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Volume 10

The 12th International Workshop on
Real and Complex Singularities,
Celebrating the 60th birthday
of Prof. Shyuichi Izumiya,
ICMC-USP, São Carlos, Brazil , 22-27th July, 2012

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Journal *of* Singularities

Shyuichi Izumiya

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Preface by proceedings editors

This 12th edition of the Workshop on Real and Complex Singularities was organized by both the Brazilian group of singularities and the Japanese researchers community, and had the great pleasure to celebrate the 60th birthday of Professor Shyuichi Izumiya from Hokkaido University, Sapporo, Japan.

For the first time, the workshop was held in two weeks, where in the first week (July 16—20, 2012) elementary and basic mini-courses were delivered for PhD students, post-docs and young researchers, by Lê D. Tráng, David Mond, Nicolas Dutertre, Hans Schönemann and Valery Romanovsky. In the second week (July 23—27, 2012), in addition to the plenary and parallel talks on specialized topics, two mini-courses on current topics of researches were also delivered by Toru Ohmoto and Anne Pichon.

We thank all members of the scientific and organizing committees for their contributions and help for building fruitful and high level school and workshop during the two weeks.

We especially thank the Fapesp agency, Capes, INCT-Mat, USP (Brazil) and JSPS (Japan) for their fundamental financial support. Without their help it would have been impossible to develop these important scientific meetings.

Also, we would like to thank all administrative staffs of ICMC-USP for their important and fundamental technical supports.

O. Saeki, V.H. Jorge Pérez, T. Nishimura and R. Araújo dos Santos

Preface by Goo Ishikawa

Professor Shyuichi Izumiya is our leader, collaborator, colleague and our friend. He has written over 100 papers and enjoyed over 460 citations by 142 authors (up to 21 July 2012). He supervised lots of graduate students (Doctors and Masters) and undergraduate students as well.

Shyuichi Izumiya was born on 7th July 1952 at Sapporo city, Hokkaido prefecture, Japan. 7th July is the day of “Tanabata” (the star festival). He lived in Takikawa city, in Hokkaido, and studied at Nishi Elementary School in Takikawa until the autumn of his 6th grade year. Then he moved back to Sapporo and graduated at Misono Elementary School in Sapporo city, Ryoyo Junior High School and Asahigaoka High School also in Sapporo.

From 1971 to 1975, he studied in Department of Mathematics, Faculty of Science, Hokkaido University, and he was awarded the degree of BSc in Mathematics. From 1976 to 1978, he studied and got the degree of MSc in Mathematics in Department of Mathematics, Faculty of Science, Hokkaido University, for the thesis entitled “Homotopy classification of regular sections which are equivariant with respect to finite group actions”, which was a work supervised by Professor Haruo Suzuki. Then from 1978 to 1984, he studied in Department of Mathematics, Faculty of Science, Hokkaido University, where he was awarded the degree of DSc in Mathematics for the thesis entitled “Generic bifurcation of varieties”, supervised by Professor Haruo Suzuki.

As for the research and professional experience of Shyuichi Izumiya, he was an assistant professor at the Department of Mathematics, Faculty of Science, Nara Women’s University, from 1978 to 1985. From 1985 to 1987, he was a lecturer at the Department of Mathematics, Faculty of Science, Hokkaido University, where from 1987 to 1995, he was an associate professor and after 1995 he became a professor. Currently, he is a professor at Research Center for Integrative Mathematics, Hokkaido University.

A partial list of academic activities follows:

1990.9 Visiting fellow at the Chinese Academy of Science.
1991.4–1992.2 Visiting fellow at the Department of Pure Mathematics, Liverpool University, UK.
1993.2 Visiting fellow at the center for non-linear analysis, Carnegie Mellon University, USA.
1995.4–5 Visiting fellow at the Banach International Mathematical Center, Warsaw, Poland.
1996.8 Visiting fellow at the Department of Pure Mathematics, Liverpool University, UK.
2000.9–12 Researcher at Isaac Newton Institute for Mathematical Sciences, University of Cambridge, UK.
2010.3 Honorary professor at Northeast Normal University, China.

A partial list of Shyuichi's students supervised during the master degree:

Asayama, Mikuri / Ashino, Takashi / Chino, Sachiko / Fusho, Takesi / Hayashi, Ryota / Ichiwara, Hisatoshi / Ito, Hiroki / Kikuchi, Makoto / Kanazawa, Sunao / Kogo, Yasuko / Kurokawa, Hitoshi / Kurokawa, Yasuhiro / Maruyama, Kunihide / (Matsuoka, Sachiko) / (Minami, Tatsuya) / Miyawaki Norio / Murata Yusuke / Nagai, Takayuki / (Nakai, Hitoshi) / Ohtani, Saki / Sano, Takashi / Sato, Takami / Takahashi, Masatomo / Takiyama, Akihiro / Tamaoki, Aiko / Torii, Erika / (Watanabe, Kazuo) / (Yamamoto, Takahiro) / (with a lot of omission).

--- Mother’s teaching ---

Shyuichi remembers that his mother said to him in his Childhood: “Be gentle to girls !” Following her saying, Shyuichi keeps to support women’s activities and women mathematicians. He is proud of that.

Shyuichi supervised the following students during the PhD degree:

Yasuhiro Kurokawa / Takashi Sano / Wei-Zhi Sun / Donghe Pei / Nobuko Takeuchi / Takaharu Tsukada / Masatomo Takahashi / Liang Chen / Masaki Kasedou / Takayuki Nagai / Yang Jiang

Shyuichi's mathematical works cover the following three major areas:

— Basic Singularity Theory
— Applications to Differential Equations
— Applications to Differential Geometry

Shyuichi started his mathematical carrier by Equivariant Topology and Singularity Theory, and then he studied on generic bifurcation of varieties and global theory, characteristic classes and obstructions. Shyuichi says that the motivation to study new topics was just to provide problems for his Master Students. Then always, he completes the joint-work by writing joint papers.

Then Shyuichi showed that stability in the tangent sense for mappings between foliated manifolds, implies infinitesimal stability in the tangent sense, (the converse of L. A. Favaro’s theorem). After that, Shyuichi began to make good connections with São Carlos’s singularity group in São Paulo, Brazil.

In the paper with Sachiko Matsuoka, Shyuichi studied functions on varieties from the viewpoint of Thom-Mather's theory. He continued to study on topology of Legendre singularities, Legendrian unfoldings and differential equations, and how to define singular solutions, Complete integrability and Clairaut-type equations and Geometric singularities of weak solutions of PDE.

In the paper with Georgios T. Kossioris, Shyuichi classified generic bifurcations of singularities of viscosity solutions to Hamilton-Jacobi equations (shock waves). In particular, he discovered the phenomena that viscosity solutions are not necessarily covered by characteristic curves starting from the initial fronts. The discovery gave shocks to specialists of PDE for several years.

Then Shyuichi started to apply singularity theory to affine differential geometry. Frederic Gauss used "Gauss maps" and height functions for his famous surface theory. Then René Thom suggested Ian Porteous to apply singularity theory to submanifold theory in Euclidean geometry: I.R. Porteous, The normal singularities of a submanifold, J. Diff. Geom. 5 (1971), 543–564. Moreover, by applying Arnol'd-Zakalyukin's Lagrange and Legendre singularity theory and its improvements to those situations, Shyuichi has found that geometric meanings of singularities of families of functions become clearer. Then Shyuichi began to develop Thom-Porteous's idea by applying to affine geometry, hyperbolic geometry, Minkowski geometry and so on.

In the joint paper with Takashi Sano, Shyuichi has found the relation of affine curvature, sextactic points etc. with singularities of affine-cubed functions or affine height functions. Then Shyuichi had many works on time-like surfaces in Minkowski space, light-cone Gauss maps, special curves and special surfaces, hyperbolic Gauss maps, and so on.

After the investigation in those paper, Shyuichi discovered a new geometry, horospherical geometry, in the hyperbolic space. In horospherical geometry, horospheres are regarded as the totally umbilic flat surfaces. Moreover, Shyuichi wrote a joint paper with S. Janeczko on gravitational lensing.

Shyuichi's co-authors are: Asayama, Mikuri / Buosi, Marcelo / Chen, Liang / Chino, Sachiko / Davydov, Aleksey / Fusho, Takesi / Hayakawa, Atsushi / Honda, Atsufumi / Ishikawa, Goo / Janeczko, Stanisław / Jiang, Yang / Katsumi, Haruyo / Kikuchi, Makoto / Kogo, Yasuko / Kossioris, Georgios T. / Kossowski, Marek / Kurokawa, Yasuhiro / Li, Bing / Makrakis, George N. / Marar, Washington Luiz / Maruyama, Kunihide / Nagai, Takayuki / Nishimori, Toshiyuki / Nuño Ballesteros, Juan José / Pei, Dong He / Romero Fuster, Maria del Carmen / Ruas, Maria Aparecida Soares / Saito, Sachiko / Saji, Kentaro / Sano, Takashi / Sato, Takami / Sun, Wei Zhi / Takahashi, Masatomo / Takeuchi, Nobuko / Takiyama, Akihiro / Tamaoki, Aiko / Tari, Farid / Torii, Erika / Watanabe, Kazuo / Yamaguchi, Keizo / Yamasaki, Takako / Yıldırım, Handan / Yu, Jian Ming / (over 43 mathematicians).

Shyuichi has projects (ongoing and in near future):

(1) To construct lightlike geometry in Lorentz-Minkowski space (with several people). To study on tightness which depends on causality.

(2) To obtain mathematical interpretations and generalisations of Randall-Sundrum model and Karch-Randall model in brane world scenario by applied singularity theory.

(3) Recurrence to applications of singularity theory to nonlinear partial differential equations.

Shyuichi Izumiya wrote the following books:

---Matrices and Systems of Linear Equations (Japanese, with Rentaro Agemi, Goo Ishikawa, Atsuro Sannami, Ungou Chin, Toshiyuku Nishimori), Kyoritsu Shuppan Co., Ltd. (1996).

---Linear Mappings and Eigen Values (Japanese, with Goo Ishikawa, Rentaro Agemi, Atsuro Sannami, Ungou Chin, Toshiyuku Nishimori) Kyoritsu Shuppan Co., Ltd. (1996).

---Applied Singularity Theory (Japanese, with Goo Ishikawa), Kyoritsu Shuppan Co., Ltd. (1998).

---Geometry and Singularities (Japanese, with Takashi Sano, Osamu Saeki, Kazuhiro Sakuma), Kyoritsu Shuppan Co., Ltd. (2001).

Mathematics on Shapes Understandable by Cutting, Looking and Touching (Japanese, with Nobuko Takeuchi), JUSE Press. Ltd. (2005).

---Elementary Linear Algebra (Japanese), Kyoritsu Shuppan Co., Ltd. (2008).

---Coordinates Geometry—An Introduction to Analytic Geometry (Japanese, with Nobuko Takeuchi, Mitsutaka Murayama), JUSE Press. Ltd (2008).

---Exercises of Coordinates Geometry (Japanese, Nobuko Takeuchi, Mitsutaka Murayama), JUSE Press. Ltd (2008).

Moreover, Shyuichi is now preparing a book on singularity theory and applications, with Maria Aparecida Soares Ruas, Maria Carmen Romero Fuster and Farid Tari.

Shyuichi is an editor of several Journals and contributes as referees of lots of papers.

Shyuichi has sent me a message on his dream (future plan):

— Shyuichi's Dream —

"I (Shyuichi) am observing the restoration of submanifold theory in physics by recent movements in brane cosmology and particle physics. I suppose that, also in mathematics, it should be the time to reconstruct the extrinsic geometry. The approach by singularity theory should be most appropriate for that. Through my recent investigations along this direction, I feel that several analogies to Gauss' idea of extrinsic geometry have appeared in theoretical physics, like AdS/CFT correspondence, covariant entropy bound, the holographic principle etc.. Then I hope to clarify, mathematically, such correspondences between extrinsic geometry and physics. It is my present dream.

I would be happy if I could continue to extend the areas and viewpoints of my investigations by the communications with worldwide mathematicians."

The 60th birthday is called "Kanreki" in Japan. "Kanreki" means a "cycle of calendar". It is regarded that one will be re-born at his/her 60th birthday. Shyuichi, please keep, even after Kanreki, being attractive, friendly, young, active, gentle and mad on Mathematics !

Happy Birthday to Shyuichi ! Thank you. Obrigado.
Goo Ishikawa

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Program of Talks

	Monday 23rd	Tuesday 24th	Wednesday 25th	Thursday 26th	Friday 27th
8:00		T. Ohmoto	T. Ohmoto		
9:00	A. Hefez	A. Parameswaran	Lê Dung Tráng	P. Popescu-Pampu	J. Seade
10:00	C. Bivià Ausina	M. Umehara	A. Fernandes	J.J.Nuño Ballesteros	R. Callejas Bedregal
11:00	**COFFEE**	**COFFEE**	**COFFEE**	**COFFEE**	**COFFEE**
11:20	1: K. Saji 2: L. Câmara 3: L. Kushner	1: Z. T. Jelonek 2: H. Aguilar 3: T. Yamamoto	1: T. Nishimura 2: A. Szücs 3: I. Laboriau	1: L. Chen 2: J.A. Moya Pérez 3: T. Sano	1: C. Mendes 2: D. Trotman 3: J. Sotomayor
11:50	1: R. Oset Sinha 2: M. Silva 3: T. Fukunaga	1: M. Fernández 2: S. Agafonov 3: M. Kobayashi	1: T. Fukui 2: J. Adachi 3: M. Roberts	1: M. Kasedou 2: I. Ahmed 3: P. Giblin	1: H. Yildirim 2: S. Trivedi 3: J.Basto-Gonçalves
12:20	**LUNCH**	**LUNCH**	**B B Q**	**LUNCH**	**LUNCH**
14:00	A. Pichon	A. Pichon		A. Pichon	1: F. Schez. Bringas 2: A. Menegon Neto 3: P. Rios
14:50	1: W. Neumann 2: D. Lehmann 3: F. Tari	1: M. Yamamoto 2: B. Oréfice 3: V. Grandjean		T. Ohmoto	1: J. J. Risler 2: H. Möller Pedersen 3: R. Garcia
15:20	Poster Session	Poster Session			
15:40	**COFFEE**	**COFFEE**		**COFFEE**	15:00 – 16:00 M. Manoel
16:00	T. Gaffney	S. Ishii		J. Vitório	**COFFEE**
17:00	M. Saia	J. F. Bobadilla		G. Ishikawa	16:30 – 17:30 S. Janeczko
18:00				**CONFERENCE DINNER**	

Poster sessions

- Monday: A. J. Miranda, A. C. Rezende, A. P. Francisco, A. Tsuchida, A. Yano, C. Wolf, C. Casonatto, C. A. Buzzi, D. Nakajo, E. C. Rizziolli, J. C. F. Costa, L. N. de Oliveira, J. V. Santos, G. Peñafort, N. R. Ribeiro, E. S. F. Ruth, A. C. Felippe, I. Oliveira, G. Miranda, J. A. Coripaca Huarcaya.

- Tuesday: L. R. dos Santos, L. Roberto, M. F. H. Iglesias, M. Milijevic, N. Hu, P. B. Riul, P. T. Cardin, R. Martins, T. Sato, T. de Carvalho, W. Yukuno, Y. Jiang, K. Takao, Y. Izumikawa, Y. Mizota, F. de M. Viríssimo, B. R. P. Sampaio, Y. Kurokawa, H. Kurokawa, R. Mendes, F. Antoneli.

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GEOMETRY AND SINGULARITIES OF THE PRONY MAPPING

DMITRY BATENKOV AND YOSEF YOMDIN

Abstract. The Prony mapping provides the global solution of the Prony system of equations

$$\Sigma_{i=1}^{n} A_i x_i^k = m_k, \ k = 0, 1, \dots, 2n-1.$$

This system appears in numerous theoretical and applied problems arising in Signal Reconstruction. The simplest example is the problem of reconstruction of linear combination of δ-functions of the form $g(x) = \sum_{i=1}^{n} a_i \delta(x - x_i)$, with the unknown parameters $a_i, \ x_i, \ i = 1, \dots, n$, from the "moment measurements" $m_k = \int x^k g(x) dx$.

The global solution of the Prony system, i.e., the inversion of the Prony mapping, encounters several types of singularities. One of the most important ones is a collision of some of the points x_i. The investigation of this type of singularities has been started in [21] where the role of finite differences was demonstrated.

In the present paper we study this and other types of singularities of the Prony mapping, and describe its global geometry. We show, in particular, close connections of the Prony mapping with the "Vieta mapping" expressing the coefficients of a polynomial through its roots, and with hyperbolic polynomials and "Vandermonde mapping" studied by V. Arnold.

1. Introduction

Prony system appears as we try to solve a very simple "algebraic signal reconstruction" problem of the following form: assume that the signal $F(x)$ is known to be a linear combination of shifted δ-functions:

$$F(x) = \sum_{j=1}^{d} a_j \delta(x - x_j). \tag{1.1}$$

We shall use as measurements the polynomial moments:

$$m_k = m_k(F) = \int x^k F(x) \, \mathrm{d}\, x. \tag{1.2}$$

After substituting F into the integral defining m_k we get

$$m_k(F) = \int x^k \sum_{j=1}^{d} a_j \delta(x - x_j) \, \mathrm{d}\, x = \sum_{j=1}^{d} a_j x_j^k.$$

Considering a_j and x_j as unknowns, we obtain equations

$$m_k(F) = \sum_{j=1}^{d} a_j x_j^k, \ k = 0, 1, \dots. \tag{1.3}$$

2000 *Mathematics Subject Classification.* 94A12 62J02, 14P10, 42C99.

Key words and phrases. Singularities, Signal acquisition, Non-linear models, Moments inversion.

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This infinite set of equations (or its part, for $k = 0, 1, \dots, 2d-1$), is called Prony system. It can be traced at least to R. de Prony (1795, [19]) and it is used in a wide variety of theoretical and applied fields. See [2] for an extensive bibligoraphy on the Prony method.

In writing Prony system (1.3) we have assumed that all the nodes $x_1, \dots, x_d$ are pairwise different. However, as the left-hand side $\mu = (m_0, \dots, m_{2d-1})$ of (1.3) is provided by the actual measurements of the signal F, we cannot guarantee a priori, that this condition is satisfied for the solution. Moreover, we shall see below that multiple nodes may naturally appear in the solution process. In order to incorporate possible collisions of the nodes, we consider "confluent Prony systems".

Assume that the signal $F(x)$ is a linear combination of shifted δ-functions and their derivatives:

$$F(x) = \sum_{j=1}^{s} \sum_{\ell=0}^{d_j-1} a_{j,\ell} \delta^{(\ell)} (x - x_j). \tag{1.4}$$

Definition 1.1. For $F(x)$ as above, the vector $D(F) \stackrel{\text{def}}{=} (d_1, \dots, d_s)$ is *the multiplicity vector* of F, $s = s(F)$ is the size of its support, $T(F) \stackrel{\text{def}}{=} (x_1, \dots, x_s)$, and $\operatorname{rank}(F) \stackrel{\text{def}}{=} \sum_{j=1}^{s} d_j$ is its rank. For avoiding ambiguity in these definitions, it is always understood that $a_{j,d_j-1} \neq 0$ for all $j = 1, \dots, s$ (i.e. d_j is the maximal index for which $a_{j,d_j-1} \neq 0$).

For the moments $m_k = m_k(F) = \int x^k F(x) \,\mathrm{d}\, x$ we now get

$$m_k = \sum_{j=1}^{s} \sum_{\ell=0}^{d_j-1} a_{j,\ell} \frac{k!}{(k-\ell)!} x_j^{k-\ell}.$$

Considering x_i and $a_{j,\ell}$ as unknowns, we obtain a system of equations

$$\sum_{j=1}^{s} \sum_{\ell=0}^{d_j-1} \frac{k!}{(k-\ell)!} a_{j,\ell} x_j^{k-\ell} = m_k, \quad k = 0, 1, \dots, 2d-1, \tag{1.5}$$

which is called a confluent Prony system of order d with the multiplicity vector $D = (d_1, \dots, d_s)$. The original Prony system (1.3) is a special case of the confluent one, with D being the vector $(1, \dots, 1)$ of length d.

The system (1.5) arises also in the problem of reconstructing a planar polygon P (or even an arbitrary semi-analytic *quadrature domain*) from its moments

$$m_k(\chi_P) = \iint_{\mathbb{R}^2} z^k \chi_P \,\mathrm{d}\, x \,\mathrm{d}\, y, \ z = x + \imath y,$$

where χ_P is the characteristic function of the domain $P \subset \mathbb{R}^2$. This problem is important in many areas of science and engineering [11]. The above yields the confluent Prony system

$$m_k = \sum_{j=1}^{s} \sum_{i=0}^{d_j-1} c_{i,j} k(k-1)\cdots(k-i+1) z_j^{k-i}, \qquad c_{i,j} \in \mathbb{C}, \ z_j \in \mathbb{C} \setminus \{0\}.$$

Definition 1.2. For a given multiplicity vector $D = (d_1, \dots, d_s)$, its *order* is $\sum_{j=1}^{s} d_j$.

As we shall see below, if we start with the measurements $\mu(F) = \mu = (m_0, \dots, m_{2d-1})$, then a natural setting of the problem of solving the Prony system is the following:

Problem 1.3 (Prony problem of order d)**.** *Given the measurements*

$$\mu = (m_0, \dots, m_{2d-1}) \in \mathbb{C}^{2d}$$

in the right hand side of (1.5)*, find the multiplicity vector* $D = (d_1, \ldots, d_s)$ *of order*

$$r = \sum_{j=1}^{s} d_j \leq d,$$

and find the unknowns x_j *and* $a_{j,\ell}$*, which solve the corresponding confluent Prony system* (1.5) *with the multiplicity vector* D *(hence, with solution of rank* r*).*

It is extremely important in practice to have *a stable method of inversion.* Many research efforts are devoted to this task (see e.g. [3, 7, 10, 17, 18, 20] and references therein). A basic question here is the following.

Problem 1.4 (Noisy Prony problem)**.** Given the *noisy* measurements

$$\tilde{\mu} = (\tilde{m_0}, \ldots, \tilde{m}_{2d-1}) \in \mathbb{C}^{2d}$$

and an estimate of the error $|\tilde{m}_k - m_k| \leq \varepsilon_k$*, solve Problem 1.3 so as to minimize the reconstruction error.*

In this paper we study the global setting of the Prony problem, stressing its algebraic structure. In Section 2 the space where the solution is to be found (Prony space) is described. It turns out to be a vector bundle over the space of the nodes $x_1, \ldots, x_d$. We define also three mappings: "Prony", "Taylor", and "Stieltjes" ones, which capture the essential features of the Prony problem and of its solution process.

In Section 3 we investigate solvability conditions for the Prony problem. The answer leads naturally to a stratification of the space of the right-hand sides, according to the rank of the associated Hankel-type matrix and its minors. The behavior of the solutions near various strata turns out to be highly nontrivial, and we present some initial results in the description of the corresponding singularities.

In Section 4, we study the multiplicity-restricted Prony problem, fixing the collision pattern of the solution, and derive simple bounds for the stability of the solution via factorization of the Jacobian determinant of the corresponding Prony map.

In Section 5 we consider the rank-restricted Prony problem, effectively reducing the dimension to $2r$ instead of $2d$, where r is precisely the rank of the associated Hankel-type matrix. In this formulation, the Prony problem is solvable in a small neighborhood of the exact measurement vector.

In Section 6 we study one of the most important singularities in the Prony problem: collision of some of the points x_i. The investigation of this type of singularities has been started in [21] where the role of finite differences was demonstrated. In the present paper we introduce global bases of finite differences, study their properties, and prove that using such bases we can resolve in a robust way at least the linear part of the Prony problem at and near colliding configurations of the nodes.

In Section 7 we discuss close connections of the Prony problem with hyperbolic polynomials and "Vandermonde mapping" studied by V.I.Arnold in [1] and by V.P.Kostov in [13, 14, 15], and with "Vieta mapping" expressing the coefficients of a polynomial through its roots. We believe that questions arising in theoretical study of Prony problem and in its practical applications justify further investigation of these connections, as well as further applications of Singularity Theory.

Finally, in Appendix A we describe a solution method for the Prony system based on Padé approximation.

2. Prony, Stieltjes and Taylor Mappings

In this section we define "Prony", "Taylor", and "Stieltjes" mappings, which capture some essential features of the Prony problem and of its solution process. The main idea behind the spaces and mappings introduced in this section is the following: associate to the signal $F(x)=\sum_{i=1}^{d}a_i\delta(x-x_i)$ the rational function $R(z)=\sum_{i=1}^{d}\frac{a_i}{z-x_i}$. (In fact, R is the Stieltjes integral transform of F). The functions R obtained in this way can be written as $R(z)=\frac{P(z)}{Q(z)}$ with $\deg P\le \deg Q-1$, and they satisfy $R(\infty)=0$. Write R as

$$R(z)=\sum_{i=1}^{d}\frac{a_i}{z(1-x_i/z)}.$$

Developing the summands into geometric progressions we conclude that $R(z)=\sum_{k=0}^{\infty}m_k(\frac{1}{z})^{k+1}$, with

$$m_k=\sum_{i=1}^{d}a_i x_i^k,$$

so the moment measurements m_k in the right hand side of the Prony system (1.3) are exactly the Taylor coefficients of $R(z)$. We shall see below that this correspondence reduces solution of the Prony system to an appropriate Padé approximation problem.

Definition 2.1. For each $w=(x_1,\dots,x_d)\in\mathbb{C}^d$, let $s=s(w)$ be the number of distinct coordinates τ_j, $j=1,\dots,s$, and denote $T(w)=(\tau_1,\dots,\tau_s)$. The multiplicity vector is

$$D=D(w)=(d_1,\dots,d_s),$$

where d_j is the number of times the value τ_j appears in $\{x_1,\dots,x_d\}$. The order of the values in $T(w)$ is defined by their order of appearance in w.

Example 2.2. For $w=(3,1,2,1,0,3,2)$, we have

$$s(w)=4,\qquad T(w)=(3,1,2,0)\,,\text{and}\qquad D(w)=(2,2,2,1)\,.$$

Remark 2.3. Note the slight abuse of notations between Definition 1.1 and Definition 2.1. Note also that the *order* of $D(w)$ equals to d ***for all*** $w\in\mathbb{C}^d$.

Definition 2.4. For each $w\in\mathbb{C}^d$, let $s=s(w)$, $T(w)=(\tau_1,\dots,\tau_s)$ and $D(w)=(d_1,\dots,d_s)$ be as in Definition 2.1.

(1) V_w is the vector space of dimension d containing the linear combinations

$$g=\sum_{j=1}^{s}\sum_{\ell=0}^{d_j-1}\gamma_{j,\ell}\delta^{(\ell)}(x-\tau_j)\tag{2.1}$$

of δ-functions and their derivatives at the points of $T(w)$. The "standard basis" of V_w is given by the distributions

$$\delta_{j,\ell}=\delta^{(\ell)}(x-\tau_j)\,,\qquad j=1,\dots,s(w)\,;\ \ell=0,\dots,d_j-1.\tag{2.2}$$

(2) W_w is the vector space of dimension d of all the rational functions with poles $T(w)$ and multiplicities $D(w)$, vanishing at ∞ :

$$R(z)=\frac{P(z)}{Q(z)},\qquad Q(z)=\prod_{j=1}^{s}(z-\tau_j)^{d_j}\,,\ \deg P(z)<\deg Q\leqslant d.$$

The "standard basis" of W_w is given by the elementary fractions

$$R_{j,\ell} = \frac{1}{(z-\tau_j)^\ell}, \qquad j=1,\dots,s;\ \ell=1,\dots,d_j.$$

Now we are ready to formally define the Prony space $\mathcal{P}_d$ and the Stieltjes space $\mathcal{S}_d$ as certain (trivial) vector bundles over $\mathbb{C}^d$.

Definition 2.5. The Prony space $\mathcal{P}_d$ is the vector bundle over $\mathbb{C}^d$, consisting of all the pairs

$$(w,g): \quad w \in \mathbb{C}^d,\ g \in V_w.$$

The topology on $\mathcal{P}_d$ is induced by the natural embedding $\mathcal{P}_d \subset \mathbb{C}^d \times \mathcal{D}$, where $\mathcal{D}$ is the space of distributions on $\mathbb{C}$ with its standard topology.

Definition 2.6. The Stieltjes space $\mathcal{S}_d$ is the vector bundle over $\mathbb{C}^d$, consisting of all the pairs

$$(w,\gamma): \qquad w \in \mathbb{C}^d,\ \gamma \in W_w.$$

The topology on $\mathcal{S}_d$ is induced by the natural embedding $\mathcal{S}_d \subset \mathbb{C}^d \times \mathcal{R}$, where $\mathcal{R}$ is the space of complex rational functions with its standard topology.

Definition 2.7. The Stieltjes mapping $\mathcal{S}M : \mathcal{P}_d \to \mathcal{S}_d$ is defined by the Stieltjes integral transform: for $(w,g) \in \mathcal{P}_d$

$$\mathcal{S}M\left((w,g)\right) = (w,\gamma), \qquad \gamma(z) = \int_{-\infty}^{\infty} \frac{g(x)\,\mathrm{d}\,x}{z-x}.$$

Sometimes we abuse notation and write for short $\mathcal{S}M(g) = \gamma$, with the understanding that $\mathcal{S}M$ is also a map $\mathcal{S}M : V_w \to W_w$ for each $w \in \mathbb{C}^d$.

The following fact is immediate consequence of the above definitions.

Proposition 2.8. *$\mathcal{S}M$ is a linear isomorphism of the bundles $\mathcal{P}_d$ and $\mathcal{S}_d$ (for each $w \in \mathbb{C}^d$, $\mathcal{S}M$ is a linear isomorphism of the vector spaces V_w and W_w). In the standard bases of V_w and W_w, the map $\mathcal{S}M$ is diagonal, satisfying*

$$\mathcal{S}M\left(\delta_{j,\ell}\right) = (-1)^\ell\, \ell! R_{j,\ell}(z).$$

Furthermore, for any $(w,g) \in \mathcal{P}_d$

$$\mathcal{S}M(g) = \underbrace{\frac{P(z)}{Q(z)}}_{irreducible}, \qquad \deg P < \deg Q = \operatorname{rank}(g) \leqslant d. \tag{2.3}$$

Definition 2.9. The Taylor space $\mathcal{T}_d$ is the space of complex Taylor polynomials at infinity of degree $2d-1$ of the form $\sum_{k=0}^{2d-1} m_k (\frac{1}{z})^{k+1}$. We shall identify $\mathcal{T}_d$ with the complex space $\mathbb{C}^{2d}$ with the coordinates $m_0,\dots,m_{2d-1}$.

Definition 2.10. The Taylor mapping $\mathcal{T}M : \mathcal{S}_d \to \mathcal{T}_d$ is defined by the truncated Taylor development at infinity:

$$\mathcal{T}M\left((w,\gamma)\right) = \sum_{k=0}^{2d-1} \alpha_k \left(\frac{1}{z}\right)^{k+1}, \qquad \text{where } \gamma(z) = \sum_{k=0}^{\infty} \alpha_k \left(\frac{1}{z}\right)^{k+1}.$$

We identify $\mathcal{T}M((w,\gamma))$ as above with $(\alpha_0,\dots,\alpha_{2d-1}) \in \mathbb{C}^{2d}$. Sometimes we write for short $\mathcal{T}M(\gamma) = (\alpha_0,\dots,\alpha_{2d-1})$.

Finally, we define the Prony mapping $\mathcal{P}M$ which encodes the Prony problem.

Definition 2.11. The Prony mapping $\mathcal{P}M : \mathcal{P}_d \to \mathbb{C}^{2d}$ for $(w,g) \in \mathcal{P}_d$ is defined as follows:

$$\mathcal{PM}((w,g)) = (m_0,\dots,m_{2d-1}) \in \mathbb{C}^{2d}, \qquad m_k = m_k(g) = \int x^k g(x)\,\mathrm{d}\,x.$$

By the above definitions, we have

$$\mathcal{P}M = \mathcal{T}M \circ \mathcal{S}M. \tag{2.4}$$

Solving the Prony problem for a given right-hand side $(m_0,\dots,m_{2d-1})$ is therefore equivalent to inverting the Prony mapping $\mathcal{P}M$. As we shall elaborate in the subsequent section, the identity (2.4) allows us to split this problem into two parts: inversion of $\mathcal{T}M$, which is, essentially, the Padé approximation problem, and inversion of $\mathcal{S}M$, which is, essentially, the decomposition of a given rational function into the sum of elementary fractions.

3. Solvability of the Prony problem

3.1. **General condition for solvability.** In this section we provde a necessary and sufficient condition for the Prony problem to have a solution (which is unique, as it turns out by Proposition 3.2). As mentioned in the end of the previous section, our method is based on inverting (2.4) and thus relies on the solution of the corresponding (diagonal) *Padé approximation problem* [4].

Problem 3.1 (Diagonal Padé approximation problem)**.** *Given $\mu = (m_0,\dots,m_{2d-1}) \in \mathbb{C}^{2d}$, find a rational function $R_d(z) = \frac{P(z)}{Q(z)} \in \mathcal{S}_d$ with $\deg P < \deg Q \leqslant d$, such that the first $2d$ Taylor coefficients at infinity of $R_d(z)$ are $\{m_k\}_{k=0}^{2d-1}$.*

Proposition 3.2. *If a solution to Problem 3.1 exists, it is unique.*

Proof. Writing $R(z) = \frac{P(z)}{Q(z)}$, $R_1(z) = \frac{P_1(z)}{Q_1(z)}$, with $\deg P < \deg Q \leqslant d$ and $\deg P_1 < \deg Q_1 \leqslant d$, we get

$$R - R_1 = \frac{PQ_1 - P_1 Q}{QQ_1},$$

and this function, if nonzero, can have a zero of order at most $2d-1$ at infinity. ☐

Let us summarize the above discussion with the following statement.

Proposition 3.3. *The tuple*

$$\left\{ s,\ D = (d_1,\dots,d_s),\ r = \sum_{j=1}^{s} d_j \le d,\ X = \{x_j\}_{j=1}^{s},\ A = \{a_{j,\ell}\}_{j=1,\dots,s;\ \ell=0,\dots,d_j-1} \right\}$$

is a (unique, up to a permutation of the nodes $\{x_j\}$) solution to Problem 1.3 with right-hand side

$$\mu = (m_0,\dots,m_{2d-1}) \in \mathbb{C}^{2d}$$

if and only if the rational function

$$R_{D,X,A}(z) = \sum_{j=1}^{s}\sum_{\ell=1}^{d_j} (-1)^{\ell-1}(\ell-1)!\frac{a_{j,\ell-1}}{(z-x_j)^{\ell}} = \sum_{k=0}^{2d-1}\frac{m_k}{z^{k+1}} + O\left(z^{-2d-1}\right)$$

is a (unique) solution to Problem 3.1 with input μ. In that case,

$$R_{D,X,A}(z) = \int_{-\infty}^{\infty} \frac{g(x)\,\mathrm{d}\,x}{z-x} \qquad \text{where } g(x) = \sum_{j=1}^{s}\sum_{\ell=0}^{d_j-1} a_{j,\ell}\delta^{(\ell)}(x-x_j),$$

i.e., $R_{D,X,A}(z)$ is the Stieltjes transform of $g(x)$.

Proof. This follows from the definitions of Section 2, (2.4), Proposition 3.2 and the fact that the problem of representing a given rational function as a sum of elementary fractions of the specified form (i.e., inverting $\mathcal{S}M$) is always uniquely solvable up to a permutation of the poles. □

The next result provides necessary and sufficient conditions for the solvability of Problem 3.1. It summarizes some well-known facts in the theory of Padé approximation, related to "normal indices" (see, for instance, [4]). However, these facts are not usually formulated in the literature on Padé approximation in the form we need in relation to the Prony problem. Consequently, we give a detailed proof of this result in Appendix A. This proof contains, in particular, some facts which are important for understanding the solvability issues of the Prony problem.

Definition 3.4. Given a vector $\mu = (m_0, \ldots, m_{2d-1})$, let $\tilde{M}_d$ denote the $d \times (d+1)$ Hankel matrix

$$\tilde{M}_d = \begin{bmatrix} m_0 & m_1 & m_2 & \ldots & m_d \\ m_1 & m_2 & m_3 & \ldots & m_{d+1} \\ \iddots & \iddots & \iddots & \iddots & \iddots \\ m_{d-1} & m_d & m_{d+1} & \ldots & m_{2d-1} \end{bmatrix}. \tag{3.1}$$

For each $e \leqslant d$, denote by $\tilde{M}_e$ the $e \times (e+1)$ submatrix of $\tilde{M}_d$ formed by the first e rows and $e+1$ columns, and let M_e denote the corresponding square matrix.

Theorem 3.5. *Let $\mu = (m_0, \ldots, m_{2d-1})$ be given, and let $r \leqslant d$ be the rank of the Hankel matrix $\tilde{M}_d$ as in (3.1). Then Problem 3.1 is solvable for the input μ if and only if the upper left minor $|M_r|$ of $\tilde{M}_d$ is non-zero.*

As an immediate consequence of Theorem 3.5 and Proposition 3.3, we obtain the following result.

Theorem 3.6. *Let $\mu = (m_0, \ldots, m_{2d-1})$ be given, and let $r \leqslant d$ be the rank of the Hankel matrix $\tilde{M}_d$ as in (3.1). Then Problem 1.3 with input μ is solvable if and only if the upper left minor $|M_r|$ of $\tilde{M}_d$ is non-zero. The solution, if it exists, is unique, up to a permutation of the nodes $\{x_j\}$. The multiplicity vector $D = (d_1, \ldots, d_s)$, of order $\sum_{j=1}^{s} d_j = r$, of the resulting confluent Prony system of rank r is the multiplicity vector of the poles of the rational function $R_{D,X,A}(z)$, solving the corresponding Padé problem.*

As a corollary we get a complete description of the right-hand side data $\mu \in \mathbb{C}^{2d}$ for which the Prony problem is solvable (unsolvable). Define for $r = 1, \ldots, d$ sets $\Sigma_r \subset \mathbb{C}^{2d}$ (respectively, $\Sigma'_r \subset \mathbb{C}^{2d}$) consisting of $\mu \in \mathbb{C}^{2d}$ for which the rank of $\tilde{M}_d = r$ and $|M_r| \neq 0$ (respectively, $|M_r| = 0$). The set Σ_r is a difference $\Sigma_r = \Sigma_r^1 \setminus \Sigma_r^2$ of two algebraic sets: Σ_r^1 is defined by vanishing of all the $s \times s$ minors of $\tilde{M}_d$, $r < s \leq d$, while Σ_r^2 is defined by vanishing of $|M_r|$. In turn, $\Sigma'_r = \Sigma_r^{'1} \setminus \Sigma_r^{'2}$, with $\Sigma_r^{'1} = \Sigma_r^1 \cap \Sigma_r^2$ and $\Sigma_r^{'2}$ defined by vanishing of all the $r \times r$ minors of $\tilde{M}_d$. The union $\Sigma_r \cup \Sigma'_r$ consists of all μ for which the rank of $\tilde{M}_d = r$, which is $\Sigma_r^1 \setminus \Sigma_r^{'2}$.

Corollary 3.7. *The set Σ (respectively, Σ') of $\mu \in \mathbb{C}^{2d}$ for which the Prony problem is solvable (respectively, unsolvable) is the union $\Sigma = \cup_{r=1}^{d}\Sigma_r$ (respectively, $\Sigma' = \cup_{r=1}^{d}\Sigma'_r$). In particular, $\Sigma' \subset \{\mu \in \mathbb{C}^{2d}, \det M_d = 0\}$.*

So for a generic right hand side μ we have $|M_d| \neq 0$, and the Prony problem is solvable. On the algebraic hypersurface of μ for which $|M_d| = 0$, the Prony problem is solvable if $|M_{d-1}| \neq 0$, etc.

Let us now consider some examples.

Example 3.8. Let us fix $d = 1, 2, \ldots$. Consider $\mu = (m_0, \ldots, m_{2d-1}) \in \mathbb{C}^{2d}$, the right hand sides of the Prony problem, to be of the form $\mu = \mu_\ell = (\delta_{k\ell}) = (0, \ldots, 0, \underbrace{1}_{\text{position } \ell+1}, 0, \ldots, 0)$, with all the $m_k = 0$ besides $m_\ell = 1$, $\ell = 0, \ldots, 2d-1$, and let $\tilde{M}_d^\ell$ be the corresponding matrix.

Proposition 3.9. *The rank of $\tilde{M}_d^\ell$ is equal to $\ell + 1$ for $\ell \leq d - 1$, and it is equal to $2d - \ell$ for $\ell \geq d$. The corresponding Prony problem is solvable for $\ell \leq d - 1$, and it is unsolvable for $\ell \geq d$.*

Proof. For $d = 5$ and $\ell = 2, 4, 5, 9$, the corresponding matrices $\tilde{M}_\ell^d$ are as follows.

$$\tilde{M}_5^2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \tilde{M}_5^4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \text{(solvable)}$$

$$\tilde{M}_5^5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \tilde{M}_5^9 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \qquad \text{(unsolvable)}$$

In general, the matrices $\tilde{M}_d^\ell$ have the same pattern as in the special cases above, so their rank is $\ell + 1$ for $\ell \leqslant d - 1$, and $2d - \ell$ for $\ell \geqslant d$, as stated above. Application of Theorem 3.6 completes the proof. ☐

In fact, μ_ℓ is a moment sequence of

$$F(x) = \frac{1}{\ell!} \delta^{(\ell)}(x),$$

and this signal belongs to $\mathcal{P}_d$ if and only if $\ell \leqslant d - 1$. In notations of Corollary 3.7 we have

$$\begin{aligned} \mu_\ell \in \quad & \Sigma_{\ell+1}, \quad \ell \leqslant d - 1, \\ \mu_\ell \in \quad & \Sigma'_{2d-\ell}, \qquad \ell \geqslant d. \end{aligned}$$

It is easy to provide various modifications of the above example. In particular, for

$$\mu = \tilde{\mu}_\ell = (0, \ldots, 0, 1, 1, \ldots, 1),$$

the result of Proposition 3.9 remains verbally true.

Example 3.10. Another example is provided by μ_{ℓ_1, ℓ_2}, with all the $m_k = 0$ besides

$$m_{\ell_1} = 1, \; m_{\ell_2} = 1, \; 0 \leq \ell_1 < d \leq \ell_2 \leq 2d - 1.$$

For $\ell_1 < \ell_2 - d + 1$ the rank of the corresponding matrix $\tilde{M}_d$ is $r = 2d + \ell_1 - \ell_2 + 1$ while $|M_r| = 0$, so the Prony problem for such μ_{ℓ_1, ℓ_2} is unsolvable. For $d = 5$ and $\ell_1 = 2, \; \ell_2 = 8$ the matrix is as follows:

$$\tilde{M}_5^{(2,8)} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

3.2. **Near-singular inversion.** The behavior of the inversion of the Prony mapping near the unsolvability stratum Σ' and near the strata where the rank of $\tilde{M}_d$ drops, turns out to be pretty complicated. In particular, in the first case at least one of the nodes tends to infinity. In the second case, depending on the way the right-hand side μ approaches the lower rank strata, the nodes may remain bounded, or some of them may tend to infinity. In this section we provide one initial result in this direction, as well as some examples. We believe that a comprehensive description of the inversion of the Prony mapping near Σ' and near the lower rank strata is important both in theoretical study and in applications of Prony-like systems, and consider it to be an important direction for future research.

Theorem 3.11. *As the right-hand side $\mu \in \mathbb{C}^{2d} \setminus \Sigma'$ approaches a finite point $\mu_0 \in \Sigma'$, at least one of the nodes $x_1, \ldots, x_d$ in the solution tends to infinity.*

Proof. By assumptions, the components $m_0, \ldots, m_{2d-1}$ of the right-hand side

$$\mu = (m_0, \ldots, m_{2d-1}) \in \mathbb{C}^{2d}$$

remain bounded as $\mu \to \mu_0$. By Theorem 6.17, the finite differences coordinates of the solution $\mathcal{P}M^{-1}(\mu)$ remain bounded as well. Now, if all the nodes are also bounded, by compactness we conclude that $\mathcal{P}M^{-1}(\mu) \to \omega \in \mathcal{P}_d$. By continuity in the distribution space (Lemma 6.9) we have $\mathcal{P}M(\omega) = \mu_0$. Hence the Prony problem with the right-hand side μ_0 has a solution $\omega \in \mathcal{P}_d$, in contradiction with the assumption that $\mu_0 \in \Sigma'$. □

Example 3.12. Let us consider an example: $d = 2$ and $\mu_0 = (0, 0, 1, 0)$. Here the rank ℓ of $\tilde{M}_2$ is 2, and $|M_2| = 0$, so by Theorem 3.6 we have $\mu_0 \in \Sigma_2' \subset \Sigma'$. Consider now a perturbation $\mu(\epsilon) = (0, \epsilon, 1, 0)$ of μ_0. For $\epsilon \neq 0$ we have $\mu(\epsilon) \in \Sigma_2 \subset \Sigma$, and the Prony system is solvable for μ_ϵ. Let us write an explicit solution: the coefficients c_0, c_1 of the polynomial $Q(z) = c_0 + c_1 z + z^2$ we find from the system (A.⋆⋆):

$$\begin{bmatrix} 0 & \epsilon \\ \epsilon & 1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix},$$

whose solution is $c_1 = -\frac{1}{\epsilon}$, $c_0 = \frac{1}{\epsilon^2}$. Hence the denominator $Q(z)$ of $R(z)$ is $Q(z) = \frac{1}{\epsilon^2} - \frac{1}{\epsilon} z + z^2$, and its roots are $x_1 = \frac{1+\imath\sqrt{3}}{2\epsilon}$, $x_2 = \frac{1-\imath\sqrt{3}}{2\epsilon}$. The coefficients b_0, b_1 of the numerator $P(z) = b_0 + b_1 z$ we find from (A.⋆):

$$\begin{bmatrix} 0 & 0 \\ 0 & \epsilon \end{bmatrix} \begin{bmatrix} -\frac{1}{\epsilon} \\ 1 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_0 \end{bmatrix},$$

i.e., $b_1 = 0$, $b_0 = \epsilon$. Thus the solution of the associated Padé problem is

$$R(z) = \frac{P(z)}{Q(z)} = \frac{\epsilon}{(z - x_1)(z - x_2)} = \frac{\epsilon^2}{\imath\sqrt{3}} \frac{1}{(z - x_1)} - \frac{\epsilon^2}{\imath\sqrt{3}} \frac{1}{(z - x_2)}.$$

Finally, the (unique up to a permutation) solution of the Prony problem for μ_ϵ is

$$a_1 = \frac{\epsilon^2}{\imath\sqrt{3}},\ a_2 = -\frac{\epsilon^2}{\imath\sqrt{3}}, \quad x_1 = \frac{1 + \imath\sqrt{3}}{2\epsilon},\ x_2 = \frac{1 - \imath\sqrt{3}}{2\epsilon}.$$

As ϵ tends to zero, the nodes x_1, x_2 tend to infinity while the coefficients a_1, a_2 tend to zero.

As it was shown above, for a given $\mu \in \Sigma$ (say, with pairwise different nodes) the rank of the matrix $\tilde{M}_d$ is equal to the number of the nodes in the solution for which the corresponding δ-function enters with a non-zero coefficients. So μ approaches a certain μ_0 belonging to a stratum of a lower rank of $\tilde{M}_d$ if and only if some of the coefficients a_j in the solution tend to zero. We do not analyze all the possible scenarios of such a degeneration, noticing just that if $\mu_0 \in \Sigma'$, i.e., the Prony problem is unsolvable for μ_0, then Theorem 3.11 remains true, with essentially the same proof. So at least one of the nodes, say, x_j, escapes to infinity. Moreover, one can show that $a_j x_j^{2d-1}$ cannot tend to zero - otherwise the remaining linear combination of δ-functions would provide a solution for μ_0.

If $\mu_0 \in \Sigma$, i.e., the Prony problem is solvable for μ_0, all the nodes may remain bounded, or some x_j may escape to infinity, but in such a way that $a_j x_j^{2d-1}$ tends to zero.

4. Multiplicity-restricted Prony problem

Consider Problem 1.4 at some point $\mu_0 \in \Sigma$. By definition, $\mu_0 \in \Sigma_{r_0}$ for some $r_0 \le d$. Let $\mu_0 = \mathcal{P}M\left((w_0, g_0)\right)$ for some $(w_0, g_0) \in \mathcal{P}_d$. Assume for a moment that the multiplicity vector $D_0 = D(g_0) = (d_1, \dots d_{s_0})$, $\sum_{j=1}^{s_0} d_j = r_0$, has a non-trivial collision pattern, i.e., $d_j > 1$ for at least one $j = 1, \dots, s_0$. It means, in turn, that the function $R_{D_0,X,A}(z)$ has a pole of multiplicity d_j. Evidently, there exists an arbitrarily small perturbation $\tilde{\mu}$ of μ_0 for which this multiple pole becomes a cluster of single poles, thereby changing the multiplicity vector to some $D' \neq D_0$. While we address this problem in Section 6 via the bases of divided differences, in this section we consider a "multiplicity-restricted" Prony problem.

Definition 4.1. Let $\mathbf{x} = (x_1, \dots, x_s) \in \mathbb{C}^s$ and $D = (d_1, \dots, d_s)$ with $d = \sum_{j=1}^s d_j$ be given. The $d \times d$ *confluent Vandermonde* matrix is

$$V = V(\mathbf{x}, D) = V(x_1, d_1, \dots, x_s, d_s) = \begin{bmatrix} \mathbf{v_{1,0}} & \mathbf{v_{2,0}} & \dots & \mathbf{v_{s,0}} \\ \mathbf{v_{1,1}} & \mathbf{v_{2,1}} & \dots & \mathbf{v_{s,1}} \\ & & \dots & \\ \mathbf{v_{1,d-1}} & \mathbf{v_{2,d-1}} & \dots & \mathbf{v_{s,d-1}} \end{bmatrix} \tag{4.1}$$

where the symbol $\mathbf{v_{j,k}}$ denotes the following $1 \times d_j$ row vector

$$\mathbf{v_{j,k}} \stackrel{\text{def}}{=} \begin{bmatrix} x_j^k, & k x_j^{k-1}, & \dots & , k(k-1)\cdots(k-d_j)\, x_j^{k-d_j+1} \end{bmatrix}.$$

Proposition 4.2. *The matrix V defines the linear part of the confluent Prony system (1.5) in the standard basis for V_w, namely,*

$$V(x_1, d_1, \dots, x_s, d_s) \begin{bmatrix} a_{1,0} \\ \vdots \\ a_{1,d_1-1} \\ \vdots \\ \\ a_{s,d_s-1} \end{bmatrix} = \begin{bmatrix} m_0 \\ m_1 \\ \vdots \\ \\ \\ m_{d-1} \end{bmatrix}. \tag{4.2}$$

Definition 4.3. Let $\mathcal{P}M(w_0, g_0) = \mu_0 \in \Sigma_{r_0}$ with $D(g_0) = D_0$ and $s(g_0) = s_0$. Let $\mathcal{P}_{D_0}$ denote the following subbundle of $\mathcal{P}_d$ of dimension $s_0 + r_0$:

$$\mathcal{P}_{D_0} = \left\{(w, g) \in \mathcal{P}_d : \quad D(g) = D_0\right\}.$$

The multiplicity-restricted Prony mapping $\mathcal{PM}^*_{D_0} : \mathcal{P}_{D_0} \to \mathbb{C}^{s_0+r_0}$ is the composition

$$\mathcal{PM}^*_{D_0} = \pi \circ \mathcal{P}M \restriction_{\mathcal{P}_{D_0}},$$

where $\pi : \mathbb{C}^{2d} \to \mathbb{C}^{s_0+r_0}$ is the projection map on the first $s_0 + r_0$ coordinates.

Inverting this $\mathcal{PM}^*_{D_0}$ represents the solution of the confluent Prony system (1.5) with fixed structure D_0 from the first $k = 0, 1, \ldots, s_0 + r_0 - 1$ measurements.

Theorem 4.4 ([7])**.** *Let $\mu_0^* = \mathcal{PM}^*_{D_0}((w_0, g_0)) \in \mathbb{C}^{s_0+r_0}$ with the unperturbed solution*

$$g_0 = \sum_{j=1}^{s_0} \sum_{\ell=0}^{d_j-1} a_{j,\ell}\delta^{(\ell)}(x - \tau_j).$$

In a small neighborhood of $(w_0, g_0) \in \mathcal{P}_{D_0}$, the map $\mathcal{PM}^*_{D_0}$ is invertible. Consequently, for small enough ε, the multiplicity-restricted Prony problem with input data $\tilde{\mu}^* \in \mathbb{C}^{r_0+s_0}$ satisfying $\|\tilde{\mu}^* - \mu_0^*\| \le \varepsilon$ has a unique solution. The error in this solution satisfies

$$\begin{aligned} |\Delta a_{j,\ell}| &\le \frac{2}{\ell!}\left(\frac{2}{\delta}\right)^{s_0+r_0}\left(\frac{1}{2} + \frac{s_0+r_0}{\delta}\right)^{d_j-\ell}\left(1 + \frac{|a_{j,\ell-1}|}{|a_{j,d_j-1}|}\right)\varepsilon, \\ |\Delta \tau_j| &\le \frac{2}{d_j!}\left(\frac{2}{\delta}\right)^{s_0+r_0}\frac{1}{|a_{j,d_j-1}|}\varepsilon, \end{aligned}$$

where $\delta \stackrel{def}{=} \min_{i\neq j}|\tau_i - \tau_j|$ (for consistency we take $a_{j,-1} = 0$ in the above formula).

Proof outline. The Jacobian of $\mathcal{PM}^*_{D_0}$ can be easily computed, and it turns out to be equal to the product

$$\mathcal{J}_{\mathcal{PM}^*_{D_0}} = V(\tau_1, d_1 + 1, \ldots, \tau_{s_0}, d_{s_0} + 1)\,\mathrm{diag}\{E_j\}$$

where V is the *confluent Vandermonde matrix* (4.1) on the nodes $(\tau_1, \ldots, \tau_{s_0})$, with multiplicity vector

$$\tilde{D}_0 = (d_1 + 1, \ldots, d_{s_0} + 1),$$

while E is the $(d_j + 1) \times (d_j + 1)$ block

$$E_j = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & a_{j,0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{j,d_j-1} \end{bmatrix}.$$

Since $\mu_0 \in \Sigma_r$, the highest order coefficients a_{j,d_j-1} are nonzero. Furthermore, since all the τ_j are distinct, the matrix V is nonsingular. Local invertibility follows. To estimate the norm of the inverse, use bounds from [6]. □

Remark 4.5. Note that as two nodes collide ($\delta \to 0$), the inversion of the multiplicity-restricted Prony mapping $\mathcal{PM}^*_{D_0}$ becomes ill-conditioned proportionally to $\delta^{-(s_0+r_0)}$.

Let us stress that we are not aware of any general method of inverting $\mathcal{PM}^*_{D_0}$, i.e., solving the multiplicity-restricted confluent Prony problem with the smallest possible number of measurements. As we demonstrate in [5], such a method exists for a very special case of a single point, i.e., $s = 1$.

5. Rank-restricted Prony problem

Recall that the Prony problem consists in inverting the Prony mapping $\mathcal{PM} : \mathcal{P}_d \to \mathcal{T}_d$. So, given $\mu = (m_0, \ldots, m_{2d-1}) \in \mathcal{T}_d$ we are looking for $(w, g) \in \mathcal{P}_d$ such that

$$m_k(g) = \int x^k g(x)dx = m_k,$$

with $k = 0, 1, \ldots, 2d - 1$. If $\mu \in \Sigma_r$ with $r < d$, then in fact any neighborhood of μ will contain points from the non-solvability set Σ'. Indeed, consider the following example.

Example 5.1. Slightly modifying the construction of Example 3.10, consider $\mu_{\ell_1,\ell_2,\epsilon} \in \mathbb{C}^{2d}$ with all the $m_k = 0$ besides $m_{\ell_1} = 1$ and $m_{\ell_2} = \epsilon$, such that $\ell_2 > \ell_1 + d - 1$. For example, if $d = 5$ and $\ell_1 = 2,\ \ell_2 = 8$, the corresponding matrix is

$$\tilde{M}_5^{(2,8,\epsilon)} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \epsilon \\ 0 & 0 & 0 & 0 & \epsilon & 0 \end{bmatrix}.$$

For $\epsilon = 0$ the Prony problem is solvable, while for any small perturbation $\epsilon \neq 0$ it becomes unsolvable. However, if we restrict the whole problem just to $d = 3$, it remains solvable for any small perturbation of the input.

We therefore propose to consider the *rank-restricted Prony problem* analogous to the construction of Section 4, but instead of fixing the multiplicity $D(g)$ we now fix the rank r (recall Definition 1.1).

Definition 5.2. Denote by $\mathcal{P}_r$ the following vector bundle:

$$\mathcal{P}_r = \{(w, g) : \quad w \in \mathbb{C}^r,\ g \in V_w\},$$

where V_w is defined exactly as in Definition 2.4, replacing d with r.

Likewise, we define the Stieltjes bundle of order r as follows.

Definition 5.3. Denote by $\mathcal{S}_r$ the following vector bundle:

$$\mathcal{S}_r = \{(w, \gamma) : \qquad w \in \mathbb{C}^r,\ \gamma \in W_w\},$$

where W_w is defined exactly as in Definition 2.4, replacing d with r.

The Stieltjes mapping acts naturally as a map $\mathcal{S}M : \mathcal{P}_r \to \mathcal{S}_r$ with exactly the same definition as Definition 2.7.

The restricted Taylor mapping $\mathcal{T}M_r : \mathcal{S}_r \to \mathbb{C}^{2r}$ is, as before, given by the truncated development at infinity to the first $2r$ Taylor coefficients.

Definition 5.4. Let $\pi : \mathbb{C}^{2d} \to \mathbb{C}^{2r}$ denote the projection operator onto the first $2r$ coordinates. Denote $\Sigma_r^* \stackrel{\text{def}}{=} \pi(\Sigma_r)$. The rank-restricted Prony mapping $\mathcal{PM}_r^* : \mathcal{P}_r \to \Sigma_r^*$ is given by by

$$\mathcal{PM}_r^*((w, g)) = (m_0, \dots, m_{2r-1}), \qquad m_k = m_k(g) = \int x^k g(x)\,\mathrm{d}\,x.$$

Remark 5.5. $\mathcal{P}_r$ can be embedded in $\mathcal{P}_d$, for example by the map $\Xi_r : \mathcal{P}_r \to \mathcal{P}_d$

$$\Xi_r :\ (w, g) \in \mathcal{P}_r \longmapsto (w', g') \in \mathcal{P}_d : \qquad w' = \left(x_1, \dots, x_r, \underbrace{0, \dots 0}_{\times (d-r)} \right),\ g' = g.$$

With this definition, $\mathcal{PM}_r^*$ can be represented also as the composition

$$\mathcal{PM}_r^* = \pi \circ \mathcal{P}M \circ \Xi_r.$$

Proposition 5.6. *The rank-restricted Prony mapping satisfies*

$$\mathcal{PM}_r^* = \mathcal{T}M_r \circ \mathcal{S}M.$$

Inverting $\mathcal{PM}_r^*$ represents the solution of the rank-restricted Prony problem. Unlike in the multiplicity-restricted setting of Section 4, here we allow two or more nodes to collide (thereby changing the multiplicty vector $D(g)$ of the solution).

The basic fact which makes this formulation useful is the following result.

Theorem 5.7. *Let $\mu_0^* \in \Sigma_r^*$. Then in a small neighborhood of $\mu_0^* \in \mathbb{C}^{2r}$, the Taylor mapping $\mathcal{T}M_r$ is continuously invertible.*

Proof. This is a direct consequence of the solution method to the Padé approximation problem described in Appendix A. Indeed, if the rank of $\tilde{M}_r$ is full, then it remains so in a small neighborhood of the *entire space* $\mathbb{C}^{2r}$. Therefore, the system (A.⋆⋆) remains continuously invertible, producing the coefficients of the denominator $Q(z)$. Consequently, the right-hand side of (A.⋆) depends continuously on the moment vector $\mu^* = (m_0, \ldots, m_{2r-1}) \in \mathbb{C}^{2r}$. Again, since the rank always remains full, the polynomials $P(z)$ and $Q(z)$ cannot have common roots, and thereby the solution $R = \frac{P}{Q} = \mathcal{T}M_r^{-1}(\mu^*)$ depends continuously on μ^* (in the topology of the space of rational functions). □

In the next section, we consider the remaining problem: how to invert $\mathcal{S}M$ in this setting.

6. Collision singularities and bases of finite differences

6.1. **Introduction.** Collision singularities occur in Prony systems as some of the nodes x_i in the signal $F(x) = \sum_{i=1}^{d} a_i \delta(x - x_i)$ approach one another. This happens for μ near the discriminant stratum $\Delta \subset \mathbb{C}^{2d}$ consisting of those $(m_0, \ldots, m_{2d-1})$ for which some of the coordinates $\{x_j\}$ in the solution collide, i.e., the function $R_{D,X,A}(z)$ has multiple poles (or, nontrivial multiplicity vector D). As we shall see below, typically, as μ approaches $\mu_0 \in \Delta$, i.e. some of the nodes x_i collide, the corresponding coefficients a_i tend to infinity. Notice, that all the moments $m_k = m_k(F)$ remain bounded. This behavior creates serious difficulties in solving "near-colliding" Prony systems, both in theoretical and practical settings. Especially demanding problems arise in the presence of noise. The problem of improvement of resolution in reconstruction of colliding nodes from noisy measurements appears in a wide range of applications. It is usually called a "super-resolution problem" and a lot of recent publications are devoted to its investigation in various mathematical and applied settings. See [8] and references therein for a very partial sample.

Here we continue our study of collision singularities in Prony systems, started in [21]. Our approach uses bases of finite differences in the Prony space $\mathcal{P}_r$ in order to "resolve" the linear part of collision singularities. In these bases the coefficients do not blow up any more, even as some of the nodes collide.

Example 6.1. Let $r = 2$, and consider the signal $F = a_1\delta(x - x_1) + a_2\delta(x - x_2)$ with

$$\begin{aligned} x_1 &= t,\ x_2 = t + \epsilon, \\ a_1 &= -\epsilon^{-1},\ a_2 = \epsilon^{-1}. \end{aligned}$$

The corresponding Prony system is

$$\left(a_1 x_1^k + a_2 x_2^k =\right) m_k = k t^{k-1} + \underbrace{\sum_{j=2}^{k} \binom{k}{j} t^{k-j} \epsilon^{j-1}}_{\stackrel{\text{def}}{=} \rho_k(t,\epsilon)}, \qquad k = 0, 1, 2, 3.$$

As $\epsilon \to 0$, the Prony system as above becomes ill-conditioned and the coefficients $\{a_j\}$ blow up, while the measurements remain bounded. Note that

$$\tilde{M}_2 = \begin{bmatrix} 0 & 1 & 2t + \rho_2(t,\epsilon) \\ 1 & 2t + \rho_2(t,\epsilon) & 3t^2 + \rho_3(t,\epsilon) \end{bmatrix},$$

therefore $\operatorname{rank} \tilde{M}_2 = 2$ and $|M_2| = 1 \neq 0$, i.e. the Prony problem with input $(m_0, \ldots, m_3)$ remains solvable for all ϵ. However, the standard basis $\{\delta(x - x_1),\ \delta(x - x_2)\}$ degenerates, and in the limit it is no more a basis. If we represent the solution

$$F_{\epsilon}(x)=-\frac{1}{\epsilon}\delta(x-t)+\frac{1}{\epsilon}\delta(x-t-\epsilon)$$

in the basis

$$\begin{aligned}\Delta_1(x_1,x_2) &= \delta(x-x_1),\\ \Delta_2(x_1,x_2) &= \frac{1}{x_1-x_2}\delta(x-x_1)+\frac{1}{x_2-x_1}\delta(x-x_2),\end{aligned}$$

then we have

$$F_{\epsilon}(x)=1\cdot\Delta_2(t,t+\epsilon),$$

i.e., the coefficients in this new basis are just $\{b_1=0,\ b_1=1\}$. As $\epsilon\to 0$, in fact we have

$$\Delta_2(t,t+\epsilon)\to\delta'(x-t),$$

where the convergence is in the topology of the bundle $\mathcal{P}_r$.

Our goal in this section is to generalize the construction of Example 6.1 and [21] to handle the general case of colliding configurations.

6.2. **Divided finite differences.** For modern treatment of divided differences, see e.g. [9, 12, 16]. We follow [9] and adopt what has become by now the standard definition.

Definition 6.2. Let an arbitrary sequence of points $w=(x_1,x_2,\dots,)$ be given (repetitions are allowed). The *(n-1)*-st *divided difference* $\Delta^{n-1}(w):\Pi\to\mathbb{C}$ is the linear functional on the space Π of polynomials in one variable x, associating to each $p\in\Pi$ its (uniquely defined) n-th coefficient in the Newton form

$$p(x)=\sum_{j=1}^{\infty}\left\{\Delta^{j-1}(x_1,\dots,x_j)\,p\right\}\cdot q_{j-1,w}(x),\qquad q_{i,w}(x)\stackrel{def}{=}\prod_{k=1}^{i}(x-x_k). \tag{6.1}$$

Example 6.3. For $n=1$, we have $\Delta^0(x_1)\,p=p(x_1)$, and the 0-th order Newton interpolation polynomial is the constant

$$P_1(x)=p(x_1)\cdot\underbrace{1}_{=q_{0,w}(x)}.$$

Example 6.4. For $n=2$ consider two cases.

(1) If $x_1\neq x_2$, we have $\Delta^1(x_1,x_2)\,p=\frac{p(x_2)-p(x_1)}{x_2-x_1}$, and the first order Newton interpolation polynomial is

$$P_2(x)=p(x_1)\cdot\underbrace{1}_{=q_{o,w}(x)}+\frac{p(x_2)-p(x_1)}{x_2-x_1}\cdot\underbrace{(x-x_1)}_{=q_{1,w}(x)}.$$

It can be readily verified that $P_2(x_k)=p(x_k)$ for $k=1,2$.

(2) If $x_1=x_2$, then $\Delta^1(x_1,x_1)\,p=p'(x_1)$, and so

$$P_2(x)=p(x_1)+p'(x_1)(x-x_1).$$

It can be readily verified that $P_2(x_1)=p(x_1)$ and $P_2'(x_1)=p'(x_1)$.

It turns out that this definition can be extended to all sufficiently smooth functions for which the interpolation problem is well-defined.

Definition 6.5 ([9])**.** For any smooth enough function f, defined at least on $x_1, \dots, x_n$, the divided finite difference $\Delta^{n-1}(x_1, \dots, x_n) f$ is the n-th coefficient in the Newton form (6.1) of the Hermite interpolation polynomial P_n, which agrees with f and its derivatives of appropriate order on $x_1, \dots, x_n$:

$$f^{(\ell)}(x_j) = P_n^{(\ell)}(x_j): \qquad 1 \leqslant j \leqslant n,\ 0 \leqslant \ell < d_j \stackrel{\text{def}}{=} \#\{i: \quad x_i = x_j\}. \tag{6.2}$$

Example 6.6. Consider the rational function depending on a parameter $z \in \mathbb{C}$:

$$f_z(x) = \frac{1}{z-x}.$$

The 0th divided difference is $\Delta^0(x_1) f = f(x_1) = \frac{1}{z-x_1}$, and the Newton interpolation polynomial is

$$P_1(x) = \frac{1}{z-x_1}.$$

For $n=2$ and $x_1 \neq x_2$, we have $\Delta^1(x_1, x_2) = \frac{1}{(z-x_1)(z-x_2)}$, and

$$P_2(x) = \frac{1}{z-x_1} + \frac{x-x_1}{(z-x_1)(z-x_2)},$$

thus $P_2(x_k) = f(x_k)$ for $k=1,2$. If $x_1 = x_2$ then $\Delta^1(x_1, x_1) = f_z'(x_1) = \frac{1}{(z-x_1)^2}$, and so

$$P_2(x) = \frac{1}{z-x_1} + \frac{x-x_1}{(z-x_1)^2}.$$

Again, $P_2(x_1) = f_z(x_1)$ and $P_2'(x_1) = f_z'(x_1)$.

Therefore, each divided difference can be naturally associated with an element of the Prony space (see Item 5 in Proposition 6.7 and Definition 6.8 below for an accurate statement).

Let us now summarize relevant properties of the functional Δ which we shall use later on.

Proposition 6.7. *For $w = (x_1, \dots, x_n) \in \mathbb{C}^n$, let $s(w)$, $T(w)$ and $D(w)$ be defined according to Definition 2.1. Let $q_{n,w}(z) = \prod_{j=1}^{s} (z-\tau_j)^{d_j}$ be defined as in (6.1).*

1. *The functional $\Delta^{n-1}(x_1, \dots, x_n)$ is a symmetric function of its arguments, i.e., it depends only on the set $\{x_1, \dots, x_n\}$ but not on its ordering.*
2. *$\Delta^{n-1}(x_1, \dots, x_n)$ is a continuous function of the vector $(x_1, \dots, x_n)$. In particular, for any test function f*
$$\lim_{(x_1,\dots,x_n)\to(t_1,\dots,t_n)} \Delta^{n-1}(x_1, \dots, x_n) f = \Delta^{n-1}(t_1, \dots, t_n) f.$$
3. *Δ may be computed by the recursive rule*
$$\Delta^{n-1}(x_1, \dots, x_n) f = \begin{cases} \frac{\Delta^{n-2}(x_2,\dots,x_n) f - \Delta^{n-2}(x_1,\dots,x_{n-1}) f}{x_n - x_1} & x_1 \neq x_n, \\ \left\{ \frac{\mathrm{d}}{\mathrm{d}\,\xi} \Delta^{n-2}(\xi, x_2, \dots, x_{n-1}) f \right\} |_{\xi = x_n}, & x_1 = x_n, \end{cases} \tag{6.3}$$
where $\Delta^0(x_1) f = f(x_1)$.
4. *(Generalization of Example 6.6) Let $f_z(x) = (z-x)^{-1}$. Then for all $z \notin \{x_1, \dots, x_n\}$*
$$\Delta^{n-1}(x_1, \dots, x_n) f_z = \frac{1}{q_{n,w}(z)}. \tag{6.4}$$
5. *By (6.2), $\Delta^{n-1}(x_1, \dots, x_n)$ is a linear combination of the functionals*
$$\delta^{(\ell)}(x-\tau_j), \qquad 1 \leqslant j \leqslant s,\ 0 \leqslant \ell < d_j.$$

In fact, using (6.4) we obtain the Chakalov's expansion (see [9])

$$\Delta^{n-1}(x_1,\dots,x_n)=\sum_{j=1}^{s}\sum_{\ell=0}^{d_j-1}a_{j,\ell}\delta^{(\ell)}(x-\tau_j), \tag{6.5}$$

where the coefficients $\{a_{j,\ell}\}$ are defined by the partial fraction decomposition[1]

$$\frac{1}{q_{n,w}(z)}=\sum_{j=1}^{s}\sum_{\ell=0}^{d_j-1}\frac{\ell! a_{j,\ell}}{(z-\tau_j)^{\ell+1}}. \tag{6.6}$$

(6) *By (6.5) and (6.6)*

$$\Delta^{n-1}\left(\underbrace{t,\dots,t}_{\times n}\right)=\frac{1}{(n-1)!}\delta^{(n-1)}(x-t). \tag{6.7}$$

(7) *Popoviciu's refinement lemma* [9, Proposition 23]*: for every index subsequence*

$$1\leqslant\sigma(1)<\sigma(2)<\dots<\sigma(k)\leqslant n,$$

there exist coefficients $\alpha(j)$ such that

$$\Delta^{k-1}\left(x_{\sigma(1)},\dots,x_{\sigma(k)}\right)=\sum_{j=\sigma(1)-1}^{\sigma(k)-k}\alpha(j)\,\Delta^{k-1}(x_{j+1},x_{j+2},\dots,x_{j+k}). \tag{6.8}$$

Based on the above, we may now identify Δ with elements of the bundle $\mathcal{P}_r$.

Definition 6.8. Let $w=(x_1,\dots,x_r)\in\mathbb{C}^r$, and $X=\{n_1,n_2,\dots,n_\alpha\}\subseteq\{1,2,\dots,r\}$ of size $|X|=\alpha$ be given. Let the elements of X be enumerated in increasing order, i.e.

$$1\leqslant n_1<n_2<\dots<n_\alpha\leqslant r.$$

Denote by w_X the vector

$$w_X\stackrel{\text{def}}{=}(x_{n_1},x_{n_2},\dots,x_{n_\alpha})\in\mathbb{C}^\alpha.$$

Then we denote

$$\Delta_X(w)\stackrel{\text{def}}{=}\Delta^{\alpha-1}(w_X).$$

We immediately obtain the following result.

Lemma 6.9. *For all $w\in\mathbb{C}^r$ and $X\subseteq\{1,2,\dots,r\}$, we have $\Delta_X(w)\in V_w$. Moreover, letting $\alpha=|X|$ we have*

$$\mathcal{S}M(\Delta_X(w))=\Delta^{\alpha-1}(w_X)\frac{1}{z-x}=\frac{1}{q_{\alpha,w_X}(z)}. \tag{6.9}$$

Finally, $(w,\Delta_X(w))$ is a continuous section of $\mathcal{P}_r$.

---

[1]The coefficients $\left\{a_{j,\ell}\right\}$ may be readily obtained by the Cauchy residue formula

$$a_{j,\ell}=\frac{1}{(d_j-1-\ell)!}\lim_{z\to\tau_j}\left(\frac{\mathrm{d}}{\mathrm{d}\,z}\right)^{d_j-1-\ell}\left\{\frac{(z-\tau_j)^{\ell+1}}{q_{n,w}(z)}\right\}.$$

6.3. **Constructing a basis.** The following result is well-known, see e.g. [9, Proposition 35].

Theorem 6.10. *Denote $N_j = \{1, 2, \dots, j\}$ for $j = 1, 2, \dots, r$. Then for every $w \in \mathbb{C}^r$, the collection*

$$\left\{\Delta_{N_j}(w)\right\}_{j=1}^{r}$$

is a basis for V_w.

There are various proofs of this statement. Below we show how to construct sets which do not necessarily remain basis for all $w \in \mathbb{C}^r$, but only for w in a small neighborhood of a given $w_0 \in \mathbb{C}^r$. Theorem 6.10 will then follow as a special case of this construction.

Informally, if two coordinates x_i and x_j can collide, then it is necessary to allow them to be glued by some element of the basis, i.e., we will need $\Delta_X(w)$ where $i, j \in X$ (in Theorem 6.10 all coordinates might be eventually glued into a single point because w is unrestricted.) In order to make this statement formal, let us introduce a notion of *configuration*, which is essentially a partition of the set of indices.

Definition 6.11. A *configuration* $\mathcal{C}$ is a partition of the set $N_r = \{1, 2, \dots, r\}$ into $s = s(\mathcal{C})$ disjoint nonempty subsets

$$\sqcup_{i=1}^{s} X_i = N_r, \qquad |X_i| = d_i > 0.$$

The multiplicity vector of $\mathcal{C}$ is

$$T(\mathcal{C}) = (d_1, \dots, d_s).$$

Every configuration defines a continuous family of divided differences as follows.

Definition 6.12. Let a configuration $\mathcal{C} = \{X_j\}_{j=1}^{s(\mathcal{C})}$. Enumerate each X_j in increasing order of its elements

$$X_j = \left\{n_1^j < n_2^j < \dots n_{d_j}^j\right\}$$

and denote for every $m = 1, 2, \dots, d_j$

$$X_{j,m} \stackrel{\text{def}}{=} \left\{n_k^j : \ k = 1, 2, \dots, m\right\}.$$

For every $w \in \mathbb{C}^r$, the collection $\mathcal{B}_C(w) \subset V_w$ is defined as follows:

$$\mathcal{B}_C(w) \stackrel{\text{def}}{=} \left\{\Delta_{X_{j,m}}(w)\right\}_{j=1,\dots,s(\mathcal{C})}^{m=1,\dots,d_j}.$$

Now we formally define when a partition is "good" with respect to a point $w \in \mathbb{C}^r$.

Definition 6.13. The point $w = (x_1, \dots, x_r) \in \mathbb{C}^r$ is *subordinated* to the configuration

$$\mathcal{C} = \{X_j\}_{j=1}^{s(\mathcal{C})}$$

if whenever $x_k = x_\ell$ for a pair of indices $k \neq \ell$, then necessarily $k, \ell \in X_j$ for some X_j.

Now we are ready to formulate the main result of this section.

Theorem 6.14. *For a given $w_0 \in \mathbb{C}^r$ and a configuration $\mathcal{C}$, the collection $\mathcal{B}_C(w_0)$ is a basis for V_{w_0} if and only if w_0 is subordinated to $\mathcal{C}$. In this case, $\mathcal{B}_C(w)$ is a continuous family of bases for V_w in a sufficiently small neighborhood of w_0.*

Let us first make a technical computation.

Lemma 6.15. *For a configuration $\mathcal{C}$ and a point $w \in \mathbb{C}^r$, consider for every fixed $j = 1, \dots, s(\mathcal{C})$ the set*

$$S_j \stackrel{\text{def}}{=} \left\{\Delta_{X_{j,m}}(w)\right\}_{m=1}^{d_j}. \tag{6.10}$$

(1) *Define for any pair of indices* $1 \leqslant k \leqslant \ell \leqslant d_j$ *the index set*

$$X_{j,k:\ell} \stackrel{def}{=} \left\{n_k^j < n_{k+1}^j < \cdots < n_\ell^j\right\} \subseteq X_j = X_{j,1:d_j} = X_{j,d_j}.$$

Then

$$\Delta_{X_{j,k:\ell}}(w) \in \operatorname{span} S_j.$$

(2) *For an arbitrary subset* $Y \subseteq X_j$ *(and not necessarily containing segments of consecutive indices), we also have*

$$\Delta_Y(w) \in \operatorname{span} S_j.$$

Proof. For clarity, we denote $y_i = x_{n_i^j}$ and $[k:\ell] = \Delta_{X_{j,k:\ell}}(w)$. By (6.3) we have in all cases (including repeated nodes)

$$(y_\ell - y_k)[k:\ell] = [k+1:\ell] - [k:\ell-1]. \tag{6.11}$$

The proof of the first statement is by backward induction on $n = \ell - k$. We start from $n = d_j$, and obviously $[1:d_j] \in S_j$. In addition, by definition of S_j we have $[1:m] \in S_j$ for all $m = 1, \dots, d_j$. Therefore, in order to obtain all $[k:\ell]$ with $\ell - k = n-1$, we apply (6.11) several times as follows.

$$\begin{aligned}
[2:n] &= (y_n - y_1)[1:n] + [1:n-1] \\
[3:n+1] &= (y_{n+1} - y_2)\underset{\longleftrightarrow}{[2:n+1]} + \underline{[2:n]} \\
&\cdots \\
[d_j - n + 2 : d_j] &= \left(y_{d_j} - y_{d_j - n + 1}\right)\underset{\longleftrightarrow}{[d_j - n + 1 : d_j]} + \underline{[d_j - n + 1 : d_j - 1]}
\end{aligned}$$

Here the symbol $\underline{\cdots}$ under a term means that the term is taken directly from the previous line, while $\underset{\longleftrightarrow}{\cdots}$ indicates that the induction hypothesis is used. In the end, the left-hand side terms are shown to belong to $\operatorname{span} S_j$.

In order to prove the second statement, we employ the first statement, (6.8) and Proposition 6.7, Item 1. □

Proof of Theorem 6.14. In one direction, assume that $w_0 = (x_1, \dots, x_r)$ is subordinated to $\mathcal{C}$. It is sufficient to show that every element of the standard basis (2.2) belongs to $\operatorname{span}\{\mathcal{B}_C(w_0)\}$.

Let $\tau_j \in T(w_0)$, let d_j be the corresponding multiplicity, and let $Y_j \subseteq N_r$ denote the index set of size d_j

$$Y_j \stackrel{\text{def}}{=} \{i : \quad x_i = \tau_j\}.$$

By the definition of subordination, there exists an element in the partition of $\mathcal{C}$, say X_k, for which $Y_j \subseteq X_k$. By Lemma 6.15 we conclude that for all subsets $Z \subseteq Y_j$,

$$\Delta_Z(w_0) \in \operatorname{span}\left\{\Delta_{X_{k,m}}(w_0)\right\}_{m=1}^{|X_k|} \subseteq \operatorname{span}\{\mathcal{B}_C(w_0)\}.$$

By (6.7), $\Delta_Z(w_0)$ is nothing else but

$$\Delta_Z(w_0) = \Delta^{|Z|-1}\left(\underbrace{\tau_j, \dots, \tau_j}_{\times |Z|}\right) = \frac{1}{(|Z|-1)!}\delta^{(|Z|-1)}(x - \tau_j).$$

This completes the proof of the necessity. In the other direction, assume by contradiction that $x_k = x_\ell = \tau$ but nevertheless there exist two distinct elements of the partition $\mathcal{C}$, say X_α and

X_β such that $k \in X_\alpha$ and $\ell \in X_\beta$. Let the sets $\{S_j\}_{j=1}^{s(\mathcal{C})}$ be defined by (6.10). Again, by Lemma 6.15 and (6.7) we conclude that

$$\delta(x-\tau) \in \operatorname{span} S_\alpha \cap \operatorname{span} S_\beta.$$

But notice that $\mathcal{B}_C(w_0) = \bigcup_{j=1}^{s(\mathcal{C})} S_j$ and $\sum_{j=1}^{s} |S_j| = d$, therefore by counting dimensions we conclude that

$$\dim \operatorname{span} \{\mathcal{B}_C(w_0)\} < d,$$

in contradiction to the assumption that $\mathcal{B}_C(w_0)$ is a basis.

Finally, one can evidently choose a sufficiently small neighborhood $U \subset \mathbb{C}^r$ of w_0 such that for all $w \in U$, no new collisions are introduced, i.e., w is still subordinated to $\mathcal{C}$. The continuity argument (Lemma 6.9) finishes the proof. □

Remark 6.16. Another possible method of proof is to consider the algebra of elementary fractions in the Stieltjes space $\mathcal{S}_r$, and use the correspondence (6.9).

As we mentioned, Theorem 6.10 follows as a corollary of Theorem 6.14 for the configuration $\mathcal{C}$ consisting of a single partition set N_r.

6.4. **Resolution of collision singularities.** Let $\mu_0^* \in \Sigma_r^* \subset \mathbb{C}^{2r}$ be given, and let $(w_0, g_0) \in \mathcal{P}_r$ be a solution to the (rank-restricted) Prony problem. The point w_0 is uniquely defined up to a permutation of the coordinates, so we just fix a particular permutation. Let $T(w_0) = (\tau_1, \dots, \tau_s)$.

Our goal is to solve the rank-restricted Prony problem for every input $\mu^* \in \mathbb{C}^{2r}$ in a small neighborhood of μ_0^*. According to Theorem 5.7, this amounts to a continuous representation of the solution $R_{\mu^*}(z) = \frac{P_{\mu^*}(z)}{Q_{\mu^*}(z)} = \mathcal{T} M_r^{-1}(\mu^*)$ to the corresponding diagonal Padé approximation problem as an element of the bundle $\mathcal{P}_r$.

Define $\delta = \min_{i \neq j} |\tau_i - \tau_j|$ to be the "separation distance" between the clusters. Since the roots of Q_{μ^*} depend continuously on μ^* and the degree of Q_{μ^*} does not drop, we can choose some μ_1^* sufficiently close to μ_0^*, for which

1. all the roots of $Q_{\mu_1^*}(z)$ are distinct, and
2. these roots can be grouped into s clusters, such that each of the elements of the j-th cluster is at most $\delta/3$ away from τ_j.

Enumerate the roots of $Q_{\mu_1^*}$ within each cluster in an arbitrary manner. This choice enables us to define locally (in a neighborhood of μ_1^*) r algebraic functions $x_1(\mu^*), \dots, x_r(\mu^*)$, satisfying

$$Q_{\mu^*}(z) = \prod_{j=1}^{s} (z - x_j(\mu^*)).$$

Then we extend these functions by analytic continuation according to the above formula into the entire neighborhood of μ_0^*. Consequently,

$$w(\mu^*) \stackrel{\text{def}}{=} (x_1(\mu^*), \dots, x_r(\mu^*))$$

is a continuous (multivalued) algebraic function in a neighborhood of μ_0^*, satisfying

$$w(\mu_0^*) = w_0.$$

After this "pre-processing" step, we can solve the rank-restricted Prony problem in this neighborhood of μ_0^*, as follows.

Algorithm 1 Solving rank-restricted Prony problem with collisions.

Let $\mu_0^* \in \Sigma_r^* \subset \mathbb{C}^{2r}$ be given, and let $(w_0, g_0) \in \mathcal{P}_r$ be a solution to the (rank-restricted) Prony problem. Let w_0 be subordinated to some configuration $\mathcal{C}$.

The input to the problem is a measurement vector $\mu^* = (m_0, \dots, m_{2r-1}) \in \mathbb{C}^{2r}$, which is in a small neighborhood of μ_0^*.

(1) Construct the function $w = w(\mu^*)$ as described above.

(2) Build the basis $\mathcal{B}_C(w) = \left\{\Delta_{X_{j,\ell}}(w)\right\}_{j=1,\dots,s(\mathcal{C})}^{\ell=1,\dots,d_j}$ for V_w.

(3) Find the coefficients $\{\beta_{j,\ell}\}_{j=1,\dots,s(\mathcal{C})}^{\ell=1,\dots,d_j}$ such that

$$\mathcal{S}M\left(\sum_{j,\ell}\beta_{j,\ell}\Delta_{X_{j,\ell}}(w)\right) = R(z),$$

by solving the linear system

$$\underbrace{\sum_{j,\ell}\beta_{j,\ell}(w)\,\Delta_{X_{j,\ell}}(w)}_{=g(w)}\left(x^k\right) = m_k\left(=\int x^k g(w)(x)\,\mathrm{d}\,x\right), \qquad k = 0,1,\dots,2r-1. \tag{6.12}$$

Theorem 6.17. *The coordinates $\{\beta_{j,\ell}\}$ of the solution to the rank-restricted Prony problem, given by Algorithm 6.4, are (multivalued) algebraic functions, continuous in a neighborhood of the point μ_0^* .*

Proof. Since the divided differences $\Delta_{j,\ell}(w)$ are continuous in w, then clearly for each

$$k = 0,1,\dots,2r-1$$

the functions

$$\nu_{j,\ell,k}(w) = \Delta_{j,\ell}(w)\left(x^k\right) = \Delta^{\ell-1}\left(w_{X_{j,\ell}}\right)\left(x^k\right)$$

are continuous[2] in w, and hence continuous, as multivalued functions, in a neighborhood of μ_0^*. Since $\mathcal{B}_C(w(\mu^*))$ remains a basis in a (possibly smaller) neighborhood of μ_0^*, the system (6.12), taking the form

$$\sum_{j,\ell}\nu_{j,\ell,k}(w)\,\beta_{j,\ell}(w) = m_k, \qquad k = 0,1,\dots,2r-1,$$

remains non-degenerate in this neighborhood. We conclude that the coefficients $\{\beta_{j,\ell}(w(\mu^*))\}$ are multivalued algebraic functions, continuous in a neighborhood of μ_0^*. □

7. Real Prony space and hyperbolic polynomials

In this section we shall restrict ourselves to the real case. Notice that in many applications only real Prony systems are used. On the other hand, considering the Prony problem over the real numbers significantly simplifies some constructions. In particular, we can easily avoid topological problems, related with the choice of the ordering of the points $x_1, \dots, x_d \in \mathbb{C}$. So in a definition of the real Prony space $R\mathcal{P}_d$ we assume that the coordinates $x_1, \dots, x_d$ are taken with their natural ordering $x_1 \le x_2 \le \dots \le x_d$. Accordingly, the real Prony space $R\mathcal{P}_d$ is defined as the bundle (w, g), $w \in \prod_d \subset \mathbb{R}^d, g \in RV_w$. Here $\prod_d$ is the prism in $\mathbb{R}^d$ defined by the inequalities $x_1 \le x_2 \le \dots \le x_d$, and RV_w is the space of linear combinations with real coefficients of δ-functions and their derivatives with the support $\{x_1, \dots, x_d\}$, as in Definition

[2] In fact, $\nu_{j,\ell,k}(w)$ are symmetric polynomials in some of the coordinates of w.

2.4. The Prony, Stieltjes and Taylor maps are the restrictions to the real case of the complex maps defined above.

In this paper we just point out a remarkable connection of the real Prony space and mapping with hyperbolic polynomials, and Vieta and Vandermonde mappings studied in Singularity Theory (see [1, 13, 14, 15] and references therein).

Hyperbolic polynomials (in one variable) are real polynomials $Q(z) = z^d + \sum_{j=1}^{d} \lambda_j z^{d-j}$, with all d of their roots real. We denote by Γ_d the space of the coefficients $\Lambda = (\lambda_1, \ldots, \lambda_d) \subset \mathbb{R}^d$ of all the hyperbolic polynomials, and by $\hat{\Gamma}_d$ the set of $\Lambda \in \Gamma_d$ with $\lambda_1 = 0,\ |\lambda_2| \leq 1$. Recalling (2.3), it is evident that all hyperbolic polynomials appear as the denominators of the irreducible fractions in the image of $R\mathcal{P}_d$ by $\mathcal{S}M$. This shows, in particular, that the geometry of the boundary $\partial\Gamma$ of the hyperbolicity domain Γ is important in the study of the real Prony map $\mathcal{P}M$: it is mapped by $\mathcal{P}M$ to the boundary of the solvability domain of the real Prony problem. This geometry has been studied in a number of publications, from the middle of 1980s. In [13] V. P. Kostov has shown that $\hat{\Gamma}$ possesses the Whitney property: there is a constant C such that any two points $\lambda_1, \lambda_2 \in \hat{\Gamma}$ can be connected by a curve inside $\hat{\Gamma}$ of the length at most $C\|\lambda_2 - \lambda_1\|$. “Vieta mapping” which associates to the nodes $x_1 \leq x_2 \leq \cdots \leq x_d$ the coefficients of $Q(z)$ having these nodes as the roots, is also studied in [13]. In our notations, Vieta mapping is the composition of the Stieltjes mapping $\mathcal{S}M$ with the projection to the coefficients of the denominator.

