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FAMILIES OF DISTRIBUTIONS AND PFAFF SYSTEMS UNDER DUALITY

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Abstract. A singular distribution on a non-singular variety X can be defined either by a subsheaf $D \subseteq TX$ of the tangent sheaf, or by the zeros of a subsheaf $D^0 \subseteq \Omega^1_X$ of 1-forms, that is, a Pfaff system. Although both definitions are equivalent under mild conditions on D, they give rise, in general, to non-equivalent notions of flat families of distributions. In this work we investigate conditions under which both notions of flat families are equivalent. In the last sections we focus on the case where the distribution is *integrable*, and we use our results to generalize a theorem of Cukierman and Pereira.

1. Introduction

Among the several motivations for studying moduli problems in geometry, there is the study of the topological and geometrical properties of differential equations on manifolds. Such problems are studied in the works of R. Thom, V. Arnold, and Kodaira-Spencer in the 1950's and were an influence in the work of Zariski about deformation of singularities; we refer to the introduction and Section 3.10 of [7] for a historical account on the subject. Algebraic geometric techniques were used in the study of (integrable) differential equations on non-singular algebraic varieties at least since Jouanolou [13], moduli and deformation problems of integrable differential equations were studied by Gomez-Mont in [9]. Since then, much has been done in the study of the geometry of integrable 1-forms on algebraic varieties, especially describing singularities of the differential equations they define (see, for instance [15]). Also, much has been done in the determination of irreducible components of the moduli space of integrable 1-forms (e.g.: [5, 6, 14] and references therein).

Many of these works were carried out in a 'naïve' way, without concern for representability of functors or formalization of deformation problems. When trying to formalize the moduli problem for integrable differential equations in non-singular algebraic varieties, an issue appears at the very beginning: although its equivalent to describe a distribution by its tangent sheaf or by the sheaf of 1-forms that vanishes on it, these two descriptions give rise to inequivalent notions of a flat family of distributions. Thus we are led to two different moduli problems, one for (involutive) sub-sheaves of the tangent sheaf, and the other for (integrable) sub-sheaves of the sheaf of 1-forms. In the present work, we prove in Proposition 6.3 and Proposition 6.5 that, under suitable assumptions, these two moduli problems are representable and, in Corollary 7.15, that their irreducible components are moreover birationally equivalent. Using this we can generalize previous theorems in the literature such as Theorem 1 of [5] and the theorems of [16, 17].

In order to better explain our results, we now introduce some notation and definitions. Let X be a non-singular projective algebraic variety. A non-singular k-dimensional distribution D on a non-singular variety X consists of a k-dimensional subspace $D_x \subseteq T_xX$ of the tangent space of X at x varying continuously with x. This notion can be formalized by saying that a k-dimensional

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distribution is a rank k sub-bundle of the tangent bundle $D \hookrightarrow TX$, and taking $D_x \subseteq T_xX$ to be $D \otimes k(x)$. Equivalently, we can say that a k-dimensional distribution is determined by a sub-bundle $I_D \hookrightarrow \Omega^1_X$, and take D_x to be $\{v \in T_xX \mid \omega(v) = 0, \forall \omega \in I_D \otimes k(x)\}$.

When X is an algebraic variety, it is often the case that there are no algebraic sub-bundles of TX or Ω^1_X of a given rank, i.e., there are no *non-singular* distributions. Nevertheless, the definition can be readily generalized to allow $D \subseteq TX$ and $I_D \subseteq \Omega^1_X$ to be subsheaves. In this way, we describe distributions D which are k-dimensional on a dense open subset, but may present singularities along proper subvarieties. Again, a singular distribution is equivalently defined either by a subsheaf $D \subseteq TX$ or by its annihilator $I_D \subseteq \Omega^1_X$. However, as was already observed by Pourcin in [16], when studying *families* of distributions parametrized by a base scheme S, it may happen that while a family of distributions $D \subseteq T_S(X \times S)$ is flat (in the sense that the quotient $T_S(X \times S)/D$ is a flat sheaf over S), its annihilator $I_D \subseteq \Omega^1_{X|S}$ may not be flat. This gives us two different notions of flat family of (singular) distributions, and therefore two different moduli problems for them.

In this work, we prove that the above two notions of flatness coincide as long as the singular set of the distribution (endowed with a convenient scheme structure) is also flat over S (Proposition 7.6 and Theorem 7.14). We focus on the case of *integrable* distributions; in Proposition 6.3, we give constructions for the moduli space Inv^X of involutive (in the sense of the Frobenius theorem) subsheaves of TX and, in Proposition 6.5, for the moduli space iPf^X of integrable subsheaf of Ω^1_X and conclude in Corollary 7.15 that taking annihilators defines a rational map between Inv^X and iPf^X that is a birational equivalence in each irreducible component of Inv^X. Moreover we give in Theorem 8.13 a sufficient criterion in terms of the singularity of the foliation to know when an involutive subsheaf $T\mathcal{F} \subseteq TX$ represents a point in the dense open set $U \subseteq \mathrm{Inv}^X$ where taking duals gives an isomorphism with an open set $V \subseteq \mathrm{iPf}^X$. Using this criterion, we can generalize the main theorem of [5] by specializing our results to the case where $X = \mathbb{P}^n$ and $T\mathcal{F} \cong \bigoplus_i \mathcal{O}_{\mathbb{P}^n}(d_i)$.

In Section 2, we show a particular example of a flat family of Pfaff systems being dual to a non-flat family of distributions.

In Section 3, we treat some preliminary general notions on sheaves and criterion for flatness that will be useful later.

In Section 4, we study the effect of applying the functor $\mathcal{H}om(-, \mathcal{O}_X)$ to a short exact sequence of sheaves. We also include in this section some observations on exterior powers, which are relevant for the study of distributions of codimension higher than 1.

Section 5 consists mainly of definitions of families of distributions and Pfaff systems, and related notions.

In Section 6, the construction of the moduli spaces of involutive distributions and integrable Pfaff systems is given as subschemes of certain Quot schemes.

In Section 7, the main results of the paper are proved. First the singular scheme of a family of distributions is defined, as well as the analogous notion for family of Pfaff systems and it is proved that the singular scheme of a family of distributions is the same as the singular scheme of its dual family (which is a family of Pfaff systems). In Section 7.1, the codimension 1 version of the main result is proved: if the singular scheme of a flat family of codimension 1 Pfaff systems is itself flat, then the dual family is flat as well. In Section 7.2, an analogous statement is proved for arbitrary codimension. In this case, however, flatness of the singular scheme is not enough to assure flatness of the dual family. To obtain a valid criterion, we define a stratification of the singular scheme, if each stratum is flat over the base, we can assure the dual family will be flat as well.

In Section 8, we give a sufficient condition to know when the singular scheme of a family of codimension 1 foliations is flat over the base. This condition is related to two of the better-studied types of singularities of foliations, the Kupka singularities and the Reeb singularities. We prove that, if the singularities of a foliation given by a distribution $T\mathcal{F} \hookrightarrow TX$ are only of these two types, then every flat family

$$0 \to T\mathcal{F}_S \to T_S(X \times S) \to N_{\mathcal{F}} \to 0,$$

such that there is an $s \in S$ with $T\mathcal{F}_s = T\mathcal{F}$, is such that $\operatorname{sing}(\mathcal{F})$ is flat in a neighborhood of s.

In Section 9, we apply the theorem of Section 8 to recover the main result of [5] as a special case of Theorem 7.8, where $X = \mathbb{P}^n$ and $T\mathcal{F}$ splits as direct sum of line bundles.

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2. An example

Set $X = \mathbb{C}^4$, $S = \mathbb{C}^2$ and consider the family of 1-forms on X

$$\omega_{(t,s)} = xdx + ydy + t \cdot zdz + s \cdot wdw,$$

parametrized by $(t, s) \in S = \mathbb{C}^2$, and the related family

$$I = I_{(t,s)} = \mathcal{O}_{X\times S} \cdot (\omega_{(t,s)}) \subseteq \Omega^1_{X\times S|S}$$

of Pfaff systems on $X = \mathbb{C}^4$.

It is easy to see that the sheaf I is flat over $S = \mathbb{C}^2$. On the other hand, if we look at the family of distributions this Pfaff systems define, we will find that it is not flat. Indeed, let D denote the annihilator of I:

$$D = \operatorname{Ann}(I) \subseteq T(X \times S).$$

To see that D is *not* flat over S, we can use Artin's criterion for flatness [2, Corollary to Proposition 3.9].

Proposition 2.1 (Artin's criterion for flatness.)**.** *Let $(A, \mathscr{M})$ and $(B, \mathscr{N})$ be local rings, $B \to A$ a flat morphism of local rings, and M a finitely generated A-module. Suppose we have generators of M, $M = (f_1, \dots, f_r)$. Let $\overline{f_i}$ be the class of f_i on $M \otimes_B B/\mathscr{N}$. Then M is flat over B if and only if every relation among $(\overline{f_1}, \dots, \overline{f_r})$ lifts to a relation among $(f_1, \dots, f_r)$.*

Note that, while this is not exactly the same context in which Artin states the proposition, his proof extends mutatis mutandi to this setting.

Now observe that $D = \operatorname{Ann}(I)$ has 6 generators:

$$f_1 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, \quad f_2 = tz\tfrac{\partial}{\partial x} - x\tfrac{\partial}{\partial z}, \quad f_3 = sw\frac{\partial}{\partial x} - x\frac{\partial}{\partial w},$$
$$f_4 = tz\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, \quad f_5 = sw\tfrac{\partial}{\partial y} - y\tfrac{\partial}{\partial w}, \quad f_6 = sw\frac{\partial}{\partial z} - tz\frac{\partial}{\partial w}.$$

These generators have, in turn, numerous relations between them, too many to list completely here. We note, however, that all relations are generated by relations of the form

$$af_i + bf_j + cf_k = 0,$$

where a, b and c are in the ideal $(x, y, tz, sw) \subseteq \mathcal{O}_{X\times S}$.

We apply Artin's criterion with $A = \mathcal{O}_{X\times S,(0,0)}$, $B = \mathcal{O}_{S,0}$ and $M = D_{(0,0)}$. In this case, the $\overline{f_i}$'s consist of evaluating the f_i on $(x, y, z, w, 0, 0) \in \mathbb{C}^4 \times (0, 0)$. We get in this way a particular relation on $M \otimes_B B/\mathscr{N}$, namely $\overline{f_6} = 0$. It is then easy to see that this relation does not lift to a relation on $(f_1, \ldots, f_6)$.

This example shows that there is, in general, no morphism between the moduli spaces $\mathrm{Quot}(X, TX)$ and $\mathrm{Quot}(X, \Omega^1_X)$ induced by taking annihilators. We will show later that taking annihilators does define a *birational* equivalence between irreducible components of $\mathrm{Quot}_X(TX)$ and those of $\mathrm{Quot}_X(\Omega^1_X)$.

The example also hints that a key aspect to understand whether flatness is preserved or not under duality is to look at how the singular set varies. Indeed, we can see that D is flat over $S \setminus \{(0, 0)\}$. For each point $(t, s) \in S \setminus \{(0, 0)\}$, the singular set of $I|_{(t,s)}$ (i.e., the set of points in $p \in X$ such that $\omega_p = 0$) is just the origin of $\mathbb{C}^4 = X$. On the other hand, the singular set of $I|_{(0,0)}$ is the plane $\{(0, 0, z, w) : z, w \in \mathbb{C}\} \subset \mathbb{C}^4$. So D ceases to be flat exactly when the singular set of I ceases to be flat as well. This illustrates the typical behavior of singular codimension 1 Pfaff systems and the distributions they determine. In the arbitrary codimension case, the situation is a bit more subtle, but the singular set still plays a decisive role. Of course, to make sense out of this, we have to define a scheme structure on the singular set of a Pfaff system. This will be done in the course of the present work.

3. Preliminaries

Here we gather known facts of algebraic geometry that will be used later. We include proofs of some of these facts for lack (to the author's best knowledge) of a better reference.

3.1. Reflexive sheaves and Serre's property S_2.

Property S_2 can be viewed as an algebraic analog of Hartog's theorem on complex holomorphic functions. For this reason, it will be extremely useful to us, for it will allow us to conclude global statements on sheaves that hold, a priori, for the restriction of these sheaves to (suitably large) open sets. Here we recall some known facts about sheaves with the S_2 property, and sheaves with the *relative* S_2 property as defined in [3].

Definition 3.1. Let $p : X \to T$ be a morphism of schemes and, for each point $x \in X$, let $d_T(x)$ equal the codimension of x in its fiber over T. We say that a sheaf $\mathscr{F}$ on X satisfies the *relative Serre condition* S_k with respect to p if and only if

$$\mathrm{depth}\mathscr{F}_x \geq \min(k, d_T(x)),$$

for all $x \in X$.

The proof of the next proposition works exactly as in the non-relative version.

