \mathcal{J}_{\nabla F}(n-1) \longrightarrow 0,$$

where the sheaf $\mathcal{J}_{\nabla F}$ (also denoted $\mathcal{J}_{\nabla D}$ in this text) is the Jacobian ideal of F. Set theoretically $\mathcal{J}_{\nabla F}$ defines the singular points of the divisor D. For instance if D consists of s generic lines then $\mathcal{J}_{\nabla D}$ defines the set of $\binom{s}{2}$ vertices of D.

Remark 1.1. A non zero section $s \in \mathrm{H}^0(\mathcal{T}_D(a))$, for some shift $a \in \mathbb{N}$, corresponds to a derivation $\delta = P_a\partial_x + Q_a\partial_y + R_a\partial_z$ verifying $\delta(F) = 0$, where $(P_a, Q_a, R_a) \in \mathrm{H}^0(\mathscr{O}_{\mathbb{P}^2}(a))^3$.

In some particular cases that can be found in [5], $\mathcal{T}_D$ is a free $\mathscr{O}_{\mathbb{P}^2}$-module; it means that there are two vector fields of degrees a and b that form a basis of $\bigoplus_n \mathrm{H}^0(\mathcal{T}_D(n))$ (D is said to be free with exponents (a, b)); it arises, for instance, when D is the union of the nine inflection lines of a smooth cubic curve. The notion of free divisor was introduced by Saito [6] for reduced divisors and studied by Terao [9] for hyperplane arrangements. Here we recall a definition of freeness for projective curves. For a more general definition we refer to Saito [6].

Definition 1.2. A reduced curve $D \subset \mathbb{P}^2$ is *free with exponents* $(a, b) \in \mathbb{N}^2$ if

$$\mathcal{T}_D \simeq \mathscr{O}_{\mathbb{P}^2}(-a) \oplus \mathscr{O}_{\mathbb{P}^2}(-b).$$

A smooth curve of degree ≥ 2 is not free, an irreducible curve of degree ≥ 3 with only nodes and cusps as singularities is not free (see [1, Example 4.5]). Actually few examples of free curves are known and of course very few families of free curves are known. One such family can be found in [8, Prop. 2.2].

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In a personal communication it was conjectured by E. Artal and J.I. Cogolludo that the union of all the singular members of a pencil of plane curves (assuming that the general one is smooth) should be free. Three different cases occur:

- the base locus is smooth (for instance the union of six lines in a pencil of conics passing through four distinct points);
- the base locus is not smooth but every curve in the pencil is reduced (for instance the four lines in a pencil of conics where two of the base points are infinitely near points);
- the base locus is not smooth and there exists exactly one non reduced curve in the pencil (for instance three lines in a pencil of bitangent conics).

In the third case the divisor of singular members is not reduced but its reduced structure is expected to be free.

We point out that if two distinct curves of the pencil are not reduced then all curves will be singular. Even in this case, we believe that a free divisor can be obtained by chosing a finite number of reduced components through all the singular points.

In this paper we prove that the union of all the singular members of a pencil of degree n plane curves with a smooth base locus (i.e. the base locus consists of n^2 distinct points) is a free divisor and we give its exponents (see theorem 2.7). More generally, we describe the vector bundle of logarithmic vector fields tangent to any union of curves of the pencil (see theorem 2.8) by studying one particular vector field "canonically tangent" to the pencil, that is introduced in the key lemma 2.1.

This gives already a new and easy method to produce free divisors.

I thank J. I. Cogolludo for his useful comments.

2. Pencil of plane curves

2.1. **Generalities and notations.** Let $\{f=0\}$ and $\{g=0\}$ be two reduced curves of degree $n\geq 1$ with no common component. For any $(\alpha,\beta)\in\mathbb{P}^1$ the curve $C_{\alpha,\beta}$ is defined by the equation $\{\alpha f+\beta g=0\}$ and $\mathcal{C}(f,g)=\{C_{\alpha,\beta}|(\alpha,\beta)\in\mathbb{P}^1\}$ is the pencil of all these curves.