In [1] V.I.Arnold introduced and studied the notion of maximal hyperbolic polynomial, relevant in description of $\hat{\Gamma}$. Furthermore, the Vandermonde mapping $\mathcal{V} : \mathbb{R}^d \to \mathbb{R}^d$ was defined there by

$$\begin{cases} y_1 = a_1 x_1 + \ldots + a_d x_d, \\ \ldots \\ y_d = a_1 x_1^d + \ldots + a_d x_d^d, \end{cases}$$

with $a_1, \ldots, a_d$ fixed. In our notations $\mathcal{V}$ is the restriction of the Prony mapping to the pairs $(w, g) \in R\mathcal{P}_d$ with the coefficients of g in the standard basis of RV_w fixed. It was shown in [1] that for $a_1, \ldots, a_d > 0$ $\mathcal{V}$ is a one-to-one mapping of $\prod_d$ to its image. In other words, the first d moments uniquely define the nodes $x_1 \leq x_2 \leq \cdots \leq x_d$. For $a_1, \ldots, a_d$ with varying signs, this is no longer true in general. This result is applied in [1] to the study of the colliding configurations.

Next, the “Vandermonde varieties” are studied in [1], which are defined by the equations

$$\begin{cases} a_1 x_1 + \ldots + a_d x_d & = \alpha_1, \\ & \ldots \\ a_1 x_1^\ell + \ldots + a_d x_d^\ell & = \alpha_\ell. \end{cases} \qquad \ell \leqslant d.$$

It is shown that for $a_1, \ldots, a_d > 0$ the intersections of such varieties with $\prod_d$ are either contractible or empty. Finally, the critical points of the next Vandermonde equation on the Vandermond variety are studied in detail, and on this base a new proof of Kostov’s theorem is given.

We believe that the results of [1, 13] and their continuation in [14, 15] and other publications are important for the study of the Prony problem over the reals, and we plan to present some results in this direction separately.

Appendix A. Proof of Theorem 3.5

Recall that we are interested in finding conditions for which the Taylor mapping $\mathcal{T}M:\ \mathcal{S}_d \to \mathcal{T}_d$ is invertible. In other words, given

$$S(z) = \sum_{k=0}^{2d-1} m_k \left(\frac{1}{z}\right)^{k+1},$$

we are looking for a rational function $R(z) \in \mathcal{S}_d$ such that

$$S(z) - R(z) = \frac{d_1}{z^{2d+1}} + \frac{d_2}{z^{2d+2}} + \dots. \tag{A.1}$$

Write $R(z) = \frac{P(z)}{Q(z)}$ with $Q(z) = \sum_{j=0}^{d} c_j z^j$ and $P(z) = \sum_{i=0}^{d-1} b_i z^i$. Multiplying (A.1) by $Q(z)$, we obtain

$$Q(z)\,S(z) - P(z) = \frac{e_1}{z^{d+1}} + \frac{e_2}{z^{d+2}} + \dots. \tag{A.2}$$

Proposition A.1. *The identity (A.2), considered as an equation on P and Q with*

$$\deg P < \deg Q \le d,$$

always has a solution.

Proof. Substituting the expressions for S, P and Q into (A.2) we get

$$\left(c_0 + c_1 z + \dots + c_d z^d\right)\left(\frac{m_0}{z} + \frac{m_1}{z^2} + \dots\right) - b_0 - \dots - b_{d-1} z^{d-1} = \frac{e_1}{z^{d+1}} + \dots. \tag{A.3}$$

The highest degree of z in the left hand side of (A.3) is $d-1$. So equating to zero the coefficients of z^s in (A.3) for $s = d-1, \dots, -d$ we get the following systems of equations:

$$\begin{bmatrix} 0 & 0 & 0 & m_0 \\ 0 & 0 & m_0 & m_1 \\ \iddots & \iddots & & \\ m_0 & m_1 & \dots & m_{d-1} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_d \end{bmatrix} = \begin{bmatrix} b_{d-1} \\ b_{d-2} \\ \vdots \\ b_0 \end{bmatrix}. \tag{A.$\star$}$$

From this point on, the equations become homogeneous:

$$\begin{bmatrix} m_0 & m_1 & \dots & m_d \\ m_1 & m_2 & \dots & m_{d+1} \\ \iddots & \iddots & & \\ m_{d-1} & m_d & \dots & m_{2d-1} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{A.$\star\star$}$$

The homogeneous system (A.$\star\star$) has the Hankel-type $d \times (d+1)$ matrix $\tilde{M}_d = (m_{i+j})$ with $0 \leqslant i \leqslant d-1$ and $0 \leqslant j \leqslant d$. This system has d equations and $d+1$ unknowns $c_0, \dots, c_d$. Consequently, it always has a nonzero solution $c_0, \dots, c_d$. Now substituting these coefficients $c_0, \dots, c_d$ of Q into the equations (A.$\star$) we find the coefficients $b_0, \dots, b_{d-1}$ of the polynomial P, satisfying (A.$\star$). Notice that if $c_j = 0$ for $j \geqslant \ell + 1$ then it follows from the structure of the equations (A.$\star$) that $b_j = 0$ for $j \ge \ell$. Hence these P, Q provide a solution of (A.2), satisfying $\deg P < \deg Q \le d$, and hence belonging to $\mathcal{S}_d$. □

However, in general (A.2) does not imply (A.1). This implication holds only if $\deg Q = d$. The following proposition describes a possible "loss of accuracy" as we return from (A.2) to (A.1) and $\deg Q < d$:

Proposition A.2. *Let (A.2) be satisfied with the highest nonzero coefficient of Q being $c_\ell,\ \ell \le d$. Then*

$$S(z) - \frac{P(z)}{Q(z)} = \frac{d_1}{z^{d+\ell+1}} + \frac{d_2}{z^{d+\ell+2}} + \dots . \tag{A.4}$$

Proof. We notice that if the leading nonzero coefficient of Q is c_ℓ then we have

$$\frac{1}{Q} = \frac{1}{z^\ell}\left(\frac{1}{c_\ell + \frac{c_{\ell-1}}{z} + \dots}\right) = \frac{1}{z^\ell}(f_0 + f_1\frac{1}{z} + \dots).$$

So multiplying (A.2) by $\frac{1}{Q}$ we get (A.4). □

Proof of Theorem 3.5. Assume that the rank of $\tilde{M}_d$ is $r \le d$, and that $|M_r| \ne 0$. Let us find a polynomial $Q(z)$ of degree r of the form $Q(z) = z^r + \sum_{j=0}^{r-1} c_j z^j$, whose coefficients satisfy system (A.⋆⋆). Put $\mathbf{c}_r = (c_0, \dots, c_{r-1}, 1)^T$ and consider a linear system $\tilde{M}_r \mathbf{c}_r = 0$. Since by assumptions $|M_r| \ne 0$, this system has a unique solution. Extend this solution by zeroes, i.e., put $\mathbf{c}_d = (c_0, \dots, c_{r-1}, 1, 0, \dots, 0)^T$. We want $\mathbf{c}_d$ to satisfy (A.⋆⋆), which is $\tilde{M}_d \mathbf{c}_d = 0$. This fact is immediate for the first r rows of $\tilde{M}_d$. But since the rank of $\tilde{M}_d$ is r by the assumption, its other rows are linear combinations of the first r ones. Hence $\mathbf{c}_d$ satisfies (A.⋆⋆).

Now the equations (A.⋆) produce a polynomial $P(z)$ of degree at most $r-1$. So we get a rational function $R(z) = \frac{P(z)}{Q(z)} \in \mathcal{S}_r \subseteq \mathcal{S}_d$ which solves the Padé problem (A.2), with $\deg Q(z) = r$. Write $R(z) = \sum_{k=0}^{\infty} \alpha_k (\frac{1}{z})^{k+1}$. By Proposition A.2 we have $m_k = \alpha_k$ till $k = d + r - 1$.

Now, the Taylor coefficients α_k of $R(z)$ satisfy a linear recurrence relation

$$m_k = -\sum_{s=1}^{r} c_s m_{k-s}, \qquad k = r, r+1, \dots . \tag{A.5}$$

Considering the rows of the system $\tilde{M}_d \mathbf{c}_d = 0$ we see that m_k satisfy the same recurrence relation (A.5) till $k = d + r - 1$ (we already know that $m_k = \alpha_k$ till $k = d + r - 1$). We shall show that in fact m_k satisfy (A.5) till $k = 2d - 1$.

Consider a $d \times r$ matrix $\bar{M}_d$ formed by the first r columns of M_d, and denote its row vectors by $\mathbf{v}_i = (m_{i,0}, \dots, m_{i,r-1}),\ i = 1, \dots, d-1$. The vectors $\mathbf{v}_i$ satisfy

$$\mathbf{v}_i = -\sum_{s=1}^{r} c_s \mathbf{v}_{i-s}, \quad i = r, \dots, d-1, \tag{A.6}$$

since their coordinates satisfy (A.5) till $k = d+r-1$. Now $\mathbf{v}_0, \dots, \mathbf{v}_{r-1}$ are linearly independent, and hence each $\mathbf{v}_i,\ i = r, \dots, d-1$, can be expressed as

$$\mathbf{v}_i = \sum_{s=0}^{r-1} \gamma_{i,s} \mathbf{v}_s. \tag{A.7}$$

Denote by $\tilde{\mathbf{v}}_i = (m_{i,0}, \dots, m_{i,d}),\ i = 1, \dots, d-1$ the row vectors of $\tilde{M}_d$. Since by assumptions the rank of $\tilde{M}_d$ is r, the vectors $\tilde{\mathbf{v}}_i$ can be expressed through the first r of them exactly in the same form as $\mathbf{v}_i$:

$$\tilde{\mathbf{v}}_i = \sum_{s=0}^{r-1} \gamma_{i,s} \tilde{\mathbf{v}}_s, \quad i = r, \dots, d-1. \tag{A.8}$$

Now the property of a system of vectors to satisfy the linear recurrence relation (A.6) depends only on the coefficients $\gamma_{i,s}$ in their representation (A.7) or (A.8). Hence from (A.6) we conclude that the full rows $\tilde{\mathbf{v}}_i$ of $\tilde{M}_d$ satisfy the same recurrence relation. Coordinate-wise this implies that m_k satisfy (A.5) till $k=2d-1$, and hence $m_k=\alpha_k$ till $k=2d-1$. So $R(z)$ solves the original Problem 3.1.

In the opposite direction, assume that $R(z)$ solves Problem 3.1, and that the representation $R(z)=\frac{P(z)}{Q(z)}\in\mathcal{S}_r\subset\mathcal{S}_d$ is irreducible, i.e., $\deg Q=r$. Write $Q(z)=z^r+\sum_{j=0}^{r-1}c_jz^j$. Then m_k, being the Taylor coefficients of $R(z)$ till $k=2d-1$, satisfy a linear recurrence relation (A.5): $m_k=-\sum_{s=1}^{r}c_sm_{k-s},\ k=r,r+1,\dots,2d-1$. Applying this relation coordinate-wise to the rows of $\tilde{M}_d$ we conclude that all the rows can be linearly expressed through the first r ones. So the rank of $\tilde{M}_d$ is at most r.

It remains to show that the left upper minor $|M_r|$ is non-zero, and hence the rank of $\tilde{M}_d$ is exactly r.

By Proposition 3.3, if the decomposition of $R(z)$ in the standard basis is

$$R(z)=\sum_{j=1}^{s}\sum_{\ell=1}^{d_j}a_{j,\ell-1}\frac{(-1)^{\ell-1}(\ell-1)!}{(z-x_j)^{\ell}},$$

where $\sum_{j=1}^{s}d_j=r$ and $\{x_j\}$ are pairwise distinct, then the Taylor coefficients of $R(z)$ are given by (1.5). Clearly, we must have $a_{j,d_j-1}\neq 0$ for all $j=1,\dots,s$, otherwise $\deg Q<r$, a contradiction. Now consider the following well-known representation of M_r as a product of three matrices (see e.g. [7]):

$$M_r=V(x_1,d_1,\dots,x_s,d_s)\times\operatorname{diag}\{A_j\}_{j=1}^{s}\times V(x_1,d_1,\dots,x_s,d_s)^T, \tag{A.9}$$

where $V(\dots)$ is the confluent Vandermonde matrix (4.1) and each A_j is the following $d_j\times d_j$ block:

$$A_j\stackrel{\text{def}}{=}\begin{bmatrix} a_{j,0} & a_{j,1} & \cdots & \cdots & a_{j,d_j-1}\\ a_{j,1} & & & \binom{d_j-1}{d_j-2}a_{j,d_j-1} & 0\\ \cdots & & & \cdots & 0\\ & \binom{d_j-1}{2}a_{j,d_j-1} & 0 & \cdots & 0\\ a_{j,d_j-1} & 0 & \cdots & \cdots & 0\end{bmatrix}.$$

The formula (A.9) can be checked by direct computation. Since $\{x_j\}$ are pairwise distinct and $a_{j,d_j-1}\neq 0$ for all $j=1,\dots,s$, we immediately conclude that $|M_r|\neq 0$.

This finishes the proof of Theorem 3.5. □

References

[1] V.I. Arnol'd. Hyperbolic polynomials and Vandermonde mappings. *Functional Analysis and Its Applications*, 20(2):125–127, 1986. DOI: 10.1007/BF01077267

[2] J.R. Auton. Investigation of Procedures for Automatic Resonance Extraction from Noisy Transient Electromagnetics Data. Volume III. Translation of Prony's Original Paper and Bibliography of Prony's Method. Technical report, Effects Technology Inc., Santa Barbara, CA, 1981.

[3] R. Badeau, B. David, and G. Richard. Performance of ESPRIT for estimating mixtures of complex exponentials modulated by polynomials. *IEEE Transactions on Signal Processing*, 56(2):492–504, 2008. DOI: 10.1109/TSP.2007.906744

[4] G.A. Baker and P. Graves-Morris. Pade approximants. Part 1: Basic theory. *Encyclopedia of Mathematics and its applications*, 1981.

[5] D. Batenkov. Complete Algebraic Reconstruction of Piecewise-Smooth Functions from Fourier Data. *To appear in Mathematics of Computation.*

[6] D. Batenkov. Numerical stability bounds for algebraic systems of Prony type and their accurate solution by decimation. September 2014. arXiv:1409.3137

[7] D. Batenkov and Y. Yomdin. On the accuracy of solving confluent Prony systems. *SIAM J. Appl. Math.*, 73(1):134–154, 2013. DOI: 10.1137/110836584

[8] Emmanuel J. Candès and Carlos Fernandez-Granda. Towards a mathematical theory of super-resolution. *Communications on Pure and Applied Mathematics*, 67(6):906–956, June 2014. DOI: 10.1002/cpa.21455

[9] C. de Boor. Divided differences. *Surv. Approx. Theory*, 1:46–69, 2005.
[10] D.L. Donoho, M. Elad, and V.N. Temlyakov. Stable recovery of sparse overcomplete representations in the presence of noise. *Information Theory, IEEE Transactions on*, 52(1):6–18, 2006.
[11] B. Gustafsson, C. He, P. Milanfar, and M. Putinar. Reconstructing planar domains from their moments. *Inverse Problems*, 16(4):1053–1070, 2000. DOI: 10.1088/0266-5611/16/4/312
[12] W. Kahan and R.J. Fateman. Symbolic computation of divided differences. *ACM SIGSAM Bulletin*, 33(2):7–28, 1999. DOI: 10.1145/334714.334716
[13] V.P. Kostov. On the geometric properties of Vandermonde's mapping and on the problem of moments. *Proceedings of the Royal Society of Edinburgh: Section A Mathematics*, 112(3-4):203–211, 1989. DOI: 10.1017/S0308210500018679
[14] V.P. Kostov. Root arrangements of hyperbolic polynomial-like functions. *Revista matemática complutense*, 19(1):197–225, 2006.
[15] V.P. Kostov. On root arrangements for hyperbolic polynomial-like functions and their derivatives. *Bulletin des sciences mathematiques*, 131(5):477–492, 2007. DOI: 10.1016/j.bulsci.2006.12.004
[16] L.M. Milne-Thompson. *The calculus of finite differences*. Macmillan, 1933.
[17] T. Peter, D. Potts, and M. Tasche. Nonlinear approximation by sums of exponentials and translates. *SIAM Journal on Scientific Computing*, 33(4):1920, 2011. DOI: 10.1137/100790094
[18] D. Potts and M. Tasche. Parameter estimation for exponential sums by approximate Prony method. *Signal Processing*, 90(5):1631–1642, 2010. DOI: 10.1016/j.sigpro.2009.11.012
[19] R. Prony. Essai experimental et analytique. *J. Ec. Polytech.(Paris)*, 2:24–76, 1795.
[20] P. Stoica and N. Arye. MUSIC, maximum likelihood, and Cramer-Rao bound. *IEEE Transactions on Acoustics, Speech and Signal Processing*, 37(5):720–741, 1989. DOI: 10.1109/29.17564
[21] Y. Yomdin. Singularities in Algebraic Data Acquisition. In M. Manoel, M.C.R. Fuster, and C.T.C. Wall, editors, *Real and Complex Singularities*. Cambridge University Press, 2010. DOI: 10.1017/CBO9780511731983.026

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NAIVE MOTIVIC DONALDSON–THOMAS TYPE HIRZEBRUCH CLASSES AND SOME PROBLEMS

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ABSTRACT. Donaldson-Thomas invariant is expressed as the weighted Euler characteristic of the so-called Behrend (constructible) function. In [2] Behrend introduced a Donaldson-Thomas type invariant for a morphism. Motivated by this invariant, we extend the motivic Hirzebruch class to naive Donaldson-Thomas type analogues. We also discuss a categorification of the Donaldson-Thomas type invariant for a morphism from a bivariant-theoretic viewpoint, and we finally pose some related questions for further investigations.

1. INTRODUCTION

The Donaldson–Thomas invariant $\chi^{DT}(\mathcal{M})$ (abbr. DT invariant) is the virtual count of the moduli space $\mathcal{M}$ of stable coherent sheaves on a Calabi–Yau threefold over k. Here k is an algebraically closed field of characteristic zero. Foundational materials for DT invariants can be found in [36], [2], [20], [23]. In [2] Behrend made the important observation that the Donaldson–Thomas invariant $\chi^{DT}(\mathcal{M})$ is described as the weighted Euler characteristic $\chi(\mathcal{M}, \nu_{\mathcal{M}})$ of the so-called Behrend (constructible) function $\nu_{\mathcal{M}}$. For a scheme X of finite type, the Donaldson–Thomas type invariant $\chi^{DT}(X)$ is defined as $\chi(X, \nu_X)$. The Euler characteristic χ defined by using the compactly-supported ℓ-adic cohomology groups (see §2 for more details) satisfies the scissor formula $\chi(X) = \chi(Z) + \chi(X \setminus Z)$ for a closed subvariety $Z \subset X$. This scissor formula implies that χ can be considered as a homomorphism from the Grothendieck group of varieties $\chi : K_0(\mathcal{V}) \to \mathbb{Z}$, and furthermore it can be extended to the relative Grothendieck group, $\chi : K_0(\mathcal{V}/X) \to \mathbb{Z}$ for each scheme X. The Grothendieck–Riemann–Roch version of the homomorphism $\chi : K_0(\mathcal{V}/X) \to \mathbb{Z}$ is the motivic Chern class transformation $T_{-1*} : K_0(\mathcal{V}/X) \to H_*^{BM}(X) \otimes \mathbb{Q}$. Namely we have that

- When X is a point, $T_{-1*} : K_0(\mathcal{V}/X) \to H_*^{BM}(X) \otimes \mathbb{Q}$ equals the homomorphism $\chi : K_0(\mathcal{V}) \to \mathbb{Z} \hookrightarrow \mathbb{Q}$.
- The composite $\int_X \circ T_{-1*} = \chi : K_0(\mathcal{V}/X) \to \mathbb{Z} \hookrightarrow \mathbb{Q}$.

Here $T_{-1*} : K_0(\mathcal{V}/X) \to H_*^{BM}(X) \otimes \mathbb{Q}$ is the specialization to $y = -1$ of the motivic Hirzebruch class transformation $T_{y*} : K_0(\mathcal{V}/X) \to H_*^{BM}(X) \otimes \mathbb{Q}[y]$ (see [5]).

On the other hand the Donaldson–Thomas type invariant $\chi^{DT}(X)$ does not in general satisfy the scissor formula $\chi^{DT}(X) \neq \chi^{DT}(Z) + \chi^{DT}(X \setminus Z)$. Namely, $\chi^{DT}(-)$ cannot be captured as a homomorphism $\chi^{DT} : K_0(\mathcal{V}) \to \mathbb{Z}$. Instead the following scissor formula holds:

$$\chi^{DT}(X \xrightarrow{\mathrm{id}_X} X) = \chi^{DT}(Z \xrightarrow{i_{Z,X}} X) + \chi^{DT}(X \setminus Z \xrightarrow{i_{X \setminus Z,X}} X). \tag{1.1}$$

Here $i_{Z,X}$ and $i_{X \setminus Z,X}$ are the inclusions. For this formula to make sense, we need a Donaldson–Thomas type invariant $\chi^{DT}(X \xrightarrow{f} Y)$ for a morphism $f : X \to Y$, which is also introduced in [2] and simply defined as $\chi(X, f^*\nu_Y)$. Then χ^{DT} can be considered as a homomorphism $\chi^{DT} : K_0(\mathcal{V}/X) \to \mathbb{Z}$. Note

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that in the case when X is a point, $\chi^{DT} : K_0(\mathcal{V}/pt) = K_0(\mathcal{V}) \to \mathbb{Z}$ is the usual Euler characteristic homomorphism $\chi : K_0(\mathcal{V}) \to \mathbb{Z}$.

In this paper we consider Grothendieck–Riemann–Roch type formulas for χ^{DT}, using the motivic Hirzebruch class transformation T_{y*} ([5]). One of the key features on constructible functions and elements of $K_0(\mathcal{V}/X)$ when we state such Grothendieck–Riemann–Roch type formulas is that they are stable under morphisms. For example, δ assigning to each variety X a constructible function δ_X is said to be *stable under a morphism* $f : X \to Y$ if $\delta_X = f^*\delta_Y$. The $\mathbb{1}$ assigning to each variety X the characteristic function $\mathbb{1}_X$ is stable under a (in fact, *any*) morphism and $\widetilde{\nu}$ assigning to each variety X the signed Behrend function $\widetilde{\nu}_X := (-1)^{\dim X}\nu_X$ is stable under a smooth morphism.

We also propose to consider a bivariant-theoretic aspect for the "categorification" of the DT invariant. By this we mean a graded vector space encoding an appropriate cohomology theory whose Euler characteristic is equal to DT invariant. Naive reasons for the latter are the following. The categorification of the Euler characteristic is nothing but

$$\chi(X) := \sum_i (-1)^i \dim_{\mathbb{R}} H^i_c(X;\mathbb{R}).$$

Note that the compact-support-cohomology $H^i_c(X;\mathbb{R})$ is isomorphic to the Borel–Moore homology $H_i^{BM}(X;\mathbb{R})$. The categorification of the Hirzebruch χ_y-genus is

$$\chi_y(X) = \sum (-1)^i \dim_{\mathbb{C}} Gr^p_F(H^i_c(X;\mathbb{C}))(-y)^p$$

with F being the Hodge filtration of the mixed Hodge structure of $H^i_c(X;\mathbb{C})$. Since the DT type invariant of a morphism satisfies the scissor formula (1.1) due to its definition, we propose to introduce some bivariant-theoretic homology theory $\Theta^*(X \xrightarrow{f} Y)$ "categorifying" $\chi^{DT}(X \xrightarrow{f} Y)$, that is $\chi^{DT}(X \xrightarrow{f} Y) = \sum_i (-1)^i \dim \Theta^i(X \xrightarrow{f} Y)$. (Here we denote it "symbolically"; as described in the case of χ_y-genus, the above alternating sum of the dimensions might be complicated involving some other ingredients such as mixed Hodge structures.)

2. Donaldson–Thomas type invariants of morphisms

Let $\mathfrak{K}$ be an algebraically closed field of characteristic p, which is not necessarily zero. Let X be a $\mathfrak{K}$-scheme of finite type. For a prime number ℓ such that $\ell \neq p$ and the field $\mathbb{Q}_\ell$ of ℓ-adic numbers, the following Euler characteristic

$$\chi(X) := \sum_i (-1)^i \dim_{\mathbb{Q}_\ell} H^i_c(X, \mathbb{Q}_\ell)$$

is independent on the choice of the prime number ℓ. In fact the following properties hold (e.g., see [17, Theorem 3.10]):

Theorem 2.1. *Let $\mathfrak{K}$ be an algebraically closed field and X, Y be separated $\mathfrak{K}$ -schemes of finite type. Then*

1. *If Z is a closed subscheme of X, then $\chi(X) = \chi(Z) + \chi(X \setminus Z)$.*
2. $\chi(X \times Y) = \chi(X)\chi(Y)$.
3. *$\chi(X)$ is independent of the choice of ℓ in the above definition*
4. *If $\mathfrak{K} = \mathbb{C}$, $\chi(X)$ is the usual Euler characteristic with the analytic topology.*
5. *$\chi(\mathfrak{K}^m) = 1$ and $\chi(\mathfrak{K}\mathbb{P}^m) = m + 1$ for $\forall m > 0$*

For a constructible function $\alpha : X \to \mathbb{Z}$ on X the weighted Euler characteristic $\chi(X, \alpha)$ is defined by

$$\chi(X, \alpha) := \sum_m m\chi(\alpha^{-1}(m)).$$

Let X be embeddable in a smooth scheme M and let $C_M X$ be the normal cone of X in M and let $\pi : C_M X \to X$ be the projection and $C_M X = \sum m_i C_i$, where $m_i \in \mathbb{Z}$ are multiplicities and C_i's are irreducible components of the cycle. Then the following cycle

$$\mathfrak{C}_{X/M} := \sum (-1)^{dim(\pi(C_i))} m_i \pi(C_i) \in \mathcal{Z}(X)$$

is in fact independent of the choice of the embedding of X into a smooth M ([1, Lemma 1.1] and [2, Proposition 1.1], also see [11, Example 4.2.6.]), thus simply denoted by $\mathfrak{C}_X$ without referring to the ambient smooth M and is called the distinguished cycle of the scheme. Then consider the isomorphism from the abelian groups $\mathcal{Z}(X)$ of cycles to the abelian group $\mathcal{F}(X)$ of constructible functions

$$\mathrm{Eu} : \mathcal{Z}(X) \xrightarrow{\cong} \mathcal{F}(X)$$

which is defined by $\mathrm{Eu}(\sum_i m_i [Z_i]) := \sum_i m_i \,\mathrm{Eu}_{Z_i}$, where Eu_Z denotes the local Euler obstruction supported on the subscheme Z_i. Then the image of the distinguished cycle $\mathfrak{C}_X$ under the above isomorphism Eu defines a canonical integer valued constructible function

$$\nu_X := \mathrm{Eu}(\mathfrak{C}_X),$$

which is called the *Behrend* function. The fundamental properties of the Behrend function are the following.

Theorem 2.2. (1) *For a smooth point x of a scheme X of dimension n, $\nu_X(x) = (-1)^n$. In particular, if X is smooth of dimension n, then $\nu_X = (-1)^n \mathbb{1}_X$.*
(2) $\nu_{X\times Y} = \nu_X \nu_Y$.
(3) *If $f : X \to Y$ is smooth of relative dimension n, then $\nu_X = (-1)^n f^* \nu_Y$.*
(4) *In particular, if $f : X \to Y$ is étale, then $\nu_X = f^* \nu_Y$.*
(5) *(see also [32]) If Y is the critical scheme of a regular function f on a smooth scheme M, i.e., $Y = Z(df)$, then for $y \in Y$*

$$\nu_Y(y) = (-1)^{\dim M}(1 - \chi(F_y)) \ (= (-1)^{\dim X}(\chi(F_y) - 1)),$$

where $X := f^{-1}(0)$ is the hypersurface, thus Y is the singularity subscheme of X defined by the partial derivatives of f, and F_y is the Milnor fiber of X at the point y.

Remark 2.3. In [1, §1 Weighted Chern–Mather Classes] Paolo Aluffi introduces *the weighted Chern–Mather class of $Y \subset M$*, denoted by $c_{\mathrm{wMa}}(Y)$, as follows:

$$c_{\mathrm{wMa}}(Y) := \sum_i (-1)^{\dim Y - \dim \pi(C_i)} m_i c_*^{Ma}(\pi(C_i)),$$

where $c_*^{Ma}(\pi(C_i))$ is the Chern–Mather class of $\pi(C_i)$, i.e. $c_*^{Ma}(\pi(C_i)) = c_*(\mathrm{Eu}_{\pi(C_i)})$. Therefore we get the following:

$$\begin{aligned} c_{\mathrm{wMa}}(Y) &:= \sum_i (-1)^{\dim Y - \dim \pi(C_i)} m_i c_*^{Ma}(\pi(C_i)) \\ &= \sum_i (-1)^{\dim Y - \dim \pi(C_i)} m_i c_*(\mathrm{Eu}_{\pi(C_i)}) \\ &= c_* \left((-1)^{\dim Y} \sum_i (-1)^{\dim \pi(C_i)} m_i \,\mathrm{Eu}_{\pi(C_i)} \right) \\ &= c_* \left((-1)^{\dim Y} \nu_Y \right). \end{aligned}$$

In other words, Aluffi introduces the distinguished constructible function, i.e. the *signed* Behrend function $(-1)^{\dim Y}\nu_Y =: \widetilde{\nu}_Y$. In [1, Theorem 1.2.] he proves that if X is defined as the zero-scheme of a nonzero section of a line bundle $\mathcal{L}$ over M, then

$$c_*(\widetilde{\nu}_Y) = (-1)^{\dim X - \dim Y} c(\mathcal{L}) \cap (c^{FJ}(X) - c_*(X)), \tag{2.4}$$

where Y is the singularity subscheme of the hypersurface X, i.e. the subscheme locally defined by the partial derivatives of an equation for X, and $c^{FJ}(X)$ is Fulton–Johnson class of X or the canonical class of X (see [11, Example 4.2.6.] and [12]). In this hypersurface case he furthermore shows the following [1, Theorem 1.5.]: As in (5) of the above Theorem 2.2, if μ_Y is the constructible function defined by $\mu_Y(y) := (-1)^{\dim X}(\chi(F_y) - 1)$, then $c_*(\widetilde{\nu}_Y) = (-1)^{\dim Y} c_*(\mu_Y)$.

It follows from (2.4) and $(-1)^{\dim Y} c_*(\widetilde{\nu}_Y) = c_*(\nu_Y)$ that we get

$$c(\mathcal{L})^{-1} \cap c_*(\nu_Y) = (-1)^{\dim X}(c^{FJ}(X) - c_*(X)).$$

The right-hand-sided invariant $(-1)^{\dim X}(c^{FJ}(X) - c_*(X))$ is the so-called *Milnor class of* X (supported on the singular locus Y). Hence, in particular, in the case when the line bundle $\mathcal{L}$ is trivial, i.e., in the case of (5) of Theorem 2.2, we have that $c_*(\nu_Y) = c_*(\mu_Y)$ is nothing but the Milnor class of X.

The weighted Euler characteristic of the above Behrend function is called the *Donaldson–Thomas type invariant* and denoted by $\chi^{DT}(X)$:

$$\chi^{DT}(X) := \chi(X, \nu_X).$$

Remark 2.5. We would like to emphasize that using the Aluffi function $\widetilde{\nu}_X$ we have that

$$\chi^{DT}(X) = \chi(X, \nu_X) = (-1)^{\dim X}\chi(X, \widetilde{\nu}_X).$$

In [2, Definition 1.7] Kai Behrend defined the following.

Definition 2.6. The *DT-invariant* or *virtual count* of a morphism $f : X \to Y$ is defined by

$$\chi^{DT}(X \xrightarrow{f} Y) := \chi(X, f^*\nu_Y),$$

where ν_Y is the Behrend function of the target scheme Y.

Remark 2.7. Here we emphasize that $\chi^{DT}(X \xrightarrow{f} Y)$ is defined by the constructible function $f^*\nu_Y$ *on the source scheme* X. From the definition we can observe the following:

(1) $\chi^{DT}(X \xrightarrow{\mathrm{id}_X} X) = \chi(X, \nu_X) = \chi^{DT}(X)$ is the DT-invariant of X.
(2) $\chi^{DT}(X \xrightarrow{\pi_X} pt) = \chi(X, f^*\nu_{pt}) = \chi(X, 1\!\!1_X) = \chi(X)$ is the topological Euler-Poincaré characteristic of X.
(3) If Y is *smooth*, whatever the morphism $f : X \to Y$ is, we have

$$\chi^{DT}(X \xrightarrow{f} Y) = (-1)^{\dim Y}\chi(X).$$

The very special case is that $Y = pt$, which is the above (2).

The Euler characteristic $\chi(-)$ satisfies the additivity $\chi(X) = \chi(Z) + \chi(X \setminus Z)$ for a closed subscheme $Z \subset X$. Hence, χ is considered as a homomorphism from the Grothendieck group of varieties $\chi : K_0(\mathcal{V}) \to \mathbb{Z}$ and furthermore as a homomorphism from the relative Grothendieck group of varieties over a fixed variety X ([28])

$$\chi : K_0(\mathcal{V}/X) \to \mathbb{Z},$$

which is defined by $\chi([V \xrightarrow{h} X]) = \chi(V) = \chi(V, \mathbb{1}_V) = \chi(V, h^*\mathbb{1}_X) = \chi(X, h_*\mathbb{1}_V)$. Moreover, the following diagram commutes:

$$\begin{array}{ccc} K_0(\mathcal{V}/X) & \xrightarrow{f_*} & K_0(\mathcal{V}/Y) \\ & {\scriptstyle\chi}\searrow \quad \swarrow{\scriptstyle\chi} & \\ & \mathbb{Z}. & \end{array} \tag{2.8}$$

On the other hand we have that $\chi^{DT}(X) \neq \chi^{DT}(Z) + \chi^{DT}(X \setminus Z)$. Thus $\chi^{DT}(-)$ cannot be captured as a homomorphism $\chi^{DT} : K_0(\mathcal{V}) \to \mathbb{Z}$. However, we have that

$$\chi^{DT}(X \xrightarrow{\mathrm{id}_X} X) = \chi^{DT}(Z \xrightarrow{i_{Z,X}} X) + \chi^{DT}(X \setminus Z \xrightarrow{i_{X\setminus Z,X}} X).$$

Lemma 2.9. *If we define $\chi^{DT}([V \xrightarrow{h} X]) := \chi(V, h^*\nu_X)$, then we get the homomorphism*

$$\chi^{DT} : K_0(\mathcal{V}/X) \to \mathbb{Z}.$$

Proof. The definition $\chi^{DT}([V \xrightarrow{h} X]) := \chi(V, h^*\nu_X)$ is independent of the choice of the representative of the isomorphism class $[V \xrightarrow{h} X]$. Indeed, let $V' \xrightarrow{h'} X$ be another representative of $[V \xrightarrow{h} X]$, i.e., we have the following commutative diagram, where $\iota : V' \xrightarrow{\cong} V$ is an isomorphism:

$$\begin{array}{ccc} V' & \xrightarrow{\iota} & V \\ & {\scriptstyle h'}\searrow \quad \swarrow{\scriptstyle h} & \\ & X. & \end{array}$$

Then we have that $\chi(V', h'^*\nu_X) = \chi(V', \iota^*(h^*\nu_X)) = \chi(V, h^*\nu_X)$.

For a closed subvariety $W \subset V$, we have

$$\begin{aligned} \chi^{DT}([V \xrightarrow{h} X] &= \chi(V, h^*\nu_X) \\ &= \chi(W, h^*\nu_X) + \chi(V \setminus W, h^*\nu_X) \\ &= \chi(W, (h_{|W})^*\nu_X) + \chi(V \setminus W, (h_{|V\setminus W})^*\nu_X) \\ &= \chi^{DT}([W \xrightarrow{h_{|W}} X]) + \chi^{DT}([V \setminus W \xrightarrow{h_{|V\setminus W}} X]). \end{aligned}$$

Thus we get the homomorphism $\chi^{DT} : K_0(\mathcal{V}/X) \to \mathbb{Z}$. □

Lemma 2.10. *If $f : X \to Y$ satisfies the condition that $\nu_X = f^*\nu_Y$ (such a morphism shall be called a "Behrend morphism") , then the following diagram commutes:*

$$\begin{array}{ccc} K_0(\mathcal{V}/X) & \xrightarrow{f_*} & K_0(\mathcal{V}/Y) \\ & {\scriptstyle\chi^{DT}}\searrow \quad \swarrow{\scriptstyle\chi^{DT}} & \\ & \mathbb{Z}. & \end{array}$$

Proof. It is straightforward:

$$\begin{aligned}
\chi^{DT} \circ f_*([V \xrightarrow{h} X]) &= \chi^{DT}([V \xrightarrow{f \circ h} X]) \\
&= \chi(V, (f \circ h)^* \nu_Y) \\
&= \chi(V, h^* f^* \nu_Y) \\
&= \chi(V, h^* \nu_X) \quad (\text{since } \nu_X = f^* \nu_Y) \\
&= \chi^{DT}([V \xrightarrow{h} X]).
\end{aligned}$$

. ∎

Remark 2.11. An étale map is a typical example of a Behrend morphism.

Remark 2.12. For a general morphism $f : X \to Y$, we have that

$$f^* \nu_Y = (-1)^{\mathrm{reldim}\, f} \nu_X + \Theta(X_{sing} \cup f^{-1}(Y_{sing})),$$

where $\mathrm{reldim}\, f := \dim X - \dim Y$ is the relative dimension of f and $\Theta(X_{sing} \cup f^{-1}(Y_{sing}))$ is some constructible functions supported on the singular locus X_{sing} of X and the inverse image of the singular locus Y_{sing} of Y. As

$$\nu_X = (-1)^{\dim X} \mathbb{1}_X + \text{some constructible function supported on } X_{sing},$$

then

$$f^* \nu_Y = (-1)^{\dim X} f^* \mathbb{1}_Y + f^*(\text{some constructible function supported on } Y_{sing}).$$

Hence in general we have

$$(\chi^{DT} \circ f_*)([V \xrightarrow{h} X]) = (-1)^{\mathrm{reldim}\, f} \chi^{DT}([V \xrightarrow{h} X]) + \text{extra terms}.$$

Here the extra terms are supported on the singular locus X_{sing}.

To avoid taking care of the sign, we use the signed Behrend function, i.e., the Aluffi function

$$\widetilde{\nu}_X = (-1)^{\dim X} \nu_X,$$

which will be used later again. Note that if X is smooth, $\widetilde{\nu}_X = \mathbb{1}_X$. Then we define the signed Donaldson–Thomas type invariant $\widetilde{\chi}^{DT}(X)$ by $\widetilde{\chi}^{DT}(X \xrightarrow{f} Y) := \chi(X, f^* \widetilde{\nu}_Y)$. (In other words, this invariant could be called an *Aluffi–Behrend–Euler characteristic of a morphism f*.) Then for a morphism $f : X \to Y$ we have $f^* \widetilde{\nu}_Y = \widetilde{\nu}_X + \widetilde{\Theta}(X_{sing} \cup f^{-1}(Y_{sing}))$. In particular the above lemma is modified as follows:

Lemma 2.13. *If $f : X \to Y$ satisfies the condition that $\widetilde{\nu}_X = f^* \widetilde{\nu}_Y$ (such a morphism shall be called a "signed Behrend morphism"; a smooth morphism is a typical example for $\widetilde{\nu}_X = f^* \widetilde{\nu}_Y$) , then the following diagram commutes:*

$$\begin{array}{ccc}
K_0(\mathcal{V}/X) & \xrightarrow{f_*} & K_0(\mathcal{V}/Y) \\
& {\scriptstyle \widetilde{\chi}^{DT}} \searrow \quad \swarrow {\scriptstyle \widetilde{\chi}^{DT}} & \\
& \mathbb{Z}. &
\end{array}$$

3. GENERALIZED DONALDSON–THOMAS TYPE INVARIANTS OF MORPHISMS

Mimicking the above definition of $\chi^{DT}(X \xrightarrow{f} Y)$ and ignoring the geometric or topological interpretation, we define the following.

Definition 3.1. For a morphism $f : X \to Y$ and a constructible function $\delta_Y \in \mathcal{F}(Y)$ we define

$$\overline{\chi^{\delta_Y}}(X \xrightarrow{f} Y) := \chi(X, f^*\delta_Y).$$

Lemma 3.2. *For a morphism $f : X \to Y$ and a constructible function $\alpha \in \mathcal{F}(X)$ we have*

$$\chi(X, \alpha) = \chi(Y, f_*\alpha).$$

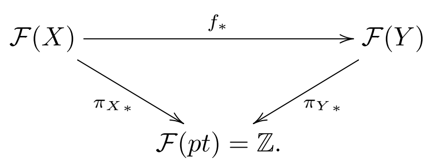
Corollary 3.3. *For a morphism $f : X \to Y$ and a constructible function $\delta_Y \in \mathcal{F}(Y)$ we have*

$$\overline{\chi^{\delta_Y}}(X \xrightarrow{f} Y) = \chi(Y, f_*f^*\delta_Y).$$

Remark 3.4. For the constant map $\pi_X : X \to pt$, the pushforward homomorphism

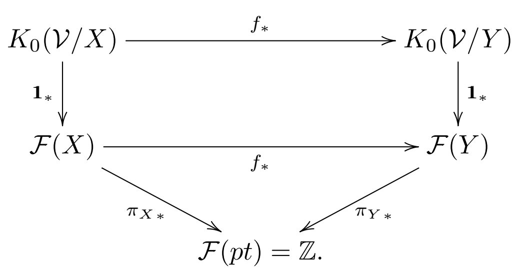
$$\pi_{X*} : \mathcal{F}(X) \to \mathcal{F}(pt) = \mathbb{Z}$$

is nothing but the fact that $\pi_{X*}(\alpha) = \chi(X, \alpha)$ (by the definition of the pushforward). Hence, the above equality $\chi(X, \alpha) = \chi(Y, f_*\alpha)$ is rephrased as the commutativity of the following diagram:



Namely, $\pi_{X*} = (\pi_Y \circ f)_* = \pi_{Y*} \circ f_*$. This might suggest that $\mathcal{F}(-)$ is a covariant functor, but we need to be a bit careful. $\mathcal{F}(-)$ is a covariant functor *provided that the ground field $\mathfrak{K}$ is of characteristic zero*. However, if it is not of characteristic zero, then it may happen that $(g \circ f)_* \neq g_* \circ f_*$, for which see Schürmann's example in [17].

Remark 3.5. If we define $1\!\!1_* : K_0(\mathcal{V}/X) \to \mathcal{F}(X)$ by $1\!\!1_*([V \xrightarrow{h} X]) := h_*1\!\!1_V$, then for a morphism $f : X \to Y$ we have the following commutative diagrams:



$(\pi_{X*} \circ 1\!\!1_*)([V \xrightarrow{h} X]) = \chi([V \xrightarrow{h} X])$ and the outer triangle is nothing but the commutative diagram (2.8) mentioned before.

Here we emphasize that the above equality $\overline{\chi^{\delta_Y}}(X \xrightarrow{f} Y) = \chi(Y, f_*f^*\delta_Y)$ have the following two aspects:

- The invariant on LHS for a morphism $f : X \to Y$ is *defined on the source space X*.
- The invariant on RHS for a morphism $f : X \to Y$ is *defined on the target space Y*.

So, in order to emphasize the distinction, we introduce the following notation:

$$\chi^{\delta_Y}(X \xrightarrow{f} Y) := \chi(Y, f_*f^*\delta_Y).$$

Since we want to deal with higher class versions of the Donaldson–Thomas type invariants and use the functoriality of the constructible function functor $\mathcal{F}(-)$, we assume that the ground field $\mathfrak{K}$ is of characteristic zero. We consider MacPherson's Chern class transformation $c_* : \mathcal{F}(X) \to H_*^{BM}(X)$, which is due to Kennedy [21].

For a morphism $h : V \to X$ and for a constructible function $\delta_X \in \mathcal{F}(X)$ on the target space X, we have

$$\int_V c_*(h^*\delta_X) = \chi(V, h^*\delta_X) = \overline{\chi^{\delta_X}}(V \xrightarrow{h} X),$$
$$\int_X c_*(h_*h^*\delta_X) = \chi(X, h_*h^*\delta_X) = \chi^{\delta_X}(V \xrightarrow{h} X).$$

Here $c_*(h^*\delta_X) \in H_*^{BM}(V)$ on the side of the source space V and $c_*(h_*h^*\delta_X) \in H_*^{BM}(X)$ on the side of the target space X. Hence when we want to deal with them as the homomorphism from $K_0(\mathcal{V}/X)$ to $H_*^{BM}(X)$, we should consider the higher analogues $c_*(h_*h^*\delta_X)$, which we denote by

$$\overline{c_*^{\delta_X}}(V \xrightarrow{h} X) := c_*(h^*\delta_X) \in H_*^{BM}(V).$$

On the other hand we denote

$$c_*^{\delta_X}(V \xrightarrow{h} X) := c_*(h_*h^*\delta_X) \in H_*^{BM}(X).$$

Note that

- $c_*^{\delta_X}(V \xrightarrow{h} X) = h_*(\overline{c_*^{\delta_X}}(V \xrightarrow{h} X))$,
- for an isomorphism $id_X : X \to X$, these two classes are identical and denoted simply by $c_*^{\delta_X}(X) := c_*(\delta_X) = c_*^{\delta_X}(X \xrightarrow{id_X} X) = \overline{c_*^{\delta_X}}(X \xrightarrow{id_X} X)$.

In the following sections we treat these two objects $c_*^{\delta_X}(V \xrightarrow{h} X)$ and $\overline{c_*^{\delta_X}}(V \xrightarrow{h} X)$ separately, since they have different natures.

4. Motivic Aluffi-type classes

In [2] the Chern class $c_*^{\nu_X}(X)$ for the Behrend function ν_X is called the Aluffi class, in which case $\int_X c_*^{\nu_X}(X) = \chi^{DT}(X)$. However, in this paper, for the signed Behrend function $\widetilde{\nu}_X$ the Chern class $c_*^{\widetilde{\nu}_X}(X)$ shall be called the Aluffi class and denoted by $c_*^{A\ell}(X)$, since this is the class which Aluffi introduced in [1] as pointed out in [2, §1.4 The Aluffi class]. Note that $\int_X c_*^{A\ell}(X) = (-1)^{\dim X}\chi^{DT}(X)$. In this sense, the Chern class $c_*^{\delta_X}(V \xrightarrow{h} X)$ defined above shall be called a *generalized Aluffi class of a morphism $h : V \to X$ associated to a constructible function $\delta_X \in \mathcal{F}(X)$*. So the original Aluffi class is $c_*^{\widetilde{\nu}_X}(X \xrightarrow{\mathrm{id}_X} X)$.

Lemma 4.1. *The following formulae hold:*

(1) *If $(V \xrightarrow{h} X) \cong (V' \xrightarrow{h'} X)$, i.e., there exists an isomorphism $k : V \xrightarrow{\cong} V'$ such that $h = h' \circ k$, then we have $c_*^{\delta_X}(V \xrightarrow{h} X) = c_*^{\delta_X}(V' \xrightarrow{h'} X)$.*

(2) *For a closed subvariety $W \subset V$,*

$$c_*^{\delta_X}(V \xrightarrow{h} X) = c_*^{\delta_X}(W \xrightarrow{h|_W} X) + c_*^{\delta_X}(V \setminus W \xrightarrow{h|_{V \setminus W}} X).$$

(3) *For morphisms $h_i : V_i \to X_i$ $(i = 1, 2)$,*

$$c_*^{\delta_{X_1} \times \delta_{X_2}}(V_1 \times V_2 \xrightarrow{h_1 \times h_2} X_1 \times X_2) = c_*^{\delta_{X_1}}(V_1 \xrightarrow{h_1} X_1) \times c_*^{\delta_{X_2}}(V_2 \xrightarrow{h_2} X_2).$$

(4) $c_*^{\delta_{pt}}(pt \to pt) = \delta_{pt}(pt) \in \mathbb{Z}$.

Corollary 4.2. *Let $\delta_X \in \mathcal{F}(X)$ be a constructible function. Then the following hold:*

(1) *The map $c_*^{\delta_X} : K_0(\mathcal{V}/X) \to H_*^{BM}(X)$ defined by*

$$c_*^{\delta_X}([V \xrightarrow{h} X]) := c_*^{\delta_X}(V \xrightarrow{h} X) = c_*(h_*h^*\delta_X)$$

and linearly extended is a well-defined homomorphism.

(2) $c_*^{\delta_X}$ *commutes with the exterior product, i.e. for constructible functions $\delta_{X_i} \in \mathcal{F}(X_i)$ and for $\alpha_i \in K_0(\mathcal{V}/X_i)$,*

$$c_*^{\delta_{X_1} \times \delta_{X_2}}(\alpha_1 \times \alpha_2) = c_*^{\delta_{X_1}}(\alpha_1) \times c_*^{\delta_{X_2}}(\alpha_2).$$

Remark 4.3. If δ_X is some function defined on X, such as the characteristic function $1\!\!1_X$, the Behrend function ν_X, the signed Behrend function $\widetilde{\nu}_X$, and if it is multiplicative, i.e. $\delta_{X\times Y} = \delta_X \times \delta_Y$, then the above Corollary 4.2 (2) can be simply rewritten as $c_*^{\delta_{X_1 \times X_2}}(\alpha_1 \times \alpha_2) = c_*^{\delta_{X_1}}(\alpha_1) \times c_*^{\delta_{X_2}}(\alpha_2)$.

Remark 4.4. If X is smooth and $h : V \to X$ is proper (here properness is required since we use the pushforward h_* of the Borel–Moore homology groups), then we have

$$c_*^{A\ell}([V \xrightarrow{h} X]) = c_*(h_*h^*\nu_X) = h_*c_*(h^*1\!\!1_X) = h_*c_*(1\!\!1_V) = h_*c_*^{SM}(V)$$

is the pushforward of the Chern–Schwartz–MacPherson class of V, thus it depends on the morphism $h : V \to X$, although the degree zero part of it, i.e. the signed Donaldson–Thomas type invariant is nothing but the Euler characteristic of V, thus it does not depend on the morphism at all. Therefore the higher class version is more subtle.

The part $h_*h^*\delta_X$ can be formulated as follows. Given a constructible function $\delta_X \in \mathcal{F}(X)$, we define

$$[\delta_X] : K_0(\mathcal{V}/X) \to \mathcal{F}(X)$$

by $[\delta_X]([V \xrightarrow{h} X]) := h_*h^*\delta_X$ and extend it linearly, i.e.,

$$[\delta_X]\left(\sum_h m_h[V \xrightarrow{h} X]\right) := \sum_h m_h\,(h_*h^*\delta_X).$$

If $(V \xrightarrow{h} X) \cong (V' \xrightarrow{h'} X)$, i.e., there exists an isomorphism $k : V \xrightarrow{\cong} V'$ such that $h = h' \circ k$, then we have

$$(h')_*(h')^*\delta_X = h_*k_*k^*h^*\delta_X = h_*h^*\delta_X$$

because $k_*k^* = \mathrm{id}_{\mathcal{F}(X)}$. For a morphism $h : V \to X$ and for a closed subvariety $W \subset V$, we have

$$h_*h^*\delta_X = (h|_W)_*(h|_W)^*\delta_X + (h|_{V\setminus W})_*(h|_{V\setminus W})^*\delta_X,$$

that is, we have that $[\delta_X]\left([V \xrightarrow{h} X] - [W \xrightarrow{h|_W} X] - [V \setminus W \xrightarrow{h|_{V\setminus W}} X]\right) = 0$. Therefore the homomorphism $[\delta_X] : K_0(\mathcal{V}/X) \to \mathcal{F}(X)$ is well-defined.

Note that $1\!\!1_* : K_0(\mathcal{V}/X) \to \mathcal{F}(X)$ is nothing but $[1\!\!1_X] : K_0(\mathcal{V}/X) \to \mathcal{F}(X)$. It is straightforward to see the following.

Lemma 4.5. *For* any *morphism $g : X \to Y$ and* any *constructible function $\delta_Y \in \mathcal{F}(Y)$, the following diagrams commute:*

$$\begin{array}{ccc} K_0(\mathcal{V}/X) & \xrightarrow{[g^*\delta_Y]} & \mathcal{F}(X) \\ {\scriptstyle g_*}\downarrow & & \downarrow{\scriptstyle g_*} \\ K_0(\mathcal{V}/Y) & \xrightarrow[{[\delta_Y]}]{} & \mathcal{F}(Y). \end{array}, \qquad \begin{array}{ccc} K_0(\mathcal{V}/Y) & \xrightarrow{[\delta_Y]} & \mathcal{F}(Y) \\ {\scriptstyle g^*}\downarrow & & \downarrow{\scriptstyle g^*} \\ K_0(\mathcal{V}/X) & \xrightarrow[{[g^*\delta_Y]}]{} & \mathcal{F}(X). \end{array}$$

The following corollary follows from MacPherson's theorem [29] and our previous results [34, 38], and here we need the properness of the morphism $g : X \to Y$, since we deal with the pushforward homomorphism for the Borel–Moore homology. $c_*^{\delta_X} : K_0(\mathcal{V}/X) \to H_*^{BM}(X)$ is the composite of

$$[\delta_X] : K_0(\mathcal{V}/X) \to \mathcal{F}(X)$$

and MacPherson's Chern class c_*, in particular $c_*^{A\ell} : K_0(\mathcal{V}/X) \to H_*^{BM}(X)$ is $c_*^{A\ell} = c_* \circ [\widetilde{\nu_X}]$. Hence we have the following corollary:

Corollary 4.6. (1) *For a proper morphism $g : X \to Y$ and* any *constructible function $\delta_Y \in \mathcal{F}(Y)$, the following diagram commutes:*

$$\begin{array}{ccc} K_0(\mathcal{V}/X) & \xrightarrow{c_*^{g^*\delta_Y}} & H_*^{BM}(X) \\ {\scriptstyle g_*}\downarrow & & \downarrow{\scriptstyle g_*} \\ K_0(\mathcal{V}/Y) & \xrightarrow[c_*^{\delta_Y}]{} & H_*^{BM}(Y). \end{array}$$

(2) *For a smooth morphism $g : X \to Y$ with $c(T_g)$ being the total Chern cohomology class of the relative tangent bundle T_g of the smooth morphism and $g^* : H_*^{BM}(Y) \to H_*^{BM}(X)$ the Gysin homomorphism (*[11, Example 19.2.1]*) , the following diagram commutes:*

$$\begin{array}{ccc} K_0(\mathcal{V}/Y) & \xrightarrow{c_*^{\delta_Y}} & H_*^{BM}(Y) \\ {\scriptstyle g^*}\downarrow & & \downarrow{\scriptstyle c(T_g)\cap g^*} \\ K_0(\mathcal{V}/X) & \xrightarrow[c_*^{g^*\delta_Y}]{} & H_*^{BM}(X). \end{array}$$

Therefore, if δ assigning to each variety X a constructible function $\delta_X \in \mathcal{F}(X)$ is stable under a proper morphism $g : X \to Y$, then we have the following commutative diagrams:

$$\begin{array}{ccc} K_0(\mathcal{V}/X) & \xrightarrow{c_*^{\delta_X}} & H_*^{BM}(X) \\ {\scriptstyle g_*}\downarrow & & \downarrow{\scriptstyle g_*} \\ K_0(\mathcal{V}/Y) & \xrightarrow[c_*^{\delta_Y}]{} & H_*^{BM}(Y), \end{array} \quad \begin{array}{ccc} K_0(\mathcal{V}/Y) & \xrightarrow{c_*^{\delta_Y}} & H_*^{BM}(Y) \\ {\scriptstyle g^*}\downarrow & & \downarrow{\scriptstyle c(T_g)\cap g^*} \\ K_0(\mathcal{V}/X) & \xrightarrow[c_*^{\delta_X}]{} & H_*^{BM}(X). \end{array}$$

In particular we get the following theorem for the Aluffi class $c_*^{A\ell} : K_0(\mathcal{V}/-) \to H_*^{BM}(-)$:

Theorem 4.7. *For a smooth proper morphism $g : X \to Y$ the following diagrams commute:*

$$\begin{array}{ccc} K_0(\mathcal{V}/X) & \xrightarrow{c_*^{A\ell}} & H_*^{BM}(X) \\ {\scriptstyle g_*}\downarrow & & \downarrow{\scriptstyle g_*} \\ K_0(\mathcal{V}/Y) & \xrightarrow[c_*^{A\ell}]{} & H_*^{BM}(Y), \end{array} \quad \begin{array}{ccc} K_0(\mathcal{V}/Y) & \xrightarrow{c_*^{A\ell}} & H_*^{BM}(Y) \\ {\scriptstyle g^*}\downarrow & & \downarrow{\scriptstyle c(T_g)\cap g^*} \\ K_0(\mathcal{V}/X) & \xrightarrow[c_*^{A\ell}]{} & H_*^{BM}(X). \end{array}$$

They are respectively Grothendieck–Riemann–Roch type and a Verdier–Riemann–Roch type formulas.

Remark 4.8. In the above theorem the smoothness of the morphism $g : X \to Y$ is crucial and the Aluffi class homomorphism $c_*^{Al} : K_0(\mathcal{V}/X) \to H_*^{BM}(X)$ cannot be captured as a natural transformation in a full generality, i.e. natural for any morphism. Indeed, if it were the case, then

$$c_*^{Al} : K_0(\mathcal{V}/-) \to H_*^{BM}(-) \hookrightarrow H_*^{BM}(-) \otimes \mathbb{Q}$$

becomes a natural transformation such that for any smooth variety X we have

$$c_*^{A\ell}([X \xrightarrow{\mathrm{id}_X} X]) = c(T_X) \cap [X].$$

Let $T_{y_*} : K_0(\mathcal{V}/-) \to H_*^{BM}(-) \otimes \mathbb{Q}[y]$ be the motivic Hirzebruch class transformation [5], which is the unique natural transformation satisfying the normalization condition that for a smooth X,

$$T_{y_*}([X \xrightarrow{\mathrm{id}_X} X]) = td_y(TX) \cap [X],$$

where $[X]$ is the fundamental class and $td_y(TX)$ is Hirzebruch characteristic cohomology class of the tangent bundle TX. Here the Hirzebruch class $td_y(E)$ of the complex or algebraic vector bundle E over X is defined to be (see [15, 16]):

$$td_y(E) := \prod_{i=1}^{\operatorname{rank} E} \left(\frac{\alpha(1+y)}{1 - e^{-\alpha(1+y)}} - \alpha y \right).$$

Here α_i's are the Chern roots of E, i.e., $c(E) = \prod_{i=1}^{\mathrm{rank(E)}} (1 + \alpha_i)$. Then $td_y(E)$ is a unification of the following three well-known characteristic cohomology classes:

- $td_{-1}(E) = \prod_{i=1}^{\mathrm{rank(E)}} (1 + \alpha) = c(E)$, the total Chern class,
- $td_0(E) = \prod_{i=1}^{\mathrm{rank(E)}} \frac{\alpha}{1 - e^{-\alpha}} = td(E)$, the total Todd class,
- $td_1(E) = \prod_{i=1}^{\mathrm{rank(E)}} \frac{\alpha}{\tanh \alpha} = L(E)$, the total Thom–Hirzebruch L-class.

Then $c_*^{A\ell}$ is equal to $T_{-1_*} : K_0(\mathcal{V}/-) \to H_*^{BM}(-) \otimes \mathbb{Q}$, since $T_{-1_*} : K_0(\mathcal{V}/-) \to H_*^{BM}(-) \otimes \mathbb{Q}$ is the unique natural transformation satisfying the normalization condition that

$$T_{-1_*}([X \xrightarrow{\mathrm{id}_X} X]) = c(T_X) \cap [X]$$

for a smooth X. Thus for any variety X, singular or non-singular, we have

$$c_*^{A\ell}([X \xrightarrow{\mathrm{id}_X} X]) = c_*^{SM}(X) = c_*(\mathbb{1}_X)$$

In particular $\int_X c_*(\mathbb{1}_X) = \chi(X)$ the topological Euler–Poincaré characteristic, which is a contradiction to the fact that

$$\int_X c_*^{A\ell}([X \xrightarrow{\mathrm{id}_X} X]) = (-1)^{\dim X} \chi^{DT}(X).$$

Remark 4.9. In fact $c_*^{\mathbf{1}_X}$ is equal to the motivic Chern class transformation

$$T_{-1_*} : K_0(\mathcal{V}/X) \to H_*^{BM}(X) \hookrightarrow H_*^{BM}(X) \otimes \mathbb{Q}.$$

$K_0(\mathcal{V}/X)$ is a ring with the following fiber product

$$[V \xrightarrow{h} X] \cdot [W \xrightarrow{k} X] := [V \times_X W \xrightarrow{h \times_X k} X].$$

Proposition 4.10. *The operation $h_* h^* \delta_X$ of pullback followed by pushforward of a constructible function makes $\mathcal{F}(X)$ a $K_0(\mathcal{V}/X)$-module with the product $[V \xrightarrow{h} X] \cdot \delta_X := h_* h^* \delta_X$. Namely, the following properties hold:*

- $[V \xrightarrow{h} X] \cdot (\delta'_X + \delta''_X) = [V \xrightarrow{h} X] \cdot \delta'_X + [V \xrightarrow{h} X] \cdot \delta''_X$.
- $([V \xrightarrow{h} X] + [W \xrightarrow{k} X]) \cdot \delta_X = [V \xrightarrow{h} X] \cdot \delta_X + [W \xrightarrow{k} X] \cdot \delta_X$.

- $([V \xrightarrow{h} X] \cdot [W \xrightarrow{k} X]) \cdot \delta_X = [V \xrightarrow{h} X] \cdot ([W \xrightarrow{k} X] \cdot \delta_X)$.
- $[X \xrightarrow{\mathrm{id}_X} X] \cdot \delta_X = \delta_X$.

Then the operation $h_* h^* \delta_X$ gives rise to a map $\Phi : K_0(\mathcal{V}/X) \otimes \mathcal{F}(X) \to \mathcal{F}(X)$ and the composition $\Phi c_* := c_* \circ \Phi : K_0(\mathcal{V}/X) \otimes \mathcal{F}(X) \to H_*^{BM}(X)$ of Φ and MacPherson's Chern class transformation c_* is a kind of extension of c_*.

Lemma 4.11. *For any morphism $g : X \to Y$ the following diagram commutes:*

$$\begin{array}{ccc} K_0(\mathcal{V}/Y) \otimes \mathcal{F}(Y) & \xrightarrow{\Phi} & \mathcal{F}(Y) \\ {\scriptstyle g^* \otimes g^*}\downarrow & & \downarrow {\scriptstyle g^*} \\ K_0(\mathcal{V}/X) \otimes \mathcal{F}(X) & \xrightarrow[\Phi]{} & \mathcal{F}(X). \end{array}$$

Corollary 4.12. *For a smooth morphism $g : X \to Y$ the following diagram commutes:*

$$\begin{array}{ccc} K_0(\mathcal{V}/Y) \otimes \mathcal{F}(Y) & \xrightarrow{\Phi c_*} & H_*^{BM}(Y) \\ {\scriptstyle g^* \otimes g^*}\downarrow & & \downarrow {\scriptstyle c(T_g) \cap g^*} \\ K_0(\mathcal{V}/X) \otimes \mathcal{F}(X) & \xrightarrow[\Phi c_*]{} & H_*^{BM}(X). \end{array}$$

Remark 4.13. Fix $\delta_Y \in \mathcal{F}(Y)$, the composite of the inclusion homomorphism

$$i_{\delta_Y} : K_0(\mathcal{V}/Y) \to K_0(\mathcal{V}/Y) \otimes \mathcal{F}(Y)$$

defined by $i_{\delta_Y}(\alpha) := \alpha \otimes \delta_Y$ and the map $\Phi : K_0(\mathcal{V}/Y) \otimes \mathcal{F}(Y) \to \mathcal{F}(Y)$ is the homomorphism $[\delta_Y]$;

$$\Phi \circ i_{\delta_Y} = [\delta_Y] : K_0(\mathcal{V}/F) \to \mathcal{F}(Y).$$

The right-hand-sided commutative diagram in Lemma 4.5 is the outer square of the following commutative diagrams:

$$\begin{array}{ccccc} K_0(\mathcal{V}/Y) & \xrightarrow{i_{\delta_Y}} & K_0(\mathcal{V}/Y) \otimes \mathcal{F}(Y) & \xrightarrow{\Phi} & \mathcal{F}(Y) \\ {\scriptstyle g^*}\downarrow & & \downarrow {\scriptstyle g^* \otimes g^*} & & \downarrow {\scriptstyle g^*} \\ K_0(\mathcal{V}/X) & \xrightarrow[i_{g^*\delta_Y}]{} & K_0(\mathcal{V}/X) \otimes \mathcal{F}(X) & \xrightarrow[\Phi]{} & \mathcal{F}(X). \end{array}$$

Furthermore, if $g : X \to Y$ is smooth, we get the following commutative diagrams:

$$\begin{array}{ccccccc} K_0(\mathcal{V}/Y) & \xrightarrow{i_{\delta_Y}} & K_0(\mathcal{V}/Y) \otimes \mathcal{F}(Y) & \xrightarrow{\Phi} & \mathcal{F}(Y) & \xrightarrow{c_*} & H_*^{BM}(Y) \\ {\scriptstyle g^*}\downarrow & & \downarrow {\scriptstyle g^* \otimes g^*} & & \downarrow {\scriptstyle g^*} & & \downarrow {\scriptstyle c(T_g) \cap g^*} \\ K_0(\mathcal{V}/X) & \xrightarrow[i_{g^*\delta_Y}]{} & K_0(\mathcal{V}/X) \otimes \mathcal{F}(X) & \xrightarrow[\Phi]{} & \mathcal{F}(X) & \xrightarrow[c_*]{} & H_*^{BM}(X), \end{array}$$

the outer square of which is the commutative diagram in Corollary 4.6 (2).

Remark 4.14. As to the pushforward we do knot know if there exists a reasonable pushforward "?" : $K_0(\mathcal{V}/X) \otimes \mathcal{F}(X) \to K_0(\mathcal{V}/Y) \otimes \mathcal{F}(Y)$ such that the following diagram commutes:

$$\begin{array}{ccc} K_0(\mathcal{V}/X) \otimes \mathcal{F}(X) & \xrightarrow{\Phi} & \mathcal{F}(X) \\ {\scriptstyle \text{"?"}}\downarrow & & \downarrow {\scriptstyle g_*} \\ K_0(\mathcal{V}/Y) \otimes \mathcal{F}(Y) & \xrightarrow[\Phi]{} & \mathcal{F}(Y). \end{array}$$

Indeed, for $[V \xrightarrow{h} X] \otimes \delta_X \in K_0(\mathcal{V}/X) \otimes \mathcal{F}(X)$ we have that $g_*\Phi([V \xrightarrow{h} X] \otimes \delta_X) = g_*h_*h^*\delta_X$. But we do not know how to define "?" : $K_0(\mathcal{V}/X) \otimes \mathcal{F}(X) \to K_0(\mathcal{V}/Y) \otimes \mathcal{F}(Y)$ such that

$$\Phi(\text{“?”}([V \xrightarrow{h} X] \otimes \delta_X)) = g_*h_*h^*\delta_X.$$

One possibility would be

$$\text{“?”} = (g_* \otimes ?_*)([V \xrightarrow{h} X] \otimes \delta_X) = [V \xrightarrow{gh} Y] \otimes ?_*(\delta_X) = (gh)_*(gh)^*(?_*(\delta_X)) = g_*h_*h^*g^*(?_*(\delta_X)),$$

but here we do not know how to define $?_* : \mathcal{F}(X) \to \mathcal{F}(Y)$ so that $g^*(?_*(\delta_X)) = \delta_X$. At the moment we can see only that the following diagrams commute:

$$\begin{array}{ccccccc}
K_0(\mathcal{V}/X) & \xrightarrow{i_{g^*\delta_Y}} & K_0(\mathcal{V}/X) \otimes \mathcal{F}(X) & \xrightarrow{\Phi} & \mathcal{F}(X) & \xrightarrow{c_*} & H_*^{BM}(X) \\
{\scriptstyle g_*}\downarrow & & & & \downarrow{\scriptstyle g_*} & & \downarrow{\scriptstyle g_*} \\
K_0(\mathcal{V}/Y) & \xrightarrow[i_{\delta_Y}]{} & K_0(\mathcal{V}/Y) \otimes \mathcal{F}(Y) & \xrightarrow[\Phi]{} & \mathcal{F}(Y) & \xrightarrow[c_*]{} & H_*^{BM}(Y)
\end{array}$$

Indeed, in the left long square, we do have that

$$(g_* \circ \Phi \circ i_{g^*\delta_Y})([V \xrightarrow{h} X]) = g_*\left(\Phi([V \xrightarrow{h} X] \otimes g^*\delta_Y)\right) = g_*(h_*h^*(g^*\delta_Y)) = (gh)_*(gh)^*\delta_Y,$$

$$(\Phi \circ i_{\delta_Y} \circ g_*)([V \xrightarrow{h} X]) = \Phi\left(i_{\delta_Y}([V \xrightarrow{gh} Y])\right) = \Phi([V \xrightarrow{gh} Y] \otimes \delta_Y) = (gh)_*(gh)^*\delta_Y.$$

Thus the left long square is commutative.

5. Naive Motivic Donaldson–Thomas type Hirzebruch classes

In this section we give a further generalization of the above generalized Aluffi class $c_*^\delta(X)$, using the motivic Hirzebruch class transformation $T_{y*} : K_0(\mathcal{V}/-) \to H_*^{BM}(-) \otimes \mathbb{Q}[y]$.

In the above argument, a key part is the operation of *pullback-followed-by-pushforward* h_*h^* for a morphism $h : V \to X$ on a fixed or chosen constructible function δ_X of the target space X. It is quite natural to do the same operation on $K_0(\mathcal{V}/X)$ itself. For that purpose we need to define a motivic element $\delta_X^{mot} \in K_0(\mathcal{V}/X)$ corresponding to the constructible function δ_X; in particular we need to define a reasonable motivic element $\nu_X^{mot} \in K_0(\mathcal{V}/X)$ corresponding to the Behrend function $\nu_X \in \mathcal{F}(X)$.

By considering the isomorphism $1\!\!1 : \mathcal{Z}(X) \xrightarrow{\cong} \mathcal{F}(X)$, $1\!\!1\left(\sum_V n_V[V]\right) := \sum_V n_V 1\!\!1_V$, we define another distinguished integral cycle: $\mathfrak{D}_X := 1\!\!1^{-1}(\nu_X)$ $\left(= 1\!\!1^{-1} \circ \mathrm{Eu}(\mathfrak{C}_X)\right)$. Then we set

$$\nu_X^{mot} := [\mathfrak{D}_X \to X].$$

This can be put in as follows. Let $\mathfrak{s} : \mathcal{F}(X) \to K_0(\mathcal{V}/X)$ be the section of $1\!\!1_* : K_0(\mathcal{V}/X) \to \mathcal{F}(X)$ defined by $\mathfrak{s}(1\!\!1_S) := [S \hookrightarrow X]$. Then $\nu_X^{mot} = \mathfrak{s}(\nu_X)$. Another way is $\nu_X^{mot} := \sum_n n[\nu_X^{-1}(n) \hookrightarrow X]$ (see [10]).

Remark 5.1. Obviously the homomorphism $[1\!\!1_X] = 1\!\!1_* : K_0(\mathcal{V}/X) \to \mathcal{F}(X)$ is not injective and its kernel is infinite. In the case when X is the critical set of a regular function $f : M \to \mathbb{C}$, then there is a notion of "motivic element" (which is called the "motivic Donaldson–Thomas invariant") corresponding to the Behrend function (which is in this case described via the Milnor fiber), using the motivic Milnor fiber, due to Denef–Loeser. In our general case, we do not have such a sophisticated machinery available, thus it seems to be natural to define a motivic element ν_X^{mot} naively as above.

Let $\Psi : K_0(\mathcal{V}/X) \otimes K_0(\mathcal{V}/X) \to K_0(\mathcal{V}/X)$ be the fiber product mentioned before:

$$\Psi\left([V \xrightarrow{h} X] \otimes [W \xrightarrow{k} X]\right) := [V \xrightarrow{h} X] \cdot [W \xrightarrow{k} X] = [V \times_X W \xrightarrow{h \times_X k} X].$$

Since $[\delta_X] = \Phi \circ i_{\delta_X} : K_0(\mathcal{V}/X) \xrightarrow{i_{\delta_X}} K_0(\mathcal{V}/X) \otimes \mathcal{F}(X) \xrightarrow{\Phi} \mathcal{F}(X)$ with $\delta_X \in \mathcal{F}(X)$, we consider its "motivic" analogue, which means the following homomorphism

$$[\gamma_X] : K_0(\mathcal{V}/X) \xrightarrow{i_{\gamma_X}} K_0(\mathcal{V}/X) \otimes K_0(\mathcal{V}/X) \xrightarrow{\Psi} K_0(\mathcal{V}/X),$$

where $\gamma_X \in K_0(\mathcal{V}/X)$ and $i_{\gamma_X} : K_0(\mathcal{V}/X) \to K_0(\mathcal{V}/X) \otimes K_0(\mathcal{V}/X)$ is defined by $i_{\gamma_X}(\alpha) := \alpha \otimes \gamma_X$.

Proposition 5.2. *Let $\gamma_X \in K_0(\mathcal{V}/X)$. Then the following diagram commutes:*

$$\begin{array}{ccc} K_0(\mathcal{V}/X) & \xrightarrow{[\gamma_X]} & K_0(\mathcal{V}/X) \\ & {\scriptstyle [\mathbf{1}_*(\gamma_X)]} \searrow \quad \swarrow {\scriptstyle \mathbf{1}_*} & \\ & \mathcal{F}(X). & \end{array}$$

Proof. Let $\gamma_X := [S \xrightarrow{h_S} X]$. Then it suffices to show the following

$$\left(\mathbb{1}_* \circ \left[[S \xrightarrow{h_S} X]\right]\right)([V \xrightarrow{h} X]) = \left[\mathbb{1}_* \left([S \xrightarrow{h_S} X]\right)\right]([V \xrightarrow{h} X]).$$

This can be proved using the fiber square $\begin{array}{ccc} V \times_X S & \xrightarrow{\widetilde{h}} & S \\ {\scriptstyle \widetilde{h_S}} \downarrow & & \downarrow {\scriptstyle h_S} \\ V & \xrightarrow[h]{} & X. \end{array}$

$$\begin{aligned} \left(\mathbb{1}_* \circ \left[[S \xrightarrow{h_S} X]\right]\right)([V \xrightarrow{h} X]) &= \mathbb{1}_* \left(\left[[S \xrightarrow{h_S} X]\right]([V \xrightarrow{h} X])\right) \\ &= \mathbb{1}_*([V \times_X S \xrightarrow{h \circ \widetilde{h_S}} X]) \\ &= (h \circ \widetilde{h_S})_* \mathbb{1}_{V \times_X S} \ \text{ (by the definition of } \mathbb{1}_*) \\ &= h_* \widetilde{h_S}_* \mathbb{1}_{V \times_X S} \\ &= h_* \widetilde{h_S}_* \widetilde{h}^* \mathbb{1}_S \\ &= h_* h^* (h_S)_* \mathbb{1}_S \quad (\text{since } \widetilde{h_S}_* \widetilde{h}^* = h^*(h_S)_*) \\ &= h_* h^* \left(\mathbb{1}_*([S \xrightarrow{h_S} X])\right) \\ &= \left[\mathbb{1}_* \left([S \xrightarrow{h_S} X]\right)\right]([V \xrightarrow{h} X]). \end{aligned}$$

□

Corollary 5.3. (1) *Let $\delta_X \in \mathcal{F}(X)$ and let $\delta_X^{mot} \in K_0(\mathcal{V}/X)$ be such that $\mathbb{1}_*(\delta_X^{mot}) = \delta_X$. Then we have*

$$\begin{array}{ccc} K_0(\mathcal{V}/X) & \xrightarrow{[\delta_X^{mot}]} & K_0(\mathcal{V}/X) \\ & {\scriptstyle [\delta_X]} \searrow \quad \swarrow {\scriptstyle \mathbf{1}_*} & \\ & \mathcal{F}(X). & \end{array}$$

The motivic element δ_X^{mot} is called a naive motivic lift of δ_X.

(2) *In particular, we have*

$$\begin{array}{ccc} K_0(\mathcal{V}/X) & \xrightarrow{[\nu_X^{mot}]} & K_0(\mathcal{V}/X) \\ & {\scriptstyle [\nu_X]}\searrow \quad \swarrow{\scriptstyle \mathbf{1}_*} & \\ & \mathcal{F}(X). & \end{array}$$

Remark 5.4. Here we emphasize that the following diagrams commutes:

$$\begin{array}{ccccc} K_0(\mathcal{V}/X) & \xrightarrow{[\nu_X^{mot}]} & K_0(\mathcal{V}/X) & & \\ & {\scriptstyle [\nu_X]}\searrow \quad \swarrow{\scriptstyle \mathbf{1}_*} & & \searrow{\scriptstyle T_{-1_*}} & \\ & \mathcal{F}(X) & \xrightarrow[c_*\otimes\mathbb{Q}]{} & & H_*^{BM}(X)\otimes\mathbb{Q}. \end{array}$$

Thus, modulo the torsion and the choices of motivic elements ν_X^{mot}, the composite $T_{-1_*} \circ [\nu_X^{mot}]$ is a higher class analogue of the Donaldson–Thomas type invariant. Thus it would be natural to generalize the Donaldson–Thomas type invariant using the motivic Hirzebruch class T_{y_*}.

Let $\gamma_X \in K_0(\mathcal{V}/X), \gamma_Y \in K_0(\mathcal{V}/Y)$. Then for any morphism $g : X \to Y$ the following diagrams commute:

$$\begin{array}{ccc} K_0(\mathcal{V}/X) & \xrightarrow{[\gamma_X]} & K_0(\mathcal{V}/X) \\ {\scriptstyle g_*}\downarrow & & \downarrow{\scriptstyle g_*} \\ K_0(\mathcal{V}/Y) & \xrightarrow[{[g_*\gamma_X]}]{} & K_0(\mathcal{V}/Y), \end{array} \quad \text{or} \quad \begin{array}{ccccc} K_0(\mathcal{V}/X) & \xrightarrow{i_{\gamma_X}} & K_0(\mathcal{V}/X)\otimes K_0(\mathcal{V}/X) & \xrightarrow{\Psi} & K_0(\mathcal{V}/X) \\ {\scriptstyle g_*}\downarrow & & \downarrow{\scriptstyle g_*\otimes g_*} & & \downarrow{\scriptstyle g_*} \\ K_0(\mathcal{V}/Y) & \xrightarrow[i_{g_*\gamma_X}]{} & K_0(\mathcal{V}/Y)\otimes K_0(\mathcal{V}/Y) & \xrightarrow[\Psi]{} & K_0(\mathcal{V}/Y) \end{array}$$

$$\begin{array}{ccc} K_0(\mathcal{V}/Y) & \xrightarrow{[\gamma_Y]} & K_0(\mathcal{V}/Y) \\ {\scriptstyle g^*}\downarrow & & \downarrow{\scriptstyle g^*} \\ K_0(\mathcal{V}/X) & \xrightarrow[{[g^*\gamma_Y]}]{} & K_0(\mathcal{V}/X), \end{array} \quad \text{or} \quad \begin{array}{ccccc} K_0(\mathcal{V}/Y) & \xrightarrow{i_{\gamma_Y}} & K_0(\mathcal{V}/Y)\otimes K_0(\mathcal{V}/Y) & \xrightarrow{\Psi} & K_0(\mathcal{V}/Y) \\ {\scriptstyle g^*}\downarrow & & \downarrow{\scriptstyle g^*\otimes g^*} & & \downarrow{\scriptstyle g^*} \\ K_0(\mathcal{V}/X) & \xrightarrow[i_{g^*\gamma_Y}]{} & K_0(\mathcal{V}/X)\otimes K_0(\mathcal{V}/X) & \xrightarrow[\Psi]{} & K_0(\mathcal{V}/X) \end{array}$$

$$\begin{array}{ccc} K_0(\mathcal{V}/X) & \xrightarrow{[g^*\gamma_Y]} & K_0(\mathcal{V}/X) \\ {\scriptstyle g_*}\downarrow & & \downarrow{\scriptstyle g_*} \\ K_0(\mathcal{V}/Y) & \xrightarrow[{[\gamma_Y]}]{} & K_0(\mathcal{V}/Y). \end{array}$$

The last commutative diagram is a bit more precisely the following

$$\begin{array}{ccccc} K_0(\mathcal{V}/X) & \xrightarrow{i_{g^*\gamma_Y}} & K_0(\mathcal{V}/X)\otimes K_0(\mathcal{V}/X) & \xrightarrow{\Psi} & K_0(\mathcal{V}/X) \\ {\scriptstyle g_*}\downarrow & & & & \downarrow{\scriptstyle g_*} \\ K_0(\mathcal{V}/Y) & \xrightarrow[i_{\gamma_Y}]{} & K_0(\mathcal{V}/Y)\otimes K_0(\mathcal{V}/Y) & \xrightarrow[\Psi]{} & K_0(\mathcal{V}/Y) \end{array}$$

Here we do not know how to define a homomorphism in the middle so that the diagrams commute, just like in the case discussed in Remark 4.14.

Corollary 5.5. (1) *Let $\gamma_X \in K_0(\mathcal{V}/X), \gamma_Y \in K_0(\mathcal{V}/Y)$. For a proper morphism $g : X \to Y$ the following diagrams commute:*

$$\begin{array}{ccc} K_0(\mathcal{V}/X) & \xrightarrow{T_{y*}\circ[\gamma_X]} & H_*^{BM}(X)\otimes\mathbb{Q}[y] \\ {\scriptstyle g_*}\downarrow & & \downarrow{\scriptstyle g_*} \\ K_0(\mathcal{V}/Y) & \xrightarrow[T_{y*}\circ[g_*\gamma_X]]{} & H_*^{BM}(Y)\otimes\mathbb{Q}[y], \end{array} \quad \begin{array}{ccc} K_0(\mathcal{V}/X) & \xrightarrow{T_{y*}\circ[g^*\gamma_Y]} & H_*^{BM}(X)\otimes\mathbb{Q}[y] \\ {\scriptstyle g_*}\downarrow & & \downarrow{\scriptstyle g_*} \\ K_0(\mathcal{V}/Y) & \xrightarrow[T_{y*}\circ[\gamma_Y]]{} & H_*^{BM}(Y)\otimes\mathbb{Q}[y], \end{array}$$

(2) *For a proper smooth morphism $g : X \to Y$ and for $\gamma_Y \in K_0(\mathcal{V}/Y)$ the following diagrams are commutative:*

$$\begin{array}{ccc} K_0(\mathcal{V}/Y) & \xrightarrow{T_{y*}\circ[\gamma_Y]} & H_*^{BM}(Y)\otimes\mathbb{Q}[y] \\ {\scriptstyle g^*}\downarrow & & \downarrow{\scriptstyle td_y(T_g)\cap g^*} \\ K_0(\mathcal{V}/X) & \xrightarrow[T_{y*}\circ[g^*\gamma_Y]]{} & H_*^{BM}(X)\otimes\mathbb{Q}[y]. \end{array}$$

(3) *Let $\widetilde{\nu}_X^{mot} := (-1)^{\dim X}\nu_X^{mot}$, the signed one. Let ${T_{y}}_*^{DT} := T_{y*}\circ[\widetilde{\nu}_X^{mot}]$. For a proper smooth morphism $g : X \to Y$ the following diagrams are commutative:*

$$\begin{array}{ccc} K_0(\mathcal{V}/X) & \xrightarrow{{T_y}_*^{DT}} & H_*^{BM}(X)\otimes\mathbb{Q}[y] \\ {\scriptstyle g_*}\downarrow & & \downarrow{\scriptstyle g_*} \\ K_0(\mathcal{V}/Y) & \xrightarrow[{T_y}_*^{DT}]{} & H_*^{BM}(Y)\otimes\mathbb{Q}[y], \end{array} \quad \begin{array}{ccc} K_0(\mathcal{V}/Y) & \xrightarrow{{T_y}_*^{DT}} & H_*^{BM}(Y)\otimes\mathbb{Q}[y] \\ {\scriptstyle g^*}\downarrow & & \downarrow{\scriptstyle td_y(T_g)\cap g^*} \\ K_0(\mathcal{V}/X) & \xrightarrow[{T_y}_*^{DT}]{} & H_*^{BM}(X)\otimes\mathbb{Q}[y]. \end{array}$$

Remark 5.6. The commutative diagram in Proposition 5.2 can be described in more details as follows:

$$\begin{array}{ccccc} K_0(\mathcal{V}/X) & \xrightarrow{i_{\gamma_X}} & K_0(\mathcal{V}/X)\otimes K_0(\mathcal{V}/X) & \xrightarrow{\Psi} & K_0(\mathcal{V}/X) \\ & & {\scriptstyle id\otimes {i_1}_X}\downarrow & & \downarrow{\scriptstyle {i_1}_X} \\ & & K_0(\mathcal{V}/X)\otimes K_0(\mathcal{V}/X)\otimes\mathcal{F}(X) & \xrightarrow{\Psi\otimes id} & K_0(\mathcal{V}/X)\otimes\mathcal{F}(X) \\ & & {\scriptstyle id\otimes\Phi}\downarrow & & \downarrow{\scriptstyle\Phi} \\ & & K_0(\mathcal{V}/X)\otimes\mathcal{F}(X) & \xrightarrow[\Phi]{} & \mathcal{F}(X) \end{array}$$

If we denote $\Phi(\alpha\otimes\delta_X)$ simply by $\alpha\cdot\delta_X$, then the bottom square on the right-hand-side commutative diagrams means that $(\alpha\cdot\beta)\cdot\delta_X = \alpha\cdot(\beta\cdot\delta_X)$, i.e. the associativity.

Remark 5.7. We remark that the following diagrams commute:

(1) for a proper marphism $g : X \to Y$

$$\begin{array}{ccccc} \overbrace{K_0(\mathcal{V}/X)\otimes\cdots\otimes K_0(\mathcal{V}/X)}^{n} & \xrightarrow{\Psi^{n-1}} & K_0(\mathcal{V}/X) & \xrightarrow{T_{y*}} & H_*^{BM}(X)\otimes\mathbb{Q}[y] \\ \downarrow{\scriptstyle g_*\otimes\cdots\otimes g_*} & & \downarrow{\scriptstyle g_*} & & \downarrow{\scriptstyle g_*} \\ \underbrace{K_0(\mathcal{V}/Y)\otimes\cdots\otimes K_0(\mathcal{V}/Y)}_{n} & \xrightarrow[\Psi^{n-1}]{} & K_0(\mathcal{V}/Y) & \xrightarrow[T_{y*}]{} & H_*^{BM}(Y)\otimes\mathbb{Q}[y], \end{array}$$

(2) for a proper smooth morphism $g : X \to Y$

$$\begin{array}{ccccc}
\overbrace{K_0(\mathcal{V}/Y) \otimes \cdots \otimes K_0(\mathcal{V}/Y)}^{n} & \xrightarrow{\Psi^{n-1}} & K_0(\mathcal{V}/Y) & \xrightarrow{T_{y_*}} & H_*^{BM}(Y) \otimes \mathbb{Q}[y] \\
\Big\downarrow g^* \otimes \cdots \otimes g^* & & \Big\downarrow g^* & & \Big\downarrow c(T_g) \cap g_* \\
\underbrace{K_0(\mathcal{V}/X) \otimes \cdots \otimes K_0(\mathcal{V}/X)}_{n} & \xrightarrow[\Psi^{n-1}]{} & K_0(\mathcal{V}/X) & \xrightarrow[T_{y_*}]{} & H_*^{BM}(X) \otimes \mathbb{Q}[y],
\end{array}$$

Here $\Psi^{n-1}([V \to X]) := [V \to X] \cdot \ \cdots \ \cdot [V \to X]$ is the fiber product of n copies of $[V \to X]$. When $n = 1$, $\Psi^0 := \mathrm{id}_{K_0(\mathcal{V}/X)}$ is understood to be the identity. Let $P(t) := \sum a_i t^i \in \mathbb{Q}[t]$ be a polynomial. Then we define the polynomial transformation $\Psi_{P(t)} : K_0(\mathcal{V}/X) \to K_0(\mathcal{V}/X)$ by

$$\Psi_{P(t)}([V \xrightarrow{h} X]) := \sum a_i \Psi^{i-1}([V \to X]).$$

Then we have the following commutative diagrams.