Proposition 3.2. *Let $p : X \to T$ be a morphism of noetherian schemes and $\mathscr{F}$ a torsion-free coherent sheaf with relative property S_2 with respect to p. Let $Y \subset X$ be a closed subset such that $d_T(Y) \geq 2$. Then the restriction map $\rho : \Gamma(X, \mathscr{F}) \to \Gamma(X \setminus Y, \mathscr{F})$ is an isomorphism.*

Corollary 3.3. *Let $p : X \to T$ be a morphism of noetherian schemes and $\mathscr{F}$ a torsion-free coherent sheaf with property S_2 with respect to p. Let $Y \subset X$ be a closed subset such that $d_T(Y) \geq 2$. Let $U = X \setminus Y$ and let $j : U \to X$ be the inclusion. Then $\mathscr{F} = j_*(\mathscr{F}|_U)$.*

This corollary motivates the following definition, which is the relative analogue of a notion due to Grothendieck [10, 5.10].

Definition 3.4. Let $p : X \to T$ be a morphism of noetherian schemes and $\mathscr{F}$ a coherent sheaf. If, for each closed subset $Y \subset X$ such that $d_T(Y) \geq 2$, with $U = X \setminus Y$ and $j : U \to X$ the inclusion, the natural map

$$\rho_U : \mathscr{F} \to j_*(\mathscr{F}|_U)$$

is an epimorphism, then we say $\mathscr{F}$ is $Z^{(2)}$*-closed relative to* p; if ρ_U is an isomorphism, we say that it is $Z^{(2)}$*-pure relative to* p.

Proposition 3.5 ([12, Proposition 1.7])**.** *Let X be a quasi-projective integral scheme. A coherent sheaf $\mathscr{F}$ is reflexive if and only if it can be included in an exact sequence*

$$0 \to \mathscr{F} \to \mathscr{E} \to \mathscr{G} \to 0,$$

where $\mathscr{E}$ is locally free and $\mathscr{G}$ is torsion-free.

Corollary 3.6. *Under the above circumstances, the dual of a coherent sheaf is always reflexive.*

Proposition 3.7 (c.f.:[12, Theorem 1.9])**.** *Let $p : X \to T$ be a morphism of noetherian schemes with normal integral fibers, and $\mathscr{F}$ a coherent sheaf on X. Then, if $\mathscr{F}$ is reflexive, it has relative property S_2 with respect to p.*

Proof. The statement being local, we can assume X is quasi-projective. Given a reflexive sheaf $\mathscr{F}$, we take an exact sequence

$$0 \to \mathscr{F} \to \mathscr{L} \to \mathscr{G} \to 0,$$

with $\mathscr{L}$ locally free and $\mathscr{G}$ torsion-free. Since p have normal fibers, $\mathcal{O}_X$ satisfies relative S_2 with respect to p, and so does $\mathscr{L}$, being locally free. Let $x \in X$ be a point of relative dimension ≥ 2 with respect to its fiber $X_{p(x)}$. Then $\operatorname{depth}\mathscr{L}_x \geq 2$ by S_2 and, as $\mathscr{G}$ is torsion-free, $\operatorname{depth}\mathscr{G}_x \geq 1$. This in turn implies $\operatorname{depth}\mathscr{F}_x \geq 2$. □

3.2. **Support of a sheaf, zeros of a section.** Recall that, given a quasi-coherent sheaf $\mathscr{F}$ on a scheme X, we define the *support* of $\mathscr{F}$, $\operatorname{supp}(\mathscr{F})$ as the closed sub-scheme defined by the ideal sheaf given locally by

$$\mathcal{I}(\mathscr{F})_x := \operatorname{Ann}(\mathscr{F}_x) \subset \mathcal{O}_{X,x}.$$

We have the following useful characterization of the support of a sheaf in terms of a universal property:

Proposition 3.8. *The support of a sheaf $\mathscr{F}$ represents the functor*

$$\begin{array}{rcl} \mathrm{S}_{\mathscr{F}} : Sch & \longrightarrow & Sets \\ T & \mapsto & \{f \in \hom(T, X) \colon \operatorname{Ann}(f^*\mathscr{F}) = 0 \subset \mathcal{O}_T\}. \end{array}$$

Proof. A morphism $f : T \to X$ factors through $\operatorname{supp}(\mathscr{F})$ if and only if the map

$$f^\sharp : f^{-1}\mathcal{O}_X \to \mathcal{O}_T$$

factors through $f^{-1}(\mathcal{O}_X/\operatorname{Ann}(\mathscr{F}))$. But this happens if and only if $f^{-1}(\operatorname{Ann}(\mathscr{F})) = 0$.

On the other hand, we have the equality

$$\operatorname{Ann}(f^*\mathscr{F}) = \mathcal{O}_T \cdot f^{-1}(\operatorname{Ann}(\mathscr{F}));$$

indeed we may check this in every localization at any point $p \in T$, so if $t \in \operatorname{Ann}(f^*\mathscr{F})_p$ in particular t annihilates every element of the form $m \otimes 1 \in \mathscr{F}_x$, so $t = f^{-1}(x)t'$ where $x \in \operatorname{Ann}(\mathscr{F})_{f(p)}$.

So $\operatorname{Ann}(f^*\mathscr{F}) = 0$ if and only if $f^{-1}(\operatorname{Ann}(\mathscr{F})) = 0$ and we are done. □

In other words, we just proved that $\mathrm{supp}(\mathscr{F})$ is the universal scheme with the property that $f^*\mathscr{F}$ is not a torsion module. This simple observation will be very useful when discussing the scheme structure on the singular set of a foliation.

A special case of support of a sheaf is the scheme-theoretic image of a morphism. Remember that the *scheme-theoretic image* of a morphism $f : X \to Y$ is the sub-scheme $\mathrm{supp}(f_*\mathcal{O}_X) \subseteq Y$.

Now we turn our attention to sections and their zeros. So let X be a scheme and $\mathscr{E}$ a locally free sheaf. Having a global section $s \in \Gamma(X, \mathscr{E})$ is the same as having a morphism (that, by abuse of notation, we also call s)

$$s : \mathcal{O}_X \longrightarrow \mathscr{E}.$$

Now, $s : \mathcal{O}_X \to \mathscr{E}$ defines a dual morphism

$$s^\vee : \mathscr{E}^\vee \longrightarrow \mathcal{O}_X^\vee = \mathcal{O}_X.$$

Definition 3.9. We define the *zero scheme* $Z(s)$ of the section s as the closed sub-scheme of X defined by the ideal sheaf $\mathrm{Im}(s^\vee) \subseteq \mathcal{O}_X$.

We'll apply this definition in the well-behaved situation where $\mathcal{O}_X$ (and therefore $\mathscr{E}$) is torsion-free.

Proposition 3.10. *Let $\mathscr{E}$ be a locally free sheaf on X and $s \in \Gamma(X, \mathscr{E})$ a global section. The scheme $Z(s)$ represents the functor*

$$\begin{array}{rcl} \mathrm{Z}_s : Sch & \longrightarrow & Sets \\ T & \mapsto & \{f \in \hom(T, X) \colon s \otimes 1 = 0 \in \Gamma(T, f^*\mathscr{E})\}. \end{array}$$

Proof. A morphism $f : T \to X$ factors through $Z(s)$ if and only if the map

$$(\mathscr{E}^\vee) \otimes \mathcal{O}_T \xrightarrow{s^\vee \otimes 1} \mathcal{O}_T$$

is identically 0. Beign locally free, we have

$$\mathscr{E}^\vee \otimes \mathcal{O}_T = \mathscr{H}om(\mathscr{E}, \mathcal{O}_X) \otimes \mathcal{O}_T \cong \mathscr{H}om(\mathscr{E} \otimes \mathcal{O}_T, \mathcal{O}_T).$$

So then we have

$$(f^*\mathscr{E})^\vee \xrightarrow{s^\vee \otimes 1} \mathcal{O}_T$$

is identically 0, as $f^*\mathscr{E}$ is locally free over T; this means $s \otimes 1 = 0 \in \Gamma(T, f^*\mathscr{E})$. □

3.3. **A criterion for flatness.** For lack of a better reference, we provide here a criterion that will become handy when dealing with both reduced and non-reduced base schemes over an algebraically closed field.

Lemma 3.11. *Let A be a ring of finite type over an algebraically closed field k, $\mathcal{M}$ a maximal ideal in A, and $f \in \mathcal{M}^n \setminus \mathcal{M}^{n+1}$. Then there is a morphism $\psi : A \to k[T]/(T^{n+1})$ such that $\psi^{-1}((T)) = \mathcal{M}$ and $\psi(f) \neq 0$.*

Proof. Fix a presentation $A \cong k[y_1, \ldots, y_r]/I$. By the Nullstelensatz, we can assume that $\mathcal{M} = (x_1, \ldots, x_r)$, where x_i is the class of $y_i \mathrm{mod} I$. Write the class of f in $\mathcal{M}^n/\mathcal{M}^{n+1}$ as

$$\overline{f} = \sum_{|\alpha|=n} a_\alpha \overline{x}^\alpha \in \mathcal{M}^n/\mathcal{M}^{n+1},$$

where $\alpha = (\alpha_1, \ldots, \alpha_r)$ and $\overline{x}^\alpha = (\overline{x_1}^{\alpha_1}, \ldots, \overline{x_r}^{\alpha_r})$.

As $f \notin \mathcal{M}^{n+1}$, the polynomial $q(y_1, \ldots, y_r) := \sum_{|\alpha|=n} a_\alpha y^\alpha$ is not in I. Now, k being algebraically closed, there is an r-tuple $(\lambda_1, \ldots, \lambda_r) \in k^r$ such that $p(\lambda_1, ..., \lambda_r) = 0$ for every $p \in I$ and $q(\lambda_1, ..., \lambda_r) \neq 0$.

Finally we can define $\psi : A \to k[T]/(T^{n+1})$ as follows:

$$\psi(x_i) = \lambda_i T.$$

The morphism is well-defined because $p(\lambda_1, ..., \lambda_r) = 0$ for every $p \in I$, moreover $\psi^{-1}(T) = \mathcal{M}$, and $\psi(f) = q(\lambda_1, ..., \lambda_r)T^n \neq 0$. □

Proposition 3.12. *Let $f : X \to Y$ a projective morphism between schemes of finite type over an algebraically closed field, $\mathscr{F}$ a coherent sheaf over X, $x \in X$ a point, and $y = f(x)$. Then $\mathscr{F}_x$ is f-flat if and only if the following conditions hold:*

(1) For every discrete valuation ring A' and every morphism $\mathcal{O}_{Y,y} \to A'$, the following holds: Taking the pull-back diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' = \operatorname{Spec}(A') \arrow[r] & Y \end{tikzcd}$$

the $\mathcal{O}_{X'}$-module $\mathscr{F}' = \mathscr{F} \otimes \mathcal{O}_{Y'}$ is f'-flat at every point $x' \in X'$ lying over x.

(2) For every $n \in \mathbb{N}$ and every morphism $\mathcal{O}_{Y,y} \to k[T]/(T^{n+1})$, if we take the diagram analogous to the one above (with $k[T]/(T^{n+1})$ instead of A'), then the $\mathcal{O}_{X'}$-module $\mathscr{F}' = \mathscr{F} \otimes \mathcal{O}_{Y'}$ is f'-flat at every point $x' \in X'$ lying over x.

Proof. Clearly Conditions 1 and 2 are necessary. Suppose then that 1 and 2 are satisfied.

Take the flattening stratification (see [8, Section 5.4.2]) of Y with respect to $\mathscr{F}$, $Y = \coprod_P Y_P$. As Condition 1 is satisfied for $\mathscr{F}$ over Y, so it is satisfied for $\iota^*\mathscr{F}$ over Y_{red}, where $\iota : Y_{\text{red}} \to Y$ is the closed immersion of the reduced structure. Then, by the valuative criterion for flatness of [10, 11.8], $\iota^*\mathscr{F}$ is flat over Y_{red}, so by the universal property of the flattening stratification, there is a factorization

$$\begin{tikzcd} Y_{\text{red}} \arrow[r, "\iota"] \arrow[d] & Y \\ \coprod_P Y_P \arrow[ur] & \end{tikzcd} .$$

As Y_{red} and Y share the same underlying topological set, the above factorization is telling us that the flattening factorization consist on a single stratum Y_P and that $Y_{\text{red}} \to Y_P$ is a closed immersion.

Assume, by way of contradiction, $Y_P \subsetneq Y$; then there is an affine open sub-scheme $U \subseteq Y$ such that $V = Y_P \cap U \neq U$. Now take the coordinate rings $k[U]$ and $k[V]$ and the morphism between them induced by the inclusion $\phi : k[U] \twoheadrightarrow k[V]$. Let's take $f \in k[U]$ such that $\phi(f) = 0$. By Lemma 3.11, there exists, for some $n \in \mathbb{N}$, a morphism $\psi : k[U] \to k[T]/(T^{n+1})$ such that $\psi(f) \neq 0$; so ψ doesn't factorize through ϕ.