In section 2 we will assume that the general member of the pencil $\mathcal{C}(f,g)$ is a smooth curve and that $C_{\alpha,\beta}$ is reduced for every $(\alpha,\beta)\in\mathbb{P}^1$.

Under these assumptions there are finitely many singular curves in $\mathcal{C}(f,g)$ but also finitely many singular points. We recall that the degree of the discriminant variety of degree n curves is $3(n-1)^2$ (it is a particular case of the Boole formula; see [10, Example 6.4]). Since the general curve in the pencil is smooth, the line defined by the pencil $\mathcal{C}(f,g)$ in the space of degree n curves meets the discriminant variety along a finite scheme of length $3(n-1)^2$ (not empty for $n\geq 2$). The number of singular points is of course related to the multiplicity of the singular curves in the pencil as we will see below.

Let us fix some notation. The scheme defined by the ideal sheaf $\mathcal{J}_{\nabla C_{\alpha_i,\beta_i}}$ is denoted by Z_{α_i,β_i}. It is well known that this scheme is locally a complete intersection (for instance, generalize to curves the lemma 2.4 in [7]). The union of all the singular members of the pencil $\mathcal{C}(f,g)$ form a divisor D^{sg}. A union of $k\geq 2$ distinct members of $\mathcal{C}(f,g)$ is denoted by D_k.

2.2. **Derivation tangent to a smooth pencil.** Let us consider the following derivation, associated "canonically" to the pencil:

Lemma 2.1. *For any union D_k of $k\geq 1$ members of the pencil there exists a non zero section $s_{\delta,k}\in \mathrm{H}^0(\mathcal{T}_{D_k}(2n-2))$ induced by the derivation*

$$\delta=(\nabla f\wedge\nabla g).\nabla=(\partial_y f\partial_z g-\partial_z f\partial_y g)\partial_x+(\partial_z f\partial_x g-\partial_x f\partial_z g)\partial_y+(\partial_x f\partial_y g-\partial_y f\partial_x g)\partial_z.$$

Proof. Since $\delta(\alpha f+\beta g)=\det(\nabla f,\nabla g,\nabla(\alpha f+\beta g))=0$ we have for any $k\geq 1$,

$$\delta(f)=\delta(g)=\delta(\alpha f+\beta g)=\delta(\prod_{i=1}^{k}(\alpha_i f+\beta_i g))=0.$$

According to the remark 1.1 it gives the desired section. □

Let us introduce a rank two sheaf $\mathcal{F}$ defined by the following exact sequence:

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^2}(2-2n) \xrightarrow{\nabla f\wedge\nabla g} \mathscr{O}_{\mathbb{P}^2}^3 \longrightarrow \mathcal{F} \longrightarrow 0.$$

If we denote by $\mathrm{sg}(\mathcal{F}):=\{p\in\mathbb{P}^2|\mathrm{rank}(\mathcal{F}\otimes\mathscr{O}_p)>2\}$ the set of singular points of $\mathcal{F}$, we have:

Lemma 2.2. *A point $p\in\mathbb{P}^2$ belongs to $\mathrm{sg}(\mathcal{F})$ if and only if two smooth members of the pencil share the same tangent line at p or one curve of the pencil is singular at p. Moreover $\mathrm{sg}(\mathcal{F})$ is a finite closed scheme with length $l(\mathrm{sg}(\mathcal{F}))=3(n-1)^2$.*

Remark 2.3. Let us precise that if two smooth members intersect then all the smooth members of the pencil intersect with the same tangency.