(1) for a proper morphism $g : X \to Y$

$$\begin{array}{ccccc}
K_0(\mathcal{V}/X) & \xrightarrow{\Psi_{P(t)}} & K_0(\mathcal{V}/X) & \xrightarrow{T_{y_*}} & H_*^{BM}(X) \otimes \mathbb{Q}[y] \\
\Big\downarrow g_* & & \Big\downarrow g_* & & \Big\downarrow g_* \\
K_0(\mathcal{V}/Y) & \xrightarrow[\Psi_{P(t)}]{} & K_0(\mathcal{V}/Y) & \xrightarrow[T_{y_*}]{} & H_*^{BM}(Y) \otimes \mathbb{Q}[y],
\end{array}$$

(2) for a proper smooth morphism $g : X \to Y$

$$\begin{array}{ccccc}
K_0(\mathcal{V}/Y) & \xrightarrow{\Psi_{P(t)}} & K_0(\mathcal{V}/Y) & \xrightarrow{T_{y_*}} & H_*^{BM}(Y) \otimes \mathbb{Q}[y] \\
\Big\downarrow g^* & & \Big\downarrow g^* & & \Big\downarrow c(T_g) \cap g_* \\
K_0(\mathcal{V}/X) & \xrightarrow[\Psi_{P(t)}]{} & K_0(\mathcal{V}/X) & \xrightarrow[T_{y_*}]{} & H_*^{BM}(X) \otimes \mathbb{Q}[y],
\end{array}$$

These are a "motivic" analogue of the corresponding case of constructible functions:

(1) for a proper morphism $g : X \to Y$

$$\begin{array}{ccccc}
\mathcal{F}(X) & \xrightarrow{\mathcal{F}_{P(t)}} & \mathcal{F}(X) & \xrightarrow{c_*} & H_*^{BM}(X) \\
\Big\downarrow g_* & & \Big\downarrow g_* & & \Big\downarrow g_* \\
\mathcal{F}(Y) & \xrightarrow[\mathcal{F}_{P(t)}]{} & \mathcal{F}(Y) & \xrightarrow[c_*]{} & H_*^{BM}(Y)
\end{array}$$

(2) for a proper smooth morphism $g : X \to Y$

$$\begin{array}{ccccc}
\mathcal{F}(Y) & \xrightarrow{\mathcal{F}_{P(t)}} & \mathcal{F}(Y) & \xrightarrow{c_*} & H_*^{BM}(Y) \\
\Big\downarrow g^* & & \Big\downarrow g^* & & \Big\downarrow c(T_g) \cap g^* \\
\mathcal{F}(X) & \xrightarrow[\mathcal{F}_{P(t)}]{} & \mathcal{F}(X) & \xrightarrow[c_*]{} & H_*^{BM}(X)
\end{array}$$

Here $\mathcal{F}_{P(t)}(\beta) := \sum a_i \beta^i$. Note also that the following diagram commutes

$$\begin{array}{ccc} K_0(\mathcal{V}/X) & \xrightarrow{\Psi_{P(t)}} & K_0(\mathcal{V}/X) \\ \Big\downarrow \mathbf{1}_* & & \Big\downarrow \mathbf{1}_* \\ \mathcal{F}(X) & \xrightarrow[\mathcal{F}_{P(t)}]{} & \mathcal{F}(X). \end{array}$$

Definition 5.8. (1) We refer to the following class

$$T_{y_*}^{DT}(X) := \left(T_{y_*}^{DT}\right)([X \xrightarrow{id_X} X]) = T_{y*}([\widetilde{\nu}_X^{mot}])$$

as the *naive motivic Donaldson–Thomas type Hirzebruch class* of X.

(2) The degree zero of the naive motivic Donaldson–Thomas type Hirzebruch class is called the *naive motivic Donaldson–Thomas type χ_y-genus* of X:

$$\chi_y^{DT}(X) := \int_X T_{y_*}^{DT}(X).$$

Remark 5.9. The cases of the three special values $y = -1, 0, 1$ are the following.

(1) For $y = -1$, $T_{-1_*}^{DT}(X) = T_{-1*}([\widetilde{\nu}_X^{mot}]) = c_*^{A\ell}(X)$.
(2) For $y = 0$, $T_{0_*}^{DT}(X) = T_{0*}([\widetilde{\nu}_X^{mot}]) =: td_*^{A\ell}(X)$, which we call an "Aluffi-type" Todd class of X.
(3) For $y = 1$, $T_{1_*}^{DT}(X) = T_{1*}([\widetilde{\nu}_X^{mot}]) =: L_*^{A\ell}(X)$, which we call an "Aluffi-type" Cappell–Shaneson L-homology class of X.

The degree zero part of these three motivic classes are respectively:

(1) for $y = -1$, $\chi_{-1}^{DT}(X) = (-1)^{\dim X}\chi^{DT}(X)$, the original Donaldson–Thomas type invariant (i.e. Euler characteristic) of X with the sign;
(2) for $y = 0$, $\chi_0^{DT}(X) =: \chi_a^{DT}(X)$, which we call a *naive Donaldson–Thomas type arithmetic genus* of X and
(3) for $y = 1$, $\chi_{-1}^{DT}(X) = \sigma^{DT}(X)$, which we call a *naive Donaldson–Thomas type signature* of X.

Remark 5.10. Since $\widetilde{\nu}_X(x) = 1$ for a smooth point $x \in X$, we have that $\widetilde{\nu}_X = \mathbb{1}_X + \alpha_{X_{sing}}$ for some constructiblee functions $\alpha_{X_{sing}}$ supported on the singular locus X_{sing}. For example, consider the simplest case that X has one isolated singularity x_0, say $\widetilde{\nu}_X = \mathbb{1}_X + a_0\mathbb{1}_{x_0}$. Then

$$\widetilde{\nu}_X^{mot} = [X \xrightarrow{id_X} X] + a_0[x_0 \xrightarrow{i_{x_0}} X] \in K_0(\mathcal{V}/X).$$

Here $x_0 \xrightarrow{i_{x_0}} X$ is the inclusion. Hence we have

$$\begin{aligned} T_{y_*}^{DT}(X) &= T_{y*}(\widetilde{\nu}_X^{mot}) \\ &= T_{y*}([X \xrightarrow{id_X} X] + a_0[x_0 \xrightarrow{i_{x_0}} X]) \\ &= T_{y*}(X) + a_0(i_{x_0})_* T_{y*}(x_0) \\ &= T_{y*}(X) + a_0. \end{aligned}$$

Thus the difference between the motivic DT type Hirzebruch class $T_{y_*}^{DT}(X)$ and the motivic Hirzebruch class $T_{y*}(X)$ is just a_0, independent of the parameter y. Of course, if $\dim X_{sing} \geq 1$, then the difference *does* depend on the parameter y. For example, for the sake of simplicity, assume that $\widetilde{\nu}_X = \mathbb{1}_X + a\mathbb{1}_{X_{sing}}$. Then the difference is

$$T_{y_*}^{DT}(X) - T_{y*}(X) = a(i_{X_{sing}})_* T_{y*}(X_{sing}),$$

which certainly depends on the parameter y, *at least for the degree zero part* $\chi_y(X_{sing})$.

If we take a different motivic element $\overline{\nu}_X^{mot} = [X \xrightarrow{id_X} X] + [V \xrightarrow{h} X]$ such that

$$1\!\!1_*([V \xrightarrow{h} X]) = a_0 1\!\!1_{x_0}$$

and $\dim V \geq 1$, then the difference ${T_{y}}_*^{DT}(X) - T_{y*}(X) = h_*(T_{y*}(V))$, thus it *does* depend on the parameter y, at least for the degree zero part, again.

In the case when X is the critical locus of a regular function $f : M \to \mathbb{C}$, the motivic DT invariant $\nu_X^{motivic}$ which DT-theory people consider, using the motivic Milnor fiber, is the latter case, simply due to the important fact that the Behrend function can be expressed using the Milnor fiber. For example, as done in [9], even for an isolated singularity x_0, the difference ${T_{y}}_*^{DT}(X) - T_{y*}(X)$ is, up to sign, the χ_y-genus of (the Hodge structure of) the Milnor fiber at the singularity x_0, so does depend on the parameter y.

So, as long as the Behrend function has some geometric or topological descriptions, e.g., such as Milnor fibers, then one could think of the corresponding motivic elements in a naive or canonical way.

We will hope to come back to properties of these two classes $td_*^{A\ell}(X)$, $L_*^{A\ell}(X)$ and $\chi_a^{DT}(X)$, $\sigma^{DT}(X)$ and discussion on some relations with other invariants of singularities.

Remark 5.11. In [9] Cappell et al. use the Hirzebruch class transformation

$$\mathrm{MHM}\, T_{y*} : K_0(\mathrm{MHM}(X)) \to H_*^{BM}(X) \otimes \mathbb{Q}[y, y^{-1}]$$

from the Grothendieck group $K_0(\mathrm{MHM}(X))$ of the category of mixed Hodge modules (introduced by Morihiko Saito), instead of the Grothendieck group $K_0(\mathcal{V}/X)$. We could do the same things on $\mathrm{MHM}\, T_{y*} : K_0(\mathrm{MHM}(X)) \to H_*^{BM}(X) \otimes \mathbb{Q}[y, y^{-1}]$ and get MHM-theoretic analogues of the above. We hope to get back to this calculation.

Remark 5.12. In [14] Göttsche and Shende made an application of the above motivic Hirzebruch class $\mathrm{MHM}\, T_{y*}$. A bit more precisely, for a family $\pi : \mathcal{C} \to B$ of plane curves they introduce certain invariants $\mathcal{N}^i_{\mathcal{C}/B} \in K_0(\mathrm{MHM}(B))$ and apply the above functor

$$\mathrm{MHM}\, T_{y*} : K_0(\mathrm{MHM}(B)) \to H_*^{BM}(B) \otimes \mathbb{Q}[y, y^{-1}]$$

to these invariant $\mathcal{N}^i_{\mathrm{C}/B}$:

$$\mathbf{N}^i_{\mathcal{C}/B}(y) := \mathrm{MHM}\, T_{y*}(\mathcal{N}^i_{\mathcal{C}/B}),$$

which are used to make some formulations and some conjectures.

Remark 5.13. In a successive paper, we intend to apply the motivic Hirzebruch transformation to the motivic vanishing cycle constructed on the Donaldson–Thomas moduli space and announced in [6, 8]. This will hopefully provide the "right" motivic Donaldson–Thomas type Hirzebruch class.

6. A BIVARIANT GROUP OF PULLBACKS OF CONSTRUCTIBLE FUNCTIONS AND A BIVARIANT-THEORETIC PROBLEM

In the above section we mainly dealt with the class $c_*^{\delta_X}(V \xrightarrow{h} X)$ of $h : V \to X$ supported on the target space X. In this section we deal with the class $\overline{c_*^{\delta_X}}(V \xrightarrow{h} X)$ of $h : V \to X$ supported on the source space V.

The class $c_*^{\delta_X}(V \xrightarrow{h} X)$ is by definition $c_*(h_*h^*\delta_X) = h_*c_*(h^*\delta_X) \in H_*^{BM}(X)$, and can be captured as the image of a homomorphism between two abelian groups assigned to the space X, as done in the previous sections. However, when it comes to the case of $\overline{c_*^{\delta_X}}(V \xrightarrow{h} X) \in H_*^{BM}(V)$, one cannot do so, i.e. one cannot capture it as the image of a homomorphism between two abelian groups assigned to the space V. So we approach this class from a bivariant-theoretic viewpoint as follows.

For a morphism $f : X \to Y$ and a constructible function $\delta_Y \in \mathcal{F}(Y)$, we define $\mathbb{F}^{\delta_Y}(X \xrightarrow{f} Y)$ as follows:

$$\mathbb{F}^{\delta_Y}(X \xrightarrow{f} Y) := \left\{ \sum_S a_S i_{S*} i_S^* f^* \delta_Y \;\middle|\; S \text{ are closed subvarieties of } X, a_S \in \mathbb{Z} \right\} \subset \mathcal{F}(X),$$

where $i_S : S \to X$ is the inclusion map. Thus, using this notation, for a morphism $h : V \to X$ and for a constructible function $\delta_X \in \mathcal{F}(X)$, we have that $h^*\delta_X \in \mathbb{F}^{\delta_X}(V \xrightarrow{h} X) \subset \mathcal{F}(V)$.

For the sake of simplicity, unless some confusion is possible, we simply denote $i_{S*}(i_S)^* f^* \delta_Y$ by $(f|_S)^*\delta_Y (= (i_S)^* f^* \delta_Y)$. In particular, let us consider the signed Behrend function $\widetilde{\nu}_Y$ as δ_Y, i.e., $\mathbb{F}^{\widetilde{\nu}_Y}(X \xrightarrow{f} Y)$, which shall be denoted by $\mathbb{F}^{Beh}(X \xrightarrow{f} Y)$. It is easy to prove the following lemma.

Lemma 6.1. (1) *If Y is smooth, then $\mathbb{F}^{Beh}(X \xrightarrow{f} Y) = \mathcal{F}(X)$.*
(2) *$\mathbb{F}^{Beh}(X \xrightarrow{\pi} pt) = \mathcal{F}(X)$.*
(3) *If X is smooth, $\mathbb{F}^{Beh}(X \xrightarrow{\mathrm{id}_X} X) = \mathcal{F}(X)$.*
(4) *If Y is singular and $f(X) \cap Y_{sing} = \emptyset$, $\mathbb{F}^{Beh}(X \xrightarrow{f} Y) = \mathcal{F}(X)$.*
(5) *If Y is singular, $f(X) \cap Y_{sing} \neq \emptyset$ and there exists a point $y \in f(X) \cap Y_{sing}$ such that $|\nu_Y(y)| > 1$, $\mathbb{F}^{Beh}(X \xrightarrow{f} Y) \subsetneqq \mathcal{F}(X)$.*

Remark 6.2. In an earlier version of the paper, in the above lemma we stated " If X is singular, then $\mathbb{F}^{Beh}(X \xrightarrow{\mathrm{id}_X} X) \subsetneqq \mathcal{F}(X)$ and in particular, the characteristic function $1\!\!1_X \notin \mathbb{F}^{Beh}(X \xrightarrow{\mathrm{id}_X} X)$." However the referee pointed out that this is not obvious, and we have realized that

$$\mathbb{F}^{Beh}(X \xrightarrow{\mathrm{id}_X} X) = \mathcal{F}(X)$$

is also possible. If X is a plane curve with a node x_0, then $\nu_X(x_0) = \mathrm{Eu}_X(x_0) = 2$, in which case we get $\mathbb{F}^{Beh}(X \xrightarrow{\mathrm{id}_X} X) \subsetneqq \mathcal{F}(X)$. Let X be the union of a reduced surface Y with an isolated singular point x_0 such that $\mathrm{Eu}_Y(x_0) = m$ with $|m| > 1$ and a reduced curve C with the isolated singular point being the same x_0 such that $\mathrm{Eu}_C(x_0) = m - 1$, where we assume that $Y \cap C = \{x_0\}$. For example, the following is such a (non-pure dimensional) surface: Let Y be a projective cone of a non-singular curve of degree $d(> 3)$ with the cone point x_0. Then $\mathrm{Eu}_Y(x_0) = 2d - d^2$ (see [29, p. 426]). Hence $\nu_Y = (-1)^2 \mathrm{Eu}_Y = \mathrm{Eu}_Y$. Now let C be a plane curve with x_0 being a $(2d - d^2 + 1)$-ple point such that $Y \cap C = \{x_0\}$. Then let us set $X = Y \cup C$. Then we have $\nu_X = (-1)^2 \mathrm{Eu}_Y + (-1)^1 \mathrm{Eu}_C$, hence $\nu_X(x_0) = 2d - d^2 - (2d - d^2 + 1) = -1$, and $\nu_X(y) = 1$ for $y \in Y - \{x_0\}$ and $\nu_X(y) = -1$ for $y \in C - \{x_0\}$. Then we have

$$1\!\!1_X = i_{Y*} i_Y{}^* \nu_X + (-1) i_{C*} i_C{}^* \nu_X + i_{x_0*} i_{x_0}{}^* \nu_X \in \mathbb{F}^{Beh}(X \xrightarrow{\mathrm{id}_X} X).$$

If $1\!\!1_X \in \mathbb{F}^{Beh}(X \xrightarrow{\mathrm{id}_X} X)$, then any constructible function belongs to $\mathbb{F}^{Beh}(X \xrightarrow{\mathrm{id}_X} X)$, thus we get $\mathbb{F}^{Beh}(X \xrightarrow{\mathrm{id}_X} X) = \mathcal{F}(X)$. In passing, at the moment we do not know an example of a pure dimensional singular variety X which satisfies $\mathbb{F}^{Beh}(X \xrightarrow{\mathrm{id}_X} X) = \mathcal{F}(X)$.

In order to show that $\mathbb{F}^{Beh}(X \xrightarrow{f} Y)$ is a bivariant theory in the sense of Fulton and MacPherson [13], first we quickly recall some basics about Fulton–MacPherson's bivariant theory.

Definition 6.3. A *bivariant theory* $\mathbb{B}$ on a category $\mathcal{C}$ assigns to each morphism $X \xrightarrow{f} Y$ in the category $\mathcal{C}$ a (graded) abelian group $\mathbb{B}(X \xrightarrow{f} Y)$.

This bivariant theory is equipped with the following three basic operations:

(i) for morphisms $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$, the *product operation*

$$\bullet : \mathbb{B}(X \xrightarrow{f} Y) \otimes \mathbb{B}(Y \xrightarrow{g} Z) \to \mathbb{B}(X \xrightarrow{gf} Z)$$

is defined;

(ii) for morphisms $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ with f proper, the *pushforward operation*

$$f_* : \mathbb{B}(X \xrightarrow{gf} Z) \to \mathbb{B}(Y \xrightarrow{g} Z)$$

is defined;

(iii) for a fiber square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y, \end{array}$$

the *pullback operation*

$$g^* : \mathbb{B}(X \xrightarrow{f} Y) \to \mathbb{B}(X' \xrightarrow{f'} Y')$$

is defined.

These three operations are required to satisfy the seven axioms which are natural properties to make them compatible each other:

(B1) product is associative;
(B2) pushforward is functorial;
(B3) pullback is functorial;
(B4) product and pushforward commute;
(B5) product and pullback commute;
(B6) pushforward and pullback commute;
(B7) projection formula.

Definition 6.4. Let $\mathbb{B}$ and $\mathbb{B}'$ be two bivariant theories on a category $\mathcal{C}$. Then a *Grothendieck transformation* from $\mathbb{B}$ to $\mathbb{B}'$

$$\gamma : \mathbb{B} \longrightarrow \mathbb{B}'$$

is a collection of morphisms

$$\mathbb{B}(X \xrightarrow{f} Y) \to \mathbb{B}'(X \xrightarrow{f} Y)$$

for each morphism $X \xrightarrow{f} Y$ in the category $\mathcal{C}$, which preserves the above three basic operations.

As to the constructible functions we recall the following fact from [40]:

Theorem 6.5. *If we define $\mathbb{F}(X \xrightarrow{f} Y) := F(X)$ (ignoring the morphism f), then it become a bivariant theory, called the "simple" bivariant theory of constructible functions with the following three bivariant operations:*

- *(bivariant product)*

$$\bullet : \mathbb{F}(X \xrightarrow{f} Y) \otimes \mathbb{F}(Y \xrightarrow{g} Z) \to \mathbb{F}(X \xrightarrow{gf} Z),$$

$$\alpha \bullet \beta := \alpha \cdot f^*\beta.$$

- *(bivariant pushforward) For morphisms $f : X \to Y$ and $g : Y \to Z$ with f* proper

$$f_\star : \mathbb{F}(X \xrightarrow{gf} Z) \to \mathbb{F}(Y \xrightarrow{g} Z)$$

$$f_\star\alpha := f_*\alpha.$$

- *(bivariant pullback) For a fiber square* $\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y, \end{array}$

$$g^\star : \mathbb{F}(X \xrightarrow{f} Y) \to \mathbb{F}(X' \xrightarrow{f'} Y')$$

$$g^\star \alpha := (g')^* \alpha.$$

Theorem 6.6. *Here we consider the category of complex algebraic varieties. Then the above group $\mathbb{F}^{Beh}(X \xrightarrow{f} Y)$ becomes a bivariant theory as a subtheory of the above simple bivariant theory $\mathbb{F}(X \xrightarrow{f} Y)$, provided that we consider smooth morphisms g for the bivariant pullback.*

Proof. All we have to do is to show that those three bivariant operations are well-defined on the subgroup $\mathbb{F}^{Beh}(X \xrightarrow{f} Y)$. Below, as to bivariant product and bivariant pushforward, we do not need the requirement that δ_Y is the signed Behrend function $\widetilde{\nu}_Y$, but we need it for bivariant pullback.

(1) (bivariant product) It suffices to show that

$$(f|_S)^*\delta_Y \bullet (g|_W)^*\delta_Z = (f|_S)^*\delta_Y \cdot f^*(g|_W)^*\delta_Z \in \mathbb{F}^{\delta_Z}(X \xrightarrow{gf} Z).$$

Since $(f|_S)^*\delta_Y$ is a constructible function on S, $(f|_S)^*\delta_Y = \sum_V a_V 1\!\!1_V$ where V's are subvarieties of S, hence subvarieties of X. Thus we get

$$\begin{aligned}(f|_S)^*\delta_Y \cdot f^*(g|_W)^*\delta_Z &= \sum_V a_V 1\!\!1_V \cdot (gf|_{f^{-1}(W)})^*\delta_Z \\ &= \sum_V a_V (gf|_{f^{-1}(W)\cap V})^*\delta_Z\end{aligned}$$

Since $f^{-1}(W) \cap V$ is a finite union of subvarieties, it follows that

$$(f|_S)^*\delta_Y \cdot f^*(g|_W)^*\delta_Z \in \mathbb{F}^{\delta_Z}(X \xrightarrow{gf} Z).$$

(2) (bivariant pushforward) It suffices to show that

$$f_*((gf|_S)^*\delta_Z) \in \mathbb{F}^{\delta_Z}(Y \xrightarrow{g} Z).$$

More precisely, $f_*((gf|_S)^*\delta_Z) = f_*(i_S)_*(f|_S)^*g^*\delta_Z) = (f|_S)_*(f|_S)^*g^*\delta_Z$. Now it follows from Verdier's result [37, (5.1) Corollaire] that the morphism $f|_S : S \to Y$ is a stratified submersion, more precisely there is a filtration of closed subvarieties $V_1 \subset V_2 \subset \cdots \subset V_m \subset Y$ such that the restriction of $f|_S$ to each strata $V_{i+1} \setminus V_i$, i.e., $(f|_S)^{-1}(V_{i+1} \setminus V_i) \to V_{i+1} \setminus V_i$ is a fiber bundle. Hence the operation $(f|_S)_*(f|_S)^*$ is the same as the multiplication $(\sum_{i=1}^m a_i 1\!\!1_{V_i})\cdot$ with some integers a_i's, i.e.,

$$(f|_S)_*(f|_S)^*g^*\delta_Z = (\sum_i a_i 1\!\!1_{V_i}) \cdot g^*\delta_Z = \sum_i a_i (g|_{V_i})^*\delta_Z \in \mathbb{F}^{\delta_Z}(Y \xrightarrow{g} Z).$$

Here we remark that the above integer a_i is expressed as follows. Let χ_i denote the Euler-Poincaré characteristic of the fiber of the above fiber bundle $(f|_S)|_{V_i \setminus V_{i-1}}$. Then

$$a_m = \chi_m \quad \text{and} \quad a_i = \chi_i - \sum_{j=i+1}^{m} \chi_j \text{ for } 1 \leqq i < m.$$

(3) (bivariant pullback) Here we show that the following is well-defined

$$g^* : \mathbb{F}^{\delta_Y}(X \xrightarrow{f} Y) \to \mathbb{F}^{g^*\delta_Y}(X' \xrightarrow{f'} Y').$$

Consider the following fiber squares:

$$\begin{array}{ccc} S' & \xrightarrow{g''} & S \\ {\scriptstyle i_{S'}}\downarrow & & \downarrow{\scriptstyle i_S} \\ X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y. \end{array}$$

Indeed,

$$\begin{aligned} g^*((f|_S)^*\delta_Y) &= (g')^*((f|_S)^*\delta_Y \quad \text{(by definition)} \\ &= (g')^*((i_S)_*(f|_S)^*\delta_Y \quad \text{(more precisely)} \\ &= (i_{S'})_*(g'')^*(i_S)^*f^*\delta_Y \\ &= (i_{S'})_*(i_{S'})^*(f')^*g^*\delta_Y \in \mathbb{F}^{g^*\delta_Y}(X' \xrightarrow{f'} Y'). \end{aligned}$$

Hence, if δ_Y is the signed Behrend function $\widetilde{\nu}_Y$, then for a smooth morphism $g : Y' \to Y$ we have $\widetilde{\nu}_{Y'} = g^*\widetilde{\nu}_Y$, thus the pullback $g^* : \mathbb{F}^{Beh}(X \xrightarrow{f} Y) \to \mathbb{F}^{Beh}(X' \xrightarrow{f'} Y')$ is well-defined. Here we note that for any constructible functions δ_Y which are preserved by smooth morphisms $g : Y' \to Y$, i.e. $\delta_{Y'} = g^*\delta_Y$, the subgroups $\mathbb{F}^{\delta_Y}(X \xrightarrow{f} Y)$ give rise to a bivariant theory.

□

Problem 6.7. *Define a "bivariant homology theory" $\widetilde{\mathbb{H}}(X \to Y)$ such that*

(1) $\widetilde{\mathbb{H}}(X \xrightarrow{f} Y) \subsetneqq H_*^{BM}(X)$ *for any morphism* $f : X \to Y$,

(2) $\widetilde{\mathbb{H}}(X \to Y) = H_*^{BM}(X)$ *for a smooth* Y,

(3) *the homomorphism*

$$c_* : \mathbb{F}^{Beh}(X \xrightarrow{f} Y) \to \widetilde{\mathbb{H}}(X \xrightarrow{f} Y)$$

defined by $c_*(i_{S*}i_S^*f^*\widetilde{\nu}_Y) := i_{S*}c_*(i_S^*f^*\widetilde{\nu}_Y) \in H_*^{BM}(X)$ *and extended linearly, becomes a Grothendieck transformation.*

(4) *if Y is a point pt, then* $c_* : F(X) = \mathbb{F}^{Beh}(X \xrightarrow{f} pt) \to \widetilde{\mathbb{H}}(X \xrightarrow{f} pt) = H_*^{BM}(X)$ *is equal to the original MacPherson's Chern class homomorphism.*

Remark 6.8. One simple-minded construction of such a "bivariant homology theory" $\widetilde{\mathbb{H}}(X \to Y)$ could be simply the image of $\mathbb{F}^{Beh}(X \xrightarrow{f} Y)$ under MacPherson's Chern class $c_* : \mathcal{F}(X) \to H_*^{BM}(X)$. It remains to see whether the image $\widetilde{\mathbb{H}}(X \to Y) := c_*(\mathbb{F}^{Beh}(X \xrightarrow{f} Y))$ gives rise to a bivariant theory.

Before closing this section, we mention a bivariant-theoretic analogue of the covariant functor $\mathcal{L}$ of conical Lagrangian cycles. For the covariant functor of conical Lagrangian cycles, see [33, 21, 22].

In [21] Kennedy proved that $Ch : F(X) \xrightarrow{\cong} \mathcal{L}(X)$ is an isomorphism. In general, suppose we have a correspondence $\mathcal{H}$ such that

- $\mathcal{H}$ assigns an abelian group $\mathcal{H}(X)$ to a variety X
- there is an isomorphism $\Theta_X : F(X) \xrightarrow{\cong} \mathcal{H}(X)$.

Then, by "transfer of structure" using the above isomorphism Θ, we can get the corresponding bivariant theory. Here we go into a bit more details. If we define the pushforward $f_* : \mathcal{H}(X) \to \mathcal{H}(Y)$ for a map $f : X \to Y$ by

$$f_*^{\mathcal{H}} := \Theta_Y \circ f_*^F \circ \Theta_X^{-1} : \mathcal{H}(X) \to \mathcal{H}(Y),$$

then the correspondence $\mathcal{H}$ becomes a covariant functor *via the covariant functor F*. Here

$$f_*^F : F(X) \to F(Y),$$

emphasizing the covariant functor F. Similary, if we define the pullback $f^* : \mathcal{H}(Y) \to \mathcal{H}(X)$ by

$$f^*_{\mathcal{H}} := \Theta_X \circ f^*_F \circ \Theta_Y^{-1} : \mathcal{H}(Y) \to \mathcal{H}(X),$$

then the correspondence $\mathcal{H}$ becomes a contravariant functor *via the contravariant functor F*. Here $f^*_F : F(Y) \to F(X)$. Furthermore, if we define

$$\mathbb{B}\mathcal{H}(X \xrightarrow{f} Y) := \mathcal{H}(X)$$

then we get the simple bivariant-theoretic version of the correspondence $\mathcal{H}$ as follows:

- (Bivariant product) $\bullet_{\mathbb{B}\mathcal{H}} : \mathbb{B}\mathcal{H}(X \xrightarrow{f} Y) \otimes \mathbb{B}\mathcal{H}(Y \xrightarrow{g} Z) \to \mathbb{B}\mathcal{H}(X \xrightarrow{gf} Z)$ is defined by
$$\alpha \bullet_{\mathbb{B}\mathcal{H}} \beta := \Theta_X\Big(\Theta_Y^{-1}(\alpha) \bullet_{\mathbb{F}} \Theta_X^{-1}(\beta)\Big).$$
- (Bivariant pushforward) $f_*^{\mathbb{B}\mathcal{H}} : \mathbb{B}\mathcal{H}(X \xrightarrow{gf} Z) \to \mathbb{B}\mathcal{H}(Y \xrightarrow{g} Z)$ is defined by
$$f_*^{\mathbb{B}\mathcal{H}} := \Theta_Y \circ f_*^{\mathbb{F}} \circ \Theta_X H^{-1}.$$
- (Bivariant pullback) $g*_{\mathbb{B}\mathcal{H}} : \mathbb{B}\mathcal{H}(X \xrightarrow{f} Y) \to \mathbb{B}\mathcal{H}(X' \xrightarrow{f'} Y')$ is defined by
$$g^*_{\mathbb{B}\mathcal{H}} := \Theta_{X'} \circ f^*_{\mathbb{F}} \circ \Theta_X^{-1}.$$

Clearly we get the canonical Grothendieck transformation

$$\gamma_\Theta = \Theta : \mathbb{F}(X \xrightarrow{f} Y) \to \mathbb{B}\mathcal{H}(X \xrightarrow{f} Y).$$

If we apply this argument to the conical Lagrangian cycle $\mathcal{L}(X)$ we get the simple bivariant theory of conical Lagrangian cycles $\mathbb{L}(X \xrightarrow{f} Y)$ and also we get the canonical Grothendieck transformation

$$\gamma_{Ch} = Ch : \mathbb{F}(X \xrightarrow{f} Y) \to \mathbb{L}(X \xrightarrow{f} Y).$$

This simple bivariant theory $\mathbb{L}(X \xrightarrow{f} Y)$ can be defined or constructed directly, which would be however harder. Indeed, it is done in [7] and one has to go through many geometric and/or topological ingredients.

Fulton–MacPherson's bivariant theory $\mathbb{F}^{FM}(X \xrightarrow{f} Y)$ is a subgroup (or a subtheory) of the simple bivariant theory $\mathbb{F}(X \xrightarrow{f} Y) = F(X)$. Then if we define

$$\mathbb{L}^{FM}(X \xrightarrow{f} Y) := \gamma_{Ch}(\mathbb{F}^{FM}(X \xrightarrow{f} Y))$$

then we can get a finer bivariant theory of conical Lagrangian cycles, putting aside the problem of how we define or describe such a finer bivariant-theoretic conical Lagrangian cycle; it would be much harder than the case of the simple one $\mathbb{L}(X \xrightarrow{f} Y)$ done in [7].

7. Some more questions and problems

7.1. **A categorification of Donaldson–Thomas type invariant of a morphism.** The cardinality $c(F)$ of a finite set F, i.e., the number of elements of F, satisfies that

(1) $X \cong X'$ (set-isomorphism) $\Longrightarrow c(X) = c(X')$,
(2) $c(X) = c(Y) + c(X \setminus Y)$ for a subset $Y \subset X$ (a *scissor relation*),
(3) $c(X \times Y) = c(X) \times c(Y)$,
(4) $c(pt) = 1$.

Now, let us suppose that there is a similar "cardinality" on a category $\mathcal{TOP}$ of certain reasonable topological spaces, satisfying the above four properties, except for the condition (1) and (2),

(1)' $X \cong X'$ ($\mathcal{TOP}$-isomorphism) $\Longrightarrow c(X) = c(X')$,
(2)' $c(X) = c(Y) + c(X \setminus Y)$ for a closed subset $Y \subset X$.
(3) $c(X \times Y) = c(X) \times c(Y)$,
(4) $c(pt) = 1$.

If such a "topological cardinality" exists, then we can show that $c(\mathbb{R}^1) = -1$, hence $c(\mathbb{R}^n) = (-1)^n$ (e.g. see [41]). Thus, for a finite CW-complex X, $c(X)$ is exactly the Euler–Poincaré characteristic $\chi(X)$. The existence of such a topological cardinality is *guaranteed by the ordinary homology theory*, more precisely

$$c(X) = \chi_c(X) := \sum (-1)^i \dim_{\mathbb{R}} H^i_c(X;\mathbb{R}) = \sum_i (-1)^i \dim_{\mathbb{R}} H^{BM}_i(X;\mathbb{R}).$$

Here $H^{BM}_*(X)$ is the Borel–Moore homology group of X.

Similarly let us suppose that there is a similar cardinality on the category $\mathcal{V}_{\mathbb{C}}$ of complex algebraic varieties:

(1)" $X \cong X'$ ($\mathcal{V}_{\mathbb{C}}$-isomorphism) $\Longrightarrow c(X) = c(X')$,
(2)" $c(X) = c(Y) + c(X \setminus Y)$ for a closed subvariety $Y \subset X$ (i.e., a closed subset in Zariski topology),
(3) $c(X \times Y) = c(X) \times c(Y)$,
(4) $c(pt) = 1$.

The complex affine line $\mathbb{C}^1$ is corresponding to the real line $\mathbb{R}^1$. But we cannot do the same trick for $\mathbb{C}^1$ as we do for $\mathbb{R}^1$. The existence of such an algebraic cardinality is *guaranteed by Deligne's theory of mixed Hodge structures*. Let u, v be two variables, then the Deligne–Hodge polynomial $\chi_{u,v}$ is defined by

$$\chi_{u,v}(X) = \sum (-1)^i \dim_{\mathbb{C}} Gr^p_F Gr^W_{p+q}(H^i_c(X;\mathbb{C}))u^p v^q.$$

In particular, $\chi_{u,v}(\mathbb{C}^1) = uv$. The particular case when $u = -y, v = 1$ is the important one for the motivic Hirzebruch class:$\chi_y(X) := \chi_{-y,1}(X) = \sum (-1)^i \dim_{\mathbb{C}} Gr^p_F(H^i_c(X;\mathbb{C}))(-y)^p$. This is called χ_y-*genus* of X.

Similarly let us consider the Donaldson–Thomas type invariant of morphisms:

(1)"' $X \xrightarrow{f} Y \cong X' \xrightarrow{f'} Y$ (isomorphism) $\Longrightarrow \chi^{DT}(X \xrightarrow{f} Y) = \chi^{DT}(X' \xrightarrow{f'} Y)$,
(2)"' $\chi^{DT}(X \xrightarrow{f} Y) = \chi^{DT}(Z \xrightarrow{f|_Z} Y) + \chi^{DT}(X \setminus Z \xrightarrow{f|_{X \setminus Z}} Y)$ for a closed subvariety $Z \subset X$.
(3)"' $\chi^{DT}(X_1 \times X_2 \xrightarrow{f_1 \times f_2} Y_1 \times Y_2) = \chi^{DT}(X_1 \xrightarrow{f_1} Y_1) \times \chi^{DT}(X_2 \xrightarrow{f_2} Y_2)$,
(4) $\chi^{DT}(pt) = 1$.

So, just like the above two cardinalities or counting $\chi_c(X)$ and $\chi_{u,v}(X)$, we pose the following problem, which is related to the above Problem 6.7:

Problem 7.1. *Is there some kind of bivariant theory $\Theta^?(X \xrightarrow{f} Y)$ such that*

(1) $\chi^{DT}(X \xrightarrow{f} Y) = \sum_i (-1)^i \dim \Theta^?(X \xrightarrow{f} Y)$?

(2) *When Y is smooth, $\Theta(X \xrightarrow{f} Y)$ is isomorphic to Borel–Moore homology theory $H_*^{BM}(X)$ (which is isomorphic to the Fulton-MacPherson bivariant homology theory $\mathbb{H}(X \xrightarrow{f} Y)$ (e.g., see* [39, 4]*)).*

Remark 7.2. (1) When Y is smooth, we have $\chi^{DT}(X \xrightarrow{f} Y) = (-1)^{\dim Y}\chi(X)$, that is

$$\chi^{DT}(X \xrightarrow{f} Y) = (-1)^{\dim Y} \sum_i (-1)^i \dim H_i^{BM}(X)$$

$$= \sum_i (-1)^{i+\dim Y} \dim \mathbb{H}^{-i}(X \xrightarrow{f} Y).$$

In the above formulation $\chi^{DT}(X \xrightarrow{f} Y) = \sum_i (-1)^i \dim \Theta^?(X \xrightarrow{f} Y)$ the sign part $(-1)^i$ should involve something of the morphism f such as $\mathrm{reldim}\, f := \dim X - \dim Y$, $\dim X$, or $\dim Y$ etc., as well.

(2) Even for the identity $X \xrightarrow{\mathrm{id}_X} X$, since $\chi^{DT}(X) \neq \chi^{DT}(Z) + \chi^{DT}(X \setminus Z)$, the cohomological part $\Theta(X \xrightarrow{\mathrm{id}_X} X)$ of such a theory (if it existed) does not satisfy the usual long exact sequence for a pair $Z \subset X$, and it should satisfy a modified one so that

$$\chi^{DT}(X) = \chi^{DT}(Z \xrightarrow{inclusion} X) + \chi^{DT}(X \setminus Z \xrightarrow{inclusion} X)$$

is correct.

7.2. **A higher class analogue of MNOP conjecture and a generalized MacMahon function.** In [27] M. Levine and R. Pandharipande proved the MNOP conjecture [30], that is, we have the homomorphism

$$M(q) : \Omega^{-3}(pt) \to \mathbb{Q}[[q]], \text{ defined by } M(q)([X]) := M(q)^{\int_X c_3(T_X \otimes K_X)},$$

where $\Omega^*(X)$ is Levine–Morel's algebraic cobordism [26] (also see [25] and [27]) and

$$M(q) := \prod_{n \leqq 1} \frac{1}{(1-q^n)^n} = 1 + q + 3q^2 + 6q^3 + 13q^4 + \cdots$$

is the MacMahon function. A naive question on the above homomorphism $M(q) : \Omega^{-3}(pt) \to \mathbb{Q}[[q]]$ is:

Question 7.3. *To what extent could one extend the homomorphism $M(q) : \Omega^{-3}(pt) \to \mathbb{Q}[[q]]$ to a higher dimensional variety Y instead of $Y = pt$? Namely, is*

$$M(q) : \Omega^*(Y) \to H_*^{BM}(Y) \otimes \mathbb{Q}[[q]]$$

defined by

$$M(q)([X \xrightarrow{f} Y]) := M(q)^{f_*(c_{\dim X - \dim Y}(T_f \otimes K_f) \cap [X])}$$

a homomorphism?

Here by the construction of algebraic cobordism X and Y are both smooth, $T_f := T_X - f^*T_Y$ and $K_f := K_X - f^*K_Y$.

Note that for $Y = pt$ the above

$$M(q) : \Omega^*(Y) \to H_*^{BM}(Y) \otimes \mathbb{Q}[[q]]$$

is nothing but $M(q) : \Omega^{-3}(pt) \to \mathbb{Q}[[q]]$ in the case when $\dim X = 3$. The MacMahon function has a combinatorial origin as the generating function for the number of 3-dimensional partitions of size n (as explained in [25]). One could conjecture that the MacMahon function is involved only in the case when $\dim X - \dim Y = 3$. If it were the case, the following more specific problem should be posed:

Problem 7.4. *Is it true that the following is a homomorphism?*

$$M(q) : \Omega^{-3}(Y) \to H_*^{BM}(Y) \otimes \mathbb{Q}[[q]] \text{ defined by } M(q)([X \xrightarrow{f} Y]) := M(q)^{f_*(c_3(T_f \otimes K_f) \cap [X])}$$

Remark 7.5. Note that the dimension d of an element

$$[X \xrightarrow{f} Y] \in \Omega^d(Y)$$

is equal to $\operatorname{codim} f = \dim Y - \dim X$, hence if $Y = pt$, then $\dim X = 3$ implies that $d = -3$. Moreover, for a general dimension d, say $d < -3$, one should come up with some other functions, i.e. "d-dimensional generalized MacMahon function $\widetilde{M(q)}_d$" such that when $d = -3$ it is the same as the original MacMahon function $M(q)$, i.e. $\widetilde{M(q)}_{-3} = M(q)$. Such a formulation would be useful in Donaldson–Thomas theory for d-Calabi–Yau manifolds with $d > 3$. However, we have to point out that the above function $\widetilde{M(q)}_d$ for the generating function of dimension d partitions is now known to be not correct, although it does appear to be asymptotically correct in dimension four [3, 31]. Following ideas from algebraic cobordism as in [27], we hope to investigate this question further in a future work.

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References

[1] P. Aluffi, *Weighted Chern-Mather classes and Milnor classes of hypersurfaces*, In *Singularities - Sapporo 1998*, Adv. Stud. Pure Math., Kinokuniya, Tokyo, 2000, 1–20.

[2] K. Behrend, *Donaldson–Thomas type invariants via microlocal geometry*, Ann. of Math. (2) **170** (2009), no. 3, 1307–1338,

[3] K. Behrend, J. Bryan and B. Szendrői, *Motivic degree zero Donaldson–Thomas invariant*, Inventiones Math., **192** (2013), 111–160. DOI: 10.1007/s00222-012-0408-1

[4] J.-P. Brasselet, J. Schürmann and S. Yokura, *On the uniqueness of bivariant Chern class and bivariant Riemann-Roch transformations*, Advances in Math. **210** (2007) 797–812. DOI: 10.1016/j.aim.2006.07.014

[5] J.-P. Brasselet, J. Schürmann and S. Yokura, *Hirzebruch classes and motivic Chern classes for singular spaces*, Journal of Topology and Analysis, Vo1. **2**, No.1 (2010), 1–55.

[6] C. Brav, V. Bussi, D. Dupont, D. Joyce and B.Szendrői, *Symmetries and stabilization for sheaves of vanishing cycles*, Nov. 2012. arXiv:1211.3259v1

[7] V. Bussi, *Donaldson–Thomas theory and its extensions*, September, 2011, preprint.

[8] V. Bussi, D. Joyce, S. Meinhardt, *Categorification in Donaldson–Thomas theory using motivic vanishing cycles*, in preparation, January 2013.

[9] S. Cappell, L. Maxim, J. Schürmann and J.Shaneson, *Characteristic classes of complex hypersurfaces*, Advances in Math. **225** (2010), no. 5, 2616–2647.

[10] B. Davison, *Orientation data in Motivic Donaldson–Thomas theory*, A thesis submitted for the degree of Ph.D, University of Oxford, 25 Oct 2011. arXiv:1006.5475v3

[11] W. Fulton, *Intersection theory*, Springer Verlag (1984) DOI: 10.1007/978-3-662-02421-8

[12] W. Fulton and K. Johnson, *Canonical classes on singular varieties*, Manuscripta Math. **32** (1980), 381–389. DOI: 10.1007/BF01299611

[13] W. Fulton and R. MacPherson, *Categorical frameworks for the study of singular spaces*, Memoirs of Amer. Math. Soc. **243**, 1981.

[14] L. Göttsche and V. Shende, *Refined curve counting on complex surfaces*, to appear in Geometry & Topology. arXiv:1208.1973
[15] F. Hirzebruch, *Topological Methods in Algebraic Geometry, 3rd ed. (1st German ed. 1956)*, Springer-Verlag, 1966.
[16] F. Hirzebruch, T. Berger and R. Jung, *Manifolds and Modular Forms*, Vieweg, 1992. DOI: 10.1007/978-3-663-14045-0
[17] D. Joyce, *Constructible functions on Artin stacks*, J. London Math. Soc. (2) **74** (2006), 583–606. DOI: 10.1112/S0024610706023180
[18] D. Joyce, *Motivic invariants of Artin stacks and 'stack functions'*, Quarterly Journal of Mathematics **58** (2007), 345–392. DOI: 10.1093/qmath/ham019
[19] D. Joyce, *Generalized Donaldson–Thomas invariants*, in *Geometry of special holonomy and related topics*, Surveys in Differential Geometry XVI (Ed. by N.C. Leung and S.-T. Yau), International Press, Cambridge, MA,(2011), 125–160.
[20] D. Joyce and Y. Song, *A theory of generalized Donaldson–Thomas invariants*, Memoirs of the AMS , **217** (2012), pages 1-216. DOI: 10.1090/S0065-9266-2011-00630-1
[21] G. Kennedy, *MacPherson's Chern classes of singular algebraic varieties*, Comm. Algebra **18**, No. 9 (1990), 2821–2839. DOI: 10.1080/00927879008824054
[22] G. Kennedy, *Specialization of MacPherson's Chern classes*, Math. Scand. **66** (1990), 12–16.
[23] M. Kontsevich and Y. Soibelman, *Stability structures, motivic Donaldson–Thomas invariants and cluster transformations*, Nov. 2008. arXiv:0811.2435v1
[24] M. Kontsevich and Y. Soibelman, *Motivic Donaldson–Thomas invariants: summary of results*, in *Mirror Symmetry and Tropical Geometry*, Contemporary Mathematics **527** (Ed. R. Castano-Bernard, Y. Soibelma and I. Zharkov), Amer. Math. Soc., Providence, RI, (2010), 55–90. DOI: 10.1090/conm/527/10400
[25] M. Levine, *A survey of algebraic cobordism*, UCLA Colloquium, January 22, 2009.
[26] M. Levine and F. Morel, *Algebraic Cobordism*, Springer Monographs in Math., Springer-Verlag, 2007.
[27] M. Levine and R.Pandharipande, *Algebraic Cobordism Revisited*, Inventiones Math., **176** (2009), 63–130. DOI: 10.1007/s00222-008-0160-8
[28] E. Looijenga, *Motivic measures*, Séminaire Bourbaki 874, Astérisque 276 (2002), 267–297.
[29] R. MacPherson, *Chern classes for singular algebraic varietes*, Ann. of Math. 100 (1974), 423-432. DOI: 10.2307/1971080
[30] D. Maulik, N. Nekrasov, A. Okounkov and R. Pandharipande, *Gromov–Witten theory and Donaldson–Thomas theory. I.*, Compositio Math. 142 (2006), 1263–1285.
[31] V. Mustonen and R. Rajesh, *Numerical estimation of the asymptotic behavior of solid partitions of an integer*, J. Physics. A, 36(24) (2003), 6651–6659.
[32] A. Parusiński and P. Pragacz, *Characteristic classes of hypersurfaces and characteristic cycles*, J. Algebraic Geomery **10**(1) (2001), 63–79,
[33] C. Sabbah, *Quelques remarques sur la géométrie des expaces conormaux*, Astérisque **130** (1985), 161–192,
[34] J. Schürmann, *A generalized Verdier-type Riemann-Roch theorem for Chern-Schwartz-MacPherson classes*, Feb. 2002. arXiv: math/0202175
[35] J. Schürmann and S. Yokura, *Motivic bivariant characteristic classes*, Advances in Math., **250** (2014), 611-649
[36] R. P. Thomas, *Gauge theories on Calabi-Yau manifolds*, D. Phil. Thesis, University of Oxford, 1997.
[37] J.-L. Verdier, *Stratifications de Whitney et thórème de Bertini–Sard*, Inventiones Math., **36** (1976), 295–312. DOI: 10.1007/BF01390015
[38] S. Yokura, *On a Verdier-type Riemann-Roch for Chern-Schwartz-MacPherson class*, Topology and its Appl. **94** (1999), 315–327.
[39] S. Yokura, *On the Uniqueness Problem of Bivariant Chern Classes*, Documenta Math.**7** (2002) 133–142.
[40] S. Yokura, *Bivariant Theories of Constructible Functions and Grothendieck Transformations*, Topology and Its Applications, Vol. **123** (2002), 283–296.
[41] S. Yokura, *Motivic characteristic classes*, in "Topology of Stratified Spaces", MSRI Publications **58**, Cambridge Univ. Press (2010), 375–418.
[42] S.Yokura, *Genera and characteristic classes of singular varieties*, Oberwolfach Reports OWR No. 56/2011 (Workshop "Stratified Spaces: Joining Analysis, Topology and Geometry", Mathematisches Forschungsinstitut Oberwolfach, December 12-16, 2011), 59 - 62.

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ON REGULARITY CONDITIONS AT INFINITY

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Abstract. Let $f\colon X \to \mathbb{K}^p$ be a restriction of a polynomial mapping on X, where $X \subset \mathbb{K}^n$ is a smooth affine variety. We prove the equivalence of regularity conditions at infinity, which are useful to control the bifurcation set of f.

1. Introduction

Let $f\colon X \to \mathbb{K}^p$ be a differentiable mapping, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, X is a smooth affine variety and $\dim X \geq p$. The *bifurcation set of* f, denoted by $B(f)$, is the smallest subset of $\mathbb{K}^p$ such that f is a locally trivial topological fibration on $\mathbb{K}^p \setminus B(f)$.

The elements of $B(f)$ may come from critical values but also from regular values of f, i.e., $B(f) \setminus (B(f) \cap f(\text{Sing} f))$ can be not empty. In the example $f\colon \mathbb{K}^2 \to \mathbb{K}$, $f(x,y) = x + x^2 y$, the value $0 \in \mathbb{K}$ is not critical but there is no trivial fibration on any neighborhood of 0.

The study of bifurcation set $B(f)$ has connections with many other topics such as problems of optimization of polynomial functions $f\colon \mathbb{R}^n \to \mathbb{R}$ (see e.g. [HP]), generalizations of Ehresmann's Theorem (see e.g. [Ga, Je3, Ra]), Jacobian Conjecture (see e.g. [LW, ST]), global Łojasiewicz exponents (see e.g. [PZ, DG]), equisingularity and Milnor numbers (see e.g. [Ga, Pa1, ST, Ti2, Ti3]), stratification theory (see e.g. [KOS, Ti1]), etc...

A complete characterization of $B(f) \setminus (B(f) \cap f(\text{Sing} f))$ is yet an open problem. In fact, a characterization of $B(f) \setminus (B(f) \cap f(\text{Sing} f))$ is available only for polynomial functions $f\colon \mathbb{K}^2 \to \mathbb{K}$, see [Su, HL] for $\mathbb{K} = \mathbb{C}$ and [TZ] for $\mathbb{K} = \mathbb{R}$.

Through the use of *regularity conditions at infinity*, one has obtained some ways to approximate $B(f)$. For polynomial functions $f\colon \mathbb{K}^n \to \mathbb{K}$, see for instance [Br, CT, NZ, Pa1, Pa2, PZ, ST, Ti2, Ti3, Ti4].

For mappings, i.e., $p \geq 1$, Rabier [Ra] considered a regularity condition, which we call here *Rabier condition*. From this condition, Rabier defined the set of *asymptotic critical values* $K_\infty(f)$ and proved that $B(f) \subset (f(\text{Sing} f) \cup K_\infty(f))$. In fact, Rabier's results apply to C^2 maps $f\colon M \to N$, where M, N are Finsler manifolds.

For polynomial mappings $f\colon \mathbb{C}^n \to \mathbb{C}^p$, Gaffney [Ga] defined the *generalized Malgrange condition*, which we call here *Gaffney condition*. This condition yields the set $A_{G_\infty}(f)$ of non-regular values at infinity and, under additional hypothesis on f, Gaffney obtained

$$B(f) \subset (f(\text{Sing} f) \cup A_{G_\infty}(f)).$$

Kurdyka, Orro and Simon [KOS] also considered Rabier condition. They obtained an equivalence between Rabier condition and another condition which depends on *Kuo function*([Kuo]) (we call this last of *Kuo-KOS condition*). They showed that, for C^2 semi-algebraic mappings $f\colon \mathbb{R}^n \to \mathbb{R}^p$ (respectively, polynomial mappings $f\colon \mathbb{C}^n \to \mathbb{C}^p$), the set $K_\infty(f)$ is a closed semi-algebraic set (respectively, a closed algebraic set) of dimension at most $p-1$.

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Jelonek [Je3] used another condition, which turns out to be equivalent to Rabier condition and to Gaffney condition. We call that condition *Jelonek condition*. Then, Jelonek [Je3] gave a more direct proof of the inclusion $B(f) \subset (f(\mathrm{Sing} f) \cup K_\infty(f))$.

The above four conditions are asymptotic conditions, which depend on the behaviours of the fibres of f and Jacobian matrix of f.

Another regularity condition at infinity is the *t-regularity*, a geometric grounded condition at infinity. The t-regularity has been introduced in [ST] for polynomial functions $f\colon \mathbb{C}^n \to \mathbb{C}$ and in [Ti3] for polynomial functions $f\colon \mathbb{R}^n \to \mathbb{R}$.

In [DRT], we considered the t-regularity for C^1 semi-algebraic mappings $f\colon \mathbb{R}^n \to \mathbb{R}^p$ and we proved that t-regularity is equivalent to the conditions of [Ra, KOS] (consequently, equivalent to the conditions of [Ga, Je3]).

In this paper, we extend the use of t-regularity to algebraic mappings $f\colon X \to \mathbb{K}^p$ and we replace $\mathbb{K}^n$ in the above results by a smooth affine variety X.

In section 4, we prove that t-regularity is equivalent to Rabier condition for $f\colon X \to \mathbb{K}^p$ (Theorem 4.1). This extends for mappings defined on X the equivalence proved in [DRT, Theorem 3.2] and the equivalence proved for $p = 1$ in [Pa2, ST].

It follows from Jelonek [Je4] that Rabier, Gaffney, Kuo-KOS and Jelonek conditions are also equivalent for mappings defined on X. Therefore, our Theorem 4.1 completes for these mappings the equivalences above mentioned in the case of mappings $f\colon \mathbb{K}^n \to \mathbb{K}^p$.

Another important set in the study of polynomial mappings is the set J_f of points at which f is not proper (see e.g. [Je1, Je2]). It was proved in [KOS, Proposition 3.1] that in the case of semi-algebraic maps $f\colon \mathbb{R}^n \to \mathbb{R}^n$, the set J_f coincides with $K_\infty(f)$. This equality is crucial in the proof of the injectivity criterion of [CDTT, CDT].

In section 5, we consider $f\colon X \to \mathbb{R}^p$, where $\dim X = p$. We prove (Proposition 5.3) that $K_\infty(f) = J_f$, which extends for mappings defined on X the equality proved in [KOS, Proposition 3.1].

2. Basic Definitions

The goal of this section is to present Lemma 2.1, which will be useful to compute the Rabier function. We also introduce here some notations.

Let V, W be normed finite dimensional vector spaces over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}, \mathbb{C}$. We denote by $\mathcal{L}(V, W)$ the set of linear mappings from V to W. For simplicity, we denote $\mathcal{L}(V, \mathbb{K})$ by V^*. Given $A \in \mathcal{L}(V, W)$, we denote by $A^* \in \mathcal{L}(W^*, V^*)$ the adjoint operator induced by A. For any linear subspace V of $\mathbb{K}^n$, we set

$$V^\perp := \{w \in \mathbb{K}^n \mid \langle w, v\rangle = 0, \forall v \in V\}.$$

We consider the following norm on $\mathcal{L}(V, W)$:

$$\|A\| := \max\{\|A(x)\|;\, x \in V \text{ and } \|x\| = 1\}, \text{ where } A \in \mathcal{L}(V, W). \tag{1}$$

We denote by e_i the vector of $\mathbb{K}^n$ with 1 in the i-th coordinate and zeros elsewhere. Let $A \in \mathcal{L}(\mathbb{K}^n, \mathbb{K})$, we denote by $\|(A(e_1), \dots, A(e_n))\|$ the Euclidean norm of the vector

$$(A(e_1), \dots, A(e_n)) \in \mathbb{K}^n.$$

Another norm on $\mathcal{L}(\mathbb{K}^n, \mathbb{K})$ can be defined as follows:

$$\|A\|_1 := \|(A(e_1), \dots, A(e_n))\|. \tag{2}$$

It is well known that norms (1) and (2) of $\mathcal{L}(\mathbb{K}^n, \mathbb{K})$ are equivalents (see e.g. [Yo, Theorem 6.8]). The next lemma will be useful in the sequel:

Lemma 2.1. *Let $V \subset \mathbb{K}^n$ be a linear subspace of $\mathbb{K}^n$. Given $A \in \mathcal{L}(\mathbb{K}^n, \mathbb{K})$, we denote by $A_{|V}$ the restriction of A to V and we set:*

$$\|A_{|V}\|_3 := \min\{\|(A(e_1), \dots, A(e_n)) + w\|; w \in V^\perp\}. \tag{3}$$

Then, the norms (1) and (3) of $A_{|V}$ are equivalent (indeed, one has $\|A_{|V}\|_3 = \|A_{|V}\|$).

Proof. Let $A \in \mathcal{L}(\mathbb{K}^n, \mathbb{K})$. For any vector $w \in V^\perp$ and $v = (v_1, \dots, v_n) \in V$, we may write $A(v) = \sum_{i=1}^n v_i A(e_i) = \langle v, (A(e_1), \dots, A(e_n))\rangle = \langle v, (A(e_1), \dots, A(e_n)) + w\rangle$, where the last equality follows from the fact that $w \in V^\perp$. These equalities and Cauchy-Schwarz inequality imply:

$$\|A(v)\| = \|\langle v, (A(e_1), \dots, A(e_n)) + w\rangle\| \le \|v\|\|(A(e_1), \dots, A(e_n)) + w\|, \tag{4}$$

If $\|v\| = 1$, the inequality (4) gives $\|A(v)\| \le \|(A(e_1), \dots, A(e_n)) + w\|$. Since v, w are arbitrary elements, this last inequality implies:

$$\|A_{|V}\| \le \|A_{|V}\|_3. \tag{5}$$

To show $\|A_{|V}\|_3 \le \|A_{|V}\|$, we write $(A(e_1), \dots, A(e_n)) = \mathrm{v}_1 + \mathrm{w}_1$, with $\mathrm{v}_1 \in V$ and $\mathrm{w}_1 \in V^\perp$ (this is possible since $\mathbb{K}^n = V \oplus V^\perp$). Then, for any $v \in V$, one obtains

$$A(v) = \langle v, (A(e_1), \dots, A(e_n))\rangle = \langle v, \mathrm{v}_1 + \mathrm{w}_1\rangle = \langle v, \mathrm{v}_1\rangle,$$

where the last equality follows from the fact that $\mathrm{w}_1 \in V^\perp$.

If $\mathrm{v}_1 = 0$ then $A_{|V} \equiv 0$ and $(A(e_1), \dots, A(e_n)) = \mathrm{w}_1$, which implies $\|A_{|V}\| = 0$ and $\|A_{|V}\|_1 = 0$. Therefore, the inequality $\|A_{|V}\|_3 \le \|A_{|V}\|$ holds if $\mathrm{v}_1 = 0$.

If $\mathrm{v}_1 \neq 0$, we set $z := \frac{\mathrm{v}_1}{\|\mathrm{v}_1\|}$. Thus, $z \in V$, $\|z\| = 1$ and $A(z) = \langle z, \mathrm{v}_1\rangle = \|\mathrm{v}_1\|$, where the last equality follows from definition of z. Since $\|z\| = 1$, one has $\|A(z)\| = \|\mathrm{v}_1\| \le \|A_{|V}\|$.

To finish, we observe that $(A(e_1), \dots, A(e_n)) - \mathrm{w}_1 = \mathrm{v}_1$, with $\mathrm{w}_1 \in V^\perp$. By definition of $\|A_{|V}\|_3$, this last equality implies $\|A_{|V}\|_3 \le \|\mathrm{v}_1\|$. Thus, we conclude $\|A_{|V}\|_3 \le \|\mathrm{v}_1\| \le \|A_{|V}\|$, which follows $\|A_{|V}\|_3 \le \|A_{|V}\|$. Therefore, from this last inequality and inequality (5), we obtain $\|A_{|V}\| = \|A_{|V}\|_3$, which finishes the proof. □

3. Regularity conditions for mappings

We introduce the main definitions leading to the notion of t-regularity and we define Rabier condition in §3.3.

3.1. t-regularity.

Let $\mathcal{X} \subset \mathbb{K}^m$ be a $\mathbb{K}$-analytic variety, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We denote the set of regular points of $\mathcal{X}$ by $\mathcal{X}_{\text{reg}}$ and the set of singular points of $\mathcal{X}$ by $\mathcal{X}_{\text{sing}}$. We assume that $\mathcal{X}$ contains at least a regular point.

Definition 3.1. Let $g : \mathcal{X} \to \mathbb{K}$ be an analytic function defined in some neighbourhood of $\mathcal{X}$ in $\mathbb{K}^m$. Let $\mathcal{X}_0$ denote the subset of $\mathcal{X}_{\text{reg}}$ where g is a submersion. The *relative conormal space of* g is defined as follows:

$$C_g(\mathcal{X}) := \text{closure}\{(x, H) \in \mathcal{X}_0 \times \check{\mathbb{P}}^{m-1} \mid T_x(g^{-1}(g(x))) \subset H\} \subset \overline{\mathcal{X}} \times \check{\mathbb{P}}^{m-1}.$$

We denote by $\pi : C_g(\mathcal{X}) \to \overline{\mathcal{X}}$ the projection $\pi(x, H) = x$.

For any $y \in \overline{\mathcal{X}}$ such that $g(y) = 0$, we define $C_{g,y}(\mathcal{X}) := \pi^{-1}(y)$. The following result shows that $C_{g,y}(\mathcal{X})$ depends on the germ of g at y only up to multiplication by some invertible analytic function germ γ.

Lemma 3.2 ([Ti4, Lemma 1.2.7])**.** *Let $\gamma : (\mathbb{K}^m, y) \to \mathbb{K}$ be an analytic function such that $\gamma(y) \neq 0$. Then $C_{\gamma g,y}(\mathcal{X}) = C_{g,y}(\mathcal{X})$.* □

We use coordinates $(x_1, \dots, x_n)$ for $\mathbb{K}^n$ and coordinates $[x_0 : x_1 : \dots : x_n]$ for the projective space $\mathbb{P}^n$. We denote by $\mathbb{H}^\infty = \{[x_0 : x_1 : \dots : x_n] \in \mathbb{P}^n \mid x_0 = 0\}$ the hyperplane at infinity.

Let $f : X \to \mathbb{K}^p$ be the restriction of a polynomial mapping to a smooth affine variety $X \subset \mathbb{K}^n$, where $\dim X \geq p$. We set $\mathbb{X} := \overline{\text{graph} f}$ as the closure of the graph of f in $\mathbb{P}^n \times \mathbb{K}^p$ and we set $\mathbb{X}^\infty := \mathbb{X} \cap (\mathbb{H}^\infty \times \mathbb{K}^p)$.

We consider the affine charts $U_j \times \mathbb{K}^p$ of $\mathbb{P}^n \times \mathbb{K}^p$, where $U_j = \{x_j \neq 0\}$ and $j = 0, 1, \dots, n$. We identify the chart U_0 with the affine space $\mathbb{K}^n$. Thus, we have $\mathbb{X} \cap (U_0 \times \mathbb{K}^p) = \mathbb{X} \setminus \mathbb{X}^\infty = \text{graph} f$ and $\mathbb{X}^\infty$ is covered by the charts $U_1 \times \mathbb{K}^p, \dots, U_n \times \mathbb{K}^p$.

If g denotes the projection to the variable x_0 in some affine chart $U_j \times \mathbb{K}^p$, then the relative conormal $C_g(\mathbb{X} \backslash \mathbb{X}^\infty \cap U_j \times \mathbb{K}^p) \subset \mathbb{X} \times \check{\mathbb{P}}^{n+p-1}$ and the projection $\pi : C_g(\mathbb{X} \backslash \mathbb{X}^\infty \cap U_j \times \mathbb{K}^p) \to \mathbb{X}$, $\pi(y, H) = y$, are well-defined.

Let us then consider the space $\pi^{-1}(\mathbb{X}^\infty)$, which is well-defined for every chart $U_j \times \mathbb{K}^p$ as a subset of $C_g(\mathbb{X} \backslash \mathbb{X}^\infty \cap U_j \times \mathbb{K}^p)$. By Lemma 3.2, the definitions coincide at the intersections of the charts and one has:

Definition 3.3. The *space of characteristic covectors at infinity* is the well-defined set

$$\mathcal{C}^\infty := \pi^{-1}(\mathbb{X}^\infty).$$

For any $z_0 \in \mathbb{X}^\infty$, we denote $\mathcal{C}^\infty_{z_0} := \pi^{-1}(z_0)$.

We denote by $\tau : \mathbb{P}^n \times \mathbb{K}^p \to \mathbb{K}^p$ the second projection. The relative conormal space $C_\tau(\mathbb{P}^n \times \mathbb{K}^p)$ is defined as in Definition 3.1, where the function g is replaced by the application τ.

Definition 3.4 (t-regularity)**.** We say that f is *t-regular* at $z_0 \in \mathbb{X}^\infty$ if $C_\tau(\mathbb{P}^n \times \mathbb{K}^p) \cap \mathcal{C}^\infty_{z_0} = \emptyset$.

3.2. ***t*-regularity interpretation.** Let $X \subset \mathbb{K}^n$ be a smooth affine variety over $\mathbb{K}$. We suppose that X is a global complete intersection. In other words,

$$X = \{x \in \mathbb{K}^n \mid h_1(x) = h_2(x) = \dots = h_r(x) = 0\}$$

and $\operatorname{rank} \mathrm{D}h(x) = r$, where $h = (h_1, \dots, h_r) : \mathbb{K}^n \to \mathbb{K}^r$ and $\mathrm{D}h(x)$ denotes the Jacobian matrix of h at x.

Let $f = (f_1, \dots, f_p) \colon X \to \mathbb{K}^p$ be the restriction of a polynomial mapping to X, where $\dim X \geq p$. Given $z_0 \in \mathbb{X}^\infty$, up to some linear change of coordinate, we may assume that $z_0 \in \mathbb{X}^\infty \cap (U_n \times \mathbb{K}^p)$. In the intersection of charts $(U_0 \cap U_n) \times \mathbb{K}^p$, we consider the change of coordinates $x_1 = y_1/y_0, \dots, x_{n-1} = y_{n-1}/y_0$, $x_n = 1/y_0$, where $(x_1, \dots, x_n)$ are the coordinates in U_0 and $(y_0, \dots, y_{n-1})$ are those in U_n. Then for $i = 1, \dots, p$ and $j = 1, \dots, r$, we define:

$$F_i(y,t) = F_i(y_0, y_1, \dots, y_{n-1}, t_1, \dots, t_p) := f_i\left(y_1/y_0, \dots, y_{n-1}/y_0, 1/y_0\right) - t_i, \tag{6}$$

$$H_j(y,t) = H_j(y_0, y_1, \dots, y_{n-1}, t_1, \dots, t_p) := h_j(y_1/y_0, \dots, y_{n-1}/y_0, 1/y_0). \tag{7}$$

Define $H(y,t) := (H_1(y,t), \dots, H_r(y,t))$ and $F(y,t) := (F_1(y,t), \dots, F_p(y,t))$. Then

$$(X \times \mathbb{K}^p) \cap ((U_0 \cap U_n) \times \mathbb{K}^p) = H^{-1}(0)$$

and $\mathbb{X} \cap ((U_0 \cap U_n) \times \mathbb{K}^p) = F^{-1}(0) \cap H^{-1}(0)$.

We denote the normal vector to the hypersurface $\{y_0 = \text{constant}\}$ by

$$\vec{n_0} = (1, 0, \dots, 0) \in \mathbb{K}^n \times \mathbb{K}^p.$$

Let us define $p + r$ normal vectors to $F^{-1}(0)$ at $(y,t) \in \mathbb{X} \cap ((U_0 \cap U_n) \times \mathbb{K}^p)$, as follows:
For $i = 1, \dots, p$, define:

$$\vec{n_i}(y,t) = \nabla F_i(y,t) = (\nabla_n F_i(y,t), \nabla_p F_i(y,t)), \tag{8}$$

where

$$\nabla_n F_i(y,t) := \left(\frac{\partial F_i}{\partial y_0}(y,t), \cdots, \frac{\partial F_i}{\partial y_{n-1}}(y,t)\right), \quad \nabla_p F_i(y,t) := \left(\frac{\partial F_i}{\partial t_1}(y,t), \cdots, \frac{\partial F_i}{\partial t_p}(y,t)\right).$$

For $j = 1, \ldots, r$, define:

$$\vec{m}_j(y,t) = \nabla H_j(y,t) = \left(\frac{\partial H_j}{\partial y_0}(y,t), \ldots, \frac{\partial H_j}{\partial y_{n-1}}(y,t), 0, \ldots, 0\right). \tag{9}$$

By Definition 3.4, f is not t-regular at $z_0 \in \mathbb{X}^\infty$ if and only if there exists a sequence $\{(y_k, t_k)\}_{k\in\mathbb{N}} \subset \mathbb{X} \cap ((U_0 \cap U_n) \times \mathbb{K}^p)$ such that $(y_k, t_k) \to z_0$ and the tangent hyperplanes to the fibres of $g_{|\mathbb{X}}$ at (y_k, t_k) tend to a hyperplane W such that its normal line has a direction of the form $[0 : \cdots : 0 : b_1 : \cdots : b_p]$ in $\mathbb{P}^{n+p-1}$. More explicitly, there exists a sequence $\{(\psi_{0k}, \psi_{1k}, \ldots, \psi_{p_k}, \varphi_{1k}, \ldots, \varphi_{r_k})\}_{k\in\mathbb{N}} \subset \mathbb{K}^{p+r+1}$ such that

$$\lim_{k\to\infty}\left(\sum_{i=0}^{p} \psi_{ik}\vec{n}_i(y_k, t_k) + \sum_{j=1}^{r} \varphi_{jk}\vec{m}_j(y_k, t_k)\right)$$

of the linear combination of normal vectors $\vec{n}_i, \vec{m}_j$ has the direction

$$\vec{n}_W = [0 : 0 : \cdots : 0 : b_1 : \cdots : b_p] \in \mathbb{P}^{n+p-1}.$$

3.3. **Rabier function and Rabier condition.**

Definition 3.5 ([Ra, p. 651])**.** Given $A \in \mathcal{L}(V, W)$. The *Rabier function at* A is defined as follows:

$$\nu(A) := \inf\{\|A^*(\varphi)\|; \varphi \in W^* \text{ and } \|\varphi\| = 1\}. \tag{10}$$

For any vector $w = (w_1, \ldots, w_m) \in \mathbb{K}^m$, we denote the line matrix associated to w by $[w]$, i.e., $[w] = \begin{bmatrix} w_1 & \ldots & w_m \end{bmatrix}$. Given $A \in \mathcal{L}(\mathbb{K}^n, \mathbb{K}^p)$, we denote by $[A]$ the matrix of A with respect to the canonical basis of $\mathbb{K}^n$ and $\mathbb{K}^p$. Thus, one has:

Lemma 3.6. *Let V be a linear subspace of $\mathbb{K}^n$. For any $A \in \mathcal{L}(\mathbb{K}^n, \mathbb{K}^p)$, if we set*

$$\nu_1(A_{|V}) := \inf\{\, \| [u][A] + [w] \| \,; w \in V^\perp, u \in \mathbb{K}^p \text{ and } \|u\| = 1\}, \tag{11}$$

then there are positive constants C_1 and C_2 such that $C_1\nu_1(A_{|V}) \le \nu(A_{|V}) \le C_2\nu_1(A_{|V})$.

Proof. The proof follows from Lemma 2.1 and Definition 3.5. □

Now, let $X \subset \mathbb{K}^n$ be a smooth affine variety over $\mathbb{K}$ and let $f : X \to \mathbb{K}^p$ be the restriction of a polynomial mapping to X, where $\dim X \ge p$. We have:

Definition 3.7 ([Ra])**.** The set of *asymptotic critical values of* f is defined as follows:

$$\begin{aligned} K_\infty(f) \quad := \quad & \{t \in \mathbb{K}^p \mid \exists\{x_j\}_{j\in\mathbb{N}} \subset X, \lim_{j\to\infty} \|x_j\| = \infty, \\ & \lim_{j\to\infty} f(x_j) = t \text{ and } \lim_{j\to\infty} \|x_j\|\nu(\mathrm{D}f(x_j)_{|T_{x_j}X}) = 0\}, \end{aligned} \tag{12}$$

where $\nu(-)$ is defined as in Definition 3.5.

We reformulate the above condition in a localized version, at some point at infinity $z_0 \in \mathbb{X}^\infty$, as follows:

Definition 3.8 (Rabier condition)**.** We say that $z_0 \in \mathbb{X}^\infty$ is an *asymptotic critical point of* f if and only if there exists $\{x_j\}_{j\in\mathbb{N}} \subset X \simeq \text{graph} f$ such that $\lim_{j\to\infty}(x_j, f(x_j)) = z_0$ and $\tau(z_0) \in K_\infty(f)$, where $\tau \colon \mathbb{P}^n \times \mathbb{K}^p \to \mathbb{K}^p$ denotes the second projection.

We say that $z_0 \in \mathbb{X}^\infty$ satisfies *Rabier condition* if z_0 is not an asymptotic critical point of f.

Remark 3.9. From Lemma 3.6, we obtain the same set of Definition 3.7 if we replace ν by the function ν_1 defined in (11).

4. Equivalence of regularity conditions

The goal of this section is to prove an equivalence between t-regularity and Rabier condition.

Let $X \subset \mathbb{K}^n$ be a smooth affine variety over $\mathbb{K}$. We suppose that X is a global complete intersection. In other words, $X = \{x \in \mathbb{K}^n \mid h_1(x) = h_2(x) = \ldots = h_r(x) = 0\}$ and $\operatorname{rank} \mathrm{D}h(x) = r$, for any $x \in X$, where $h = (h_1, \ldots, h_r) : \mathbb{K}^n \to \mathbb{K}^r$ and $\mathrm{D}h(x)$ denotes the Jacobian matrix of h at x (see Remark 4.2). With above definitions and statements, we have:

Theorem 4.1. *Let $f : X \to \mathbb{K}^p$ be a non-constant polynomial mapping, with $\dim X \geq p$. Let $z_0 \in \mathbb{X}^\infty$. Then f is t-regular at z_0 if and only if z_0 is not an asymptotic critical point of f.*

Proof. We may assume (eventually after some linear change of coordinates) that

$$z_0 \in \mathbb{X}^\infty \cap (U_n \times \mathbb{R}^p)$$

and that $|x_n| \geq |x_i|$, $i = 1, \ldots, n-1$, for x in some neighbourhood of z_0.

"$\Rightarrow$". Let z_0 be an asymptotic critical point of f. By Definition 3.8 and Remark 3.9, this means that there exist sequences $\{(\psi_k, \varphi_k) = ((\psi_{1k}, \ldots, \psi_{pk}), (\varphi_{1k}, \ldots, \varphi_{r_k}))\}_{k\in\mathbb{N}} \subset \mathbb{K}^{p+r}$ and $\{x_k := (x_{1k}, \ldots, x_{nk})\}_{k\in\mathbb{N}} \subset X$, where $\|\psi_k\| = 1$ and $\lim_{k\to\infty}(\psi_k, \varphi_k) = (\psi, \varphi)$, such that $\lim_{k\to\infty} \psi_k = \psi = (\psi_1, \ldots, \psi_p) \neq (0, \ldots, 0)$, $\lim_{k\to\infty}(x_k, f(x_k)) = z_0$ and:

$$\|x_k\| \left\| \left(\sum_{i=1}^{p} \psi_{ik} \frac{\partial f_i}{\partial x_1}(x_k) + \sum_{j=1}^{r} \varphi_{jk} \frac{\partial h_j}{\partial x_1}(x_k), \ldots, \sum_{i=1}^{p} \psi_{ik} \frac{\partial f_i}{\partial x_n}(x_k) + \sum_{j=1}^{r} \psi_{jk} \frac{\partial h_j}{\partial x_n}(x_k) \right) \right\| \to 0. \tag{13}$$

Since for large enough k we have $|x_{nk}| \geq |x_{ik}|$, $i = 1, \ldots, n-1$, we may replace in (13) $\|x_k\|$ by $|x_{nk}|$ and then multiply the sums of (13) by x_{nk}.

In the notations of §3.2, by changing coordinates within $U_0 \cap U_n$, one has $y_0 = 1/x_n, y_i = x_i/x_n$ and the relations:

$$\begin{cases} \frac{\partial F_j}{\partial y_i}(y,t) = x_n \frac{\partial f_j}{\partial x_i}(x), & 1 \leq i \leq n-1,\ 1 \leq j \leq p, \\ \frac{\partial F_j}{\partial t_l}(y,t) = -\delta_{l,j}, & 1 \leq j, l \leq p, \\ \frac{\partial F_j}{\partial y_0}(y,t) = -x_n(x_1 \frac{\partial f_j}{\partial x_1}(x) + \ldots + x_n \frac{\partial f_j}{\partial x_n}(x)), & 1 \leq j \leq p. \end{cases} \tag{14}$$

$$\begin{cases} \frac{\partial H_j}{\partial y_i}(y,t) = x_n \frac{\partial h_j}{\partial x_i}(x), & 1 \leq i \leq n-1,\ 1 \leq j \leq r, \\ \frac{\partial H_j}{\partial t_l}(y,t) = 0, & 1 \leq j \leq r,\ 1 \leq l \leq p, \\ \frac{\partial H_j}{\partial y_0}(y,t) = -x_n(x_1 \frac{\partial h_j}{\partial x_1}(x) + \ldots + x_n \frac{\partial h_j}{\partial x_n}(x)), & 1 \leq j \leq r. \end{cases} \tag{15}$$

The condition (13) yields:

$$\left\| \left(\left(\sum_{i=1}^{p} \psi_{ik} \frac{\partial F_i}{\partial y_1} + \sum_{j=1}^{r} \varphi_{jk} \frac{\partial H_j}{\partial y_1} \right)(y_k, t_k), \ldots, \left(\sum_{i=1}^{p} \psi_{ik} \frac{\partial F_i}{\partial y_{n-1}} + \sum_{j=1}^{r} \varphi_{jk} \frac{\partial H_j}{\partial y_{n-1}} \right)(y_k, t_k) \right) \right\| \to 0. \tag{16}$$

We set $\vec{n}_{W_k} := (0, \omega_k, -\psi_{1k}, \ldots, -\psi_{pk})$, where ω_k is the vector of equation (16). Let W_k be the hyperplane defined by $\vec{n}_{W_k}$. Let $\vec{n}_i$ and $\vec{m}_j$ be the vectors defined in §3.2. Then, the vectors

$\{\vec{n}_{W_k}\}$ are linear combinations of $\vec{n}_i$ and $\vec{m}_j$ with coefficients $\{\psi_{ik}, \varphi_{jk}\}$, and the hyperplanes W_k are tangent to the levels of the function $g_{|\mathbb{X}}$. Since we have supposed

$$\lim_{k\to\infty}(\psi_{1k},\ldots,\psi_{pk}) = (\psi_1,\ldots,\psi_p) \neq (0,\ldots,0),$$

it follows from definition of $\vec{n}_{W_k}$ and equation (16) that:

$$\lim_{k\to\infty}\vec{n}_{W_k} = [0:0:\ldots:0:\psi_1:\ldots:\psi_p].$$

Denote by W the hyperplane defined by $[0:0:\ldots:0:\psi_1:\ldots:\psi_p]$. Then $W = \lim_{k\to\infty} W_k$, which implies that W belongs to $\mathcal{C}^{\infty}_{z_0}$ and consequently f is not t-regular at z_0 (see §3.2).

"$\Leftarrow$". Let $z_0 \in \mathbb{X}^\infty$ be not t-regular. By Definition 3.4, this means that there exist a sequence of points $\{(\mathrm{y}_k,\mathrm{t}_k)\}_{k\in\mathbb{N}} \subset \mathbb{X}\cap((U_0\cap U_n)\times\mathbb{K}^p)$ tending to z_0, and a sequence of hyperplanes W_k tangent to the levels of g at $(\mathrm{y}_k,\mathrm{t}_k)$, such that $W_k \to W \in \mathcal{C}^{\infty}_{z_0}$.

Let $\vec{n}_i$ and $\vec{m}_j$ be the vectors defined in §3.2. From §3.2, if f is not t-regular at z_0 then there exist sequences $\{\tilde{\psi}_k = (\tilde{\psi}_{1k},\ldots,\tilde{\psi}_{pk})\}_{k\in\mathbb{N}} \subset \mathbb{K}^p$, $\{\tilde{\varphi}_k = (\tilde{\varphi}_{1k},\ldots,\tilde{\varphi}_{rk})\}_{k\in\mathbb{N}} \subset \mathbb{K}^r$ and $\{\lambda_k\}_{k\in\mathbb{N}} \subset \mathbb{K}$ such that $\vec{n}_{W_k} = \lambda_k\vec{n}_0(\mathrm{y}_k,\mathrm{t}_k) + \sum_i \tilde{\psi}_{ik}\vec{n}_i(\mathrm{y}_k,\mathrm{t}_k) + \sum_j \tilde{\varphi}_{jk}\vec{m}_j(\mathrm{y}_k,\mathrm{t}_k)$ and that $\lim_{k\to\infty}\vec{n}_{W_k} = \left[0:0:\ldots:0:\tilde{\psi}_1:\ldots:\tilde{\psi}_p\right]$, where $\left(\tilde{\psi}_1,\ldots,\tilde{\psi}_p\right) \neq (0,\ldots,0)$. By assumption, the vector $\vec{n}_{W_k}$ has the following expression:

(a) In the first coordinate of $\vec{n}_{W_k}$ one has: $\lambda_k + \left(\sum_{i=1}^{p}\tilde{\psi}_{ik}\frac{\partial F_i}{\partial y_0} + \sum_{j=1}^{r}\tilde{\varphi}_{jk}\frac{\partial H_i}{\partial y_0}\right)(\mathrm{y}_k,\mathrm{t}_k)$.

(b) In the l-th coordinate, with $2 \le l \le n$, one has: $\left(\sum_{i=1}^{p}\tilde{\psi}_{ik}\frac{\partial F_i}{\partial y_l} + \sum_{j=1}^{r}\tilde{\varphi}_{jk}\frac{\partial H_j}{\partial y_l}\right)(\mathrm{y}_k,\mathrm{t}_k)$.

(c) In the q-th coordinate, with $n+1 \le q \le n+p$, one has: $-\tilde{\psi}_{qk}$.

We may take $\lambda_k := -\sum_{i=1}^{p}\tilde{\psi}_{ik}\frac{\partial F_i}{\partial y_0}(\mathrm{y}_k,\mathrm{t}_k) - \sum_{j=1}^{r}\tilde{\varphi}_{jk}\frac{\partial H_i}{\partial y_0}(\mathrm{y}_k,\mathrm{t}_k)$. After, we divide out by $\mu_k := \|(\tilde{\psi}_{1k},\ldots,\tilde{\psi}_{pk})\|$. Then, we replace $\tilde{\psi}_{ik}$ and $\tilde{\varphi}_{jk}$ by $\psi_{ik} := \frac{\tilde{\psi}_{ik}}{\mu_k}$ and $\varphi_{jk} := \frac{\tilde{\varphi}_{jk}}{\mu_k}$, respectively. This implies that $\|(\psi_{1k},\ldots,\psi_{pk})\| = 1$ and $\lim_{k\to\infty}\vec{n}_{W_k} = [0:\ldots:0:\psi_1:\ldots:\psi_p]$, where $(\psi_1,\ldots,\psi_p) \neq (0,\ldots,0)$. Therefore,

$$\lim_{k\to\infty}\sum_{i=1}^{p}\psi_{ik}\frac{\partial F_i}{\partial y_l}(\mathrm{y}_k,\mathrm{t}_k) + \sum_{j=1}^{r}\varphi_{jk}\frac{\partial H_j}{\partial y_l}(\mathrm{y}_k,\mathrm{t}_k) = 0, \text{ for any } 1 \le l \le n-1. \tag{17}$$

By using (14) and (15), this is equivalent to:

$$\lim_{k\to\infty} x_{nk}\left(\sum_{i=1}^{p}\psi_{ik}\frac{\partial f_i}{\partial x_l}(x_k) + \sum_{j=1}^{r}\varphi_{jk}\frac{\partial h_j}{\partial x_l}(x_k)\right) = 0, \tag{18}$$

for $1 \le l \le n-1$, and one has $|x_{nk}| \ge \frac{1}{\sqrt{n}}\|x_k\|$ for large enough k. Therefore, in order to get the limit (13) it remains to prove that (18) is true for $l = n$. The rest of our argument is devoted to this proof.

From relations (14) and (15), we obtain $x_n\frac{\partial f_i}{\partial x_n}(x) = -\sum_{j=0}^{n-1} y_j\frac{\partial F_i}{\partial y_j}(y,t)$ and

$$x_n\frac{\partial h_i}{\partial x_n}(x) = -\sum_{j=0}^{n-1} y_j\frac{\partial H_i}{\partial y_j}(y,t).$$

Therefore:

$$x_{nk}\sum_{i=1}^{p}\psi_{ik}\frac{\partial f_i}{\partial x_n}(x_k)=-\sum_{j=1}^{n-1}\sum_{i=1}^{p}y_{jk}\psi_{ik}\frac{\partial F_i}{\partial y_j}(\mathrm{y}_k,\mathrm{t}_k)-\sum_{i=1}^{p}\psi_{ik}y_{0k}\frac{\partial F_i}{\partial y_0}(\mathrm{y}_k,\mathrm{t}_k). \tag{19}$$

$$x_{nk}\sum_{i=1}^{r}\varphi_{ik}\frac{\partial h_i}{\partial x_n}(x_k)=-\sum_{j=1}^{n-1}\sum_{i=1}^{r}y_{jk}\varphi_{ik}\frac{\partial H_i}{\partial y_j}(\mathrm{y}_k,\mathrm{t}_k)-\sum_{i=1}^{r}\varphi_{ik}y_{0k}\frac{\partial H_i}{\partial y_0}(\mathrm{y}_k,\mathrm{t}_k). \tag{20}$$

We will show that the following two terms tend to zero:

$$\sum_{j=1}^{n-1}\sum_{i=1}^{p}y_{jk}\psi_{ik}\frac{\partial F_i}{\partial y_j}(\mathrm{y}_k,\mathrm{t}_k)+\sum_{j=1}^{n-1}\sum_{i=1}^{r}y_{jk}\varphi_{ik}\frac{\partial H_i}{\partial y_j}(\mathrm{y}_k,\mathrm{t}_k),\ \text{and} \tag{21}$$

$$\sum_{i=1}^{p}\psi_{ik}y_{0k}\frac{\partial F_i}{\partial y_0}(\mathrm{y}_k,\mathrm{t}_k)+\sum_{i=1}^{r}\varphi_{ik}y_{0k}\frac{\partial H_i}{\partial y_0}(\mathrm{y}_k,\mathrm{t}_k). \tag{22}$$

First, we have:

$$\left\|\sum_{j=1}^{n-1}\sum_{i=1}^{p}y_{jk}\psi_{ik}\frac{\partial F_i}{\partial y_j}(\mathrm{y}_k,\mathrm{t}_k)+\sum_{j=1}^{n-1}\sum_{i=1}^{r}y_{jk}\varphi_{ik}\frac{\partial H_i}{\partial y_j}(\mathrm{y}_k,\mathrm{t}_k)\right\|\leq \tag{23}$$

$$\left\|\frac{x_k}{x_{nk}}\right\|\left\|((\sum_{i=1}^{p}\psi_{ik}\frac{\partial F_i}{\partial y_1}+\sum_{i=1}^{r}\varphi_{ik}\frac{\partial H_i}{\partial y_1})(\mathrm{y}_k,\mathrm{t}_k),\dots,(\sum_{i=1}^{p}\psi_{ik}\frac{\partial F_i}{\partial y_{n-1}}+\sum_{i=1}^{r}\varphi_{ik}\frac{\partial H_i}{\partial y_{n-1}})(\mathrm{y}_k,\mathrm{t}_k))\right\|,$$

since by hypothesis $|y_{jk}|=|\frac{x_{jk}}{x_{nk}}|\leq 1$ for large enough k. Then we obtain from (17) that the right hand side of (23) tends to zero as $k\to\infty$, which shows that (21) tends to zero.

To show that (22) tends to zero, let us assume that the following inequality holds for large enough $k\gg 1$, the proof of which will be given below:

$$\left\|\sum_{i=1}^{p}\psi_{ik}y_{0k}\frac{\partial F_i}{\partial y_0}+\sum_{j=1}^{r}\varphi_{jk}y_{0k}\frac{\partial H_j}{\partial y_0}\right\|\ll \tag{24}$$

$$\left\|(\sum_{i=1}^{p}\psi_{ik}\frac{\partial F_i}{\partial y_1}+\sum_{j=1}^{r}\varphi_{jk}\frac{\partial H_j}{\partial y_1},\dots,\sum_{i=1}^{p}\psi_{ik}\frac{\partial F_i}{\partial y_{n-1}}+\sum_{j=1}^{r}\varphi_{jk}\frac{\partial H_j}{\partial y_{n-1}},\sum_{i=1}^{p}\psi_{ik}\frac{\partial F_i}{\partial t_1},\dots,\sum_{i=1}^{p}\psi_{ik}\frac{\partial F_i}{\partial t_p})\right\|.$$

Then, by using (17), (24) and the equality $\sum_{i=1}^{p}\psi_{ik}\frac{\partial F_i}{\partial t_l}=-\psi_{lk}$ for any $1\leq l\leq p$ (implied by (14)), we have:

$$\left\|\sum_{i=1}^{p}\psi_{ik}y_{0k}\frac{\partial F_i}{\partial y_0}+\sum_{j=1}^{r}\varphi_{jk}y_{0k}\frac{\partial H_j}{\partial y_0}\right\|\ll\|\psi_k\|=1.$$

This implies $\lim_{k\to\infty}\|(\sum_{i=1}^{p}\psi_{ik}y_{0k}\frac{\partial F_i}{\partial y_0}+\sum_{j=1}^{r}\varphi_{jk}y_{0k}\frac{\partial H_j}{\partial y_0})(\mathrm{y}_k,\mathrm{t}_k)\|=0$, which shows that (22) tends to zero as $k\to\infty$.

We have shown that (21) and (22) tend to zero as $k\to\infty$. From the equations (19) and (20), we have that the sum (21) + (22) is equal to equation of (18) with $l=n$. These imply that (18) is also true for $l=n$. This completes our proof of relation (13) showing that z_0 is an asymptotic critical point of f.