On the other hand, let $Z = \operatorname{Spec}(k[T]/(T^{n+1}))$ and $g : Z \to Y$ be the morphism induced by ψ. As Condition 2 is satisfied, the pull-back $g^*\mathscr{F}$ is flat over $Z = \operatorname{Spec}(k[T]/(T^{n+1}))$. So, by the universal property of the flattening stratification, g factors as

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d] & Y \\ Y_P \arrow[ur] & \end{tikzcd} ,$$

contradicting the statement of the above paragraph, thus proving the proposition. □

Note that the hypotheses of this property on X and Y (aside from reducedness) are quite stronger than the ones of the original theorem of Grothendieck (the valuative criterion for flatness in [10]); such is the price we have paid to allow a criterion for possibly non-reduced schemes. The price paid is okay with us anyway, considering that we will work mostly with schemes of finite type over $\mathbb{C}$.

Next we provide a criterion for a $k[T]/(T^{n+1})$-module to be flat.

Proposition 3.13. *Let $A = k[T]/(T^{n+1})$ and M an A-module. Then M is flat if and only if, for every $m \in M$ such that $T^n \cdot m = 0$, there exists $m' \in M$ such that $m = T \cdot m'$.*

Proof. Flatness of M is equivalent to the injectivity of the map $M \otimes I \to M$ for every ideal $I \subset A$ (see e.g.:[11, IV.1]). In this case, there are finitely many ideals:

$$\mathcal{M} = (T),\ \mathcal{M}^2, \dots,\ \mathcal{M}^n.$$

If M is flat, it is easy to see that the second condition in our statement hold.

Suppose that, for every $m \in M$ such that $T^n \cdot m = 0$, there exists $m' \in M$ such that $m = T \cdot m'$. Let $a \in M \otimes \mathcal{M}^{n-i}$ be in the kernel of $M \otimes \mathcal{M}^{n-i} \to M$. When $i = 0$, we have $a = m \otimes T^n$, and m is such that $T^n \cdot m = 0$; so, by hypothesis, $m = T \cdot m'$ and then $m \otimes T^n = m' \otimes T^{n+1} = 0$.

When $i > 0$, we have $a = \sum_{j=n-i}^{i} m_j \otimes T^j$, so $T^i \cdot a = m_{n-i} \otimes T^n \in M \otimes \mathcal{M}^n$. By hypothesis, $m_{n-i} = T \cdot m'$. So $a \in M \otimes \mathcal{M}^{n-i+1}$ and we are done by induction. □

The following will be useful in the study of foliations of codimension greater than 1.

Proposition 3.14. *Let $p : X \to S$ a projective morphism between schemes of finite type over an algebraically closed field k, $f : S \to Y$ another morphism, with Y of finite type over k, and $\mathscr{F}$ a coherent sheaf over X. Take a stratification $\coprod_i S_i \subseteq S$ of S such that $\mathscr{F}|_{S_i} := \mathscr{F} \otimes_S \mathcal{O}_{S_i}$ is flat for all i. If the composition $\coprod_i S_i \hookrightarrow S \xrightarrow{f} Y$ is a flat morphism, then $\mathscr{F}$ is flat over Y.*

Proof. Invoking Proposition 3.12 we can, after applying base change, reduce to the case where Y is either the spectrum of a DVR or $Y = \mathrm{Spec}(k[T]/(T^{n+1}))$.

(i) Case $Y = \mathrm{Spec}(A)$ with A a DVR. Suppose there is, for some point $x \in X$, a section $s \in \mathscr{F}_x$ that is of torsion over A. Consider $Z = \mathrm{supp}_S(s) \subseteq S$, the support of s over S, that is the support of s as an element of $\mathscr{F}_x$ considered as an $\mathcal{O}_{S,p(x)}$-module. Now take any stratum S_i and suppose $Z \cap S_i \neq \emptyset$. Then there is a section of the pullback $\mathscr{F}_{S_i}$ that is of torsion over A. But $\mathscr{F}_{S_i}$ is flat over S_i which is in turn flat over A, so $\mathscr{F}_{S_i}$ is flat and $Z \cap S_i$ must be empty for every stratum S_i, i.e., $s = 0$.

(ii) Case $Y = \mathrm{Spec}(k[T]/(T^{n+1}))$. One can essentially repeat the argument above, now taking the section s to be such that $T^n s = 0$ but $s \notin T \cdot \mathscr{F}_x$. □

Corollary 3.15. *Take the flattening stratification $\coprod_P S_P \subseteq S$, of S with respect to $\mathscr{F}$. If the composition $\coprod_P S_P \hookrightarrow S \xrightarrow{f} Y$ is a flat morphism, then $\mathscr{F}$ is flat over Y.*

4. Families of sub-sheaves and their dual families

Definition 4.1. Given a short exact sequence of sheaves

$$0 \to \mathscr{G} \xrightarrow{\iota} \mathscr{F} \to \mathscr{H} \to 0,$$

we apply to it the left-exact contravariant functor $\mathscr{F} \mapsto \mathscr{F}^\vee := \mathcal{H}om_X(\mathscr{F}, \mathcal{O}_X)$ to obtain exact sequences:

$$0 \to \mathscr{H}^\vee \to \mathscr{F}^\vee \to \mathrm{Im}(\iota^\vee) \to 0, \tag{1}$$

$$0 \to \mathrm{Im}(\iota^\vee) \to \mathscr{G}^\vee \to \mathcal{E}xt^1_X(\mathscr{H}, \mathcal{O}_X) \to \mathcal{E}xt^1_X(\mathscr{F}, \mathcal{O}_X).$$

We say that the exact sequence (1), is the *dual exact sequence* of $0 \to \mathscr{G} \xrightarrow{\iota} \mathscr{F} \to \mathscr{H} \to 0$.

Lemma 4.2. *Let $0 \to N \xrightarrow{\iota} T \xrightarrow{\pi} M \to 0$ be a short exact sequence of R-modules such that T is reflexive and M is torsion free. Then $\mathrm{Im}(\iota^\vee)^\vee = N$ and $M = \mathrm{Im}(\pi^{\vee\vee})$.*

Proof. First we take the duals in the short exact sequence to get a sequence

$$0 \to \hom_R(M, R) \xrightarrow{\pi^\vee} \hom_R(T, R) \xrightarrow{\iota^\vee} \mathrm{Im}(\iota^\vee) \to 0.$$

Then we take duals one more time and, given that T is reflexive and that M is torsion-free, we get the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Im}(\iota^\vee)^\vee \arrow[r] & T^{\vee\vee} \arrow[r, "\pi^{\vee\vee}"] \arrow[d, equal] & M^{\vee\vee} \arrow[r] & \mathrm{ext}^1_R(\mathrm{Im}(\iota^\vee), R) \\
0 \arrow[r] & N \arrow[r] \arrow[u] & T \arrow[r] & M \arrow[r] \arrow[u, hook] & 0,
\end{tikzcd}$$

whose rows are exact.

Chasing arrows, we readily see that the leftmost vertical arrow must be an isomorphism. Indeed, since the monomorphism $N \to T^{\vee\vee}$ factors as

$$N \to \mathrm{Im}(\iota^\vee)^\vee \to T^{\vee\vee},$$

the second arrow being a monomorphism, so must $N \to \mathrm{Im}(\iota^\vee)^\vee$ be. On the other hand, given $a \in \mathrm{Im}(\iota^\vee)^\vee$, we can regard it, via the inclusion, as an element in $T^{\vee\vee} = T$, so we can compute $\pi(a)$. As the canonical map $\theta : M \to M^{\vee\vee}$ is an inclusion we have that, $\theta \circ \pi(a) = \pi^{\vee\vee}(a) = 0$, then $\pi(a) = 0$, so $a \in N$. From this, we have $N \cong \mathrm{Im}(\iota^\vee)^\vee$, wich implies $M = \mathrm{Im}(\pi^{\vee\vee})$. □

4.1. **Exterior Powers.** When dealing with foliations of codimension/dimension greater than 1 is usually convenient to work with p-forms. We'll need then to compare sub-sheaves $I \subset \Omega^1_X$ with their exterior powers $\wedge^p I \subset \Omega^p_X$. In order to do that we include the following statements, valid in a wider context.

We'll concentrate on flat modules and their exterior powers. This will be important when dealing with flat families of Pfaff systems of codimension higher than 1 (see Remark 5.5).

Lemma 4.3. *Let A be a ring containing the field $\mathbb{Q}$ of rational numbers, and let M be a flat A-module. Then, for every p, $\wedge^p M$ is also flat.*

Proof. If tensoring with M is an exact functor, so is its iterate $- \otimes M \otimes \cdots \otimes M$. So $M^{\otimes p}$ is flat. As A contains $\mathbb{Q}$, there is an anti-symmetrization operator

$$M^{\otimes p} \to \wedge^p M,$$

which is a retraction of the canonical inclusion $\wedge^p M \subset M^{\otimes p}$. This makes $\wedge^p M$ a direct summand of $M^{\otimes p}$; set $M^{\otimes p} = \wedge^p M \oplus R$ for some module R. As the tensor power distributes over direct sums (i.e., $(\wedge^p M \oplus R) \otimes N \cong (\wedge^p M \otimes N) \oplus (R \otimes N)$), so does its derived functors. In particular we have, for every module N,

$$0 = \mathrm{Tor}_1(M^{\otimes p}, N) = \mathrm{Tor}_1(\wedge^p M, N) \oplus \mathrm{Tor}_1(R, N).$$

So $\wedge^p M$ is flat. □

Finally, we draw some conclusions regarding flat quotients. When dealing with Pfaff systems, we'll be interested in short exact sequences of the form

$$0 \to \wedge^p I \to \Omega^p_X \to \mathcal{G} \to 0,$$

arising from short exact sequences of flat modules

$$0 \to I \to \Omega^1_X \to \Omega \to 0.$$

Note that, in general $\mathcal{G} \neq \wedge^p \Omega$. Nevertheless, we can state:

Proposition 4.4. *Let A be a ring containing $\mathbb{Q}$. Given an exact sequence*

$$0 \to M \to P \to N \to 0$$

of flat A-modules, we have an associated exact sequence

$$0 \to \wedge^p M \to \wedge^p P \to Q \to 0.$$

Then Q is also flat.

Proof. Q inherits a filtration from $\wedge^p P$:

$$Q = \wedge^p P / \wedge^p M = \overline{F}^0 \supseteq \overline{F}^1 \supseteq \cdots \supseteq \overline{F}^p = 0,$$

with quotients

$$F^i / F^{i+1} \cong \wedge^i M \otimes \wedge^{p-i} N.$$

Then Q has a filtration all of whose quotients are flat, so Q itself is flat. □

5. Families of distributions and Pfaff systems

We will consider subsheaves of the relative tangent sheaf $T_S X$ and the relative differentials $\Omega^1_{X|S}$.

Definition 5.1. A *family of distributions* is a short exact sequence

$$0 \to T\mathcal{F} \to T_S X \to N_{\mathcal{F}} \to 0.$$

The family is called flat if $N_{\mathcal{F}}$ is flat over the base S. A family of distributions is called *involutive* if it's closed under the Lie bracket operation, that is, if for every pair of local sections X, $Y \in T\mathcal{F}(V)$, we have $[X, Y] \in T\mathcal{F}(V)$, where $[-,-]$ is the Lie bracket in $T_S X(V)$.

Likewise, a *family of Pfaff systems* is just a short exact sequence

$$0 \to I(\mathcal{F}) \to \Omega^1_{X|S} \to \Omega^1_{\mathcal{F}} \to 0.$$

It's called flat if $\Omega^1_{\mathcal{F}}$ is flat.

We will say that a family of Pfaff systems is *integrable* if

$$d(I(\mathcal{F})) \wedge \bigwedge^r I(\mathcal{F}) = 0 \subset \Omega^{r+2}_{X|S};$$

where $d : \Omega^j_{X|S} \to \Omega^{j+1}_{X|S}$ is the <u>relative</u> de Rham differential, and r is the generic rank of the sheaf $\Omega^1_{\mathcal{F}}$.

Remark 5.2. Observe that the relative differential $d : \Omega^j_{X|S} \to \Omega^{j+1}_{X|S}$ is not an $\mathcal{O}_X$-linear morphism. It is, however, $f^{-1}\mathcal{O}_S$-linear, so the sheaf $d(I(\mathcal{F})) \wedge \bigwedge^r I(\mathcal{F})$, whose annihilation encodes the integrability of the Pfaff system, is actually a sheaf of $f^{-1}\mathcal{O}_S$-modules.

In particular, the dual to a family of distributions is a family of Pfaff systems and vice-versa.

Remark 5.3. The dual of an <u>involutive</u> family of distributions is an <u>integrable</u> family of Pfaff systems. Reciprocally, the dual of an integrable family of Pfaff systems is a family of involutive distributions. This is just a consequence of the Cartan-Eilenberg formula for the de Rham differential of a 1-form applied to vector fields

$$d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]).$$

Indeed, as involutiveness and integrability can be checked locally over sections, we can proceed as in [19, Prop. 2.30].