Remark 2.4. If the base locus of $\mathcal{C}(f,g)$ consists of n^2 distinct points then two curves of the pencil meet transversaly at the base points and $p\in\mathrm{sg}(\mathcal{F})$ if and only if p is a singular point for a unique curve $C_{\alpha,\beta}$ in the pencil and does not belong to the base locus. One can assume that $p\in\mathrm{sg}(\mathcal{F})$ is singular for $\{f=0\}$. Then, locally at p, the curve $\{g=0\}$ can be assumed to be smooth and the local ideals $(\nabla f\wedge\nabla g)_p$ and $(\nabla f)_p$ coincide. In other words, when the base locus is smooth, we have

$$\mathrm{sg}(\mathcal{F})=\sqcup_{i=1,\dots,s}Z_{\alpha_i,\beta_i}.$$

Proof. The singular locus of $\mathcal{F}$ is also defined by $\mathrm{sg}(\mathcal{F}):=\{p\in\mathbb{P}^2|(\nabla f\wedge\nabla g)(p)=0\}$. The zero scheme defined by $\nabla f\wedge\nabla g$ and $\nabla(\alpha f+\beta g)\wedge\nabla g$ are clearly the same; it means that the singular points of any member in the pencil is a singular point for $\mathcal{F}$. One can also obtain $(\nabla f\wedge\nabla g)(p)=0$ at a smooth point when the vectors $(\nabla f)(p)$ and $(\nabla g)(p)$ are proportional i.e. when two smooth curves of the pencil share the same tangent line at p.

Since every curve of the pencil is reduced $\mathrm{sg}(\mathcal{F})$ is finite, its length can be computed by writing the resolution of the ideal $\mathcal{J}_{\mathrm{sg}(\mathcal{F})}$ (for a sheaf of ideal $\mathcal{J}_Z$ defining a finite scheme Z of length $l(Z)$, we have $c_2(\mathcal{J}_Z)=l(Z)$). Indeed, if we dualize the following exact sequence

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^2}(2-2n) \xrightarrow{\nabla f\wedge\nabla g} \mathscr{O}_{\mathbb{P}^2}^3 \longrightarrow \mathcal{F} \longrightarrow 0$$

we find, according to Hilbert-Burch theorem,

$$0 \longrightarrow \mathcal{F}^\vee \xrightarrow{(\nabla f,\nabla g)} \mathscr{O}_{\mathbb{P}^2}^3 \xrightarrow{\nabla f\wedge\nabla g} \mathscr{O}_{\mathbb{P}^2}(2n-2).$$

It proves that $\mathcal{F}^\vee=\mathscr{O}_{\mathbb{P}^2}(1-n)^2$ and that the image of the last map is $\mathcal{J}_{\mathrm{sg}(\mathcal{F})}(2n-2)$.

Then $l(\mathrm{sg}(\mathcal{F}))=3(n-1)^2$. We point out that this number is the degree of the discriminant variety of degree n curves. □

Now let us call D_k the divisor defined by $k\geq 2$ members of the pencil and let us consider the section $s_{\delta,k}\in \mathrm{H}^0(\mathcal{T}_{D_k}(2n-2))$ corresponding (see remark 1.1) to the derivation δ. Let $Z_k:=Z(s_{\delta,k})$ be the zero locus of $s_{\delta,k}$.

Lemma 2.5. *The section $s_{\delta,k}$ vanishes in codimension at least two.*

Proof. Let us consider the following commutative diagram

$$\begin{tikzcd} & & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & \mathscr{O}_{\mathbb{P}^2}(2-2n) \arrow[r, "s_{\delta,k}"] \arrow[d, equal] & \mathcal{T}_{D_k} \arrow[r] \arrow[d] & \mathcal{Q} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathscr{O}_{\mathbb{P}^2}(2-2n) \arrow[r, "\nabla f \wedge \nabla g"] & \mathscr{O}_{\mathbb{P}^2}^3 \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & 0 \\ & & \mathcal{J}_{\nabla D_k}(nk-1) \arrow[r, equal] \arrow[d] & \mathcal{J}_{\nabla D_k}(nk-1) \arrow[d] & \\ & & 0 & 0 & \end{tikzcd}$$