Let us now give the proof of (24). Suppose not; this means that there exists $\delta > 0$ such that for $k \gg 1$ we have:

(25)
$$\frac{\left\|\sum_{i=1}^{p}\psi_{ik}y_{0k}\frac{\partial F_i}{\partial y_0}+\sum_{j=1}^{r}\varphi_{jk}y_{0k}\frac{\partial H_j}{\partial y_0}\right\|}{\left\|(\sum_{i=1}^{p}\psi_{ik}\frac{\partial F_i}{\partial y_1}+\sum_{j=1}^{r}\varphi_{jk}\frac{\partial H_j}{\partial y_1},\dots,\sum_{i=1}^{p}\psi_{ik}\frac{\partial F_i}{\partial y_{n-1}}+\sum_{j=1}^{r}\varphi_{jk}\frac{\partial H_j}{\partial y_{n-1}},-\psi_{1k},\dots,-\psi_{pk})\right\|} > \delta,$$

where, by relations (14), we have $-\psi_{lk} = \sum_{i=1}^{p}\psi_{ik}\frac{\partial F_i}{\partial t_l}$, for $1 \le l \le p$. The set:

$$\mathcal{W} = \{((y,t),\psi,\varphi) \in ((U_n \cap U_0)\times \mathbb{K}^p \times \mathbb{K}^p \times \mathbb{K}^r) \cap (\mathbb{X} \times S_1^{p-1} \times \mathbb{K}^r) \mid (25) \text{ holds for } ((y,t),\psi,\varphi)\}$$

is a semi-algebraic set and we have $((\mathrm{y}_k,\mathrm{t}_k),\psi_k,\varphi_k) \in \mathcal{W}$ for $k \gg 1$. We observe that if $((y,t),\psi,\varphi) \in \mathcal{W}$ then $((y,t),\gamma\psi,\gamma\varphi) \in \mathcal{W}$, for any $\gamma \in \mathbb{K}^*$. This last observation implies that $((\mathrm{y}_k,\mathrm{t}_k),\tilde{\psi_k},\tilde{\varphi_k}) \in \mathcal{W}$, where $\tilde{\psi_k} := \frac{\psi_k}{\|(\psi_k,\varphi_k)\|}$ and $\tilde{\varphi_k} := \frac{\varphi_k}{\|(\psi_k,\varphi_k)\|}$.

Since $\lim_{k\to\infty}\psi_k \to \psi \neq 0$, one may suppose that $\lim_{k\to\infty}(\tilde{\psi_k},\tilde{\varphi_k}) \to (\tilde{\psi},\tilde{\varphi})$, with $(\tilde{\psi},\tilde{\varphi}) \neq 0$. Then $\lim_{k\to\infty}((\mathrm{y}_k,\mathrm{t}_k),\tilde{\psi_k},\tilde{\varphi_k}) = (z_0,\tilde{\psi},\tilde{\varphi})$ and by the curve selection lemma [Mi] there exists an analytic curve $\lambda = (\phi,\psi,\varphi) : [0,\varepsilon[\to \overline{\mathcal{W}}$ such that $\lambda(]0,\varepsilon[) \subset \mathcal{W}$ and $\lambda(0) = (z_0,\psi,\varphi)$. We denote

$$\phi(s) = (y_0(s),y_1(s),\dots,y_{n-1}(s),t_1(s),\dots,t_p(s)), \qquad \psi(s) = (\psi_1(s),\dots,\psi_p(s)), \text{ and}$$
$$\varphi(s) = (\varphi_1(s),\dots,\varphi_r(s)).$$

Since $(F,H)(\phi(s)) \equiv 0$, we have:

$$0 = \frac{d}{ds}(F,H)(\phi(s)) = y_0'(s)\frac{\partial(F,H)}{\partial y_0}(\phi(s)) + \sum_{i=1}^{n-1}y_i'(s)\frac{\partial(F,H)}{\partial y_i}(\phi(s)) + \sum_{i=1}^{p}t_i'(s)\frac{\partial(F,H)}{\partial t_i}(\phi(s)),$$

where $\frac{\partial(F,H)}{\partial y_i} = (\frac{\partial F_1}{\partial y_i},\dots,\frac{\partial F_p}{\partial y_i},\frac{\partial H_1}{\partial y_i},\dots,\frac{\partial H_r}{\partial y_i})$.

Multiplying by $(\psi(s),\varphi(s))$ we obtain:

$$(26) \quad -y_0'(s)\left(\left(\sum_{i=1}^{p}\psi_i(s)\frac{\partial F_i}{\partial y_0}+\sum_{j=1}^{r}\varphi_j\frac{\partial H_j}{\partial y_0}\right)(\phi(s))\right) =$$
$$\sum_{l=1}^{n-1}y_l'(s)\left(\left(\sum_{i=1}^{p}\psi_i(s)\frac{\partial F_i}{\partial y_l}+\sum_{j=1}^{r}\varphi_j\frac{\partial H_j}{\partial y_l}\right)(\phi(s))\right) + \sum_{l=1}^{p}t_l'(s)\sum_{i=1}^{p}\psi_i(s)\frac{\partial F_i}{\partial t_l}(\phi(s)).$$

Since ϕ is analytic, thus bounded at $s = 0$, by applying the Cauchy-Schwarz inequality one finds a constant $C > 0$ such that:

$$(27) \quad \left|y_0'(s)\left(\sum_{i=1}^{p}\psi_i(s)\frac{\partial F_i}{\partial y_0}+\sum_{j=1}^{r}\varphi_j\frac{\partial H_j}{\partial y_0}\right)(\phi(s))\right| \le$$
$$C\left\|\left((\sum_{i=1}^{p}\psi_i\frac{\partial F_i}{\partial y_1}+\sum_{j=1}^{r}\varphi_j\frac{\partial H_j}{\partial y_1})(\phi),\dots,(\sum_{i=1}^{p}\psi_i\frac{\partial F_i}{\partial y_{n-1}}+\sum_{j=1}^{r}\varphi_j\frac{\partial H_j}{\partial y_{n-1}})(\phi),\psi_1,\dots,\psi_p\right)(s)\right\|.$$

We have $l := \mathrm{ord}_s y_0'(s) \ge 0$ and $\mathrm{ord}_s y_0(s) = l+1 \ge 1$ since $y_0(0) = 0$. Thus $\left|y_0(s)(\sum_{i=1}^{p}\psi_i(s)\frac{\partial F_i}{\partial y_0}+\sum_{j=1}^{r}\psi_j\frac{\partial H_j}{\partial y_0})(\phi(s))\right| \ll \left|y_0'(s)(\sum_{i=1}^{p}\psi_i(s)\frac{\partial F_i}{\partial y_0}+\sum_{j=1}^{r}\psi_j\frac{\partial H_j}{\partial y_0})(\phi(s))\right|$.

This and (27) give:

$$\left\| y_0(s)(\sum_{i=1}^{p}\psi_i(s)\frac{\partial F_i}{\partial y_0}+\sum_{j=1}^{r}\psi_j\frac{\partial H_j}{\partial y_0})(\phi(s))\right\| \ll$$
$$\left\| \left((\sum_{i=1}^{p}\psi_i\frac{\partial F_i}{\partial y_1}+\sum_{j=1}^{r}\varphi_j\frac{\partial H_j}{\partial y_1})(\phi),\dots,(\sum_{i=1}^{p}\psi_i\frac{\partial F_i}{\partial y_{n-1}}+\sum_{j=1}^{r}\varphi_j\frac{\partial H_j}{\partial y_{n-1}})(\phi),\psi_1,\dots,\psi_p\right)(s)\right\|,$$

which contradicts our assumption that $(\phi(s),\psi(s),\varphi(s)) \in \mathcal{W}$, for $s \in]0,\varepsilon[$. Therefore, we conclude that (24) holds, which completes the proof of Theorem 4.1. □

The above theorem extends for mappings defined on X the equivalence proved in [DRT, Theorem 3.2]. It also extends an equivalence proved for $p = 1$ in [Pa2, ST].

Remark 4.2. In Theorem 4.1 we suppose that $X \subset \mathbb{K}^n$ is a complete intersection. It is well known that any manifold is a locally complete intersection (see e.g [GP, p. 18]). So, in the general case of a smooth affine variety X, one may take a locally finite cover $\{U_i\}$ of $\mathbb{K}^n$ such that the manifold $X_i := X \cap U_i$ is a complete intersection. Then we consider the normal vector fields on each X_i as in §3.2 and we use a partition of unity subordinate to the cover $\{U_i\}$ to obtain normal vector fields defined on X. Then the proof of Theorem 4.1 in the general case is the same as above.

5. t-regularity and Jelonek set

In this section, we consider $f\colon X \to \mathbb{R}^p$, where $\dim X = p$. We prove that, in this case, t-regularity is related with the Jelonek set J_f ([Je1]). We begin with:

Definition 5.1 ([Je1, Definition 3.3])**.** Let $f\colon M \to N$ be a continuous mapping, where M, N are manifolds. We say that f is proper at a point $t_0 \in N$ if there exists an open neighbourhood U of t_0 such that the restriction $f_{|f^{-1}(U)}\colon f^{-1}(U) \to U$ is a proper mapping. We denote by J_f the set of points at which f is not proper.

See for instance [Je1, Je2] for applications and related problems with J_f.

Definition 5.2. Let $f : X \to \mathbb{K}^p$ be the restriction of a polynomial mapping to a smooth variety X, where $\dim X \geq p$. We set

$$\mathcal{NT}_\infty(f) := \{t_0 = \tau(z_0) \in \mathbb{K}^p \mid z_0 \in \mathbb{X}^\infty \text{ and } z_0 \text{ is not } t\text{-regular}\}. \tag{28}$$

When $\dim X = p$, we have:

Proposition 5.3. *Let $X \subset \mathbb{R}^n$ be a smooth affine variety over $\mathbb{R}$. We suppose that X is a global complete intersection. In other words $X = \{x \in \mathbb{R}^n \mid h_1(x) = h_2(x) = \ldots = h_r(x) = 0\}$ and $\operatorname{rank} \mathrm{D}h(x) = r$, for any $x \in X$, where $h = (h_1, \ldots, h_r) : \mathbb{R}^n \to \mathbb{R}^r$ and $\mathrm{D}h(x)$ denotes the Jacobian matrix of h at x.*

Let $f = (f_1, \ldots, f_p) : X \to \mathbb{R}^p$ be the restriction of a polynomial mapping to X, where $\dim X = n - r = p$. Then $\mathcal{NT}_\infty(f) = K_\infty(f) = J_f$.

Proof. The equality $\mathcal{NT}_\infty(f) = K_\infty(f)$ follows directly from Theorem 4.1. Thus, we need only show the equality $K_\infty(f) = J_f$.

The inclusion $K_\infty(f) \subset J_f$ follows directly from Definitions 3.7 and 5.1. On the other hand, let $t_0 \in J_f$. By the curve selection lemma [Mi], there exists an analytic path

$$\phi = (\phi_1, \ldots, \phi_n) :]0, \epsilon[\to X \subset \mathbb{R}^n$$

such that $\lim_{s\to 0}\|\phi(s)\| = \infty$ and $\lim_{s\to 0} f(\phi(s)) = t_0$.

Consider

$$\frac{\partial f_i}{\partial x}(x) := \left(\frac{\partial f_i}{\partial x_1}(x), \ldots, \frac{\partial f_i}{\partial x_n}(x)\right), \text{ for } i = 1, \ldots, p, \tag{29}$$

$$\frac{\partial h_j}{\partial x}(x) := \left(\frac{\partial h_j}{\partial x_1}(x), \ldots, \frac{\partial h_j}{\partial x_n}(x)\right), \text{ for } j = 1, \ldots, r. \tag{30}$$

Since $n = h + r$, there exist analytic curves $\tilde{\lambda}(s), \tilde{\varphi}_1(s), \ldots, \tilde{\varphi}_p(s), \tilde{\psi}_1(s), \ldots, \tilde{\psi}_r(s)$, from $]0, \epsilon[$ to $\mathbb{R}$, such that $(\tilde{\lambda}(s), \tilde{\varphi}_1(s), \ldots, \tilde{\varphi}_p(s), \tilde{\psi}_1(s), \ldots, \tilde{\psi}_r(s)) \neq (0, \ldots, 0)$, for any $s \in]0, \epsilon[$, and the following equality holds:

$$\tilde{\lambda}(s)(\phi_1(s), \ldots, \phi_n(s)) = \sum_{i=1}^{p} \tilde{\varphi}_i(s)\frac{\partial f_i}{\partial x}(\phi(s)) + \sum_{j=1}^{r} \tilde{\psi}_j(s)\frac{\partial h_j}{\partial x}(\phi(s)). \tag{31}$$

Let $\tilde{\varphi}(s) := (\tilde{\varphi}_1(s), \ldots, \tilde{\varphi}_p(s))$. Let us assume that there exists $0 < \epsilon_1 \leq \epsilon$ such that $\tilde{\varphi}(s) \neq 0$, for any $s \in]0, \epsilon_1[$, the proof of which will be given below.

We consider the curves $\lambda(s)$, $\varphi(s) := (\varphi_1(s), \ldots, \varphi_p(s))$ and $\psi(s) := (\psi_1(s), \ldots, \psi_r(s))$, where $\lambda(s) := \frac{\tilde{\lambda}(s)}{\|\tilde{\varphi}(s)\|}$, $\varphi_i(s) := \frac{\tilde{\varphi}_i(s)}{\|\tilde{\varphi}(s)\|}$, $i = 1, \ldots, p$, and $\psi_j(s) = \frac{\tilde{\psi}_j(s)}{\|\tilde{\varphi}(s)\|}$, $j = 1, \ldots, r$.

Then $\|\varphi(s)\| = 1$ and we can rewrite equation (31) as follows:

$$\lambda(s)(\phi_1(s), \ldots, \phi_n(s)) = \sum_{i=1}^{p} \varphi_i(s)\frac{\partial f_i}{\partial x}(\phi(s)) + \sum_{j=1}^{r} \psi_j(s)\frac{\partial h_j}{\partial x}(\phi(s)). \tag{32}$$

By chain rule and from (32), we obtain the following equalities:

$$\sum_{i=1}^{p} \varphi_i(s)\frac{d}{ds}f_i(\phi(s)) + \sum_{j=1}^{r} \psi_j(s)\frac{d}{ds}h_j(\phi(s)) = \tag{33}$$
$$\left\langle \sum_{i=1}^{p} \varphi_i(s)\frac{\partial f_i}{\partial x}(\phi(s)) + \sum_{j=1}^{r} \psi_j(s)\frac{\partial h_j}{\partial x}(\phi(s)); \frac{d}{ds}\phi(s)\right\rangle = \frac{1}{2}\lambda(s)\left(\frac{d}{ds}\|\phi(s)\|^2\right).$$

Since $\lim_{s\to 0} f(\phi(s)) = t_0$ and $h(\phi(s)) \equiv 0$, we have that $\mathrm{ord_s}\left(\frac{d}{ds}f_i(\phi(s))\right) \geq 0$, for $i = 1, \ldots, p$, and $\frac{d}{ds}h_j(\phi(s)) \equiv 0$, for $j = 1, \ldots, r$. These and (33) imply:

$$0 \leq \mathrm{ord_s}\left(\lambda(s)\left(\frac{d}{ds}\|\phi(s)\|^2\right)\right) < \mathrm{ord_s}\left(\lambda(s)\|\phi(s)\|^2\right). \tag{34}$$

On the other hand, the equality (32) yields:

$$\mathrm{ord_s}\left(|\lambda(s)|\|\phi(t)\|^2\right) = \mathrm{ord_s}\left(\|\phi(s)\|\left\|\sum_{i=1}^{p} \varphi_i(s)\frac{\partial f_i}{\partial x}(\phi(s)) + \sum_{j=1}^{r} \psi_j(s)\frac{\partial h_j}{\partial x}(\phi(s))\right\|\right). \tag{35}$$

From (34), we conclude that (35) is positive, which implies:

$$\lim_{s\to 0}\|\phi(s)\|\left\|\sum_{i=1}^{p} \varphi_i(s)\frac{\partial f_i}{\partial x}(\phi(s)) + \sum_{j=1}^{r} \psi_j(s)\frac{\partial h_j}{\partial x}(\phi(s))\right\| = 0. \tag{36}$$

Therefore, since $\lim_{s\to 0} f(\phi(s)) = t_0$, $\|\varphi(s)\| = 1$, $\sum_{j=1}^{r}\psi_j(s)\frac{\partial h_j}{\partial x}(\phi(s)) \in (T_{\phi(s)}X)^{\perp}$, we conclude from (36), Definition 3.7 and Lemma 3.6 that $t_0 \in K_\infty(f)$.

Let us now show that there exists $0 < \epsilon_1 \leq \epsilon$ such that $\tilde{\varphi}(s) \neq 0$, for any $s \in]0, \epsilon_1[$. Suppose not; this means that there exists a sequence $\{s_k\}_{k\in\mathbb{N}} \subset]0, \epsilon[$ such that $\lim_{k\to\infty} s_k = 0$ and $\tilde{\varphi}(s_k) = (0, \dots, 0)$. This and (31) yield the following equality:

$$\tilde{\lambda}(s_k)(\phi_1(s_k), \dots, \phi_n(s_k)) = \sum_{j=1}^{r} \tilde{\psi}_j(s_k) \frac{\partial h_j}{\partial x}(\phi(s_k)), \text{ for any } k \in \mathbb{N}. \tag{37}$$

We remember that $(\tilde{\lambda}(s), \tilde{\varphi}_1(s), \dots, \tilde{\varphi}_p(s), \tilde{\psi}_1(s), \dots, \tilde{\psi}_r(s)) \neq (0, \dots, 0)$, for any $s \in]0, \epsilon[$. Consequently, the condition on $\tilde{\varphi}$ implies $(\tilde{\lambda}(s_k), \tilde{\psi}_1(s_k), \dots, \tilde{\psi}_r(s_k)) \neq (0, \dots, 0)$, for any $k \in \mathbb{N}$. Moreover, since $\lim_{k\to\infty} s_k = 0$, we have $\lim_{k\to\infty} \|\phi(s_k)\| = \infty$ and $\lim_{k\to\infty} f(\phi(s_k)) = t_0$. From these conditions, equality (37) and curve selection lemma, we can obtain new analytic curves $\lambda(s), \psi_1(s), \dots, \psi_r(s)$ and an analytic curve $\alpha = (\alpha_1, \dots, \alpha_n) :]0, \epsilon[\to X \subset \mathbb{R}^n$ such that $\lim_{s\to 0} \|\alpha(s)\| = \infty$, $\lim_{s\to 0} f(\alpha(s)) = t_0$, $(\lambda(s), \psi_1(s), \dots, \psi_r(s)) \neq (0, \dots, 0)$, for any s, and the following equality holds:

$$\lambda(s)(\alpha_1(s), \dots, \alpha_n(s)) = \sum_{j=1}^{r} \psi_j(s) \frac{\partial h_j}{\partial x}(\phi(s)). \tag{38}$$

Since $\alpha(s) \in X$, we have $h_j(\alpha(s)) \equiv 0$, which implies $\frac{d}{ds} h_j(\alpha(s)) \equiv 0$, for $j = 1, \dots, r$. These and chain rule give:

$$0 \equiv \sum_{j=1}^{r} \psi_j(s) \frac{d}{ds} h_j(\alpha(s)) = \left\langle \sum_{j=1}^{r} \psi_j(s) \frac{\partial h_j}{\partial x}(\alpha(s)), \frac{d}{ds}\alpha(s) \right\rangle = \frac{1}{2}\lambda(s) \left(\frac{d}{ds} \|\alpha(s)\|^2 \right). \tag{39}$$

Since λ and α are analytic curves, equality (39) gives $\lambda(s) \equiv 0$ or $\frac{d}{ds}\|\alpha(s)\|^2 \equiv 0$. If $\lambda(s) \equiv 0$ then, from (38) and statements on $\lambda, \psi_1, \dots, \psi_r$, we obtain that $\sum_{j=1}^{r} \psi_j(s) \frac{\partial h_j}{\partial x}(\phi(s)) \equiv 0$, with $(\psi_1(s), \dots, \psi_r(s)) \neq (0, \dots, 0)$. But this contradicts the hypothesis that X is a global intersection. If $\frac{d}{ds}\|\alpha(s)\|^2 \equiv 0$ then $\|\alpha(s)\|^2$ is constant, which contradicts the assumption $\lim_{s\to 0} \|\alpha(s)\| = \infty$. Therefore, we have shown by contradiction that the assertion "there exists $0 < \epsilon_1 \leq \epsilon$ such that $\tilde{\varphi}(s) \neq 0$, for any $s \in]0, \epsilon_1[$," is true, which completes the proof of Proposition 5.3. □

The above proposition extends for mappings defined on X the equality proved in [KOS, Proposition 3.1].

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References

[Br] S. A. Broughton, *Milnor number and the topology of polynomial hypersurfaces*, Inventiones Math. 92 (1988), 217–241. DOI: 10.1007/BF01404452

[CDTT] Y. Chen, L. R. G. Dias, K. Takeuchi, M. Tibăr, *Invertible polynomial mappings via Newton non-degeneracy*, Ann. Inst. Fourier (Grenoble), to appear.

[CDT] Y. Chen, L. R. G. Dias, M. Tibăr, *On Newton non-degeneracy of polynomial mappings*, (2012). arXiv:1207.1612

[CT] Y. Chen, M. Tibăr, *Bifurcation values and monodromy of mixed polynomials*, Math. Res. Lett. 19(1) (2012), 59–79. DOI: 10.4310/MRL.2012.v19.n1.a6

[DG] D. D'Acunto, V. Grandjean, *On gradient at infinity of semialgebraic functions*, Ann. Polon. Math. 87 (2005), 39–49. DOI: 10.4064/ap87-0-4

[DRT] L. R. G. Dias, M. A. S. Ruas, M. Tibăr, *Regularity at infinity of real mappings and a Morse-Sard theorem*, Journal of Topology 5 (2012), 323–340. DOI: 10.1112/jtopol/jts005

[Ga] T. Gaffney, *Fibers of polynomial mappings at infinity and a generalized Malgrange condition,* Compositio Math. 119 (1999), 157–167. DOI: 10.1023/A:1001702518505

[GP] V. Guillemin, A. Pollack, *Differential topology*, Prentice-Hall Inc., Englewood Cliffs, N.J., 1974.

[HL] Hà H. V., Lê D. T., *Sur la topologie des polynômes complexes,* Acta Math. Vietnam. 9(1) (1984), 21–32 (1985).

[HP] Hà H. V. and Pham T. S., *Minimizing polynomial functions*, Acta Math. Vietnam. 32 (2007), 71–82.

[Je1] Z. Jelonek, *Testing sets for properness of polynomial mappings*, Math. Ann. 315(1) (1999), 1–35.

[Je2] Z. Jelonek, *Geometry of real polynomial mappings*, Math. Z. 239(2) (2002), 321–333.

[Je3] Z. Jelonek, *On the generalized critical values of a polynomial mapping*, Manuscripta Math. 110(2) (2003), 145–157.

[Je4] Z. Jelonek, *On asymptotic critical values and the Rabier theorem*, Geometric singularity theory, Banach Center Publ. 65 (Polish Academy of Sciences, Warsaw, 2004), 125–133.

[Kuo] T. Z. Kuo, *Characterizations of v-sufficiency of jets.* Topology 11 (1972), 115–131.

[KOS] K. Kurdyka, P. Orro, S. Simon, *Semialgebraic Sard theorem for generalized critical values,* J. Differential Geometry 56 (2000), 67–92.

[LW] Lê D. T., C. Weber, *Polynômes à fibres rationnelles et conjecture jacobienne à 2 variables*, C. R. Acad. Sci. Paris Sér. I Math. 320(5) (1995), 581–584.

[Mi] J. Milnor, *Singular points of complex hypersurfaces*, Ann. of Math. Studies 61, Princeton 1968.

[NZ] A. Némethi, A. Zaharia, *On the bifurcation set of a polynomial and Newton boundary*, Publ. Res. Inst. Math. Sci. 26 (1990), 681–689.

[Pa1] A. Parusiński, *On the bifurcation set of complex polynomial with isolated singularities at infinity*, Compositio Math. 97(3) (1995), 369–384.

[Pa2] A. Parusiński, *A note on singularities at infinity of complex polynomials*, Simplectic singularities and geometry of gauge fields (Warsaw, 1995), Banach Center Publication 39 (1997), 131–141.

[PZ] A. L. Păunescu, A. Zaharia, *On the Łojasiewicz exponent at infinity for polynomial functions*, Kodai Math. J. 20(3) (1997), 269–274.

[Ra] P.J. Rabier, *Ehresmann fibrations and Palais-Smale conditions for morphisms of Finsler manifolds*, Ann. of Math. (2) 146(3) (1997), 647–691.

[ST] D. Siersma, M. Tibăr, *Singularities at infinity and their vanishing cycles*, Duke Math. Journal 80 (3) (1995), 771-783.

[Su] M. Suzuki, *Propriétés topologiques des polynômes de deux variables complexes, et automorphismes algébriques de l'espace* $\mathbf{C}^2$, J. Math. Soc. Japan 26 (1974), 241–257. DOI: 10.2969/jmsj/02620241

[Ti1] M. Tibăr, *Topology at infinity of polynomial mappings and Thom regularity condition*, Compositio Math. 111(1) (1998), 89–109.

[Ti2] M. Tibăr, *Asymptotic equisingularity and topology of complex hypersurfaces*, Internat. Math. Res. Notices no.18 (1998), 979–990.

[Ti3] M. Tibăr, *Regularity at infinity of real and complex polynomial maps*, Singularity Theory, the C.T.C. Wall Anniversary Volume, LMS Lecture Notes Series 263 (1999), 249–264.

[Ti4] M. Tibăr, *Polynomials and Vanishing Cycles*, Cambridge Tracts in Mathematics 170, Cambridge University Press (2007).

[TZ] M. Tibăr, A. Zaharia, *Asymptotic behaviour of families of real curves*, Manuscripta Math. 99(3) (1999), 383-393.

[Yo] N. Young, *An introduction to Hilbert space*, Cambridge Mathematical Textbooks. Cambridge University Press, Cambridge, (1988).

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SINGULARITIES OF AFFINE EQUIDISTANTS: PROJECTIONS AND CONTACTS

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Abstract. Using standard methods for studying singularities of projections and of contacts, we classify the stable singularities of affine λ-equidistants of n-dimensional closed submanifolds of $\mathbb{R}^q$, for $q \leq 2n$, whenever $(2n, q)$ is a pair of nice dimensions [12].

1. Introduction

When M is a smooth closed curve on the affine plane $\mathbb{R}^2$, the set of all midpoints of chords connecting pairs of points on M with parallel tangent vectors is called the *Wigner caustic* of M, or the *area evolute* of M, or still, the *affine* $1/2$*-equidistant* of M, denoted $E_{1/2}(M)$.

The $1/2$-equidistant is generalized to any λ-equidistant, denoted $E_\lambda(M)$, $\lambda \in \mathbb{R}$, by considering all chords connecting pairs of points of M with parallel tangent vectors and the set of all points of these chords which stand in the λ-proportion to their corresponding pair of points on M. In this case, when M is a curve on $\mathbb{R}^2$, the local classification of stable singularities of $E_\lambda(M)$ is well known [2, 5].

The definition of the affine λ-equidistant of M is generalized to the cases when M is an n-dimensional closed submanifold of $\mathbb{R}^q$, with $q \leq 2n$, by considering the set of all λ-points of chords connecting pairs of points on M whose direct sum of tangent spaces do not coincide with $\mathbb{R}^q$, the so-called *weakly parallel pairs* on M.

In addition to curves in $\mathbb{R}^2$, the possible stable singularities of $E_\lambda(M)$ have been previously studied in the general setting when M is a hypersurface [5, 6], or when M is a surface in $\mathbb{R}^4$ [7]. The cases of curves in $\mathbb{R}^2$ and surfaces in $\mathbb{R}^4$ have also been studied in the particular setting of Lagrangian submanifolds of affine symplectic spaces [3].

In this paper, we classify the possible stable singularities of $E_\lambda(M)$ in a quite more general circumstance, namely, when the double dimension of M, $2n$, and the dimension of the ambient affine space, q, form a pair of *nice dimensions* [12], see Theorem 5.3 below.

In order to obtain such a classification, we start in Section 2 by defining an affine λ-equidistant of $M^n \subset \mathbb{R}^q$ as the set of critical values of the λ-point map (projection)

$$\pi_\lambda : \mathbb{R}^q \times \mathbb{R}^q \to \mathbb{R}^q, (x^+, x^-) \mapsto \lambda x^+ + (1-\lambda)x^-$$

restricted to $M \times M$, thus locally a map

$$\tilde{\pi}_\lambda : \mathbb{R}^{2n} \to \mathbb{R}^q \ ,$$

see Definition 2.8, Remark 2.9 and equation (5.2), below. Then, we also present the characterization of affine equidistants by a contact map, extending previous construction for the Wigner caustic ([14, 7]).

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In Section 3 we review the standard $\mathcal{K}$-equivalence and the classification of $\mathcal{K}$-simple singularities [10, 12], Theorem 3.9 below. Then, in Section 4 we combine the study of singularities of projections and of contacts, in view of Theorem 4.6 below ([12, 11]), with emphasis on contact reduction to rank 0 map-germs, Proposition 4.14.

Our main result is obtained in Section 5. First, in Theorem 5.2 we apply the Multijet Transversality Theorem [8] to a $\mathcal{K}$-invariant stratification of the jet space. When $(2n, q)$ is a pair of nice dimensions, the relevant strata of this stratification are the $\mathcal{K}$-simple orbits in jet space. Then, we use the results of Section 4 in the context of affine equidistants: Proposition 5.4 and Corollary 5.5, as well as equations (5.8)-(5.12). The following table summarizes our main result, Theorem 5.6, which is presented more extensively as subsection 5.1. The normal forms for the $\mathcal{A}$-stable singularities of the map $\tilde{\pi}_\lambda$ follow the notation of [10] (see Theorem 3.9 below) for the $\mathcal{K}$-simple rank-0 contact map-germ

$$\theta_\lambda : (\mathbb{R}^k, 0) \to (\mathbb{R}^{k-(2n-q)}, 0) ,$$

where k is the degree of parallelism of the pair of points on M joined by the chord (cf. Definition 2.1 and Tables I, II, III in Theorem 3.9).

(n , q)	Stable $E_\lambda(M)$, $M^n \subset \mathbb{R}^q$	Restrictions
(1 , 2)	A_μ	$\mu \leq 2$
(2 , 3)	A_μ	$\mu \leq 3$
(2 , 4)	$A_\mu, C^{\pm}_{2,2}$	$\mu \leq 4$
(3 , 4)	$A_\mu, D^{\pm}_4$	$\mu \leq 4$
(3 , 5)	$A_\mu, D^{\pm}_4, D^{\pm}_5, S_5$	$\mu \leq 5$
(3 , 6)	$A_\mu, C^{\pm}_{\rho,\tau}, C_6$	$\mu \leq 6$, $2 \leq \rho \leq \tau$, $\rho + \tau \leq 6$
(4 , 5)	$A_\mu, D^{\pm}_4, D^{\pm}_5$	$\mu \leq 5$
(4 , 7)	$A_\mu, D^{\pm}_\nu, E_6, E_7, S_\beta, T_7, \tilde{T}_7$	$\mu \leq 7$, $4 \leq \nu \leq 7$, $5 \leq \beta \leq 7$
(4 , 8)	$A_\mu, C^{\pm}_{\rho,\tau}, C_6, C_8, F_7, F_8$	$\mu \leq 8$, $2 \leq \rho \leq \tau$, $\rho + \tau \leq 8$
(5 , 6)	$A_\mu, D^{\pm}_\nu, E_6$	$\mu \leq 6$, $4 \leq \nu \leq 6$

We note that the case $M^4 \subset \mathbb{R}^6$ is absent from the table of results. This is due to the fact that $(2n = 8, q = 6)$ is not a pair of nice dimensions (see Theorem 5.3 below). Similarly, $(2n, q > 6)$ is not a pair of nice dimensions, for all $n \geq 5$. Classification of stable singularities of $E_\lambda(M)$, in these cases, lies outside the scope of this paper.

As mentioned before, the cases in the table of results when

$$(n, q) \in \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 6)\}$$

correspond to hypersurfaces and have been previously studied in [5, 6], and the case $(n, q) = (2, 4)$ was partially studied in [7]. On the other hand, the results for the cases when

$$(n, q) \in \{(3, 5), (3, 6), (4, 7), (4, 8)\}$$

are entirely new.

We emphasize that, in all of the above, we are excluding the cases of *vanishing chords*, that is, when the λ-point of the chord connecting two points on M touches M because the pair of points on M lies in the diagonal of $M \times M$. Such "diagonal singularities" or *singularities on shell* for $E_\lambda(M)$ possess additional symmetries when $\lambda = 1/2$ and these have been studied for the cases of curves on the plane and surfaces in $\mathbb{R}^4$, both in the general setting [7] and in the more particular setting of Lagrangian submanifolds of affine symplectic space [4]. In this paper, we don't study such singularities on shell for $E_\lambda(M)$.

2. Affine equidistants

2.1. **Definition of affine equidistants.** Let M be a smooth closed n-dimensional submanifold of the affine space $\mathbb{R}^q$, with $q \leq 2n$. Let a, b be points of M and denote by

$$\tau_{a-b} : \mathbb{R}^q \ni x \mapsto x + (a - b) \in \mathbb{R}^q$$

the translation by the vector $(a - b)$.

Definition 2.1. A pair of points $a, b \in M$ $(a \neq b)$ is called a **weakly parallel** pair if

$$T_aM + \tau_{a-b}(T_bM) \neq \mathbb{R}^q.$$

$\operatorname{codim}(T_aM + \tau_{a-b}(T_bM))$ in $T_a\mathbb{R}^q$ is called the **codimension of a weakly parallel pair** a, b. We denote it by $\operatorname{codim}(a, b)$.

A weakly parallel pair $a, b \in M$ is called **k-parallel** if

$$\dim(T_aM \cap \tau_{b-a}(T_bM)) = k. \tag{2.1}$$

If $k = n$ the pair $a, b \in M$ is called **strongly parallel**, or just **parallel**. We also refer to k as the **degree of parallelism** of the pair (a, b) and denote it by $\deg(a, b)$. The degree of parallelism and the codimension of parallelism are related in the following way:

$$2n - \deg(a, b) = q - \operatorname{codim}(a, b). \tag{2.2}$$

Definition 2.2. A **chord** passing through a pair a, b, is the line

$$l(a, b) = \{x \in \mathbb{R}^q | x = \lambda a + (1 - \lambda)b, \lambda \in \mathbb{R}\}.$$

Definition 2.3. For a given λ, an **affine λ-equidistant** of M, $E_\lambda(M)$, is the set of all $x \in \mathbb{R}^q$ such that $x = \lambda a + (1-\lambda)b$, for all weakly parallel pairs $a, b \in M$. $E_\lambda(M)$ is also called a (affine) **momentary equidistant** of M. Whenever M is understood, we write E_λ for $E_\lambda(M)$.

Note that, for any λ, $E_\lambda(M) = E_{1-\lambda}(M)$ and in particular $E_0(M) = E_1(M) = M$. Thus, the case $\lambda = 1/2$ is special:

Definition 2.4. $E_{1/2}(M)$ is called the **Wigner caustic** of M [2, 14].

2.2. **Characterization of affine equidistants by projection.** Consider the product affine space: $\mathbb{R}^q \times \mathbb{R}^q$ with coordinates (x_+, x_-) and the tangent bundle to $\mathbb{R}^q$: $T\mathbb{R}^q = \mathbb{R}^q \times \mathbb{R}^q$ with coordinate system $(x, \dot{x})$ and standard projection $\pi : T\mathbb{R}^q \ni (x, \dot{x}) \to x \in \mathbb{R}^q$.

Definition 2.5. For $\lambda \in \mathbb{R}$, a **λ-chord transformation**

$$\Gamma_\lambda : \mathbb{R}^q \times \mathbb{R}^q \to T\mathbb{R}^q\ ,\ (x^+, x^-) \mapsto (x, \dot{x})$$

is a linear diffeomorphism defined by the *λ-point equation*:

$$x = \lambda x^+ + (1 - \lambda)x^-\ , \tag{2.3}$$

for the λ-point x, and a *chord equation*:

$$\dot{x} = x^+ - x^-. \tag{2.4}$$

Remark 2.6. For our purposes, the choice (2.4) for a chord equation is not unique, but is the simplest one. Among other possibilities, the choice $\dot{x} = \lambda x^+ - (1 - \lambda)x^-$ is particularly well suited for the study of affine equidistants of *Lagrangian* submanifolds in symplectic space [3].

Now, let M be a smooth closed n-dimensional submanifold of the affine space $\mathbb{R}^q$ $(2n \geq q)$ and consider the product $M \times M \subset \mathbb{R}^q \times \mathbb{R}^q$. Let $\mathcal{M}_\lambda$ denote the image of $M \times M$ by a λ-chord transformation,

$$\mathcal{M}_\lambda = \Gamma_\lambda(M \times M) ,$$

which is a $2n$-dimensional smooth submanifold of $T\mathbb{R}^q$.

Then we have the following general characterization:

Theorem 2.7 ([3]). *The set of critical values of the standard projection $\pi : T\mathbb{R}^q \to \mathbb{R}^q$ restricted to $\mathcal{M}_\lambda$ is $E_\lambda(M)$.*

Definition 2.8. For $\lambda \in \mathbb{R}$, the **λ-point map** is the projection

$$\pi_\lambda : \mathbb{R}^q \times \mathbb{R}^q \to \mathbb{R}^q \ , \ (x^+, x^-) \mapsto x = \lambda x^+ + (1-\lambda)x^- \ .$$

Remark 2.9. Because $\pi_\lambda = \pi \circ \Gamma_\lambda$ we can rephrase Theorem 2.7: *the set of critical values of the projection π_λ restricted to $M \times M$ is $E_\lambda(M)$.*

2.3. **Characterization of affine equidistants by contact.** In the literature, if $M \subset \mathbb{R}^2$ is a smooth curve, the Wigner caustic $E_{1/2}(M)$ has been described in various ways. A particular description says that, if $\mathcal{R}_a : \mathbb{R}^2 \to \mathbb{R}^2$ denotes reflection through $a \in \mathbb{R}^2$, then $a \in E_{1/2}(M)$ when M and $\mathcal{R}_a(M)$ are not transversal [2, 14]. This description has also been used in [14] for the case of Lagrangian surfaces in symplectic $\mathbb{R}^4$ and, more recently [7], for the case of general surfaces in $\mathbb{R}^4$.

We now generalize this description for every λ-equidistant of submanifolds of more arbitrary dimensions.

Definition 2.10. For $\lambda \in \mathbb{R} \setminus \{0, 1\}$, a **$\lambda$-reflection** through $a \in \mathbb{R}^q$ is the map

$$\mathcal{R}_a^\lambda : \mathbb{R}^q \to \mathbb{R}^q \ , \ x \mapsto \mathcal{R}_a^\lambda(x) = \frac{1}{\lambda}a - \frac{1-\lambda}{\lambda}x \tag{2.5}$$

Remark 2.11. A λ-reflection through a is not a reflection in the strict sense because

$$\mathcal{R}_a^\lambda \circ \mathcal{R}_a^\lambda \neq id : \mathbb{R}^q \to \mathbb{R}^q,$$

instead,

$$\mathcal{R}_a^{1-\lambda} \circ \mathcal{R}_a^\lambda = id : \mathbb{R}^q \to \mathbb{R}^q \ ,$$

so that, if $a = a_\lambda = \lambda a^+ + (1-\lambda)a^-$ is the λ-point of $(a^+, a^-) \in \mathbb{R}^{2q}$,

$$\mathcal{R}_{a_\lambda}^\lambda(a^-) = a^+ \ , \ \mathcal{R}_{a_\lambda}^{1-\lambda}(a^+) = a^- \ .$$

Of course, for $\lambda = 1/2$, $\mathcal{R}_a^{1/2} \equiv \mathcal{R}_a$ is a reflection in the strict sense.

Now, let M be a smooth n-dimensional submanifold of $\mathbb{R}^q$, with $2n \geq q$, and let

$$a = a_\lambda = \lambda a^+ + (1-\lambda)a^-$$

be the λ-point of $(a^+, a^-) \in M \times M \subset \mathbb{R}^q \times \mathbb{R}^q$. Also, let M^+ be a germ of submanifold M around a^+ and M^- be a germ of submanifold M around a^-. We have:

Proposition 2.12. *The following statements are equivalent:*
(i) The λ-point a belongs to $E_\lambda(M)$.
(ii) M^+ and $\mathcal{R}_a^\lambda(M^-)$ are not transversal at a^+.
(iii) M^- and $\mathcal{R}_a^{1-\lambda}(M^+)$ are not transversal at a^-.

Remark 2.13. Furthermore, from Remark 2.9 we see that the study of the singularities of affine equidistants is the study of the singularities of π_λ. But this is the same as the study of the singularities at $a = 0$ of

$$(x^+, x^-) \to x^+ + \frac{1-\lambda}{\lambda} x^- = x^+ - \mathcal{R}_0^\lambda(x^-) \ .$$

In other words, *the study of the singularities of* $E_\lambda(M) \ni 0$ *can be proceeded via the study of the contact between* M^+ *and* $\mathcal{R}_0^\lambda(M^-)$ *or, equivalently, the contact between* M^- *and* $\mathcal{R}_0^{1-\lambda}(M^+)$.

3. $\mathcal{K}$-EQUIVALENCE

We recall some basic definitions and results (for details, see [1]).

Henceforth, $\mathcal{E}_s$ denotes the local ring of smooth function-germs on $\mathbb{R}^s$, and $\mathfrak{m}_s$ its maximal ideal.

Definition 3.1. Map-germs $f, \widetilde{f} : (\mathbb{R}^s, y_0) \to (\mathbb{R}^t, 0)$ are $\mathcal{K}$**-equivalent** if there exists a diffeomorphism-germ $\phi : (\mathbb{R}^s, y_0) \to (\mathbb{R}^s, y_0)$ and a map-germ $A : (\mathbb{R}^s, y_0) \to GL(\mathbb{R}^t)$ such that $\tilde{f} = A \cdot (f \circ \phi)$.

Theorem 3.2 ([1])**.** *For the* $\mathcal{K}$*-equivalence of two map-germs it is necessary and sufficient that two ideals generated by the components of these map-germs may be mapped one to the other by an isomorphism of* $\mathcal{E}_s$ *induced by a diffeomorphism-germ of the source space* $(\mathbb{R}^s, y_0)$.

Definition 3.3. A map-germ $F : (\mathbb{R}^s \times \mathbb{R}^p, (y_0, z_0)) \to \mathbb{R}^t$ is a **deformation** of a map-germ $f : (\mathbb{R}^s, y_0) \to \mathbb{R}^t$ if $F|_{\mathbb{R}^s \times \{z_0\}} = f$, where p is the number of parameters of deformation F.

Definition 3.4. A diffeomorphism-germ $\Phi : (\mathbb{R}^s \times \mathbb{R}^p, (y_0, z_0)) \to (\mathbb{R}^s \times \mathbb{R}^p, (y_0, z_0))$ is called **fiber-preserving** if $\Phi(y, z) = (Y(y, z), Z(z))$ for a smooth map-germ

$$Y : (\mathbb{R}^s \times \mathbb{R}^p, (y_0, z_0)) \to (\mathbb{R}^s, y_0)$$

and a diffeomorphism-germ $Z : (\mathbb{R}^p, z_0) \to (\mathbb{R}^p, z_0)$. It means that Φ preserves the fibers of the projection $pr : (\mathbb{R}^s \times \mathbb{R}^p, (y_0, z_0)) \to (\mathbb{R}^p, z_0)$.

Definition 3.5. Deformations $F, \widetilde{F} : (\mathbb{R}^s \times \mathbb{R}^p, (y_0, z_0)) \to (\mathbb{R}^t, 0)$ of respective map-germs $f, \widetilde{f} : (\mathbb{R}^s, y_0) \to (\mathbb{R}^t, 0)$ are **fiber** $\mathcal{K}$**-equivalent** if there is a fiber-preserving diffeomorphism-germ $\Phi : (\mathbb{R}^s \times \mathbb{R}^p, (y_0, z_0)) \to (\mathbb{R}^s \times \mathbb{R}^p, (y_0, z_0))$, i.e. $\Phi(y, z) = (Y(y, z), Z(z))$, and a map-germ $\mathbb{A} : (\mathbb{R}^s \times \mathbb{R}^p, (y_0, z_0)) \to GL(\mathbb{R}^t)$ such that $\widetilde{F} = \mathbb{A} \cdot (F \circ \Phi)$.

Corollary 3.6. *For the fiber* $\mathcal{K}$*-equivalence of two deformations it is necessary and sufficient that the two ideals of* $\mathcal{E}_{s+p}$ *generated by the components of these deformations may be mapped one to the other by an isomorphism of* $\mathcal{E}_{s+p}$ *induced by a fiber-preserving diffeomorphism-germ of the source space* $(\mathbb{R}^s \times \mathbb{R}^p, (y_0, z_0))$.

Definition 3.7. The germ $f : (\mathbb{R}^s, 0) \to (\mathbb{R}^t, 0)$ is said to be $\mathcal{K}$**-simple** if its k-jet, for any k, has a neighborhood in the jet space $J^k_{0,0}(\mathbb{R}^s, \mathbb{R}^t)$ that intersects only a finite number of $\mathcal{K}$-equivalence classes (bounded by a constant independent of k).

Definition 3.8. The p-parameter **suspension** of the map-germ $f : (\mathbb{R}^s, 0) \to (\mathbb{R}^t, 0)$ is the map germ

$$F : (\mathbb{R}^s \times \mathbb{R}^p, 0) \ni (y, z) \mapsto (f(y), z) \in (\mathbb{R}^t \times \mathbb{R}^p, 0).$$

Theorem 3.9 ([10])**.** $\mathcal{K}$*-simple map-germs* $(\mathbb{R}^s, 0) \to (\mathbb{R}^t, 0)$ *with* $s \geq t$ *belong, up to* $\mathcal{K}$*-equivalence and suspension, to one of the following three lists in Tables 1-3:*

Notation	*Normal form*	*Restrictions*
A_μ	$y_1^{\mu+1} + Q_{s-1}$	$\mu \geq 1$
D_μ	$y_1^2 y_2 \pm y_2^{\mu-1} + Q_{s-2}$	$\mu \geq 4$
E_6	$y_1^3 + y_2^4 + Q_{s-2}$	-
E_7	$y_1^3 + y_1 y_2^3 + Q_{s-2}$	-
E_8	$y_1^3 + y_2^5 + Q_{s-2}$	-

TABLE 1. $\mathcal{K}$-simple germs $\mathbb{R}^s \to \mathbb{R}$. $Q_{s-i} = \pm y_{i+1}^2 \pm \cdots \pm y_s^2$.

Notation	*Normal form*	*Restrictions*
$C_{k,l}^{\pm}$	$(y_1 y_2, y_1^k \pm y_2^l)$	$l \geq k \geq 2$
$\widetilde{C}_{2k}$	$(y_1^2 + y_2^2, y_2^k)$	$k \geq 3$
F_{2m+1}	$(y_1^2 + y_2^3, y_2^m)$	$m \geq 3$
F_{2m+4}	$(y_1^2 + y_2^3, y_1 y_2^m)$	$m \geq 2$
G_{10}^*	(y_1^2, y_2^4)	-
$H_{m+5}^{\pm}$	$(y_1^2 \pm y_2^m, y_1 y_2^2)$	$m \geq 4$

TABLE 2. $\mathcal{K}$-simple germs $\mathbb{R}^2 \to \mathbb{R}^2$.

Notation	*Normal form*	*Restrictions*
S_μ	$(\pm y_1^2 \pm y_2^2 + y_3^{\mu-3}, y_2 y_3)$	$\mu \geq 5$
T_7	$(y_1^2 + y_2^3 + y_3^3, y_2 y_3)$	-
$\widetilde{T}_7$	$(y_1^2 + y_2^2, y_2^2 + y_3^2)$	-
T_8	$(y_1^2 + y_2^3 \pm y_3^4, y_2 y_3)$	-
T_9	$(y_1^2 + y_2^3 + y_3^5, y_2 y_3)$	-
U_7	$(y_1^2 + y_2 y_3, y_1 y_2 + y_3^3)$	-
U_8	$(y_1^2 + y_2 y_3 + y_3^3, y_1 y_2)$	-
U_9	$(y_1^2 + y_2 y_3, y_1 y_2 + y_3^4)$	-
W_8	$(y_1^2 + y_2^3, y_2^2 + y_1 y_3)$	-
W_9	$(y_1^2 + y_2 y_3^2, y_2^2 + y_1 y_3)$	-
Z_9	$(y_1^2 + y_3^3, y_2^2 + y_3^3)$	-
Z_{10}	$(y_1^2 + y_2 y_3^2, y_2^2 + y_3^3)$	-

TABLE 3. $\mathcal{K}$-simple germs $\mathbb{R}^3 \to \mathbb{R}^2$.

Definition 3.10. A deformation

$$F : (\mathbb{R}^s \times \mathbb{R}^p, (0,0)) \to (\mathbb{R}^t, 0)$$

of a map-germ $f : (\mathbb{R}^s, 0) \to (\mathbb{R}^t, 0)$ is $\mathcal{K}$**-versal** if any other deformation

$$\widetilde{F} : (\mathbb{R}^s \times \mathbb{R}^q, (0,0)) \to (\mathbb{R}^t, 0)$$

of f is of the form

$$\widetilde{F}(y, z) = \mathbb{A}(y, z) \cdot F(g(y, z), h(z)),$$

where $\mathbb{A} : \mathbb{R}^s \times \mathbb{R}^q \to GL(\mathbb{R}^t)$, $g : (\mathbb{R}^s \times \mathbb{R}^q, (0,0)) \to (\mathbb{R}^s, 0)$, $h : (\mathbb{R}^q, 0) \to (\mathbb{R}^p, 0)$ are map-germs such that $\mathbb{A}(0,0)$ is nondegerate matrix and $g(y, 0) = y$.

Theorem 3.11 ([1])**.** *$\mathcal{K}$-versal deformations of $\mathcal{K}$-equivalent germs with the same number of parameters are fiber $\mathcal{K}$-equivalent.*

4. Singularities of projection and of contact

4.1. **Singularities of projection.** In view of Theorem 2.7, let M and $\widetilde{M}$ be smooth closed n-dimensional submanifolds of $\mathbb{R}^q$, $q \leq 2n$, and

$$\mathcal{M}_\lambda = \Gamma_\lambda(M \times M)\ ,\ \widetilde{\mathcal{M}}_\lambda = \Gamma_\lambda(\widetilde{M} \times \widetilde{M})\ ,$$

where Γ_λ is the λ-chord transformation.

For local classification of singularities, we introduce the definition:

Definition 4.1. $E_\lambda(M)$ and $E_\lambda(\widetilde{M})$ are **λ-chord equivalent** if there exists a fiber-preserving diffeomorphism-germ of $T\mathbb{R}^q$ that maps the germ of $\mathcal{M}_\lambda$ to the germ of $\widetilde{\mathcal{M}}_\lambda$ i.e. if the following diagram commutes (vertical arrows indicate diffeomorphism-germs):

$$\begin{array}{ccccc} M \times M & \overset{\Gamma_\lambda|_{M\times M}}{\longrightarrow} & T\mathbb{R}^q & \overset{\pi}{\longrightarrow} & \mathbb{R}^q \\ \downarrow & & \downarrow & & \downarrow \\ \widetilde{M} \times \widetilde{M} & \overset{\Gamma_\lambda|_{\widetilde{M}\times\widetilde{M}}}{\longrightarrow} & T\mathbb{R}^q & \overset{\pi}{\longrightarrow} & \mathbb{R}^q \end{array}$$

The λ-chord equivalence of E_λ is a special case of equivalence of projections studied by V. Goryunov ([9], [10]), as outlined below.

Definition 4.2. A **projection** of a (smooth) submanifold S from a total space E to the base B of the bundle $p : E \to B$ is a triple

$$S \overset{\iota}{\hookrightarrow} E \overset{p}{\to} B$$

where ι is an embedding. A projection is called a **projection "onto"** if the dimension of S is not less than the dimension of the base B.

Definition 4.3. Two projections $S_i \hookrightarrow E_i \to B_i$ for $i = 1, 2$ are **equivalent** if the following diagram commutes

$$\begin{array}{ccccc} S_1 & \overset{\iota_1}{\hookrightarrow} & E_1 & \overset{p_1}{\to} & B_1 \\ \downarrow & \iota_2 & \downarrow & p_2 & \downarrow \\ S_2 & \hookrightarrow & E_2 & \to & B_2 \end{array}$$

where vertical arrows indicate diffeomorphisms.

A projection of S onto B defines a family of subvarieties in the fibers of the bundle $p : E \to B$ parameterized by B: $S_b = S \cap p^{-1}(b)$ for any $b \in B$. A germ of the projection

$$(S, q_0) \hookrightarrow (E, e_0) \to (B, b_0)$$

can be considered in a natural way as a deformation of the subvariety S_{b_0}.

The germ of a bundle $E \to B$ can be identified with the germ of the trivial bundle

$$\mathbb{R}^s \times \mathbb{R}^p \to \mathbb{R}^p.$$

A germ of an embedded smooth submanifold S can be described by the germ of the variety of zeros of some mapping-germ $F : (\mathbb{R}^s \times \mathbb{R}^p, (y_0, z_0)) \to \mathbb{R}^t$. Then S_{z_0} can be identified with the germ of the variety of zeros of $F|_{\mathbb{R}^s\times\{z_0\}}$.

If deformations $F, \widetilde{F} : (\mathbb{R}^s \times \mathbb{R}^p, (y_0, z_0)) \to (\mathbb{R}^t, 0)$ of map-germs $f, \widetilde{f} : (\mathbb{R}^s, y_0) \to (\mathbb{R}^t, 0)$ (respectively) are fiber $\mathcal{K}$-equivalent then the following diagram commutes (Φ, Z indicate diffeomorphism-germs and pr indicate the projection):

$$\begin{array}{ccccc} F^{-1}(0) & \hookrightarrow & \mathbb{R}^s \times \mathbb{R}^p & \overset{pr}{\longrightarrow} & \mathbb{R}^p \\ \downarrow & & \downarrow \Phi & & \downarrow Z \\ \widetilde{F}^{-1}(0) & \hookrightarrow & \mathbb{R}^s \times \mathbb{R}^p & \overset{pr}{\longrightarrow} & \mathbb{R}^p \end{array}$$

If the ideal of function-germs vanishing on $F^{-1}(0)$ is generated by the components of F, then by Corollary 3.6 the inverse result is also true.

We remind that the group $\mathcal{A} = \mathrm{Diff}(\mathbb{R}^m, 0) \times \mathrm{Diff}(\mathbb{R}^p, 0)$ acts on map-germs $(\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$ by composition on source and target, with corresponding definitions for $\mathcal{A}$-equivalent and $\mathcal{A}$-simple (refer to Definitions 3.1 and 3.7 for the group $\mathcal{K}$). Then, from the above we have the following results:

Proposition 4.4 ([9, 10])**.** *F and $\widetilde{F}$ are fiber $\mathcal{K}$-equivalent if and only if the projections of $F^{-1}(0)$ and $\widetilde{F}^{-1}(0)$ onto $\mathbb{R}^p$ are $\mathcal{A}$-equivalent.*

Theorem 4.5 ([9])**.** *If the germ of a projection $(F^{-1}(0), (0, 0)) \hookrightarrow (\mathbb{R}^s \times \mathbb{R}^p, (0, 0)) \to (\mathbb{R}^p, 0)$ is $\mathcal{A}$-simple then $f = F|_{\mathbb{R}^s \times \{0\}}$ is $\mathcal{K}$-simple.*

Theorem 4.6 ([11, 12])**.** *The map-germ $F : \mathbb{R}^s \times \mathbb{R}^p \to \mathbb{R}^t$ is a $\mathcal{K}$-versal deformation of a rank-0 map-germ $f : \mathbb{R}^s \to \mathbb{R}^t$ of finite $\mathcal{K}$-codimension if and only if the projection-germ of $F^{-1}(0)$ onto $\mathbb{R}^p$ is $\mathcal{A}$-stable (infinitesimally stable).*

By Theorems 4.5 and 4.6, in order to classify stable singularities of projections one considers deformations of three classes of singularities: simple singularities of hypersurfaces (Table 1), simple singularities of curves in a 3-dimensional space (Table 3), simple singularities of a multiple point on a plane (Table 2). We are interested in projections "onto" when the projected submanifold $S = F^{-1}(0)$ is smooth and the dimension of the base B of the bundle is greater than 1.

In order to see in a more clear way how these three tables are applied to the classification of singularities of affine equidistants, we now turn to the contact viewpoint.

4.2. **Singularities of contact.** Let N_1, N_2 be germs at x of smooth n-dimensional submanifolds of the space $\mathbb{R}^q$, with $2n \geq q$. We describe N_1, N_2 in the following way:

- $N_1 = f^{-1}(0)$, where $f : (\mathbb{R}^q, x) \to (\mathbb{R}^{q-n}, 0)$ is a submersion-germ,
- $N_2 = g(\mathbb{R}^n)$, where $g : (\mathbb{R}^n, 0) \to (\mathbb{R}^q, x)$ is an embedding-germ.

Let $\tilde{N_1}, \tilde{N_2}$ be another pair of germs at $\tilde{x}$ of smooth n-dimensional submanifolds of the space $\mathbb{R}^q$, described in the same way.

Definition 4.7. The contact of N_1 and N_2 at x is of the same **contact-type** as the contact of $\tilde{N_1}$ and $\tilde{N_2}$ at $\tilde{x}$ if there exists a diffeomorphism-germ $\Phi : (\mathbb{R}^q, x) \to (\mathbb{R}^q, \tilde{x})$ such that $\Phi(N_1) = \tilde{N_1}$ and $\Phi(N_2) = \tilde{N_2}$. We denote the contact-type of N_1 and N_2 at x by $\mathcal{K}(N_1, N_2, x)$.

Definition 4.8. A **contact map** between submanifold-germs N_1, N_2 is the following map-germ $f \circ g : (\mathbb{R}^n, 0) \to (\mathbb{R}^{q-n}, 0)$.

Theorem 4.9 ([13])**.** *$\mathcal{K}(N_1, N_2, x) = \mathcal{K}(\tilde{N_1}, \tilde{N_2}, \tilde{x})$ if and only if the contact maps $f \circ g$ and $\tilde{f} \circ \tilde{g}$ are $\mathcal{K}$-equivalent.*

Remark 4.10. If N_1 and N_2 are transversal at x then it is obvious that the contact map $f \circ g : (\mathbb{R}^n, 0) \to (\mathbb{R}^{q-n}, 0)$ is a submersion-germ or a diffeomorphism-germ (when $q = 2n$).

The interesting cases are when N_1 and N_2 are not transversal at x_0

$$T_{x_0}N_1 + T_{x_0}N_2 \neq T_{x_0}\mathbb{R}^q.$$

Definition 4.11. We say that N_1 and N_2 are k-**tangent** at x_0 if

$$\dim(T_{x_0}N_1 \cap T_{x_0}N_2) = k \ .$$

If k is maximal, that is

$$k = n = \dim(T_{x_0}N_1) = \dim(T_{x_0}N_2) \ ,$$

we say that N_1 and N_2 are **tangent** at x_0.

Remark 4.12. In order to bring this definition into the context of affine equidistants, $E_\lambda(M)$, note that $N_1 = M^+$ and $N_2 = \mathcal{R}_0^\lambda(M^-)$ are k-**tangent** at 0 if and only if T_aM^+ and T_bM^- are k-**parallel**, where $\lambda a + (1-\lambda)b = 0 \in E_\lambda(M)$.

If N_1 and N_2 are k-tangent then we can describe germs of N_1 and N_2 at 0 in the following way:

$$N_1 = \{(y, z, u, v) \in \mathbb{R}^q : u = \phi(y, z),\ v = \psi(y, z)\}, \tag{4.1}$$

$$N_2 = \{(y, z, u, v) \in \mathbb{R}^q : z = \eta(y, v),\ u = \zeta(y, v)\}, \tag{4.2}$$

where $y = (y_1, \cdots, y_k)$, $z = (z_1, \cdots, z_{n-k})$, $u = (u_1, \cdots, u_{q+k-2n})$, $v = (v_1, \cdots, v_{n-k})$ and (y, z, u, v) is a coordinate system on the affine space $\mathbb{R}^q$,

$$\phi = (\phi_1, \cdots, \phi_{q+k-2n}), \qquad \psi = (\psi_1, \cdots, \psi_{n-k}),$$

$$\eta = (\eta_1, \cdots, \eta_{n-k}), \quad \zeta = (\zeta_1, \cdots, \zeta_{q+k-2n}), \quad \text{and} \quad \phi_i, \psi_j, \eta_j, \zeta_i \in \mathcal{M}_q^2,$$

for $i = 1, \cdots, q+k-2n$ and $j = 1, \cdots, n-k$.

Then, the contact map $\kappa_{N_1,N_2} : (\mathbb{R}^n, 0) \to (\mathbb{R}^{q-n}, 0)$ is given by:

$$\kappa_{N_1,N_2}(y, z) = (z - \eta(y, \psi(y, z)), \phi(y, z) - \zeta(y, \psi(y, z))) \tag{4.3}$$

From the form of κ_{N_1,N_2} we easily obtain the following fact

Proposition 4.13. *If N_1 and N_2 are k-tangent at 0 then the corank of the contact map κ_{N_1,N_2} is k.*

We can interpret the contact between two k-tangent n-dimensional submanifolds N_1, N_2 of $\mathbb{R}^q$ as the contact between tangent k-dimensional submanifolds P_{N_1} and P_{N_2} of N_1 and N_2, respectively, in a smooth $q - 2n + 2k$-dimensional submanifold S of $\mathbb{R}^q$. These submanifolds are constructed in the following way:

Let H be a smooth $q + k - n$-dimensional submanifold-germ on $\mathbb{R}^q$ which contains N_1 and is transversal to N_2 at 0. Then $P_{N_2} = H \cap N_2$ is a smooth k-dimensional submanifold on N_2.

Let G be a smooth $q + k - n$-dimensional submanifold-germ on $\mathbb{R}^q$ which contains N_2 and is transversal to N_1 at 0. Then $P_{N_1} = G \cap N_1$ is a smooth k-dimensional submanifold on N_1.

P_{N_1} and P_{N_2} are tangent at 0 and they are contained in the smooth $q - 2n + 2k$-dimensional submanifold-germ $S = H \cap G$.

The contact between N_1 and N_2 at 0 can now be described as the contact between P_{N_1} and P_{N_2} at 0, which defines a rank-0 map

$$\kappa_{P_{N_1}, P_{N_2}} : (\mathbb{R}^k, 0) \to (\mathbb{R}^{k-(2n-q)}, 0) \ . \tag{4.4}$$

Although in general P_{N_1} and P_{N_2} depend on the choices of H and G, the contact type of P_{N_1} and P_{N_2} does not depend on these choices. This means that if $\tilde{N}_1, \tilde{N}_2$ is another pair of germs at 0 of smooth n-dimensional submanifold of $\mathbb{R}^q$ then we have the following result.

Proposition 4.14. *$\mathcal{K}(N_1, N_2, 0) = \mathcal{K}(\tilde{N}_1, \tilde{N}_2, 0)$ if and only if*

$$\mathcal{K}(P_{N_1}, P_{N_2}, 0) = \mathcal{K}(P_{\tilde{N}_1}, P_{\tilde{N}_2}, 0).$$

Proof. It is easy to see that in general H can be described in the following way:

$$v = \psi(y, z) + A(y, z, u, v)(u - \phi(y, z)), \tag{4.5}$$

and G can be described in the following way:

$$z = \eta(y, v) + B(y, z, u, v)(u - \zeta(y, v)), \tag{4.6}$$

where $A = (a_{ij})_{i=1,\cdots,q+k-2n}^{j=1,\cdots,n-k}, B = (b_{ij})_{i=1,\cdots,q+k-2n}^{j=1,\cdots,n-k}$ and a_{ij}, b_{ij} are smooth function-germs on $\mathbb{R}^q$.

Thus $S = H \cap G$ is given by (4.5) and (4.6).

P_{N_1} is given by (4.5), (4.6), and $u = \phi(y, z)$ and P_{N_2} is given by (4.5), (4.6) and $u = \zeta(y, v)$.

On the other hand we can also describe N_1 by (4.5) and $u = \phi(y, z)$ and N_2 by (4.6) and $u = \zeta(y, v)$. Then it is easy to see that contact maps are the same after a suitable suspension. □

In view of Proposition 4.14, it is enough to classify the rank-0 map-germs of the form (4.4) with respect to the group $\mathcal{K}$.

5. Stable singularities of affine equidistants

Since our goal is to classify singularities of affine equidistants of n-dimensional submanifold M of $\mathbb{R}^q$, we substitute submanifold-germs N_1 and N_2 of the previous section by $N_1 = M^+$ and $N_2 = \mathcal{R}_0^\lambda(M^-)$, or equivalently by $N_1 = M^-$ and $N_2 = \mathcal{R}_0^{1-\lambda}(M^+)$, where M^+ and M^- are germs of $M \subset \mathbb{R}^q$ at points $a^+ \neq a^- \in M \subset \mathbb{R}^q$, such that $\lambda a^+ + (1 - \lambda)a^- = 0$.

First, we state the following definition and theorem:

Definition 5.1. A mapping $\psi : N^m \to \mathbb{R}^q$ is *locally stable* at $p \in N^m$ if there exists a neighbourhood W_p of ψ in the space $C^\infty(N^m, \mathbb{R}^q)$ of C^∞-mappings from N^m into $\mathbb{R}^q$ with the Whitney C^∞-topology, and neighbourhoods U_p around p and V_p around $\psi(p)$ such that for all $\phi \in W_p$, it follows that $\phi : U_p \to V_p$ is $\mathcal{A}$- equivalent to $\psi : U_p \to V_p$, where $\mathcal{A} = \text{Diff}(U_p) \times \text{Diff}(V_p)$ (see [8]).

Theorem 5.2. *For a residual set of embeddings $\iota : M^n \to \mathbb{R}^q$ the map*

$$\pi_\lambda \circ (\iota \times \iota) : M \times M \setminus \Delta \to \mathbb{R}^q$$

is locally stable whenever the pair $(2n, q)$ is a pair of nice dimensions, where Δ is the diagonal in $M \times M$.

Proof. From the diagram of maps

$$M \times M \overset{\iota \times \iota}{\longrightarrow} \mathbb{R}^q \times \mathbb{R}^q \overset{\pi_\lambda}{\to} \mathbb{R}^q\ ,$$

we obtain the diagram of r-jet maps

$$M \times M \overset{j^r(\iota \times \iota)}{\longrightarrow} J^r(M \times M, \mathbb{R}^q \times \mathbb{R}^q) \overset{(\pi_\lambda)_*}{\to} J^r(M \times M, \mathbb{R}^q)\ .$$

A typical fiber of $J^r(M \times M, \mathbb{R}^q)$ is $J^r_0(M \times M, \mathbb{R}^q)$, the space of (degree $\leq r$)-polynomial map-germs $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^q$, vanishing at 0.

Let $\{W_1, \dots, W_s\}$ be the finite set of all $\mathcal{K}$ simple orbits in $J^r(M\times M, \mathbb{R}^q)$; let $\{W_{s+1}, \dots, W_t\}$ be a finite stratification of the complement of the union of simple orbits $W_1 \cup \dots \cup W_s$. This stratification exists because these are semialgebraic sets. We denote by $\mathcal{S} = \{W_j\}_{1\le j\le t}$ the resulting stratification of $J^r(M \times M, \mathbb{R}^q)$. Because $(\pi_\lambda)_*$ is a submersion, $(\pi_\lambda)_*^{-1} W_j = W_j^*$ is a submanifold of $J^r(M\times M, \mathbb{R}^q\times\mathbb{R}^q)$, for all $j = 1, \dots, t$, so that $\mathcal{S}^* = \{W_j^*\}_{1\le j\le t}$ is a stratification of this space.

Furthermore,

$$j^r(\iota\times\iota) \pitchfork \mathcal{S}^* \iff j^r(\pi_\lambda\circ(\iota\times\iota)) \pitchfork \mathcal{S} \,, \tag{5.1}$$

where transversality to $\mathcal{S}$ (respectively to $\mathcal{S}^*$) means transversality of $j^r(\iota\times\iota)$ (respectively $j^r(\pi_\lambda\circ(\iota\times\iota))$) to each stratum of the corresponding stratification.

On the other hand, under the natural identification

$$j^r(\iota\times\iota)|_{M\times M\setminus\Delta} \simeq {}_2j^r\iota \subset {}_2J^r(M,\mathbb{R}^q) \,,$$

where ${}_2J^r(M, \mathbb{R}^q)$ is the space of double r-jets, we can apply the Multijet Transversality Theorem [8] to get that, for each W_j^* in ${}_2J^r(M, \mathbb{R}^q)$, the set of immersions

$$\mathcal{R}_{W_j} = \{\iota : M \to \mathbb{R}^q \,|{}_2j^r\iota \pitchfork W_j^*\}$$

is residual. Then, the set

$$\mathcal{R} = \cap_{j=1}^t \mathcal{R}_{W_j}$$

is also residual.

Now, it follows from equation (5.1) that $j^r(\pi_\lambda\circ(\iota\times\iota)) \pitchfork W_j$, for all $\iota\in\mathcal{R}$, for all $j = 1, \dots, t$. When $(2n, q)$ is a pair of nice dimensions, this implies that $j^r(\pi_\lambda\circ(\iota\times\iota))$ is transversal to all $\mathcal{K}$ orbits in $J^r(M \times M, \mathbb{R}^q)$, which says that this mapping is locally stable (see [8, 12]). □

Theorem 5.3 ([12]). *The nice dimensions for pairs* $(2n, q)$ *are:*
(i) $n < q = 2n,\ n \le 4$
(ii) $n < q = 2n-1,\ n \le 4$
(iii) $n < q = 2n-2,\ n \le 3$
(iv) $n < q \le 2n-3, q \le 6$

Thinking locally, denote two distinct germs of embedding $\iota : M^n \to \mathbb{R}^q$ by

$$\iota^+ : (\mathbb{R}^n, 0) \to (\mathbb{R}^q, a^+) \quad \text{and} \quad \iota^- : (\mathbb{R}^n, 0) \to (\mathbb{R}^q, a^-),$$

and by

$$\tilde{\pi}_\lambda = \pi_\lambda\circ(\iota^+\times\iota^-) : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^q, 0) \,, \tag{5.2}$$

the restriction of π_λ to $M^+ \times M^-$. Then, recalling the notation of (4.1)-(4.2), $\tilde{\pi}_\lambda$ is given by

$$\tilde{\pi}_\lambda : (y, z, \tilde{y}, v) \mapsto (\tilde{\pi}^1_\lambda(y,\tilde{y}), \tilde{\pi}^2_\lambda(z,\tilde{y},v), \tilde{\pi}^3_\lambda(y,z,\tilde{y},v), \tilde{\pi}^4_\lambda(y,z,v)) \tag{5.3}$$

where $y, \tilde{y} \in \mathbb{R}^k$, $z, v \in \mathbb{R}^{n-k}$, and

$$\tilde{\pi}^1_\lambda(y,\tilde{y}) = \lambda y + (1-\lambda)\tilde{y}, \tag{5.4}$$

$$\tilde{\pi}^2_\lambda(z,\tilde{y},v) = \lambda z + (1-\lambda)\eta(\tilde{y},v), \tag{5.5}$$

$$\tilde{\pi}^3_\lambda(y,z,\tilde{y},v) = \lambda\phi(y,z) + (1-\lambda)\zeta(\tilde{y},v), \tag{5.6}$$

$$\tilde{\pi}^4_\lambda(y,z,v) = \lambda\psi(y,z) + (1-\lambda)v. \tag{5.7}$$

Let

$$\kappa_\lambda : (\mathbb{R}^n, 0) \to (\mathbb{R}^{q-n}, 0)$$

denote the the contact-map between M^+ and $\mathcal{R}^\lambda_0(M^-)$. We have:

Proposition 5.4. *Local rings* $\dfrac{\mathcal{E}_{2n}}{\tilde{\pi}_\lambda^*(\mathfrak{m}_q)}$ *and* $\dfrac{\mathcal{E}_n}{\kappa_\lambda^*(\mathfrak{m}_{q-n})}$ *are isomorphic.*

Proof. From (5.3), we have that

$$\frac{\mathcal{E}_{2n}}{\tilde{\pi}_\lambda^*(\mathfrak{m}_q)} \simeq \frac{\mathcal{E}_{(y,z,\tilde{y},v)}}{\langle \tilde{\pi}_\lambda^1(y,\tilde{y}), \tilde{\pi}_\lambda^2(z,\tilde{y},v), \tilde{\pi}_\lambda^3(y,z,\tilde{y},v), \tilde{\pi}_\lambda^4(y,z,v)\rangle}$$

so that, using (5.4)-(5.7), this is isomorphic to

$$\frac{\mathcal{E}_{(y,z)}}{\langle z + \frac{(1-\lambda)}{\lambda}\eta(-\frac{\lambda}{(1-\lambda)}y, -\frac{\lambda}{(1-\lambda)}\psi(y,z)), \phi(y,z) + \frac{(1-\lambda)}{\lambda}\zeta(-\frac{\lambda}{(1-\lambda)}y, -\frac{\lambda}{(1-\lambda)}\psi(y,z))\rangle}$$

and, using (4.3) for $N_1 = M^+$ and $N_2 = \mathcal{R}_0^\lambda(M^-)$, we see that the above local ring is isomorphic to $\dfrac{\mathcal{E}_n}{\kappa_\lambda^*(\mathfrak{m}_{q-n})}$. □

On the other hand, we remind from Remark 4.12 that k is the degree of tangency of M^+ and $\mathcal{R}_0^\lambda(M^-)$ and therefore k is the degree of parallelism of $T_{a^+}M^+$ and $T_{a^-}M^-$, where

$$\lambda a^+ + (1-\lambda)a^- = 0 \in E_\lambda(M),$$

so that, denoting by

$$\theta_\lambda : (\mathbb{R}^k, 0) \to (\mathbb{R}^{k-(2n-q)}, 0)$$

the reduced (rank-0) contact map $\theta_\lambda = \kappa_{P_{N_1}, P_{N_2}}$, for $N_1 = M^+$ and $N_2 = \mathcal{R}_0^\lambda(M^-)$, from Proposition 4.14 we have the following

Corollary 5.5. *The local rings* $\dfrac{\mathcal{E}_n}{\kappa_\lambda^*(\mathfrak{m}_{q-n})}$ *and* $\dfrac{\mathcal{E}_k}{\theta_\lambda^*(\mathfrak{m}_{k-(2n-q)})}$ *are isomorphic.*

Thus, by Theorems 4.6 and 5.2, Proposition 5.4 and Corollary 5.5, for the local classification of stable singularities of affine equidistants, we need to determine every rank-0 $\mathcal{K}$-simple map-germ

$$\theta_\lambda : (\mathbb{R}^k, 0) \to (\mathbb{R}^l, 0) \; , \tag{5.8}$$

that admits a $\mathcal{K}$-versal deformation $F_\lambda : \mathbb{R}^k \times \mathbb{R}^q \to \mathbb{R}^l$, so that

$$\tilde{\pi}_\lambda : (F_\lambda)^{-1}(0) = (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^q, 0) \tag{5.9}$$

is an $\mathcal{A}$-stable map. Here, $\theta_\lambda = \kappa_{P_{N_1}, P_{N_2}}$, for $N_1 = M^+$ and $N_2 = \mathcal{R}_0^\lambda(M^-)$, and $\tilde{\pi}_\lambda$ is the restriction of π_λ to $M^+ \times M^-$, so that

$$l = k - (2n - q) \quad , \quad 1 \le k \le n \quad , \quad 2n \ge q > n \quad , \tag{5.10}$$

for any pair $(2n, q)$ in the nice dimensions (Theorem 5.3).

In other words, we unfold the map-germ θ_λ with m parameters,

$$\tilde{\pi}_\lambda : (\mathbb{R}^m \times \mathbb{R}^k, 0) \to (\mathbb{R}^m \times \mathbb{R}^l, 0) \quad , \quad (w, y) \mapsto (w, u(w, y)) \; , \tag{5.11}$$

where $m = 2n - k$, so that $\tilde{\pi}_\lambda$ is $\mathcal{A}$-stable. Thus, in each case, we look for the rank-0 $\mathcal{K}$-simple map-germs θ_λ that can be unfolded with $m = 2n - k$ parameters so that its $\mathcal{K}_e$-codimension μ is such that

$$\mu \le l + m = q \; . \tag{5.12}$$

The list of $\mathcal{K}$-simple map-germs θ_λ is presented in Tables 1, 2 and 3, in section 2 above. Thus, for classifying the stable singularities of affine equidistants of smooth submanifolds $M^n \subset \mathbb{R}^q$, all we have to do is read those Tables with respect to the numbers k, l and μ, subject to conditions (5.10) and (5.12) for each pair $(2n, q)$ in the nice dimensions.

In this way, we arrive at our main result, as follows.

5.1. **All possible stable singularities in the nice dimensions.** First, remind the definition of k-parallelism, cf. (2.1). Then, we have:

Theorem 5.6. *Let $M^n \subset \mathbb{R}^q$ be a smooth closed submanifold of the affine space, such that $2n \geq q$ and $(2n, q)$ is a pair of nice dimensions, as listed in Theorem 5.3. Then, the possible stable singularities of the λ-affine equidistant $E_\lambda(M) \subset \mathbb{R}^q$ are listed case by case, as below.*

Curves:

In this case, we have curves in $\mathbb{R}^2$ and the rank-0 contact map is $\theta_\lambda : \mathbb{R} \to \mathbb{R}$, $\mu \leq 2$. From Table 1, the stable singularities of affine equidistants can be of type A_1 and A_2.

Surfaces:

(1) $M^2 \subset \mathbb{R}^3$.
2-parallelism. $\theta_\lambda : \mathbb{R}^2 \to \mathbb{R}$, $\mu \leq 3$.
$E_\lambda(M)$ with stable singularities of types A_1, A_2 and A_3.

(2) $M^2 \subset \mathbb{R}^4$.
(i) 1-parallelism. $\theta_\lambda : \mathbb{R} \to \mathbb{R}$, $\mu \leq 4$.
$E_\lambda(M)$ with stable singularities of types A_1, A_2, A_3 and A_4.
(ii) 2-parallelism. $\theta_\lambda : \mathbb{R}^2 \to \mathbb{R}^2$, $\mu \leq 4$.
$E_\lambda(M)$ with stable singularities of types $C^{\pm}_{2,2}$.

3-*manifolds*:

(1) $M^3 \subset \mathbb{R}^4$.
3-parallelism. $\theta_\lambda : \mathbb{R}^3 \to \mathbb{R}$, $\mu \leq 4$.
$E_\lambda(M)$ with stable singularities of types A_1, ..., A_4 and $D^{\pm}_4$.

(2) $M^3 \subset \mathbb{R}^5$.
(i) 2-parallelism. $\theta_\lambda : \mathbb{R}^2 \to \mathbb{R}$, $\mu \leq 5$.
$E_\lambda(M)$ with stable singularities of types A_1, ..., A_5, $D^{\pm}_4$, $D^{\pm}_5$.
(ii) 3-parallelism. $\theta_\lambda : \mathbb{R}^3 \to \mathbb{R}^2$, $\mu \leq 5$.
$E_\lambda(M)$ with stable singularites of types S_5.

(3) $M^3 \subset \mathbb{R}^6$.
(i) 1-parallelism. $\theta_\lambda : \mathbb{R} \to \mathbb{R}$, $\mu \leq 6$.
$E_\lambda(M)$ with stable singularities of types A_1, ..., A_6.
(ii) 2-parallelism. $\theta_\lambda : \mathbb{R}^2 \to \mathbb{R}^2$, $\mu \leq 6$.
$E_\lambda(M)$ with stable singularities of types $C^{\pm}_{2,2}$, $C^{\pm}_{2,3}$, $C^{\pm}_{2,4}$, $C^{\pm}_{3,3}$, C_6.
(iii) 3-parallelism. No stable singularities for $E_\lambda(M)$.

4-*manifolds*:

(1) $M^4 \subset \mathbb{R}^5$.
4-parallelism. $\theta_\lambda : \mathbb{R}^4 \to \mathbb{R}$, $\mu \leq 5$.
$E_\lambda(M)$ with stable singularities of types A_1, ..., A_5, $D^{\pm}_4$, $D^{\pm}_5$.

(2) $M^4 \subset \mathbb{R}^6$: The map $\tilde{\pi}_\lambda : \mathbb{R}^8 \to \mathbb{R}^6$ is not in nice dimensions.

(3) $M^4 \subset \mathbb{R}^7$.
(i) 2-parallelism. $\theta_\lambda : \mathbb{R}^2 \to \mathbb{R}$, $\mu \leq 7$.
$E_\lambda(M)$ with stable singularities A_1, ..., A_7, $D_4^\pm$, ..., $D_7^\pm$, E_6, E_7.
(ii) 3-parallelism. $\theta_\lambda : \mathbb{R}^3 \to \mathbb{R}^2$, $\mu \leq 7$.
$E_\lambda(M)$ with stable singularities of types S_5, S_6, S_7, T_7, $\tilde{T}_7$.
(iii) 4-parallelism. No stable singularities for $E_\lambda(M)$.

(4) $M^4 \subset \mathbb{R}^8$.
(i) 1-parallelism. $\theta_\lambda : \mathbb{R} \to \mathbb{R}$, $\mu \leq 8$.
$E_\lambda(M)$ with stable singularities of types A_1, ..., A_8.
(ii) 2-parallelism. $\theta_\lambda : \mathbb{R}^2 \to \mathbb{R}^2$, $\mu \leq 8$.
$E_\lambda(M)$ with stable singularities of types
$C_{2,2}^\pm$, $C_{2,3}^\pm$, $C_{2,4}^\pm$, $C_{2,5}^\pm$, $C_{2,6}^\pm$, $C_{3,3}^\pm$, $C_{3,4}^\pm$, $C_{3,5}^\pm$, $C_{4,4}^\pm$, C_6, C_8, F_7, F_8.
(iii) 3-parallelism, 4-parallelism. No stable singularities for $E_\lambda(M)$.

5-*manifolds*:

(1) $M^5 \subset \mathbb{R}^6$.
5-parallelism. $\theta_\lambda : \mathbb{R}^5 \to \mathbb{R}$, $\mu \leq 6$.
$E_\lambda(M)$ with stable singularities A_1, ..., A_6, $D_4^\pm$, $D_5^\pm$, $D_6^\pm$, E_6.

(2) For all other embeddings $M^5 \subset \mathbb{R}^q$, no map $\tilde{\pi}_\lambda$ in nice dimensions.

n-manifolds, $n \geq 6$: No map $\tilde{\pi}_\lambda$ in nice dimensions.

References

[1] V. I. Arnol'd, S. M. Gusein-Zade, A. N. Varchenko, *Singularities of Differentiable Maps*, Vol. 1, Birhauser, Boston (1985). DOI: 10.1007/978-1-4612-5154-5

[2] M. V. Berry, *Semi-classical mechanics in phase space: A study of Wigner's function*, Philos. Trans. R. Soc. Lond., A **287**, 237-271 (1977). DOI: 10.1098/rsta.1977.0145

[3] W. Domitrz, P. de M. Rios, *Singularities of equidistants and Global Centre Symmetry sets of Lagrangian submanifolds*, Geom. Dedicata **169** , 361-382 (2014). DOI: 10.1007/s10711-013-9861-2

[4] W. Domitrz, M. Manoel, P. de M. Rios, *The Wigner caustic on shell and singularities of odd functions*, Journal of Geometry and Physics **71**, 58-72 (2013). DOI: 10.1016/j.geomphys.2013.04.005

[5] P. J. Giblin, V. M. Zakalyukin, *Singularities of systems of chords*, Funct. Anal. Appl. **36**, 220-224 (2002). DOI: 10.1023/A:1020154222764

[6] P. J. Giblin, V. M. Zakalyukin, *Singularities of centre symmetry sets*, Proc. London Math. Soc. **90**, no. 3, 132-166 (2005). DOI: 10.1112/S0024611504014923

[7] P. Giblin, S. Janeczko, *Geometry of curves and surfaces through the contact map*, Topology Appl. **159**, no. 2, 466-475 (2012). DOI: 10.1016/j.topol.2011.09.021

[8] M. Golubitsky, V. Guillemin, *Stable mappings and their singularities*, Springer, Berlin-N.Y. (1973). DOI: 10.1007/978-1-4615-7904-5

[9] V. V. Goryunov, *Simple projecting maps*, Sib. Math. J. **25**, 50-56 (1984). DOI: 10.1007/BF00969508

[10] V. V. Goryunov, *Singularities of projections of full intersections*, J. Sov. Math. **27**, 2785-2811 (1984). DOI: 10.1007/BF01084821

[11] J. Martinet, *Déploiements versels des applications differentiables et classification des applications stables*, Lecture Notes in Math. **535**, 1-44, Springer, Berlin-N.Y. (1976).

[12] J. N. Mather, *Stability of C^∞ mappings VI: the nice dimensions*, Lecture Notes in Math. **192**, 207-253, Springer, Berlin-N.Y. (1971).

[13] J. A. Montaldi, *On contact between submanifolds*, Michigan Math. J. **33**, no. 2, 195-199 (1986). DOI: 10.1307/mmj/1029003348

[14] A. M. Ozorio de Almeida, J. Hannay, *Geometry of Two Dimensional Tori in Phase Space: Projections, Sections and the Wigner Function*, Annals of Physics **138**, 115-154 (1982). DOI: 10.1016/0003-4916(82)90177-4

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SOME NOTES ON THE EULER OBSTRUCTION OF A FUNCTION

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Abstract. In this paper, we present an alternative proof of the Brasselet, Massey, Parameswaran and Seade formula for the Euler obstruction of a function [5] using Ebeling and Gusein-Zade's results on the radial index and the Euler obstruction of 1-forms [11].

1. Introduction

Let $(X,0) \subset (\mathbb{C}^N,0)$ be an equidimensional reduced complex analytic germ. The Euler obstruction $\mathrm{Eu}_X(0)$ was defined by MacPherson [20] as a tool to prove the conjecture about existence and unicity of Chern classes in the singular case. Since that the Euler obstruction has been deeply investigated by many authors as Brasselet, Schwartz, Seade, Sebastiani, Gonzalez-Sprinberg, Lê, Teissier, Sabbah, Dubson, Kato and others. For an overview about the Euler obstruction see [2, 3].

In [4] a Lefschetz type formula for the Euler obstruction was given by Brasselet, Lê and Seade. This formula relates the Euler obstruction $\mathrm{Eu}_X(0)$ to the topology of the Milnor fibre of a generic linear form $l : (X,0) \to (\mathbb{C},0)$. It shows that the Euler obstruction, as a constructible function, satisfies the Euler condition relatively to generic linear forms (Theorem 2.3).

In [5], the authors studied how far the equality given in the above theorem is from being true if we replace the generic linear form l with some other analytic function on X with at most an isolated stratified critical point at 0. For this, they defined the Euler obstruction $\mathrm{Eu}_{f,X}(0)$ of a function f on a complex analytic variety X, which can be seen as a generalization of the Milnor number, and they established a Lefschetz type formula for this new invariant (Theorem 2.5).

The definition of the Euler obstruction of a function was extended by Ebeling and Gusein-Zade in [11] to the case of complex 1-forms. When the 1-form is the differential of a holomorphic function f, they recovered the Euler obstruction of the function (up to sign). They also define the radial index of a 1-form, which is a generalization to the singular case of the classical Poincaré-Hopf index. Then they established relations between the local Euler obstruction of a 1-form, the radial index and Euler characteristics of complex links.

In this paper, we use the results of Ebeling and Gusein-Zade to give an alternative proof of the Brasselet, Massey, Parameswaran and Seade formula for the Euler obstruction of a function (Theorem 2.5).

The main idea of the original proof of Theorem 2.5 was to construct a vector field that combines all the properties needed to prove the result, essentially using Poincaré-Hopf type theorems. Let us say some words about our proof, which uses combinatorial techniques and is a less extensive and less constructive proof than the original one in [5]. We first give an expression of the Euler obstruction of a 1-form in terms of the radial indices of this form on the closures of the strata of X and Euler characteristics of complex links (this relation appears first in [11],

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Corollary 1, with a different proof). As a corollary, we obtain a formula for $\mathrm{Eu}_X(0) - \mathrm{Eu}_{f,X}(0)$ in terms of Euler characteristics of complex links and the Euler characteristics of the Milnor fibre of f on the closures of the stata of X. Then we use the addivity of the Euler characteristic to get a relation between $\mathrm{Eu}_{f,X}(0)$ and the Euler characteristics of the Milnor fibres of f on the strata of X.

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2. The Euler obstruction

Let us now introduce some objects in order to define the Euler obstruction.

Let $(X, 0) \subset (\mathbb{C}^N, 0)$ be an equidimensional reduced complex analytic germ of dimension d in an open set $U \subset \mathbb{C}^N$. We consider a complex analytic Whitney stratification $\{V_i\}$ of U adapted to X and we assume that $\{0\}$ is a stratum. We choose a small representative of $(X, 0)$ such that 0 belongs to the closure of all the strata. We denote it by X and we write $X = \cup_{i=0}^{q} V_i$ where $V_0 = \{0\}$ and $V_q = X_{\text{reg}}$, the set of smooth points of X. We assume that the strata $V_0, \dots, V_{q-1}$ are connected and that the analytic sets $\overline{V_0}, \dots, \overline{V_{q-1}}$ are reduced. We set $d_i = \dim V_i$ for $i \in \{1, \dots, q\}$ (note that $d_q = d$).

Let $G(d, N)$ denote the Grassmanian of complex d-planes in $\mathbb{C}^N$. On the regular part X_{reg} of X the Gauss map $\phi : X_{\text{reg}} \to U \times G(d, N)$ is well defined by $\phi(x) = (x, T_x(X_{\text{reg}}))$.

Definition 2.1. *The Nash transformation (or Nash blow-up) $\widetilde{X}$ of X is the closure of the image $\mathrm{Im}(\phi)$ in $U \times G(d, N)$. It is a (usually singular) complex analytic space endowed with an analytic projection map $\nu : \widetilde{X} \to X$ which is a biholomorphism away from $\nu^{-1}(\mathrm{Sing}(X))$.*

The fiber of the tautological bundle $\mathcal{T}$ over $G(d, N)$, at the point $P \in G(d, N)$, is the set of vectors v in the d-plane P. We still denote by $\mathcal{T}$ the corresponding trivial extension bundle over $U \times G(d, N)$. Let $\widetilde{T}$ be the restriction of $\mathcal{T}$ to $\widetilde{X}$, with projection map π. The bundle $\widetilde{T}$ on $\widetilde{X}$ is called *the Nash bundle* of X.