Definition 5.4. The *dimension* of a family of distributions is the generic rank of $T\mathcal{F}$. Likewise, the *dimension* of a family of Pfaff systems is the generic rank of $\Omega^1_{\mathcal{F}}$.

If $p : X \to S$ is moreover projective, S is connected, and the family is flat, so $T\mathcal{F}$ is a flat sheaf over S. Then for every $s \in S$ the Hilbert polynomial of $T\mathcal{F}_s$ is the same, and so is its generic rank (being encoded in the principal coeficient of the polynomial). The same occurs with families of Pfaff systems.

Remark 5.5. Frequently, in the study of foliations of codimension higher than 1, it is more convenient and better adapted to calculations to work with an alternative description of foliations. Namely, one can define a codimension q foliation on a variety X as in [15], with a global section ω of $\Omega^q_X \otimes \mathcal{L}$ such that:

- ω is locally decomposable, i.e., there is, for all $x \in X$, an open set such that
$$\omega = \eta_1 \wedge \cdots \wedge \eta_q,$$
with $\eta_i \in \Omega^1_X$.
- ω is integrable, i.e., $\omega \wedge d\eta_i = 0,\ 1 \leq i \leq q$.

With this setting, studying flat families of codimension q foliations (meaning here families of integrable Pfaff systems) as in [5] and [6], parametrized by a scheme S, amounts to studying short exact sequences of flat sheaves

$$0 \to \mathcal{L}^{-1} \to \Omega^q_{X|S} \to \mathcal{G} \to 0$$

that are locally decomposable and integrable. By the results of Section 4.1, a flat family of codimension q Pfaff systems given as a sub-sheaf of $\Omega^1_{X|S}$ gives rise to a flat family in the above sense.

6. Universal families

Now let's take a non-singular projective scheme X, a polynomial $P \in \mathbb{Q}[t]$, and consider the following functor

$$\begin{array}{rcl} \mathfrak{Inv}^P(X) : Sch & \longrightarrow & Sets \\ S & \mapsto & \left\{ \begin{array}{l} \text{flat families } 0 \to T\mathcal{F} \to T_S(X \times S) \to N_{\mathcal{F}} \to 0 \text{ of in-} \\ \text{volutive distributions such that } N_{\mathcal{F}} \text{ have Hilbert} \\ \text{polynomial } P(t). \end{array} \right\}. \end{array}$$

Say $p : X \times S \to X$ is the projection, so $T_S(X \times S) = p^*TX$. Clearly one has $\mathfrak{Inv}^P(X)$ is a sub-functor of $\mathfrak{Quot}^P(X, TX)$. We are going to show that $\mathfrak{Inv}^P(X)$ is actually a *closed* sub-functor of $\mathfrak{Quot}^P(X, TX)$ and therefore also representable.

So take the smooth morphism given by the projection

$$p_1 : \mathrm{Quot}_P(X, TX) \times X \to \mathrm{Quot}_P(X, TX).$$

Here we are taking as base scheme $S = \mathrm{Quot}_P(X, TX)$, then on the total space,

$$S \times X = \mathrm{Quot}_P(X, TX) \times X,$$

we have the natural short exact sequence

$$0 \to \mathscr{F} \to p_2^* TX = T_S(S \times X) \to \mathscr{Q} \to 0.$$

Now we consider the push-forward of these sheaves by p_1; as X is proper, these push-forwards are coherent sheaves over S. In particular, we have maps of coherent sheaves over $\mathrm{Quot}^P(X, TX)$

$$p_{1*}\mathscr{F} \otimes_S p_{1*}\mathscr{F} \xrightarrow{[-,-]} p_{1*}T_S(S \times X) \to p_{1*}\mathscr{Q}$$

induced by the maps over $S \times X$. Note that while the Lie bracket on $T_S(S \times X)$ is only $p_1^{-1}\mathcal{O}_S$-linear, the map induced on the push-forwards is $\mathcal{O}_S$-linear, and so is a morphism of coherent sheaves. We then also have for all $m, n \in \mathbb{Z}$ the twisted morphisms

$$p_{1*}\mathscr{F}(m) \otimes_S p_{1*}\mathscr{F}(n) \xrightarrow{[-,-]} p_{1*}T_S(S \times X)(m+n) \to p_{1*}\mathscr{Q}(m+n).$$

Note also that, as p_1 is a projective morphism, then there exists an $n \in \mathbb{Z}$ such that, for every $m \geq n$, the natural sheaves morphism over $S \times X$, $p_1^* p_{1*}(\mathscr{F})(m) \to \mathscr{F}(m)$ is an epimorphism. So, if for some $f : Z \to S$ and some $m \geq n$, one has that the composition

$$f^* p_{1*}\mathscr{F}(m) \otimes_Z f^* p_{1*}\mathscr{F}(m) \xrightarrow{[-,-]} f^* p_{1*}T_S(S \times X)(2m) \to f^* p_{1*}\mathscr{Q}(2m)$$

is zero, then the map

$$(f \times id)^*\mathscr{F}(m) \otimes_{\pi_1^{-1}\mathcal{O}_Z} (f \times id)^*\mathscr{F}(m) \xrightarrow{[-,-]} T_Z(Z \times X)(2m) \to (f \times id)^*\mathscr{Q}(2m)$$

is zero as well. Here $\pi_1 : Z \times X \to Z$ is the projection, which is by the way the pull-back of p_1.

Now to conclude the representability of $\mathfrak{Inv}^P(X)$ we need one important lemma.

Lemma 6.1. *Let S be a noetherian scheme, $p : X \to S$ a projective morphism, and $\mathscr{F}$ a coherent sheaf on X. Then $\mathscr{F}$ is flat over S if and only if there exists some integer N such that, for all $m \geq N$, the push-forwards $p_*\mathscr{F}(m)$ are locally free.*

Proof. This is [8, Lemma 5.5]. □

We can then take $m \in Z$ big enough so $p_{1*}\mathscr{F}(m)$ and $p_{1*}\mathscr{Q}(2m)$ are locally free and the morphism $p_1^* p_{1*}(\mathscr{F})(m) \to \mathscr{F}(m)$ is epimorphism. Then we can regard the composition

$$p_{1*}\mathscr{F}(m) \otimes_S p_{1*}\mathscr{F}(m) \xrightarrow{[-,-]} p_{1*}T_S(S \times X)(2m) \to p_{1*}\mathscr{Q}(2m)$$

as a global section σ of the locally free sheaf $\mathcal{H}om_S(p_{1*}\mathscr{F}(m) \otimes_S p_{1*}\mathscr{F}(m), \mathscr{Q}(2m))$. We can then make the following definition.

Definition 6.2. We define the scheme $\mathrm{Inv}^P(X)$ to be the zero scheme $Z(\sigma)$ (cf.: Definition 3.9) of the section σ defined above.

A direct application of Proposition 3.10 to this definition together with the discussion so far immediately gives us the following.

Proposition 6.3. *The subscheme $\mathrm{Inv}^P(X) \subseteq \mathrm{Quot}^P(X, TX)$ represents the functor $\mathfrak{Inv}^P(X)$.*

Similarly we can consider the sub-functor $\mathfrak{iPf}^P(X)$ of $\mathfrak{Quot}^P(X, \Omega_X^1)$.

$$\begin{array}{rcl} \mathfrak{i}\mathfrak{P}\mathfrak{f}^P(X) : Sch & \longrightarrow & Sets \\ S & \mapsto & \left\{\begin{array}{l}\text{flat families } 0 \to I(\mathcal{F}) \to \Omega^1 X|S \to \Omega^1_{\mathcal{F}} \to 0 \text{ of inte-}\\ \text{grable Pfaff systems such that } \Omega^1_{\mathcal{F}} \text{ have Hilbert poly-}\\ \text{nomial } P(t).\end{array}\right\}. \end{array}$$

Then as before we take $S = \mathrm{Quot}^P(X, \Omega^1_X)$ and consider the map

$$p_{1*}\big(d(\mathscr{I}) \wedge \bigwedge^r \mathscr{I}\big)(m) \longrightarrow p_{1*}\Omega^{r+2}_{S\times X|S}(m),$$

which is, for large enough m, a morphism between locally free sheaves on S.

Definition 6.4. We define the scheme $\mathrm{iPf}^P(X)$ to be the zero scheme of the above morphism, viewed as a global section of the locally free sheaf $\mathcal{H}om(p_{1*}(d(\mathscr{I}) \wedge \bigwedge^r \mathscr{I})(m), p_{1*}\Omega^{r+2}_{S\times X|S}(m))$.

And then, by Proposition 3.10, we have representability.

Proposition 6.5. *The subscheme* $\mathrm{iPf}^P(X) \subset \mathrm{Quot}^P(X, \Omega^1_X)$ *represents the functor* $\mathfrak{i}\mathfrak{P}\mathfrak{f}^P(X)$.

7. Duality

Definition 7.1. The *singular locus* of a family of distributions $0 \to T\mathcal{F} \to T_S X \to N_{\mathcal{F}} \to 0$ is the (scheme-theoretic) support of $\mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X)$. Intuitively, its points are the points where $N_{\mathcal{F}}$ fails to be a fiber bundle.

Similarly, for a family of Pfaff systems $0 \to I(\mathcal{F}) \to \Omega^1_{X|S} \to \Omega^1_{\mathcal{F}} \to 0$, its singular locus is $\mathrm{supp}(\mathcal{E}xt^1_X(\Omega^1_{\mathcal{F}}, \mathcal{O}_X))$.

Remark 7.2. Call $i : T\mathcal{F} \to T_S X$ the inclusion. We have an open non-empty set U where, for every $x \in U$, $\dim(Im(i \otimes k(x)))$ is maximal. More precisely, U is the open set where $Tor^X_1(N_{\mathcal{F}}, k(x)) = 0$, which is the maximal open set such that $N_{\mathcal{F}}|_U$ is locally free, and therefore so is $T\mathcal{F}$. Then, when restricted to U, $T\mathcal{F}$ can be given locally as the subsheaf of $T_S X$ generated by k linearly independent relative vector fields, i.e., $T\mathcal{F}$ defines a family of non-singular foliations. In U, one has that $\mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X) = 0$. Then, the underlying topological space of the singular locus of the family given by $T\mathcal{F}$ is the singular set of the foliation in a classical (topologial space) sense.

The above discussion translates verbatim to families of Pfaff systems.

Proposition 7.3. *Let*

$$0 \to I(\mathcal{F}) \to \Omega^1_{X|S} \to \Omega^1_{\mathcal{F}} \to 0$$

be a family of Pfaff systems such that $\Omega^1_{\mathcal{F}}$ *is torsion-free. Its singular locus and the singular locus of the dual family*

$$0 \to T\mathcal{F} \to T_S X \to N_{\mathcal{F}} \to 0$$

are the same sub-scheme of X*. We denote this sub-scheme by* $\mathrm{sing}(\mathcal{F})$

Proof. We are going to show that the immersions

$$Y_1 := \mathrm{supp}(\mathcal{E}xt^1_X(\Omega^1_{\mathcal{F}}, \mathcal{O}_X)) \subseteq X \quad \text{and} \quad Y_2 := \mathrm{supp}(\mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X)) \subseteq X$$

represent the same sub-functor of $\mathrm{Hom}(-, X)$, thus proving the proposition.

First note that, if $\mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X) = 0$, then $\mathcal{E}xt^1_X(\Omega^1_{\mathcal{F}}, \mathcal{O}_X) = 0$. Indeed, if $\mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X) = 0$, $N_{\mathcal{F}}$ is locally free and then so is $T\mathcal{F}$. Moreover, since $\Omega^1_{\mathcal{F}}$ is torsion free, we can dualize the short exact sequence $0 \to T\mathcal{F} \to T_S X \to N_{\mathcal{F}} \to 0$ and, by Lemma 4.2, obtain the equality

$\Omega^1_{\mathcal{F}} = T\mathcal{F}^\vee$. So $\Omega^1_{\mathcal{F}}$ is locally free and $\mathcal{E}xt^1_X(\Omega^1_{\mathcal{F}}, \mathcal{O}_X) = 0$.

Now, given a quasi-coherent sheaf $\mathscr{G}$ of X, its support $\mathrm{supp}(\mathscr{G}) \subseteq X$ represents the following sub-functor of $\mathrm{Hom}(-, X)$:

$$T \longmapsto \{f : T \to X \quad \text{s.t.: } f^*\mathscr{G} \text{ is not a torsion sheaf}\} \subseteq \mathrm{Hom}(T, X).$$

So, let's take a morphism $f : T \to Y_1 \subseteq X$.