where $\mathcal{Q} = \operatorname{coker}(s_{\delta,k})$. Assume that Z_k contains a divisor H. Tensor now the last vertical exact sequence of the above diagram by $\mathscr{O}_p$ for a general point $p \in H$. Since p does not belong to the Jacobian scheme defined by $\mathcal{J}_{\nabla D_k}$ we have $\mathcal{J}_{\nabla D_k} \otimes \mathscr{O}_p = \mathscr{O}_p$ and $\operatorname{Tor}_1(\mathcal{J}_{\nabla D_k}, \mathscr{O}_p) = 0$. Since $p \in H \subset Z_k$ we have $\operatorname{rank}(\mathcal{Q} \otimes \mathscr{O}_p) \geq 2$; it implies $\operatorname{rank}(\mathcal{F} \otimes \mathscr{O}_p) \geq 3$ in other words that $p \in \operatorname{sg}(\mathcal{F})$; this contradicts $\operatorname{codim}(\operatorname{sg}(\mathcal{F}), \mathbb{P}^2) = 2$, proved in lemma 2.2.

Then $\mathcal{Q}$ is the ideal sheaf of the codimension two scheme Z_k, i.e. $\mathcal{Q} = \mathcal{J}_{Z_k}(n(2-k)-1)$ and we have an exact sequence

$$0 \longrightarrow \mathcal{J}_{Z_k}(n(2-k)-1) \longrightarrow \mathcal{F} \longrightarrow \mathcal{J}_{\nabla D_k}(nk-1) \longrightarrow 0.$$

□

From this commutative diagram we obtain the following lemma.

Lemma 2.6. *Let D_k be a union of $k \geq 2$ members of $\mathcal{C}(f,g)$. Then*

$$c_2(\mathcal{J}_{\nabla D_k}) + c_2(\mathcal{J}_{Z_k}) = 3(n-1)^2 + n^2(k-1)^2.$$

Proof. According to the above commutative diagram we compute $c_2(\mathcal{F})$ in two different ways. The horizontal exact sequence gives $c_2(\mathcal{F}) = 4(n-1)^2$ when the vertical one gives

$$c_2(\mathcal{F}) = c_2(\mathcal{J}_{\nabla D_k}) + c_2(\mathcal{J}_{Z_k}) + (n-1)^2 - n^2(k-1)^2.$$

The lemma is proved by eliminating $c_2(\mathcal{F})$. □

2.3. **Free divisors in the pencil.** When D_k contains the divisor D^{sg} of all the singular members of the pencil we show now that it is free with exponents $(2n-2, n(k-2)+1)$.

Theorem 2.7. *Assume that the base locus of the pencil $\mathcal{C}(f,g)$ is smooth. Then,*

$$D_k \supseteq D^{\mathrm{sg}} \Leftrightarrow \mathcal{T}_{D_k} = \mathscr{O}_{\mathbb{P}^2}(2-2n) \oplus \mathscr{O}_{\mathbb{P}^2}(n(2-k)-1).$$

Proof. Assume first that $D_k \supseteq D^{\mathrm{sg}}$. Then the singular locus of D_k defined by the Jacobian ideal consists of the base points of the k curves in the pencil and, since it contains all the singular members, of the whole set of singularities of the curves in the pencil. This last set has length $3(n-1)^2$ by lemma 2.2 and remark 2.4. Moreover the subscheme supported by the base points in the scheme defined by $\mathcal{J}_{\nabla D_k}$ has length $n^2(k-1)^2$ since at each point among the n^2 points of the base locus of the pencil, the k curves meet transversaly and define k different directions (i.e. the local ring at the point $(0,0,1)$ is isomorphic to $\boldsymbol{k}[x,y]/(x^{k-1},y^{k-1})$). Then

$$c_2(\mathcal{J}_{\nabla D_k}) = 3(n-1)^2 + n^2(k-1)^2,$$

which implies that $Z_k = \emptyset$. In other words there is an exact sequence

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^2}(2-2n) \xrightarrow{s_{\delta,k}} \mathcal{T}_{D_k} \longrightarrow \mathscr{O}_{\mathbb{P}^2}(n(2-k)-1) \longrightarrow 0.$$

And such an exact sequence splits.