Let us recall the original definition of the Euler obstruction, due to MacPherson [20]. Let $z = (z_1, \dots, z_N)$ be local coordinates in $\mathbb{C}^N$ around $\{0\}$, such that $z_i(0) = 0$. We denote by B_ε and S_ε the ball and the sphere centered at $\{0\}$ and of radius ε in $\mathbb{C}^N$. Let us consider the norm $\|z\| = \sqrt{z_1\overline{z}_1 + \cdots + z_N\overline{z}_N}$. Then the differential form $\omega = d\|z\|^2$ defines a section of the real vector bundle $T(\mathbb{C}^N)^*$, cotangent bundle on $\mathbb{C}^N$. Its pull-back restricted to $\widetilde{X}$ becomes a section of the dual bundle $\widetilde{T}^*$ which we denote by $\widetilde{\omega}$. For ε small enough, the section $\widetilde{\omega}$ is nonzero over $\nu^{-1}(z)$ for $0 < \|z\| \leq \varepsilon$. The obstruction to extend $\widetilde{\omega}$ as a nonzero section of $\widetilde{T}^*$ from $\nu^{-1}(S_\varepsilon)$ to $\nu^{-1}(B_\varepsilon)$, denoted by $Obs(\widetilde{T}^*, \widetilde{\omega})$ lies in $H^{2d}(\nu^{-1}(B_\varepsilon), \nu^{-1}(S_\varepsilon); \mathbb{Z})$. Let us denote by $\mathcal{O}_{\nu^{-1}(B_\varepsilon), \nu^{-1}(S_\varepsilon)}$ the orientation class in $H_{2d}(\nu^{-1}(B_\varepsilon), \nu^{-1}(S_\varepsilon); \mathbb{Z})$.

Definition 2.2. *The local Euler obstruction of X at 0 is the evaluation of $Obs(\widetilde{T}^*, \widetilde{\omega})$ on $\mathcal{O}_{\nu^{-1}(\mathbb{B}_\varepsilon), \nu^{-1}(\mathbb{S}_\varepsilon)}$, i.e.:*

$$\mathrm{Eu}_X(0) = \langle Obs(\widetilde{T}^*, \widetilde{\omega}), \mathcal{O}_{\nu^{-1}(B_\varepsilon), \nu^{-1}(S_\varepsilon)} \rangle.$$

An equivalent definition of the Euler obstruction was given by Brasselet and Schwartz in the context of vector fields [6].

The idea of studying the Euler obstruction using hyperplane sections appears in the works of Dubson [8] and Kato [13], but the approach we follow here comes from [4, 5].

Theorem 2.3 ([4]). *Let $(X,0)$ and $\{V_i\}$ be given as before, then for each generic linear form l, there is ε_0 such that for any ε with $0<\varepsilon<\varepsilon_0$ and $\delta \neq 0$ sufficiently small, the Euler obstruction of $(X,0)$ is equal to:*

$$\mathrm{Eu}_X(0) = \sum_{i=1}^{q} \chi\big(V_i \cap B_\varepsilon \cap l^{-1}(\delta)\big) \cdot \mathrm{Eu}_X(V_i),$$

where χ denotes the Euler-Poincaré characteristic, $\mathrm{Eu}_X(V_i)$ is the value of the Euler obstruction of X at any point of V_i, $i=1,\dots,q$, and $0<|\delta| \ll \varepsilon \ll 1$.

We define now an invariant introduced by Brasselet, Massey, Parameswaran and Seade in [5], which measures in a way how far the equality given in Theorem 2.3 is from being true if we replace the generic linear form l with some other function on X with at most an isolated stratified critical point at 0. Let $f : X \to \mathbb{C}$ be a holomorphic function which is the restriction of a holomorphic function $F : U \to \mathbb{C}$. A point x in X is a critical point of f if it is a critical point of $F_{|V(x)}$, where $V(x)$ is the stratum containing x. We assume that f has an isolated singularity (or an isolated critical point) at 0, i.e. that f has no critical point in a punctured neighborhood of 0 in X. In order to define this new invariant, the authors constructed in [5] a stratified vector field on X, denoted by $\overline{\nabla}_X f$. This vector field is homotopic to $\overline{\nabla} F|_X$ and one has $\overline{\nabla}_X f(x) \neq 0$ unless $x=0$.

Let $\tilde{\zeta}$ be the lifting of $\overline{\nabla}_X f$ as a section of the Nash bundle $\widetilde{T}$ over $\widetilde{X}$ without singularity over $\nu^{-1}(X \cap S_\varepsilon)$. Let $\mathcal{O}(\tilde{\zeta}) \in H^{2n}\big(\nu^{-1}(X \cap B_\varepsilon), \nu^{-1}(X \cap S_\varepsilon)\big)$ be the obstruction cocycle to the extension of $\widetilde{\zeta}$ as a nowhere zero section of $\widetilde{T}$ inside $\nu^{-1}(X \cap B_\varepsilon)$.

Definition 2.4. *The local Euler obstruction $\mathrm{Eu}_{f,X}(0)$ is the evaluation of $\mathcal{O}(\tilde{\zeta})$ on the fundamental class of the pair $(\nu^{-1}(X \cap B_\varepsilon), \nu^{-1}(X \cap S_\varepsilon))$.*

The following result is the Brasselet, Massey, Parameswaran and Seade formula [5] that compares the Euler obstruction of the space X with that of a function on X.

Theorem 2.5. *Let $(X,0)$ and $\{V_i\}$ be given as before and let $f : (X,0) \to (\mathbb{C},0)$ be a function with an isolated singularity at 0. For $0<|\delta| \ll \varepsilon \ll 1$ we have:*

$$\mathrm{Eu}_X(0) - \mathrm{Eu}_{f,X}(0) = \left(\sum_{i=1}^{q} \chi\big(V_i \cap B_\varepsilon \cap f^{-1}(\delta)\big) \cdot \mathrm{Eu}_X(V_i) \right).$$

In this paper, we present an alternative proof for this result using Ebeling and Gusein-Zade's work [11] . In order to do this, let us consider the Nash bundle $\widetilde{T}$ on $\widetilde{X}$. The corresponding dual bundles of complex and real 1-forms are denoted, respectively, by $\widetilde{T}^* \to \widetilde{X}$ and $\widetilde{T}^*_{\mathbb{R}} \to \widetilde{X}$.

Definition 2.6. *Let $(X,0)$ and $\{V_\alpha\}$ be given as before. Let ω be a (real or complex) 1-form on X, i.e. a continuous section of either $T^*_{\mathbb{R}}\mathbb{C}^N|_X$ or $T^*\mathbb{C}^N|_X$. A singularity of ω in the stratified sense means a point x where the kernel of ω contains the tangent space of the corresponding stratum.*

This means that the pull-back of the form to V_α vanishes at x. Given a section η of $T^*_{\mathbb{R}}\mathbb{C}^N|_A$, $A \subset V$, there is a canonical way of constructing a section $\tilde{\eta}$ of $\widetilde{T}^*_{\mathbb{R}}|_{\tilde{A}}$, $\tilde{A} = \nu^{-1}A$, such that if η has an isolated singularity at the point $0 \in X$ (in the stratified sense), then we have a never-zero section $\widetilde{\eta}$ of the dual Nash bundle $\widetilde{T}^*_{\mathbb{R}}$ over $\nu^{-1}(S_\varepsilon \cap X) \subset \widetilde{X}$. Let

$$o(\eta) \in H^{2d}(\nu^{-1}(B_\varepsilon \cap X), \nu^{-1}(S_\varepsilon \cap X); \mathbb{Z})$$

be the cohomology class of the obstruction cycle to extend this to a section of $\widetilde{T}^*_{\mathbb{R}}$ over $\nu^{-1}(B_\varepsilon \cap X)$. Then we can define (c.f. [7]):

Definition 2.7. *The local Euler obstruction of the real differential form* η *at an isolated singularity is the integer* $\mathrm{Eu}_{X,0}\ \eta$ *obtained by evaluating the obstruction cohomology class* $o(\eta)$ *on the orientation fundamental cycle* $[\nu^{-1}(B_\varepsilon \cap X), \nu^{-1}(S_\varepsilon \cap X)]$.

In the complex case, one can perform the same construction, using the corresponding complex bundles. If ω is a complex differential form, section of $T^*\mathbb{C}^N|_A$ with an isolated singularity, one can define the local Euler obstruction $\mathrm{Eu}_{X,0}\ \omega$. Notice that, as explained in [7] p.151, it is equal to the local Euler obstruction of its real part up to sign:

$$\mathrm{Eu}_{X,0}\ \omega \ = \ (-1)^d \mathrm{Eu}_{X,0} \mathrm{Re}\ \omega.$$

This is an immediate consequence of the relation between the Chern classes of a complex vector bundle and those of its dual. Remark also that when we consider the differential of a function f, we have the following equality (see [11]):

$$\mathrm{Eu}_{X,0}\ df = (-1)^d \mathrm{Eu}_{f,X}(0).$$

3. The complex link, radial index and Euler obstruction

In this section, we recall the definition of the complex link and of the radial index. We also present a formula of Ebeling and Gusein-Zade which expresses the radial index of a 1-form in terms of Euler characteristics of complex links and Euler obstructions.

The complex link is an important object in the study of the topology of complex analytic sets. It is analogous to the Milnor fibre and was studied first in [15]. It plays a crucial role in complex stratified Morse theory (see [12]) and appears in general bouquet theorems for the Milnor fibre of a function with isolated singularity (see [16, 17, 22, 23]). It is related to the multiplicity of polar varieties and also the local Euler obstruction (see [8, 9, 18, 19]). Let us recall briefly its definition. Let M be a complex analytic manifold equipped with a Riemannian metric and let $Y \subset M$ be a complex analytic variety equipped with a Whitney stratification. Let V be a stratum of Y and let p be a point in V. Let N be a complex analytic submanifold of M which meets V transversally at the single point p. By choosing local coordinates on N, in some neighborhood of p we can assume that N is an Euclidian space $\mathbb{C}^k$.

Definition 3.1. *The complex link of* V *in* Y *is the set denoted by* $\mathrm{lk}^{\mathbb{C}}(V, Y)$ *and defined as follows:*

$$\mathrm{lk}^{\mathbb{C}}(V, Y) = Y \cap N \cap B_\varepsilon \cap l^{-1}(\delta),$$

where $l : N \to \mathbb{C}$ *is a generic linear form and* $0 < |\delta| \ll \varepsilon \ll 1$.

The fact that the complex link of a stratum is well-defined, i.e. independent of all the choices made to define it, is explained in [19, 9, 12]. It is also independent of the embedding of the analytic variety Y (see [19]).

In [11], Ebeling and Gusein-Zade established relations between the local Euler obstruction of a 1-form, its radial index and Euler characteristics of complex links. The radial index is a generalization to the singular case of the Poincaré-Hopf index.

This index for 1-forms is a natural extension of the equivalent notion for vector fields, a notion first introduced by King and Trotman in a 1995 preprint only recently published [14] and then studied by Ebeling and Gusein-Zade in [10] and by Aguilar, Seade and Verjovsky in [1].

In order to define this index, let us consider first the real case. Let $Z \subset \mathbb{R}^n$ be a closed subanalytic set equipped with a Whitney stratification $\{S_\alpha\}_{\alpha \in \Lambda}$. Let ω be a continuous 1-form defined on $\mathbb{R}^n$. We say that a point P in Z is a zero (or a singular point) of ω on Z if it is a zero of $\omega_{|S}$, where S is the stratum that contains P. In the sequel, we define the radial index of ω at P, when P is an isolated zero of ω on Z. We can assume that $P = 0$ and we denote by S_0 the stratum that contains 0.

Definition 3.2. *A 1-form ω is radial on Z at 0 if, for an arbitrary non-trivial subanalytic arc $\varphi : [0,\nu[\to Z$ of class C^1, the value of the form ω on the tangent vector $\dot{\varphi}(t)$ is positive for t small enough.*

Let $\varepsilon > 0$ be small enough so that in the closed ball B_ε, the 1-form has no singular points on $Z \setminus \{0\}$. Let $S_0, \ldots, S_r$ be the strata that contain 0 in their closure. Following Ebeling and Gusein-Zade, there exists a 1-form $\tilde{\omega}$ on $\mathbb{R}^n$ such that:

(1) The 1-form $\tilde{\omega}$ coincides with the 1-form ω on a neighborhood of S_ε.
(2) The 1-form $\tilde{\omega}$ is radial on Z at the origin.
(3) In a neighborhood of each zero $Q \in Z \cap B_\varepsilon \setminus \{0\}$, $Q \in S_i$, $\dim S_i = k$, the 1-form $\tilde{\omega}$ looks as follows. There exists a local subanalytic diffeomorphism $h : (\mathbb{R}^n, \mathbb{R}^k, 0) \to (\mathbb{R}^n, S_i, Q)$ such that $h^*\tilde{\omega} = \pi_1^*\tilde{\omega}_1 + \pi_2^*\tilde{\omega}_2$ where π_1 and π_2 are the natural projections $\pi_1 : \mathbb{R}^n \to \mathbb{R}^k$ and $\pi_2 : \mathbb{R}^n \to \mathbb{R}^{n-k}$, $\tilde{\omega}_1$ is a 1-form on a neighborhood of 0 in $\mathbb{R}^k$ with an isolated zero at the origin and $\tilde{\omega}_2$ is a radial 1-form on $\mathbb{R}^{n-k}$ at 0.

Definition 3.3. *The radial index $\mathrm{ind}_{Z,0}^{\mathbb{R}}\ \omega$ of the 1-form ω on Z at 0 is the sum:*

$$1 + \sum_{i=0}^{r} \sum_{Q \mid \tilde{\omega}_{|S_i}(Q)=0} \mathrm{ind}_{PH}(\tilde{\omega}, Q, S_i),$$

where $\mathrm{ind}_{PH}(\tilde{\omega}, Q, S_i)$ is the Poincaré-Hopf index of the form $\tilde{\omega}_{|S_i}$ at Q and where the sum is taken over all zeros of the 1-form $\tilde{\omega}$ on $(Z \setminus \{0\}) \cap B_\varepsilon$. If 0 is not a zero of ω on Z, we put $\mathrm{ind}_{Z,0}^{\mathbb{R}}\ \omega = 0$.

A straightforward corollary of this definition is that the radial index satisfies the law of conservation of number (see Remark 9.4.6 in [7] or the remark before Proposition 1 in [11]).

Let us go back to the complex case. As in Section 2, $(X,0) \subset (\mathbb{C}^N, 0)$ is an equidimensional reduced complex analytic germ of dimension d in an open set $U \subset \mathbb{C}^N$. Let ω be a complex 1-form on U with an isolated singular point on X at the origin.

Definition 3.4. *The complex radial index $\mathrm{ind}_{X,0}^{\mathbb{C}}\ \omega$ of the complex 1-form ω on X at the origin is $(-1)^d$ times the index of the real 1-form given by the real part of ω.*

Let us write $n_i = (-1)^{d-d_i-1}\Big(\chi\big(\mathrm{lk}^{\mathbb{C}}(V_i, X)\big) - 1\Big)$, where $\{V_i\}$ is the Whitney stratification of $(X,0)$ considered before. In particular for an open stratum V_i of X, $\mathrm{lk}^{\mathbb{C}}(V_i, X)$ is empty and so $n_i = 1$. Let us define the Euler obstruction $\mathrm{Eu}_{Y,0}\ \omega$ to be equal to 1 for a zero-dimensional connected variety Y. Under this conditions Ebeling and Gusein-Zade proved in [11] the following result which relates the radial index of a 1-form to Euler obstructions.

Theorem 3.5. *Let $(X,0) \subset (\mathbb{C}^N, 0)$ be the germ of a reduced complex analytic space at the origin, with a Whitney stratification $\{V_i\}$, $i = 0, \ldots, q$, where $V_0 = \{0\}$ and V_q is the regular part of X. Then:*

$$\mathrm{ind}_{X,0}^{\mathbb{C}}\ \omega = \sum_{i=0}^{q} n_i \cdot \mathrm{Eu}_{\overline{V_i},0}\ \omega.$$

4. Corollaries of Theorem 3.5 and alternative proof of Theorem 2.5

In this section, we give some corollaries of Theorem 3.5, among them an alternative proof of Theorem 2.5.

As in the previous sections, $(X,0) \subset (\mathbb{C}^N, 0)$ is an equidimensional reduced complex analytic germ of dimension d in an open set U, equipped with a Whitney stratification $\{V_i\}$ such that 0 belongs to the closure of all the strata. We write $X = \cup_{i=0}^{q} V_i$ where $V_0 = \{0\}$ and $V_q = X_{\text{reg}}$. We assume that the strata $V_0, \dots, V_{q-1}$ are connected and that the analytic sets $\overline{V_0}, \dots, \overline{V_{q-1}}$ are reduced. We set $d_i = \dim V_i$ for $i \in \{1, \dots, q\}$. Let $f : X \to \mathbb{C}$ be a holomorphic function which is the restriction of a holomorphic function $F : U \to \mathbb{C}$. We assume that f has an isolated singularity at 0.

Let us see what happens when we apply Theorem 3.5 to the form $\sum \overline{z_k} dz_k$. Let us consider $(z_1, z_2, \dots, z_N)$ as complex coordinates of $\mathbb{C}^N$, where $z_k = u_k + \sqrt{-1} v_k$. This implies that $(u_1, v_1, \dots, u_N, v_N)$ are real coordinates of $\mathbb{R}^{2N}$. Let ω be a 1-form defined by $\omega = \sum_k \overline{z_k} dz_k$, it means that:

$$\omega = \sum_k (u_k - \sqrt{-1} v_k)(du_k + \sqrt{-1} dv_k),$$

and so that:

$$\omega = \sum_k (u_k du_k + v_k dv_k) + \sqrt{-1} \sum (u_k dv_k - v_k du_k).$$

In this case, the real 1-form $\text{Re}\, \omega = \sum (u_k du_k + v_k dv_k)$ is a radial 1-form, and $\text{ind}_{X,0}^{\mathbb{R}} \text{Re}\, \omega = 1$. Since $\text{ind}_{X,0}^{\mathbb{C}} \omega = (-1)^d \text{ind}_{X,0}^{\mathbb{R}} \text{Re}\, \omega$, we find that:

$$\text{ind}_{X,0}^{\mathbb{C}} \omega = (-1)^d \text{ind}_{X,0}^{\mathbb{R}} \text{Re}\, \omega = (-1)^d.$$

As it was remarked before,

$$\text{Eu}_{X,0}\, \omega = (-1)^d \text{Eu}_{X,0} \text{ Re}\, \omega.$$

Using this information and the definition of n_i given in Section 3, we have the next equality:

$$n_i \text{Eu}_{\overline{V_i},0}\, \omega = (-1)^{d-d_i-1} \left(\chi\big(\text{lk}^{\mathbb{C}}(V_i, X)\big) - 1 \right) (-1)^{d_i} \text{Eu}_{\overline{V_i}}(0).$$

Therefore, by Theorem 3.5 we conclude that:

$$(-1)^d = (-1)^d \left[\sum_{i=0}^{q-1} \left(1 - \chi\big(\text{lk}^{\mathbb{C}}(V_i, X)\big) \right) \text{Eu}_{\overline{V_i}}(0) + \text{Eu}_X(0) \right],$$

and so we arrive to the following lemma:

Lemma 4.1. *We have:*

$$\text{Eu}_X(0) = 1 + \sum_{i=0}^{q-1} \left(\chi\big(\text{lk}^{\mathbb{C}}(V_i, X)\big) - 1 \right) \text{Eu}_{\overline{V_i}}(0). \tag{1}$$

When we apply Theorem 3.5 to the form df, we obtain a similar result for the Euler obstruction of the function f.

Lemma 4.2. *We have:*

$$1 - \chi(f^{-1}(\delta) \cap X \cap B_\varepsilon) = \sum_{i=0}^{q} \left(1 - \chi\big(\text{lk}^{\mathbb{C}}(V_i, X)\big) \right) \text{Eu}_{f,\overline{V_i}}(0).$$

Proof. On the one hand, applying Theorem 3.5 to the form df, we have:

$$\text{ind}_{X,0}^{\mathbb{C}}\, df = \sum_{i=0}^{q} n_i \text{Eu}_{\overline{V_i},0}\, df = (-1)^{d-d_i-1} \left(\chi\big(\text{lk}^{\mathbb{C}}(V_i, X)\big) - 1 \right) (-1)^{d_i} \text{Eu}_{f,\overline{V_i}}(0).$$

On the other hand, by Theorem 3 of [11] we have:

$$\text{ind}_{X,0}^{\mathbb{C}}\, df = (-1)^d \left(1 - \chi(f^{-1}(\delta) \cap X \cap B_\varepsilon) \right).$$

It follows that:

$$1-\chi(f^{-1}(\delta)\cap X\cap B_\varepsilon)=\sum_{i=0}^{q}\Big(1-\chi\big(\mathrm{lk}^{\mathbb{C}}(V_i,X)\big)\Big)\,\mathrm{Eu}_{f,\overline{V_i}}(0).$$

□

Before stating the next result, let us set $\mathrm{B}_{f,X}(0)=\mathrm{Eu}_X(0)-\mathrm{Eu}_{f,X}(0)$.

Corollary 4.3. *We have:*

$$\chi(f^{-1}(\delta)\cap X\cap B_\varepsilon)=\sum_{i=0}^{q}\Big(1-\chi\big(\mathrm{lk}^{\mathbb{C}}(V_i,X)\big)\Big)\,\mathrm{B}_{f,\overline{V_i}}(0).$$

Proof. By the previous lemma, we have the following equation:

$$\mathrm{Eu}_{f,X}(0)=1-\chi(f^{-1}(\delta)\cap X\cap B_\varepsilon)+\sum_{i=0}^{q-1}\Big(\chi\big(\mathrm{lk}^{\mathbb{C}}(V_i,X)\big)-1\Big)\,\mathrm{Eu}_{f,\overline{V_i}}(0). \tag{2}$$

By the difference (1) − (2) we arrive at:

$$\mathrm{B}_{f,X}(0)=\chi(f^{-1}(\delta)\cap X\cap B_\varepsilon)+\sum_{i=0}^{q-1}\Big(\chi\big(\mathrm{lk}^{\mathbb{C}}(V_i,X)\big)-1\Big)\,\mathrm{B}_{f,\overline{V_i}}(0). \tag{3}$$

Hence we find:

$$\chi(f^{-1}(\delta)\cap X\cap B_\varepsilon)=\sum_{i=0}^{q}\Big(1-\chi\big(\mathrm{lk}^{\mathbb{C}}(V_i,X)\big)\Big)\,\mathrm{B}_{f,\overline{V_i}}(0).$$

□

In [11, Corollary 1], Ebeling and Gusein-Zade give an "inverse" of the formula of Theorem 3.5. They use combinatorial theory (Möbius inverse). In the sequel, we give an inductive proof of that result. Let us recall the notations of [11]. The strata V_i of X are partially ordered: $V_i\prec V_j$ (we shall write $i\prec j$) if $V_i\subset\overline{V_j}$ and $V_i\neq V_j$. For two strata V_i and V_j with $V_i\preceq V_j$ (we shall write $i\preceq j$), let N_{ij} be the normal slice of the variety $\overline{V_j}$ to the stratum V_i at a point of it and let $M_{l|N_{ij}}$ be the complex link of V_i in $\overline{V_j}$. We denote $\chi(Z)-1$ by $\overline{\chi}(Z)$. For $i\prec j$, let m_{ij} be defined as follows:

$$m_{ij}=(-1)^{\dim\,X-\dim\,V_i}\sum_{i=k_0\prec\cdots\prec k_r=j}\overline{\chi}(M_{l|N_{k_0k_1}})\cdots\overline{\chi}(M_{l|N_{k_{r-1}k_r}}),$$

and let us set $m_{ii}=1$.

Corollary 4.4. *Let ω be a complex 1-form with an isolated zero on X at the origin. We have:*

$$\mathrm{Eu}_{X,0}\;\omega=\sum_{i=0}^{q}m_{iq}\cdot\mathrm{ind}^{\mathbb{C}}_{\overline{V_i},0}\;\omega.$$

Proof. This is clearly true if $\dim X=0$. Let us assume that $\dim X=d\geq 1$ and prove the result by induction on the depth of the stratification. The first step is to consider the case when X has an isolated singularity at the origin. In this case, the stratification will be $\{V_0=\{0\},V_1=X_{\mathrm{reg}}\}$ and

$$n_0=(-1)^{d-1}(\chi(\mathrm{lk}^{\mathbb{C}}(V_0,X)-1)=(-1)^{d-1}\overline{\chi}(M_{l|N_{01}}),$$

$\mathrm{Eu}_{X,0}\;\omega=1$, $n_1=1$ and $\mathrm{Eu}_{\overline{V_1},0}\;\omega=\mathrm{Eu}_{X,0}\;\omega$. Applying Theorem 3.5, we get:

$$\mathrm{ind}^{\mathbb{C}}_{X,0}\;\omega=(-1)^{d-1}\overline{\chi}(M_{l|N_{01}})+\mathrm{Eu}_{X,0}\;\omega,$$

and so:

$$\mathrm{Eu}_{X,0}\ \omega = \mathrm{ind}^{\mathbb{C}}_{X,0}\ \omega + (-1)^d \overline{\chi}(M_{l|N_{01}}).$$

This is exactly the expected formula because $\mathrm{ind}^{\mathbb{C}}_{V_0,0}\ \omega = 1$ and $m_{01} = (-1)^d \overline{\chi}(M_{l|N_{01}})$.

Let us prove the general case. By the induction hypothesis, for each $k \in \{0, \ldots, d-1\}$, we have:

$$\mathrm{Eu}_{\overline{V_k},0}\ \omega = \sum_{j \ | \ V_j \subset \overline{V_k}} m_{ik} \cdot \mathrm{ind}^{\mathbb{C}}_{\overline{V_j},0}\ \omega.$$

But we know by Theorem 3.5 that:

$$\mathrm{Eu}_{X,0}\ \omega = \mathrm{ind}^{\mathbb{C}}_{X,0}\ \omega - \sum_{k=0}^{d-1} n_k \cdot \mathrm{Eu}_{\overline{V_k},0}\ \omega.$$

Replacing $\mathrm{Eu}_{\overline{V_k},0}\ \omega$ by its above value, we obtain:

$$\mathrm{Eu}_{X,0}\ \omega = \mathrm{ind}^{\mathbb{C}}_{X,0}\ \omega - \sum_{k=0}^{d-1} n_k \left(\mathrm{ind}^{\mathbb{C}}_{\overline{V_k},0}\ \omega + \sum_{j \ | \ V_j \subset \partial\overline{V_k}} m_{jk} \cdot \mathrm{ind}^{\mathbb{C}}_{\overline{V_j},0}\ \omega \right).$$

We see that each $\mathrm{ind}_{\overline{V_j},0}\mathbb{C}\ \omega$ appears in each term

$$n_k \left(\mathrm{ind}^{\mathbb{C}}_{\overline{V_k},0}\ \omega + \sum_{j \ | \ V_j \subset \partial\overline{V_k}} m_{jk} \cdot \mathrm{ind}^{\mathbb{C}}_{\overline{V_j},0}\ \omega \right),$$

for which $V_j \subset \overline{V_k}$. Therefore we can write:

$$\mathrm{Eu}_{X,0}\ \omega = \mathrm{ind}^{\mathbb{C}}_{X,0}\ \omega - \sum_{j=0}^{d-1} \mathrm{ind}^{\mathbb{C}}_{\overline{V_j},0}\ \omega \left(n_j + \sum_{k \ | \ V_j \subset \partial\overline{V_k}} m_{jk} \cdot n_k \right).$$

Let us examine $A_j = n_j + \sum_{k \ | \ V_j \subset \partial\overline{V_k}} m_{jk} \cdot n_k$. We have:

$$A_j = (-1)^{d-d_j-1}\overline{\chi}(M_{l|N_{jq}}) +$$
$$\sum_{k \ | \ V_j \subset \partial\overline{V_k}} \Big((-1)^{d_k-d_j-1} \sum_{j=k_0 \prec \cdots \prec k_r = k} \overline{\chi}(M_{l|N_{k_0k_1}}) \cdots \overline{\chi}(M_{l|N_{k_{r-1}k_r}}) \times (-1)^{d-d_k-1}\overline{\chi}(M_{l|N_{kq}}) \Big).$$

Therefore, we see that:

$$A_j = (-1)^{d-d_j-1}\overline{\chi}(M_{l|N_{jq}}) +$$
$$\sum_{k \ | \ V_j \subset \partial\overline{V_k}} \left((-1)^{d-d_j-1} \sum_{j=k_0 \prec \cdots \prec k_{r+1} = q} \overline{\chi}(M_{l|N_{k_0k_1}}) \cdots \overline{\chi}(M_{l|N_{k_rk_{r+1}}}) \right),$$

i.e. $A_j = -m_{jq}$. We get the desired result. □

When we apply this to the form $\omega = \sum \overline{z_k} dz_k$, we get:

$$\mathrm{Eu}_X(0) = \sum_{i=0}^{q} \sum_{i=k_0 \prec \cdots \prec k_r = q} \overline{\chi}(M_{l|N_{k_0k_1}}) \cdots \overline{\chi}(M_{l|N_{k_{r-1}k_r}}). \qquad (*)$$

This formula is still valid if $V_0 \neq \{0\}$. In this case, we can introduce the stratum $V_{-1} = \{0\}$. The above formula becomes:

$$\mathrm{Eu}_X(0) = \sum_{i=-1}^{q} \sum_{i=k_0 \prec \cdots \prec k_r = q} \overline{\chi}(M_{l|N_{k_0 k_1}}) \cdots \overline{\chi}(M_{l|N_{k_{r-1} k_r}}).$$

But since generically the linear form l has no singularity at 0 on V_0, the Milnor fibre $M_{l|\overline{V_k}}$ of $l : \overline{V_k} \to \mathbb{C}$ is contractible for $k \geq 0$, which implies that $\overline{\chi}(M_{l|N_{-1k}}) = 0$ for $k \geq 0$.

Applied to the form df, Corollary 4.4 gives:

$$\mathrm{Eu}_{f,X}(0) = -\sum_{i=0}^{q} \overline{\chi}(M_{f|\overline{V_i}}) \sum_{i=k_0 \prec \cdots \prec k_r = q} \overline{\chi}(M_{l|N_{k_0 k_1}}) \cdots \overline{\chi}(M_{l|N_{k_{r-1} k_r}}), \qquad (**)$$

where $M_{f|\overline{V_i}}$ denotes the Milnor fibre of $f : \overline{V_i} \to \mathbb{C}$, because $\mathrm{Eu}_{f,X}(0) = (-1)^d \mathrm{Eu}_{X,0}\ df$ and $\mathrm{ind}^{\mathbb{C}}_{\overline{V_i},0}\ df = (-1)^{d_i - 1} \overline{\chi}(M_{f|\overline{V_i}})$.

We are now in position to give the alternative proof of Theorem 2.5.

Proof. Using the two equalities $(*)$ and $(**)$ above, we find:

$$\mathrm{Eu}_X(0) - \mathrm{Eu}_{f,X}(0) = \sum_{i=0}^{q} \chi(M_{f|\overline{V_i}}) \sum_{i=k_0 \prec \cdots \prec k_r = q} \overline{\chi}(M_{l|N_{k_0 k_1}}) \cdots \overline{\chi}(M_{l|N_{k_{r-1} k_r}}).$$

By the additivity of the Euler characteristic, for each $i \in \{0, \ldots, q\}$ we have:

$$\chi(M_{f|\overline{V_i}}) = \sum_{j\ |V_j \subset \overline{V_i}} \chi(M_{f|V_j}).$$

Therefore, we have:

$$\mathrm{Eu}_X(0) - \mathrm{Eu}_{f,X}(0) = \sum_{i=0}^{q} \left(\sum_{j\ |V_j \subset \overline{V_i}} \chi(M_{f|V_j}) \right) \sum_{i=k_0 \prec \cdots \prec k_r = q} \overline{\chi}(M_{l|N_{k_0 k_1}}) \cdots \overline{\chi}(M_{l|N_{k_{r-1} k_r}}).$$

As in the proof of the previous corollary, we see that each $\chi(M_{f|V_j})$ appears in an expression

$$\left(\sum_{j\ |V_j \subset \overline{V_i}} \chi(M_{f|V_j}) \right) \sum_{i=k_0 \prec \cdots \prec k_r = q} \overline{\chi}(M_{l|N_{k_0 k_1}}) \cdots \overline{\chi}(M_{l|N_{k_{r-1} k_r}}),$$

when $V_j \subset \overline{V_i}$. We can factorize $\chi(M_{f|V_j})$ in the above equality and get:

$$\mathrm{Eu}_X(0) - \mathrm{Eu}_{f,X}(0) = \sum_{j=0}^{q} \chi(M_{f|V_j}) \left(\sum_{i\ |V_j \subset \overline{V_i}} \sum_{i=k_0 \prec \cdots \prec k_r = q} \overline{\chi}(M_{l|N_{k_0 k_1}}) \cdots \overline{\chi}(M_{l|N_{k_{r-1} k_r}}) \right).$$

But by the equality $(*)$ and the remark that follows it, we see that:

$$\sum_{i\ |V_j \subset \overline{V_i}} \sum_{i=k_0 \prec \cdots \prec k_r = q} \overline{\chi}(M_{l|N_{k_0 k_1}}) \cdots \overline{\chi}(M_{l|N_{k_{r-1} k_r}}),$$

is exactly $\mathrm{Eu}_X(V_j)$. □

References

[1] AGUILAR, M., SEADE, J. and VERJOVSKY, A.: Indices of vector fields and topological invariants or real analytic singularities, *J. Reine Angew. Math.* **504** (1998), 159-176.

[2] BRASSELET, J.-P.: Local Euler obstruction, old and new, *XI Brazilian Topology Meeting (Rio Claro, 1998)*, 140-147, World Sci. Publishing, River Edge, NJ, 2000.

[3] BRASSELET, J.-P. and GRULHA JR., N. G.: Local Euler obstruction, old and new II, *London Mathematical Society - Lectures Notes Series 380 - Real and Complex Singularities*, Cambridge University Press, 2010, v., p. 23-45.

[4] BRASSELET, J.-P., LÊ, D. T. and SEADE, J.: Euler obstruction and indices of vector fields, *Topology*, **6** (2000) 1193-1208. DOI: 10.1016/S0040-9383(99)00009-9

[5] BRASSELET, J.-P., MASSEY, D. and PARAMESWARAN, A. and SEADE, J.: Euler obstruction and defects of functions on singular varieties, *Journal London Math. Soc (2)* **70** (2004), no.1, 59-76.

[6] BRASSELET, J.-P. and SCHWARTZ, M.-H.: Sur les classes de Chern d'un ensemble analytique complexe, *Astérisque* **82-83** (1981) 93-147.

[7] BRASSELET, J.-P., SEADE, J. and SUWA, T.: Vector fields on singular varieties, *Lecture Notes in Mathematics*, **1987**, Springer-Verlag, Berlin (2009).

[8] DUBSON, A.: Classes caractéristiques des variétés singulières, *C. R. Acad. Sci. Paris Sr. A-B* **287** (1978), no. 4, 237-A240.

[9] DUBSON, A.: Calcul des invariants numériques des singularités et applications, *Sonderforschungsbereich 40 Theoretische Mathematik* Universitaet Bonn (1981).

[10] EBELING, W. and GUSEIN-ZADE, S.M .: On the index of a vector field at an isolated singularity, *The Arnoldfest (Toronto, ON, 1997)*, 141-152, Fields Inst. Commun., 24, Amer. Math. Soc., Providence, RI, 1999.

[11] EBELING, W. and GUSEIN-ZADE, S.M .: Radial index and Euler obstruction of a 1-form on a singular variety, *Geom. Dedicata* **113** (2005), 231-241. DOI: 10.1007/s10711-005-2184-1

[12] GORESKY, M. and MACPHERSON, R.: Stratified Morse theory, *Springer-Verlag*, Berlin, 1988. DOI: 10.1007/978-3-642-71714-7

[13] KATO, M.: Singularities and some global topological properties, *Proc. R.I.M.S. Singularities Symposium*, Kyoto University (1978).

[14] KING, H. and TROTMAN, D.: Poincaré-Hopf theorems on singular spaces, *Proc. London Math. Soc.*, 108 (3), (2014), 682-703. DOI: 10.1112/plms/pdt039

[15] LÊ D.T.: Vanishing cycles on complex analytic sets, *Proc. Sympos., Res. Inst. Math. Sci., Kyoto, Univ. Kyoto, 1975* Sûrikaisekikenkyûsho Kókyûroku, no. 266 (1976), 299-318.

[16] LÊ D. T.: Le concept de singularité isolée de fonction analytique, *Adv. Stud. Pure Math.* **8** (1986) 215–227.

[17] LÊ D.T.: Complex analytic functions with isolated singularities, *J. Algebraic Geom.* **1** (1992), no. 1, 83-99.

[18] LÊ D. T. and TEISSIER, B.: Variétés polaires Locales et classes de Chern des variétés singulières, *Ann. of Math.* **114**, (1981), 457-491. DOI: 10.2307/1971299

[19] LÊ D. T. and TEISSIER, B.: Cycles évanescents, sections planes et conditions de Whitney II, Singularities, Part 2 (Arcata, Calif., 1981), 65-103, *Proc. Sympos. Pure Math.* **40**, Amer. Math. Soc., Providence, RI, 1983.

[20] MACPHERSON, R. D.: Chern classes for singular algebraic varieties, *Ann. of Math.* **100** (1974), 423-432. DOI: 10.2307/1971080

[21] SEADE, J., TIBĂR, M. and VERJOVSKY A.: Milnor Numbers and Euler obstruction, *Bull. Braz. Math. Soc. (N.S)* **36** (2005), no. 2, 275-283.

[22] SIERSMA, D.: A bouquet theorem for the Milnor fibre, *J. Algebraic Geom.* **4** (1995), no. 1, 51-66.

[23] TIBĂR, M.: Bouquet decomposition of the Milnor fibre, *Topology* **35** (1996), no. 1, 227-241.

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EVOLUTES OF FRONTS IN THE EUCLIDEAN PLANE

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Dedicated to Professor Shyuichi Izumiya on the occasion of his 60th birthday

Abstract. The evolute of a regular curve in the Euclidean plane is given by not only the caustics of the regular curve, envelope of normal lines of the regular curve, but also the locus of singular points of parallel curves. In general, the evolute of a regular curve has singularities, since such points correspond to vertices of the regular curve and there are at least four vertices for simple closed curves. If we repeat an evolute, we cannot define the evolute at a singular point. In this paper, we define an evolute of a front and give properties of such an evolute by using a moving frame along a front and the curvature of the Legendre immersion. As applications, repeated evolutes are useful to recognize the shape of curves.

1. Introduction

The evolute of a regular plane curve is a classical object (cf. [5, 8, 9]). It is useful for recognizing the vertex of a regular plane curve as a singularity (generically, a $3/2$ cusp singularity) of the evolute. The caustics (evolutes) are related to general relativity theory, see for instance [6, 10]. The properties of evolutes are discussed by using distance squared functions and the theories of Lagrangian and Legendrian singularities (cf. [1, 2, 3, 13, 14, 17, 20]). Moreover, the singular points of parallel curves of a regular curve sweep out the evolute. By using this property, we define an evolute of a front in §2. In order to consider properties of an evolute of a front, we introduce a moving frame along a front (a Legendre immersion) (cf. [7]). In [7], we give existence and uniqueness for a Legendre curve in the unit tangent bundle like for regular plane curves. It is quite useful to analyze a Legendre curve (or, a frontal) in the unit tangent bundle. In §3, we give another representation for the evolute of a front by using the moving frame and the curvature of the Legendre immersion (Theorem 3.3). By the representation, we give properties of the evolutes of fronts, for example, the evolute of a front is also a front. It follows that we can consider the repeated evolutes, namely, the evolute of an evolute of a front, see Theorem 4.1 in §4. Moreover, we extend the notion of the vertex for a front (or, a Legendre immersion) and give a kind of four vertex theorem for a front, see Proposition 3.11. Furthermore, the evolute of a front is also given by the envelope of normal lines of the front. A singular point of the evolute of the evolute of a regular curve measure to the contact of an involute of a circle. We give the n-th evolute of a front in §5. In §6, we give examples of the evolutes of fronts. In the appendix, we give the condition of contact between regular curves.

All maps and manifolds considered here are differentiable of class C^∞.

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2. Definitions and basic concepts

Let I be an interval or $\mathbb{R}$. Suppose that $\gamma : I \to \mathbb{R}^2$ is a regular plane curve, that is, $\dot{\gamma}(t) \neq 0$ for any $t \in I$. If s is the arc-length parameter of γ, we denote $\boldsymbol{t}(s)$ by the unit tangent vector $\boldsymbol{t}(s) = \gamma'(s) = (d\gamma/ds)(s)$ and $\boldsymbol{n}(s)$ by the unit normal vector $\boldsymbol{n}(s) = J(\boldsymbol{t}(s))$ of $\gamma(s)$, where J is the anticlockwise rotation by $\pi/2$. Then we have the Frenet formula as follows:

$$\begin{pmatrix} \boldsymbol{t}'(s) \\ \boldsymbol{n}'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) \\ -\kappa(s) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{t}(s) \\ \boldsymbol{n}(s) \end{pmatrix},$$

where $\kappa(s) = \boldsymbol{t}'(s) \cdot \boldsymbol{n}(s)$ is the curvature of γ and $\cdot$ is the inner product on $\mathbb{R}^2$.

Even if t is not the arc-length parameter, we have the unit tangent vector $\boldsymbol{t}(t) = \dot{\gamma}(t)/|\dot{\gamma}(t)|$, the unit normal vector $\boldsymbol{n}(t) = J(\boldsymbol{t}(t))$ and the Frenet formula

$$\begin{pmatrix} \dot{\boldsymbol{t}}(t) \\ \dot{\boldsymbol{n}}(t) \end{pmatrix} = \begin{pmatrix} 0 & |\dot{\gamma}(t)|\kappa(t) \\ -|\dot{\gamma}(t)|\kappa(t) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{t}(t) \\ \boldsymbol{n}(t) \end{pmatrix},$$

where $\dot{\gamma}(t) = (d\gamma/dt)(t)$, $|\dot{\gamma}(t)| = \sqrt{\dot{\gamma}(t) \cdot \dot{\gamma}(t)}$ and $\kappa(t) = \det(\dot{\gamma}(t), \ddot{\gamma}(t))/|\dot{\gamma}(t)|^3 = \dot{\boldsymbol{t}}(t) \cdot \boldsymbol{n}(t)/|\dot{\gamma}(t)|$. Note that $\kappa(t)$ is independent of the choice of a parametrization.

In this paper, we consider evolutes of plane curves. The *evolute* $Ev(\gamma) : I \to \mathbb{R}^2$ *of a regular plane curve* γ is given by

$$Ev(\gamma)(t) = \gamma(t) + \frac{1}{\kappa(t)}\boldsymbol{n}(t),$$

away from the points where $\kappa(t) = 0$ (cf. [5, 8, 9]).

If γ is not a regular curve, then we cannot define the evolute as above, since the curvature may diverge at a singular point. However, we define an evolute of a front in the Euclidean plane, see Definition 2.10 and Theorem 3.3. It is a generalization of the evolute of regular plane curves.

We say that $\gamma : I \to \mathbb{R}^2$ is a *front* (or, a *wave front*) in the Euclidean plane, if there exists a smooth map $\nu : I \to S^1$ such that the pair $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ is a Legendre immersion, namely, $(\dot{\gamma}(t), \dot{\nu}(t)) \neq (0, 0)$ and $(\gamma(t), \nu(t))^*\theta = 0$ for each $t \in I$. Here θ is the canonical contact structure on $T_1\mathbb{R}^2 = \mathbb{R}^2 \times S^1$, and S^1 is the unit circle. We remark that the second condition is equivalent to $\dot{\gamma}(t) \cdot \nu(t) = 0$ for each $t \in I$ (cf. [1, 2, 3]).

Throughout the paper, we assume that the pair (γ, ν) is co-orientable, the singular points of γ are finite and γ has no inflection points. The first and second conditions can be removed, see Remarks 3.4 and 3.5. However, we add these conditions for the sake of simplicity.

We give examples of fronts. See [1, 4, 11] for other examples.

Example 2.1. One of the typical examples of a front is a regular plane curve. Let $\gamma : I \to \mathbb{R}^2$ be a regular plane curve. In this case, we may take $\nu : I \to S^1$ by $\nu(t) = \boldsymbol{n}(t)$. Then it is easy to check that (γ, ν) is a Legendre immersion.

Example 2.2. Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be a 3/2 cusp (A_2-singularity) given by $\gamma(t) = (t^2, t^3)$. In this case, 0 is a singular point of γ. If we take $\nu : \mathbb{R} \to S^1$ by $\nu(t) = (1/\sqrt{9t^2 + 4})(-3t, 2)$, then we can show that (γ, ν) is a Legendre immersion. Hence the 3/2 cusp is an example of a front. The 3/2 cusp is the generic singularity of fronts and also evolutes in the Euclidean plane.

Example 2.3. Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be a 4/3 cusp (E_6-singularity) given by $\gamma(t) = (t^3, t^4)$. In this case, 0 is also a singular point of γ. If we take $\nu : \mathbb{R} \to S^1$ by $\nu(t) = (1/\sqrt{16t^2 + 9})(-4t, 3)$, then we can show that (γ, ν) is a Legendre immersion. Hence the 4/3 cusp is also an example of a front, see Example 6.3.

Example 2.4. Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be a 5/2 cusp (A_4-singularity) given by $\gamma(t) = (t^2, t^5)$. In this case, 0 is also a singular point of γ. However, the 5/2 cusp is not a front. By the condition

$\dot{\gamma}(t) \cdot \nu(t) = 0$, we take $\nu : \mathbb{R} \to S^1$ by $\nu(t) = \pm(1/\sqrt{25t^6+4})(-5t^3, 2)$. Then (γ, ν) is not an immersion at $t = 0$ and hence γ is not a front (but γ is a frontal, see [7]).

Remark 2.5. By the definition of the Legendre immersion, if (γ, ν) is a Legendre immersion, then $(\gamma, -\nu)$ is also.

We have the following Lemma (cf. [4, 11, 12]).

Lemma 2.6. *Let $\gamma : I \to \mathbb{R}^2$ be a front and $t_0 \in I$. If $\gamma^{(i)}(t_0) = 0$ for each $1 \leq i \leq k-1$ and $\gamma^{(k)}(t_0) \neq 0$, then γ at t_0 is diffeomorphic to the curve $(t^k, t^{k+1} + o(t^{k+1}))$ at $t = 0$. Moreover, if $k = 2$ (respectively, $k = 3$), the curve at t_0 is diffeomorphic to a $3/2$ (respectively, $4/3$) cusp.*

Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre immersion. We define a *parallel curve* $\gamma_\lambda : I \to \mathbb{R}^2$ of γ by $\gamma_\lambda(t) = \gamma(t) + \lambda\nu(t)$ for each $\lambda \in \mathbb{R}$. Then we have following results.

Proposition 2.7. *For a Legendre immersion $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$, the parallel curve $\gamma_\lambda : I \to \mathbb{R}^2$ is a front for each $\lambda \in \mathbb{R}$.*

Proof. We take $\nu_\lambda : I \to S^1$ by $\nu_\lambda(t) = \nu(t)$. Since $\gamma_\lambda(t) = \gamma(t) + \lambda\nu(t)$, it holds that $\dot{\gamma}_\lambda(t) = \dot{\gamma}(t) + \lambda\dot{\nu}(t)$. If $\dot{\gamma}_\lambda(t_0) = 0$ at a point $t_0 \in I$, then we have $\dot{\gamma}(t_0) + \lambda\dot{\nu}(t_0) = 0$. If $\dot{\nu}_\lambda(t_0) = \dot{\nu}(t_0) = 0$, then $\dot{\gamma}(t_0) = 0$. It contradicts the fact that (γ, ν) is an immersion. Hence $(\gamma_\lambda, \nu_\lambda)$ is an immersion. By $\nu(t) \cdot \nu(t) = 1$, we have $\dot{\nu}(t) \cdot \nu(t) = 0$. Then

$$\dot{\gamma}_\lambda(t) \cdot \nu_\lambda(t) = (\dot{\gamma}(t) + \lambda\dot{\nu}(t)) \cdot \nu(t) = \dot{\gamma}(t) \cdot \nu(t) + \lambda\dot{\nu}(t) \cdot \nu(t) = 0$$

holds. It follows that $(\gamma_\lambda, \nu_\lambda)$ is a Legendre immersion and hence γ_λ is a front. □

We denote the curvature of the parallel curve $\gamma_\lambda(t)$ by $\kappa_\lambda(t)$, when γ_λ is a regular curve.

Proposition 2.8. *Let (γ, ν) be a Legendre immersion. If γ is a regular curve and $\lambda \neq 1/\kappa(t)$, then a parallel curve γ_λ is also regular and $Ev(\gamma_\lambda)(t)$ is consistent with $Ev(\gamma)(t)$.*

Proof. Since $\gamma_\lambda(t) = \gamma(t) + \lambda\boldsymbol{n}(t)$, it holds that $\dot{\gamma}_\lambda(t) = |\dot{\gamma}(t)|(1 - \lambda\kappa(t))\boldsymbol{t}(t)$. By the assumption $\lambda \neq 1/\kappa(t)$, γ_λ is a regular curve. By a direct calculation, we have

$$\kappa_\lambda(t) = \frac{\kappa(t)}{|1 - \lambda\kappa(t)|}, \quad \boldsymbol{n}_\lambda(t) = \frac{1 - \lambda\kappa(t)}{|1 - \lambda\kappa(t)|}\boldsymbol{n}(t).$$

Hence we have

$$\begin{aligned} Ev(\gamma_\lambda)(t) &= \gamma_\lambda(t) + \frac{1}{\kappa_\lambda(t)}\boldsymbol{n}_\lambda(t) = \gamma(t) + \lambda\boldsymbol{n}(t) + \frac{|1 - \lambda\kappa(t)|}{\kappa(t)}\frac{1 - \lambda\kappa(t)}{|1 - \lambda\kappa(t)|}\boldsymbol{n}(t) \\ &= \gamma(t) + \frac{1}{\kappa(t)}\boldsymbol{n}(t) = Ev(\gamma)(t) \end{aligned}$$

□

Remark 2.9. Let (γ, ν) be a Legendre immersion. If t_0 is a singular point of the front γ, then $\lim_{t \to t_0} |\kappa(t)| = \infty$. By the equality $\kappa_\lambda(t) = \kappa(t)/|1 - \lambda\kappa(t)|$, we have $\lim_{t \to t_0} \kappa_\lambda(t) \neq 0$, see also Remark 3.2.

We now define an evolute of a front in the Euclidean plane.

Definition 2.10. Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre immersion. We define an *evolute* $\mathcal{E}v(\gamma) : I \to \mathbb{R}^2$ *of the front* γ as follows:

$$\mathcal{E}v(\gamma)(t) = \begin{cases} \gamma(t) + \frac{1}{\kappa(t)}\boldsymbol{n}(t) & \text{if} \quad t \text{ is a regular point}, \\ \gamma_\lambda(t) + \frac{1}{\kappa_\lambda(t)}\boldsymbol{n}_\lambda(t) & \text{if} \quad t \in (t_0 - \delta, t_0 + \delta),\ t_0 \text{ is a singular point of } \gamma, \end{cases}$$

where δ is a sufficiently small positive real number, $\lambda \in \mathbb{R}$ is satisfied the condition $\lambda \neq 1/\kappa(t)$ and $\kappa(t) \neq 0$.

Remark 2.11. By the assumption of the finiteness of singularities of a front, there exists $\lambda \in \mathbb{R}$ with the condition $\lambda \neq 1/\kappa(t)$. Moreover, by Proposition 2.8, we can glue on the regular interval of γ and γ_λ. Then the evolute of a front is well-defined. Furthermore, by definition, the evolute of a front $\mathcal{E}v(\gamma)$ is a C^∞ map.

In order to consider properties of the evolute of a front, we need a moving frame along a front (or, a Legendre immersion) (cf. [7]). Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre immersion. If γ is a regular curve around a point t_0, then we have the Frenet formula of γ in §2. On the other hand, if γ is singular at a point t_0, then we don't define such a frame. However, ν is always defined even if t is a singular point of γ. Therefore, we have the Frenet formula of a front as follows. We put $\boldsymbol{\mu}(t) = J(\nu(t))$. We call the pair $\{\nu(t), \boldsymbol{\mu}(t)\}$ is a *moving frame along a front* $\gamma(t)$ in $\mathbb{R}^2$ and we have the Frenet formula of a front which is given by

$$\begin{pmatrix} \dot{\nu}(t) \\ \dot{\boldsymbol{\mu}}(t) \end{pmatrix} = \begin{pmatrix} 0 & \ell(t) \\ -\ell(t) & 0 \end{pmatrix} \begin{pmatrix} \nu(t) \\ \boldsymbol{\mu}(t) \end{pmatrix}, \tag{1}$$

where $\ell(t) = \dot{\nu}(t) \cdot \boldsymbol{\mu}(t)$. Moreover, if $\dot{\gamma}(t) = \alpha(t)\nu(t) + \beta(t)\boldsymbol{\mu}(t)$ for some smooth functions $\alpha(t), \beta(t)$, then $\alpha(t) = 0$ follows from the condition $\dot{\gamma}(t) \cdot \nu(t) = 0$. Hence, there exists a smooth function $\beta(t)$ such that

$$\dot{\gamma}(t) = \beta(t)\boldsymbol{\mu}(t). \tag{2}$$

Since (γ, ν) is an immersion, we have $(\ell(t), \beta(t)) \neq (0, 0)$ for each $t \in I$. The pair (ℓ, β) is an important invariant of Legendre curves (or, frontals) in the unit tangent bundle like as the curvature of a regular plane curve, for more detail, see [7]. We call the pair (ℓ, β) *the curvature of the Legendre curve*. Since we assume that (γ, ν) is a Legendre immersion, so that we call (ℓ, β) *the curvature of the Legendre immersion*. For the related properties, see [15, 16].

3. Properties of the evolutes of fronts

In this section, we consider properties of the evolutes of fronts. Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre immersion with the curvature of the Legendre immersion (ℓ, β).

First we give a relationship between the curvature of the Legendre immersion $(\ell(t), \beta(t))$ and the curvature $\kappa(t)$ if γ is a regular curve.

Lemma 3.1. (1) *If γ is a regular curve, then $\ell(t) = |\beta(t)|\kappa(t)$.*
(2) *If γ_λ is a regular curve, then $\ell(t) = |\beta(t) + \lambda\ell(t)|\kappa_\lambda(t)$.*

Proof. (1) By a direct calculation, $\dot{\gamma}(t) = \beta(t)\boldsymbol{\mu}(t)$, $\ddot{\gamma}(t) = \dot{\beta}(t)\boldsymbol{\mu}(t) - \beta(t)\ell(t)\nu(t)$ and

$$\kappa(t) = \frac{\det(\dot{\gamma}(t), \ddot{\gamma}(t))}{|\dot{\gamma}(t)|^3} = \frac{\det\Big(\beta(t)\boldsymbol{\mu}(t), \dot{\beta}(t)\boldsymbol{\mu}(t) - \beta(t)\ell(t)\nu(t)\Big)}{|\beta(t)|^3} = \frac{\beta(t)^2\ell(t)}{|\beta(t)|^3} = \frac{\ell(t)}{|\beta(t)|}.$$

Therefore we have $\ell(t) = |\beta(t)|\kappa(t)$.
(2) We can also prove by the same calculations of (1). □

Remark 3.2. Since $(\ell(t), \beta(t)) \neq (0, 0)$, if t_0 is a singular point of γ, then γ_λ is a regular curve. By Lemma 3.1 (2), $\ell(t_0) = |\lambda\ell(t_0)|\kappa_\lambda(t_0)$. It follows from $\lambda\ell(t_0) \neq 0$ that $\kappa_\lambda(t_0) \neq 0$.

We give another representation of the evolute of a front by using the moving frame of a front $\{\nu(t), \boldsymbol{\mu}(t)\}$ and the curvature of the Legendre immersion $(\ell(t), \beta(t))$.

Theorem 3.3. *Under the above notations, the evolute of a front* $\mathcal{E}v(\gamma)(t)$ *is represented by*

$$\mathcal{E}v(\gamma)(t) = \gamma(t) - \frac{\beta(t)}{\ell(t)}\nu(t), \tag{3}$$

and $\mathcal{E}v(\gamma)$ *is a front. More precisely,* $(\mathcal{E}v(\gamma)(t), J(\nu(t)))$ *is a Legendre immersion with the curvature*

$$\left(\ell(t),\ \frac{d}{dt}\left(\frac{\beta(t)}{\ell(t)}\right)\right).$$

Proof. First suppose that γ is a regular curve. Since $\dot\gamma(t) = \beta(t)\boldsymbol{\mu}(t)$, we have $|\beta(t)| \neq 0$ and

$$\boldsymbol{t}(t) = \frac{\beta(t)}{|\beta(t)|}\boldsymbol{\mu}(t), \quad \boldsymbol{n}(t) = -\frac{\beta(t)}{|\beta(t)|}\nu(t).$$

By Lemma 3.1 (1), $\kappa(t) = \ell(t)/|\beta(t)|$ and $\ell(t) \neq 0$. Then

$$\mathcal{E}v(\gamma)(t) = \gamma(t) + \frac{1}{\kappa(t)}\boldsymbol{n}(t) = \gamma(t) + \frac{|\beta(t)|}{\ell(t)}\left(-\frac{\beta(t)}{|\beta(t)|}\right)\nu(t) = \gamma(t) - \frac{\beta(t)}{\ell(t)}\nu(t).$$

Second suppose that t_0 is a singular point of γ and γ_λ is a regular curve with $\lambda \neq 1/\kappa(t)$. Since $\dot\gamma_\lambda(t) = (\beta(t) + \lambda\ell(t))\boldsymbol{\mu}(t)$, we have $|\beta(t) + \lambda\ell(t)| \neq 0$ and

$$\boldsymbol{t}_\lambda = \frac{\beta(t) + \lambda\ell(t)}{|\beta(t) + \lambda\ell(t)|}\boldsymbol{\mu}(t), \quad \boldsymbol{n}_\lambda = -\frac{\beta(t) + \lambda\ell(t)}{|\beta(t) + \lambda\ell(t)|}\nu(t).$$

By Lemma 3.1 (2), $\kappa_\lambda(t) = \ell(t)/|\beta(t) + \lambda\ell(t)|$ and $\ell(t) \neq 0$. Then

$$\begin{aligned}\mathcal{E}v(\gamma_\lambda)(t) &= \gamma_\lambda(t) + \frac{1}{\kappa_\lambda(t)}\boldsymbol{n}_\lambda(t) = \gamma(t) + \lambda\nu(t) + \frac{|\beta(t) + \lambda\ell(t)|}{\ell(t)}\left(-\frac{\beta(t) + \lambda\ell(t)}{|\beta(t) + \lambda\ell(t)|}\right)\nu(t)\\ &= \gamma(t) - \frac{\beta(t)}{\ell(t)}\nu(t).\end{aligned}$$

If we take $\widetilde{\nu}(t) = J(\nu(t)) = \boldsymbol{\mu}(t)$, then $(\mathcal{E}v(\gamma)(t), \widetilde{\nu}(t))$ is a Legendre immersion. In fact, $\dot{\widetilde{\nu}}(t) = \ell(t)J(\boldsymbol{\mu}(t)) \neq 0$ and by the form of

$$\dot{\mathcal{E}v}(\gamma)(t) = -\frac{\dot\beta(t)\ell(t) - \beta(t)\dot\ell(t)}{\ell(t)^2}\nu(t) = \frac{d}{dt}\left(\frac{\beta(t)}{\ell(t)}\right)J(\boldsymbol{\mu}(t)), \tag{4}$$

we have $\dot{\mathcal{E}v}(\gamma)(t) \cdot \widetilde{\nu}(t) = 0$. It follows that $(\mathcal{E}v(\gamma)(t), J(\nu(t)))$ is a Legendre immersion with the curvature $(\ell(t), (d/dt)(\beta(t)/\ell(t)))$ and hence $\mathcal{E}v(\gamma)$ is a front. This completes the proof of the Theorem. □

Remark 3.4. By the representation (3), we may define the evolute of a front even if γ have non-isolated singularities, under the condition $\ell(t) \neq 0$.

By Lemma 3.1 and Remark 3.4, for a Legendre immersion (γ, ν) with the curvature of the Legendre immersion (ℓ, β), we say that t_0 is *an inflection point of the front* γ (or, *the Legendre immersion* (γ, ν)) if $\ell(t_0) = 0$. Since $\beta(t_0) \neq 0$ and Proposition 3.1, $\ell(t_0) = 0$ is equivalent to the condition $\kappa(t_0) = 0$.

Remark 3.5. Let (γ, ν) be a Legendre immersion, then $(\gamma, -\nu)$ is also (Remark 2.5). However, $\mathcal{E}v(t)$ does not change. It follows that we can define an evolute of a non co-orientable front, by taking double covering of γ.

Remark 3.6. By Definition 2.10, the evolute of a front is independent on the parametrization of (γ, ν). The curvature of the Legendre immersion (ℓ, β) is depended on the parametrization of (γ, ν), see [7]. If $s = s(t)$ is a parameter changing on I to $\overline{I}$, then $\ell(t) = \ell(s(t))\dot{s}(t)$ and $\beta(t) = \beta(s(t))\dot{s}(t)$. It also follows from the representation (3) that the evolute of a front is independent on the parametrization of (γ, ν).

If t_0 is a singular point of γ, then $\beta(t_0) = 0$. As a corollary of Theorem 3.3, we have the following.

Corollary 3.7. *If t_0 is a singular point of γ, then $\mathcal{E}v(\gamma)(t_0) = \gamma(t_0)$.*

Proposition 3.8. *Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre immersion without inflection points. Suppose that t_0 is a singular point of γ. Then t_0 is a regular point of $\mathcal{E}v(\gamma)(t)$ if and only if $\ddot{\gamma}(t_0) \neq 0$.*

Proof. By the assumption, $\beta(t_0) = 0$. Let t_0 be a regular point of $\mathcal{E}v(\gamma)(t)$. Since (4) and $\ell(t_0) \neq 0$, we have $\dot{\beta}(t_0) \neq 0$. By the differentiate of $\dot{\gamma}(t) = \beta(t)\boldsymbol{\mu}(t)$, we have

$$\ddot{\gamma}(t) = \dot{\beta}(t)\boldsymbol{\mu}(t) - \beta(t)\ell(t)\nu(t)$$

It follows that $\dot{\gamma}(t_0) = 0$ and $\ddot{\gamma}(t_0) = \dot{\beta}(t_0)\boldsymbol{\mu}(t_0) \neq 0$. The converse is also holded by reversing the arguments. □

Note that by Lemma 2.6 and Proposition 3.8, the conditions follows that γ is diffeomorphic to the 3/2 cusp at t_0. Hence, we can recognize the 3/2 cusp of original curve by the regularity of the evolute of a front, see Examples 6.2 and 6.3.

The most degenerate case of the evolute of a front, we have classified as follows:

Proposition 3.9. *If $\dot{\mathcal{E}v}(\gamma)(t) \equiv 0$, then γ is a part of a circle or a point.*

Proof. By the condition $\dot{\mathcal{E}v}(\gamma)(t) \equiv 0$, there exists a constant $c \in \mathbb{R}$ such that $\beta(t)/\ell(t) \equiv c$, if and only if $\beta(t) = c\ell(t)$. If $c = 0$, then $\dot{\gamma}(t) = \beta(t)\boldsymbol{\mu}(t) = 0$. It follows that γ is a point. Suppose that $c \neq 0$. By the existence and the uniqueness of a front in [7], we take

$$\nu(t) = \left(\cos\left(\int \ell(t)dt\right), \ \sin\left(\int \ell(t)dt\right)\right), \ \boldsymbol{\mu}(t) = \left(-\sin\left(\int \ell(t)dt\right), \ \cos\left(\int \ell(t)dt\right)\right).$$

By $\dot{\gamma}(t) = \beta(t)\boldsymbol{\mu}(t)$, we have

$$\begin{aligned}\gamma(t) &= \left(-c\int \ell(t)\sin\left(\int \ell(t)dt\right)dt + a, \ c\int \ell(t)\cos\left(\int \ell(t)dt\right)dt + b\right) \\ &= \left(c\cos\left(\int \ell(t)dt\right) + a, \ c\sin\left(\int \ell(t)dt\right) + b\right)\end{aligned}$$

for some constants $a, b \in \mathbb{R}$. Therefore, γ is a part of a circle. □

As a well-known result, a singular point of $Ev(\gamma)$ of a regular plane curve γ is corresponding to a vertex of γ, namely, $\dot{\kappa}(t) = 0$ (cf. [5, 8, 18, 19]).

We extend the notion of vertex. For a Legendre immersion (γ, ν) with the curvature of the Legendre immersion (ℓ, β), t_0 is *a vertex of the front* γ (or *a Legendre immersion* (γ, ν)) if $(d/dt)(\beta/\ell)(t_0) = 0$, namely, $(d/dt)\mathcal{E}v(t_0) = 0$. Note that if t_0 is a regular point of γ, the definition of the vertex coincides with usual vertex for regular curves. Therefore, this is a generalization of the notion of the vertex of regular plane curves.

Remark 3.10. Let (γ,ν) be a Legendre immersion. If t_0 is a singular point of γ which degenerate more than $3/2$ cusp, then t_0 is a vertex of a front γ. In fact,

$$\frac{d}{dt}\left(\frac{\beta}{\ell}\right)(t_0)=\frac{\dot{\beta}(t_0)\ell(t_0)-\beta(t_0)\dot{\ell}(t_0)}{\ell(t_0)^2}=0,$$

since $\beta(t_0)=\dot{\beta}(t_0)=0$ by Proposition 3.8.

In this paper, a Legendre immersion $(\gamma,\nu):[a,b]\to\mathbb{R}^2\times S^1$ is a *closed* Legendre immersion if $(\gamma^{(n)}(a),\nu^{(n)}(a))=(\gamma^{(n)}(b),\nu^{(n)}(b))$ for all $n\in\mathbb{N}\cup\{0\}$ where $\gamma^{(n)}(a)$, $\nu^{(n)}(a)$, $\gamma^{(n)}(b)$ and $\nu^{(n)}(b)$ means one-sided differential. If $(\gamma,\nu):[a,b]\to\mathbb{R}^2\times S^1$ is a closed Legendre immersion, then both a and b are regular points or both a and b are singular points of γ. When a and b are singular points of γ, we treat these singular points as one singular point.

Proposition 3.11. *Let $(\gamma,\nu):[a,b]\to\mathbb{R}^2\times S^1$ be a closed Legendre immersion without inflection points.*

(1) *If γ has at least two singular points which degenerate more than $3/2$ cusp, then γ has at least four vertices.*

(2) *If γ has at least four singular points, then γ has at least four vertices.*

Proof. (1) Suppose that γ has at least two singular points which degenerate more than $3/2$ cusp. By Remark 3.10, these singularities are vertices of γ, therefore it is sufficient to show that there is at least one vertex between two adjacent singular points. Since γ has no inflection points, the sign of the curvature of γ on regular points is constant. Therefore, either $\lim_{t\to t_0}\kappa(t)=\infty$ for all $t_0\in\Sigma(\gamma)$ or $\lim_{t\to t_0}\kappa(t)=-\infty$ for all $t_0\in\Sigma(\gamma)$, where $\Sigma(\gamma)$ is the set of singular points of γ. This concludes there exist $t\in(t_1,t_2)$ such that $\dot{\kappa}(t)=0$ for singular points t_1 and t_2 of γ.

Suppose that a and b are singular points which degenerate more than $3/2$ cusp. Since we treat a and b as the one singular point, there exists at least one singular point $t_1\in(a,b)$ which degenerate more than $3/2$ cusp by the assumption. In this case, there exist at least two vertices $v_1\in(a,t_1)$ and $v_2\in(t_1,b)$. Moreover, a and t_1 are also vertices. Therefore, there exist at least four vertices.

Next, suppose that a and b are regular points or $3/2$ cusps. Then there exist at least two singular points t_1 and t_2 (we assume $t_1<t_2$) in (a,b) which degenerate more than $3/2$ cusp. In this case, there exists at least one vertex $v_1\in(t_1,t_2)$. Moreover, since (γ,ν) is closed, there exists a point $v_2\in[a,t_1)\cup(t_2,b]$ such that $\dot{\kappa}(v_2)=0$. Therefore, γ has at least four vertices.

(2) Suppose that γ has at least four singular points. Since γ has no inflection points, the sign of the curvature of γ on regular points is constant. Therefore, either $\lim_{t\to t_0}\kappa(t)=\infty$ for all $t_0\in\Sigma(\gamma)$ or $\lim_{t\to t_0}\kappa(t)=-\infty$ for all $t_0\in\Sigma(\gamma)$. This concludes there exist $t\in(t_1,t_2)$ such that $\dot{\kappa}(t)=0$, that is, there is at least one vertex between two adjacent singular points.

Suppose that a and b are singular points of γ. Since we treat a and b as the one singular point, there exist at least three singular points t_1, t_2 and t_3 of γ in (a,b), which we assume to be ordered so that $a<t_1<t_2<t_3<b$. Since there is at least one vertex between two adjacent singular points, there exist at least four vertices $v_1\in(a,t_1)$, $v_2\in(t_1,t_2)$, $v_3\in(t_2,t_3)$ and $v_4\in(t_3,b)$.

Next, suppose that a and b are regular points of γ. Let t_1, t_2, t_3 and t_4 be singular points of γ (we assume $a<t_1<t_2<t_3<t_4<b$). Since there is at least one vertex between two adjacent singular points, there exist at least three vertices $v_1\in(t_1,t_2)$, $v_2\in(t_2,t_3)$, $v_3\in(t_3,t_4)$. Moreover, since (γ,ν) is closed, there exists a point $v_4\in[a,t_1)\cup(t_4,b]$ such that $\dot{\kappa}(v_4)=0$. Therefore, γ has at least four vertices. □

Finally, in this section, we consider the evolute of a front as a (wave) front of a Legendre immersion by using a family of functions.

We define a family of functions

$$F : I \times \mathbb{R}^2 \to \mathbb{R}$$

by $F(t, x, y) = (\gamma(t) - (x, y)) \cdot \boldsymbol{\mu}(t)$.

Proposition 3.12. *Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre immersion with the curvature of the Legendre immersion (ℓ, β).*

(1) *$F(t, x, y) = 0$ if and only if there exists a real number λ such that $(x, y) = \gamma(t) - \lambda\nu(t)$.*

(2) *$F(t, x, y) = (\partial F/\partial t)(t, x, y) = 0$ if and only if $\ell(t) \neq 0$ and $(x, y) = \gamma(t) - (\beta(t)/\ell(t))\nu(t)$.*

Proof. (1) $(\gamma(t) - (x, y)) \cdot \boldsymbol{\mu}(t) = 0$ if and only if there exists $\lambda \in \mathbb{R}$ such that $\gamma(t) - (x, y) = \lambda\nu(t)$.

(2) $(\partial F/\partial t)(t, x, y) = \dot\gamma(t) \cdot \boldsymbol{\mu}(t) + (\gamma(t) - (x, y)) \cdot \dot{\boldsymbol{\mu}}(t) = \beta(t) - \lambda\ell(t)$. If $\ell(t) = 0$, then $\beta(t) = 0$. This is a contradiction for $(\ell(t), \beta(t)) \neq (0, 0)$. It follows that $\lambda = \beta(t)/\ell(t)$. The converse is also holded. □

One can show that F is a *Morse family*, in the sense of Legendrian singularity theory (cf. [1, 14, 20]), namely, $(F, \partial F/\partial t) : I \times \mathbb{R}^2 \to \mathbb{R} \times \mathbb{R}$ is a submersion at $(t, x, y) \in D(F)$, where

$$D(F) = \{(t, x, y) \mid F(t, x, y) = (\partial F/\partial t)(t, x, y) = 0\}.$$

It follows that the evolute of a front $\mathcal{E}v(\gamma)$ is a (wave) front of a Legendre immersion and is given by the envelope of normal lines of the front.

4. Evolutes of the evolutes of fronts

By Theorem 3.3, the evolute of a front is also a front without inflection points. We consider a repeated evolute of an evolute of a front and give properties of a singular point of it. Let (γ, ν) be a Legendre immersion with the curvature of the Legendre immersion (ℓ, β) and without inflection points.

Theorem 4.1. *The evolute of an evolute of a front is given by*

$$\mathcal{E}v(\mathcal{E}v(\gamma))(t) = \mathcal{E}v(\gamma)(t) - \frac{\dot\beta(t)\ell(t) - \beta(t)\dot\ell(t)}{\ell(t)^3}\boldsymbol{\mu}(t). \tag{5}$$

Proof. At this proof, we denote $\widetilde\gamma(t) = \mathcal{E}v(\gamma)(t)$. By the proof of Theorem 3.3,

$$(\widetilde\gamma(t), \widetilde\nu(t)) = (\mathcal{E}v(\gamma)(t), \boldsymbol{\mu}(t))$$

is a Legendre immersion. Since $\widetilde{\boldsymbol{\mu}}(t) = J(\widetilde\nu(t)) = -\nu(t)$ and the derivative of the evolute of the front (4), we have

$$\widetilde\beta(t) = \frac{\dot\beta(t)\ell(t) - \beta(t)\dot\ell(t)}{\ell(t)^2},$$

where $\dot{\widetilde\gamma}(t) = \widetilde\beta(t)\widetilde{\boldsymbol{\mu}}(t)$. Moreover $\widetilde\ell(t) = \ell(t)$ by the Frenet formula of a front (1). It follows that

$$\mathcal{E}v(\mathcal{E}v(\gamma))(t) = \mathcal{E}v(\widetilde\gamma)(t) = \widetilde\gamma(t) - \frac{\widetilde\beta(t)}{\widetilde\ell(t)}\widetilde\nu(t) = \mathcal{E}v(\gamma)(t) - \frac{\dot\beta(t)\ell(t) - \beta(t)\dot\ell(t)}{\ell(t)^3}\boldsymbol{\mu}(t).$$

□

We can also prove Theorem 4.1 by a direct calculation of the definition of the evolute of a front (Definition 2.10). We need to divide into four cases, that is, γ is a regular or a singular, and $\mathcal{E}v(\gamma)$ is a regular or a singular. All cases coincide with (5). We also call $\mathcal{E}v(\mathcal{E}v(\gamma))$ the *second evolute of a front.*

Now we consider a geometric meaning of a singular point of $\mathcal{E}v(\mathcal{E}v(\gamma))(t)$.

Lemma 4.2. *Suppose that γ and $\mathcal{E}v(\gamma)$ are both regular curves. If $\dot{\mathcal{E}v}(\mathcal{E}v(\gamma))(t) \equiv 0$, then γ is an involute of a circle.*

Proof. We may assume that t is the arc-length parameter of γ. It follows that $|\beta(t)| = 1$ and hence $\ell(t) = \kappa(t)$ by Lemma 3.1. Moreover, we have $\beta(t)^2 = 1$ and $\dot{\beta}(t) = 0$. Since $\boldsymbol{t}(t) = \beta(t)\boldsymbol{\mu}(t)$ and $\boldsymbol{n}(t) = -\beta(t)\nu(t)$, we have $\boldsymbol{\mu}(t) = \beta(t)\boldsymbol{t}(t)$ and $\nu(t) = -\beta(t)\boldsymbol{n}(t)$. Then

$$\mathcal{E}v(\gamma)(t) = \gamma(t) - \frac{\beta(t)}{\ell(t)}\nu(t) = \gamma(t) - \frac{\beta(t)}{\kappa(t)}(-\beta(t)\boldsymbol{n}(t)) = \gamma(t) + \frac{1}{\kappa(t)}\boldsymbol{n}(t)$$

and

$$\mathcal{E}v(\mathcal{E}v(\gamma))(t) = \mathcal{E}v(\gamma)(t) + \frac{\beta(t)\dot{\kappa}(t)}{\kappa(t)^3}\beta(t)\boldsymbol{t}(t) = \mathcal{E}v(\gamma)(t) + \frac{\dot{\kappa}(t)}{\kappa(t)^3}\boldsymbol{t}(t)$$

hold. It follows that

$$\dot{\mathcal{E}v}(\gamma)(t) = -\frac{\dot{\kappa}(t)}{\kappa(t)^2}\boldsymbol{n}(t),\ \dot{\mathcal{E}v}(\mathcal{E}v(\gamma))(t) = \frac{\ddot{\kappa}(t)\kappa(t) - 3\dot{\kappa}(t)^2}{\kappa(t)^4}\boldsymbol{t}(t).$$

By the assumptions, $\kappa(t) \neq 0, \dot{\kappa}(t) \neq 0$ and $\ddot{\kappa}(t)\kappa(t) - 3\dot{\kappa}(t)^2 \equiv 0$, it follows that

$$\frac{d}{dt}\left(\frac{\dot{\kappa}(t)}{\kappa(t)}\right) = 2\left(\frac{\dot{\kappa}(t)}{\kappa(t)}\right)^2.$$

Solving this differential equation, there exist constants $C_1, C_2 \in \mathbb{R}$ with $C_2 \neq 0$ such that

$$\kappa(t) = C_2\frac{1}{\sqrt{2t + C_1}}.$$

A curve having the curvature $1/\sqrt{2ct}$ for a constant $c \in \mathbb{R} \setminus \{0\}$ is an involute of a circle with radius c. By the existence and the uniqueness theorems of regular plane curves, see for example [8, 9], γ is an involute of a circle (cf. [9, P.138]). □

Let $\gamma : I \to \mathbb{R}^2$ be a regular curve and $t_0 \in I$. The *involute of a regular curve* is defined by $Inv(\gamma, t_0) : I \to \mathbb{R}^2$;

$$Inv(\gamma, t_0)(t) = \gamma(t) - \left(\int_{t_0}^{t} |\dot{\gamma}(u)|\, du\right)\boldsymbol{t}(t).$$

Note that $\mathcal{E}v(Inv(\gamma, t_0))(t) = \gamma(t)$, for more detail see [5, 8, 9].

Theorem 4.3. *Suppose that γ and $\mathcal{E}v(\gamma)$ are regular curves. If t_0 is a singular point of $\mathcal{E}v(\mathcal{E}v(\gamma))$, then γ is at least 4-th order contact to an involute of a circle at the point $t = t_0$ up to congruent.*

Proof. We may assume that t is the arc-length parameter of γ. By the same arguments in the proof of Lemma 4.2, we have $\kappa(t_0) \neq 0, \dot{\kappa}(t_0) \neq 0$ and $\ddot{\kappa}(t_0)\kappa(t_0) - 3\dot{\kappa}(t_0)^2 = 0$. We set $\kappa(t_0) = a$ and $\dot{\kappa}(t_0) = b$. Then we define a curve $\widetilde{\gamma}(t)$ whose curvature is given by

$$\widetilde{\kappa}(t) = a\sqrt{\frac{a}{b}}\frac{1}{\sqrt{-2t + 2t_0 + \frac{a}{b}}}, \quad \left(\text{respectively, } \widetilde{\kappa}(t) = a\sqrt{-\frac{a}{b}}\frac{1}{\sqrt{2t - 2t_0 - \frac{a}{b}}}\right)$$

if $ab > 0$ (respectively, $ab < 0$). Then $\kappa(t_0) = \widetilde{\kappa}(t_0) = a$ and $\dot{\kappa}(t_0) = \dot{\widetilde{\kappa}}(t_0) = b$. Since $\ddot{\kappa}(t_0)\kappa(t_0) - 3\dot{\kappa}(t_0)^2 = 0$ and $\ddot{\widetilde{\kappa}}(t)\widetilde{\kappa}(t) - 3\dot{\widetilde{\kappa}}(t)^2 \equiv 0$, we have $\ddot{\kappa}(t_0) = \ddot{\widetilde{\kappa}}(t_0)$. By the Theorem A.1 in the appendix, γ and $\widetilde{\gamma}$ are at least 4-th order contact at the point $t = t_0$ up to congruent. It follows that γ and an involute of a circle are at least 4-th order contact at the point $t = t_0$ up to congruent. This completes the proof of Theorem. □

Remark 4.4. Suppose that γ is a regular curve. If t_0 is a singular point of $\mathcal{E}v(\gamma)(t)$ and $\mathcal{E}v(\mathcal{E}v(\gamma))(t)$, then $\dot{\kappa}(t_0) = \ddot{\kappa}(t_0) = 0$ by the same calculations of the proof of Lemma 4.2. It follows that γ and the osculating circle are at least 4-th order contact at the point $t = t_0$.

Proposition 4.5. *Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre immersion without inflection points. Suppose that t_0 is a singular point of both γ and $\mathcal{E}v(\gamma)$. Then t_0 is a regular point of $\mathcal{E}v(\mathcal{E}v(\gamma))$ if and only if $\dddot{\gamma}(t_0) \neq 0$.*

Proof. Let t_0 be a regular point of $\mathcal{E}v(\mathcal{E}v(\gamma))$. By Proposition 3.8, $\beta(t_0) = \dot{\beta}(t_0) = 0$ and $\ell(t_0) \neq 0$. Since

$$\frac{d}{dt}\mathcal{E}v(\mathcal{E}v(\gamma))(t) = -\frac{\ddot{\beta}(t)\ell(t)^2 - \beta(t)\ell(t)\ddot{\ell}(t) - 3\dot{\beta}(t)\ell(t)\dot{\ell}(t) + 3\beta(t)\dot{\ell}(t)^2}{\ell(t)^4}\boldsymbol{\mu}(t),$$

it holds that $(d/dt)\mathcal{E}v(\mathcal{E}v(\gamma))(t_0) = -\ddot{\beta}(t_0)\ell(t_0)^{-2}\boldsymbol{\mu}(t_0) \neq 0$ if and only if $\ddot{\beta}(t_0) \neq 0$. By the differentiate of $\dot{\gamma}(t) = \beta(t)\boldsymbol{\mu}(t)$, we have

$$\dddot{\gamma}(t) = (\ddot{\beta}(t) - \beta(t)\ell(t)^2)\boldsymbol{\mu}(t) - (2\dot{\beta}(t)\ell(t) + \beta(t)\dot{\ell}(t))\nu(t)$$

It follows that $\dddot{\gamma}(t_0) = \ddot{\beta}(t_0)\boldsymbol{\mu}(t_0) \neq 0$. The converse is also shown by reversing the arguments. □

Note that by Lemma 2.6 and Proposition 4.5, the conditions follows that γ is diffeomorphic to the 4/3 cusp at t_0.

5. The n-th evolutes of fronts

Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre immersion with the curvature (ℓ, β) and without inflection points. We give the form of the n-th evolute of a front, where n is a natural number. We denote $\mathcal{E}v^0(\gamma)(t) = \gamma(t)$ and $\mathcal{E}v^1(\gamma)(t) = \mathcal{E}v(\gamma)(t)$ for convenience. We define

$$\mathcal{E}v^n(\gamma)(t) = \mathcal{E}v(\mathcal{E}v^{n-1}(\gamma))(t), \quad \beta_0(t) = \beta(t), \quad \text{and} \quad \beta_n(t) = \frac{d}{dt}\left(\frac{\beta_{n-1}(t)}{\ell(t)}\right)$$

inductively.

Theorem 5.1. *$(\mathcal{E}v^n(\gamma), J^n(\nu)) : I \to \mathbb{R}^2 \times S^1$ is a Legendre immersion with the curvature (ℓ, β_n), where the n-th evolute of the front is given by*

$$\mathcal{E}v^n(\gamma)(t) = \mathcal{E}v^{n-1}(\gamma)(t) - \frac{\beta_{n-1}(t)}{\ell(t)}J^{n-1}(\nu(t)),$$

where J^n is n-times operations of J.

Proof. Let $n = 1$ and $n = 2$, then

$$\mathcal{E}v^1(\gamma)(t) = \mathcal{E}v^0(\gamma)(t) - \frac{\beta_0(t)}{\ell(t)}J^0(\nu(t)) = \gamma(t) - \frac{\beta(t)}{\ell(t)}\nu(t)$$

and

$$\begin{aligned}\mathcal{E}v^2(\gamma)(t) &= \mathcal{E}v^1(\gamma)(t) - \frac{\beta_1(t)}{\ell(t)}J^1(\nu(t)) = \mathcal{E}v(\gamma)(t) - \frac{d}{dt}\left(\frac{\beta(t)}{\ell(t)}\right)\frac{1}{\ell(t)}J(\nu(t)) \\ &= \mathcal{E}v(\gamma)(t) - \frac{\dot{\beta}(t)\ell(t) - \beta(t)\dot{\ell}(t)}{\ell(t)^3}\boldsymbol{\mu}(t).\end{aligned}$$

These are nothing but the evolute of a front (3) and the second evolute of a front (5).

Next suppose that $1 \leq j \leq k$ is holded, namely,

$$\mathcal{E}v^j(\gamma)(t) = \mathcal{E}v^{j-1}(\gamma)(t) - \frac{\beta_{j-1}(t)}{\ell(t)} J^{j-1}(\nu(t))$$

for $1 \leq j \leq k$. We consider $\mathcal{E}v(\mathcal{E}v^k(\gamma))(t)$. Suppose that $(\mathcal{E}v^k(\gamma)(t), J^k(\nu(t)))$ is a Legendre immersion with the curvature $(\ell(t), \beta_k(t))$. By Theorem 3.3, we have $(k+1)$-th evolute of the front

$$\mathcal{E}v^{k+1}(\gamma)(t) = \mathcal{E}v^k(\gamma)(t) - \frac{\beta_k(t)}{\ell(t)} J^k(\nu(t)).$$

Since

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}v^{k+1}(\gamma)(t) &= \frac{d}{dt}\mathcal{E}v^k(\gamma)(t) - \frac{d}{dt}\left(\frac{\beta_k(t)}{\ell(t)}\right) J^k(\nu(t)) - \frac{\beta_k(t)}{\ell(t)} J^k(\dot{\nu}(t)) \\
&= \beta_k(t) J^{k+1}(t) + \beta_{k+1}(t) J^{k+2}(\nu(t)) - \beta_k(t) J^{k+1}(\nu(t)) \\
&= \beta_{k+1}(t) J^{k+2}(\nu(t)), \\
\frac{d}{dt} J^{k+1}(\nu(t)) &= J^{k+1}(\dot{\nu}(t)) = J^{k+1}(\ell(t)\boldsymbol{\mu}(t)) = \ell(t) J^{k+1}(J(\nu(t))) \\
&= \ell(t) J^{k+2}(\nu(t)),
\end{aligned}$$

it holds that $(\mathcal{E}v^{k+1}(\gamma), J^{k+1}(\nu))$ is a Legendre immersion with the curvature $(\ell(t), \beta_{k+1}(t))$. By the induction, this completes the proof of Theorem. □

As a generalization of Propositions 3.8 and 4.5, we have the following result:

Proposition 5.2. *Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre immersion with the curvature of the Legendre immersion (ℓ, β) and without inflection points. Suppose that t_0 is a singular point of γ. Then the following are equivalent:*

(1) *t_0 is a singular point of $\mathcal{E}v^i(\gamma)(t)$ for $i = 1, \dots, n$.*
(2) *$(d^i\beta/dt^i)(t_0) = 0$ for $i = 1, \dots, n$.*
(3) *$(d^i\gamma/dt^i)(t_0) = 0$ for $i = 2, \dots, n+1$.*

Proof. First, we show that $\beta_i(t)$ is given by the form $\beta^{(i)}(t)$ and lower terms of $\beta^{(i)}(t)$, namely,

$$\beta_i(t) = \frac{\beta^{(i)}(t)}{\ell(t)^i} + L(\beta(t), \dots, \beta^{(i-1)}(t)) \tag{6}$$

for some smooth function L which contain $\ell(t)$ and derivatives of $\ell(t)$.

Since

$$\beta_1(t) = \frac{d}{dt}\left(\frac{\beta(t)}{\ell(t)}\right) = \frac{\dot{\beta}(t)}{\ell(t)} + \beta(t)\frac{d}{dt}\left(\frac{1}{\ell(t)}\right),$$

the case of $i = 1$ is holded. Suppose that $i = k$ is holded, namely, there exists a smooth function L such that

$$\beta_k(t) = \frac{\beta^{(k)}(t)}{\ell(t)^k} + L(\beta(t), \dots, \beta^{(k-1)}(t)).$$

Then

$$\beta_{k+1}(t) = \frac{d}{dt}\left(\frac{\beta_k(t)}{\ell(t)}\right) = \frac{\beta^{(k+1)}(t)}{\ell(t)^{k+1}} + \widetilde{L}(\beta(t), \dots, \beta^{(k)}(t)),$$

for some smooth function $\widetilde{L}$. By the induction, we conclude the assertion.

Second, assume that t_0 is a singular point of $\mathcal{E}v^i(\gamma)(t)$ for $i = 1, \dots, n$. By Theorem 5.1, $(d/dt)\mathcal{E}v^i(\gamma)(t_0) = 0$ if and only if $\beta_i(t_0) = 0$. Since (6) and $\beta(t_0) = 0$, it holds that $\beta_i(t_0) = 0$ for $i = 1, \dots, n$ if and only if $\beta^{(i)}(t_0) = 0$ for $i = 1, \dots, n$. It follows that (1) implies (2). By the reversing arguments, the converse (1) follows from (2).

Finally, since $\dot{\gamma}(t) = \beta(t)\boldsymbol{\mu}(t)$, we can also show that (2) is equivalent to (3) by the induction.
□

6. Examples

We give examples to understand the phenomena for evolutes of fronts.