(i) $f : T \to Y_1$ is an immersion: Suppose $f^*\mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X)$ is a torsion sheaf. Then there's a point $t \in T$ such that

$$\mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X) \otimes k(t) = 0.$$

By Nakayama's lemma this implies that there's an open subset $U \subseteq X$ containing t such that $\mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X)|_U = 0$. This in turn implies $\mathcal{E}xt^1_X(\Omega^1_{\mathcal{F}}, \mathcal{O}_X)|_U = 0$, contradicting the fact that $t \in T \subseteq Y_1$. Then $T \subseteq Y_2$.
Similarly one proves that if $T \subseteq Y_2$ then $T \subseteq Y_1$.

(ii) General case: Taking the scheme-theoretic image of f, we can reduce to the above case where T is a sub-scheme of X.

□

7.1. **The codimension 1 case.** We now treat the case of families of codimension 1 foliations. From now on, we'll suppose that $X \to S$ is a smooth morphism.

Definition 7.4. A family of involutive distributions

$$0 \to T\mathcal{F} \to T_S X \to N_{\mathcal{F}} \to 0,$$

is of *codimension* 1 iff $N_{\mathcal{F}}$ is a sheaf of generic rank 1.

Likewise a family of Pfaff systems

$$0 \to I(\mathcal{F}) \to \Omega^1_{X|S} \to \Omega^1_{\mathcal{F}} \to 0,$$

is of codimension 1 if the sheaf $I(\mathcal{F})$ have generic rank 1.

Lemma 7.5. *Consider a family of codimension* 1 *Pfaff systems*

$$0 \to I(\mathcal{F}) \to \Omega^1_{X|S} \to \Omega^1_{\mathcal{F}} \to 0,$$

over an integral scheme X, such that $\Omega^1_{\mathcal{F}}$ is torsion-free. Then $I(\mathcal{F})$ is a line-bundle over X.

Proof. If $\Omega^1_{\mathcal{F}}$ is torsion-free, by Lemma 4.2 we have $I(\mathcal{F}) \cong N^\vee_{\mathcal{F}}$. In particular $I(\mathcal{F})$ is the dual of a sheaf, and thus is reflexive and observes property S_2. Write $I = I(\mathcal{F})$ and consider now the sheaf $I^\vee \otimes I$ together with the canonical morphism

$$I^\vee \otimes I \to \mathcal{O}_X.$$

The generic rank of $I^\vee \otimes I$ is 1. As I is reflexive, $I^\vee \otimes I$ is self-dual. So the canonical morphism above induces the dual morphism $\mathcal{O}_X \to I^\vee \otimes I$. The composition

$$\mathcal{O}_X \to I^\vee \otimes I \to \mathcal{O}_X$$

must be invertible, otherwise the image of $I^\vee \otimes I$ in $\mathcal{O}_X$ would be a torsion sub-sheaf. Then I is an invertible sheaf. □

Proposition 7.6. *In the case of codimension* 1 *Pfaff systems, if $\Omega^1_{\mathcal{F}}$ is torsion-free over X and the inclusion $I(\mathcal{F}) \to \Omega^1_{X|S}$ is nowhere trivial on S (meaning that $I(\mathcal{F}) \otimes \mathcal{O}_T \to \Omega^1_{X|S} \otimes \mathcal{O}_T$ is never the zero morphism for any $T \to S$), then the family is automatically flat.*

Proof. Indeed, $\Omega^1_{\mathcal{F}}$ being torsion free implies that the rank 1 sheaf $I(\mathcal{F})$ must be a line bundle. Then, if we take any morphism $f : T \to S$ and take pull-backs, we'll have an exact sequence

$$0 \to \mathrm{Tor}_1^S(\Omega^1_{\mathcal{F}}, T) \to f^*I(\mathcal{F}) \to f^*\Omega^1_{X|S} \to f^*\Omega^1 \to 0.$$

Now, as $I(\mathcal{F})$ is a line bundle, the cokernel $f^*I(\mathcal{F})/\mathrm{Tor}_1^S(\Omega^1_{\mathcal{F}}, T)$ must be a torsion sheaf over X_T. But, X being smooth over S, the annihilator $f^*\Omega^1_{X|S}$ is of the form $p^*(J)$, with $J \subset \mathcal{O}_T$, so $f^*I(\mathcal{F}) \to f^*\Omega^1_{X|S}$ must be the zero morphism when restricted to $\mathcal{O}_T/J$, contradicting the nowhere triviality assumption. □

Remark 7.7. In the codimension 1 case, we can calculate $\mathrm{sing}(\mathcal{F})$ by noting that $\mathcal{E}xt^1_X(\Omega^1_{\mathcal{F}}, \mathcal{O}_X)$ is the cokernel in the exact sequence

$$T_SX \to I(\mathcal{F})^\vee \to \mathcal{E}xt^1_X(\Omega^1_{\mathcal{F}}, \mathcal{O}_X) \to 0.$$

We can then tensor the sequence by $I(\mathcal{F})$ and obtain

$$T_SX \otimes I(\mathcal{F}) \to \mathcal{O}_X \to \mathcal{E}xt^1_X(\Omega^1_{\mathcal{F}}, \mathcal{O}_X) \otimes I(\mathcal{F}) \to 0.$$

Now, $I(\mathcal{F})$ being a line bundle, the support of $\mathcal{E}xt^1_X(\Omega^1_{\mathcal{F}}, \mathcal{O}_X)$ and that of $\mathcal{E}xt^1_X(\Omega^1_{\mathcal{F}}, \mathcal{O}_X) \otimes I(\mathcal{F})$ are exactly the same. Note then that, in the second exact sequence, the cokernel is the scheme-theoretic zero locus of the twisted 1-form given by

$$\mathcal{O}_X \xrightarrow{\omega} \Omega^1_{X|S} \otimes I(\mathcal{F})^\vee,$$

as defined in Section 3.2. So, if we have a family of codimension 1 Pfaff systems given locally by a twisted form

$$\omega = \sum_{i=1}^n f_i(x)dx_i,$$

then $\mathrm{sing}(\mathcal{F})$ is the scheme defined by the ideal $(f_1, \ldots, f_n)$.

The above proposition and remark tell us that our definition of flat family for Pfaff systems of codimension 1 is essentially the same as the one used in the now classical works of Lins-Neto, Cerveau, et.al.

Theorem 7.8. *Assume we have two families*

$$0 \to I(\mathcal{F}) \to \Omega^1_{X|S} \to \Omega^1_{\mathcal{F}} \to 0 \tag{2}$$

$$0 \to T\mathcal{F} \to T_SX \to N_{\mathcal{F}} \to 0, \tag{3}$$

satisfying the following conditions:

- *The families 2 and 3 are dual to each other.*
- *$N_{\mathcal{F}}$ is torsion free (or, equivalently, $\Omega^1_{\mathcal{F}}$ is torsion free).*
- *They are codimension 1 families.*
- $\mathrm{sing}(\mathcal{F})$ *is flat over S.*

Then 2 is flat if and only if 3 is flat.

Proof. Let $\Sigma = \mathrm{sing}(\mathcal{F})$.

Let's suppose first that the family

$$0 \to I(\mathcal{F}) \to \Omega^1_{X|S} \to \Omega^1_{\mathcal{F}} \to 0$$

is flat. We have to prove that $N_{\mathcal{F}}$ is also flat. To do this we note that applying the functor $\mathcal{H}om_X(-,\mathcal{O}_X)$ to the family of distributions not only gives us the family of Pfaff systems but also the exact sequence

$$0 \to N_{\mathcal{F}} \to I(\mathcal{F})^\vee \to \mathcal{E}xt^1_X(\Omega^1_{\mathcal{F}},\mathcal{O}_X) \to 0.$$

Being $\Omega^1_{\mathcal{F}}$ torsion-free, $I(\mathcal{F})$ must be a line bundle, and so must $I(\mathcal{F})^\vee$; let's call $I(\mathcal{F})^\vee = \mathcal{L}$ to ease the notation. Now $\mathcal{L}$ has $N_{\mathcal{F}}$ as a sub-sheaf generically of rank 1, so $N_{\mathcal{F}} = \mathcal{I}\cdot\mathcal{L}$ for some ideal sheaf $\mathcal{I}$. Then $\mathcal{E}xt^1_X(\Omega^1_{\mathcal{F}},\mathcal{O}_X) \cong \mathcal{L}\otimes\mathcal{O}_X/\mathcal{I}$. As $\Sigma = \mathrm{supp}(\mathcal{E}xt^1_X(\Omega^1_{\mathcal{F}},\mathcal{O}_X))$, one necessarily has $\mathcal{E}xt^1_X(\Omega^1_{\mathcal{F}},\mathcal{O}_X) \cong \mathcal{L}_\Sigma$. Then $\mathcal{L}_\Sigma$, being a locally free sheaf over Σ which is flat over S, is itself flat over S. Therefore, as $\mathcal{L}$ is also flat over S, flatness for $N_{\mathcal{F}}$ follows.

Let's suppose now that the family

$$0 \to T\mathcal{F} \to T_S X \to N_{\mathcal{F}} \to 0$$

is flat. We have to prove that $\Omega^1_{\mathcal{F}}$ is also flat. By the above proposition, it's enough to show that the morphism $I(\mathcal{F}) \xrightarrow{\iota} \Omega^1_{X|S}$ is nowhere zero. Suppose there is $T \to S$ such that $\iota_T = 0$. Take an open set $U \subset X$ where $\Omega^1_{\mathcal{F}}$ is locally free. In that open set, we can apply base change with respect to the functor $\mathcal{H}om_X(-,\mathcal{O}_X)$ ([1] or [4]); so, restricting everything to U we have $(\iota_T)^\vee \cong (\iota^\vee)_T$. But, in U, $\iota^\vee$ is the morphism $T_S X \to N_{\mathcal{F}}$ and so it cannot become the zero morphism under any base change. □

7.2. **The arbitrary codimension case.** To give an analogous theorem to 7.8 in arbitrary codimension, we'll have to deal with finer invariants than the singular locus of the foliation. In the scheme X, we'll consider a stratification naturally associated with $\mathcal{F}$. This stratification have been already studied and described by Suwa in [18]. To deal with flatness issues, we have to provide a scheme structure to Suwa's stratification; this will be a particular case of a flattening stratification. Before going into that, we begin with some generalities. Remember that we are working over a smooth morphism $X \to S$.

Lemma 7.9. *Let $X \to S$ be a smooth morphism and $\mathscr{F}$ a coherent sheaf on X that is relatively $Z^{(2)}$-closed over S. Then, for all $s \in S$, the sheaf $\mathscr{F}_s = \mathscr{F}\otimes k(s)$ is $Z^{(2)}$-closed over X_s.*

Proof. We have to show that, for every $U \subset X_s$ such that $\mathrm{codim}(X\setminus U) \geq 2$, the restriction

$$\mathscr{F}_s \xrightarrow{\rho_U} \mathscr{F}_s|_U$$

is surjective. As the formal completion $\widehat{\mathcal{O}}_{X_s,x}$ of $\mathcal{O}_{X_s}$ with respect to any closed point x is faithfully flat over $\mathcal{O}_{X_s}$ [11, IV.3.2], we can check surjectivity of ρ_U by looking at every formal completion. As $X \to S$ is smooth formally around a point x, we have $\mathcal{O}_X \cong \mathcal{O}_S \otimes_k k[z_1,\dots,z_d]$ and so we can take an open subset $V \subseteq X$ to be $V = U\times S$. Then, with this choice of V, we have an epimorphism

$$\widehat{\mathscr{F}}|_V \to \widehat{\mathscr{F}_s}|_U \to 0.$$

Then we have a diagram with exact rows and columns

$$\begin{tikzcd}
\widehat{\mathscr{F}} \arrow[r] \arrow[d, "\rho_V"] & \widehat{\mathscr{F}_s} \arrow[r] \arrow[d, "\rho_U"] & 0\,. \\
\widehat{\mathscr{F}}|_V \arrow[r] \arrow[d] & \widehat{\mathscr{F}_s}|_U \arrow[r] & 0 \\
0 & &
\end{tikzcd}$$

So ρ_U must be an epimorphism as well.

□

Lemma 7.10. *Let $p : X \to S$ be a smooth morphism. Consider a family of distributions*

$$0 \to T\mathcal{F} \to T_S X \to N_{\mathcal{F}} \to 0.$$

If the codimension of $\mathrm{sing}(\mathcal{F})$ *with respect to* $X_{p(\mathrm{sing}(\mathcal{F}))}$ *is greater than* 2*, then, for every map* $T \to S$*, one has*

$$\mathcal{H}om_X(T\mathcal{F}, \mathcal{O}_X) \otimes \mathcal{O}_T \cong \mathcal{H}om_{X_T}(T\mathcal{F}_T, \mathcal{O}_T).$$

The analogous statement is true for $I(\mathcal{F})^\vee$ in a flat family of Pfaff systems.