Conversely, assume that there is a singular member C that does not belong to D_k. Let $p \in C$ be one of its singular point. Since $p \notin D_k$ we have $\mathcal{J}_{\nabla D_k} \otimes \mathscr{O}_p = \mathscr{O}_p$ and $\mathrm{Tor}_1(\mathcal{J}_{\nabla D_k}, \mathscr{O}_p) = 0$. Consider the following exact sequence that comes from the commutative diagram above (in the proof of lemma 2.5) when $Z_k = \emptyset$:

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^2}(n(2-k)-1) \longrightarrow \mathcal{F} \longrightarrow \mathcal{J}_{\nabla D_k}(nk-1) \longrightarrow 0.$$

If we tensor this exact sequence by $\mathscr{O}_p$ we find $\mathrm{rank}(\mathcal{F} \otimes \mathscr{O}_p) = 2$. This contradicts $p \in \mathrm{sg}(\mathcal{F})$ that was proved in lemma 2.2. $\square$

2.4. **Singular members ommitted.** When $D_k \supset D^{\mathrm{sg}}$ we have seen in theorem 2.7 that $Z_k = \emptyset$ by computing the length of the scheme defined by the Jacobian ideal of D_k. More generally we can describe, at least when the base locus is smooth, the scheme Z_k for any union of curves of the pencil.

Theorem 2.8. *Assume that the base locus of the pencil $\mathcal{C}(f,g)$ is smooth. Assume also that $D_k \supset D^{\mathrm{sg}} \setminus \bigcup_{i=1,\dots,r} C_{\alpha_i,\beta_i}$, $C_{\alpha_i,\beta_i} \not\subseteq D_k$ and C_{α_i,β_i} is a singular curve for $i=1,\dots,r$. Then,*

$$\mathcal{J}_{Z_k} = \mathcal{J}_{\nabla C_{\alpha_1,\beta_1}} \otimes \cdots \otimes \mathcal{J}_{\nabla C_{\alpha_r,\beta_r}}.$$

Remark 2.9. When $r=0$ we obtain the freeness again.

Proof. Since the set of singular points of two disctinct curves are disjoint, it is enough to prove it for $r=1$ (i.e. $D_k \supset D^{\mathrm{sg}} \setminus C_{\alpha_1,\beta_1}$ and $C_{\alpha_1,\beta_1} \not\subseteq D_k$). Recall that $\mathcal{E}xt^1(\mathcal{J}_Z, \mathscr{O}_{\mathbb{P}^2}) = \omega_Z$ where Z is a finite scheme and ω_Z its dualizing sheaf (see [2, Chapter III, section 7]); it is well known that, since the finite scheme Z is locally complete intersection, $\omega_Z = \mathscr{O}_Z$.

Then the dual exact sequence of

$$0 \longrightarrow \mathcal{J}_{Z_k}(n(2-k)-1) \longrightarrow \mathcal{F} \longrightarrow \mathcal{J}_{\nabla D_k}(nk-1) \longrightarrow 0$$

is the long exact sequence

$$\begin{aligned} 0 \longrightarrow \mathscr{O}_{\mathbb{P}^2}(1-nk) \longrightarrow \mathscr{O}_{\mathbb{P}^2}(1-n)^2 \longrightarrow \mathscr{O}_{\mathbb{P}^2}(n(k-2)+1) \longrightarrow \\ \longrightarrow \omega_{\nabla D_k} \longrightarrow \mathscr{O}_{\mathrm{sg}(\mathcal{F})} \longrightarrow \omega_{Z_k} \longrightarrow 0. \end{aligned}$$