Example 6.1. Let $\gamma(t) = (a\cos t, b\sin t)$ be an ellipse with $a, b > 0$ and $a \neq b$. Since

$$\nu(t) = \frac{1}{\sqrt{a^2\sin^2 t + b^2\cos^2 t}}(-b\cos t, a\sin t),\ \boldsymbol{\mu}(t) = \frac{1}{\sqrt{a^2\sin^2 t + b^2\cos^2 t}}(-a\sin t, -b\cos t),$$

we have

$$\ell(t) = \frac{ab}{a^2\sin^2 t + b^2\cos^2 t},\ \beta(t) = -\sqrt{a^2\sin^2 t + b^2\cos^2 t}.$$

The evolute, the second evolute and the third evolute of the ellipse γ are given by

$$\begin{aligned}
\mathcal{E}v(\gamma)(t) &= \left(\frac{a^2-b^2}{a}\cos^3 t, -\frac{a^2-b^2}{b}\sin^3 t\right), \\
\mathcal{E}v(\mathcal{E}v(\gamma))(t) &= \Big(\frac{a^2-b^2}{ab^2}\cos t\left(b^2\cos^4 t + 3a^2\sin^4 t + b^2\sin^2 2t\right), \\
&\qquad -\frac{a^2-b^2}{a^2b}\sin t\left(a^2\sin^4 t + 3b^2\cos^4 t + a^2\sin^2 2t\right)\Big),
\end{aligned}$$

and $\mathcal{E}v^3(\gamma)(t) =$

$$\begin{aligned}
&\Big(\frac{a^2-b^2}{8a^3b^2}\cos^3 t\left(45a^4 - 10a^2b^2 - 3b^4 + 12(-5a^4 + 4a^2b^2 + b^4)\cos 2t + 15(a^2-b^2)^2\cos 4t\right), \\
&\ \frac{a^2-b^2}{8a^2b^3}\sin^3 t\left(3a^4 + 10a^2b^2 - 45b^4 + 12(a^4 + 4a^2b^2 - 5b^4)\cos 2t - 15(a^2-b^2)^2\cos 4t\right)\Big).
\end{aligned}$$

The ellipse γ and its evolute (red curve) are showed in Figures 1 left and 2 center. Moreover, the second evolute (yellow curve), see Figure 1 center, and the third evolute (green curve), see Figures 1 right and 2 right.

The evolute is useful to recognize the difference of the sharp of curves. In Figure 2, the left is a circle and the center is an ellipse and its evolute. We can observe the evolute of the ellipse, however, it is very small (red curve). If we consider the repeated evolute, we can easy to observe it. The right in Figure 2 is the second and the third evolute of the ellipse.

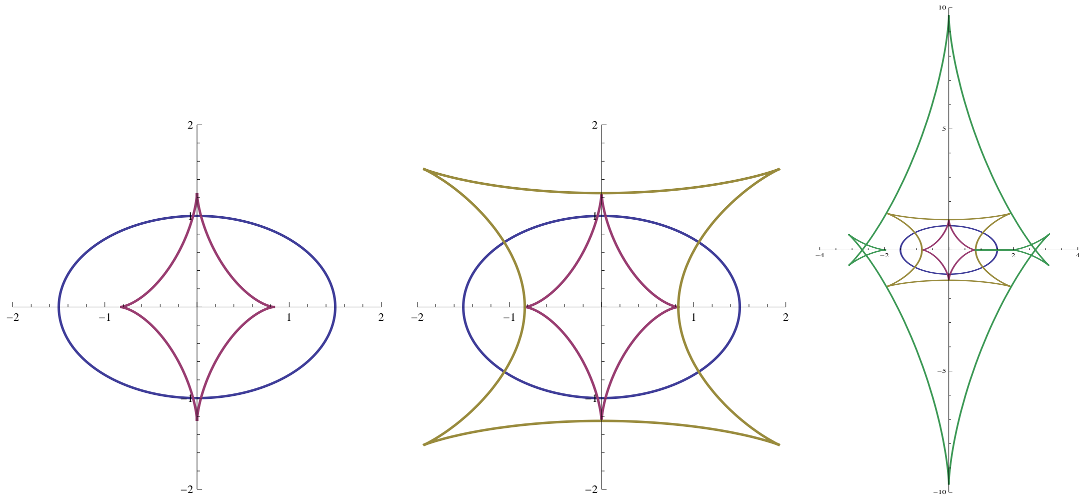

Figure 1. The ellipse and evolutes.

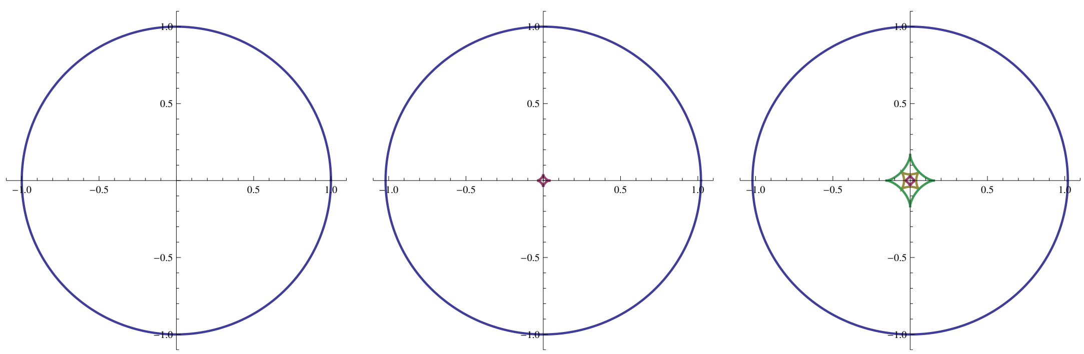

Figure 2.

Example 6.2. Let $\gamma(t) = (3\cos t - \cos 3t, 3\sin t - \sin 3t) = (6\cos t - 4\cos^3 t, 4\sin^3 t)$ be the nephroid, see Figure 3 left. Since $\nu(t) = (-\sin 2t, \cos 2t)$ and $\boldsymbol{\mu}(t) = (-\cos 2t, \sin 2t)$, we have $\ell(t) = 2, \beta(t) = -6\sin t$. The evolute and the second evolute of the nephroid are as follows, see Figure 3 center and right:

$$\begin{aligned}
\mathcal{E}v(\gamma)(t) &= \left(2\cos^3 t, 3\sin t - 2\sin^2 t\right), \\
\mathcal{E}v(\mathcal{E}v(\gamma))(t) &= \left(\frac{3}{2}\cos t - \cos^3 t, \sin^3 t\right).
\end{aligned}$$

We can observe that $\gamma(t)/4 = \mathcal{E}v(\mathcal{E}v(\gamma))(t)$.

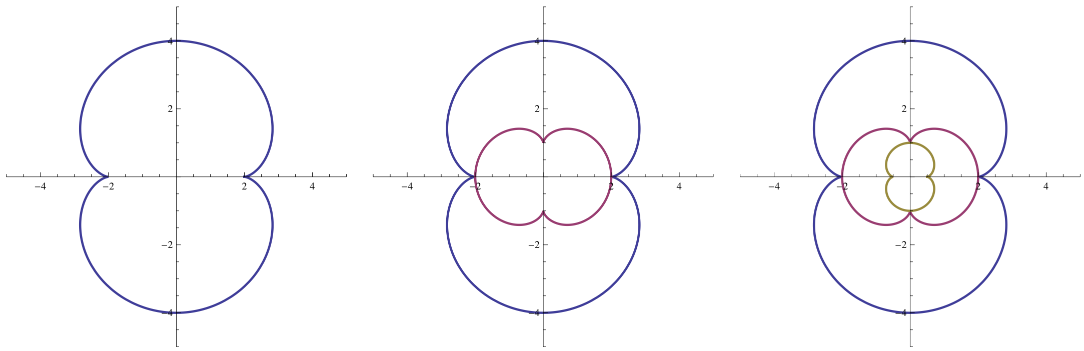

Figure 3. The nephroid and evolutes.

Example 6.3. Let $\gamma(t) = (t^3, t^4)$ be the 4/3 cusp, Figure 4 left. Since $\nu(t) = (1/\sqrt{16t^2+9})(-4t, 3)$ and $\boldsymbol{\mu}(t) = (1/\sqrt{16t^2+9})(-3, -4t)$, we have $\ell(t) = 12/(16t^2+9), \beta(t) = -t^2\sqrt{16t^2+9}$. The evolute and the second evolute of the 4/3 cusp are as follows, see Figure 4 center and right:

$$\begin{aligned} \mathcal{E}v(\gamma)(t) &= \left(-2t^3 - \frac{16}{3}t^5, \frac{9}{4}t^2 + 5t^4\right), \\ \mathcal{E}v(\mathcal{E}v(\gamma))(t) &= \left(-\frac{27}{8}t - 23t^3 - 32t^5, -\frac{9}{4}t^2 - 23t^4 - \frac{320}{9}t^6\right). \end{aligned}$$

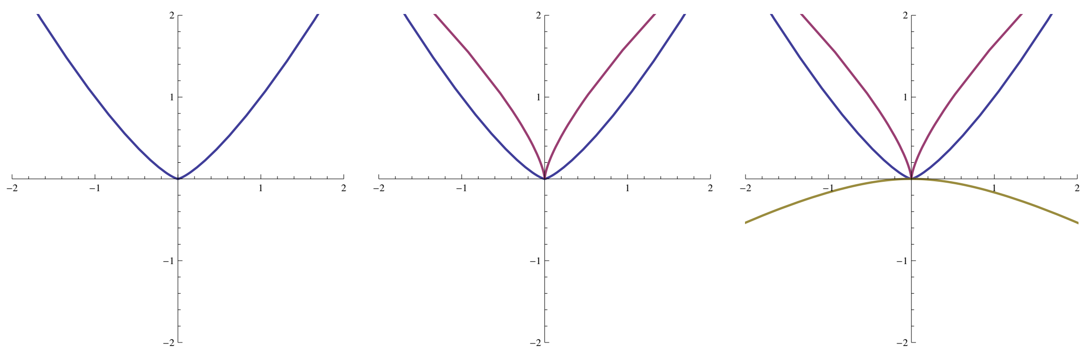

Figure 4. The 4/3 cusp and evolutes.

Appendix A. Contact between regular curves

In this appendix, we discuss contact between regular curves. Let $\gamma : I \to \mathbb{R}^2; t \mapsto \gamma(t)$ and $\widetilde{\gamma} : \widetilde{I} \to \mathbb{R}^2; u \mapsto \widetilde{\gamma}(u)$ be regular plane curves, respectively. We say that γ and $\widetilde{\gamma}$ have *k-th order contact at $t = t_0, u = u_0$* if

$$\gamma(t_0) = \widetilde{\gamma}(u_0),\ \frac{d\gamma}{dt}(t_0) = \frac{d\widetilde{\gamma}}{du}(u_0),\ \cdots,\ \frac{d^k\gamma}{dt^k}(t_0) = \frac{d^k\widetilde{\gamma}}{du^k}(u_0),\ \frac{d^{k+1}\gamma}{dt^{k+1}}(t_0) \neq \frac{d^{k+1}\widetilde{\gamma}}{du^{k+1}}(u_0).$$

Moreover, we say that γ and $\widetilde{\gamma}$ have *at least k-th order contact at $t = t_0, u = u_0$* if

$$\gamma(t_0) = \widetilde{\gamma}(u_0),\ \frac{d\gamma}{dt}(t_0) = \frac{d\widetilde{\gamma}}{du}(u_0),\ \cdots,\ \frac{d^k\gamma}{dt^k}(t_0) = \frac{d^k\widetilde{\gamma}}{du^k}(u_0).$$

Let $\gamma_1, \gamma_2 : I \to \mathbb{R}^2$ be regular plane curves. We say that γ_1 and γ_2 are *congruent* if there exists a congruence C such that $\gamma_2(t) = C(\gamma_1(t)) = A(\gamma_1(t)) + \boldsymbol{b}$ for all $t \in I$, where the congruence is given by a rotation A and a translation $\boldsymbol{b}$ on $\mathbb{R}^2$.

Let $\gamma : I \to \mathbb{R}^2; t \mapsto \gamma(t)$ and $\widetilde{\gamma} : \widetilde{I} \to \mathbb{R}^2; u \mapsto \widetilde{\gamma}(u)$ be regular plane curves. We take the arc-length parameter for $\gamma(t)$ and $\widetilde{\gamma}(u)$, respectively. In general, we may assume that $\gamma(t)$ and

$\widetilde{\gamma}(u)$ have at least first order contact at any point $t = t_0, u = u_0$ up to congruent. We denote the curvatures $\kappa(t)$ of $\gamma(t)$ and $\widetilde{\kappa}(u)$ of $\widetilde{\gamma}(u)$, respectively.

Theorem A.1. *Let $\gamma : I \to \mathbb{R}^2$ and $\widetilde{\gamma} : \widetilde{I} \to \mathbb{R}^2$ be regular plane curves. If $\gamma(t)$ and $\widetilde{\gamma}(u)$ have at least $(k+2)$-th order contact at $t = t_0, u = u_0$ then*

$$\kappa(t_0) = \widetilde{\kappa}(u_0),\ \frac{d\kappa}{dt}(t_0) = \frac{d\widetilde{\kappa}}{du}(u_0),\ \cdots,\ \frac{d^k\kappa}{dt^k}(t_0) = \frac{d^k\widetilde{\kappa}}{du^k}(u_0). \tag{7}$$

Conversely, if t and u are the arc-length parameter of γ and $\widetilde{\gamma}$ respectively, and the condition (7) holds, then γ and $\widetilde{\gamma}$ have at least $(k+2)$-th order contact at $t = t_0, u = u_0$ up to congruent.

Proof. We may assume that t and u are the arc-length parameter of γ and $\widetilde{\gamma}$ respectively. Suppose that γ and $\widetilde{\gamma}$ have at least third order contact. Since the Frenet formula, we have $(d\gamma/dt)(t) = \boldsymbol{t}(t), (d^2\gamma/dt^2)(t) = \kappa(t)\boldsymbol{n}(t)$ and $(d\widetilde{\gamma}/du)(u) = \widetilde{\boldsymbol{t}}(u), (d^2\widetilde{\gamma}/du^2)(u) = \widetilde{\kappa}(u)\widetilde{\boldsymbol{n}}(u)$. It follows that $\boldsymbol{t}(t_0) = \widetilde{\boldsymbol{t}}(u_0), \boldsymbol{n}(t_0) = \widetilde{\boldsymbol{n}}(u_0)$ and $\kappa(t_0) = \widetilde{\kappa}(u_0)$. Hence, the case of $k = 1$ holds.

Suppose that γ and $\widetilde{\gamma}$ have at least $(k+2)$-th order contact and

$$\kappa(t_0) = \widetilde{\kappa}(u_0),\ \frac{d\kappa}{dt}(t_0) = \frac{d\widetilde{\kappa}}{du}(u_0), \cdots,\ \frac{d^{k-1}\kappa}{dt^{k-1}}(t_0) = \frac{d^{k-1}\widetilde{\kappa}}{du^{k-1}}(u_0)$$

hold. Since $(d^3\gamma/dt^3)(t) = (d\kappa/dt)(t)\boldsymbol{n}(t) - \kappa(t)^2\boldsymbol{t}(t)$, the form of $(d^{k+1}\gamma/dt^{k+1})(t)$ is given by

$$\frac{d^{k-1}\kappa}{dt^{k-1}}(t)\boldsymbol{n}(t) + f\left(\kappa(t), \cdots, \frac{d^{k-2}\kappa}{dt^{k-2}}(t)\right)\boldsymbol{t}(t) + g\left(\kappa(t), \cdots, \frac{d^{k-2}\kappa}{dt^{k-2}}(t)\right)\boldsymbol{n}(t),$$

for some smooth functions f and g. Then

$$\frac{d^{k+2}\gamma}{dt^{k+2}}(t) = \frac{d^k\kappa}{dt^k}(t)\boldsymbol{n}(t) + F\left(\kappa(t), \cdots, \frac{d^{k-1}\kappa}{dt^{k-1}}(t)\right)\boldsymbol{t}(t) + G\left(\kappa(t), \cdots, \frac{d^{k-1}\kappa}{dt^{k-1}}(t)\right)\boldsymbol{n}(t)$$

for some smooth functions F and G. By the same calculations, we have

$$\frac{d^{k+2}\widetilde{\gamma}}{du^{k+2}}(u) = \frac{d^k\widetilde{\kappa}}{du^k}(u)\widetilde{\boldsymbol{n}}(u) + F\left(\widetilde{\kappa}(u), \cdots, \frac{d^{k-1}\widetilde{\kappa}}{du^{k-1}}(u)\right)\widetilde{\boldsymbol{t}}(u) + G\left(\widetilde{\kappa}(u), \cdots, \frac{d^{k-1}\widetilde{\kappa}}{du^{k-1}}(u)\right)\widetilde{\boldsymbol{n}}(u).$$

It follows that $(d^k\kappa/dt^k)(t_0) = (d^k\widetilde{\kappa}/du^k)(u_0)$. By the induction, we have the first assertion.

By the reversing arguments, we can prove the converse assertion up to congruent. □

References

[1] V. I. Arnol'd, S. M. Gusein-Zade and A. N. Varchenko, *Singularities of Differentiable Maps vol.* I. Birkhäuser (1986).

[2] V. I. Arnol'd, *Singularities of Caustics and Wave Fronts.* Mathematics and Its Applications **62** Kluwer Academic Publishers (1990).

[3] V. I. Arnol'd, Topological properties of Legendre projections in contact geometry of wave fronts.. *St. Petersburg Math. J.* **6** (1995), 439–452.

[4] J. W. Bruce and T. J. Gaffney, Simple singularities of mappings $\mathbb{C}, 0 \to \mathbb{C}^2, 0$. *J. London Math. Soc. (2).* **26** (1982), 465–474. DOI: 10.1112/jlms/s2-26.3.465

[5] J. W. Bruce and P. J. Giblin, *Curves and singularities. A geometrical introduction to singularity theory. Second edition.* Cambridge University Press, Cambridge, 1992. DOI: 10.1017/CBO9781139172615

[6] J. Ehlers and E. T. Newman, The theory of caustics and wave front singularities with physical applications, *J. Math. Physics.* **41** (2000), 3344–3378. DOI: 10.1063/1.533316

[7] T. Fukunaga and M. Takahashi, Existence and uniqueness for Legendre curves. *J. Geom.* **104** (2013), 287–307. DOI: 10.1007/s00022-013-0162-6

[8] C. G. Gibson, *Elementary geometry of differentiable curves. An undergraduate introduction.* Cambridge University Press, Cambridge, 2001. DOI: 10.1017/CBO9781139173377

[9] A. Gray, E. Abbena and S. Salamon, *Modern differential geometry of curves and surfaces with Mathematica. Third edition.* Studies in Advanced Mathematics. Chapman and Hall/CRC, Boca Raton, FL, 2006.

[10] W. Hasse, M. Kriele and V. Perlick, Caustics of wavefronts in general relativity, *Class. Quantum Grav.* **13** (1996), 1161–1182. DOI: 10.1088/0264-9381/13/5/027

[11] G. Ishikawa, Zariski's moduli problem for plane branches and the classification of Legendre curve singularities. *Real and complex singularities, World Sci. Publ. Hackensack, NJ,* (2007), 56–84.

[12] G. Ishikawa and S. Janeczko, Symplectic bifurcations of plane curves and isotropic liftings. *Q. J. Math.* **54** (2003), 73–102. DOI: 10.1093/qjmath/54.1.73

[13] S. Izumiya, D. Pei, T. Sano and E. Torii, Evolutes of hyperbolic plane curves. *Acta Math. Sin.* (Engl. Ser.) **20** (2004), 543–550. DOI: 10.1007/s10114-004-0301-y

[14] S. Izumiya and M. Takahashi, Caustics and wave front propagations: applications to differential geometry. Geometry and topology of caustics–CAUSTICS '06, *Banach Center Publ.*, **82** (2008), 125–142. DOI: 10.4064/bc82-0-9

[15] K. Saji, M. Umehara and K. Yamada, The geometry of fronts. *Ann. of Math.* **169** (2009), 491–529. DOI: 10.4007/annals.2009.169.491

[16] K. Saji, M. Umehara and K. Yamada, The duality between singular points and inflection points on wave fronts. *Osaka J. Math.* **47** (2010), 591–607.

[17] A. Saloom and F. Tari, Curves in the Minkowski plane and their contact with pseudo-circles. *Geom. Dedicata.* **159** (2012), 109–124. DOI: 10.1007/s10711-011-9649-1

[18] M. Umehara, A computation of the basic invariant J^+ for closed 2-vertex curves, *J. Knot Theory Ramifications* **6** (1997), 105–113. DOI: 10.1142/S0218216597000091

[19] M. Umehara, A unified approach to the four vertex theorems. I. Differential and symplectic topology of knots and curves, *Amer. Math. Soc. Transl. Ser. 2*, **190** (1999), 185–228.

[20] V. M. Zakalyukin, Reconstructions of fronts and caustics depending on a parameter and versality of mappings. *J. Soviet Math.* **27** (1983), 2713–2735. DOI: 10.1007/BF01084818

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THE GENERICITY OF THE INFINITESIMAL LIPSCHITZ CONDITION FOR HYPERSURFACES

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ABSTRACT. We continue the development of the theory of infinitesimal Lipschitz equivalence, showing the genericity of the condition for families of hypersurfaces with isolated singularities.

1. Introduction

In an earlier paper [7], we introduced a candidate for a theory of infinitesimal Lipschitz equisingularity for families of complex analytic hypersurfaces with isolated singularities. The definition given there has an equivalent formulation, using the theory of integral closure of modules. This alternate form is easier to work with in many situations. In this paper we show that a slightly evolved version of this condition is *generic*. More precisely, we show, in the case of two strata, considered here, that the condition holds on a Zariski open subset of the parameter stratum Y. Proving that a stratification property is generic is essential for an equisingularity condition to have any value.

In preparation for using the integral closure formulation of our condition, we review some elements of the theory of integral closure of modules in section 2.

In section 3, we review the definition of the Lipschitz saturation of an ideal, give its alternate formulation using the theory of integral closure and define two infinitesimal Lipschitz conditions, one which we denote by iL_{m_Y} which is the analogue of the Whitney conditions and one which is the analogue of the Whitney A or the a_f condition which we denote by iL_A. We also give a geometric interpretation of these conditions on the family X.

We also introduce an invariant coming from the integral formulation of the Lipschitz condition. We use this invariant to show when two different ideals have the same Lipschitz saturation. We also use it to characterize generic hyperplanes in section 4.

In section 4, we come to the heart of this paper. As mentioned earlier, proving a genericity theorem is an important step in developing the theory attached to an equisingularity condition. Not only is this result necessary to ensure the condition is widely applicable, but the fact of genericity implies a strong connection with the geometry of the family. For example, Teissier proved that condition C held on a Zariski open and dense subset of the parameter space Y^k, of a k parameter family of isolated hypersurface singularities in $\mathbb{C}^{n+k}$ in [16]. Condition C later was seen to be equivalent to Verdier's condition W_f for the pair of strata $\{\mathbb{C}^{n+k} - Y^k, Y^k\}$, where f defined the family. Condition C was the keystone of Teissier's work on the Whitney equisingularity of families of hypersurfaces with isolated singularities. We use Teissier's proof in [16] as a model in developing a similar theorem for the iL_A condition. Currently a proof for the genericity of the iL_{m_Y} remains unknown.

In section 4, we state and prove the genericity theorem for the iL_A condition for the case of families of isolated hypersurface singularities. For the proof, we work in the module setting. Analogous results exist in the general case for families of isolated singularities, but requires further work in developing the definition of the infinitesimal Lipschitz condition; since you start with modules in the general case instead of ideals, a further layer of complexity is added in passing to the module theoretic version of the definition.

Also in section 4, we give an application of the genericity theorem. Given an equisingularity condition it is natural to ask if it passes to the family of generic plane sections of the singularity. We use the genericity theorem to show that it does for the iL_A condition. We then use the invariant introduced in section 3, and the multiplicity polar theorem, discussed in section 2, to give a condition for a hyperplane to be generic.

Ultimately, we hope to use the stratification condition defined here to prove that for a family of isolated hypersurface singularities, the iL_A condition gives a necessary and sufficient condition for the family to have a bi-Lipschitz stratification which includes Y as a stratum. This would give an infinitesimal criterion for the existence of a bi-Lipschitz stratification of such a family. It is known by work of Mostowski, [13] that bi-Lipschitz stratifications exist in the complex analytic setting, but not much is known about them besides their existence.

Using the conditions of this paper to characterize the "thick" and "thin" zones of Birbrair, Neumann and Pichon [1], developed by them for normal surface singularities, would open an avenue to generalizing these notions to higher dimensions, as well as linking them with Mostowski's work on showing the existence of these stratifications.

I am happy to acknowledge the impetus to this work given by the beautiful paper of Birbrair, Neumann and Pichon [1] and the stimulation afforded from conversation with them.

2. The theory of the Integral closure of modules

Let (X, x) be a germ of a complex analytic space and X a small representative of the germ and let $\mathcal{O}_X$ denote the structure sheaf on a complex analytic space X. One of the formulations of the definition of the infinitesimal Lipschitz condition uses the theory of integral closure of modules, which we now review. This theory will also provide the tools for working with the condition.

Definition 2.1. Suppose (X, x) is the germ of a complex analytic space, M a submodule of $\mathcal{O}^p_{X,x}$. Then $h \in \mathcal{O}^p_{X,x}$ is in the integral closure of M, denoted $\overline{M}$, if for all analytic $\phi : (\mathbb{C}, 0) \to (X, x)$, $h \circ \phi \in (\phi^* M)\mathcal{O}_1$. If M is a submodule of N and $\overline{M} = \overline{N}$ we say that M is a reduction of N.

To check the definition it suffices to check along a finite number of curves whose generic point is in the Zariski open subset of X along which M has maximal rank. (Cf. [3].)

If a module M has finite colength in $\mathcal{O}^p_{X,x}$, it is possible to attach a number to the module, its Buchsbaum-Rim multiplicity, $e(M, \mathcal{O}^p_{X,x})$. We can also define the multiplicity $e(M, N)$ of a pair of modules $M \subset N$, M of finite colength in N, as well, even if N does not have finite colength in $\mathcal{O}^p_X$.

We recall how to construct the multiplicity of a pair of modules using the approach of Kleiman and Thorup [9]. Given a submodule M of a free $\mathcal{O}_{X^d}$ module F of rank p, we can associate a subalgebra $\mathcal{R}(M)$ of the symmetric $\mathcal{O}_{X^d}$ algebra on p generators. This is known as the Rees algebra of M. If $(m_1, \cdots, m_p)$ is an element of M then $\sum m_i T_i$ is the corresponding element of $\mathcal{R}(M)$. Then Projan$(\mathcal{R}(M))$, the projective analytic spectrum of $\mathcal{R}(M)$ is the closure of the projectivised row spaces of M at points where the rank of a matrix of generators of M is maximal. Denote the projection to X^d by c. If M is a submodule of N or h is a section of N, then h and M generate ideals on Projan$\mathcal{R}(N)$; denote them by $\rho(h)$ and $\rho(\mathcal{M})$. If we can express h in terms of a set of generators $\{n_i\}$ of N as $\sum g_i n_i$, then in the chart in which $T_1 \neq 0$, we can express a generator of $\rho(h)$ by $\sum g_i T_i/T_1$. Having defined the ideal sheaf $\rho(\mathcal{M})$, we blow it up.

On the blow up $B_{\rho(\mathcal{M})}(\text{Projan}\mathcal{R}(N))$ we have two tautological bundles. One is the pullback of the bundle on $\text{Projan}\mathcal{R}(N)$. The other comes from $\text{Projan}\mathcal{R}(M)$. Denote the corresponding Chern classes by c_M and c_N, and denote the exceptional divisor by $D_{M,N}$. Suppose the generic rank of N (and hence of M) is g.

Then the multiplicity of a pair of modules M, N is:

$$e(M,N) = \sum_{j=0}^{d+g-2} \int D_{M,N} \cdot c_M^{d+g-2-j} \cdot c_N^j.$$

Kleiman and Thorup show that this multiplicity is well defined at $x \in X$ as long as $\overline{M} = \overline{N}$ on a deleted neighborhood of x. This condition implies that $D_{M,N}$ lies in the fiber over x, hence is compact. Notice that when $N = F$ and M has finite colength in F then $e(M,N)$ is the Buchsbaum-Rim multiplicity $e(M, \mathcal{O}^p_{X,x})$. There is a fundamental result due to Kleiman and Thorup, the principle of additivity [9], which states that given a sequence of $\mathcal{O}_{X,x}$-modules $M \subset N \subset P$ such that the multiplicity of the pairs is well defined, then

$$e(M,P) = e(M,N) + e(N,P).$$

Also if $\overline{M} = \overline{N}$ then $e(M,N) = 0$ and the converse also holds if X is equidimensional. Combining these two results we get thet if $\overline{M} = \overline{N}$ then $e(M,N) = e(N,P)$. These results will be used in Section 5.

In studying the geometry of singular spaces, it is natural to study pairs of modules. In dealing with non-isolated singularities, the modules that describe the geometry have non-finite colength, so their multiplicity is not defined. Instead, it is possible to define a decreasing sequence of modules, each with finite colength inside its predecessor, when restricted to a suitable complementary plane. Each pair controls the geometry in a particular codimension.

We also need the notion of the polar varieties of M. The *polar variety of codimension k* of M in X, denoted $\Gamma_k(M)$, is constructed by intersecting $\text{Projan}\mathcal{R}(M)$ with $X \times H_{g+k-1}$ where H_{g+k-1} is a general plane of codimension $g + k - 1$, then projecting to X.

Setup: We suppose we have families of modules $M \subset N$, M and N submodules of a free module F of rank p on an equidimensional family of spaces with equidimensional fibers $\mathcal{X}^{d+k}$, $\mathcal{X}$ a family over a smooth base Y^k. We assume that the generic rank of M, N is $g \leq p$. Let $P(M)$ denote $\text{Projan}\mathcal{R}(M)$, π_M the projection to $\mathcal{X}$.

We will be interested in computing, as we move from the special point 0 to a generic point, the change in the multiplicity of the pair (M,N), denoted $\Delta(e(M,N))$. We will assume that the integral closures of M and N agree off a set C of dimension k which is finite over Y, and assume we are working on a sufficiently small neighborhood of the origin, so that every component of C contains the origin in its closure. Then $e(M,N,y)$ is the sum of the multiplicities of the pair at all points in the fiber of C over y, and $\Delta(e(M,N))$ is the change in this number from 0 to a generic value of y. If we have a set S which is finite over Y, then we can project S to Y, and the degree of the branched cover at 0 is $\text{mult}_y S$. (Of course, this is just the number of points in the fiber of S over our generic y.)

Let $C(M)$ denote the locus of points where M is not free, *i.e.*, the points where the rank of M is less than g, $C(\text{Projan}\mathcal{R}(M))$ its inverse image under π_M.

We can now state the Multiplicity Polar Theorem. The proof in the ideal case appears in [5]; the general proof appears in [6].

Theorem 2.2. *(Multiplicity Polar Theorem) Suppose in the above setup we have that $\overline{M} = \overline{N}$ off a set C of dimension k which is finite over Y. Suppose further that*

$$C(\text{Projan}\mathcal{R}(M))(0) = C(\text{Projan}\mathcal{R}(M(0))),$$

except possibly at the points which project to $0 \in \mathcal{X}(0)$.

Then, for y *a generic point of* Y,

$$\Delta(e(M,N)) = \text{mult}_y \Gamma_d(M) - \text{mult}_y \Gamma_d(N),$$

where $C(\text{Projan}\mathcal{R}(M))(0)$ *is the fiber of* $C(\text{Projan}\mathcal{R}(M))$ *over* 0, $\mathcal{X}(0)$ *is the fiber over* 0 *of the family* $\mathcal{X}^{d+k}$, *and* $M(0)$ *is the restriction of the module* M *to* $\mathcal{X}(0)$.

3. The Lipschitz saturation of an ideal and the definition of the iL conditions

The construction of the integral closure of an ideal is an example of a general approach to constructing closure operations on sheaves of ideals and modules given a closure operation on a sheaf of rings. Here is the idea. Denote the closure operation on the ring R by $C(R)$. Given a ring, R, blow-up R by an ideal I. (If we have a module M which is a submodule of a free module F, form the blow-up $B_{\rho(\mathcal{M})}(\text{Projan}\mathcal{R}(F))$, as in the last section.) Use the projection map of the blow-up to the base to pullback I to the blow-up. Now apply the closure operation to the structure sheaf of the blow-up, and look at the sheaf of ideals generated by the pull back of I. The elements of the structure sheaf on the base which pull back to elements of the ideal sheaf are the elements of $C(I)$.

Two examples of this are given by the normalization of a ring and the semi-normalization of a ring. (In the normalization, all of the bounded meromorphic functions become regular, while in the semi-normalization only those which are continuous become regular. Cf [8] for details on this construction.) Consider $B_I(X)$, the blow-up of X by I. If we pass to the normalization of the blow-up, then h is in $\bar{I}$ iff and only if the pull back of h to the normalization is in the ideal generated by the pullback of I [11]. If we pass to the semi-normalization of the blow-up, then h is in the weak sub-integral closure of I denoted ${}^*I$, iff the pullback of h to the semi-normalization is in the ideal generated by the pullback of I. (For a proof of this and more details on the weak subintegral closure cf. [8]).

There is another way to look at the closure operation defined above; in the case of the integral closure of an ideal, we are looking at an open cover of the co-support of an ideal sheaf, and choosing locally bounded meromorphic functions on each open set, and seeing if we can write a regular function locally in terms of generators of the ideal using our locally bounded meromorphic functions as coefficients. This suggests, that in the Lipschitz case, we use locally bounded meromorphic functions which satisfy a Lipschitz condition. The closure operation on rings that this indicates is the Lipschitz saturation of a space, as developed by Pham-Teissier ([15]).

In the approach of Pham-Teissier, let A be a commutative local ring over $\mathbb{C}$, and $\bar{A}$ its normalization. (We can assume A is the local ring of an analytic space X at the origin in $\mathbb{C}^n$.) Let I be the kernel of the inclusion $\bar{A} \otimes_{\mathbb{C}} \bar{A} \to \bar{A} \otimes_A \bar{A}$.

In this construction, the tensor product is the analytic tensor product which has the right universal property for the category of analytic algebras, and which gives the analytic algebra for the analytic fiber product.

Pham and Teissier then defined the Lipschitz saturation of A, denoted $\tilde{A}$, to consist of all elements $h \in \bar{A}$ such that $h \otimes 1 - 1 \otimes h \in \bar{A} \otimes_{\mathbb{C}} \bar{A}$ is in the integral closure of I. (For related results see [12].)

The connection between this notion and that of Lipschitz functions is as follows. If we pick generators $(z_1, \dots, z_n)$ of the maximal ideal of the local ring A, then $z_i \otimes 1 - 1 \otimes z_i \in \bar{A} \otimes_{\mathbb{C}} \bar{A}$ give a set of generators of I. Choosing z_i so that they are the restriction of coordinates on the ambient space, the integral closure condition is equivalent to

$$|h(z_1, \dots, z_n) - h(z'_1, \dots, z'_n)| \leq C \sup_i |z_i - z'_i|$$

holding on some neighborhood U, of $(0,0)$ on $X \times X$. This last inequality is what is meant by the meromorphic function h being Lipschitz at the origin on X. (Note that the integral closure condition is equivalent to the inequality holding on a neighborhood U for some C for any set of generators of the maximal ideal of the local ring A. The constant C and the neighborhood U will depend on the choice.)

If X, x is normal, then passing to the Lipschitz saturation doesn't add any functions. Denote the saturation of the blow-up by $SB_I(X)$, and the map to X by π_S. Then we make the definition:

Definition 3.1. let I be an ideal in $\mathcal{O}_{X,x}$, then the **Lipschitz saturation** of the ideal I, denoted I_S, is the ideal $I_S = \{h \in \mathcal{O}_{X,x} | \pi_S^*(h) \in \pi_S^*(I)\}$.

Since the normalization of a local ring A contains the seminormalization of A, and the seminomalization contains the Lipschitz saturation of A, it follows that $\bar{I} \supset {}^*I \supset I_S \supset I$. In particular, if I is integrally closed, all three sets are the same.

Here is a viewpoint on the Lipschitz saturation of an ideal I, which will be useful later. Given an ideal, I, and an element h that we want to check for inclusion in I_S, we can consider $(B_I(X), \pi)$, $\pi^*(I)$ and $h \circ \pi$. Since $\pi^*(I)$ is locally principal, working at a point z on the exceptional divisor E, we have a local generator $f \circ \pi$ of $\pi^*(I)$. Consider the quotient $(h/f) \circ \pi$. Then $h \in I_S$ if and only if at the generic point of any component of E, $(h/f) \circ \pi$ is Lipshitz with respect to a system of local coordinates. If this holds we say $h \circ \pi \in (\pi^*(I))_S$.

We can also work on the normalized blow-up, $(NB_I(X), \pi_N)$. Then we say $h \circ \pi_N \in (\pi_N^*(I))_S$ if $(h/f) \circ \pi_N$ satisfies a Lipschitz condition at the generic point of each component of the exceptional divisor of $(NB_I(X), \pi_N)$ with respect to the pullback to $(NB_I(X), \pi_N)$ of a system of local coordinates on $B_I(X)$ at the corresponding points of $B_I(X)$. As usual, the inequalities at the level of $NB_I(X)$ can be pushed down and are equivalent to inequalities on a suitable collection of open sets on X.

This definition can be given an equivalent statement using the theory of integral closure of modules. Since Lipschitz conditions depend on controlling functions at two different points as the points come together, we should look for a sheaf defined on $X \times X$. We describe a way of moving from a sheaf of ideals on X to a sheaf on $X \times X$. Let $h \in \mathcal{O}_{X,x}$; define h_D in $\mathcal{O}^2_{X \times X, x, x}$, as $(h \circ \pi_1, h \circ \pi_2)$, π_i the projection to the i-th factor of the product. Let I be an ideal in $\mathcal{O}_{X,x}$; then I_D is the submodule of $\mathcal{O}^2_{X \times X, x, x}$ generated by the h_D where h is an element of I.

If I is an ideal sheaf on a space X then intuitively, $h \in \bar{I}$ if h tends to zero as fast as the elements of I do as you approach a zero of I. If h_D is in $\overline{I_D}$ then the element defined by $(1,-1) \cdot (h \circ \pi_1, h \circ \pi_2) = h \circ \pi_1 - h \circ \pi_2$ should be in the integral closure of the ideal generated by applying $(1,-1)$ to the generators of I_D, namely the ideal generated by $g \circ \pi_1 - g \circ \pi_2$, g any element of I. This implies the difference of h at two points goes to zero as fast as the difference of elements of I at the two points go to zero as the points approach each other. It is reasonable that elements in I_S should have this property. In fact we have:

Theorem 3.2. *Suppose (X, x) is a complex analytic set germ, $I \subset \mathcal{O}_{X,x}$. Then $h \in I_S$ if and only if $h_D \in \overline{I}_D$.*

Proof. This is theorem 2.3 of [7], and is proved there under the additional assumption that $h \in \bar{I}$. However, as we have noted if $h \in I_S$, then $h \in \bar{I}$. If $h_D \in \overline{I}_D$, it follows that $(1,0) \cdot h_D$ is in the integral closure of $\pi_1^*(I)$ on $X \times X$, which clearly implies $h \in \bar{I}$. □

Here is an example showing the difference between the integral closure of the Jacobian ideal and its saturation. Consider $f(x,y) = x^2 + y^p$, $p > 3$ odd. Denote the plane curve defined by f by X. Then X has a normalization given by $\phi = (t^p, t^2)$. The elements in the integral closure of the Jacobian ideal are just those ring elements h such that $h \circ \phi \in \phi^*(J(f)) = (t^p)$. Now

$y^q \circ \phi = t^{2q}$, so $y^q \in \overline{J(f)}$ for $q > p/2$. Denote a matrix of generators for $J(f)_D$ by $[J(f)_D]$. Consider the curve mapping into $X \times X$ given by $\Phi(t) = (t^p, t^2, t^p, ct^2)$, where c is a p-th root of unity different from 1. Now consider the ideal generated by the entries of the vector

$$< 1, -1 > [J(f)_D] \circ \Phi(t).$$

This ideal is generated by $(y^{p-1} - y'^{p-1}, (x, x', y^{p-1}, y'^{p-1})(y - y')) \circ \Phi(t) = (t^{p+2})$. Meanwhile the order in t of $< 1, -1 > (y^q, y'^q) \circ \Phi(t) = 2q$. If $p < 2q < p+2$ ie. $q = (p+1)/2$, then (y^q, y'^q) cannot be in $\overline{J(f)_D}$, hence $y^q \notin J(f)_S$ but y^q is in $\overline{J(f)}$.

Because we have re-cast the Lipschitz saturation of an ideal in integral closure terms, the invariants associated with integral closure become available to describe/control the Lipschitz saturation of an ideal. Notice first that the multiplicity of an ideal doesn't help, because the multiplicity of I_S is same as the multiplicity of I since they have the same integral closure.

Even if X is an isolated hypersurface singularity, $J(f)_D$ will not have finite colength, even in the plane curve case. The co-support will be $X \times 0 \cup 0 \times X \cup \Delta X$ in $X \times X$. However the multiplicity of the pair offers a way around this. The module $J(f)_D$ has a simple description, as we will see, off the origin in each of these three sets, and any integral closure condition we wish to use is easily checked because of this structure. This suggests looking for the largest module whose integral closure agrees with $J(f)_D$ off the origin, and using the multiplicity of the pair as our invariant. In the notation of [4], this module is denoted $H_{2n-3}(J(f)_D)$. This is the integral hull of $J(f)_D$ of codimension $2n-3$, which means the integral closure of $J(f)_D$ and $H_{2n-3}(J(f)_D)$ agree off a set of codimension $2n-2$, ie. off $(0,0)$ in $X^{n-1} \times X^{n-1}$. The next lemma identifies $H_{2n-3}(J(f)_D)$.

Lemma 3.3. *Suppose X^{n-1} is an isolated hypersurface singularity, defined by f. Then*

$$H_{2n-3}(J(f)_D) = \overline{J(f)}_D.$$

Proof. We'll show that the integral closure of $J(f)_D$ and $\overline{J(f)}_D$ agree off the origin in $X \times X$.

Suppose $p = (x, x') \notin X \times 0 \cup 0 \times X \cup \Delta X$. Then for some i, j, k, $f_j(x)(z_i - z'_i)$ and $f_k(x')(z_i - z'_i)$ are not zero at p. This implies that both modules have rank 2 at p, hence are equal.

Suppose $p \in \Delta_X$, $p \neq (0,0)$; then for some i, $f_i(x) \neq 0$. This implies $I_\Delta \oplus I_\Delta$ is in both modules. Further by adding elements of the form $(0, f_i(z) - f_i(z'))$ which are in $I_\Delta \oplus I_\Delta$ to $(f_i(z), f_i(z'))$, we see both modules contain $(1,1)$. Since both modules are contained in the module generated by $(1,1)$ and $I_\Delta \oplus I_\Delta$, and this module is integrally closed, the result is checked on $\Delta_X - (0,0)$.

Suppose $p = (x, 0), \neq 0$. Since $x \neq 0$, $J(f)_D$ contains $(1,0)$ and $(0, J(f))$.Thus

$$\overline{J(f)_D} = \mathcal{O}_{X,x} \oplus \overline{J(f)} = \overline{J(f)}_D.$$

□

The lemma suggests that it is interesting to consider the multiplicity of the pair $J(f)_D, \overline{J(f)}_D$, and we will use this invariant in the last section in the study of hyperplane sections of X. For now we remark as a corollary of the proof of the lemma, we have for any I an ideal of finite colength in any $\mathcal{O}^d_X$, that $H_{2d-1}(I) = (\overline{I})_D$. As a corollary we have:

Corollary 3.4. *Suppose $I \subset J \subset \overline{I}$ are ideals in $\mathcal{O}_{X,x}$, with X, x equidimensional, then $e(I_D, \overline{I}_D) = e(J_D, \overline{I}_D)$ if and only if $\overline{I}_D = \overline{J}_D$.*

Proof. From the additivity of multiplicity of pairs [9] it follows that $e(I_D, J_D) = 0$ which is equivalent to their integral closures being the same. □

Corollary 3.5. *Suppose $I \subset J \subset \overline{I}$ are ideals in $\mathcal{O}_{X,x}$, with X, x equidimensional, then $e(I_D, \overline{I}_D) = e(J_D, \overline{I}_D)$ if and only if $I_S = J_S$.*

Proof. This follows from the connection between the Lipschitz saturation of an ideal and integral closure. □

Now we add the necessary structure to deal with families of spaces.

Just as Pham-Teissier extended their original definition to a family of spaces, we can do the same. Suppose $X^{d+k}, 0$ is an analytic space containing a smooth subset $Y^k, 0$, and (X^{d+k}, p) is a family of spaces over Y, X, Y embedded in $\mathbb{C}^{n+k}, 0$, so that p is the projection on the last k factors of $\mathbb{C}^{n+k}, 0$, where $Y^k = 0 \times \mathbb{C}^k$.

Then, in the definition of the Lipschitz saturation rel Y of the local ring of $X^{d+k}, 0$, we use a set of local coordinates on the ambient space which restrict to generators of the maximal ideals of the fibers of X over Y. This amounts to looking at the fiber product of the normalization of X with itself over Y, and asking that locally $h \circ p_1 - h \circ p_2$ is in the integral closure of the double of the ideal generated by these coordinates.

Given an ideal sheaf I on $X^{d+k}, 0$, using the relative saturation, we can define the Lipschitz saturation of I relative to Y. When we are working in the context of a family of spaces we will also use I_S to denote this saturation. In a similar way, we can develop an equivalent integral closure condition using modules as before, just working on $X \times_Y X$ instead of $X \times X$.

In practice we will be working with ideal sheaves on a family of spaces, where the ideals vanish on Y, and our local coordinates at points of $B_I(X^{n+k})$ consist of the pullbacks of a set of generators of m_Y and local coordinates on the projective space(s) in the blow-up.

It is not difficult to check that Theorem 2.3 of [7] continues to hold in this new context.

Having constructed the necessary infinitesimal objects we now develop our condition.

Setup Let $X^{n+k}, 0 \subset \mathbb{C}^{n+1+k}, 0$ be a hypersurface, containing a smooth subset Y embedded in $\mathbb{C}^{n+1+k}$ as $0 \times \mathbb{C}^k$, with p_Y the projection to Y. Assume $Y = S(X)$, the singular set of X. Suppose F is the defining equation of X, (z, y) coordinates on $\mathbb{C}^{n+1+k}$. Denote by $f_y(z) = F(z, y)$ the family of functions of defined by F, and by X_y, $f_y^{-1}(0)$. Assume f_y has an isolated singularity at the origin. Let m_Y denote the ideal defining Y, and $J(F)_Y$, the ideal generated by the partial derivatives with respect to the y coordinates, $J_z(F)$, those with respect to the z coordinates.

Definition 3.6. The pair (X, Y) satisfy the iL_{m_Y} condition at the origin if either of the two equivalent conditions hold:

1) $J(F)_Y \subset (m_Y J_z(F))_S$
2) $(J(F)_Y)_D \subset \overline{(m_Y J_z(F))_D}$.

An analogous condition for iL_{m_Y} is $J(F)_Y \subset \overline{m_Y J_z(F)}$. This is the equivalent to the Verdier's condition W or the Whitney conditions.

Next we give the definition of iL_A.

Definition 3.7. The pair (X, Y) satisfy the iL_A, at the origin if either of the two equivalent conditions hold:

1) $J(F)_Y \subset (J_z(F))_S$
2) $(J(F)_Y)_D \subset \overline{J_z(F))_D}$.

The analogous condition is $J(F)_Y \subset \overline{J_z(F)}$. If one works on the ambient space, then this is equivalent to the A_F condition. Working on X, it is equivalent to asking that the X has no vertical tangent plane at the origin, so this is weaker than Whitney A. However, suppose l is a linear form on the ambient space. Let $J(F)_l$ denote the ideal generated by applying tangent vectors in the kernel of l to F. So $J_z(F) = J(F)_y$ in the case dim $Y = 1$. Working in the one dimensional parameter case, if there exist a pencil of forms l_s including y such that $J(F) \subset \overline{J(F)_{l_s}}$ then not only does Whitney A hold but the total space has no relative polar curve. This follows because if the dimension of the fiber of the limiting tangent hyperplanes over the origin is not maximal then the fiber over the origin must be in the closure of the fiber over the parameter space with $y \neq 0$, and all of these hyperplanes contain Y. Because the dimension of the fiber over the origin is less than maximal this also implies the polar curve is empty. The condition with the pencil of forms ensures that no hyperplane defined by an element of the pencil can be a limiting tangent hyperplane, hence the pencil of hyperplanes has no intersection with the fiber over zero, which must therefore have less than maximal dimension.

Since there are different ways in which the total space X^{n+k} can be made into a family of spaces, it is natural to ask if the conditions we have defined depend on the projection to Y which defines the family. We now show that the condition iL_{m_Y} does not depend on the projection to Y.

Proposition 3.8. *In the above set-up the following conditions are equivalent:*
1) $(J(F)_Y)_D \subset \overline{(m_Y J_z(F))_D}$,
2) $(J(F)_Y)_D \subset \overline{(m_Y J(F))_D}$.

The analogous result for W is quite easy. The Lipschitz case is more technical. We first show:

Lemma 3.9. In the above setup if $(J(F)_Y)_D \subset \overline{(m_Y J(F))_D}$, then $J(F)_Y \subset \overline{m_Y J(F)}$, hence condition W holds for the pair $(X-Y, Y)$ at the origin (and hence on some Z-open subset of Y containing the origin.)

Proof. We use the curve criterion. We can choose a curve $\Phi = (\phi_1, \phi_2)$, where ϕ_1 maps $\mathbb{C}, 0$ to 0, and ϕ_2 is arbitrary. Then the curve criterion for this curve becomes $\phi_2^*(J(F)_Y) \subset \phi_2^*(m_Y J(F))$. Here an easy argument using Nakayama's lemma implies that $\phi_2^*(J(F)_Y) \subset \phi_2^*(m_Y J_z(F))$, which implies the W condition. □

Now we prove our proposition.

Proof. We use the curve criterion again. Let $\Phi = (\phi_1, \phi_2)$. It is enough to prove it in the case where Y is one dimensional, since the notation is the only part of the proof which is harder in general. It is also clear that 1) implies 2), so we assume 2). By the given we have:

$$\left(\frac{\partial F}{\partial y}\right)_D \circ \Phi = \sum g_{i,j}(t)\left(z_i \frac{\partial F}{\partial z_j}\right)_D \circ \Phi + \sum g_{i,j,k}(t)(z_k \circ \phi_1 - z_k \circ \phi_2)\left(0, z_i \frac{\partial F}{\partial z_j}\right) \circ \phi_2$$
$$+ \sum h_i(t)\left(z_i \frac{\partial F}{\partial y}\right)_D \circ \Phi.$$

We now work mod $m_1 \Phi^*(m_Y J(F)_D)$ and we call the left side of the above equation $*$. Subtract $\sum h_i(t) z_i \circ \phi_1 *$ from both sides of the above equation. This sum is in $m_1 \Phi^*(m_Y J(F)_D)$, so we get:

$$\left(\frac{\partial F}{\partial y}\right)_D \circ \Phi = \sum g_{i,j}(t)\left(z_i \frac{\partial F}{\partial z_j}\right)_D \circ \Phi + \sum g_{i,j,k}(t)(z_k \circ \phi_1 - z_k \circ \phi_2)\left(0, z_i \frac{\partial F}{\partial z_j}\right) \circ \phi_2$$
$$+ \sum h_i(t)(z_i \circ \phi_2 - z_i \circ \phi_1)\left(0, \frac{\partial F}{\partial y} \circ \phi_2\right).$$

Now we use the lemma to write $\frac{\partial F}{\partial y} \circ \phi_2$ as an element of $\phi_2^*(m_Y J_z(F))$. Making the substitution into the line above shows that the terms there are 0 mod $m_1 \Phi^*(m_Y J(F)_D)$, hence we have $\frac{\partial F}{\partial y} \circ \Phi$ is an element of $(m_Y J_z(F))_D$ mod $m_1 \Phi^*(m_Y J(F)_D)$. Hence by Nakayama's lemma, $\Phi^* m_Y J_z(F))_D = \Phi^* m_Y J(F))_D$ and the proposition follows. □

While a similar result for iL_A doesn't make sense, if we ask that $(J(F)_Y)_D$ is strictly dependent on $J_z(F))_D$ then an analogous result holds. (Recall that an element $h \in \mathcal{O}^p_{X,x}$ is strictly dependent on $M \subset \mathcal{O}^p_{X,x}$, if for each curve ϕ $h \circ \phi \in m_1\phi^*(M)$. The set of elements strictly dependent on M are denoted M^+.)

We give a geometric interpretation of these conditions at the level of the family X^{n+k}. We make some preliminary constructions to do this. Denote the coordinates on $\mathbb{P}^n$ by T_i, for $1 \leq i \leq n+1$, let V_i be the subset of $\mathbb{P}^n$ defined by $T_i \neq 0$, and let U_i denote

$$B_{J_z(F)}(X^{n+k}) \cap (X \times V_i).$$

At each point of U_i, $\frac{\partial F}{\partial z_i} \circ \pi$ is a local generator of the principal ideal sheaf $\pi^*(J_z(F))$. The condition that $\frac{\partial F}{\partial y_j}$ be in the Lipschitz saturation of $J_z(F))$ means that at each point of U_i, $\frac{\frac{\partial F}{\partial y_j}}{\frac{\partial F}{\partial z_i}} \circ \pi$ is Lipschitz rel Y with respect to the local coordinates, which are $z_k \circ \pi$, $1 \leq k \leq n+1$, and T_j/T_i, $1 \leq j \leq n+1$, $j \neq i$. Since $\frac{\frac{\partial F}{\partial z_j}}{\frac{\partial F}{\partial z_i}} \circ \pi = \frac{T_j}{T_i}$, this implies that $\frac{\frac{\partial F}{\partial y_j}}{\frac{\partial F}{\partial z_i}}$ is Lipschitz with respect to z_k, $1 \leq k \leq n+1$, and $\frac{\frac{\partial F}{\partial z_j}}{\frac{\partial F}{\partial z_i}}$, $1 \leq j \leq n+1$, $j \neq i$ on $\pi(U_i)$.

This implies the existence of k vectorfields tangent to X defined on each $\pi(U_i)$ of the form

$$\vec{v}_{j,i} = \frac{\partial}{\partial y_j} - \frac{\frac{\partial F}{\partial y_j}}{\frac{\partial F}{\partial z_i}} \frac{\partial}{\partial z_i},$$

each vectorfield Lipschitz relative to Y,with respect to z_k, $1 \leq k \leq n+1$, and $\frac{\frac{\partial F}{\partial z_j}}{\frac{\partial F}{\partial z_i}}$, $1 \leq j \leq n+1$, $j \neq i$. Since every element of $J_z(X)$ is in the Lipschitz saturation of $J_z(X)$ it is not true apriori that these vectorfields are extensions of the constant fields on Y. However, if we assume the A_F condition holds for $(X-Y, Y)$, then the quotients $\frac{\frac{\partial F}{\partial y_j}}{\frac{\partial F}{\partial z_i}} \circ \pi$ will vanish on the exceptional divisor, and the $\vec{v}_{j,i}$ will be extensions of the constant fields on Y.

There is another useful interpretation which we can make. Recall the following definition of distance between two linear subspaces A, B at the origin in $\mathbb{C}^N$, then

$$\text{dist }(A, B) = \sup_{\substack{u \in B^{\perp} - \{0\} \\ v \in A - \{0\}}} \frac{|(u, v)|}{\|u\| \, \|v\|}.$$

If p, p' are smooth points in the same fiber y over Y in $\pi(U_i)$, we claim that the distance between the tangent spaces to X at p and p' is commensurate with the maximum of the distance between the tangent spaces to X_y at p and p' and the distance between the points.

We first relate the distance defined above to a notion of distance closer to our Lipschitz condition.

Suppose $\mathbf{a} = (a_0, \ldots, a_n)$, $\mathbf{b} = (b_0, \ldots, b_n)$ define hyperplanes A and B in $\mathbb{C}^{n+1}$. We will use the supnorm on $\mathbb{C}^{n+1}$; suppose $||\mathbf{a}|| = a_i$, and $||\mathbf{b}|| = b_i$, same index for both, for simplicity take $i = 0$.

We can then also measure the distance between A and B by using the $\sup\limits_{i, i \leq i \leq n} ||a_i/a_0 - b_i/b_0||$. The a_i/a_0 are just the coordinates of the hyperplane A regarded as a point of $\hat{\mathbb{P}}^n$. We compare this notion of distance with the usual one.

Lemma 3.10. *Suppose* $\mathbf{a} = (a_0, \dots, a_n)$, $\mathbf{b} = (b_0, \dots, b_n)$ *define hyperplanes* A *and* B *in* $\mathbb{C}^{n+1}$, $||\mathbf{a}|| = a_0$, *and* $||\mathbf{b}|| = b_0$. *Then*

$$\text{dist }(A, B) = \sup_{i, 1 \leq i \leq n} ||a_i/a_0 - b_i/b_0||.$$

Proof. A basis for the vectors in A are given by $a_0 e_i - a_i e_0$ where e_k is the k-th standard basis vector in $\mathbb{C}^{n+1}$. Since we are using the supnorm, the terms $\frac{|(u,v)|}{\|u\|\|v\|}$ become

$$\frac{|(a_0 e_i - a_i e_0, \bar{\mathbf{b}})|}{\|a_0\| \, \|b_0\|} = ||a_i/a_0 - b_i/b_0||.$$

□

Now we return to our geometric interpretation. Since the $\frac{\partial F}{\partial y_i}$ are in the integral closure of $J_z(F)$, we may work in a system of neighborhoods U_i on X where we may assume for each $p \in U_i$ the values of the elements of $J(F)$ are bounded in norm by $|\frac{\partial F}{\partial z_i}(p)|$. Then, applying the above lemma, we see that the distance between tangent planes to X at points p_1, p_2 in the same U_i is the sup over

$$\left\{ \left\| \frac{\frac{\partial F}{\partial y_k}(p_1)}{\frac{\partial F}{\partial z_i}(p_1)} - \frac{\frac{\partial F}{\partial y_k}(p_2)}{\frac{\partial F}{\partial z_i}(p_2)} \right\| \ , \ \left\| \frac{\frac{\partial F}{\partial z_j}(p_1)}{\frac{\partial F}{\partial z_i}(p_1)} - \frac{\frac{\partial F}{\partial z_j}(p_2)}{\frac{\partial F}{\partial z_i}(p_2)} \right\| \right\}.$$

Then condition iL_A implies that this is the same as the sup over

$$\left\{ \left\| \frac{\frac{\partial F}{\partial z_j}(p_1)}{\frac{\partial F}{\partial z_i}(p_1)} - \frac{\frac{\partial F}{\partial z_j}(p_2)}{\frac{\partial F}{\partial z_i}(p_2)} \right\| \ , \ ||p_1 - p_2|| \right\},$$

which is the same as the maximum of the distance between the tangent spaces to X_y at p_1 and p_2 and the distance between the points, p_1 and p_2.

We can say something similar for the iL_W condition. First, since iL_W implies iL_A, the same interpretation applies to the iL_W condition. But more is true, and we develop some material related to the Lipschitz saturation of the product of two ideals to explain it.

Lemma 3.11. *(Product lemma) Given* h,g *in* $\mathcal{O}_{X,x}$, p_1,$p_2 \in X$, *then*

$$\|(hg)(p_1) - (hg)(p_2)\| \leq \|h(p_1)\|\|g(p_1) - g(p_2)\| + \\ \|g(p_2)\|\|h(p_1) - h(p_2)\|.$$

Proof. We have

$$\begin{aligned} \|(hg)(p_1) - (hg)(p_2)\| &= \|(hg)(p_1) - h(p_1)g(p_2) + h(p_1)g(p_2) - (hg)(p_2)\| \\ &= \|h(p_1)(g(p_1) - g(p_2)) + g(p_2)(h(p_1) - h(p_2))\| \\ &\leq \|h(p_1)\|\|g(p_1) - g(p_2)\| + \|g(p_2)\|\|h(p_1) - h(p_2)\|. \end{aligned}$$

□

Note that we can always choose one of the terms, say $\|g(p_i)\|$, to be the minimum of the $\|g(p_i)\|$. (You cannot, in general, minimize both h and g terms.)

We apply this lemma to the condition for $h \in \mathcal{O}_{X,x}$ to be in the Lipschitz saturation of IJ, I,J two ideals of $\mathcal{O}_{X,x}$.

Suppose $I = (f_1, \dots, f_p)$, $J = (g_1, \dots, g_q)$. Work on the Zariski open subset $U_{m,n}$ of $(B_{IJ}(X), \pi)$ in which $(f_m g_n) \circ \pi$ is a local generator of $\pi^*(IJ)$. Local coordinates are given by the pullback of coordinates at x, and by $T_{i,j}$ where $(i, j) \neq (m, n)$, $1 \leq i \leq p$,$1 \leq j \leq q$, and where $T_{i,j} = \frac{(f_i g_j) \circ \pi}{f_m g_n \circ \pi}$.

Note that

$$T_{m,j}=\frac{(f_mg_j)\circ\pi}{f_mg_n\circ\pi}=\frac{g_j\circ\pi}{g_n\circ\pi},\quad\text{while}\quad T_{i,n}=\frac{(f_ig_n)\circ\pi}{f_ig_n\circ\pi}=\frac{f_i\circ\pi}{f_m\circ\pi}.$$

The next lemma shows that among all the $T_{i,j}$, on $U_{m,n}$ we need only consider the $T_{m,j}$ and $T_{i,n}$ to define the Lipschitz saturation of IJ. As usual, π_N denotes the normalization map, while p_1 and p_2 are projection maps from the product of the normalization of $B_{IJ}(X)$ with itself.

Lemma 3.12. *Let $U_{m,n}$ be as above, then the ideal generated by*

$$\{T_{j,n}\circ\pi_N\circ p_1-T_{j,n}\circ\pi_N\circ p_2, T_{m,i}\circ\pi_N\circ p_1-T_{m,i}\circ\pi_N\circ p_2\},$$

for $1\le j\le p, j\ne m, 1\le i\le q, i\ne n$, is a reduction of the ideal generated by

$$\{T_{j,i}\circ\pi_N\circ p_1-T_{j,i}\circ\pi_N\circ p_2\}$$

at points of $\pi_N^{-1}(U_{m,n})\times\pi_N^{-1}(U_{m,n})$.

Proof. By the product lemma we have

$$\begin{aligned}&\|\frac{f_ig_j}{f_mg_n}\circ\pi\circ\pi_N\circ p_1(z_1',z_2')-\frac{f_ig_j}{f_mg_n}\circ\pi\circ\pi_N\circ p_2(z_1',z_2')\|\\
\le&\|\frac{f_i\circ\pi}{f_m\circ\pi}\circ\pi_N\circ p_1(z_1',z_2')\|\|\frac{g_j\circ\pi}{g_n\circ\pi}\circ\pi_N\circ p_1(z_1',z_2')-\frac{g_j\circ\pi}{g_n\circ\pi}\circ\pi_N\circ p_2(z_1',z_2')\|\\
+&\|\frac{g_j\circ\pi}{g_n\circ\pi}\circ\pi_N\circ p_1(z_1',z_2')\|\|\frac{f_i\circ\pi}{f_m\circ\pi}\circ\pi_N\circ p_1(z_1',z_2')-\frac{f_i\circ\pi}{f_n\circ\pi}\circ\pi_N\circ p_2(z_1',z_2')\|.\end{aligned}$$

Now we can bound the terms $\|\frac{f_i\circ\pi}{f_m\circ\pi}\circ\pi_N\circ p_1(z_1',z_2')\|$ and $\|\frac{g_j\circ\pi}{g_n\circ\pi}\circ\pi_N\circ p_1(z_1',z_2')\|$ locally by constants because the ideal IJ is principal on $U_{m,n}$. The result follows from this. □

We apply the above results to say something about the local vectorfields $\vec{v}_{i,j}$ defined above. Since $\frac{\partial F}{\partial y_j}\in(m_YJ_z(F)_S)$, we can usefully re-write $\vec{v}_{i,j}$ as

$$\vec{v}_{i,j,k}=\frac{\partial}{\partial y_j}-\frac{\frac{\partial F}{\partial y_j}}{z_k\frac{\partial F}{\partial z_i}}z_k\frac{\partial}{\partial z_i}.$$

Denote the coefficient of $\frac{\partial}{\partial z_i}$ in $\vec{v}_{i,j,k}$ by $v_{i,j,k}$.

Then for pairs of points $(t,p_1),(t,p_2)$ in $\pi(U_{i,k})$ we have:

$$\begin{aligned}\|v_{i,j,k}(t,p_1)-v_{i,j,k}(t,p_2)\|\le&\left\|\frac{\frac{\partial F}{\partial y_j}}{z_k\frac{\partial F}{\partial z_i}}(t,p_1)\right\|||z_k(p_1)-z_k(p_2)||\\
&+\|z_k(p_2)\|\left\|\frac{\frac{\partial F}{\partial y_j}}{z_k\frac{\partial F}{\partial z_i}}(t,p_1)-\frac{\frac{\partial F}{\partial y_j}}{z_k\frac{\partial F}{\partial z_i}}(t,p_2)\right\|.\end{aligned}$$

Hence,

$$\begin{aligned}&\|v_{i,j,k}(t,p_1)-v_{i,j,k}(t,p_2)\|\le C\|z_k(p_1)-z_k(p_2)\|\\
&+\|z_k(p_1)\|\sup\left\{\left\|\frac{\frac{\partial F}{\partial z_j}}{\frac{\partial F}{\partial z_i}}(t,p_1)-\frac{\frac{\partial F}{\partial z_j}}{\frac{\partial F}{\partial z_i}}(t,p_2)\right\|,\left\|\frac{z_j}{z_k}(p_1)-\frac{z_j}{z_k}(p_2)\right\|\right\}.\end{aligned}$$

Here we may assume that $\|z_k(p_2)\|$ is the smaller of $\|z_k(p_1)\|$, $\|z_k(p_2)\|$. So, if the local fields are not Lipschitz on $U_{i,k}$ with respect to the distance between points, then they are Lipschitz with respect the distance between planes or secant lines to the origin and in this case the Lipschitz constant goes to zero as one of the points goes to the origin.

4. Genericity Theorem

Although at present we can't give a complete proof that the iL_{m_Y} condition is generic, we can do both conditions at once in some of the cases. We first determine the different cases in which it is necessary to check the conditions. These cases are the different ways in which $J_z(F)_D$ can fail to have maximal rank.

Proposition 4.1. *The co-supports of $(m_Y J_z(F))_D$ or $J_z(F)_D$ on $X \times_Y X$ consist of*
1) $Y \times (0,0)$,
2) $\Delta(X \times_Y X)$, and
3) $(0 \times_Y X) \cup (X \times_Y 0)$.

Proof. Suppose (x, x') does not lie in one of the sets. Then, since some $z_i \circ p_1$ and some $z_j \circ p_2$ are not zero at (x, x'), $(m_Y J_z(F))_D = J_z(F)_D$ locally. Then $J_z(F)_D$ contains terms of the form $(0, \frac{\partial F}{\partial z_j} \circ p_2)$, $(\frac{\partial F}{\partial z_j} \circ p_1, 0)$, which implies that the rank of $(m_Y J_z(F))_D$ is 2 and (x, x') are not in the cossupport.

□

The reader may have noted that $Y \times (0,0)$ is a subset of both $\Delta(X \times_Y X)$ and

$$(0 \times_Y X) \cup (X \times_Y 0).$$

We will next show that generically both conditions hold at points of $\Delta(X \times_Y X) - Y \times (0,0)$, and of $(0 \times_Y X) \cup (X \times_Y 0) - Y \times (0,0)$. Since we are working on a Z-open set of Y, and we are working with families of isolated singularities, we may assume that the only singular point of X_y is at $(y, 0)$, that $(X - Y, Y)$ satisfies W at $(y, 0)$. We will show that checking the conditions at points of the form $(y, 0, x)$, $x \neq 0$ amounts to checking W at $(y, 0)$ for $(X - Y, Y)$, while checking the conditions at points of $\Delta(X \times_Y X)$, $x \neq 0$ is trivial. Thus it will suffice to look at components of the appropriate exceptional divisor that surject onto $Y \times (0,0)$.

Proposition 4.2. *In the set-up of this section, iL_A and iL_{m_Y} hold at all points of*

$$\Delta(X \times_Y X) - Y \times (0,0),$$

and both conditions hold at all points of $(0 \times_Y X) \cup (X \times_Y 0) - Y \times (0,0)$ such that $(X - Y, Y)$ satisfies W at $(y, 0)$.

Proof. Work at (y, x, x), $x \neq 0$. Then since $x \neq 0$, $(m_Y J_z(F))_D = J_z(F)_D$ locally. Since f_y is a submersion at x, and $J_z(F)_D$ contains elements of the form $(0, (z_i \circ p_1 - z_i \circ p_2)(\frac{\partial F}{\partial z_j} \circ p_2))$, $((z_i \circ p_1 - z_i \circ p_2)(\frac{\partial F}{\partial z_j} \circ p_1), 0)$, it follows that $J_z(F)_D$ contains $I_\Delta \mathcal{O}^2_{X \times_Y X,(x,x)}$. By adding elements of the form $(0, \frac{\partial F}{\partial y} \circ p_1 - \frac{\partial F}{\partial y} \circ p_2)$ to $(\frac{\partial F}{\partial y} \circ p_1, \frac{\partial F}{\partial y} \circ p_2)$ and elements of the form $(0, \frac{\partial F}{\partial z_j} \circ p_1 - \frac{\partial F}{\partial z_j} \circ p_2)$ to $(\frac{\partial F}{\partial z_j} \circ p_1, \frac{\partial F}{\partial z_j} \circ p_2)$, this part of the proof is finished since $\frac{\partial F}{\partial y}$ is in the ideal $J_z(F)$ at x since f_y is a submersion.

Now work at $(x, 0)$, $x \neq 0$. Since f_y is a submersion at x, and $x \neq 0$ it follows that $(m_Y J_z(F))_D$ contains elements of the form $(1, 0)$, so it suffices to show that $\frac{\partial F}{\partial y}$ is in the integral closure of $m_Y J_z(F))$ and this is equivalent to W. This ends the second part of the proof.

□

Theorem 4.3. *In the set-up of this section, there exists a Zariski open subset of U of Y such that iL_A holds for the pair $(X - Y, U \cap Y)$ along Y.*

Proof. We will follow the lines of the proof of the Idealistic Bertini Theorem given in [16] p591-598. We prove that the il_A condition is generic using the module criterion. We will work on the normalized blow-up of $X \times_Y X \times \mathbb{P}^1$ by the ideal sheaf induced from the submodule

$J_z(F)_D$, denoting $NB_{(J_z(F))_D}(X\times_Y X\times\mathbb{P}^1)$ by N. We need to check that on each component of the exceptional divisor that the pullback of the element induced from $(\frac{\partial F}{\partial y})_D$ to the normalized blowup is in the pullback of $(J_z(F))_D$. Denote the projection to Y by p. By the previous lemmas we need only consider those components of the exceptional divisor which project to Y under the map to $X\times_Y X$. Since we are working over a Zariski open subset of Y we may assume that every such component maps surjectively onto Y. Since we are working on the normalization, we can work at a point q of the exceptional divisor such that E is smooth at q, N is smooth at q and the projection to Y is a submersion at q. Thus, we can choose coordinates at q, (y', u', x'), such that $y' = y\circ p$, and u' defines E locally with reduced structure. The key point is that $\frac{\partial u'}{\partial y'} = 0$.

Let π_i denote the composition of π, the projection from N to $X\times_Y X\times\mathbb{P}^1$ with the projection p_i to the i-th factor of $X\times_Y X\times\mathbb{P}^1$, $i = 1, 2$.

We have that $F\circ p_1 + sF\circ p_2$ is identically zero on $X\times_Y X\times\mathbb{P}^1$. Pull this back to N by π and take the partial derivative with respect to y' at q. We get by the chain rule:

$$0 = \frac{\partial F}{\partial y}\circ\pi_1 + s\frac{\partial F}{\partial y}\circ\pi_2 + \sum_{i=1}^{n}\frac{\partial F}{\partial z_i}\circ\pi_1\frac{\partial z_i\circ\pi_1}{\partial y'} + s\frac{\partial F}{\partial z_i}\circ\pi_2\frac{\partial z_i\circ\pi_2}{\partial y'}.$$

Notice that there is no term involving the derivative of s. This is because the coefficient of this partial by the product rule would be zero, since $F\circ\pi_i = 0$.

Now we work to re-shape the above term to prove the theorem. Notice that since z_i all vanish along Y, $z_i\circ\pi_j$ all vanish along E at q. We can assume the order of vanishing of $z_1\circ\pi_j$ is minimal among $\{z_i\circ\pi_j\}$, and that the strict transforms of $z_1\circ\pi_j$ do not pass through q.

We have:

$$\begin{aligned}\frac{\partial F}{\partial y}\circ\pi_1 + s\frac{\partial F}{\partial y}\circ\pi_2 \;=\; & -\left(\sum_{i=1}^{n}\left(\frac{\partial F}{\partial z_i}\circ\pi_1\right)\left(\frac{\partial z_i\circ\pi_1}{\partial y'}\right)+\right.\\ & \left. s\left(\left(\frac{\partial F}{\partial z_i}\circ\pi_2\right)\left(\frac{\partial z_i\circ\pi_1}{\partial y'}\right) - \left(\frac{\partial F}{\partial z_i}\circ\pi_2\right)\left[\frac{\partial z_i\circ\pi_1}{\partial y'} - \frac{\partial z_i\circ\pi_2}{\partial y'}\right]\right)\right).\end{aligned}$$

We want to show that the terms on the right hand side in the above expression are in the ideal generated by the pullback of the ideal sheaf on $X\times_Y X\times\mathbb{P}^1$ induced by $J_z(F))_D$. For this we use the curve criterion. We use a test curve to show that the order of vanishing of $\frac{\partial F}{\partial y}\circ\pi_1 + s\frac{\partial F}{\partial y}\circ\pi_2$ along a component is same as the order of vanishing of the ideal $(J_z(F))_D$. This will imply that $\frac{\partial F}{\partial y}\circ\pi_1 + s\frac{\partial F}{\partial y}\circ\pi_2$ is in the ideal along the component. We can choose a curve $\tilde\Phi$ such that $\tilde\Phi$ is the lift of a curve $\Phi = (\psi, \phi_1, \phi_2)$, $\Phi : \mathbb{C} :\to \mathbb{P}^1\times X\times_Y X$. Further $\tilde\Phi(0)$ is a smooth point of the component and the ambient space, $\tilde\Phi$ transverse to the component so that $u'\circ\tilde\Phi = t$, where t is a coordinate in the local ring of $\mathbb{C}$ at the origin. This implies that if an ideal is generated by u'^p, that the pullback is generated by t^p. Since the pullback of the ideal $(J_z(F))_D$ is locally principal, we can choose $\tilde\Phi(0)$ so that $(J_z(F))_D$ is generated by a power of u'.

Then we have

$$\tilde\Phi^*\left(\frac{\partial F}{\partial y}\circ\pi_1 + s\frac{\partial F}{\partial y}\circ\pi_2\right) =$$

$$\begin{aligned}-\left(\sum_{i=1}^{n}\left(\frac{\partial F}{\partial z_i}\circ\pi_1\circ\tilde\phi_1\right)\left(\frac{\partial z_i\circ\pi_1}{\partial y'}\right)\circ\tilde\phi_1 + \psi_2/\psi_1\left(\left(\frac{\partial F}{\partial z_i}\circ\pi_2\right)\circ\tilde\phi_2\left(\frac{\partial z_i\circ\pi_1}{\partial y'}\right)\circ\tilde\phi_1\right.\right.\\ \left.\left. - \left(\frac{\partial F}{\partial z_i}\circ\pi_2\right)\circ\tilde\phi_2\left[\frac{\partial z_i\circ\pi_1}{\partial y'}\circ\tilde\phi_1 - \frac{\partial z_i\circ\pi_2}{\partial y'}\tilde\phi_2\right]\right)\right).\end{aligned}$$

The right hand side will clearly be in the ideal $\Phi^*(J_z(F))_D)$, provided the pullback of

$$\left(\frac{\partial F}{\partial z_i} \circ \pi_2\right)\left(\frac{\partial z_i \circ \pi_1}{\partial y'} - \frac{\partial z_i \circ \pi_2}{\partial y'}\right)$$

is. However, by construction, since y' and u' are independent coordinates, the order of

$$\frac{\partial z_i \circ \pi_1}{\partial y'} - \frac{\partial z_i \circ \pi_2}{\partial y'}$$

in u' will be the same as the order of $z_i \circ \pi_1 - z_i \circ \pi_2$. Hence the pullback of $(\frac{\partial F}{\partial z_i} \circ \pi_2)(\frac{\partial z_i \circ \pi_1}{\partial y'} - \frac{\partial z_i \circ \pi_2}{\partial y'})$ does vanish to the desired order in t, which finishes the proof. □

We describe an application of this result. Given X an isolated hypersurface singularity we can consider the sections of X by hyperplanes. It is natural to ask if there is a generic set of hyperplanes for which the associated family of hyperplane sections satisfies the iL_A condition. We will show this is true after recalling the ideas necessary to make precise statements. (For more details on this material see [2].) We first need the notion of the Grassman modification of X, which we describe in the hyperplane case. Let E_{n-1} denote the canonical bundle over $\mathbb{P}^{n-1}$, which we view as hyperplanes though the origin in $\mathbb{C}^n$. Denote the projection of E_{n-1} to $\mathbb{C}^n$ by β_{n-1}. If X^{n-1} is a subset of $\mathbb{C}^n$, we call $\tilde{X} = {\beta_{n-1}}^{-1}(X)$, the G_{n-1} modification of X. In this paper we will simply refer to the G_{n-1} modification as the Grassman modification of X^{n-1}. Note that $\mathbb{P}^{n-1}$ is embedded in E_{n-1} as the zero section of E_{n-1}. This means that we can think of $0 \times \mathbb{P}^{n-1}$ as a stratum of $\tilde{X}$; note that the projection to $0 \times \mathbb{P}^{n-1}$ makes $\tilde{X}$ a family of analytic sets with $0 \times \mathbb{P}^{n-1}$ as the parameter space which we denote by Y. The members of this family are just $\{P \cap X\}$ as P varies through the points of $\mathbb{P}^{n-1}$.

The set of hyperplanes which are limiting tangent planes to X at the origin form a Zariski closed set. It is known that on the complement of this set, $(\widetilde{X} - Y, Y)$ are a pair of strata which satisfy the Whitney conditions. We can now apply Theorem 4.3 to this situation.

Theorem 4.4. *Suppose $X^n, 0$ is the germ of an analytic hypersurface in $\mathbb{C}^n$, then there exists a Zariski open subset U of $\mathbb{P}^{n-1}$, such that condition iL_A holds for the pair $\widetilde{X} - U, U$ along U.*

Proof. We can view $\widetilde{X}$ locally as a family of hypersurfaces parameterized by $\mathbb{P}^{n-1}$. The fiber of the family over the plane P is just the intersection $P \cap X$. The existence of U follows from 4.3. □

We can use the ideas of [2] to describe these generic hyperplanes. We work in the chart U_n given by planes P with equation $z_n = \sum_i a_i z_i$. Then we have local coordinates on E_{n-1} given by $(z_1, ...z_n, a_1, ..., a_{n-1})$. In these coordinates we have

$$\beta(z_1, ...z_n, a_1, ..., a_{n-1}) = (z_1, ...z_n, \sum_i a_i z_i).$$

If $\phi : \mathbb{C}, 0 \to \tilde{X}, P \times \{0\}$, then $\beta \circ \phi$ is tangent to P at the origin. If $\phi : \mathbb{C}, 0 \to X, 0$ is tangent to P at 0, then ϕ lifts to $\tilde{X}, P \times \{0\}$, and we say ϕ is liftable. It follows from [2], that since F defines X, $G := F \circ \beta$ defines $\tilde{X}$. From the chain rule we note that

$$\frac{\partial G}{\partial a_i} = z_i \frac{\partial F}{\partial z_n} \circ \beta, \qquad J_z(G) = \left(\frac{\partial F}{\partial z_j} \circ \beta + \sum_i a_i \frac{\partial F}{\partial z_n} \circ \beta\right),$$

for $1 \leq j \leq n - 1$.

Corollary 4.5. *Suppose $X^n, 0$ is the germ of an analytic hypersurface in $\mathbb{C}^n$, then, for $P \in U_n$, P is a point in the Z-open set of the last theorem, if and only if $z_i \frac{\partial F}{\partial z_n} \circ \beta \in (J_z(G))_S$ for $1 \leq i \leq n-1$ at $P, 0$.*

Proof. In the framework of the corollary, the condition of the corollary is exactly the iL_A condition. □

The corollary says that to check a plane is generic, it suffices to check that for all curves ϕ_i $i = 1, 2$ on X, tangent to P at the origin, with lifts $\widetilde{\phi}_i$ for ϕ_i, and $\Phi := (\phi_1, \phi_2)$, $\widetilde{\Phi} := (\widetilde{\phi}_1, \widetilde{\phi}_2)$, that

$$\left(z_i \frac{\partial F}{\partial z_n}\right)_D \circ \Phi \in \left(\left(\frac{\partial F}{\partial z_j}\right)_D \circ \Phi + \left(\sum_i a_i \frac{\partial F}{\partial z_n} \circ \beta\right)_D \circ \widetilde{\Phi}\right).$$

We will give a description using analytic invariants of these generic hyperplanes. For the rest of this section we will assume that the planes we consider are not limiting tangent hyperplanes to $X, 0$. This condition is equivalent to $\overline{J(F)_H} = \overline{J(F)}$ in $\mathcal{O}_{X,0}$.

The invariant we will use appeared earlier in section 3. It is the multiplicity of the pair $J(X \cap H)_D, \overline{J(X \cap H)_D}$, which we denote $e(J(X \cap H)_D, \overline{J(X \cap H)_D})$.

Similar invariants have been used in this setting before. In the case of ICIS singularities, to test for whether or not a hyperplane is in the generic set of planes for which the hyperplane sections form a Whitney equisingular family, you use the multiplicity of the pair $(JM(X \cap H), \mathcal{O}^p_X)$, which is $e(JM(X \cap H))$. The plane is generic if this multiplicity is minimal, and the minimal number is the sum of the Milnor numbers of $X \cap H$, and $X \cap H \cap G$, where H and G are generic hyperplanes.

The proof that the minimal value of $e(J(X \cap H)_D, \overline{J(X \cap H)_D})$ again identifies generic hyperplanes will be done in the context of the multiplicity polar theorem, so we identify the modules we will use.

We will work in $\widetilde{X} \times_{\mathbb{P}^{n-1}} \widetilde{X} \subset X \times \mathbb{P}^{n-1} \times X$. The module N will be $(\beta^* \overline{J(F)})_D$, and the module M will be $J_z(G)_D$. Notice that M restricted to the fiber of the family over the plane H is just $J(X \cap H)_D$, while N restricted to H is $(\overline{J(X)}|_H)_D$; because we are assuming H is not a limiting tangent hyperplane, we have that $\overline{J(X)}|_H = \overline{J(X \cap H)}$, hence N restricted to H is $\overline{J(X \cap H)_D}$, so the multiplicity of the pair $M(H), N(H)$ is the same as $e(J(X \cap H)_D, \overline{J(X \cap H)_D})$. At this time we do not have a geometric interpretation of this number.

Theorem 4.6. *Suppose $X^{n-1}, 0$ is an isolated singularity hypersurface and U the set of hyperplanes which are limiting tangent hyperplanes to X at 0. Then*

1) $e(J(X \cap H)_D, \overline{J(X \cap H)_D})$ is upper semicontinuous on U.

2) The iL_A condition holds along U at a hyperplane H for which the value of

$$e(J(X \cap H)_D, \overline{J(X \cap H)_D})$$

is minimal.

Proof. The condition on U implies that $\overline{J(X \cap H)_D})$ is the restriction of N to the fiber. Essentially since N is independent of H, N has no polar variety of the same codimension as U. The multiplicity polar theorem then implies $e(J(X \cap H)_D, \overline{J(X \cap H)_D})$ is upper semicontinuous on U.

Suppose we are at H which gives the minimal value of the multiplicity. Since the value of the multiplicity cannot go down, it must be constant, which implies that the polar variety of M of the same dimension as U must be empty. The emptiness of the polar variety puts restrictions on the size of the fiber of $\operatorname{Proj} \mathcal{R}(M)$. Now we know that generically the $\frac{\partial G}{\partial a_i}$ are in $\overline{M}$; coupling this with the bound on the dimension of the fiber of Proj $\mathcal{R}(M)$, by Theorem A1 of [10], it follows that the $\frac{\partial G}{\partial a_i}$ are in the integral closure of M at H as well, which finishes the proof. □

References

[1] L. Birbrair, W. D Neumann and A. Pichon, *The thick-thin decomposition and the bilipschitz classification of normal surface singularities*, 2011. arXiv:1105.3327v2

[2] T.Gaffney, *Equisingularity of Plane Sections, t^1 Condition, and the Integral Closure of Modules*, Real and Complex Singularities, ed.by W.L.Marar, Pitman Research Series in Math. 333, Longman,1995.

[3] T. Gaffney, *Integral closure of modules and Whitney equisingularity,* Inventiones, 107 (1992) 301–22. DOI: 10.1007/BF01231892

[4] T. Gaffney, Generalized Buchsbaum-Rim Multiplicities and a Theorem of Rees Communications in Algebra, 31 (2003) 3811-3828.

[5] T. Gaffney, *Polar methods, invariants of pairs of modules and equisingularity*, Real and Complex Singularities (São Carlos, 2002), Ed. T. Gaffney and M. Ruas, Contemp. Math., 354, Amer. Math. Soc., Providence, RI, June (2004), 113–136.

[6] T. Gaffney, *The Multiplicity polar theorem*, 2007. arXiv:math/0703650v1

[7] T. Gaffney, *Bi-Lipschitz equivalence, integral closure and invariants*, Proceedings of the 10th International Workshop on Real and Complex Singularities. Edited by: M. Manoel, Universidade de So Paulo, M. C. Romero Fuster, Universitat de Valncia, Spain, C. T. C. Wall, University of Liverpool, London Mathematical Society Lecture Note Series (No. 380) November 2010. DOI: 10.1017/CBO9780511731983.010

[8] T. Gaffney, and M. Vitulli, *Weak subintegral closure of ideals.* Advances in Mathematics, Volume 226, Issue 3, 15 February 2011, Pages 2089-2117. DOI: 10.1016/j.aim.2010.09.020

[9] S. Kleiman and A. Thorup, *A geometric theory of the Buchsbaum-Rim multiplicity*, J. Algebra 167 (1994), 168-231. DOI: 10.1006/jabr.1994.1182

[10] S. Kleiman and A. Thorup, *The exceptional fiber of a generalized conormal space,* in "New Developments in Singularity Theory." D. Siersma, C.T.C. Wall and V. Zakalyukin (eds.), Nato Science series, II Mathematics, Physics and Chemistry-Vol. 21 2001 401-404.

[11] M. Lejeune-Jalabert and B. Teissier, Clôture intégrale des idéaux et equisingularité, Séminaire Lejeune-Teissier, Centre de Mathématiques École Polytechnique, (1974) Publ. Inst. Fourier St. Martin d'Heres, F–38402 (1975).

[12] J. Lipman, *Relative Lipschitz-saturation*, Amer. J. Math. 97 (1975), no. 3, 791–813.

[13] T. Mostowski, *Lipschitz equisingularity*, Dissertationes Math., CC XLIII, 1985, PWN, Warsawa.

[14] F. Pham, *Fractions lipschitziennes et saturation de Zariski des algèbres analytiques complexes.* Exposé d'un travail fait avec Bernard Teisser. Fractions lipschitziennes d'une algèbre analytique complexe et saturation de Zariski, Centre Math. lÉcole Polytech., Paris, 1969. Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 2, pp. 649–654. Gauthier-Villars, Paris, 1971.

[15] F. Pham and B. Teissier, *Fractions lipschitziennes d'une algébre analytique complexe et saturation de Zariski*, Centre de Mathématiques de l'Ecole Polytechnique (Paris), http://people.math.jussieu.fr/ teissier/old-papers.html, June 1969.

[16] B. Teissier, *The hunting of invariants in the geometry of discriminants*, Real and complex singularities (Proc. Ninth Nordic Summer School NAVF Sympos. Math., Oslo, 1976), pp. 565–678. Sijthoff and Noordhoff, Alphen aan den Rijn, 1977.

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AXIUMBILIC SINGULAR POINTS ON SURFACES IMMERSED IN $\mathbb{R}^4$ AND THEIR GENERIC BIFURCATIONS

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Abstract. Here are described the *axiumbilic* points that appear in generic one parameter families of surfaces immersed in $\mathbb{R}^4$. At these points the ellipse of curvature of the immersion, Little [7], Garcia - Sotomayor [11], has equal axes.

A review is made on the basic preliminaries on axial curvature lines and the associated axiumbilic points which are the singularities of the fields of *principal*, *mean axial lines*, *axial crossings* and the quartic differential equation defining them.

The Lie-Cartan vector field suspension of the quartic differential equation, giving a line field tangent to the Lie-Cartan surface (in the projective bundle of the source immersed surface which quadruply covers a punctured neighborhood of the axiumbilic point) whose integral curves project regularly on the lines of axial curvature.

In an appropriate Monge chart the configurations of the generic axiumbilic points, denoted by E_3, E_4 and E_5 in [11] [12], are obtained by studying the integral curves of the Lie-Cartan vector field.

Elementary bifurcation theory is applied to the study of the transition and elimination between the axiumbilic generic points. The two generic patterns E^1_{34} and E^1_{45} are analysed and their axial configurations are explained in terms of their qualitative changes (bifurcations) with one parameter in the space of immersions, focusing on their close analogy with the saddle-node bifurcation for vector fields in the plane [1], [10].

This work can be regarded as a partial extension to $\mathbb{R}^4$ of the umbilic bifurcations in Garcia - Gutierrez - Sotomayor [5], for surfaces in $\mathbb{R}^3$. With less restrictive differentiability hypotheses and distinct methodology it has points of contact with the results of Gutierrez - Guiñez - Castañeda [3].

Introduction

In this work are described the axiumbilic singularities, at which the ellipse of curvature, as defined in Little [7] and Garcia - Sotomayor [11], has equal axes. The focus here are the axiumbilic points that appear generically in one parameter families of surfaces immersed in $\mathbb{R}^4$. It can be regarded as an extension from $\mathbb{R}^3$ to $\mathbb{R}^4$, as target spaces for immersed surfaces, and from umbilic to axiumbilic points as singularities, of results obtained by Gutierrez - Garcia - Sotomayor in [5]. It is also a continuation, in the direction of bifurcations of axiumbilic singularities, of the study of the structural stability of global axial configurations started in Garcia - Sotomayor [11].