Proof. By [1, Theorem 1.9], we only have to prove that, for every closed point $s \in S$, the natural map

$$\mathcal{H}om_X(T\mathcal{F}, \mathcal{O}_X) \otimes k(s) \to \mathcal{H}om_{X_s}(T\mathcal{F} \otimes k(s), \mathcal{O}_X \otimes k(s))$$

is surjective. Being the dual of some sheaves, both $\mathcal{H}om_X(T\mathcal{F}, \mathcal{O}_X)$ and

$$\mathcal{H}om_{X_s}(T\mathcal{F} \otimes k(s), \mathcal{O}_X \otimes k(s))$$

possess the relative property S_2 with respect to p (Proposition 3.7), and so are relatively $Z^{(2)}$-closed with respect to p, and so is $\mathcal{H}om_X(T\mathcal{F}, \mathcal{O}_X) \otimes k(s)$ by the above lemma.

Let $U = X \setminus \mathrm{sing}\mathcal{F}$ and $j : U \hookrightarrow X$ be the inclusion. As $T\mathcal{F}|_U$ is locally free over U, so is $T\mathcal{F}^\vee|_U$. Then, in U, we have

$$\mathcal{E}xt^1(T\mathcal{F}|_U, \mathcal{O}_X|_U \otimes_S \mathcal{G}) = 0,$$

for every $\mathcal{G} \in Coh(S)$. Then, from the exchange property for local Ext's [1, Theorem 1.9], we get surjectivity on

$$\mathcal{H}om_X(T\mathcal{F}|_U, \mathcal{O}_X|_U) \otimes k(s) \to \mathcal{H}om_{X_s}(T\mathcal{F}|_U \otimes k(s), \mathcal{O}_X|_U \otimes k(s)).$$

But, as $\mathrm{codim}(\mathrm{sing}(\mathcal{F})) > 1$ and both sheaves are S_2, then surjectivity holds in all of X_s. □

Lemma 7.11. *Suppose that a flat family*

$$0 \to T\mathcal{F} \to T_S X \to N_{\mathcal{F}} \to 0$$

is such that the relative codimension $d_S(\mathrm{sing}(\mathcal{F}))$ of $\mathrm{sing}(\mathcal{F})$ over S verifies $d_S(\mathrm{sing}(\mathcal{F})) \geq 2$. Suppose further that the flattening stratification of X over $T\mathcal{F}$ is flat over S (c.f., Proposition 3.14). Then $T\mathcal{F}^\vee$ is also a flat $\mathcal{O}_S$-module.

The analogous statement is true for $I(\mathcal{F})^\vee$ in a flat family of Pfaff systems.

Proof. The proof works exactly the same for distributions of Pfaff systems mutatis mutandi.

Take $\coprod_P X_P$ to be the flattening stratification of X with respect to $T\mathcal{F}$. The restriction $T\mathcal{F}_{X_P}$ (being coherent and flat over X_P) is locally free over X_P, and hence so is its dual $\mathcal{H}om_{X_P}(T\mathcal{F}_{X_P}, \mathcal{O}_{X_P})$. By Lemma 7.10, in each stratum X_P, we have the isomorphism

$$\mathcal{H}om_{X_P}(T\mathcal{F}_{X_P}, \mathcal{O}_{X_P}) \cong \mathcal{H}om_X(T\mathcal{F}, \mathcal{O}_X) \otimes \mathcal{O}_{X_P} = T\mathcal{F}^\vee \otimes \mathcal{O}_{X_P}.$$

So $T\mathcal{F}^\vee$ is flat when restricted to the filtration $\coprod_P X_P$, which is in turn flat over S. Then, by [8, Section 5.4.2], $T\mathcal{F}^\vee$ is flat over S. □

Definition 7.12. For a family of distributions consider the flattening stratification

$$\coprod_{P(\mathcal{F})} X_{P(\mathcal{F})} \subseteq X$$

of X with respect to $T\mathcal{F} \oplus N_{\mathcal{F}}$. We call this the *rank stratification* of X with respect to $T\mathcal{F}$.

Remark 7.13. Note that the flattening stratification of $T\mathcal{F} \oplus N_{\mathcal{F}}$ is the (scheme-theoretic) intersection of the flattening stratification of $T\mathcal{F}$ with that of $N_{\mathcal{F}}$. This is because

$$(T\mathcal{F} \oplus N_{\mathcal{F}}) \otimes \mathcal{O}_Y$$

is flat if and only if both $T\mathcal{F} \otimes \mathcal{O}_Y$ and $N_{\mathcal{F}} \otimes \mathcal{O}_Y$ are.

This tells us, in particular, that each stratum is indexed by two natural numbers r and k such that

$$x \in X_{r,k} \Longleftrightarrow \dim(T\mathcal{F} \otimes k(x)) = r \text{ and } \dim(N_{\mathcal{F}} \otimes k(x)) = k.$$

In [18], Suwa studied a related stratification associated to a foliation. Given a distribution $D \subset TM$ on a complex manifold M, he defines the strata $M^{(l)}$ as

$$M^{(l)} = \{x \in M \text{ such that } D_x \subset T_x M \text{ is a sub-space of dimension } l\ \}.$$

Here D is spanned point-wise by vector fields $v_1, \ldots, v_r$, and $D_x =< v_i(x) >$. Clearly if D is of generic rank r, the open stratum is $M^{(r)}$.

Note that, in the setting of distribution as sub-sheafs $i : T\mathcal{F} \hookrightarrow TX$ of the tangent sheaf of a variety, the vector space $T_x\mathcal{F}$ is actually the image of the map

$$T\mathcal{F} \otimes k(x) \xrightarrow{i \otimes k(x)} TX \otimes k(x),$$

whose kernel is $\mathrm{Tor}_1^X(N_{\mathcal{F}}, k(x))$. Moreover we have the exact sequence

$$0 \to T_x\mathcal{F} = \mathrm{Im}(i \otimes k(x)) \to TX \otimes k(x) \to N_{\mathcal{F}} \otimes k(x) \to 0.$$

In particular, in a variety X of dimension n, if $\dim(T_x\mathcal{F}) = l$, then $\dim(N_{\mathcal{F}} \otimes k(x)) = n - l$. So what we call the rank stratification of X is actually a refinement of the stratification studied in [18].

Our main motivation for defining this refinement of the stratification of [18] is the following result.

Theorem 7.14. *Assume we have dual families*

$$0 \to T\mathcal{F} \to T_S X \to N_{\mathcal{F}} \to 0, \tag{4}$$

$$0 \to I(\mathcal{F}) \to \Omega^1_{X|S} \to \Omega^1_{\mathcal{F}} \to 0, \tag{5}$$

parametrized by a scheme S of finite type over an algebraically closed field, such that

- *$N_{\mathcal{F}}$ is torsion free*
- *The relative codimension of* $\mathrm{sing}(\mathcal{F})$ *over S (that is $d_S(\mathrm{sing}(\mathcal{F}))$) verifies $d_S(\mathrm{sing}(\mathcal{F}) \geq 2$.*
- *Each stratum $X_{r,k}$ of the rank stratification is flat over S.*

Then 4 is flat over S if and only if 5 is. Moreover, for each point $s \in S$, we have

$$I(\mathcal{F})_s = (N_{\mathcal{F}s})^\vee,$$

in other terms "the dual family is the family of the duals".

Proof. We prove one of the implications; the proof of the other is identical.

Consider the exact sequence

$$0 \to \Omega^1_{\mathcal{F}} \to T\mathcal{F}^\vee \to \mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X) \to 0.$$

It is clear that, to prove flatness of the dual family, it's enough to show $\mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X)$ is flat over S.

Also, by Lemma 7.10, we have, for every $s \in S$, the diagram with exact rows and columns,

$$\begin{tikzcd}
\mathcal{H}om(I(\mathcal{F}), \mathcal{O}_X) \otimes k(s) \arrow[r] \arrow[d] & \mathcal{H}om(I(\mathcal{F})_s, \mathcal{O}_{X_s}) \arrow[r] \arrow[d] & 0 \,. \\
\mathcal{E}xt^1_X(\Omega^1_{\mathcal{F}}, \mathcal{O}_X) \otimes k(s) \arrow[r] \arrow[d] & \mathcal{E}xt^1_X(\Omega^1_{\mathcal{F}} \otimes k(s), \mathcal{O}_{X_s}) \arrow[d] & \\
0 & 0 &
\end{tikzcd}$$

So $\mathcal{E}xt^1_X(\Omega^1_{\mathcal{F}}, \mathcal{O}_X) \otimes k(s) \to \mathcal{E}xt^1_X(\Omega^1_{\mathcal{F}} \otimes k(s), \mathcal{O}_{X_s})$ is surjective for every $s \in S$ and so, by [1, Theorem 1.9], the exchange property is valid for $\mathcal{E}xt^1_X(\Omega^1_{\mathcal{F}}, \mathcal{O}_X)$. If moreover $\mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X)$ is flat over S, then, again by [1, Theorem 1.9],

$$I(\mathcal{F})_s = \mathcal{H}om_X(N_{\mathcal{F}}, \mathcal{O}_X) \otimes k(s) \cong \mathcal{H}om_X(N_{\mathcal{F}s}, \mathcal{O}_{X_s}) = (N_{\mathcal{F}s})^\vee.$$

By Proposition 3.14, it is enough to show the restriction of $\mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X)$ to every rank stratum is flat over S. So let $Y \subseteq X$ be a rank stratum. If we can show that $\mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X) \otimes \mathcal{O}_Y$ is locally free, then we're set.

By hypothesis, one has the isomorphism $\mathcal{H}om_X(T\mathcal{F}, \mathcal{O}_X) \otimes \mathcal{O}_Y \cong \mathcal{H}om_Y(T\mathcal{F}_Y, \mathcal{O}_Y)$. So we can express $\mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X) \otimes \mathcal{O}_Y$ as the cokernel in the $\mathcal{O}_Y$-modules exact sequence

$$\mathcal{H}om_Y(TX_Y, \mathcal{O}_Y) \to \mathcal{H}om_Y(T\mathcal{F}_Y, \mathcal{O}_Y) \to \mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X)_Y \to 0.$$

So, localizing at a point $y \in Y$, we can realize the local $\mathcal{O}_{Y,y}$-module $\mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X)_{Y,y}$ as the set of maps $T\mathcal{F}_{Y,y} \to \mathcal{O}_{Y,y}$ modulo the ones that factors as

$$\begin{tikzcd}
T\mathcal{F}_{Y,y} \arrow[r, dashed] \arrow[d] & \mathcal{O}_{Y,y} \,. \\
TX_{Y,y} \arrow[ur] &
\end{tikzcd}$$

To study $\mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X)_{Y,y}$ this way, note that we have the following exact sequence.

$$0 \to \mathrm{Tor}^X_1(N_{\mathcal{F}}, \mathcal{O}_{Y,y}) \to T\mathcal{F}_{Y,y} \to TX_{Y,y} \to (N_{\mathcal{F}})_{Y,y} \to 0.$$

which we split into two short exact sequences,

$$0 \to \mathcal{K} \to \quad TX_{Y,y} \quad \to (N_{\mathcal{F}})_{Y,y} \to 0 \quad \text{and} \tag{6}$$

$$0 \to \mathrm{Tor}^X_1(N_{\mathcal{F}}, \mathcal{O}_{Y,y}) \to \quad T\mathcal{F}_{Y,y} \quad \to \mathcal{K} \to 0. \tag{7}$$

Now, as Y is a rank stratum, $\mathcal{Q}_{\mathcal{Y}}$ and $T\mathcal{F}_Y$ are flat over Y, and coherent, and so they are locally free. As a consequence, short exact sequence (6) splits, so $TX_{Y,y} \cong (N_{\mathcal{F}})_{Y,y} \oplus \mathcal{K}$. So

$$\mathcal{H}om_Y(TX_Y, \mathcal{O}_Y)_y \cong \mathcal{H}om_Y(\mathcal{K}, \mathcal{O}_{Y,y}) \oplus \mathcal{H}om_Y((N_{\mathcal{F}})_Y, \mathcal{O}_Y)_y$$

and we get $\mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X)_{Y,y}$ as the cokernel in

$$\mathcal{H}om_Y(\mathcal{K}, \mathcal{O}_{Y,y}) \to \mathcal{H}om_Y(T\mathcal{F}_Y, \mathcal{O}_Y)_y \to \mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X)_{Y,y} \to 0. \tag{8}$$

Since $(N_{\mathcal{F}})_Y$ and TX_Y are locally free over Y, so is $\mathcal{K}$. Then short exact sequence (7) splits, so $T\mathcal{F}_{Y,y} \cong \mathrm{Tor}^X_1(N_{\mathcal{F}}, \mathcal{O}_{Y,y}) \oplus \mathcal{K}$. Also, as $T\mathcal{F}_Y$ and $\mathcal{K}$ are locally free over Y, so is $\mathrm{Tor}^X_1(N_{\mathcal{F}}, \mathcal{O}_Y)$. Sequence (8) now reads

$$\mathcal{H}om_Y(\mathcal{K}, \mathcal{O}_{Y,y}) \to \mathcal{H}om_Y(\mathrm{Tor}^X_1(N_{\mathcal{F}}, \mathcal{O}_{Y,y}), \mathcal{O}_Y)_y \oplus \mathcal{H}om_Y(\mathcal{K}, \mathcal{O}_{Y,y}) \to \mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X)_{Y,y} \to 0.$$

So we have

$$\mathcal{H}om_Y(\mathrm{Tor}_1^X(N_{\mathcal{F}}, \mathcal{O}_{Y,y}), \mathcal{O}_Y)_y \cong \mathcal{E}xt_X^1(N_{\mathcal{F}}, \mathcal{O}_X)_{Y,y}.$$

Now, as $\mathrm{Tor}_1^X(N_{\mathcal{F}}, \mathcal{O}_{Y,y})$ is locally free over Y, so is its dual. In other words, we just proved $\mathcal{E}xt_X^1(N_{\mathcal{F}}, \mathcal{O}_X)_Y$ is locally free over Y, which settles the theorem. □

Now, by generic flatness, we can conclude the following:

Corollary 7.15. *Every irreducible component of the scheme* Inv_P *is birationally equivalent to an irreducible component of* iPf_P.