The map $\mathcal{F} \longrightarrow \mathcal{J}_{\nabla D_k}(nk-1)$ can be described by composition; indeed it is given by two polynomials (U,V) such that

$$(U,V).(\nabla f, \nabla g) = \nabla(\prod_i (\alpha_i f + \beta_i g)).$$

We find, $U = \sum_i \alpha_i \prod_{j\neq i}(\alpha_j f + \beta_j g)$ and $V = \sum_i \beta_i \prod_{j \neq i}(\alpha_j f + \beta_j g)$. If a point p belongs to one curve C_{α_1,β_1} in D_k and does not belong to the base locus, then $U(p) \neq 0$ and $V(p) \neq 0$. It shows that these two polynomials vanish simultaneously and precisely along the base locus. Then the complete intersection $T = \{U=0\} \cap \{V=0\}$ of length $n^2(k-1)^2$ is supported exactly by the base points. We have

$$0 \longrightarrow \omega_{\nabla D_k}/\mathscr{O}_T \longrightarrow \mathscr{O}_{\mathrm{sg}(\mathcal{F})} \longrightarrow \omega_{Z_k} \longrightarrow 0.$$

We have already seen that the subscheme supported by the base points in the scheme defined by $\mathcal{J}_{\nabla D_k}$ has length $n^2(k-1)^2$. It implies that $\omega_{\nabla D_k}/\mathcal{O}_T = \oplus_{i=2,\dots,s}\mathcal{O}_{Z_{\alpha_i,\beta_i}}$. According to remark 2.4, $\mathcal{O}_{\mathrm{sg}(\mathcal{F})} = \oplus_{i=1,\dots,s}\mathcal{O}_{Z_{\alpha_i,\beta_i}}$. This proves $\omega_{Z_k} = \mathcal{O}_{Z_{\alpha_1,\beta_1}}$.

$\square$

There are exact sequences relating the vector bundles $\mathcal{T}_{D_k}$ and $\mathcal{T}_{D_k\setminus C}$ when $C \subset D_k$.

Proposition 2.10. *We assume that the base locus of the pencil $\mathcal{C}(f,g)$ is smooth and that D_k contains D^{sg}. Let C be a singular member in $\mathcal{C}(f,g)$ and Z its scheme of singular points. Then there is an exact sequence*

$$0 \longrightarrow \mathcal{T}_{D_k} \longrightarrow \mathcal{T}_{D_k\setminus C} \longrightarrow \mathcal{J}_{Z/C}(n(3-k)-1) \longrightarrow 0,$$

where $\mathcal{J}_{Z/C} \subset \mathcal{O}_C$ defines Z into C.

Proof. The derivation $(\nabla f \wedge \nabla g).\nabla$ is tangent to D_k then also to $D_k \setminus C$. It induces the following commutative diagram which proves the proposition:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}(2-2n) & \longrightarrow & \mathcal{T}_{D_k} & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}(n(2-k)-1) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}(2-2n) & \longrightarrow & \mathcal{T}_{D_k\setminus C} & \longrightarrow & \mathcal{J}_Z(n(3-k)-1) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{J}_{Z/C}(n(3-k)-1) & = & \mathcal{J}_{Z/C}(n(3-k)-1) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

$\square$

3. The pencil contains a non-reduced curve

When the pencil $\mathcal{C}(f,g)$ contains a non-reduced curve, the arguments used in the previous sections are not valid since the scheme defined by the jacobian ideal contains a divisor. We have to remove this divisor somehow. Remember that if two curves of the pencil are multiple then the general curve is singular. So let us consider that there is only one curve that is not reduced. Let $hh_1^{r_1}\cdots h_s^{r_s} = 0$ be the equation of this unique non-reduced curve where $h=0$ is reduced, $\deg(h_i) = m_i \geq 1$ and $r_i \geq 2$. Since the derivation $\frac{1}{\prod_i h_i^{r_i}}(\nabla f \wedge \nabla g).\nabla$ is still tangent to all curves of the pencil, we believe that the following statement is true:

Conjecture. *Let $hh_1^{r_1}\cdots h_s^{r_s} = 0$ be the equation of the unique non-reduced curve where $\{h=0\}$ is reduced, $\deg(h_i) = m_i \geq 1$ and $r_i \geq 2$. Then,*

$$D_k \supseteq D^{\mathrm{sg}} \Leftrightarrow \mathcal{T}_{D_k} = \mathcal{O}_{\mathbb{P}^2}(2-2n+\sum_{i=1}^{i=s}(r_i-1)m_i) \oplus \mathcal{O}_{\mathbb{P}^2}(n(2-k)-1).$$

4. Examples

Let us call $\Sigma_3 \subset \mathbb{P}^9 = \mathbb{P}(\mathrm{H}^0(\mathscr{O}_{\mathbb{P}^2}(3)))$ the hypersurface of singular cubics. It is well known that its degree is 12 (see [3], for instance).

- **Pappus arrangement freed by nodal cubics**: Let us consider the divisor of the nine lines appearing in the Pappus arrangement; this divisor is the union of three triangles T_1, T_2, T_3 with nine base points. The pencil generated by T_1 and T_2 contains 3 triangles (each one represents a triple point in Σ_3); since $9 < 12$, singular cubics are missing in the pencil. There is no other triangle and no smoth conic+line in the Pappus pencil, when it is general enough. We can conclude that the missing cubics are, in general, nodal cubics C_1, C_2, C_3.

  Let $D = T_1 \cup T_2 \cup T_3 \cup C_1 \cup C_2 \cup C_3$ be the union of all singular fibers in the pencil generated by T_1 and T_2. Then, according to theorem 2.7 we have
  $$\mathcal{T}_D = \mathscr{O}_{\mathbb{P}^2}(-4) \oplus \mathscr{O}_{\mathbb{P}^2}(-13).$$

- **Pappus arrangement**: Let $T_1 \cup T_2 \cup T_3$ be the divisor consisting of the nine lines of the projective Pappus arrangement and $D = T_1 \cup T_2 \cup T_3 \cup C_1 \cup C_2 \cup C_3$ be the union of all singular fibers in the pencil generated by two triangles among the T_i's. Let us call $K := C_1 \cup C_2 \cup C_3$ the union of the nodal cubics and z_1, z_2, z_3 their nodes. Then we have, according to theorem 2.8, $\mathcal{T}_D = \mathscr{O}_{\mathbb{P}^2}(-4) \oplus \mathscr{O}_{\mathbb{P}^2}(-13)$ and an exact sequence
  $$0 \longrightarrow \mathscr{O}_{\mathbb{P}^2}(-4) \longrightarrow \mathcal{T}_{D\setminus K} \longrightarrow \mathcal{I}_{z_1,z_2,z_3}(-4) \longrightarrow 0.$$
  The logarithmic bundle $\mathcal{T}_{D\setminus K}$ associated to the Pappus configuration is semi-stable and its divisor of jumping lines is the triangle $z_1^\vee \cup z_2^\vee \cup z_3^\vee$ as it is proved by retricting the above exact sequence to any line through one of the zeroes.

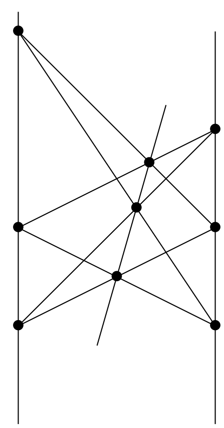

Figure 1. Pappus arrangement

- **Hesse arrangement**: Let us consider the pencil generated by a smooth cubic C and its hessian $\mathrm{Hess}(C)$. The pencil contains 4 triangles T_1, T_2, T_3, T_4 and since the degree of Σ_3 is 12, no other singular cubic can be present. Let us call D the union of these four triangles. Then, according to theorem 2.7 we have
  $$\mathcal{T}_D = \mathscr{O}_{\mathbb{P}^2}(-4) \oplus \mathscr{O}_{\mathbb{P}^2}(-7).$$

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