An outline of the organization of this paper follows:

Section 1 deals with geometric preliminaries and a review of axial lines and axiumblic points in order to define the *principal* and *mean curvature* configurations and their quartic differential equations.

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In Section 2, locally presenting a surface M immersed into $\mathbb{R}^4$ with a Monge chart, are studied the axiumbilic points and the transversality conditions in terms of which are defined the generic axiumbilic points are made explicit.

Section 3 establishes the axial principal and mean configurations in a neighborhood of generic axiumbilic points, denoted E_3, E_4 and E_5. This description uses the suspension of Lie-Cartan, giving rise to a line field tangent to a surface, which quadruply covers a punctured neighborhood of the axiumbilic point, and whose integral lines project regularly on the lines of axial curvature. This follows the approach of Garcia and Sotomayor in [11] and [12], chap. 8. After this review follow two subsections devoted to describe the behaviors of the axial lines near the axiumbilic points denoted E^1_{34} and E^1_{45}, which are the transversal transitions between the generic axiumbilic points.

In fact, the axiumbilic point E^1_{34} (Figure 7) characterizes the transition between an axiumbilic point of type E_3 and one of type E_4, which is explained by the variation of one parameter family in the space of immersions $\mathcal{C}^r,\ r \geq 5$ of a surface M into $\mathbb{R}^4$ (Proposition 11), in a first analogy with the saddle-node bifurcation of vector fields [1], [10].

The axiumbilic point E^1_{45} (Figure 11) is characterized by the collision and subsequent elimination between one point of type E_4 and other of type E_5. Here also, this bifurcation phenomenon is explained by means of a parameter variation in the space of immersions (Proposition 17), in a second analogy with the saddle-node bifurcations in the plane [1] [10].

Section 4 establishes the genericity of the axiumbilic bifurcations studied in this paper.

This work is related to the papers by Guíñez-Gutiérrez [2] and Guíñez-Gutiérrez-Castañeda [3] where a description, in class $\mathcal{C}^\infty$ and in the context of quartic differential forms, of the points E^1_{34} and E^1_{45} (using the notation H_{34} and H_{45}), can be found.

Here was adopted a different approach, using the Lie-Cartan suspension as established in Garcia-Sotomayor [11], for immersions of class $\mathcal{C}^r, 5 \leq r \leq \infty$. This leads to an interpretation of these points with less restrictive differentiability hypotheses and allows proofs with techniques closer to those of elementary bifurcation theory as in [1] and [10].

Section 5 closes the paper with related comments on its results and their connection with others found in the literature.

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1. Differential Equation of Axial Lines

Let $\alpha : M \longrightarrow \mathbb{R}^4$ be an immersion of class $\mathcal{C}^r$, $r \geq 5$, of an oriented smooth surface in $\mathbb{R}^4$, with the canonical orientation. Assume that (x, y) is a positive chart of M and that $\{\alpha_x, \alpha_y, N_1, N_2\}$ is a smooth positive frame in $\mathbb{R}^4$, where for $\mathfrak{p} \in M$, $\{\alpha_x = \partial\alpha/\partial x, \alpha_y = \partial\alpha/\partial y\}_\mathfrak{p}$ is the the standard basis of $T_\mathfrak{p}M$ in the chart (x, y) and $\{N_1, N_2\}_\mathfrak{p}$ is an orthonormal basis of the normal plane $N_\mathfrak{p}M$.

In the chart (x, y), the first fundamental form is expressed by

$$I_\alpha = \langle D\alpha, D\alpha\rangle = Edx^2 + 2Fdxdy + Gdy^2$$

where, $E = \langle\alpha_x, \alpha_x\rangle, F = \langle\alpha_x, \alpha_y\rangle$ and $G = \langle\alpha_y, \alpha_y\rangle$ and the second fundamental form is given by $II_\alpha = II^1_\alpha N_1 + II^2_\alpha N_2$ where $II^i_\alpha, i = 1, 2,$ is

$$II^i_\alpha := \langle D^2\alpha, N_i\rangle = e_i dx^2 + 2f_i dxdy + g_i dy^2,$$

with $e_i = \langle\alpha_{xx}, N_i\rangle, f_i = \langle\alpha_{xy}, N_i\rangle$ and $g_i = \langle\alpha_{yy}, N_i\rangle$.

The *mean curvature vector* is defined by $H = h_1 N_1 + h_2 N_2$ with

$$h_i = \frac{Eg_i - 2Ff_i + Ge_i}{2(EG - F^2)}.$$

For $v \in T_{\mathfrak{p}}M$, the *normal curvature vector in the direction* v is defined by:

$$k_n = k_n(\mathfrak{p}, v) = \frac{II_\alpha(v)}{I_\alpha(v)} = \frac{II_\alpha^1(v)}{I_\alpha(v)} N_1 + \frac{II_\alpha^2(v)}{I_\alpha(v)} N_2. \tag{1}$$

The image of k_n restricted to the unitary circle $S^1_{\mathfrak{p}}$ of $T_{\mathfrak{p}}M$ describes in $N_{\mathfrak{p}}M$ an ellipse centered in $H(\mathfrak{p})$, which is called *ellipse of curvature* of α at $\mathfrak{p}$, and it will be denoted by $\varepsilon_\alpha(\mathfrak{p})$. When $(e_1 - g_1)f_2 - (e_2 - g_2)f_1 \neq 0$, it is an actual non-degenerate ellipse, which can be a circle. Otherwise it can be a segment or a point. As $k_n|_{S^1_{\mathfrak{p}}}$ is quadratic, the pre-image of each point of the ellipse is formed of two antipodal points on $S^1_{\mathfrak{p}}$, and therefore each point of $\varepsilon_\alpha(\mathfrak{p})$ is associated to a direction in $T_{\mathfrak{p}}M$. Moreover, for each pair of points in $\varepsilon_\alpha(\mathfrak{p})$ antipodally symmetric with respect to $H(\mathfrak{p})$, it is associated two orthogonal directions in $T_{\mathfrak{p}}M$, defining a pair of *lines* in T_pM [7], [8], [9].

Consider the function:

$$\begin{aligned} \|k_n - H\|^2 \;:=\; & \left[\frac{e_1 dx^2 + 2f_1 dxdy + g_1 dy^2}{Edx^2 + 2Fdxdy + Gdy^2} - \frac{Eg_1 - 2Ff_1 + Ge_1}{2(EG - F^2)}\right]^2 \\ + & \left[\frac{e_2 dx^2 + 2f_2 dxdy + g_2 dy^2}{Edx^2 + 2Fdxdy + Gdy^2} - \frac{Eg_2 - 2Ff_2 + Ge_2}{2(EG - F^2)}\right]^2 \end{aligned}$$

For each $\mathfrak{p} \in M$ in which $\varepsilon_\alpha(\mathfrak{p})$ is not a circle, the points maximum and minimum of this function determine four points over the ellipse of curvature $\varepsilon_\alpha(\mathfrak{p})$, which are their vertices, located at the large and small axes.

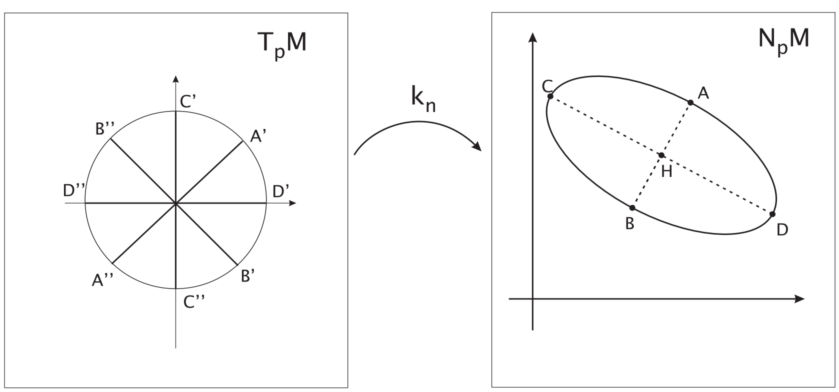

FIGURE 1. Ellipse of curvature $\varepsilon_\alpha(\mathfrak{p})$ and lines of axial curvature

As illustrated in Figure 1, to the small axis AB is associated the crossing $A'A''B'B''$ and to the large axis CD is associated the crossing $C'C''D'D''$. Thus, for each $\mathfrak{p} \in M$ at which the non-degenerate ellipse is not a circle or a point, *two crossings* are defined in $T_{\mathfrak{p}}M$, one associated to the large axis and the other to the small axis of the ellipse of curvature. These *fields of 2-crossings* in M are called *fields of axial curvature*.

Outside the set $\mathcal{U}_\alpha$ of points at which the ellipse of curvature is a circle (i.e. has equal axes), called *axiumbilic points*, the lines and crossings are said to be *lines and crossings of axial* *curvature*. Those related to the large (respectively small) axis of the ellipse of curvature are called *lines and crossings of principal (respectively mean) axial curvature.*

From the considerations above, the axial directions are defined by the equationm

$$Jac(\|k_n - H\|^2, I_\alpha) = 0$$

which has four solutions for $\mathfrak{p} \notin \mathcal{U}_\alpha$ and is singular at $\mathfrak{p} \in \mathcal{U}_\alpha$. According to [11] and [12], the *differential equation of axial lines* is given by:

$$a_4 dy^4 + a_3 dy^3 dx + a_2 dy^2 dx^2 + a_1 dy dx^3 + a_0 dx^4 = 0, \tag{2}$$

where

$$\begin{aligned}
a_4 &= -4F(EG-2F^2)(g_1^2+g_2^2) + 4G(EG-4F^2)(f_1g_1+f_2g_2),\\
&+ 8FG^2(f_1^2+f_2^2) + 4FG^2(e_1g_1+e_2g_2) - 4G^3(e_1f_1+e_2f_2)\\[2ex]
a_3 &= -4E(EG-4F^2)(g_1^2+g_2^2) - 32EFG(f_1g_1+f_2g_2),\\
&+ 16EG^2(f_1^2+f_2^2) - 4G^3(e_1^2+e_2^2) + 8EG^2(e_1g_1+e_2g_2)\\[2ex]
a_2 &= -12FG^2(e_1^2+e_2^2) + 12E^2F(EG-4F^2)(g_1^2+g_2^2),\\
&+ 24EG^2(e_1f_1+e_2f_2) - 24E^2G(f_1g_1+f_2g_2)\\[2ex]
a_1 &= 4E^3(g_1^2+g_2^2) + 4G(EG-4F^2)(e_1^2+e_2^2)\\
&+ 32EFG(e_1f_1+e_2f_2) - 16E^2G(f_1^2+f_2^2) - 8E^2G(e_1g_1+e_2g_2),\\[2ex]
a_0 &= 4F(EG-2F^2)(e_1^2+e_2^2) - 4E(EG-4F^2)(e_1f_1+e_2f_2)\\
&+ -8E^2F(f_1^2+f_2^2) - 4E^2F(e_1g_1+e_2g_2) + 4E^3(f_1g_1+f_2g_2).
\end{aligned}$$

Proposition 1 ([11], [12])**.** Let $\alpha : M \longrightarrow \mathbb{R}^4$ be an immersion of class $\mathcal{C}^r,\ r \geq 5$, of an oriented and smooth surface. Denote the first fundamental form of α by

$$I_\alpha = Edx^2 + 2Fdxdy + Gdy^2$$

and the second fundamental form by:

$$II_\alpha = (e_1dx^2 + 2f_1dxdy + g_1dy^2)N_1 + (e_2dx^2 + 2f_2dxdy + g_2dy^2)N_2$$

where $\{N_1, N_2\}$ is an orthonormal frame.

i) The differential equation of axial lines is given by:

$$\begin{aligned}
\mathcal{G} &= [a_0G(EG-4F^2) + a_1F(2F^2-EG)]dy^4\\
&+ [-8a_0EFG + a_1E(4F^2-EG)]dy^3dx\\
&+ [-6a_0GE^2 + 3a_1FE^2]dy^2dx^2 + a_1E^3dydx^3 + a_0E^3dx^4 = 0,
\end{aligned}$$

where

$$\begin{aligned}
a_1 &= 4G(EG-4F^2)(e_1^2+e_2^2) + 32EFG(e_1f_1+e_2f_2)\\
&+ 4E^3(g_1^2+g_2^2) - 8E^2G(e_1g_1+e_2g_2) - 16E^2G(f_1^2+f_2^2)
\end{aligned}$$

and

$$\begin{aligned}
a_0 &= 4F(EG-2F^2)(e_1^2+e_2^2) - 4E(EG-4F^2)(e_1f_1+e_2f_2)\\
&+ 4E^3(f_1g_1+f_2g_2) - 4E^2F(e_1g_1+e_2g_2) - 8E^2F(f_1^2+f_2^2).
\end{aligned}$$

ii) The axiumbilic points of α are characterized by $a_0 = a_1 = 0$.

The axiumbilic points are defined by the intersection of the curves $a_0(x,y) = 0$ and $a_1(x,y) = 0$. Assume, with no lost of generality, that they intersect at $(x,y) = (0,0)$. In this work it will be considered the case where the intersection is transversal or quadratic at $(0,0)$.

Figure 2 illustrates the generic contact of the curves $a_0(x,y) = 0$ and $a_1(x,y) = 0$, whose intersection characterizes the axiumbilic points.

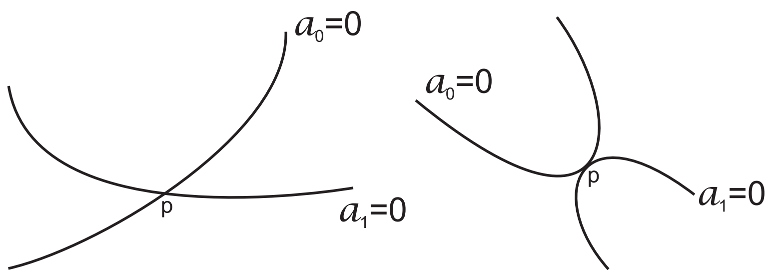

FIGURE 2. Transversal and quadratic contact between the curves $a_0 = 0$ and $a_1 = 0$ at an axiumbilic point $\mathfrak{p}$.

An axiumbilic point given by $(x,y) = (0,0)$ is called *transversal* if

$$\frac{\partial(a_0,a_1)}{\partial(x,y)}\bigg|_{(0,0)} = \begin{vmatrix} \frac{\partial a_0}{\partial x}(0,0) & \frac{\partial a_0}{\partial y}(0,0) \\ \frac{\partial a_1}{\partial x}(0,0) & \frac{\partial a_1}{\partial y}(0,0) \end{vmatrix} \neq 0. \tag{3}$$

The axiumbilic point given by $(x,y) = (0,0)$ is said to be of *quadratic type* if the matrix

$$\frac{\partial(a_0,a_1)}{\partial(x,y)}\bigg|_{(0,0)} = \begin{bmatrix} \frac{\partial a_0}{\partial x}(0,0) & \frac{\partial a_0}{\partial y}(0,0) \\ \frac{\partial a_1}{\partial x}(0,0) & \frac{\partial a_1}{\partial y}(0,0) \end{bmatrix} \tag{4}$$

has rank 1 and, assuming $\frac{\partial a_0}{\partial y}(0,0) \neq 0$, it follows from the implicit function theorem that $y(x)$ is a local solution of $a_0(x,y(x)) = 0$. Writing $s(x) = a_1(x,y(x))$ it follows that $s'(0) = 0$ and $s''(0) \neq 0$.

A similar analysis can be carried out if other element of the matrix $\frac{\partial(a_0,a_1)}{\partial(x,y)}\Big|_{(0,0)}$ is non zero.

Remark 2 ([11]). In isothermic coordinates, where $E = G$ and $F = 0$, it follows that

$$a_1 = -a_3 = E^3[e_1^2 + e_2^2 + g_1^2 + g_2^2 - 4(f_1^2 + f_2^2) - 2(e_1g_1 + e_2g_2)]$$

$$a_0 = a_4 = -\frac{a_2}{6} = 4E^3[f_1g_1 + f_2g_2 - (e_1f_1 + e_2f_2)]$$

and the differential equation of axial lines is simplified to

$$a_0(x,y)(dx^4 - 6dx^2dy^2 + dy^4) + a_1(x,y)(dx^2 - dy^2)dxdy = 0. \tag{5}$$

1.1. **Axial configurations of immersed surfaces in $\mathbb{R}^4$.** Let $\mathcal{I}^r = \mathcal{I}^r(M,\mathbb{R}^4)$ the set of immersions of class $\mathcal{C}^r$. For $\alpha \in \mathcal{I}^r$, the differential equation of axial lines is well defined (equation (2)):

$$\mathcal{G}(x,y,dx,dy) = a_4dy^4 + a_3dy^3dx + a_2dy^2dx^2 + a_1dydx^3 + a_0dx^4 = 0 \tag{6}$$

in the projective bundle PM of M.

For each $\alpha \in \mathcal{I}^r$, define the *Lie-Cartan surface* of the immersion α by $\mathbb{L}_\alpha := \mathcal{G}_\alpha^{-1}(0)$, which is of class $\mathcal{C}^{r-2}$, regular in $M - \mathcal{U}_\alpha$ and may present singularities at $\mathcal{U}_\alpha$.

Moreover, as the set defined by the quartic equation (6) contains the projective lines at $\mathcal{U}_\alpha$, it follows that $\mathbb{L}_\alpha$ is a ramified covering of degree 4 in $M - \mathcal{U}_\alpha$ and contains the projective line $\pi^{-1}(\mathfrak{p})$ for each $\mathfrak{p} \in \mathcal{U}_\alpha$.

In the chart (x, y, p) of PM, with $p = \frac{dy}{dx}$, equation (6) is given by

$$\mathcal{G}(x, y, p) = a_4 p^4 + a_3 p^3 + a_2 p^2 + a_1 p + a_0 = 0. \tag{7}$$

Consider the *Lie-Cartan vector field* X_α, of class $\mathcal{C}^{r-3}$, tangent to the surface $\mathcal{G} = 0$

$$X_\alpha := \mathcal{G}_p \frac{\partial}{\partial x} + p\mathcal{G}_p \frac{\partial}{\partial y} - (\mathcal{G}_x + p\mathcal{G}_y)\frac{\partial}{\partial p}. \tag{8}$$

The *axial curvature lines* are the projections by $\pi : PM \longrightarrow M$ restricted to $\mathbb{L}_\alpha$, of the integral curves of X_α.

See illustration in Figure 3. For each $\mathfrak{p} \in M - \mathcal{U}_\alpha$ there are 4 well defined axial directions, given the four roots of equation (7).

Two *axial configurations* are given: the *principal axial configuration* $\mathcal{P}_\alpha = \{\mathcal{U}_\alpha, \mathcal{X}_\alpha\}$ defined by the axiumbilic points $\mathcal{U}_\alpha$ and by the net $\mathcal{X}_\alpha$ (related to the crossing of principal axial curvature), in $M - \mathcal{U}_\alpha$ and the *mean axial configuration* $\mathcal{Q}_\alpha = \{\mathcal{U}_\alpha, \mathcal{Y}_\alpha\}$, defined by the axiumbilic points $\mathcal{U}_\alpha$ and the net $\mathcal{Y}_\alpha$ (related to the crossing of mean axial curvature), in $M - \mathcal{U}_\alpha$.

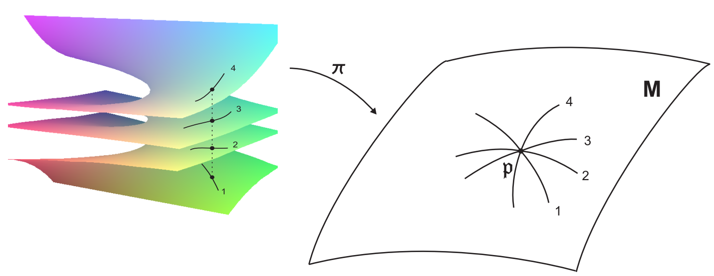

FIGURE 3. Projection on M of the integral curves of the Lie-Cartan vector field tangent to $\mathbb{L}_\alpha$ in a neighborhood of $\mathfrak{p} \in M - \mathcal{U}_\alpha$. For each point in M pass four lines, associated, in pairs, to the axis of the ellipse.

2. Differential Equation of Axial Lines in a Monge Chart

The surface M will be locally parametrized by a Monge chart near an axiumbilic point $\mathfrak{p}$ as follows

$$z = R(x, y), \qquad \text{and} \qquad w = S(x, y),$$

where

$$\begin{aligned} R(x, y) = &\frac{r_{20}}{2}x^2 + r_{11}xy + \frac{r_{02}}{2}y^2 + \frac{r_{30}}{6}x^3 + \frac{r_{21}}{2}x^2y + \frac{r_{12}}{2}xy^2 + \frac{r_{03}}{6}y^3 \\ &+ \frac{r_{40}}{24}x^4 + \frac{r_{31}}{6}x^3y + \frac{r_{22}}{4}x^2y^2 + \frac{r_{13}}{6}xy^3 + \frac{r_{04}}{24}y^4 + h.o.t., \end{aligned} \tag{9}$$

$$\begin{aligned} S(x, y) = &\frac{s_{20}}{2}x^2 + s_{11}xy + \frac{s_{02}}{2}y^2 + \frac{s_{30}}{6}x^3 + \frac{s_{21}}{2}x^2y + \frac{s_{12}}{2}xy^2 + \frac{s_{03}}{6}y^3 \\ &+ \frac{s_{40}}{24}x^4 + \frac{s_{31}}{6}x^3y + \frac{s_{22}}{4}x^2y^2 + \frac{s_{13}}{6}xy^3 + \frac{s_{04}}{24}y^4 + h.o.t. \end{aligned} \tag{10}$$

At the point $(x, y, R(x, y), S(x, y))$, the tangent plane to the surface is generated by $\{t_1, t_2\}$, where $t_1 = (1, 0, R_x, S_x)$ and $t_2 = (0, 1, R_y, S_y)$. The normal plane is generated by $\{N_1, N_2\}$, where $N_1 = \frac{\widetilde{N_1}}{|\widetilde{N_1}|}$ and $N_2 = \frac{\widetilde{N_2}}{|\widetilde{N_2}|}$ are defined by $\widetilde{N_1} = (-R_x, -R_y, 1, 0)$ and $\widetilde{N_2} = t_1 \wedge t_2 \wedge \widetilde{N_1}$.

Here $\wedge$ is the *exterior* or *wedge* product $v_1 \wedge v_2 \wedge v_3$ of three vectors v_1, v_2, v_3 in $\mathbb{R}^4$ is defined by equation $\det(v_1, v_2, v_3, v) = \langle v_1 \wedge v_2 \wedge v_3, v\rangle$ for all $v \in \mathbb{R}^4$.

Therefore it follows that:

$$\det(t_1, t_2, \widetilde{N_1}, \bullet) = \langle \widetilde{N_2}, \bullet\rangle.$$

From the expressions of R and S given by equations (9) and (10), it follows that:

$$E = 1 + O(2), \qquad F = O(2), \qquad G = 1 + O(2),$$

and

$$\begin{array}{ll} e_1 = r_{20} + r_{30}x + r_{21}y + O(2), & e_2 = s_{20} + s_{30}x + s_{21}y + O(2), \\ f_1 = r_{11} + r_{21}x + r_{12}y + O(2), & f_2 = s_{11} + s_{21}x + s_{12}y + O(2), \\ g_1 = r_{02} + r_{12}x + r_{03}y + O(2), & g_2 = s_{02} + s_{12}x + s_{03}y + O(2). \end{array}$$

The axiumbilic points are defined by $a_0(x, y) = 0$ and $a_1(x, y) = 0$. So, in a neighborhood of $(0, 0)$, it follows that

$$a_0(x, y) = a^0_{00} + a^0_{10}x + a^0_{01}y + O(2) \tag{11}$$

and

$$a_1(x, y) = a^1_{00} + a^1_{10}x + a^1_{01}y + O(2), \tag{12}$$

where

$$\begin{array}{rcl} a^0_{00} & = & r_{11}(r_{02} - r_{20}) + s_{11}(s_{02} - s_{20}), \\ a^0_{10} & = & r_{21}(r_{02} - r_{20}) + r_{11}(r_{12} - r_{30}) + s_{11}(s_{12} - s_{30}) + s_{21}(s_{02} - s_{20}), \\ a^0_{01} & = & r_{12}(r_{02} - r_{20}) + r_{11}(r_{03} - r_{21}) + s_{11}(s_{03} - s_{21}) + s_{12}(s_{02} - s_{20}) \end{array}$$

and

$$\begin{array}{rcl} a^1_{00} & = & (r_{02} - r_{20})^2 + (s_{02} - s_{20})^2 - 4(r^2_{11} + s^2_{11}), \\ a^1_{10} & = & 2(r_{12} - r_{30})(r_{02} - r_{20}) + 2(s_{12} - s_{30})(s_{02} - s_{20}) - 8(r_{21}r_{11} + s_{21}s_{11}), \\ a^1_{01} & = & 2(r_{03} - r_{21})(r_{02} - r_{20}) + 2(s_{03} - s_{21})(s_{02} - s_{20}) - 8(r_{12}r_{11} + s_{12}s_{11}). \end{array}$$

Therefore a point $\mathfrak{p}$, expressed in a Monge chart by $(0, 0)$, is an axiumbilic point when the following relations hold.

$$\left\{ \begin{array}{l} a^0_{00} = r_{11}(r_{02} - r_{20}) + s_{11}(s_{02} - s_{20}) = 0, \\ a^1_{00} = (r_{02} - r_{20})^2 + (s_{02} - s_{20})^2 - 4(r^2_{11} + s^2_{11}) = 0. \end{array} \right. \tag{13}$$

Algebraic manipulations of the equations above, see [2], show that $(0, 0)$ is an axiumbilic point when the following equations hold

$$\left\{ \begin{array}{l} 2r_{11} = (s_{02} - s_{20}), \\ 2s_{11} = -(r_{02} - r_{20},) \end{array} \right. \quad \text{or} \quad \left\{ \begin{array}{l} 2r_{11} = -(s_{02} - s_{20}), \\ 2s_{11} = (r_{02} - r_{20}). \end{array} \right. \tag{14}$$

Remark 3. Let $r_{02} = r_{20} + r$ and $s_{02} = s_{20} + s$, $\rho^2 = r^2_{11} + s^2_{11}$. Then condition for $(0, 0)$ to be an axiumbilic point, see equation (13), is given by

$$\left\{ \begin{array}{l} r_{11} \cdot r + s_{11} \cdot s = 0, \\ r^2 + s^2 = 4\rho^2. \end{array} \right. \tag{15}$$

These condition for being an axiumbilic point can be interpreted as the intersection of a circle and a straight line in the plane (r, s). The intersections are given by

$$(16)\qquad \begin{cases} r_{11} = \frac{s}{2}, \\ s_{11} = -\frac{r}{2}, \end{cases} \qquad \text{or} \qquad \begin{cases} r_{11} = -\frac{s}{2}, \\ s_{11} = \frac{r}{2}, \end{cases}$$

and therefore equation (16) is another form of equation (14).

Let

$$\alpha_1 = s_{12} - s_{30} + 2r_{21},\ \alpha_2 = r_{30} - r_{12} + 2s_{21},$$
$$\alpha_3 = s_{03} - s_{21} + 2r_{12},\ \alpha_4 = r_{21} - r_{03} + 2s_{12}.$$

The discussion above is synthesized in the following lemma.

Lemma 4. Let $\mathfrak{p}$ be an axiumbilic point with coordinates $(0, 0)$ in a Monge chart. The differential equation of axial lines in a neighborhood of $(0, 0)$ is given by

$$(17)\qquad \tilde{a}_0(x, y)(dx^4 - 6dx^2dy^2 + dy^4) + \tilde{a}_1(x, y)(dx^2 - dy^2)dxdy + H(x, y, dx, dy) = 0,$$

where

$$(18)\qquad \begin{aligned} \tilde{a}_0(x, y) &= \frac{1}{2}(r\alpha_1 + s\alpha_2)x + \frac{1}{2}(r\alpha_3 + s\alpha_4)y + a^0_{20}x^2 + a^0_{11}xy + a^0_{02}y^2, \\ \tilde{a}_1(x, y) &= 2(s\alpha_1 - r\alpha_2)x + 2(s\alpha_3 - r\alpha_4)y + a^1_{20}x^2 + a^1_{11}xy + a^1_{02}y^2 \end{aligned}$$

and H contains terms of order greater than or equal to 3 in (x, y).

With the notation in equation (17), the *condition of transversality* between the curves $a_0 = 0$ and $a_1 = 0$ is given by

$$\begin{vmatrix} a^0_{10} & a^0_{01} \\ a^1_{10} & a^1_{01} \end{vmatrix} \neq 0.$$

The determinant above has the following expression:

$$[\alpha_2\alpha_3 - \alpha_1\alpha_4] \cdot (r^2 + s^2),$$

where $r = r_{02} - r_{20}$ and $s = s_{02} - s_{20}$. If $(r^2 + s^2)$ is zero, it follows that $a^0_{10} = a^0_{01} = a^1_{10} = a^1_{01} = 0$, and therefore the matrix

$$\begin{bmatrix} a^0_{10} & a^0_{01} \\ a^1_{10} & a^1_{01} \end{bmatrix}$$

is identically zero. Thus the axiumbilic points with $r = s = 0$ form a set of codimension at least four.

Therefore, the *condition of transversality*, supposing $r^2 + s^2 \neq 0$, is given by:

$$(19)\qquad T := \alpha_2\alpha_3 - \alpha_1\alpha_4 \neq 0.$$

Long, but straightforward calculations show that condition (19) is invariant by positive rotations in the tangent and in the normal planes.

Lemma 5. Consider the quartic differential equation

$$(a_{10}x + a_{01}y)(dx^4 - 6dx^2dy^2 + dy^4) + (b_{10}x + b_{01}y)dxdy(dx^2 - dy^2) = 0.$$

Consider a rotation $x = \cos\theta u + \sin\theta v,\ y = -\sin\theta u + \cos\theta v$, where θ is a real root of the system of equations

$$-a_{01}t^5 + (a_{10} - b_{01})t^4 + (6a_{01} + b_{10})t^3 + (b_{01} - 6a_{10})t^2 - (a_{01} + b_{10})t + a_{10} = 0,\ t = \tan\theta.$$

Then it follows that

$$\bar{a_{01}}v(du^4 - 6du^2dv^2 + dv^4) + (\bar{b_{10}}u + \bar{b_{01}}v)dudv(du^2 - dv^2) = 0, \text{where}$$

$$\begin{aligned}\bar{a_{01}} =&(t^2+1)[a_{01}(t^4-6t^2+1)+b_{01}t(t^2-1)t]\\ \bar{b_{10}} =&-16t(t^2-1)(a_{10}-a_{01}t)+(t^4-6t^2+1)(b_{10}-b_{01}t)\\ \bar{b_{01}} =&-16(t^2-1)(ta_{10}+a_{01})+(t^4-6t^2+1)(b_{10}t+b_{01})\end{aligned}$$

Proof. The result follows from straightforward calculations. Observe that when $a_{01}=0$ a rotation of angle $\pi/2$ is sufficient to obtain the result stated. □

Proposition 6. Let $\mathfrak{p}$ be an axiumbilic point. Then there exists a Monge chart and a homothety in $\mathbb{R}^4$ such that the differential equation of axial lines is given by

$$y(dy^4-6dx^2dy^2+dx^4)+(ax+by)dxdy(dx^2-dy^2)+H(x,y,dx,dy)=0 \tag{20}$$

where H contains terms of order greater than or equal to 2 in (x,y). Moreover, the axiumbilic point $\mathfrak{p}$ is transversal if and only if $a\neq 0$.

Proof. Consider a parametrization $X(x,y)=(x,y,R(x,y),S(x,y))$ given by equations (9) and (10) such that 0 is an axiumbilic point. By equation (18) it follows that:

$$\begin{aligned}a_0(x,y) =&\frac{1}{2}(r\alpha_1+s\alpha_2)x+\frac{1}{2}(r\alpha_3+s\alpha_4)y+O(2),\\ a_1(x,y) =&2(s\alpha_1-r\alpha_2)x+2(s\alpha_3-r\alpha_4)y+O(2).\end{aligned}$$

By an appropriate choice of the rotation in the plane $\{x,y\}$ given by Lemma 5 and a homothety in $\mathbb{R}^4$, it is possible to make $2a_{10}=r\alpha_1+s\alpha_2=0$ and, when $(\alpha_1\alpha_4-\alpha_2\alpha_3)(r^2+s^2)\neq 0$, also $a_{01}=\frac{1}{2}(r\alpha_3+s\alpha_4)=1$. So the result is established, $a=\frac{4(s\alpha_1-r\alpha_2)}{r\alpha_3+s\alpha_4}$ when $r\alpha_1+s\alpha_2=0$ and $b=\frac{4(s\alpha_3-r\alpha_4)}{r\alpha_3+s\alpha_4}$. If $r\neq 0$ it follows that $a=-\frac{4(r^2+s^2)\alpha_2}{r(r\alpha_3+s\alpha_4)}$ and $a=\frac{4\alpha_1}{\alpha_4}$ when $s\neq 0$ and $r=0$. □

Remark 7. Let $p=\frac{dy}{dx}$. Then the differential equation (20) is given by:

$$y(p^4-6p^2+1)+(ax+by)p(1-p^2)+H(x,y,p)=0, \tag{21}$$

where H contains terms of order greater than or or equal to 2 in (x,y).

3. Axial configuration in the neighborhood of axiumbilic points

Let $\mathfrak{p}$ be an axiumbilic point whose neighborhood is parametrized by a Monge chart and assume the notation established at the beginning of Section 2.

When it is a transversal axiumbilic point, which is determined by transversal intersection of the curves $a_0=0$ and $a_1=0$ (see equation (3)), it results from Proposition 6 and Remark 7 that the differential equation of axial lines is given by

$$\mathcal{G}(x,y,p)=y(p^4-6p^2+1)+(ax+by)p(1-p^2)+H(x,y,p)=0, \tag{22}$$

where $H(x,y,p)$ contains higher order terms greater or equal to 2 in (x,y).

The Lie-Cartan surface $\mathbb{L}_\alpha$ in PM is defined implicitly by

$$\mathcal{G}(x,y,p)=0. \tag{23}$$

In the case that $\mathfrak{p}$ is a transversal axiumbilic point the surface defined above is regular and of class $\mathcal{C}^{r-2}$ in the neighborhood of the projective axis p.

In the coordinates (x,y,p), the Lie-Cartan vector field X, is of class $\mathcal{C}^{r-3}$, (equation (8)):

$$X:=\mathcal{G}_p\frac{\partial}{\partial x}+p\mathcal{G}_p\frac{\partial}{\partial y}-(\mathcal{G}_x+p\mathcal{G}_y)\frac{\partial}{\partial p} \tag{24}$$

and the projections of the integral curves of $X\Big|_{\mathcal{G}=0}$ are the axial lines in a neighborhood of $\mathfrak{p}$ (Figure 3).

Restricted to the projective axis p, defined by $x = 0$, $y = 0$, the Lie-Cartan vector field is given by

$$X = -p[(p^4 - 6p^2 + 1) + (1 - p^2)(a + bp)]\frac{\partial}{\partial p}.$$

Therefore, the singular points of the Lie-Cartan vector field in the projective line are given by the equation:

$$P(p) = pR(p) = p[(p^4 - 6p^2 + 1) + (1 - p^2)(a + bp)] = 0. \tag{25}$$

The discriminant of $R(p) = (p^4 - 6p^2 + 1) + (1 - p^2)(a + bp)$ is

$$\begin{aligned} \Delta(a, b) =& 16a^5 + 4(b^2 + 68)a^4 + 16(b^2 + 144)a^3 \\ & -8(b^2 - 80)(16 + b^2)a^2 + 96(16 + b^2)^2 a + 4(16 + b^2)^3. \end{aligned} \tag{26}$$

Furthermore, $R(\pm 1) = -4$, $R(0) = 1 + a$ and $\lim_{p\longrightarrow\pm\infty} R(p) = +\infty$, thus R has at least two simple real roots, one is less than -1 and the other is greater than 1.

The derivative of X at $(0, 0, p)$ is given by:

$$DX(0, 0, p) = \begin{bmatrix} a(1 - 3p^2) & 4p^3 + b(1 - 3p^2) - 12p & 0 \\ a(1 - 3p^2)p & p[4p^3 + b(1 - 3p^2) - 12p] & 0 \\ 0 & 0 & -P'(p) \end{bmatrix}$$

whose eigenvalues are 0 and

$$\begin{aligned} \lambda_1(p) &= a(1 - 3p^2) + p[4p^3 + b(1 - 3p^2) - 12p], \\ \lambda_2(p) &= -P'(p). \end{aligned}$$

Recall that $P(p) = pR(p)$, and so $P'(p) = R(p) + pR'(p)$. Therefore at the roots of R, it follows that $-P'(p) = -pR'(p)$. Also, as ± 1 are not roots of R, it follows that

$$a = \frac{(-p^4 + 6p^2 - 1) + bp(1 - p^2)}{1 - p^2}.$$

Substituting the equation above into the expression of $\lambda_1(p)$, p being a root of $R(p)$ (a singular point of X), it follows that

$$\begin{cases} \lambda_1(p) = \frac{(p^2+1)^3}{(p^2-1)}, \\ \lambda_2(p) = -pR'(p). \end{cases}$$

Therefore, the eigenvalues of DX, at the singular points $(0, 0, p_0) = (0, 0, 0)$ and $(0, 0, p_i)$, $p_i \neq 0$, on the tangent space to $\mathcal{G} = 0$, are as follows:

$$p_0 = 0 : \quad \begin{cases} \lambda_1 = a, \\ \lambda_2 = -(a + 1), \end{cases} \tag{27}$$

$$p_i \neq 0 : \quad \begin{cases} \lambda_1 = \frac{(p_i^2+1)^3}{(p_i^2-1)}, \\ \lambda_2 = -p_i R'(p_i). \end{cases} \tag{28}$$

The eigenspace associated to the eigenvalue λ_1 is transversal to the axis p and the eigenvalue λ_2 has the projective axis as the associated eigenspace.

In [11] the axial configuration near an axiumbilic point was established in the following situation:

- $\Delta(a, b) < 0$,
- $\Delta(a, b) > 0,\ a < 0,\ a \neq -1$,
- $\Delta(a, b) > 0,\ a > 0$.

When $\Delta(a, b) < 0$, R has two simple real roots, and the Lie-Cartan vector field has three hyperbolic saddles in the projective axis. This axiumbilic point is called of type E_3.

When $\Delta(a, b) > 0$, $a < 0$, $a \neq -1$, R has four simple real roots, and the Lie-Cartan vector field has 5 singular points in the projective line. Four are hyperbolic saddles and one is a hyperbolic node. This axiumbilic point is called of type E_4.

When $\Delta(a, b) > 0$, $a > 0$, the Lie-Cartan vector field has 5 hyperbolic saddles in the projective line. This axiumbilic point is called of type E_5.

In Figure 4 the Lie-Cartan surfaces and the integral curves of the Lie-Cartan vector field are sketched in the three cases E_3, E_4 and E_5. The projections of the integral curves by $\pi : PM \longrightarrow M$ are the axial lines near the axiumbilic points (see Figure 5) E_3, E_4 and E_5.

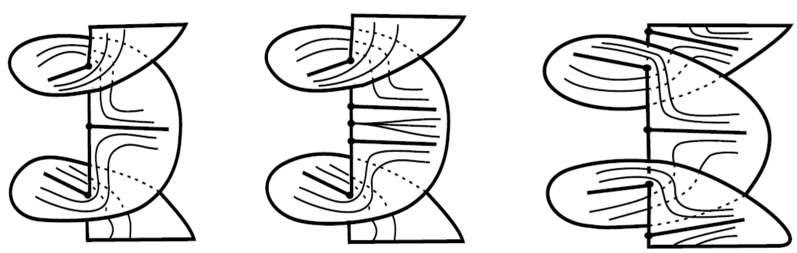

FIGURE 4. Lie-Cartan vector field and its integral curves in the cases E_3, E_4 and E_5.

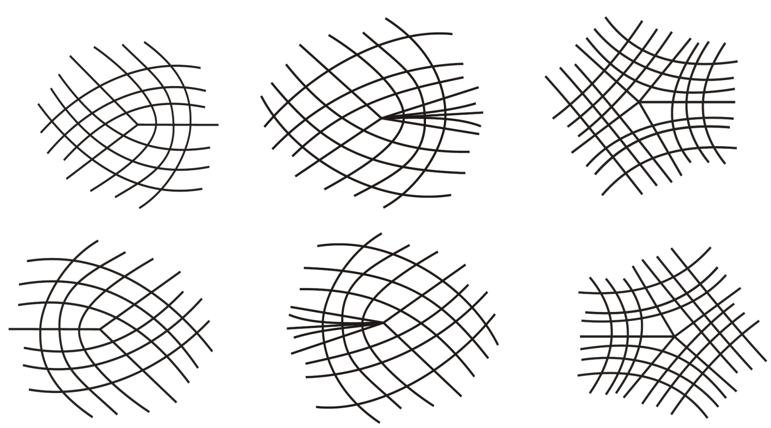

FIGURE 5. Axial configurations near axiumbilic points E_3 (left), E_4 (center) and E_5 (right).

For an immersion α of a surface M into $\mathbb{R}^4$, the axiumbilic singularities $\mathcal{U}_\alpha$ and the lines of axial curvature are assembled into two *axial configurations*: the *principal axial configuration* $\mathcal{P}_\alpha = \{\mathcal{U}_\alpha,\ \mathcal{X}_\alpha\}$ and the *mean axial configuration* $\mathcal{Q}_\alpha = \{\mathcal{U}_\alpha,\ \mathcal{Y}_\alpha\}$.

An immersion $\alpha \in \mathcal{I}^r$ is said to be *principal axial stable* if it has a C^r neighborhood $\mathcal{V}(\alpha)$ such that, for any $\beta \in \mathcal{V}(\alpha)$ there exists a homeomorphism $h : M \to M$ mapping $\mathcal{U}_\alpha$ onto $\mathcal{U}_\beta$ and mapping the integral net of $\mathcal{X}_\alpha$ onto that of $\mathcal{X}_\beta$. Analogous definition is given for *mean axial stability*.

In Proposition 8 are described the axiumbilic points which are principal axial stable. In Figure 6 are sketched the curves $\Delta(a, b) = 0$, $a = -1$ and $a = 0$ in the plane a, b, which bound the open regions corresponding to the three types of axiumbilic points of principal axial stable type.

Proposition 8 ([11], [12] p. 209)**.** Let $\mathfrak{p}$ be an axiumbilic point of $\alpha \in \mathcal{I}^r,\ r \geq 5$. Then, α is locally principal axial stable and locally mean axial stable at $\mathfrak{p}$ if and only if $\mathfrak{p}$ is of type $E_3,\ E_4$ or E_5. The curve $\Delta(a, b) = 0$ has three connected components, is contained in the region $a \leq -1$ and it is regular outside the points $(-\frac{27}{2}, \pm\frac{5\sqrt{5}}{2})$ which are of cuspidal type.

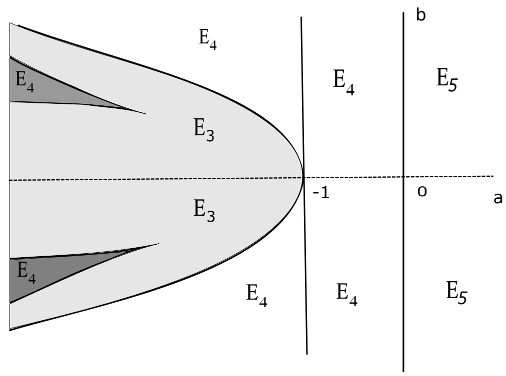

FIGURE 6. Diagram of stable axiumbilic points, E_3, E_4 and E_5.

Proof. The function $\Delta(a, b)$ defined by equation (26) is symmetric in b. The polynomials $\Delta(a, b)$ and $\frac{\partial \Delta}{\partial b}$ in the variable b have resultant equal to a positive multiple of

$$(1 + a)(a^2 + 8a + 32)^2 a^{16}(2a + 27)^6.$$

The critical points $p_\pm = (-\frac{27}{2}, \pm\frac{5\sqrt{5}}{2})$ of Δ are contained in $\Delta(a, b) = 0$.

Near the point p_+ it follows that:

$$\Delta(a, b) = -\,54675 \left[\left(a + \frac{27}{2}\right)^2 + 5\left(b - \frac{5\sqrt{5}}{2}\right)^2 + 2\sqrt{5}\left(a + \frac{27}{2}\right)\left(b - \frac{5\sqrt{5}}{2}\right)\right] + h.o.t.$$

Further analysis shows $p_\pm$ are Whitney cuspidal points.

Also the curve $\Delta(a, b) = 0$ is contained in the region $a \leq -1$ and near $(-1, 0)$ it is given by $a = -\frac{1}{20}b^2 + O(3)$. In fact, for $a > -1$ all the roots of $\Delta(a, b)$ are complex.

By the classification of axiumbilic points E_3, E_4 and E_5 by the sign of $\Delta(a, b)$ and of a, the diagram of stable axiumbilic points, see [11], [12] p. 209, is as shown in Fig. 6. □

3.1. **The axiumbilic point E^1_{34}.**

Definition 9. Let $\alpha : M \longrightarrow \mathbb{R}^4$ be an immersion of class $\mathcal{C}^r, r \geq 5$, of a smooth and oriented surface. An axiumbilic point $\mathfrak{p}$ is said to be of type E^1_{34} if a defined in Proposition 6 does not vanish and:

i) $\Delta(a,b) = 0$, $(a,b) \neq (-1,0)$ and $(a,b) \neq (-\frac{27}{2}, \pm\frac{5}{2}\sqrt{5})$, or
ii) $b \neq 0$ if $a = -1$.

Proposition 10. Let $\alpha : M \longrightarrow \mathbb{R}^4$ be an immersion of class $\mathcal{C}^r$, $r \geq 5$ of a smooth and oriented surface having an axiumbilic point $\mathfrak{p}$ of type E^1_{34}. Then the axial configuration, defined in subsection 1.1, of α in a neighborhood of $\mathfrak{p}$ is as shown in Figure 7.

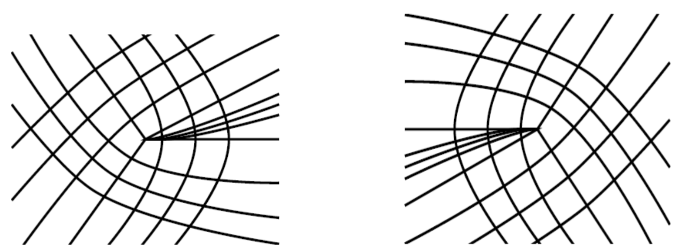

FIGURE 7. Axial configurations in a neighborhood of an axiumbilic point of type E^1_{34}.

Proof. Since the condition of transversality ($a \neq 0$) is preserved at an axiumbilic point of type E^1_{34} the implicit surface defined by equation (23) is regular in a neighborhood of the projective line. From the hypotheses $\Delta(a,b) = 0$, $(a,b) \neq (-1,0)$ and $(a,b) \neq (-\frac{27}{2}, \pm\frac{5}{2}\sqrt{5})$ or $b \neq 0$, if $a = -1$, the polynomial $P(p) = p[(p^4 - 6p^2 + 1) + (1-p^2)(a+bp)] = pR(p)$, which defines the singularities of the Lie-Cartan vector field, has one double root and three real simple roots.
With no loss of generality, we can consider the case $a = -1$ and $b \neq 0$, where $p = 0$ is a double root of the polynomial $P(p)$. In this case, we have $P(p) = p^2(p^3 - bp^2 - 5p + b)$.
The eigenvalues of DX at $(0,0,p)$ are given by:

$$\lambda_1 = 4p^4 - 3bp^3 - 9p^2 + bp - 1 \quad \text{and} \quad \lambda_2 = p(-5p^3 + 4bp^2 + 15p - 2b).$$

Therefore, at the singular points $(0,0,p)$, $p \neq 0$, of X it follows that:

$$\lambda_1 = \frac{(p^2+1)^3}{p^2-1} \quad \text{and} \quad \lambda_2 = -\frac{p^2(p^4+2p^2+5)}{p^2-1}.$$

Then, $\lambda_1\lambda_2 < 0$ when $p \neq 0$ and these three singular points of X are hyperbolic saddles. At $p = 0$, double root of P, it follows that $\lambda_1 = -1$, $\lambda_2 = 0$. Recall that the eigenspace associated to λ_1 is transversal to the axis p and that one associated to λ_2 is the projective axis itself.
Since $\mathcal{G}_y(0,0,0) = 1$, it follows from the implicit function theorem that $y(x,p) = xp + O(3)$ is defined in a neighborhood of $(0,0,0)$ such that $\mathcal{G}(x, y(x,p), p) = 0$. In this case, the Lie-Cartan vector field in the chart (x,p) is given by:

$$\begin{cases} \dot{x} = -x + bxp + O(3) \\ \dot{p} = -bp^2 + O(3) \end{cases} \tag{29}$$

with $b \neq 0$. Therefore, $(0,0,0)$ is a quadratic saddle-node with the center manifold tangent to the projective line. The phase portrait is sketched in Figure 8, and the projections of the integral curves are the axial lines shown in Figure 7.

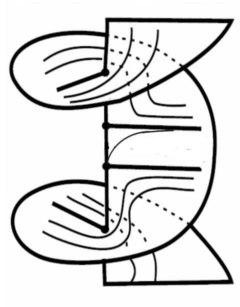

FIGURE 8. Integral curves of $X|_{\mathcal{G}=0}$ in the neighborhood of the projective line in the case of an axiumbilic point of type E_{34}^1

When $(a,b) \neq (-1,0)$, $(a,b) \neq (-\frac{27}{2}, \pm\frac{5}{2}\sqrt{5})$ and $\Delta(a,b) = 0$ the polynomial

$$P(p) = p[(p^4 - 6p^2 + 1) + (1 - p^2)(a + bp)]$$

has a double root $p_0 \neq 0$ and three real simple roots. This case is reduced to the case when $p = 0$ is a double root, making an appropriate rotation of coordinates in the plane $\{x, y\}$ so that, in the new coordinates, the double root p_0 is located at $p = 0$. □

Proposition 11. Let $\alpha \in \mathcal{I}^r$, $r \geq 5$, be an immersion such that $\mathfrak{p}$ is axiumbilic point of type E_{34}^1. Then, there is a neighborhood V of $\mathfrak{p}$, a neighborhood $\mathcal{V}$ of α and a function $\mathcal{F} : \mathcal{V} \longrightarrow \mathbb{R}$ of class $\mathcal{C}^{r-3}$ such that for each $\mu \in \mathcal{V}$ there is an unique axiumbilic point $\mathfrak{p}_\mu \in V$ such that:

i) $d\mathcal{F}_\alpha \neq 0$,
ii) $\mathcal{F}(\mu) < 0$ if and only if $\mathfrak{p}_\mu$ is of type E_3,
iii) $\mathcal{F}(\mu) > 0$ if and only if $\mathfrak{p}_\mu$ of type E_4,
iv) $\mathcal{F}(\mu) = 0$ if, and only if, $\mathfrak{p}_\mu$ is of type E_{34}^1.

Proof. Since $\mathfrak{p}$ is a transversal axiumbilic point of α, the existence of the neighborhoods $\mathcal{V}$ and V follows from the Implicit Function Theorem. For $\mu \in \mathcal{V}$ with an axiumbilic point $\mathfrak{p}_\mu \in V$, after a rigid motion Γ_μ in $\mathbb{R}^4$, locally the immersion $\mu \in \mathcal{V}$ can be parametrized in terms of a Monge chart $(x, y, R_\mu(x,y), S_\mu(x,y))$, with the origin being the axiumbilic point p_μ and

$$\begin{aligned}
R_\mu(x,y) =& \frac{r_{20}(\mu)}{2}x^2 + r_{11}(\mu)xy + \frac{r_{02}(\mu)}{2}y^2 + \frac{r_{30}(\mu)}{6}x^3 + \frac{r_{31}(\mu)}{2}x^2y \\
&+ \frac{r_{13}(\mu)}{2}xy^2 + \frac{r_{03}(\mu)}{6}y^3 + h.o.t., \\
S_\mu(x,y) =& \frac{s_{20}(\mu)}{2}x^2 + s_{11}(\mu)xy + \frac{s_{02}(\mu)}{2}y^2 + \frac{s_{03}(\mu)}{6}x^3 + \frac{s_{21}(\mu)}{2}x^2y \\
&+ \frac{s_{12}(\mu)}{2}xy^2 + \frac{s_{03}(\mu)}{6}y^3 + h.o.t.
\end{aligned}$$

For μ, performing rotations and homoteties as described in Section 2, the coefficients a_μ and b_μ can be expressed in function of the coefficients of the surface presented in a Monge chart, as was done in Proposition 6, considering the coefficients in function of the parameter $\mu \in \mathcal{V}$.

Define $\mathcal{F}(\mu) = \Delta(a(\mu), b(\mu))$ whose zeros define locally the manifold of immersions with an E_{34}^1 axiumbilic point. Here, $\Delta(a,b)$, given by equation (26), is the discriminant of the polynomial $R(p) = (p^4 - 6p^2 + 1) + (1 - p^2)(a + bp)$.

Notice that due to the particular representation of the 3-jets taken here, the condition $a(\mu) = -1$ in Definition 9, the jet extension of the immersion is not transversal, but tangent, to the manifold of jets with E^1_{34} axiumbilic points. It is always possible, by an appropriate rotation in the plane $\{x,y\}$ to suppose that $a(\alpha)\notin\{-\frac{27}{2},-1\}$. See Section 2.

Assertions (ii), (iii) and (iv) follow from the definition of $\mathcal{F}$ and the previous analysis on the sign of the discriminant $\Delta(a_\mu,b_\mu)$.

Moreover, the derivative of $\mathcal{F}(\mu)$ in the direction of the coordinate a does not vanish, leading to conclude that $d\mathcal{F}_\alpha\neq 0$.

In fact, assuming $s_{11}(\alpha)=\frac{1}{2}r\neq 0$, it follows that $a_0(\mu)=y+0(2)$,

$$a_1(\mu)(x,y) = -\frac{4(r(\mu)^2+s(\mu)^2)\alpha_2(\mu)}{r(\mu)\,(r(\mu)\alpha_3(\mu)+s(\mu)\alpha_4(\mu))}x+\frac{4(s(\mu)\alpha_3(\mu)-r(\mu)\alpha_4(\mu))}{r(\mu)\alpha_3(\mu)+s(\mu)\alpha_4(\mu)}y+O(2)$$
$$= a(\mu)x+b(\mu)y+O(2),$$

$\alpha_1=s_{12}-s_{30}+2r_{21},\ \alpha_2=r_{30}-r_{12}+2s_{21},\ \alpha_3=s_{03}-s_{21}+2r_{12},$ and $\alpha_4=r_{21}-r_{03}+2s_{12}$.

Consider the deformation

$$\mu=(x,y,R_\alpha(x,y),S_\alpha(x,y))+\left(0,0,t(\frac{1}{6}x^3-\frac{1}{2}xy^2),tx^2y\right).$$

Then, as $\alpha_2=r_{30}-r_{12}+2s_{21}$, it follows that $a(\mu)=-\frac{4(r^2+s^2)(\alpha_2+t)}{r(r\alpha_3+s\alpha_4)}$ and

$$\frac{d}{dt}\left(\Delta(a(\mu),b(\mu))\right)\Big|_{t=0}=\frac{\partial\Delta}{\partial a}\cdot\frac{da}{dt}=\frac{\partial\Delta}{\partial a}\cdot\left(-\frac{4(r^2+s^2)}{r(r\alpha_3+s\alpha_4)}\right)\neq 0.$$

In the case where $s_{11}(\alpha)=0$ it follows that $r_{11}(\alpha)=-\frac{1}{2}s\neq 0$, $\alpha_1\alpha_4\neq 0$ and $\alpha_2(\mu)=0$. Now consider the deformation

$$\mu=(x,y,R_\alpha(x,y),S_\alpha(x,y))+\left(0,0,tx^2y,t(-\frac{1}{6}x^3+\frac{1}{2}xy^2)\right).$$

Then, $a(\mu)=\frac{4(\alpha_1+t)}{\alpha_4}$ and

$$\frac{d}{dt}\left(\Delta(a(\mu),b(\mu))\right)\Big|_{t=0}=\frac{\partial\Delta}{\partial a}\cdot\frac{da}{dt}=\frac{\partial\Delta}{\partial a}\cdot\left(\frac{4}{\alpha_4}\right)\neq 0.$$

□

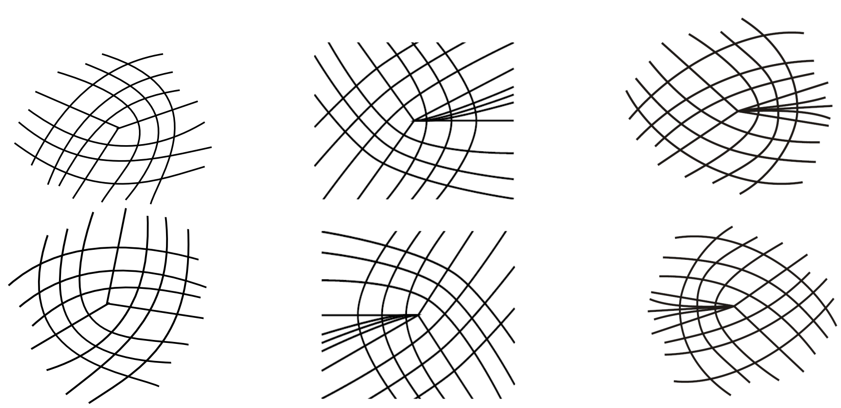

FIGURE 9. Axial configuration near axiumbilic points. E_3 (left), E^1_{34} (center) and E_4 (right).

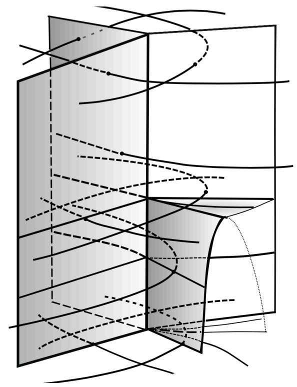

FIGURE 10. Bifurcation diagram of the axial configuration near an axiumbilic point E^1_{34} and the structure of separatrices.

3.2. **The axiumbilic point** $E^1_{4,5}$**.** Consider the Monge chart described by equations (9) and (10). Suppose that the origin is an axiumbilic point, which is expressed by

$$R(x,y) = \frac{r_{20}}{2}x^2 + r_{11}xy + \frac{r_{02}}{2}y^2 + \frac{r_{30}}{6}x^3 + \frac{r_{21}}{2}x^2y + \frac{r_{12}}{2}xy^2 + \frac{r_{03}}{6}y^3 + \frac{r_{40}}{24}x^4 + \frac{r_{31}}{6}x^3y + \frac{r_{22}}{4}x^2y^2 + \frac{r_{13}}{6}xy^3 + \frac{r_{04}}{24}y^4 + h.o.t., \tag{30}$$

$$S(x,y) = \frac{s_{20}}{2}x^2 + s_{11}xy + \frac{s_{02}}{2}y^2 + \frac{s_{30}}{6}x^3 + \frac{s_{21}}{2}x^2y + \frac{s_{12}}{2}xy^2 + \frac{s_{03}}{6}y^3 + \frac{s_{40}}{24}x^4 + \frac{s_{31}}{6}x^3y + \frac{s_{22}}{4}x^2y^2 + \frac{s_{13}}{6}xy^3 + \frac{s_{04}}{24}y^4 + h.o.t., \tag{31}$$

where, $r_{02} = r_{20} + r,\ r_{11} = -\frac{1}{2}s,\ s_{02} = s_{20} + s,\ s_{11} = \frac{1}{2}r.$

Let $\alpha_1 = s_{12} - s_{30} + 2r_{21}$, $\alpha_2 = r_{30} - r_{12} + 2s_{21}$, $\alpha_3 = s_{03} - s_{21} + 2r_{12}$, $\alpha_4 = r_{21} - r_{03} + 2s_{12}$, $\beta_1 = s_{22} - s_{40} + 2r_{31}$, $\beta_2 = r_{40} - r_{22} + 2s_{31}$, $\beta_3 = s_{13} - s_{31} + 2r_{22}$, $\beta_4 = r_{31} - r_{13} + 2s_{22}$, $\beta_5 = s_{04} - s_{22} + 2r_{13}$, and $\beta_6 = r_{22} - r_{04} + 2s_{13}$.

The functions a_0 and a_1 (see Proposition 1) are given by

$$a_0(x,y) = a_{10}x + a_{01}y + \frac{1}{2}a_{20}x^2 + a_{11}xy + \frac{1}{2}a_{02}y^2 + h.o.t. \qquad \text{and} \tag{32}$$

$$a_1(x,y) = b_{10}x + b_{01}y + \frac{1}{2}b_{20}x^2 + b_{11}xy + \frac{1}{2}b_{02}y^2 + h.o.t., \tag{33}$$

where

$$a_{10} = \frac{1}{2}(r\alpha_1 + s\alpha_2), \qquad a_{01} = \frac{1}{2}(r\alpha_3 + s\alpha_4),$$

$$a_{20} = -\alpha_2 r_{21} + \alpha_1 s_{21} + \left[\tfrac{\beta_1}{4} + \tfrac{s_{20}}{2}(r_{20}^2 + s_{20}^2)\right]r + \left[\tfrac{\beta_2}{4} - \tfrac{r_{20}}{2}(r_{20}^2 + s_{20}^2)\right]s + (r_{20}^2 - s_{20}^2)sr - \tfrac{3}{8}(r^2+s^2)(s_{20}r - r_{20}s) + r_{20}s_{20}(s^2 - r^2),$$

$$a_{11} = -r_{12}\alpha_2 + s_{12}\alpha_1 - r_{21}\alpha_4 + s_{21}\alpha_3 - \left[\tfrac{\beta_3}{2} + r_{20}(r_{20}^2 + s_{20}^2)\right]r + \left[\tfrac{\beta_4}{2} - s_{20}(r_{20}^2 + s_{20}^2)\right]s - 2s_{20}r_{20}rs - \tfrac{1}{2}(3s_{20}^2 + r_{20}^2)s^2 - \tfrac{1}{2}(3r_{20}^2 + s_{20}^2)r^2 - \tfrac{3}{8}(r^2+s^2)^2 - \tfrac{5}{4}(r^2+s^2)(r_{20}r + s_{20}s),$$

$$a_{02} = -r_{12}\alpha_4 + s_{12}\alpha_3 + \left[\tfrac{\beta_5}{2} - \tfrac{s_{20}}{2}(r_{20}^2 + s_{20}^2)\right]r + \left[\tfrac{\beta_6}{2} + \tfrac{r_{20}}{2}(r_{20}^2 + s_{20}^2)\right]s + (-2s_{20}^2 + 2r_{20}^2)sr + 2s_{20}r_{20}(s^2 - 2r^2) + -\tfrac{9}{8}(r^2+s^2)(rs_{20} - sr_{20}),$$

$$b_{10} = 2(s\alpha_1 - r\alpha_2), \qquad b_{01} = 2(s\alpha_3 - r\alpha_4),$$

$$b_{20} = \alpha_1^2 + \alpha_2^2 - 4(s_{21}\alpha_2 + r_{21}\alpha_1) + \left[-\beta_2 + 2r_{20}(r_{20}^2 + s_{20}^2)\right]r + \left[\beta_1 + 2s_{20}(r_{20}^2 + s_{20}^2)\right]s - \tfrac{1}{2}(r^2+s^2)(s_{20}s + r_{20}r) + 4(r_{20}s - s_{20}r)^2,$$

$$b_{11} = 2(\alpha_3\alpha_1 + \alpha_2\alpha_4) - 4(\alpha_1 r_{12} + \alpha_2 s_{12} + \alpha_3 r_{21} + \alpha_4 s_{21}) + 2\left[-\beta_4 + 2s_{20}(r_{20}^2 + s_{20}^2)\right]r + 2\left[\beta_3 - 2r_{20}(r_{20}^2 + s_{20}^2)\right]s + 4(s_{20}^2 - r_{20}^2)rs + 4r_{20}s_{20}(r^2 - s^2),$$

$$b_{02} = \alpha_3^2 + \alpha_4^2 + 4(r_{12}^2 + s_{12}^2) + 4s_{12}(r_{21} - r_{03}) + 4r_{12}(s_{03} - s_{21}) + [-\beta_6 - 2r_{20}(r_{20}^2 + s_{20}^2)]r + [\beta_5 - 2s_{20}(r_{20}^2 + s_{20}^2)]s + 2(r_{20}^2 - 3s_{20}^2)s^2 + 2(s_{20}^2 - r_{20}^2)r^2.$$

Definition 12. An axiumbilic point is said to be of type $E^1_{4,5}$ if the variety $\mathbb{L}_\alpha$ has exactly 4 singular points which are of Morse type located along the projective line over the point.

Proposition 13. Consider a Monge chart and a homothety such that the differential equation of axial lines is written as

$$a_0(x,y)(dx^4 - 6dx^2dy^2 + dy^4) + a_1(x,y)dxdy(dx^2 - dy^2) + 0(3) = 0,$$

where

$$a_0(x,y) = y + \frac{1}{2}a_{20}x^2 + a_{11}xy + \frac{1}{2}a_{02}y^2 + h.o.t.,$$
$$a_1(x,y) = b_{01}y + \frac{1}{2}b_{20}x^2 + b_{11}xy + \frac{1}{2}b_{02}y^2 + h.o.t.$$

Then the following conditions are equivalent:

i) the curves $a_0 = 0$ and $a_1 = 0$ are regular and have quadratic contact at 0,
ii) the axiumbilic point 0 is of type $E^1_{4,5}$,
iii) the Lie-Cartan vector field defined in $\mathbb{L}_\alpha$ has a quadratic saddle-node in the projective axis with the center manifold transversal to the projective line.

Proof. The differential equation of axial lines can be written as

$$a_0(x,y)(dx^4 - 6dx^2dy^2 + dy^4) + a_1(x,y)dxdy(dx^2 - dy^2) + 0(3) = 0,$$

where

$$a_0(x,y) = a_{10}x + a_{01}y + \frac{1}{2}a_{20}x^2 + a_{11}xy + \frac{1}{2}a_{02}y^2 + h.o.t.$$
$$a_1(x,y) = b_{10}x + b_{01}y + \frac{1}{2}b_{20}x^2 + b_{11}xy + \frac{1}{2}b_{02}y^2 + h.o.t.$$

where the coefficients of a_0 and a_1 are given by equations (32) and (33). Here $O(3)$ means terms of order greater than or equal to 3 in the variables x and y.

In what follows it will be considered a Monge chart such that $a_{10} = 0$. This is possible as shown in Lemma 5 and Proposition 6. Since the contact between $a_0 = 0$ and $a_1 = 0$ is supposed to be quadratic it results that $b_{10} = 0$ and $a_{01} \cdot b_{01} \neq 0$. Also by a homothety it is possible to obtain $a_{01} = 1$.

So, it results that:

$$a_0(x, y) = y + \frac{1}{2}a_{20}x^2 + a_{11}xy + \frac{1}{2}a_{02}y^2 + h.o.t. \tag{34}$$

$$a_1(x, y) = b_{01}y + \frac{1}{2}b_{20}x^2 + b_{11}xy + \frac{1}{2}b_{02}y^2 + h.o.t. \tag{35}$$

Therefore, the condition of quadratic contact between the two regular curves is expressed by $\chi = b_{20} - a_{20}b_{01} \neq 0$.

Notice that this amounts to establish the implication $i) \to ii)$.

Claim 14. In the neighborhood of $(0, 0, 0)$, the Lie-Cartan vector field restricted to the surface $\mathcal{G} = 0$, can be expressed in the chart (x, p) by

$$\begin{cases} \dot{x} = \frac{\chi}{2}x^2 + O(3), \\ \dot{p} = -p + \frac{3}{2}a_{11}a_{20}x^2 - (a_{11} + \chi)p - b_{01}p^2 + 0(3) \end{cases} \tag{36}$$

and $(0, 0, 0)$ is a saddle-node when $\chi \neq 0$.

Since $\mathcal{G}_y(0, 0, 0) = 1 \neq 0$, it follows from implicit function theorem that locally $y = y(x, p)$ and $\mathcal{G}(x, y(x, p), p) = 0$.

The Taylor expansion of $y(x, p)$ in the neighborhood of $(x, p) = (0, 0)$ is given by:

$$y(x, p) = -\frac{1}{2}a_{20}x^2 + O(3). \tag{37}$$

The Lie-Cartan vector field restricted to the surface $\mathcal{G} = 0$ is given by

$$\begin{cases} \dot{x} = \mathcal{G}_p(x, y(x, p), p) = \frac{1}{2}\chi x^2 + O(3) \\ \dot{p} = -(\mathcal{G}_x + p\mathcal{G}_y)(x, y(x, p), p) = -p + \frac{3}{2}a_{11}a_{20}x^2 - (\chi + a_{11})p - b_{01}p^2 + 0(3) \end{cases}$$

The eigenvalues of the vector field (36) at 0 are $\lambda_1 = 0$ and $\lambda_2 = -1$ with respective associated eigenspaces $\ell_1 = (1, -a_{20})$ and $\ell_2 = (0, 1)$. By Invariant Manifold Theory the center manifold is tangent to ℓ_1 and is given by $W^c = \{(x, -a_{20}x + \frac{3}{2}a_{20}(\chi + a_{11})x^2 + O(3))\}$.

The restriction of the vector field (36) to the center manifold is given by $[\frac{1}{2}\chi x^2 + 0(3)]\frac{\partial}{\partial x}$.

This establishes that $ii) \to iii)$.

Claim 15. The function $\mathcal{G}$ has exactly 4 critical points in the projective line, and they are of Morse-type of index 1 or 2 if and only if $\chi \neq 0$.

The critical points of $\mathcal{G}$ along the projective line are determined by

$$S(p) = \mathcal{G}_v(0, 0, p) = (p^4 - 6p^2 + 1) + b_{01}p(1 - p^2) = 0, \tag{38}$$

which has for 4 simple real roots located in the intervals $(-\infty, -1), (-1, 0), (0, 1)$ and $(1, \infty)$. This follows from $S(\pm 1) = -4$, $S(0) = 1$ and from the discriminant $\Delta(S) = 4(16 + b_{01}^2)^3 > 0$. Along the projective line, the determinant of the Hessian of $\mathcal{G}$ is given by

$$\operatorname{Hess}\mathcal{G}(0, 0, p) = -(a_{20}(1 - 6p^2 + p^4) + b_{20}p(1 - p^2))(b_{01} - 12p - 3b_{01}p^2 + 4p^3)^2. \tag{39}$$

The resultant of $S(p)$ and $\operatorname{Hess}\mathcal{G}(0, 0, p)$ is given by $256\chi^4(16 + b_{01}^2)^6$ and therefore $\operatorname{Hess}\mathcal{G}(0, 0, p) \neq 0$ at the critical points of $\mathcal{G}$. This implies that the critical points are of Morse type. As $\mathcal{G}(0, 0, p) = 0$ it follows that the index of a critical point is 1 or 2 and so locally the level set $\mathcal{G} = 0$ is a cone.

The eigenvalues of the derivative of the Lie-Cartan vector field at a point $(0, 0, p)$ are given by:

$$\lambda_1 = -p(-4p^3 + 3b_{01}p^2 + 12p - b_{01}), \quad \lambda_2 = -1 + 18p^2 - 5p^4 - 2b_{01}p + 4b_{01}p^3.$$

At the critical points p_i (satisfying $S(p_i) = 0$) it follows that $\lambda_1 = -\lambda_2 = \frac{p^6+3p^4+3p^2+1}{p^2-1}$, then $\lambda_1^i \lambda_2^i < 0$, for $i = 1..4$.

Therefore, these 4 points are saddles of the Lie-Cartan vector field. As the projective line is invariant it is follows that the other invariant manifold (stable or stable) of a singular point is transversal to the projective line.

This amounts to prove that $iii) \to i)$. □

Proposition 16. Let $\alpha \in \mathcal{I}^r$, $r \geq 5$ and $\mathfrak{p}$ be an axiumbilic point. Suppose, in the Monge chart expressed by equations (30) and (31), that $\alpha_1 = \alpha_3 = 0$ and $\chi \neq 0$. Then $\mathfrak{p}$ is an axiumbilic point of type $E^1_{4,5}$ and the axial configurations of α in a neighborhood of $\mathfrak{p}$ is as shown in Figure 11.

FIGURE 11. Axial configurations in a neighborhood of an axiumbilic point of type $E^1_{4,5}$.

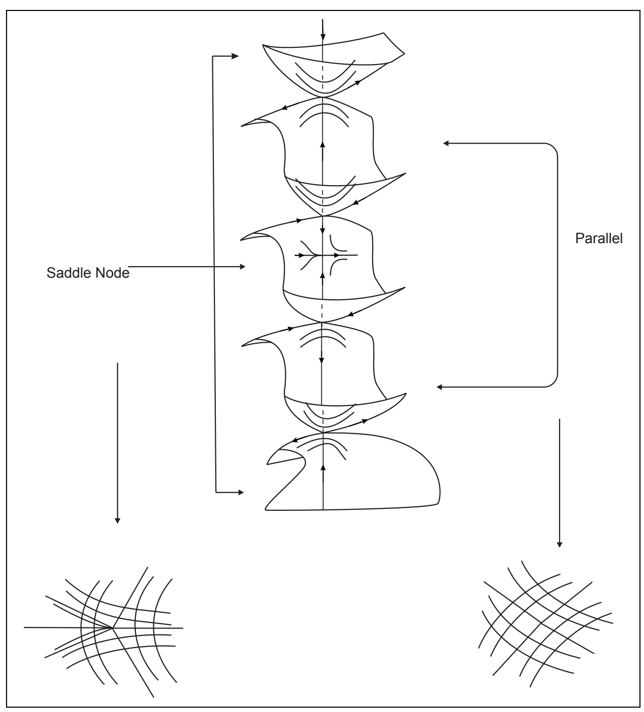

FIGURE 12. Lie-Cartan vector field near an axiumbilic point E^1_{45} and the axial configuration (principal and mean).

Proof. Condition $\alpha_1 = \alpha_3 = 0$ implies the non-transversal contact of the curves $a_0 = 0$ and $a_1 = 0$ at the axiumbilic point $\mathfrak{p}$ expressed in the Monge chart by $(0,0)$. By Lemma 5 and Proposition 6, it is possible to express these curves as in equation (34). Assuming $\chi \neq 0$, we have the quadratic contact of the curves at the axiumbilic point.
Proposition 13 implies that over the axiumbilic point we have five equilibria of the Lie-Cartan vector field. One of them is a regular point of the Lie-Cartan surface, and this is an equilibrium of saddle-node type with center manifold transversal to the axis p (see Claim 14).
The remaining equilibria are critical points of Morse type of the Lie-Cartan surface. In the neighborhood of these points, the level set $\mathcal{G} = 0$ are locally cones, and the 4 points are saddles of the Lie-Cartan vector field (see Claim 15).
Therefore, we conclude that the configuration is as described in Figure 12, whose projection of the saddle-node and parallel sectors describe the principal axial and mean axial configurations close to the axiumbilic point $\mathfrak{p}$ of type E^1_{45} (Figure 11). □

Proposition 17. Let $\alpha \in \mathcal{I}^r$, $r \geq 5$, be an immersion having an axiumbilic point $\mathfrak{p}$ of type $E^1_{4,5}$. Then, there exist a neighborhood V of $\mathfrak{p}$, a neighborhood $\mathcal{V}$ of α and a function $F : \mathcal{V} \longrightarrow \mathbb{R}$ of class $\mathcal{C}^{r-3}$ such that:

i) $dF_\alpha \neq 0$,
ii) $F(\mu) = 0$ if, and only if, $\mu \in \mathcal{V}$ has just one axiumbilic point in V, which is of type $E^1_{4,5}$,
iii) $F(\mu) < 0$ if, and only if, μ has exactly two axiumbilic points in V, one of type E_4 and the other of type E_5,
iv) $F(\mu) > 0$ if, and only if, μ has no axiumbilic points in V.

Proof. By Proposition 13, α being an immersion having an axiumbilic point $\mathfrak{p}$ of type E^1_{45}, the curves $a^\alpha_0 = 0$ and $a^\alpha_1 = 0$ have quadratic contact at $\mathfrak{p}$.
Since $\frac{\partial a^\alpha_0}{\partial y}(0,0) = a_{01} \neq 0$, if follows from Implicit Function Theorem that locally, for μ in a neighborhood $\mathcal{V}$ of α, $y = y_\mu(x)$ and $a^\mu_0(x, y_\mu(x)) = 0$.
Moreover, $\frac{\partial^2 a^\alpha_1}{\partial x^2}(0,0) = b_{20} \neq 0$, and so $x = x_\mu$ is a local solution of $\frac{\partial a^\mu_1}{\partial x}(x_\mu, y_\mu(x_\mu)) = 0$.
Define $F(\mu) = a^\mu_1(x_\mu, y_\mu(x_\mu))$. Consider the variation

$$h_t(x,y) = (x, y, R(x,y) + txy, S(x,y) + txy).$$

It follows that $\left.\frac{dF(t)}{dt}\right|_{t=0} \neq 0$, and so $dF_\alpha \neq 0$. Therefore, the result follows from the Implicit Function Theorem.

The axiumbilic point of type E^1_{45} is therefore the transition between zero and two axiumbilic points, one of type E_4 and the other of type E_5.

In Figures 13 and 15 are illustrated this transition, with the axial configurations sketched in two different styles. See also Figure 15 for an illustration of transition in the Lie - Cartan surface. □

Proposition 18. In the space of smooth mappings of $M \times \mathbb{R} \longrightarrow \mathbb{R}^4$ which are immersions relative to the first variable, those which have all their axiumbilic points either generic (of types E_3, E_4 and E_5) or of types E^1_{34} and E^1_{45}, crossed transversally, is open and dense. Furthermore, for such families the axiumbilic points describe a regular curve in $M \times \mathbb{R}$ whose projection into $\mathbb{R}$ has only non-degenerate critical points at E^1_{45} and the regular points of the projection is a collection of arcs bounded by E^1_{34} points, which a the common boundary points of the arcs consisting of points of types E_3 and E_4.

Proposition 18 follows from the analysis in Propositions 11 and 17 and an application of Thom Transversality Theorem to the submanifold of four jets of immersions at axiumbilic points,

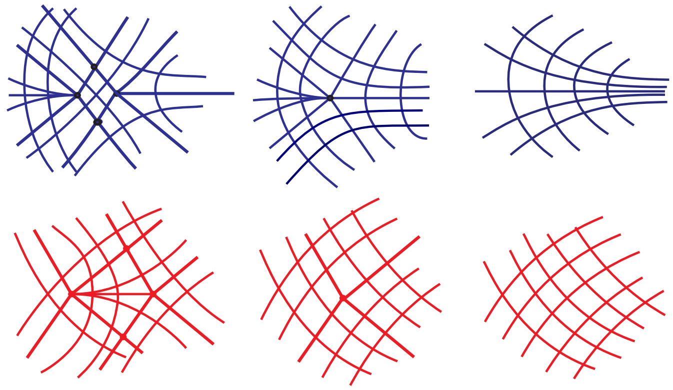

FIGURE 13. Axiumbilic point E_{45}^1. The axiumbilic points E_4 and E_5 collapse in an axiumbilic point E_{45}^1, and after they are eliminated and there are no axiumbilic points.

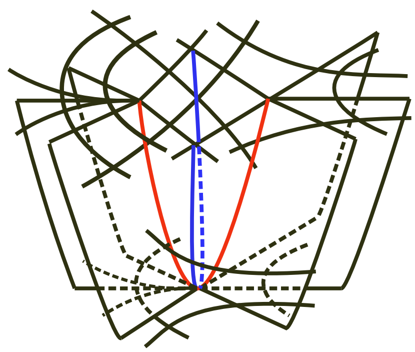

FIGURE 14. Bifurcation diagram of the axial configuration near an axiumbilic point of type E_{45}^1 and the structure of separatrices

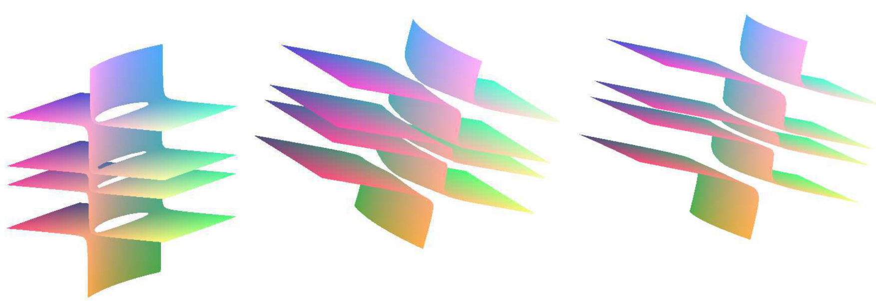

FIGURE 15. The Lie-Cartan surface. In the left, with two axiumbilic point, in the center with four singular points, and in the right the four regular levels.

stratified by the generic axiumbilic points of types E_3, E_4 and E_5, by those of types E_{34}^1 and E_{45}^1, and by their complement which has codimension larger than 3. See Section 4.

4. Transversality and Stratification

Consider the space $\mathbb{J}^k(M, \mathbb{R}^4)$ of k-jets of immersions α of a compact oriented surface M into $\mathbb{R}^4$, endowed with the structure of Principal Fiber Bundle. The base is M; the fiber is the space $\mathbb{R}^4 \times \mathbb{J}^k(2,4)$, where $\mathcal{J}^k(2,4)$ is the space of k-jets of immersions of $\mathbb{R}^2$ to $\mathbb{R}^4$, preserving the respective origins. The structure group, $\mathbb{A}^k_+$, is the product of the group $\mathcal{L}^k_+(2,2)$ of k-jets of origin and orientation preserving diffeomorphisms of $\mathbb{R}^2$, acting on the right by coordinate changes, and by the group of positive isometries of $\mathbb{R}^4$, acting on the left. This group is generated by the groups of translations and that of positive rotations, $\mathcal{O}_+(4)$, of $\mathbb{R}^4$.

Denote by $\Pi_{k,l}, k \leq l$ the projection of $\mathcal{J}^l(2,4)$ to $\mathcal{J}^k(2,4)$. It is well known that the group action commutes with projections.

Definition 19. We define below the *canonic axiumbilic stratification* of $\mathcal{J}^4(2,4)$. The term *canonic* means that the strata are invariant under the action of the group $\mathbb{A}^k_+ = \mathcal{O}_+(4) \times \mathcal{L}^k_+(2,2)$.

1) *Axiumbilic Jets*: $\mathcal{U}^4$, those in the orbit of $j^4(x, y, R(x,y), S(x,y))$, where R and S are as in equations (9) and (10) satisfying the axiumbilic conditions defined in terms of $j^2R(0)$ and $j^2S(0)$. It is a closed variety of codimension 2.
2) *Non-axiumbilic Jets*: $(\mathcal{NU})^4$ is the complement of $\mathcal{U}^4$. It is an open submanifold of codimension 0.
3) *Non-stable axumbilic Jets*: $(\mathcal{NE})^4$, in the orbit of the axiumbilic jets for which:
   - $T = (\alpha_1\alpha_4 - \alpha_2\alpha_3)(r^2 + s^2) = 0$ or
   - $T \neq 0$ and conditions that characterize E_3 or E_4 axiumbilic points in Proposition 8 fail.

$\mathcal{E}^1_{45}$ is a closed variety of codimension 3, which can be expressed as the union of the following invariant strata:

3.1) *Non-Transversal jets*: $\mathcal{E}^1_{45}$ for which $T = 0$ and $\chi \neq 0$. It has codimension 3.

3.2) *Transversal-double jets*: $(\mathcal{E}^1_{34})^4$, The Lie-Cartan field has a quadratic saddle-node in the projective line which is characterized by Proposition 11. It has codimension 3.

4) The *stable axumbilic jets*: $\mathcal{UE}^4$, the complement in $\mathcal{U}^4$ of $\mathcal{NE}^4$.

Proposition 20. In the space of 1-parameter families of immersions, those whose 4-jet extension are transversal to the canonical axiumbilic stratification is open and dense.

Proof. Follows from Thom Transversality Theorem [6]. □

5. Concluding Comments

In this work was established the principal axial and the mean axial configurations in a neighborhood of the axiumbilc points of types E^1_{34} and E^1_{45}. The approach concerning methods and class of differentiability requirements is distinct from that presented in the work of Gutiérrez-Guínez-Castañeda in [3]. The use of the Lie-Cartan suspension method made possible the study of these points by means the classic theory of differential equations, in clear analogy with the saddle-node bifurcation of vector fields in the plane, following [1], [10] and [5].

The type E^1_{34} satisfies the transversality condition of the curves a_0 and a_1, Proposition 6, which amounts to the fact the Lie-Cartan surface remains regular in a neighborhood of the projective axis over the axiumbilic point. In this case there is a saddle-node equilibrium point of the Lie-Cartan vector field whose central separatrix is along the projective axis itself. The axial configurations are established in Proposition 10 and the qualitative change (bifurcation) between the types E_3 and E_4, with the variation of a parameter in the space of immersions, is explained in Proposition 7. See Figure 10.

In the case E^1_{45} the transversality condition fails, since curves a_0 and a_1, Proposition 13, have quadratic contact at the axiumbilic point. Here the Lie-Cartan surface is not regular along the projective axis. It is established in Proposition 13 that there are four conic critical points of Morse type on the $p-$axis. At these points there are partially hyperbolic equilibria of the Lie-Cartan vector field. There is also a saddle-node equilibrium in the regular part of the surface whose central separatrix is transversal to the projective axis. The integral curves of the Lie - Cartan vector field on the regular components of the Lie - Cartan surface (which are four bi-punctured disks) are illustrated in Figure 12. Their projections on the plane give the axial configurations in a neighborhood of the axiumbilic point.

In Proposition 18 is established the one parameter variation (bifurcation) in the space of immersions. This leads to the fact that for small perturbations of an immersion with an axiumbilic point of this type it holds that two axiumbilic points, one of type E_4 and the other of type E_5, bifurcate form E^1_{45} or disappear leaving a neighborhood free from axiumbilic points, in full analogy with the saddle-node bifurcation [1] and [10]. See Figure 14.

In Proposition 20 the genericity of the points E^1_{34} and E^1_{45} is established in terms of stratification and transversality.

References

[1] Andronov, A. Gordon, I. Leontovich, E. Maier, G. Theory of Bifurcations of Dynamical Systems on a Plane. Israel Program for Scientific Translations, John Wiley, New York, 1973.

[2] Gutiérrez, C., Guíñez, V. Simple Umbilic Points on Surfaces Immersed in $\mathbb{R}^4$. Discrete Contin. Dyn. Syst. **9** (2003), 877-900. DOI: 10.3934/dcds.2003.9.877

[3] Gutiérrez, C., Guíñez V., Castañeda, A. Quartic Differential Forms and Transversal Nets with Singularities. Discrete Contin. Dyn. Syst.**26** (2010), 225-249.

[4] Gutiérrez, C., Guadalupe, I., Tribuzy, R., Guíñez, V. Lines of Curvature on Surfaces immersed in $\mathbb{R}^4$. Bol. Soc. Bras. Mat., **28** (1997), 233-251. DOI: 10.1007/BF01233393

[5] Gutiérrez, C., Garcia, R. , Sotomayor, J. Bifurcations of Umbilic Points and Related Principal Cycles. Journ. Dyn. and Diff. Eq., **16** (2004), 321-346.

[6] H.T. Levine, Singularities of Differentiable Mappings, Lect. Notes in Math. **192**, 1971.

[7] Little, J. A. On Singularities of Submanifolds of Higher Dimensional Euclidean Spaces. Ann. Mat. Pura Appl., **83** (1969), 261-335. DOI: 10.1007/BF02411172

[8] Mello, L. F. Mean Directionally Curved Lines on Surfaces Immersed in $\mathbb{R}^4$. Publ. Mat., **47**,(2003), 415-440. DOI: 10.5565/PUBLMAT_47203_06

[9] Mello, L. F. Orthogonal asymptotic lines on surfaces immersed in $\mathbb{R}^4$. Rocky Mountain J. Math. 39 (2009), 1597-1612. DOI: 10.1216/RMJ-2009-39-5-1597

[10] Sotomayor, J. Generic one-parameter families of vector fields on two-dimensional manifolds. Inst. Hautes Études Sci. Publ. Math. **43** (1974), 5-46. DOI: 10.1007/BF02684365

[11] Sotomayor, J., Garcia, R. Lines of Axial Curvature on Surfaces Immersed in $\mathbb{R}^4$. Diff. Geom. and its Applications. **12** (2000), 253-269.

[12] Sotomayor, J., Garcia, R. Differential Equations of Classical Geometry, a Qualitative Theory. Publicações Matemáticas. IMPA. 2009.

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ON THE EULER CHARACTERISTIC OF REAL MILNOR FIBRES

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ABSTRACT. We study the Milnor fibres of a real analytic mapping defined on a real analytic space which has an isolated critical point. In particular we look at the Euler characteristic. We discuss the global case, too.

0. INTRODUCTION

Mappings $f : \mathbb{R}^n \to \mathbb{R}^k$ with an isolated singularity have been already studied by J. Milnor [M]. It is not important whether one works in the real algebraic or real analytic category, here we prefer the real analytic one. We replace $\mathbb{R}^n$ by a germ of a real analytic space with an isolated singularity, introduce a kind of Milnor fibration and study the Euler characteristic of its fibres. Finally we pass shortly to the global case.

Part of the results has been announced in [H].

1. THE REAL MILNOR FIBRATION

Let $f : (X, 0) \to (\mathbb{R}^k, 0)$ be a real analytic mapping between real analytic space germs with an isolated singularity, which means that $f : X \setminus \{0\} \to \mathbb{R}^k$ is a submersion between manifolds. Let X be purely n-dimensional. We may suppose that $(X, 0)$ is embedded in $(\mathbb{R}^N, 0)$. Let $D_\epsilon := \{x \in \mathbb{R}^N \mid \|x\| \leq \epsilon\}$, $S_\epsilon := \partial D_\epsilon$. Let $L := X \cap S_\epsilon$ and $K := f^{-1}(\{0\}) \cap S_\epsilon$, $0 < \epsilon \ll 1$, be the links of $(X, 0)$ and $(f^{-1}(\{0\}), 0)$. Note that $X \setminus \{0\}, L$ and K are manifolds which are not necessarily orientable!