Remark 7.16. During the proof of Theorem 7.14, we have actually obtained this result:

Proposition 7.17. $\mathcal{E}xt_X^1(N_{\mathcal{F}}, \mathcal{O}_X)$ *is flat over the rank stratification.*

In particular, if $\coprod X_Q$ denotes the flattening stratification of $\mathcal{E}xt_X^1(N_{\mathcal{F}}, \mathcal{O}_X)$, there is a morphism

$$\coprod_{P(\mathcal{F})} X_{P(\mathcal{F})} \to \coprod_Q X_Q.$$

Now, by the construction of the flattening stratification, $\coprod X_Q$ consist of an open stratum U such that $\mathcal{E}xt_X^1(N_{\mathcal{F}}, \mathcal{O}_X)|_U = 0$, and closed strata whose closure is $\mathrm{sing}(\mathcal{F})$. So the morphism $\coprod_{P(\mathcal{F})} X_{P(\mathcal{F})} \to \coprod_Q X_Q$ actually defines a stratification of $\mathrm{sing}(\mathcal{F})$.

8. Singularities

Theorem 7.8 gives a condition for a flat family of integrable Pfaff systems to give rise to a flat family of involutive distributions in terms of the flatness of the singular locus. We have then to be able to decide when can we apply the theorem. More precisely, say

$$0 \to I(\mathcal{F}) \to \Omega^1_{X|S} \to \Omega^1_{\mathcal{F}} \to 0$$

is a flat family of codimension 1 integrable Pfaff systems, and let $s \in S$. How do we know when $\mathrm{sing}(\mathcal{F})$ is flat around s? In this section, we address this question and give a sufficient condition for $\mathrm{sing}(\mathcal{F})$ to be flat at s in terms of the classification of singular points of the Pfaff system $0 \to I(\mathcal{F})_s \to \Omega^1_{X_s} \to \Omega^1_{\mathcal{F}_s} \to 0$.

From now on, we will consider only Pfaff systems such that $\Omega^1_{\mathcal{F}}$ is torsion-free.

Remember that, if we have a Pfaff system of codimension 1, $0 \to I(\mathcal{F})_s \to \Omega^1_{X_s} \to \Omega^1_{\mathcal{F}_s} \to 0$, such that $\Omega^1_{\mathcal{F}_s}$ is torsion-free, we can consider, locally on X, that it is given by a single 1-form ω and that it is integrable iff $\omega \wedge d\omega = 0$.

Definition 8.1. Let $0 \to I(\mathcal{F})_s \to \Omega^1_{X_s} \to \Omega^1_{\mathcal{F}_s} \to 0$ be a codimension 1 integrable Pfaff system, and let $x \in X$. We say that x is a *Kupka singularity* of the Pfaff system if, for some 1-form ω locally defining $I(\mathcal{F})$ around x, we have $\omega_x = 0$ and $d\omega_x \neq 0$.

We also say that x is a Kupka singularity of the foliation defined by $I(\mathcal{F})$.

Proposition 8.2. *The set of Kupka singularities of a codimension* 1 *foliation, if non-void, has a natural structure of a codimension* 2 *sub-scheme of* X.

Proof. See [14] □

Definition 8.3. Let $0 \to I(\mathcal{F})_s \to \Omega^1_{X_s} \to \Omega^1_{\mathcal{F}_s} \to 0$ be as before. Say X has dimension n over $\mathbb{C}$. Then we call $x \in X$ a *Reeb singularity* if there exists an analytic neighborhood U of x such that $I(\mathcal{F})$ may be generated by a form ω with the property that ω can be written, locally

in U, in the form $\omega = \sum_{i=1}^{n} f_i dz_i$ with $f_i(x) = 0$ for all i, and $(df_1)_x, \ldots, (df_n)_x$ are linearly independent in $T_x^* X$.

Remark 8.4. Note that Kupka singularities and Reeb singularities are singularities in the sense of 7.1, i.e., they are points in $\mathrm{sing}(\mathcal{F})$.

We now give a version for families of the fundamental result of Kupka.

Proposition 8.5. *Let*

$$0 \to I(\mathcal{F}) \to \Omega^1_{X|S} \to \Omega^1_{\mathcal{F}} \to 0$$

be a flat family of integrable Pfaff systems of codimension 1*, and let* $\Sigma = \mathrm{sing}(\mathcal{F}) \subset X$*. Let* $s \in S$*, and* $x \in \Sigma_s$ *be such that* x *is a Kupka singularity of* $0 \to I(\mathcal{F})_s \to \Omega^1_{X_s} \to \Omega^1_{\mathcal{F}_s} \to 0$*. Then, locally around* x*,* $I(\mathcal{F})$ *can be given by a* relative 1*-form* $\omega(z,s) \in \Omega^1_{X|S}$ *such that*

$$\omega = f_1(z,s)dz_1 + f_2(z,s)dz_2,$$

that is, ω *is locally the pull-back of a relative form* $\eta \in \Omega^1_{Y|S}$*, where* $Y \to S$ *is of relative dimension* 2*.*

The proof is essentially the same as the proof of the classical Kupka theorem, as in [15]. One only needs to note that every ingredient there can be generalized to a relative setup.

For this we note that, as $p : X \to S$ is a smooth morphism, the relative tangent sheaf $T_S X$ is locally free and is the dual sheaf of the locally free sheaf $\Omega^1_{X|S}$. We note also that, if $v \in T_S X(U)$, and $\omega \in \Omega^1_{X|S}(U)$, the relative Lie derivative $L_v(\omega)$ is well-defined by Cartan's formula

$$L_v^S = d_S \iota_v(\omega) + \iota_v(d_S \omega),$$

where $\iota_v(\omega) =< v, \omega >$ is the pairing of dual spaces (and by extension also the map $\Omega^q_{X|S} \to \Omega^{q-1}_{X|S}$ determined by v), and d_S is the *relative* de Rham differential. Also $\Omega^q_{X|S} = \wedge^q \Omega^1_{X|S}$.

Finally we observe that, if $p : X \to S$ is of relative dimension d and X is a regular variety over $\mathbb{C}$ of total dimension n, a family of integrable Pfaff systems gives rise to a foliation on X whose leaves are tangent to the fibers of p. Indeed, we can pull back the subsheaf $I(\mathcal{F}) \subset \Omega^1 X|S$ by the natural epimorphism

$$f^* \Omega^1_S \to \Omega^1_X \to \Omega^1_{X|S} \to 0,$$

and get $J = I(\mathcal{F}) + f^* \Omega^1_S \subset \Omega^1_X$, which is an integrable Pfaff system in X, determining a foliation $\hat{\mathcal{F}}$. As $f^* \Omega^1_S \subset J$, the leaves of the foliation $\hat{\mathcal{F}}$ are contained in the fibers X_s of p.

In the general case, where p is smooth but S and X need not to be regular over $\mathbb{C}$, the Frobenius theorem still gives foliations $\mathcal{F}_s$ in each fiber X_s. Indeed, as $p : X \to S$ is smooth, each fiber X_s is regular over $\mathbb{C}$ and, $\Omega^1_{\mathcal{F}}$ being flat, $I(\mathcal{F})_s \subset \Omega^1_{X_s}$ is an integrable Pfaff system on X_s.

Proposition 8.6. *Let* $p : X \to S$ *a smooth morphism over* $\mathbb{C}$ *and*

$$0 \to I(\mathcal{F}) \to \Omega^1 X|S \to \Omega^1_{\mathcal{F}} \to 0$$

a codimension 1 *flat family of Pfaff systems. Let* $\omega \in \Omega^1_{X|S}(U)$ *be an integrable* 1*-form such that* $I(\mathcal{F})(U) = (\omega)$ *in a neighborhood* U *of a point* $x \in X$*. Then* $d\omega$ *is locally decomposable.*

Proof. As $T_S X = (\Omega^1 X|S)^\vee$ and $\Omega^q_{X|S} = \wedge^q \Omega^1_{X|S}$, we can apply Plücker relations to determine if $d\omega$ is locally decomposable and proceed as in [15, Lemma 2.5]. □

Lemma 8.7. *Suppose that* $d\omega_x \neq 0$*. Consider* $\mathcal{G}_s$ *the codimension* 2 *foliations defined by* $d\omega$ *in* X_s*. In the neighborhood* V *of* $x \in X$ *where* $\mathcal{G}_s$ *is non-singular for every* s*, we have that the leaves of* $\mathcal{G}_s$ *are integral manifolds of* $\omega|_{X_s}$*.*

Proof. We have only to prove, for every $v \in T_S X$ such that $\iota_v(d\omega) = 0$, that $\iota_v(\omega) = 0$. We can do this exactly as in [15, Lemma 2.6]. □

Lemma 8.8. *Assume the same hypothesis as Lemma 8.7. Let v be a vector field tangent to $\mathcal{G}$. Then the* relative *Lie derivative of ω with respect to v is zero.*

Proof. Like the proof of [15, Lemma 2.7]. □

Lemma 8.9. *Under the same hypothesis as Lemma 8.7 and 8.8,* $\mathrm{sing}(\omega)$ *is saturated by leaves of $(\mathcal{G}_s)_{s\in S}$ (i.e., take $y \in V$ a zero of ω such that $p(y) = s$, and L the leaf of $\mathcal{G}_s$ going through y. Then the inclusion $L \to V$ factors through* $\mathrm{sing}(\omega)$*).*

Proof. We can do this entirely on X_s. Then this reduces to [15, Lemma 2.7]. □

Proof of Proposition 8.5. We take an analytic neighborhood V of $x \in X$ such that $V \cong U \times D^d$ with $U \subseteq S$ an open set, D^d a complex polydisk, and

$$\begin{array}{rcl} p|V : V \cong U \times D^d & \longrightarrow & U. \\ (s, z_1, \ldots, z_d) & \mapsto & s \end{array}$$

By the Frobenius theorem, we can choose the coordinates z_i in such a way that

$$v_i = \frac{\partial}{\partial z_i} \in T_S X(V), \qquad 3 \leq i \leq d,$$

are tangent to $d\omega$. Then, as $L^S_{v_i}\omega = 0$ and $\iota_{v_i}\omega = 0$, we can write ω as

$$\omega = f_1(z, s)dz_1 + f_2(z, s)dz_2.$$

□

Proposition 8.10. *Let*

$$0 \to I(\mathcal{F}) \to \Omega^1_{X|S} \to \Omega^1_{\mathcal{F}} \to 0$$

be a flat family of integrable Pfaff systems of codimension 1*, and let* $\Sigma = \mathrm{sing}(\mathcal{F}) \subset X$*. Let $s \in S$, and $x \in \Sigma_s$ be such that x is a Kupka singularity of $0 \to I(\mathcal{F})_s \to \Omega^1_{X_s} \to \Omega^1_{\mathcal{F}_s} \to 0$. Then $\Sigma \to S$ is smooth around x.*

Proof. By 8.5 above, we can determine Σ around x as the common zeroes of $f_1(z, s)$ and $f_2(z, s)$. The condition $d\omega \neq 0$ implies Σ is *smooth* over S (remember that we are using the *relative* de Rham differential and that means the variable s counts as a constant). □

Proposition 8.11. *Let*

$$0 \to I(\mathcal{F}) \to \Omega^1_{X|S} \to \Omega^1_{\mathcal{F}} \to 0$$

be a flat family of integrable Pfaff systems of codimension 1*,* $\Sigma = \mathrm{sing}(\mathcal{F}) \subset X$*, and $s \in S$. Suppose $x \in \Sigma_s$ is such that x is a* Reeb *singularity of $0 \to I(\mathcal{F})_s \to \Omega^1_{X_s} \to \Omega^1_{\mathcal{F}_s} \to 0$. Then $\Sigma \to S$ is étalé around x.*

Proof. The condition on x means we can actually give $I(\mathcal{F})$ locally by a relative 1-form $\omega \in \Omega^1_{X|S}$, $\omega = \sum_{i=1}^n f_i(z, s)dz_i$, with n the relative dimension of X over S and the df_i's linearly independent on x. Then Σ is given by the equations $f_1 = \cdots = f_n = 0$ and is therefore étalé over S. □

With these two propositions, we are almost in a position to state our condition for flatness of the dual family, we just need a general:

Lemma 8.12. *Let $X \xrightarrow{p} S$ be a morphism between schemes of finite type over an algebraically closed field k. Let $U \subseteq X$ be the maximal open sub-scheme such that $U \xrightarrow{p} S$ is flat, and $s \in S$ a point such that X_s is without embedded components. If $U_s \subseteq X_s$ is dense, then $U_s = X_s$.*

Proof. By Proposition 3.12, we must check that, for A either a discrete valuation domain or an Artin ring of the form $k[T]/(T^{n+1})$, and every arrow $\mathrm{Spec}(A) \to S$, the pull-back scheme $X_{\mathrm{Spec}}(A)$ is flat over $\mathrm{Spec}(A)$. In this way, the problem reduces to the case where $S = \mathrm{Spec}(A)$.

(i) Case A is a DVD. In this case, A being a principal domain, flatness of X over $\mathrm{Spec}(A)$ is equivalent to the local rings $\mathcal{O}_{X,x}$ being torsion-free $A_{p(x)}$-modules for every point $x \in X$ ([11, IV.1.3] , so it suffices to consider the case $A_{p(x)} = A$.

Now, let $f \in \mathcal{O}_{X,x}$ and $J = \mathrm{Ann}_A(f) \subseteq A$. Suppose $J \neq (0)$ and consider $V(J) \subseteq \mathrm{Spec}(A)$; clearly $\mathrm{supp}(f) \subseteq p^{-1}(V(J)) \subseteq X$. So $U \cap \mathrm{supp}(f) = \varnothing$. But then the restriction $f|s$ of f to X_s has support disjoint with U_s. On the other hand

$$\mathrm{supp}(f|_s) = \overline{\{\mathfrak{P}_1\}} \cup \cdots \cup \overline{\{\mathfrak{P}_m\}} \subseteq X_s,$$

where $\{\mathfrak{P}_1, \ldots, \mathfrak{P}_m\} = \mathrm{Ass}(\mathcal{O}_{X_s,x}/(f|_s)) \subseteq \mathrm{Ass}(\mathcal{O}_{X_s,x})$.
As X_s is without immersed components, the $\mathfrak{P}_i$'s are all minimal, so $X_k \cap \overline{\mathfrak{P}_i}$ is an irreducible component of X_k. But

$$U_s \cap \overline{\mathfrak{P}_i} = \varnothing,$$

contradicting the hypothesis that U_s is dense in X_s.

(ii) Case $A = k[T]/(T^{n+1})$. Using Proposition 3.13, this works just as the first case taking $f \in \mathcal{O}_{X,x}$ as a section such that $T^n f = 0$ but $f \notin T\mathcal{O}_{X,x}$.

□

We have already said that, in a Pfaff system, Kupka singularities, if they exist, form a codimension 2 sub-scheme of X. We will call $\mathcal{K}(\mathcal{F})$ this sub-scheme, and $\overline{\mathcal{K}}(\mathcal{F})$ its closure.

Theorem 8.13. *Let*

$$0 \to I(\mathcal{F}) \to \Omega^1_{X|S} \to \Omega^1_{\mathcal{F}} \to 0$$

be a flat family of integrable Pfaff systems, for $s \in S$. Consider the Pfaff system

$$0 \to I(\mathcal{F})_s \to \Omega^1_{X_s} \to \Omega^1_{\mathcal{F}_s} \to 0.$$

If $\mathrm{sing}(\mathcal{F}_s)$ *is without embedded components and* $\mathrm{sing}(\mathcal{F}_s) = \overline{\mathcal{K}}(\mathcal{F}_s) \cup \{p_1, \ldots, p_m\}$*, where the p_i's are Reeb-type singularities, then* $\mathrm{sing}(\mathcal{F}) \to S$ *is flat in a neighborhood of $s \in S$.*

Proof. Indeed, by Proposition 8.10, $\mathrm{sing}(\mathcal{F})$ is flat in a neighborhood of $\mathcal{K}(\mathcal{F}_s)$, and, as $\mathrm{sing}(\mathcal{F}_s)$ is without embedded components, we can apply Lemma 8.12 to conclude that $\mathrm{sing}(\mathcal{F})$ is flat in a neighborhood of $\overline{\mathcal{K}(\mathcal{F}_s)}$.

Lastly, from Proposition 8.11, it follows that $\mathrm{sing}(\mathcal{F})$ is flat in a neighborhood of $\{p_1, \ldots, p_m\}$.

□

9. Applications

Let $X = \mathbb{P}^n(\mathbb{C})$. It's well-known that the class of sheaves $\mathscr{F}$ that split as a direct sum of line bundles $\mathscr{F} \cong \bigoplus_i \mathcal{O}(k_i)$ has no non-trivial deformations. Indeed, as deformation theory teaches us, first-order deformations of $\mathscr{F}$ are parametrized by $\mathrm{Ext}^1(\mathscr{F}, \mathscr{F})$, in this case we have

$$\mathrm{Ext}^1(\mathscr{F}, \mathscr{F}) \cong \bigoplus_{i,j} \mathrm{Ext}^1(\mathcal{O}(k_i), \mathcal{O}(k_j)) \cong$$

$$\bigoplus_{i,j} \mathrm{Ext}^1(\mathcal{O}, \mathcal{O}(k_j - k_i)) \cong \bigoplus_{i,j} H^1(\mathbb{P}^n, \mathcal{O}(k_j - k_i)) = 0$$

In particular, given a flat family of distributions

$$0 \to T\mathcal{F} \to T_S(\mathbb{P}^n \times S) \to N_{\mathcal{F}} \to 0,$$

such that, for some $s \in S$, $T\mathcal{F}_s \cong \bigoplus_i \mathcal{O}(k_i)$, the same decomposition holds true for the rest of the members of the family.

When we deal with codimension 1 foliations, it's more common, however, to work with Pfaff systems or, more concretely, with integrable twisted 1-forms $\omega \in \Omega^1_{\mathbb{P}^n}(d)$, $\omega \wedge d\omega = 0$ (see [14]). It's then than the following question emerged: Given a form $\omega \in \Omega^1_{\mathbb{P}^n}(d)$ such that the vector fields that annihilate ω generate a split sheaf (i.e., a sheaf that decomposes as direct sum of line bundles), will the same feature hold for every deformation of ω? Such a question was addressed by Cukierman and Pereira in [5]. Here, we use our results to recover the theorem of Cukierman-Pereira as a special case.

As was observed before, every time we have a codimension 1 Pfaff system

$$0 \to I(\mathcal{F}) \to \Omega^1_X \to \Omega^1_{\mathcal{F}} \to 0$$

such that $\Omega^1_{\mathcal{F}}$ is torsion-free, it follows that $I(\mathcal{F})$ must be a line bundle. In the case $X = \mathbb{P}^n(\mathbb{C})$, we thus have $I(\mathcal{F}) \cong \mathcal{O}_{\mathbb{P}^n}(-d)$ for some $d \in \mathbb{Z}$. It is then equivalent to give a Pfaff system and to give a morphism $0 \to \mathcal{O}_{\mathbb{P}^n}(-d) \to \Omega^1_{\mathbb{P}^n}$ which is in turn equivalent to $0 \to \mathcal{O}_{\mathbb{P}^n} \to \Omega^1_{\mathbb{P}^n}(d)$, that is, to give a global section ω of the sheaf $\Omega^1_{\mathbb{P}^n}(d)$.

We can explicitly write such a global section as

$$\omega = \sum_{i=0}^{n} f_i(x_0, \ldots, x_n) dx_i,$$

with f_i a homogeneous polynomial of degree $d-1$ and such that $\sum_i x_i f_i = 0$.

Such an expression gives rise to a foliation with split tangent sheaf if and only if there are $n-1$ polynomial vector fields

$$\begin{aligned} X_1 &= g_1^0 \frac{\partial}{\partial x_0} + \cdots + g_1^n \frac{\partial}{\partial x_n}, \\ &\vdots \\ X_{n-1} &= g_{n-1}^0 \frac{\partial}{\partial x_0} + \cdots + g_{n-1}^n \frac{\partial}{\partial x_n} \end{aligned}$$

such that $\omega(X_i) = 0$, for all $1 \leq i \leq n-1$; moreover, on a generic point, the vector fields must be linearly independent.

The singular set of this foliation is given by the ideal $I = (f_0, \ldots, f_n)$. The condition $\omega(X_i) = 0$ means that the ring $\mathbb{C}[x_0, \ldots, x_n]/I$ admits a syzygy of the form

$$0 \to \mathbb{C}[x_0, ..., x_n]^n \xrightarrow{\begin{pmatrix} x_0 & \cdots & x_n \\ g_1^0 & \cdots & g_1^n \\ \vdots & \ddots & \vdots \\ g_{n-1}^0 & \cdots & g_{n-1}^n \end{pmatrix}} \mathbb{C}[x_0, \ldots, x_n]^{n+1}$$

$$\xrightarrow{(f_0, \ldots, f_n)} \mathbb{C}[x_0, \ldots, x_n] \to \mathbb{C}[x_0, \ldots, x_n]/I \to 0.$$

For such rings a theorem of Hilbert and Schaps tells us the following:

Theorem 9.1 (Hilbert, Schaps)**.** *Let $A = k[x_0, ...x_n]/I$ be such that there is a 3-step resolution of A as above by free modules. Then the ring A is Cohen-Macaulay; in particular, it is equidimensional.*

Proof. This is theorem 5.1 in [2] □

We thus recover the theorem of Cukierman and Pereira ([5, Theorem 1]).

Theorem 9.2 ([5])**.** *Let*

$$0 \to I(\mathcal{F}) \to \Omega^1_{\mathbb{P}^n \times S|S} \to \Omega^1_{\mathcal{F}} \to 0$$

be a flat family of codimension 1 *integrable Pfaff systems, and suppose*

$$0 \to I(\mathcal{F})_s \to \Omega^1_{\mathbb{P}^n} \to \Omega^1_{\mathcal{F}} \to 0$$

defines a foliation with split tangent sheaf. If $\operatorname{sing}(\mathcal{F}_s) \setminus \overline{\mathcal{K}(\mathcal{F}_s)}$ *has codimension greater than* 2*, then every member of the family defines a split tangent sheaf foliation.*

Proof. By the above theorem, $\operatorname{sing}(\mathcal{F}_s)$ is equidimensional. The singular locus of a foliation on $\mathbb{P}^n$ always has an irreducible component of codimension 2 (see [14, Teorema 1.13]). If $\operatorname{sing}(\mathcal{F}_s) \setminus \overline{\mathcal{K}(\mathcal{F}_s)}$ has codimension greater than 2, then it must be empty. So $\operatorname{sing}(\mathcal{F}_s) = \overline{\mathcal{K}(\mathcal{F}_s)}$ and we can then apply Theorem 8.13. So the flat family

$$0 \to I(\mathcal{F}) \to \Omega^1_{\mathbb{P}^n \times S|S} \to \Omega^1_{\mathcal{F}} \to 0$$

gives rise to a flat family

$$0 \to T\mathcal{F} \to T_S X \to N_{\mathcal{F}} \to 0,$$

and so $T\mathcal{F}$ must be flat over S, and then $T\mathcal{F}_s$ splits for every $s \in S$. □

Remark 9.3. In [5], Theorem 2, a similar statement is proved for an arbitrary codimensional distribution (not necessarily involutive). More concretely, they prove:

Let D *be a singular holomorphic distribution on* $\mathbb{P}^n$*. If* $\operatorname{codim} \operatorname{sing}(D) \geq 3$ *and*

$$TD \cong \bigoplus_{i=1}^{d} \mathcal{O}(e_i), \qquad e_i \in \mathbb{Z},$$

then there exists a Zariski-open neighborhood $\mathcal{U}$ *of the space of Pfaff systems such that, for every Pfaff system* $\mathcal{I}' \subset \Omega^1_{\mathbb{P}^n}$ *in* $\mathcal{U}$*, its annihilator* TD' *splits as* $TD' \cong \bigoplus_{i=1}^{d} \mathcal{O}(e_i)$*.*

When we try to arrive at the above statement as a particular case of the the theory hereby exposed, we run into some difficulties, which we will explain now.

Note that generic (non-involutive) singular distributions have isolated singularities. So, in order to apply an analogue of Theorem 8.13 to non-involutive distributions of arbitrary codimension, we should be able to conclude, from the hypothesis $\operatorname{codim} \operatorname{sing}(D) \geq 3$, that D has isolated singularities. The author was unable to do so. It seems that a keener knowledge of the singular set of arbitrary codimensional distributions is necessary to correctly understand and generalize Theorem 2 in [5].

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