Similarly, let $B_\alpha := \{t \in \mathbb{R}^k \mid \|t\| \leq \alpha\}$.

Theorem 1.1:

a) Let $0 < \alpha \ll \epsilon \ll 1$. Then $f : X \cap D_\epsilon \cap f^{-1}(B_\alpha \setminus \{0\}) \to B_\alpha \setminus \{0\}$ is a locally trivial fibration ("Milnor fibration").

b) The mapping $f : X \cap S_\epsilon \cap f^{-1}(B_\alpha) \to B_\alpha$ is a locally trivial, hence a trivial fibration, so ∂F_t is diffeomorphic to K for every "Milnor fibre" $F_t = f^{-1}(\{t\}) \cap D_\epsilon$.

Proof. Note that we have supposed that 0 is an isolated singularity of f. In particular $f^{-1}(0)$ has an isolated singularity at 0, and S_ϵ is transversal to $f^{-1}(0)$, $0 < \epsilon \ll 1$. Hence S_ϵ is transversal to $f^{-1}(t)$ for $\|t\| \leq \alpha, 0 < \alpha \ll \epsilon \ll 1$. □

The base space in a) is connected if $k \geq 2$ but not if $k = 1$, so we treat these cases separately.

Note that we have a lemma which goes back to Milnor ([M] Lemma 11.3) in the case $X = \mathbb{R}^n$:

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Lemma 1.2: For $0 < \alpha \ll \epsilon \ll 1$ we have a homeomorphism
$$X \cap \big((D_\epsilon \cap f^{-1}(\partial B_\alpha)) \cup (S_\epsilon \cap f^{-1}(B_\alpha \setminus \{0\}))\big) \approx L \setminus K\,,$$
hence a homotopy equivalence $X \cap D_\epsilon \cap f^{-1}(\partial B_\alpha) \sim L \setminus K$.

So we use the symbol $\approx$ in the case of a homeomorphism and $\sim$ in the case of a homotopy equivalence.

Proof. We have assumed $X \subset \mathbb{R}^N$. Put $\phi, \psi : X \to \mathbb{R} : \phi(x) := \|f(x)\|^2, \psi(x) := \|x\|^2$. By the Curve Selection Lemma we know that there are no $x \in D_\epsilon \cap X \setminus f^{-1}(0)$ such that there is a $\lambda \leq 0$ with $d\psi_x = \lambda\, d\phi_x$ if $0 < \epsilon \ll 1$. Therefore we can find on $X \setminus f^{-1}(0)$ a vector field v such that $d\psi_x(v(x)) > 0$, $d\phi_x(v(x)) = 1$ for $\|x\| \leq \epsilon$. Using the flow we can construct the desired homeomorphism. Furthermore $X \cap D_\epsilon \cap f^{-1}(\partial B_\alpha)$ is a deformation retract of $X \cap ((D_\epsilon \cap f^{-1}(\partial B_\alpha)) \cup (S_\epsilon \cap f^{-1}(B_\alpha \setminus \{0\})))$. □

According to Milnor [M], p. 99, the homotopy equivalence can in general not be chosen as to be fibre preserving with respect to $x \mapsto \frac{f(x)}{\|f(x)\|}$.

2. The Milnor fibre of a real analytic mapping ($k \geq 2$)

First we suppose $k \geq 2$. Then we can speak of the typical Milnor fibre F because all Milnor fibres are diffeomorphic.

Standard example: $k = 2, n = 2m$, $f : (\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$ with isolated singularity. For the more general case see e.g. [M] p. 103, and [CL].

In this paper we look at cohomology with integral coefficients.

Theorem 2.1: We have long exact sequences:
$$\ldots \to H^m(L \setminus K) \to H^m(F) \to H^{m+2-k}(F) \to H^{m+1}(L \setminus K) \to \ldots \text{ (Wang sequence)},$$
$$\ldots \to H^{m-1}(K) \to H^m(F, \partial F) \to H^m(F) \to H^m(K) \to \ldots,$$
$$\ldots \to H^m(L) \to H^m(F) \to H^{m-k+2}(F, \partial F) \to H^{m+1}(L) \to \ldots$$
Note that the second and third long exact sequences are the ones for the pair $(F, \partial F)$ and the pair (L, F): we can embed F in L.

For $k = 2$ the first and third sequences read:
$$\ldots \to H^m(L \setminus K) \to H^m(F) \overset{h^* - id}{\to} H^m(F) \to H^{m+1}(L \setminus K) \to \ldots \text{ and}$$
$$\ldots \to H^m(L) \to H^m(F) \overset{Var^*}{\to} H^m(F, \partial F) \to H^{m+1}(L) \to \ldots$$
Here $h : F \to F$ is "the" monodromy.

Proof. (i) For the Wang sequence, see Spanier [S] p. 456.

Note that $L \setminus K$ may be replaced by $X \cap D_\epsilon \cap f^{-1}(\partial B_\alpha)$, see Lemma 1.2, and
$$f : X \cap D_\epsilon \cap f^{-1}(\partial B_\alpha) \to \partial B_\alpha$$
is a locally trivial fibration.

(ii) In the second exact sequence we may replace K by ∂F; see Theorem 1.1b).

(iii) As for the third exact sequence, note that we may replace L by

$$X \cap \big((D_\epsilon \cap f^{-1}(\partial B_\alpha)) \cup (S_\epsilon \cap f^{-1}(B_\alpha))\big)$$

Let D be an open "ball" in ∂B_α, $t \in D$. Then:

$$H^{m+1}(L, F) \simeq H^{m+1}\left(X \cap ((D_\epsilon \cap f^{-1}(\partial B_\alpha)) \cup (S_\epsilon \cap f^{-1}(B_\alpha))), X \cap D_\epsilon \cap f^{-1}(\bar{D})\right)$$

$$\simeq H^{m+1}\left(X \cap ((D_\epsilon \cap f^{-1}(\partial B_\alpha)) \cup (S_\epsilon \cap f^{-1}(B_\alpha))), X \cap ((D_\epsilon \cap f^{-1}(\bar{D})) \cup (S_\epsilon \cap f^{-1}(B_\alpha)))\right)$$

$$\simeq H^{m+1}\left(X \cap D_\epsilon \cap f^{-1}(\partial B_\alpha \setminus D), X \cap ((D_\epsilon \cap f^{-1}(\partial D)) \cup (S_\epsilon \cap f^{-1}(\partial B_\alpha \setminus D)))\right)$$

$$\simeq H^{m+1}((F, \partial F) \times (\partial B_\alpha \setminus D, \partial D)) \simeq H^{m+2-k}(F, \partial F)\,.$$

In fact, for the first isomorphism note that $L \approx X \cap ((D_\epsilon \cap f^{-1}(\partial B_\alpha)) \cup (S_\epsilon \cap f^{-1}(B_\alpha)))$, similarly as in Lemma 1.2. Furthermore, F is a deformation retract of $X \cap D_\epsilon \cap f^{-1}(\bar{D})$.
For the second one, note that $f|S_\epsilon \cap f^{-1}(B_\alpha)$ is trivial, see Theorem 1.1b), so $S_\epsilon \cap f^{-1}(D)$ is a strong deformation retract of $S_\epsilon \cap f^{-1}(B_\alpha)$.
The third isomorphism is established by excision, the fourth one is due to the fact that

$$f : D_\epsilon \cap f^{-1}(\partial B_\alpha \setminus D) \to \partial B_\alpha \setminus D$$

is a trivial fibration. The last one follows from the Künneth formula. □

Since one cannot expect good connectivity properties in the real case, let us look at the Euler characteristic.

Corollary 2.2:

a) $\chi(L) = 0$ if n is even, $\chi(L) = 2\chi(F)$ if n is odd,

b) $\chi(K) = 0$ if $n - k$ is even, $\chi(K) = 2\chi(F)$ if $n - k$ is odd.

Proof. First let us observe the following: Suppose that M is a compact manifold with boundary of dimension m. Then $\chi(M, \partial M) = (-1)^m \chi(M)$. In particular, $\chi(M) = 0$ if M is closed and m is odd.
This is obvious by Poincaré duality, in the non-orientable case with coefficients in $\mathbb{Z}/2\mathbb{Z}$.
a) Suppose that n is even. Then L is a closed manifold of odd dimension, hence $\chi(L) = 0$. Therefore we assume now that n is odd. By the third exact sequence and Poincaré duality we have

$$\chi(L) = \chi(F) - (-1)^k \chi(F, \partial F) = \chi(F) - (-1)^n \chi(F) = 2\chi(F)$$

b) Similarly, $\chi(K) = 0$ if $n - k$ is even. So suppose that $n - k$ is odd. Then

$$\chi(K) = \chi(F) - \chi(F, \partial F) = \chi(F) - (-1)^{n-k} \chi(F) = 2\chi(F).$$

□

So $\chi(F)$ can be expressed by the Euler characteristic of a link except if k and n are both even.

3. The Milnor fibres of a real analytic function ($k = 1$)

Now let us switch to the case $k = 1$. Then we have two typical Milnor fibres: F_+ (resp. F_-), corresponding to F_t with $t > 0$ (resp. $t < 0$).

Theorem 3.1:
a) $H^m(L \setminus K) \simeq H^m(F_+) \oplus H^m(F_-)$.
b) We have long exact sequences:

$$\ldots \to H^{m-1}(K) \to H^m(F_+, \partial F_+) \to H^m(F_+) \to H^m(K) \to \ldots,$$
$$\ldots \to H^m(L) \to H^m(F_+) \oplus H^m(F_-) \to H^m(K) \to \ldots \text{ and}$$
$$\ldots \to H^m(L) \to H^m(F_+) \to H^{m+1}(F_-, \partial F_-) \to \ldots$$

The middle exact sequence is a Mayer-Vietoris sequence, of course. As a consequence,

$$\chi(L) + \chi(K) = \chi(F_+) + \chi(F_-).$$

Proof. $H^m(L, F_+) \simeq H^m(F_-, \partial F_-)$ by excision. The rest is clear. □

Corollary 3.2: If n is even, we have

$$\chi(F_+) = \chi(F_-), \chi(L) = 0, \chi(K) = 2\chi(F_+).$$

If n is odd,

$$\chi(L) = \chi(F_+) + \chi(F_-), \chi(K) = 0.$$

Proof. If n is even, $\chi(L) = 0$, hence $\chi(F_+) = -\chi(F_-, \partial F_-) = \chi(F_-)$. If n is odd, $\chi(K) = 0$. The rest is clear. □

It is difficult to calculate individual cohomology groups but:

Corollary 3.3: a) Suppose that $n = 2m + 1, m \geq 1$ and that F_+ and F_- have the homotopy type of a bouquet of m-spheres. Then $H^0(L) = \mathbb{Z}$, $H^l(L) = 0$ for $l \neq 0, m, 2m$, and $H^m(L)$ is free abelian. Furthermore $H^{2m}(L) \simeq \mathbb{Z}/2\mathbb{Z}$ if $m = 1$ and L is non-orientable, otherwise $H^{2m}(L) \simeq \mathbb{Z}$.
b) Suppose that $n = 2m + 2, m \geq 1$ and that F_+ or F_- has the homotopy type of a bouquet of m-spheres. Then $H^0(K) = \mathbb{Z}$, $H^l(K) = 0$ for $l \neq 0, m, 2m$, and $H^m(K)$ is free abelian. Furthermore $H^{2m}(K) \simeq \mathbb{Z}/2\mathbb{Z}$ if $m = 1$ and K is non-orientable, otherwise $H^{2m}(K) \simeq \mathbb{Z}$.

Proof. We know that $K \neq \emptyset$, otherwise F_+ and F_- are compact which gives the wrong homology.
a) The exact sequence

$$0 \to H^0(L) \to H^0(F_+) \oplus H^0(F_-) \to H^0(K)$$

shows that L is connected. This implies the statement for $m = 1$.
In the case $m \geq 2$ we know that F_+ and F_- are simply connected, hence orientable. So we have for $0 < l < 2m$ an exact sequence

$$H^{l-1}(F_+) \to H_{2m-l}(F_-) \to H^l(L) \to H^l(F_+) \to H_{2m-1-l}(F_-)$$

because $H_{2m-l}(F_-) \simeq H^l(F_-, \partial F_-)$.
For $l \neq m$ we deduce $H^l(L) = 0$. For $l = m$ we obtain

$$0 \to H_m(F_-) \to H^m(L) \to H^m(F_+) \to 0,$$

so $H^m(L)$ is free abelian. Of course, $H^{2m}(L) \simeq \mathbb{Z}$.
b) Assume that the hypothesis is true for F_+. Again, F_+ is orientable if $m \geq 2$. Note that $H_{2m-l}(F_+) \simeq H^{l+1}(F_+, \partial F_+)$.

Suppose first that F_+ is orientable. Then we have an exact sequence

$$H^0(F_+) \to H^0(K) \to H_{2m}(F_+).$$

Since $H_{2m}(F_+) = 0$ we obtain that K is connected.

If F_+ is non-orientable we have that $m = 1$, and the universal covering of F_+ is contractible. Therefore the orientation covering of F_+ has the homotopy type of a bouquet of 1-spheres, too. We conclude as before that its boundary is connected. So K is connected, too.

So we must only look at the case $m \geq 2$. For $0 < l < 2m$, we have an exact sequence

$$H_{2m+1-l}(F_+) \to H^l(F_+) \to H^l(K) \to H_{2m-l}(F_+) \to H^{l+1}(F_+)$$

For $l \neq m$ we have

$$H^l(F_+) = H_{2m-l}(F_+) = 0$$

hence $H^l(K) = 0$.

For $l = m$ we have an exact sequence

$$0 \to H^m(F_+) \to H^m(K) \to H_m(F_+) \to 0$$

which implies that $H^m(K)$ is free abelian. □

Example 3.4: a) $g : (\mathbb{C}^{m+1}, 0) \to (\mathbb{C}, 0)$ holomorphic with isolated singularity,

$$X := \mathbb{C}^{m+1} \cap \{Im\, g = 0\}, \quad f := Re\, g, \text{ and } \quad n = 2m + 1.$$

We obtain that $L := S_\epsilon \cap \{Im\, g = 0\}$ is a compact manifold of dimension $2m$,

$$H^0(L) = H^{2m}(L) = \mathbb{Z},$$

$H^m(L)$ free abelian of rank 2μ, $\mu =$ Milnor number of g.

b) $X = \mathbb{C}^{m+1}, f = Im\, g$, which leads with K instead of L to the same result as before, because the Milnor fibres of f and g have the same homotopy type. See Lemma 5.1 below.

4. Euler characteristic of the real Milnor fibre

Using resolution of singularities we can calculate the Euler characteristic of the Milnor fibre(s).

In the situation of section 2, we can put $Y := X \cap \{f_1 = \ldots = f_{k-1} = 0\}$. Then the Milnor fibres of $f_k : (Y, 0) \to (\mathbb{R}, 0)$ coincide with the one of $f : (X, 0) \to (\mathbb{R}^k, 0)$, so we can reduce to the case $k = 1$ with $F_+ \approx F_-$. So it is sufficient to look at the case $k = 1$ (cf. Example 3.4a).

Let us assume $k = 1$. Choose an embedded resolution $\pi : X' \to X$ of $f^{-1}(\{0\}) \subset X$. Then $(f \circ \pi)^{-1}(\{0\})$ is a divisor with normal crossing, it has a natural stratification. Let S_{li}, l being the codimension of the stratum, be the strata contained in $\pi^{-1}(\{0\})$. Locally at a point of this stratum, $f \circ \pi = \varepsilon x_1^{\nu_1} \cdot \ldots \cdot x_l^{\nu_l}$ with respect to suitable local coordinates, $\varepsilon = \pm 1$.

Put:

$\alpha_{li} := 2^{l-1}$ if there is a j such that ν_j is odd,

$\alpha_{li} := 2^l$ if $\nu_1, \ldots, \nu_l$ are even, $\varepsilon = 1$,

$\alpha_{li} := 0$ if $\nu_1, \ldots, \nu_l$ are even, $\varepsilon = -1$.

Theorem 4.1: $\chi(F_+) = \sum_{l,i} \alpha_{li}(-1)^{n-l}\chi(S_{li})$.

Proof. (cf. [CF] in the case $X = \mathbb{R}^n$) Let U_l be a suitable closed neighbourhood of the union of strata of codimension $\geq l$. More precisely, put

$$U_l := \{x \in X' \mid \psi_l(x) \leq \epsilon_l\},$$

where $\psi_l : X' \to [0, \infty[$ is a real analytic function whose zero set is the union of strata of codimension $\geq l$, and where $0 < \epsilon_1 \ll \epsilon_2 \ll \ldots \ll \epsilon_n \ll 1$, and suppose $0 < t \ll \epsilon_1$. Put $U^l := U_l \cup \ldots \cup U_n$. Then each connected component of $U_l \setminus U^{l+1}$ is the total space of a topological fibre bundle over $S_{li} \setminus U^{l+1}$, the fibre being the normal slice with respect to S_{li} for some i. Note that $S_{li} \setminus U^{l+1}$ has the same homotopy type as S_{li}. The normal slice N of S_{li} at p is homeomorphic to $\mathbb{R}^l$. Near p we can write $f \circ \pi$ as above. Then

$$N \cap \{f \circ \pi = t\} = \{x \in \mathbb{R}^l \,|\, \varepsilon x_1^{\nu_1} \cdot \ldots \cdot x_l^{\nu_l} = t\}, \quad 0 < t \ll 1.$$

If there is a j such that ν_j is odd, we may assume $j = l$, then we can write the right hand side as the graph of a function defined on $(\mathbb{R}^*)^{l-1}$. This set is the disjoint union of 2^{l-1} contractible components.

If all ν_j are even, $\varepsilon = -1$, the set is empty.

If all ν_j are even, $\varepsilon = 1$, we get the disjoint union of two graphs of functions defined on the same set as above, so we obtain 2^l contractible components.

Therefore the Euler characteristic of $N \cap \{f \circ \pi = t\}$, $t > 0$, is α_{li}.

Now

$$F_+ \sim D_\epsilon \cap X \cap \{f > 0\} \sim \pi^{-1}(D_\epsilon \cap X \cap \{f > 0\})$$

If we vary ϵ (resp. $\epsilon_1, \ldots, \epsilon_n$) we see that

$$\pi^{-1}(D_\epsilon \cap X \cap \{f > 0\}) \sim U^1 \cap \{f \circ \pi > 0\} \sim U^1 \cap \{f \circ \pi = t\}$$

Furthermore, $U^1 = \dot{\bigcup} (U_l - U^{l+1})$, hence

$$\chi(F_+) = \chi(\{f \circ \pi = t\} \cap U^1) = \sum_l \chi_c(\{f \circ \pi = t\} \cap (U_l \setminus U^{l+1}))$$

$$= \sum_{l,i} \alpha_{li} \chi_c(S_{li}) = \sum_{l,i} \alpha_{li} (-1)^{n-l} \chi(S_{li})$$

Here χ_c is the Euler characteristic with compact support. □

It is easier to calculate $\chi(F_+) + \chi(F_-)$:

Corollary 4.2: $\chi(F_+) + \chi(F_-) = \sum_{l,i} 2^l (-1)^{n-l} \chi(S_{li})$, and so, if $\chi(F_+) = \chi(F_-)$ (in particular if n is even), then

$$\chi(F_+) = \sum_{l,i} 2^{l-1} (-1)^{n-l} \chi(S_{li}).$$

The first statement of the corollary can also be proved directly, without using the local description of $f \circ \pi$: note that $(\mathbb{R}^*)^l$ has 2^l contractible components.

By the way, we can calculate $\chi(K)$ and $\chi(L)$ using the same resolution:

Let us denote by S'_{li} those strata S_{li} which are contained in the strict transform of $f^{-1}(0)$, i.e., in the closure of $\pi^{-1}(f^{-1}(\{0\}) \setminus \{0\})$, S''_{li} the remaining ones. Then:

$$\chi(K) = \sum 2^{l-1} (-1)^{n-l} \chi(S'_{li}),$$

$$\chi(L) = \sum 2^{l-1} (-1)^{n-l} \chi(S'_{li}) + \sum 2^l (-1)^{n-l} \chi(S''_{li})$$

which agrees with the formula $\chi(L) + \chi(K) = \chi(F_+) + \chi(F_-)$ proved before (Theorem 3.1).

In the case of L, note that in the normal slice we have to look at $N \setminus \pi^{-1}(0)$ which differs from $N \setminus (f \circ \pi)^{-1}(0)$ if we are at a point of the strict transform of $f = 0$: then we have 2^{l-1} instead of 2^l contractible components.

Using the formula for $\chi(K)$ we obtain an easier formula for $\chi(F_+)$ if n is even: Then

$$\chi(F_+) = \chi(F_-) = \sum 2^{l-2}(-1)^{n-l}\chi(S'_{li}),$$

because $\chi(K) = 2\chi(F_+)$.

5. Comparison of Milnor fibres of mappings (resp. functions)

There is another connection between the cases $k \geq 2$ and $k = 1$ in section 2 (resp. 3):

Let us take up the assumptions of section 2 (in particular, $k \geq 2$) and write $\chi(f)$ instead of $\chi(F)$. Similarly in 3: $\chi(f)_+$ instead of $\chi(F_+)$.

Lemma 5.1: For $0 < \alpha \ll \epsilon \ll 1$, the inclusion of $X \cap D_\epsilon \cap \{f_1 = \ldots = f_{k-1} = 0, f_k = \alpha\}$ in $X \cap D_\epsilon \cap f_k^{-1}(\alpha)$ is a homotopy equivalence.

Proof. Let ϕ, ψ be defined as in the proof of Lemma 1.2. Compare

$$X \cap D_\epsilon \cap \{\|f\| \leq \alpha, f_k > 0\}$$

with $X \cap B_\epsilon \cap \{f_k > 0\}$. Choose a vector field v such that, on $X \cap D_\epsilon \cap \{\|f\| \geq \alpha\}$:

$$d\phi_x(v(x)) = 1, d\psi_x(v(x)) > 0,$$

and near $f_k = 0 : (df_k)_x(v(x)) = 0$. This is possible: assume that we have a point p such that $d\psi_p = \lambda d\phi_p$ with $\lambda < 0$, we get a contradiction because of the Curve Selection Lemma. Similarly, suppose that near $f_k = 0$ there is a p, $\|f(p)\| \geq \alpha$, such that $d\psi_p = \lambda d\phi_p + \mu(df_k)_p$ with $\lambda \leq 0$ we would get also such a point with $f_k(p) = 0$, which contradicts the Curve Selection Lemma. So we obtain that

$$X \cap D_\epsilon \cap \{\|f\| \leq \alpha, f_k > 0\} \sim X \cap D_\epsilon \cap \{f_k > 0\}$$

Moreover, $f : X \cap D_\epsilon \cap \{\|f\| \leq \alpha, f_k > 0\} \to \{t \in B_\alpha \,|\, t_k > 0\}$ is a trivial fibration, so

$$X \cap D_\epsilon \cap \{\|f\| \leq \alpha, f_k > 0\} \sim X \cap D_\epsilon \cap \{f = (0, \ldots, 0, \alpha)\}$$

Now we can find a vector field w on $\{f_k > 0\}$ such that, on $X \cap D_\epsilon \cap \{f_k > 0\}$:

$$(df_k)_x(w(x)) = 1, d\psi_x(w(x)) > 0,$$

because of the Curve Selection Lemma. Therefore

$$X \cap D_\epsilon \cap \{f_k > 0\} \sim X \cap D_\epsilon \cap \{0 < f_k \leq \alpha\}.$$

Finally, X has an isolated singularity at 0, so $f_k : X \cap D_\epsilon \cap \{0 < f_k \leq \alpha\} \to]0, \alpha]$ is a trivial fibration, hence

$$X \cap D_\epsilon \cap \{0 < f_k \leq \alpha\} \sim X \cap D_\epsilon \cap \{f_k = \alpha\}.$$

□

In the case of $X = \mathbb{R}^n$ this is a consequence of a conjecture by J.Milnor [M], p. 100, see also [ADD].

Corollary 5.2: $\chi(f) = \chi(f_1)_+ = \chi(f_1)_- = \ldots = \chi(f_k)_+ = \chi(f_k)_-$.

Now let us turn to the special case $X = \mathbb{R}^n$. Then we have the following formula:

Theorem 5.3: (G.Khimshiashvili [K]) If $k=1$, $\chi(f)_+ = 1-(-1)^n deg_0 \nabla f$, where ∇f is the gradient of f and $deg_0\nabla f$ is the topological degree of $\frac{\nabla f}{|\nabla f|}: S_\epsilon \to S_1$.

Replacing f by $-f$ we obtain that $\chi(f)_- = 1 - deg_0\nabla f$
Note that L is a sphere in our case. This implies altogether:

Corollary 5.4: ([ADD])
a) $\chi(f) = 1 - deg_0\nabla f_1 = \ldots = 1 - deg_0 \nabla f_k$.
b) If n is odd, $deg_0\nabla f_1 = \ldots = deg_0\nabla f_k = 0$, so $\chi(f)=1$.

Proof. b) By the Corollary before, $\chi(f_i)_+ = \chi(f_i)_-$, so according to Khimshiashvili: $deg_0\nabla f_i = 0$, so $\chi(f) = \chi(f_i)_+ = 1$. □

6. Global analogue

a) Now let us pass to the global case. Let X be a compactifiable real analytic (e.g. a real algebraic) subspace of $\mathbb{R}^N$ which is purely n-dimensional, $f: X \to \mathbb{R}^k$ a compactifiable real analytic mapping. Let C be the set of critical points of f; recall that singular points of X are automatically critical points of f. Assume that
(i) the set of critical points of f which are contained in $f^{-1}(\{0\})$ is compact,
(ii) for $0 < \alpha \ll 1$ the set $C \cap f^{-1}(B_\alpha \setminus \{0\})$ is closed in X, i.e. there is no convergent sequence $(p_n) \to p^*$ of critical points of f such that $f(p_n) \neq 0$ for all n, $p^* \in X$, $f(p^*) = 0$.

Then we get that for $0 < \alpha \ll \frac{1}{R} \ll 1$ the mapping

$$f: X \cap D_R \cap f^{-1}(B_\alpha \setminus \{0\}) \to B_\alpha \setminus \{0\}$$

is a locally trivial fibration:
Assume $R \gg 0$. Then X is smooth along $X \cap S_R$, S_R intersects X transversally, and $f|X\cap S_R$ has no critical point which is mapped to 0. Therefore $f|X \cap S_R$ has no critical points above B_α. Finally, f has no critical points in $X \cap D_R \cap f^{-1}(B_\alpha \setminus \{0\})$.

As at the beginning of section 4 we may reduce to the case $k=1$. So assume $k=1$; then we get fibres F_+ and F_-.
Let us fix a compactification $\bar f : \bar X \to \mathbb{R}$ and let $\pi : \bar X' \to \bar X$ be an embedded resolution of $\bar f^{-1}(0) \cup C \cup X_\infty \subset \bar X$ where $X_\infty := \bar X \setminus X$. We can achieve that

$$\pi : \pi^{-1}(f^{-1}(\{0\}) \setminus C) \to f^{-1}(\{0\}) \setminus C$$

is an isomorphism. Put $X' := \pi^{-1}(X)$. We have a natural stratification of $(\bar f \circ \pi)^{-1}(0)$ such that $\pi^{-1}(f^{-1}(\{0\}))$ is a union of strata. Locally at a point of such a stratum of codimension l in $\bar X'$, $\bar f \circ \pi = \varepsilon x_1^{\nu_1} \cdot \ldots \cdot x_\lambda^{\nu_\lambda}$, $\lambda \le l$, with respect to suitable local coordinates, $\varepsilon = \pm 1$.
Put:
$\alpha_{li} := 2^{l-1}$ if there is a j such that ν_j is odd,
$\alpha_{li} := 2^l$ if $\nu_1, \ldots, \nu_\lambda$ are even, $\varepsilon = 1$,
$\alpha_{li} := 0$ if $\nu_1, \ldots, \nu_\lambda$ are even, $\varepsilon = -1$.
Then we have, similarly as in section 4:

Theorem 6.1: $\chi(F_+) = \sum_{l,i} \alpha_{li}(-1)^{l+1}\chi(S_{li})$, where the sum extends over all strata contained in $\pi^{-1}(f^{-1}(\{0\}))$.

Proof. Let U_l be a suitable closed neighbourhood of the union of $(\bar{X}' \setminus X') \cap (\bar{f} \circ \pi)^{-1}(\{0\})$ and all strata of $\pi^{-1}(f^{-1}(\{0\}))$ of codimension $\geq l$, $U^l := U_l \cup \ldots U_{n+1}$. Then

$$\chi(F_+) = \chi((\bar{f} \circ \pi)^{-1}(\{t\}) \setminus U_{n+1}) = (-1)^{n-1}\chi_c((\bar{f} \circ \pi)^{-1}(\{t\}) \setminus U_{n+1}),$$

and

$$\chi_c((\bar{f} \circ \pi)^{-1}(\{t\}) \setminus U_{n+1}) = \sum_{l=1}^{n} \chi_c((\bar{f} \circ \pi)^{-1}(\{t\}) \cap U_l \setminus U^{l+1})$$

We continue similarly as in the proof of Theorem 4.1. □

We have a similar formula for $K := f^{-1}(0) \cap S_R, R \gg 0$:
$\chi(K) = \sum_{l,i} 2^{l-1}(-1)^{n-l}\chi(S_{li})$, where the sum extends to all strata contained in

$$(\bar{X}' \setminus X') \cap \overline{X' \cap (\bar{f} \circ \pi)^{-1}(0)}.$$

If n is even, this implies a simpler formula for $\chi(F_+) = \chi(F_-)$ because

$$\chi(K) = 2\chi(F_+) = 2\chi(F_-):$$

$$\chi(K) = \chi(\partial F_+) = \chi(F_+) - \chi(F_+, \partial F_+) = 2\chi(F_+)$$

because of Poincaré duality. Similarly for F_-.

b) The fibration studied in **a**) is not so natural because it ignores vanishing cycles at infinity.

So let us suppose instead that X is a compactifiable real analytic space, $f : X \to \mathbb{R}^k$ compactifiable real analytic, and that f is a submersive mapping between smooth spaces above $B_\alpha \setminus \{0\}$ for $0 < \alpha \ll 1$. Let $\bar{f} : \bar{X} \to \mathbb{R}^k$ be a compactification of f. Put $X_\infty := \bar{X} \setminus X$. We can stratify $\bar{X}$ and $\mathbb{R}^k$ subanalytically so that X is a union of strata and $\bar{f}$ is a stratified mapping.

Let T be a stratum of $\mathbb{R}^k$ such that $T \neq \{0\}$, $0 \in \overline{T}$. Because of Thom's first isotopy lemma we know that $f : f^{-1}(T) \to T$ defines a locally trivial fibration.

We want to calculate the Euler characteristic of the typical fibre F of this fibration. Since T is subanalytic we can find by the Curve Selection Lemma a real analytic curve $p :]-c, c[\to \mathbb{R}^k$ such that $p(0) = 0$, $p(t) \in T$ for $t > 0$. We apply base change to f with respect to p. In this way we reduce to the case $k = 1$. We need only to look at F_+.

So let us look at the case $k = 1$. Then we obtain that f is a locally trivial fibration above $B_\alpha \setminus \{0\}$, we have two typical fibres F_+, F_-. Let π and α_{li} be defined as in subsection a).

Theorem 6.2: $\chi(F_+) = \sum_{l,i} \alpha_{li}(-1)^{l+1}\chi(S_{li})$, where the sum extends over all strata of $(\bar{f} \circ \pi)^{-1}(\{0\})$ which are not contained in the closure of $\pi^{-1}(X_\infty \setminus (\bar{f} \circ \pi)^{-1}(\{0\}))$.

Proof. Analogous to the one of Theorem 6.1. □

Again we can find a simpler formula if n is even. First fix t, $0 < t \leq \alpha$. For $R \gg \frac{1}{\alpha}$ we have that $f^{-1}(\{t\}) \cap D_R$ is a deformation retract of $f^{-1}(\{t\})$. Now we have a formula for the boundary:

$$\chi(f^{-1}(\{t\}) \cap S_R) = \sum_{l,i} \alpha_{li}(-1)^{n-l}\chi(S_{li}),$$

where the sum extends over all strata of $(\bar{f} \circ \pi)^{-1}(\{0\})$ which are contained in the closure of $\pi^{-1}(X_\infty \setminus (\bar{f} \circ \pi)^{-1}(\{0\}))$.

If n is even we have that $\chi(f^{-1}(\{t\}) \cap S_R) = 2\chi(f^{-1}(\{t\}) \cap D_R) = 2\chi(F_+)$.

c) Assume that hypothesis (i) of part a) as well as the hypothesis of b) are given. Then we have hypothesis (ii) of part a), too. The fibrations in a) and b) may be different due to the presence of vanishing cycles at infinity, as shown by the real version of the Broughton example. Here is a different example where the fibres F_+ and F_- in b) have a different Euler characteristic:

Put $X := \mathbb{R}^2$, $f : X \to \mathbb{R}$: $f(x,y) := -x(xy^2-1)$.

Then $f^{-1}(\{0\})$ is the disjoint union of $\{x=0\}$, $\{y<0, x=\frac{1}{y^2}\}$, $\{y>0, x=\frac{1}{y^2}\}$;

for $t>0$,

$$f^{-1}(\{t\}) = \left\{x \geq t, y = \pm\frac{\sqrt{x-t}}{x}\right\};$$

for $t<0$, $f^{-1}(\{t\})$ is the disjoint union of $\left\{x>0, y=\frac{\sqrt{x-t}}{x}\right\}$, $\left\{x>0, y=-\frac{\sqrt{x-t}}{x}\right\}$ and $\left\{t \leq x < 0, y = \pm\frac{\sqrt{x-t}}{x}\right\}$.

So $\chi(f^{-1}(\{0\}))=3$, $\chi(f^{-1}(\{t\}))=1$ for $t>0$ and $\chi(f^{-1}(\{t\}))=3$ for $t<0$. Note that f has no critical points, so the fibre in a) has the same Euler characteristic as $f^{-1}(\{0\})$. Altogether, 0 is not a critical value but an atypical one.

References:

[ADD] R. Araújo dos Santos, D. Dreibelbis, N. Dutertre: Topology of the real Milnor fiber for isolated singularities. In: Contemp. Math. **569**, 2012, pp. 67-75. DOI: 10.1090/conm/569/11242

[CF] G. Comte, G. Fichou: Grothendieck ring of semialgebraic formulas and motivic real Milnor fibres. arXiv: 1111.3181v4 (2012)

[CL] P.T. Church, K. Lamotke: Nontrivial polynomial isolated singularities. Indag. Math. **37**, 149-154 (1975) DOI: 10.1016/1385-7258(75)90027-X

[H] H.A. Hamm: On the Euler characteristic of the real Milnor fibre(s). Oberwolfach Reports (to appear)

[K] G. Khimshiashvili: On the local degree of a smooth map. Soobshch. Akad. Nauk Gruz. SSR **85**, 309-311 (1977)

[M] J. Milnor: Singular points of complex hypersurfaces. Princeton Univ. Press: Princeton 1968.

[S] E.H. Spanier: Algebraic topology. McGraw-Hill: New York 1966.

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LIGHTLIKE HYPERSURFACES ALONG SPACELIKE SUBMANIFOLDS IN DE SITTER SPACE

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Abstract. We consider the singularities of lightlike hypersurfaces along spacelike submanifolds with general codimension in de Sitter space. As an application of the theory of Legendrian singularities, we investigate the geometric meanings of the singularities of lightlike hypersurfaces from the viewpoint of the contact of spacelike submanifolds with de Sitter lightcones.

1. Introduction

One of the important objects in theoretical physics is the notion of lightlike hypersurfaces because they provide good models for different types of horizons [3, 5, 20, 23]. The lightlike hypersurfaces are constructed as ruled hypersurfaces along spacelike submanifolds whose rulings are the lightlike geodesics. A lightlike hypersurface is also called a *light sheet* in theoretical physics (cf., [2]), which plays a principal role in the quantum theory of gravity. In this paper, we consider the singularities of lightlike hypersurfaces along spacelike submanifolds in de Sitter space which is one of the Lorentz space forms. There are three kinds of Lorentz space forms: Lorentz-Minkowski space is a flat Lorentz space form, de Sitter space is a positively curved one, and anti-de Sitter space is a negatively curved one.

On the other hand, tools in the theory of singularities have proven to be useful in the description of geometrical properties of submanifolds immersed in different ambient spaces, from both the local and global viewpoint [6, 7, 9, 10, 11, 13, 16, 18]. The natural connection between geometry and singularities relies on the basic fact that the contacts of a submanifold with the models of the ambient space can be described by means of the analysis of the singularities of appropriate families of contact functions, or equivalently, of their associated Legendrian maps ([1, 21, 24]). When working in a Lorentz space form, the properties associated to the contacts of a given submanifold with lightcones have a special relevance. In [4, 8, 11, 17], a framework for the study of spacelike submanifolds with codimension two in Lorentz space forms was constructed, and a Lorentz invariant concerning their contacts with models related to lightlike hyperplanes was discovered. The geometry described in this framework is called the *lightlike geometry* of spacelike submanifolds with codimension two. By using the invariants of lightlike geometry, the singularities of lightlike hypersurfaces along spacelike submanifolds with codimension two in Lorentz-Minkowski space or de Sitter space were investigated in [10, 12, 16]. However, the situation is rather complicated for the general codimensional case. The main difference from the Euclidean space (or, Hyperbolic space) case is the fiber of the canal hypersurface of a spacelike submanifold is neither connected nor compact. In order to avoid the above difficulty, we arbitrarily choose a timelike future directed unit normal vector field along the spacelike submanifold, which always exists for an orientable submanifold (cf., [13, 14, 15]). Then we construct the unit spherical normal bundle relative to the above timeline unit normal vector field, which can be considered as a codimension two spacelike canal submanifold of the ambient Lorentz space form.

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Therefore, we can apply the idea of the lightlike geometry of spacelike submanifolds with codimension two in Lorentz space-forms. Recently, we have applied this framework and investigated the geometric meanings of the singularities of lightlike hypersurfaces along spacelike submanifolds in Lorentz-Minkowski space or anti-de Sitter space from the viewpoint of the theory of Legendrian singularities [14, 15]. In this paper, we consider spacelike submanifolds with general codimensions in de Sitter space applying an idea similar to [14, 15].

In §2 the basic notions of Lorentz-Minkowski space are described. We explain the differential geometry of spacelike submanifolds with general codimension in de Sitter space in §3. The notion of lightlike hypersurfaces is introduced in §4 and investigated the basic properties. In §5 we investigate the geometric meanings of the singularities of lightlike hypersurfaces in de Sitter space from the viewpoint of the theory of contact with de Sitter lightcones and the theory of Legendrian singularities. We review the classification result of Kasedou [17] on singularities of lightlike hypersurfaces along spacelike surfaces in de Sitter 4-space in §5.

2. Basic notions

In this section we prepare basic notions on Lorentz-Minkowski space. Let $\mathbb{R}^{n+1}$ be an $(n+1)$-dimensional cartesian space. For any vectors $\boldsymbol{x} = (x_0, x_1, \dots, x_n)$, $\boldsymbol{y} = (y_0, y_1, \dots, y_n) \in \mathbb{R}^{n+1}$, the *pseudo scalar product* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined by $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_0 y_0 + \sum_{i=1}^{n} x_i y_i$. The space $(\mathbb{R}^{n+1}, \langle , \rangle)$ is called *Lorentz-Minkowski* $(n+1)$*-space* and denoted by $\mathbb{R}^{n+1}_1$. We say that a vector $\boldsymbol{x}$ in $\mathbb{R}^{n+1}_1 \setminus \{\boldsymbol{0}\}$ is *spacelike*, *lightlike* or *timelike* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0, = 0$ or < 0 respectively. The norm of the vector $\boldsymbol{x} \in \mathbb{R}^{n+1}_1$ is defined by $\|\boldsymbol{x}\| = \sqrt{|\langle \boldsymbol{x}, \boldsymbol{x} \rangle|}$. We define a *hyperplane with pseudo normal* $\boldsymbol{v}$ by $HP(\boldsymbol{v}, c) = \{ \boldsymbol{x} \in \mathbb{R}^{n+1}_1 \mid \langle \boldsymbol{x}, \boldsymbol{v} \rangle = c \}$, where $\boldsymbol{v} \in \mathbb{R}^{n+1}_1 \setminus \{\boldsymbol{0}\}$ and c is a real number. We call $HP(\boldsymbol{v}, c)$ a *spacelike hyperplane*, a *timelike hyperplane* or a *lightlike hyperplane* if $\boldsymbol{v}$ is timelike, spacelike or lightlike respectively. We have the following three kinds of pseudo-spheres in $\mathbb{R}^{n+1}_1$: The *hyperbolic* n*-space* is defined by

$$H^n(-1) = \{ \boldsymbol{x} \in \mathbb{R}^{n+1}_1 | \ \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1 \},$$

the *de Sitter* n*-space* by

$$S^n_1 = \{ \boldsymbol{x} \in \mathbb{R}^{n+1}_1 | \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1 \ \}$$

and the (*open*) *lightcone* by

$$LC^* = \{ \boldsymbol{x} \in \mathbb{R}^{n+1}_1 \setminus \{\boldsymbol{0}\} | \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0 \ \}.$$

We also define $LC_{\boldsymbol{\lambda}_0} = \{ \boldsymbol{x} \in \mathbb{R}^{n+1}_1 \ | \langle \boldsymbol{x} - \boldsymbol{\lambda}_0, \boldsymbol{x} - \boldsymbol{\lambda}_0 \rangle = 0 \ \}$ which is called a *lightcone with the vertex* $\boldsymbol{\lambda}_0$.

For any $\boldsymbol{x}^1, \boldsymbol{x}^2, \dots, \boldsymbol{x}^n \in \mathbb{R}^{n+1}_1$, we define a vector $\boldsymbol{x}^1 \wedge \boldsymbol{x}^2 \wedge \cdots \wedge \boldsymbol{x}^n$ by

$$\boldsymbol{x}^1 \wedge \boldsymbol{x}^2 \wedge \cdots \wedge \boldsymbol{x}^n = \begin{vmatrix} -\boldsymbol{e}_0 & \boldsymbol{e}_1 & \cdots & \boldsymbol{e}_n \\ x^1_0 & x^1_1 & \cdots & x^1_n \\ x^2_0 & x^2_1 & \cdots & x^2_n \\ \vdots & \vdots & \cdots & \vdots \\ x^n_0 & x^n_1 & \cdots & x^n_n \end{vmatrix},$$

where $\boldsymbol{e}_0, \boldsymbol{e}_1, \dots, \boldsymbol{e}_n$ is the canonical basis of $\mathbb{R}^{n+1}_1$ and $\boldsymbol{x}^i = (x^i_0, x^i_1, \dots, x^i_n)$.

3. Differential geometry on spacelike submanifolds in de Sitter space

In [16] Kasedou has investigated differential geometry of spacelike submanifolds in de Sitter space from the viewpoint of contact with de Sitter hyperhorospheres. Here we construct another framework on differential geometry of spacelike submanifolds in de Sitter space. Let $\mathbb{R}^{n+1}_1$ be an oriented and time-oriented space. We choose $\boldsymbol{e}_0 = (1, 0, \dots, 0)$ as a future timelike vector field. We consider de Sitter n-space $S^n_1 \subset \mathbb{R}^{n+1}_1$. Let $\boldsymbol{X} : U \longrightarrow S^n_1$ be a spacelike embedding of codimension k, where $U \subset \mathbb{R}^s$ $(s+k=n)$ is an open subset. We also write $M = \boldsymbol{X}(U)$ and identify M and U through the embedding $\boldsymbol{X}$ as usual. Since M is a spacelike submanifold with codimension $k+1$ in $\mathbb{R}^{n+1}_1$, $N_p(M)$ is a $(k+1)$-dimensional Lorentzian subspace of $T_p\mathbb{R}^{n+1}_1$ (cf.,[22]). On the pseudo-normal space $N_p(M)$, we have two kinds of k-dimensional pseudo spheres:

$$\begin{aligned} N_p(M;-1) &= \{\boldsymbol{v} \in N_p(M) \mid \langle \boldsymbol{v},\boldsymbol{v}\rangle = -1\ \} \\ N_p(M;1) &= \{\boldsymbol{v} \in N_p(M) \mid \langle \boldsymbol{v},\boldsymbol{v}\rangle = 1\ \}, \end{aligned}$$

so that we have two unit pseudo-spherical normal bundles over M:

$$N(M;-1) = \bigcup_{p\in M} N_p(M;-1) \text{ and } N(M;1) = \bigcup_{p\in M} N_p(M;1).$$

Since $M = \boldsymbol{X}(U)$ is spacelike, $\boldsymbol{e}_0 \notin T_pM$. For any $\boldsymbol{v} \in T_p\mathbb{R}^{n+1}_1|M$, we have $\boldsymbol{v} = \boldsymbol{v}_1 + \boldsymbol{v}_2$, where $\boldsymbol{v}_1 \in T_pM$ and $\boldsymbol{v}_2 \in N_p(M)$. If $\boldsymbol{v}$ is timelike, then $\boldsymbol{v}_2$ is timelike. Let

$$\pi_{N(M)} : T\mathbb{R}^{n+1}_1|M \longrightarrow N(M)$$

be the canonical projection. Then $\pi_{N(M)}(\boldsymbol{e}_0)$ is a future directed timelike normal vector field along M. If we project $\pi_{N(M)}(\boldsymbol{e}_0)$ onto the normal space of T_pM in $T_pS^n_1$, then we have a future directed unit timelike normal vector field in TS^n_1 along M (even globally). We now arbitrarily choose a future directed unit timelike normal vector field $\boldsymbol{n}^T(u) \in N_p(M;-1) \cap T_pS^n_1$, where $p = \boldsymbol{X}(u)$. Therefore we have the pseudo-orthonormal compliment $(\langle \boldsymbol{n}^T(u)\rangle_{\mathbb{R}})^{\perp}$ in $N_p(M) \cap T_pS^n_1$ which is a $(k-1)$-dimensional subspace of $N_p(M)$. We define a $(k-2)$-dimensional spacelike unit sphere in $N_p(M)$ by $N^{dS}_1(M)_p[\boldsymbol{n}^T] = \{\boldsymbol{\xi} \in N_p(M;1) \mid \langle \boldsymbol{\xi}, \boldsymbol{n}^T(p)\rangle = \langle \boldsymbol{\xi}, \boldsymbol{X}(u)\rangle = 0\ \}$. Then we have a *spacelike unit* $(k-2)$*-spherical bundle over* M *with respect to* $\boldsymbol{n}^T$ defined by

$$N^{dS}_1(M)[\boldsymbol{n}^T] = \bigcup_{p\in M} N^{dS}_1(M)_p[\boldsymbol{n}^T].$$

Since we have $T_{(p,\xi)}N^{dS}_1(M)[\boldsymbol{n}^T] = T_pM \times T_\xi N^{dS}_1(M)_p[\boldsymbol{n}^T]$, we have the canonical Riemannian metric on $N^{dS}_1(M)[\boldsymbol{n}^T]$ which is denoted by $(G_{ij}(p,\boldsymbol{\xi}))_{1\leqslant i,j\leqslant n-2}$.

On the other hand, we define a map $\mathbb{LG}(\boldsymbol{n}^T) : N^{dS}_1(M)[\boldsymbol{n}^T] \longrightarrow LC^*$ by

$$\mathbb{LG}(\boldsymbol{n}^T)(u,\boldsymbol{\xi}) = \boldsymbol{n}^T(u) + \boldsymbol{\xi},$$

which we call the *de Sitter lightcone Gauss image* of $N^{dS}_1(M)[\boldsymbol{n}^T]$. This map leads us to the notions of curvatures. Let $T_{(p,\xi)}N^{dS}_1(M)[\boldsymbol{n}^T]$ be the tangent space of $N^{dS}_1(M)[\boldsymbol{n}^T]$ at $(p,\boldsymbol{\xi})$. Under the canonical identification

$$(\mathbb{LG}(\boldsymbol{n}^T)^*T\mathbb{R}^{n+1}_1)_{(p,\boldsymbol{\xi})} = T_{(\boldsymbol{n}^T(p)+\boldsymbol{\xi})}\mathbb{R}^{n+1}_1 \equiv T_p\mathbb{R}^{n+1}_1,$$

we have

$$T_{(p,\boldsymbol{\xi})}N^{dS}_1(M)[\boldsymbol{n}^T] = T_pM \oplus T_\xi S^{k-2} \subset T_pM \oplus N_p(M) = T_p\mathbb{R}^{n+1}_1,$$

where $T_\xi S^{k-2} \subset T_\xi N_p(M) \equiv N_p(M)$ and $p = \boldsymbol{X}(u)$. Let

$$\Pi^t : \mathbb{LG}(\boldsymbol{n}^T)^*T\mathbb{R}^{n+1}_1 = TN_1(M)[\boldsymbol{n}^T] \oplus \mathbb{R}^{k+1} \longrightarrow TN^{dS}_1(M)[\boldsymbol{n}^T]$$

be the canonical projection. Then we have a linear transformation

$$S_\ell(\boldsymbol{n}^T)_{(p,\boldsymbol{\xi})} = -\Pi^t_{\mathbb{LG}(n^T)(p,\xi)} \circ d_{(p,\xi)}\mathbb{LG}(\boldsymbol{n}^T) : T_{(p,\xi)}N^{dS}_1(M)[\boldsymbol{n}^T] \longrightarrow T_{(p,\xi)}N^{dS}_1(M)[\boldsymbol{n}^T],$$

which is called the *de Sitter lightcone shape operator* of $N^{dS}_1(M)[\boldsymbol{n}^T]$ at $(p,\boldsymbol{\xi})$. Consider the eigenvalues of $S_\ell(\boldsymbol{n}^T)_{(p,\boldsymbol{\xi})}$, $(i=1,\ldots,n-2)$. Then we write $\kappa_\ell(\boldsymbol{n}^T)_i(p,\boldsymbol{\xi})$, $(i=1,\ldots,s)$ for the eigenvalues whose eigenvectors belong to T_pM and $\kappa_\ell(\boldsymbol{n}^T)_i(p,\boldsymbol{\xi})$, $(i=s+1,\ldots n-2)$ for the eigenvalues whose eigenvectors belong to the tangent space of the fiber of $N_1^{dS}(M)[\boldsymbol{n}^T]$. By exactly the same arguments as those in [13, 15], we have $\kappa_\ell(\boldsymbol{n}^T)_i(p,\boldsymbol{\xi}) = -1$, $(i = s+1, \dots n-2)$. We call $\kappa_\ell(\boldsymbol{n}^T)_i(p,\boldsymbol{\xi})$, $(i = 1,\dots,s)$ the *de Sitter lightcone principal curvatures* of M with respect to $(\boldsymbol{n}^T,\boldsymbol{\xi})$ at $p \in M$.

We deduce now the lightcone Weingarten formula. Since $\boldsymbol{X}$ is a spacelike embedding, we have a Riemannian metric (the *first fundamental form*) on $M = \boldsymbol{X}(U)$ defined by

$$ds^2 = \sum_{i=1}^{s} g_{ij} du_i du_j,$$

where $g_{ij}(u) = \langle \boldsymbol{X}_{u_i}(u), \boldsymbol{X}_{u_j}(u)\rangle$ for any $u \in U$. Let $\boldsymbol{n}^S$ be a local section of $N_1^{dS}(M)[\boldsymbol{n}^T]$. Clearly, the vectors $\boldsymbol{n}^T(u) \pm \boldsymbol{n}^S(u)$ are lightlike. Here we choose $\boldsymbol{n}^T + \boldsymbol{n}^S$ as a lightlike normal vector field along M. We define a mapping $\mathbb{LG}(\boldsymbol{n}^T,\boldsymbol{n}^S) : U \longrightarrow LC^*$ by

$$\mathbb{LG}(\boldsymbol{n}^T,\boldsymbol{n}^S)(u) = \boldsymbol{n}^T(u) + \boldsymbol{n}^S(u).$$

We call it the *lightcone Gauss image* of $M = \boldsymbol{X}(U)$ with respect to $(\boldsymbol{n}^T,\boldsymbol{n}^S)$. Under the identification of M and U through $\boldsymbol{X}$, we have the linear mapping provided by the derivative of the lightcone Gauss image $\mathbb{LG}(\boldsymbol{n}^T,\boldsymbol{n}^S)$ at each point $p \in M$,

$$d_p\mathbb{LG}(\boldsymbol{n}^T,\boldsymbol{n}^S) : T_pM \longrightarrow T_p\mathbb{R}^{n+1}_1 = T_pM \oplus N_p(M).$$

Consider the orthogonal projection $\pi^t : T_pM \oplus N_p(M) \to T_p(M)$. We define

$$d_p\mathbb{LG}(\boldsymbol{n}^T,\boldsymbol{n}^S)^t = \pi^t \circ d_p(\boldsymbol{n}^T + \boldsymbol{n}^S).$$

We call the linear transformation $S_p(\boldsymbol{n}^T,\boldsymbol{n}^S) = -d_p\mathbb{LG}(\boldsymbol{n}^T,\boldsymbol{n}^S)^t$ the $(\boldsymbol{n}^T,\boldsymbol{n}^S)$*-shape operator* of $M = \boldsymbol{X}(U)$ at $p = \boldsymbol{X}(u)$. Let $\{\kappa_i(\boldsymbol{n}^T,\boldsymbol{n}^S)(p)\}_{i=1}^s$ be the eigenvalues of $S_p(\boldsymbol{n}^T,\boldsymbol{n}^S)$, which are called the *lightcone principal curvatures with respect to* $(\boldsymbol{n}^T,\boldsymbol{n}^S)$ at $p = \boldsymbol{X}(u)$. Then we have a *lightcone second fundamental invariant with respect to* $(\boldsymbol{n}^T,\boldsymbol{n}^S)$ defined by

$$h_{ij}(\boldsymbol{n}^T,\boldsymbol{n}^S)(u) = \langle -(\boldsymbol{n}^T + \boldsymbol{n}^S)_{u_i}(u), \boldsymbol{X}_{u_j}(u)\rangle$$

for any $u \in U$. By the similar arguments to those in the proof of [11, Proposition 3.2], we have the following proposition.

Proposition 3.1. *Let $\{\boldsymbol{X},\boldsymbol{n}^T,\boldsymbol{n}^S_1,\dots,\boldsymbol{n}^S_{k-2}\}$ be a pseudo-orthonormal frame of $N(M)$ with $\boldsymbol{n}^S_{k-2} = \boldsymbol{n}^S$. Then we have the following lightcone Weingarten formula :*

(a) $\mathbb{LG}(\boldsymbol{n}^T,\boldsymbol{n}^S)_{u_i} = \langle \boldsymbol{n}^T_{u_i},\boldsymbol{n}^S\rangle(\boldsymbol{n}^T+\boldsymbol{n}^S) + \sum_{\ell=1}^{k-3}\langle(\boldsymbol{n}^T+\boldsymbol{n}^S)_{u_i},\boldsymbol{n}^S_\ell\rangle\boldsymbol{n}^S_\ell - \sum_{j=1}^{s} h_i^j(\boldsymbol{n}^T,\boldsymbol{n}^S)\boldsymbol{X}_{u_j}$

(b) $\pi^t \circ \mathbb{LG}(\boldsymbol{n}^T,\boldsymbol{n}^S)_{u_i} = -\sum_{j=1}^{s} h_i^j(\boldsymbol{n}^T,\boldsymbol{n}^S)\boldsymbol{X}_{u_j}$.

Here $\left(h_i^j(\boldsymbol{n}^T,\boldsymbol{n}^S)\right) = \left(h_{ik}(\boldsymbol{n}^T,\boldsymbol{n}^S)\right)\left(g^{kj}\right)$ *and* $\left(g^{kj}\right) = \left(g_{kj}\right)^{-1}$.

Since $\langle -(\boldsymbol{n}^T+\boldsymbol{n}^S)(u), \boldsymbol{X}_{u_j}(u)\rangle = 0$, we have $h_{ij}(\boldsymbol{n}^T,\boldsymbol{n}^S)(u) = \langle \boldsymbol{n}^T(u)+\boldsymbol{n}^S(u), \boldsymbol{X}_{u_iu_j}(u)\rangle$. Therefore the lightcone second fundamental invariant at a point $p_0 = \boldsymbol{X}(u_0)$ depends only on the values $\boldsymbol{n}^T(u_0)+\boldsymbol{n}^S(u_0)$ and $\boldsymbol{X}_{u_iu_j}(u_0)$, respectively. Thus, the lightcone curvatures also depend only on $\boldsymbol{n}^T(u_0)+\boldsymbol{n}^S(u_0)$, $\boldsymbol{X}_{u_i}(u_0)$ and $\boldsymbol{X}_{u_iu_j}(u_0)$, independent of the derivation of the vector fields $\boldsymbol{n}^T$ and $\boldsymbol{n}^S$. We write $\kappa_i(\boldsymbol{n}^T_0,\boldsymbol{n}^S_0)(p_0)$ $(i = 1,\dots,s)$ as the lightcone principal curvatures at $p_0 = \boldsymbol{X}(u_0)$ with respect to $(\boldsymbol{n}^T_0,\boldsymbol{n}^S_0) = (\boldsymbol{n}^T(u_0),\boldsymbol{n}^S(u_0))$. So we write that

$$h_{ij}(\boldsymbol{n}^T,\boldsymbol{\xi})(u_0) = h_{ij}(\boldsymbol{n}^T,\boldsymbol{n}^S)(u_0)$$

and $\kappa_\ell(\boldsymbol{n}^T)_i(\boldsymbol{\xi},p_0) = \kappa_i(\boldsymbol{n}^T_0,\boldsymbol{n}^S_0)(p_0)$, where $\boldsymbol{\xi} = \boldsymbol{n}^S(u_0)$ for some local extension $\boldsymbol{n}^T(u)$ of $\boldsymbol{\xi}$. Let $\kappa_\ell(\boldsymbol{n}^T)_i(p,\boldsymbol{\xi})$ be the eigenvalues of $S_\ell(\boldsymbol{n}^T)_{(p,\boldsymbol{\xi})}$, $(i = 1,\dots,n-1)$. Here, we write

$$\kappa_\ell(\boldsymbol{n}^T)_i(p,\boldsymbol{\xi}),\ \ (i = 1,\dots,s)$$

for the eigenvalues belonging to the eigenvectors on T_pM and

$$\kappa_\ell(\boldsymbol{n}^T)_i(p,\boldsymbol{\xi}),\ (i=s+1,\dots n-1)$$

for the eigenvalues belonging to the eigenvectors on the tangent space of the fiber of $N_1(M)[\boldsymbol{n}^T]$. Then we have the following proposition.

Proposition 3.2. *We choose a (local) pseudo-orthonormal frame* $\{\boldsymbol{X},\boldsymbol{n}^T,\boldsymbol{n}^S_1,\dots,\boldsymbol{n}^S_{k-2}\}$ *of* $N(M)$ *with* $\boldsymbol{n}^S_{k-2}=\boldsymbol{n}^S$*. For* $p_0=\boldsymbol{X}(u_0)$ *and* $\boldsymbol{\xi}_0=\boldsymbol{n}^S(u_0)$*, we have*

$$\kappa_\ell(\boldsymbol{n}^T)_i(p_0,\boldsymbol{\xi}_0)=\kappa_i(\boldsymbol{n}^T,\boldsymbol{n}^S)(u_0),\ (i=1,\dots,s)$$

and $\kappa_\ell(\boldsymbol{n}^T)_i(p_0,\boldsymbol{\xi}_0)=-1$*,* $(i=s+1,\dots n-1)$*.*

Proof. Since $\{\boldsymbol{X},\boldsymbol{n}^T,\boldsymbol{n}^S_1,\dots,\boldsymbol{n}^S_{k-2}\}$ is a pseudo-orthonormal frame of $N(M)$, we have

$$\langle \boldsymbol{X}(u_0),\boldsymbol{\xi}_0\rangle=\langle \boldsymbol{n}^T(u_0),\boldsymbol{\xi}_0\rangle=\langle \boldsymbol{n}^S_i(u_0),\boldsymbol{\xi}_0\rangle=0.$$

Therefore, we have

$$T_{\boldsymbol{\xi}}S^{k-2}=\langle \boldsymbol{n}^S_1(u_0),\dots,\boldsymbol{n}^S_{k-2}(u_0)\rangle.$$

Using this orthonormal basis of $T_{\boldsymbol{\xi}_0}S^{k-2}$, the canonical Riemannian metric $G_{ij}(p_0,\boldsymbol{\xi}_0)$ is represented by

$$(G_{ij}(p_0,\boldsymbol{\xi}))=\begin{pmatrix} g_{ij}(p_0) & 0\\ 0 & I_{k-2}\end{pmatrix},$$

where $g_{ij}(p_0)=\langle \boldsymbol{X}_{u_i}(u_0),\boldsymbol{X}_{u_j}(u_0)\rangle$.

On the other hand, by Proposition 3.1, we have

$$-\sum_{j=1}^{s}h^j_i(\boldsymbol{n}^T,\boldsymbol{n}^S)(u_0)\boldsymbol{X}_{u_j}=\mathbb{LG}(\boldsymbol{n}^T,\boldsymbol{n}^S)_{u_i}(u_0)=d_{p_0}\mathbb{LG}(\boldsymbol{n}^T,\boldsymbol{n}^S)\left(\frac{\partial}{\partial u_i}\right),$$

so that we have

$$S_\ell(\boldsymbol{n}^T)_{(p_0,\boldsymbol{\xi}_0)}\left(\frac{\partial}{\partial u_i}\right)=\sum_{j=1}^{s}h^j_i(\boldsymbol{n}^T,\boldsymbol{n}^S)(u_0)\boldsymbol{X}_{u_j}.$$

Therefore, the representation matrix of $S_\ell(\boldsymbol{n}^T)_{(p_0,\boldsymbol{\xi}_0)}$ with respect to the basis

$$\{\boldsymbol{X}_{u_1}(u_0),\dots,\boldsymbol{X}_{u_s}(u_0),\boldsymbol{n}^S_1(u_0),\dots,\boldsymbol{n}^S_{k-2}(u_0)\}$$

of $T_{(p_0,\boldsymbol{\xi}_0)}(N^{dS}_1(M)[\boldsymbol{n}^T])$ is of the form

$$\begin{pmatrix} h^j_i(\boldsymbol{n}^T,\boldsymbol{n}^S)(u_0) & *\\ 0 & -I_{k-2}\end{pmatrix}.$$

It follows that the eigenvalues of this matrix are $\lambda_i=\kappa_i(\boldsymbol{n}^T,\boldsymbol{n}^S)(u_0)$, $(i=1,\dots,s)$ and $\lambda_i=-1$, $(i=s+1,\dots,n-1)$. This completes the proof. □

We call $\kappa_\ell(\boldsymbol{n}^T)_i(p,\boldsymbol{\xi})$, $(i=1,\dots,s)$ the *lightcone principal curvatures* of M with respect to $(\boldsymbol{n}^T,\boldsymbol{\xi})$ at $p\in M$.

4. Lightlike hypersurfaces in de Sitter space

We define a hypersurface $\mathbb{LH}_M(\boldsymbol{n}^T) : N_1^{dS}(M)[\boldsymbol{n}^T] \times \mathbb{R} \longrightarrow S_1^n$ by

$$\mathbb{LH}_M((p,\boldsymbol{\xi}),\mu) = \boldsymbol{X}(u) + \mu(\boldsymbol{n}^T + \boldsymbol{\xi})(u) = \boldsymbol{X}(u) + \mu\mathbb{LG}(\boldsymbol{n}^T)(u,\boldsymbol{\xi}),$$

where $p = \boldsymbol{X}(u)$, which is called the *de Sitter lightlike hypersurface* along M relative to $\boldsymbol{n}^T$. We introduce the notion of height functions on spacelike submanifold, which is useful for the study of singularities of de Sitter lightlike hypersurfaces. We define a family of functions $H : M \times S_1^n \longrightarrow \mathbb{R}$ on a spacelike submanifold $M = \boldsymbol{X}(U)$ by

$$H(p,\boldsymbol{\lambda}) = H(u,\boldsymbol{\lambda}) = \langle \boldsymbol{X}(u), \boldsymbol{\lambda}\rangle - 1,$$

where $p = \boldsymbol{X}(u)$. We call H the *de Sitter height function* (briefly, *dS-height function*) on the spacelike submanifold M. For any fixed $\boldsymbol{\lambda}_0 \in S_1^n$, we write $h_{\boldsymbol{\lambda}_0}(p) = H(p,\boldsymbol{\lambda}_0)$ and have the following proposition.

Proposition 4.1. *Suppose that $p_0 = \boldsymbol{X}(u_0) \neq \boldsymbol{\lambda}_0$. Then we have the following*:

(1) *$h_{\boldsymbol{\lambda}_0}(p_0) = \partial h_{\boldsymbol{\lambda}_0}/\partial u_i(p_0) = 0$, $(i = 1,\dots,s)$ if and only if there exist $\boldsymbol{\xi}_0 \in N_1^{dS}(M)_{p_0}[\boldsymbol{n}^T]$ and $\mu_0 \in \mathbb{R}\setminus\{0\}$ such that*

$$\boldsymbol{\lambda}_0 = \boldsymbol{X}(u_0) + \mu_0\mathbb{LG}(\boldsymbol{n}^T)(u_0,\boldsymbol{\xi}_0) = \mathbb{LH}_M(\boldsymbol{n}^T)((p_0,\boldsymbol{\xi}_0),\mu_0).$$

(2) *$h_{\boldsymbol{\lambda}_0}(p_0) = \partial h_{\boldsymbol{\lambda}_0}/\partial u_i(p_0) = \det\mathcal{H}(h_{\boldsymbol{\lambda}_0})(p_0) = 0$ $(i = 1,\dots,s)$ if and only if there exist $\boldsymbol{\xi}_0 \in N_1^{dS}(M)_{p_0}[\boldsymbol{n}^T]$ and $\mu_0 \in \mathbb{R}\setminus\{0\}$ such that*

$$\boldsymbol{\lambda}_0 = \mathbb{LH}_M(\boldsymbol{n}^T)((p_0,\boldsymbol{\xi}_0),\mu_0)$$

and $1/\mu$ is one of the non-zero lightcone principal curvatures $\kappa_\ell(\boldsymbol{n}^T)_i(p_0,\boldsymbol{\xi}_0), (i = 1,\dots,s)$.

(3) *With condition* (2), *$\operatorname{rank}\mathcal{H}(h_{\boldsymbol{\lambda}_0})(p_0) = 0$ if and only if $p_0 = \boldsymbol{X}(u_0)$ is a non-flat $(\boldsymbol{n}^T(u_0),\boldsymbol{\xi}_0)$-umbilical point.*

Proof. (1) We write that $p = \boldsymbol{X}(u)$. The condition $h_{\boldsymbol{\lambda}_0}(p) = \langle \boldsymbol{X}(u),\boldsymbol{\lambda}_0\rangle - 1 = 0$ means that

$$\begin{aligned}\langle \boldsymbol{X}(u) - \boldsymbol{\lambda}_0, \boldsymbol{X}(u) - \boldsymbol{\lambda}_0\rangle &= \langle \boldsymbol{X}(u),\boldsymbol{X}(u)\rangle - 2\langle \boldsymbol{X}(u),\boldsymbol{\lambda}_0\rangle + \langle \boldsymbol{\lambda}_0,\boldsymbol{\lambda}_0\rangle \\ &= -2(-1 + \langle \boldsymbol{X}(u),\boldsymbol{\lambda}_0\rangle) = 0,\end{aligned}$$

so that $\boldsymbol{X}(u) - \boldsymbol{\lambda}_0 \in LC^*$. Since $\partial h_{\boldsymbol{\lambda}_0}/\partial u_i(p) = \langle \boldsymbol{X}_{u_i}(u),\boldsymbol{\lambda}_0\rangle$ and $\langle \boldsymbol{X}_{u_i},\boldsymbol{X}\rangle = 0$, we have $\langle \boldsymbol{X}_{u_i}(u),\boldsymbol{\lambda}_0\rangle = -\langle \boldsymbol{X}_{u_i}(u),\boldsymbol{X}(u) - \boldsymbol{\lambda}_0\rangle$. Therefore, $\partial h_{\boldsymbol{\lambda}_0}/\partial u_i(p) = 0$ if and only if

$$\boldsymbol{X}(u) - \boldsymbol{\lambda}_0 \in N_pM.$$

On the other hand, the condition $h_{\boldsymbol{\lambda}_0}(p) = \langle \boldsymbol{X}(u),\boldsymbol{\lambda}_0\rangle - 1 = 0$ implies that

$$\langle \boldsymbol{X}(u), \boldsymbol{X}(u) - \boldsymbol{\lambda}_0\rangle = 0.$$

This means that $\boldsymbol{X}(u) - \boldsymbol{\lambda}_0 \in T_pS_1^n$. Hence $h_{\boldsymbol{\lambda}_0}(p_0) = \partial h_{\boldsymbol{\lambda}_0}/\partial u_i((p_0) = 0$ $(i = 1,\dots,s)$ if and only if $\boldsymbol{X}(u_0) - \boldsymbol{\lambda}_0 \in N_{p_0}M \cap LC^* \cap T_{p_0}S_1^n$. Let

$$\boldsymbol{v} = \boldsymbol{X}(u_0) - \boldsymbol{\lambda}_0 \in N_{p_0}M \cap LC^* \cap T_{p_0}S_1^n.$$

If $\langle \boldsymbol{n}^T(u_0),\boldsymbol{v}\rangle = 0$, then $\boldsymbol{n}^T(u_0)$ belongs to a lightlike hyperplane in the Lorentz space $T_{p_0}S_1^n$, so that $\boldsymbol{n}^T(u_0)$ is lightlike or spacelike. This contradicts the fact that $\boldsymbol{n}^T(u_0)$ is a timelike unit vector. Thus, $\langle \boldsymbol{n}^T(u_0),\boldsymbol{v}\rangle \neq 0$. We set

$$\boldsymbol{\xi}_0 = \frac{-1}{\langle \boldsymbol{n}^T(u_0),\boldsymbol{v}\rangle}\boldsymbol{v} - \boldsymbol{n}^T(u_0).$$

Then we have

$$\begin{aligned}\langle \boldsymbol{\xi}_0, \boldsymbol{\xi}_0 \rangle &= -2\frac{-1}{\langle \boldsymbol{n}^T(u_0), \boldsymbol{v}\rangle}\langle \boldsymbol{n}^T(u_0), \boldsymbol{v}\rangle - 1 = 1 \\ \langle \boldsymbol{\xi}_0, \boldsymbol{n}^T(u_0)\rangle &= \frac{-1}{\langle \boldsymbol{n}^T(u_0), \boldsymbol{v}\rangle}\langle \boldsymbol{n}^T(u_0), \boldsymbol{v}\rangle + 1 = 0,\end{aligned}$$

and $\langle \boldsymbol{\xi}_0, \boldsymbol{X}(u_0)\rangle = 0$. This means that $\boldsymbol{\xi}_0 \in N_1^{dS}(M)_{p_0}(M)[\boldsymbol{n}^T]$.

Since $-\boldsymbol{v} = \langle \boldsymbol{n}^T(u_0), \boldsymbol{v}\rangle(\boldsymbol{n}^T(u_0) + \boldsymbol{\xi}_0)$, we have $\boldsymbol{\lambda}_0 = \boldsymbol{X}(u_0) + \mu_0\mathbb{LG}(\boldsymbol{n}^T)(p_0, \boldsymbol{\xi}_0)$, where $p_0 = \boldsymbol{X}(u_0)$ and $\mu_0 = \langle \boldsymbol{n}^T(u_0), \boldsymbol{v}\rangle$. For the converse assertion, suppose that

$$\boldsymbol{\lambda}_0 = \boldsymbol{X}(u_0) + \mu_0\mathbb{LG}(\boldsymbol{n}^T)(p_0, \boldsymbol{\xi}_0).$$

Then $\boldsymbol{\lambda}_0 - \boldsymbol{X}(u_0) \in N_{p_0}(M) \cap LC^*$ and

$$\langle \boldsymbol{\lambda}_0 - \boldsymbol{X}(u_0), \boldsymbol{X}(u_0)\rangle = \langle \mu_0\mathbb{LG}(\boldsymbol{n}^T)(p_0, \boldsymbol{\xi}_0), \boldsymbol{X}(u_0)\rangle = 0.$$

Thus we have $\boldsymbol{\lambda}_0 - \boldsymbol{X}(u_0) \in N_{p_0}(M) \cap LC^* \cap T_{p_0}S^n_1$. By the previous arguments, these conditions are equivalent to the condition that $h_{\boldsymbol{\lambda}_0}(p_0) = \partial h_{\boldsymbol{\lambda}_0}/\partial u_i((p_0) = 0$ $(i = 1, \dots, s)$.

(2) By a straightforward calculation, we have

$$\frac{\partial^2 h_{\boldsymbol{\lambda}_0}}{\partial u_i \partial u_j}(u) = \langle \boldsymbol{X}_{u_i u_j}, \boldsymbol{\lambda}_0\rangle.$$

Under the condition that $\boldsymbol{\lambda}_0 = \boldsymbol{X}(u_0) + \mu_0(\boldsymbol{n}^T(u_0) + \boldsymbol{\xi}_0)$, we have

$$\frac{\partial^2 h_{\boldsymbol{\lambda}_0}}{\partial u_i \partial u_j}(u_0) = \langle \boldsymbol{X}_{u_i u_j}(u_0), \boldsymbol{X}(u_0)\rangle + \mu_0\langle \boldsymbol{X}_{u_i u_j}(u_0), (\boldsymbol{n}^T(u_0) + \boldsymbol{\xi}_0)\rangle.$$

Since $\langle \boldsymbol{X}_{u_i}, \boldsymbol{X}\rangle = 0$, we have $\langle \boldsymbol{X}_{u_i u_j}, \boldsymbol{X}\rangle = -\langle \boldsymbol{X}_{u_i}, \boldsymbol{X}_{u_j}\rangle$. Thus, we have

$$\left(\frac{\partial^2 h_{\boldsymbol{\lambda}_0}}{\partial u_i \partial u_\ell}(u_0)\right)\left(g^{j\ell}(u_0)\right) = \left(\mu_0 h^j_i(\boldsymbol{n}^T, \boldsymbol{\xi}_0)(p_0) - \delta^j_i\right).$$

It follows that $\det\mathcal{H}(g)(p_0) = 0$ if and only if $1/\mu_0$ is an eigenvalue of $(h^i_j(\boldsymbol{n}^T, \boldsymbol{\xi}_0)(p_0))$, which is equal to one of the lightcone principal curvatures $\kappa_\ell(\boldsymbol{n}^T)_i(p_0, \boldsymbol{\xi}_0), (i = 1, \dots, s)$.

(3) By the above calculation, $\operatorname{rank}\mathcal{H}(h_{\boldsymbol{\lambda}_0})(p_0) = 0$ if and only if $(h^i_j(\boldsymbol{n}^T)(p_0, \boldsymbol{\xi}_0)) = \frac{1}{\mu_0}(\delta^j_i)$, where $1/\mu_0 = \kappa_\ell(\boldsymbol{n}^T)_i(p_0, \boldsymbol{\xi}_0)$, $(i = 1, \dots, s)$. This means that $p_0 = \boldsymbol{X}(u_0)$ is an $(\boldsymbol{n}^T(u_0), \boldsymbol{\xi}_0)$-umbilical point. □

In order to understand the geometric meanings of the assertions of Proposition 4.1, we briefly review the theory of Legendrian singularities For detailed expressions, see [1, 24]. Let $\pi : PT^*(\mathbb{R}^{n+1}) \longrightarrow \mathbb{R}^{n+1}$ be the projective cotangent bundle with its canonical contact structure. We next review the geometric properties of this bundle. Consider the tangent bundle $\tau : TPT^*(\mathbb{R}^{n+}) \to PT^*(\mathbb{R}^{n+1})$ and the differential map $d\pi : TPT^*(\mathbb{R}^{n+1}) \to T\mathbb{R}^{n+1}$ of π. For any $X \in TPT^*(\mathbb{R}^{n+1})$, there exists an element $\alpha \in T^*(\mathbb{R}^{n+1}_1$ such that $\tau(X) = [\alpha]$. For an element $V \in T_x(\mathbb{R}^{n+1})$, the property $\alpha(V) = 0$ does not depend on the choice of representative of the class $[\alpha]$. Thus we can define the canonical contact structure on $PT^*(\mathbb{R}^{n+1})$ by

$$K = \{X \in TPT^*(\mathbb{R}^{n+1}) \mid \tau(X)(d\pi(X)) = 0\}.$$

We have the trivialization $PT^*(\mathbb{R}^{n+1}) \cong \mathbb{R}^{n+1} \times P^n(\mathbb{R})^*$, and call

$$((v_0, v_1, \dots, v_n), [\xi_0 : \xi_1 : \cdots : \xi_n])$$

homogeneous coordinates of $PT^*(\mathbb{R}^{n+1})$, where $[\xi_0:\xi_1:\cdots:\xi_n]$ are the homogeneous coordinates of the dual projective space $P^n(\mathbb{R})^*$. It is easy to show that $X\in K_{(x,[\xi])}$ if and only if

$$\sum_{i=0}^n \mu_i\xi_i=0,$$

where $d\tilde{\pi}(X)=\sum_{i=0}^n \mu_i\partial/\partial v_i$. An immersion $i:L\to PT^*(\mathbb{R}^{n+1})$ is said to be *a Legendrian immersion* if $\dim L=n$ and $di_q(T_qL)\subset K_{i(q)}$ for any $q\in L$. The map $\pi\circ i$ is also called *the Legendrian map* of i and the set $W(i)=\operatorname{image}\pi\circ i$, the *wave front set* of i. Moreover, i (or, the image of i) is called the *Legendrian lift* of $W(i)$.

Let $F:(\mathbb{R}^k\times\mathbb{R}^{n+1},\mathbf{0})\longrightarrow(\mathbb{R},\mathbf{0})$ be a function germ. We say that F is *a Morse family of hypersurfaces* if the map germ

$$\Delta^*F=\left(F,\frac{\partial F}{\partial q_1},\ldots,\frac{\partial F}{\partial q_k}\right):(\mathbb{R}^k\times\mathbb{R}^{n+1},\mathbf{0})\longrightarrow(\mathbb{R}\times\mathbb{R}^k,\mathbf{0})$$

is submersive, where $(q,x)=(q_1,\ldots,q_k,x_0,\ldots,x_n)\in(\mathbb{R}^k\times\mathbb{R}^{n+1},\mathbf{0})$. In this case we have a smooth n-dimensional submanifold

$$\Sigma_*(F)=\left\{(q,x)\in(\mathbb{R}^k\times\mathbb{R}^{n+1},\mathbf{0})\ \middle|\ F(q,x)=\frac{\partial F}{\partial q_1}(q,x)=\cdots=\frac{\partial F}{\partial q_k}(q,x)=0\right\}$$

and the map germ $\mathscr{L}_F:(\Sigma_*(F),\mathbf{0})\longrightarrow PT^*\mathbb{R}^{n+1}$ defined by

$$\mathscr{L}_F(q,x)=\left(x,\left[\frac{\partial F}{\partial x_0}(q,x):\cdots:\frac{\partial F}{\partial x_n}(q,x)\right]\right)$$

is a Legendrian immersion. We call F *a generating family* of $\mathscr{L}_F(\Sigma_*(F))$, and the wave front set is given by $W(\mathscr{L}_F)=\pi_n(\Sigma_*(F))$, where $\pi_n:\mathbb{R}^k\times\mathbb{R}^n\longrightarrow\mathbb{R}^n$ is the canonical projection. In the theory of unfoldings of function germs, the wave front set $W(\mathscr{L}_F)$ is called a *discriminant set* of F, which is also denoted by $\mathcal{D}_F$.

By the assertion (2) of Proposition 4.1, a singular point of the de Sitter lightlike hypersurface is a point $\boldsymbol{\lambda}_0=\boldsymbol{X}(u_0)+\mu_0(\boldsymbol{n}^T+\boldsymbol{\xi}_0)(u_0)$ for $p_0=\boldsymbol{X}(u_0)$ and $\mu_0=1/\kappa_\ell(\boldsymbol{n}^T)_i(p_0,\boldsymbol{\xi}_0)$, $i=1,\ldots.s)$. Then we have the following corollary.

Corollary 4.2. *The critical value of* $\mathbb{LH}_M(\boldsymbol{n}^T)$ *is the point*

$$\boldsymbol{\lambda}=\boldsymbol{X}(u)+\frac{1}{\kappa_\ell(\boldsymbol{n}^T)_i(p,\boldsymbol{\xi})}\mathbb{LG}(\boldsymbol{n}^T)(u,\boldsymbol{\xi}),$$

where $p=\boldsymbol{X}(u)$ *and* $\kappa_\ell(\boldsymbol{n}^T)_i(p,\boldsymbol{\xi})\neq 0$.

For a non-zero lightcone principal curvature $\kappa_\ell(\boldsymbol{n}^T)_i(p_0,\boldsymbol{\xi}_0)\neq 0$, we have an open subset $O_i\subset N_1^{dS}(M)[\boldsymbol{n}^T]$ such that $\kappa_\ell(\boldsymbol{n}^T)_i(p,\boldsymbol{\xi})\neq 0$. Therefore, we have a non-zero lightcone principal curvature function $\kappa_\ell(\boldsymbol{n}^T)_i:O_i\longrightarrow\mathbb{R}$. We define a mapping $\mathbb{LF}_{\kappa_\ell(\boldsymbol{n}^T)_i}:O_i\longrightarrow AdS^{n+1}$ by

$$\mathbb{LF}_{\kappa_\ell(\boldsymbol{n}^T)_i}(p,\boldsymbol{\xi})=\boldsymbol{X}(u)+\frac{1}{\kappa_\ell(\boldsymbol{n}^T)_i(p,\boldsymbol{\xi})}\mathbb{NG}(\boldsymbol{n}^T)(u,\boldsymbol{\xi}),$$

where $p=\boldsymbol{X}(u)$. We also define

$$\mathbb{LF}_M(\boldsymbol{n}^T)=\bigcup_{i=1}^s\left\{\mathbb{LF}_{\kappa_\ell(\boldsymbol{n}^T)_i}(p,\boldsymbol{\xi})\mid(p,\boldsymbol{\xi})\in N_1^{dS}(M)[\boldsymbol{n}^T]\text{ s.t. }\kappa_\ell(\boldsymbol{n}^T)_i(p,\boldsymbol{\xi})\neq 0\right\}.$$

We call $\mathbb{LF}_M(\boldsymbol{n}^T)$ the *de Sitter lightlike focal set* of $M=\boldsymbol{X}(U)$ relative to $\boldsymbol{n}^T$, which is the critical value set of the de Sitter lightlike hypersurface $\mathbb{LH}_M(\boldsymbol{n}^T)(N_1^{dS}(M)[\boldsymbol{n}^T]\times\mathbb{R})$ along M relative to $\boldsymbol{n}^T$.

By Proposition 4.1, the image of the lightlike hypersurface along M relative to $\boldsymbol{n}^T$ is the discriminant set of the AdS-height function H on M. Moreover, the focal set is the critical value set of the lightlike hypersurface along M relative to $\boldsymbol{n}^T$. Since H is independent of the choice of $\boldsymbol{n}^T$, we have shown the following corollary.

Corollary 4.3. *Let $\boldsymbol{n}^T$ and $\overline{\boldsymbol{n}}^T$ be future directed timelike unit normal fields along M. Then we have*

$$\mathbb{LH}_M(\boldsymbol{n}^T)(N_1(M)[\boldsymbol{n}^T]\times\mathbb{R}) = \mathbb{LH}_M(\overline{\boldsymbol{n}}^T)(N_1(M)[\overline{\boldsymbol{n}}^T]\times\mathbb{R}) \text{ and } \mathbb{LF}_M(\boldsymbol{n}^T) = \mathbb{LF}_M(\overline{\boldsymbol{n}}^T).$$

We have the following proposition.

Proposition 4.4. *For any point $(u,\boldsymbol{\lambda})\in\Sigma_*(F)=\Delta^*H^{-1}(0)$, the germ of the dS-height function H at $(u,\boldsymbol{\lambda})$ is a Morse family of hypersurfaces.*

Proof. We write

$$\boldsymbol{X}(u)=(X_0(u),X_1(u),\dots,X_n(u)) \text{ and } \boldsymbol{\lambda}=(\lambda_0,\lambda_1,\dots,\lambda_n).$$

We define an open subset $U_n^+=\{\boldsymbol{\lambda}\in S_1^n \mid \lambda_n>0\ \}$. For any $\boldsymbol{\lambda}\in U_n^+$, we have

$$\lambda_n=\sqrt{\lambda_0^2-\sum_{i=1}^{n-1}\lambda_i^2+1}.$$

Thus, we have local coordinates on S_1^n given by $(\lambda_0,\lambda_1,\dots,\lambda_{n-1})$ on U_n^+. By definition, we have

$$H(u,\boldsymbol{\lambda})=-X_0(u)\lambda_0+X_1(u)\lambda_1+\cdots+X_{n-1}(u)\lambda_{n-1}+X_n(u)\sqrt{\lambda_0^2-\sum_{i=1}^{n-1}\lambda_i^2+1}-1.$$

We now prove that the mapping

$$\Delta^*H=\left(H,\frac{\partial H}{\partial u_1},\dots,\frac{\partial H}{\partial u_s}\right)$$

is non-singular at $(u,\boldsymbol{\lambda})\in\Sigma_*(F)$. Indeed, the Jacobian matrix of Δ^*H is given by

$$\left(\begin{array}{ccccc} & X_n\dfrac{\lambda_0}{\lambda_n}-X_0 & -X_n\dfrac{\lambda 1}{\lambda_n}+X_1 & \cdots & -X_n\dfrac{\lambda_{n-1}}{\lambda_n}+X_{n-1} \\ \mathrm{A} & X_{nu_1}\dfrac{\lambda_0}{\lambda_n}-X_{0u_1} & -X_{nu_1}\dfrac{\lambda_1}{\lambda_n}+X_{1u_1} & \cdots & -X_{nu_1}\dfrac{\lambda_{n-1}}{\lambda_n}+X_{n-1u_1} \\ & \vdots & \vdots & \ddots & \vdots \\ & X_{nu_s}\dfrac{\lambda_0}{\lambda_n}-X_{0u_s} & -X_{nu_s}\dfrac{\lambda_1}{\lambda_n}+X_{1u_s} & \cdots & -X_{nu_s}\dfrac{\lambda_{n-1}}{\lambda_n}+X_{n-1u_s}\end{array}\right),$$

where

$$\mathrm{A}=\begin{pmatrix}\langle\boldsymbol{X}_{u_1},\boldsymbol{\lambda}\rangle & \cdots & \langle\boldsymbol{X}_{u_s},\boldsymbol{\lambda}\rangle \\ \langle\boldsymbol{X}_{u_1u_1},\boldsymbol{\lambda}\rangle & \cdots & \langle\boldsymbol{X}_{u_1u_s},\boldsymbol{\lambda}\rangle \\ \vdots & \ddots & \vdots \\ \langle\boldsymbol{X}_{u_su_1},\boldsymbol{\lambda}\rangle & \cdots & \langle\boldsymbol{X}_{u_su_s},\boldsymbol{\lambda}\rangle\end{pmatrix}.$$

We now show that the rank of

$$\mathrm{B}=\begin{pmatrix} X_n\dfrac{\lambda_0}{\lambda_n}-X_0 & -X_n\dfrac{\lambda_1}{\lambda_n}+X_1 & \cdots & -X_n\dfrac{\lambda_{n-1}}{\lambda_n}+X_{n-1} \\ X_{nu_1}\dfrac{\lambda_0}{\lambda_n}-X_{0u_1} & -X_{nu_1}\dfrac{\lambda_1}{\lambda_n}+X_{1u_1} & \cdots & -X_{nu_1}\dfrac{\lambda_{n-1}}{\lambda_n}+X_{n-1u_1} \\ \vdots & \vdots & \ddots & \vdots \\ X_{nu_s}\dfrac{\lambda_0}{\lambda_n}-X_{0u_s} & -X_{nu_s}\dfrac{\lambda_1}{\lambda_n}+X_{1u_s} & \cdots & -X_{nu_s}\dfrac{\lambda_{n-1}}{\lambda_n}+X_{n-1u_s} \end{pmatrix}$$

is $s+1$ at $(u,\boldsymbol{\lambda})\in\Sigma_*(H)$. Since $(u,\boldsymbol{\lambda})\in\Sigma_*(H)$, we have

$$\boldsymbol{\lambda}=\boldsymbol{X}(u)+\mu\left(\boldsymbol{n}^T(u)+\sum_{i=1}^{k-1}\xi_i\boldsymbol{n}_i(u)\right)$$

with $\sum_{i=1}^{k-1}\xi_i^2=1$, where $\{\boldsymbol{X},\boldsymbol{n}^T,\boldsymbol{n}_1^S,\dots,\boldsymbol{n}_{k-1}^S\}$ is a pseudo-orthonormal (local) frame of $N(M)$. Without loss of generality, we assume that $\mu\neq 0$ and $\xi_{k-1}\neq 0$. We write

$$\boldsymbol{n}^T(u)={}^t(n_0^T(u),\dots n_n^T(u)),\ \ \boldsymbol{n}_i^S(u)={}^t(n_0^i(u),\dots n_n^i(u)).$$

It is enough to show that the rank of the matrix

$$\mathrm{C}=\begin{pmatrix} X_n\dfrac{\lambda_0}{\lambda_n}-X_0 & -X_n\dfrac{\lambda 1}{\lambda_n}+X_1 & \cdots & -X_n\dfrac{\lambda_{n-1}}{\lambda_n}+X_{n-1} \\ X_{nu_1}\dfrac{\lambda_0}{\lambda_n}-X_{0u_1} & -X_{nu_1}\dfrac{\lambda_1}{\lambda_n}+X_{1u_1} & \cdots & -X_{nu_1}\dfrac{\lambda_{n-1}}{\lambda_n}+X_{n-1u_1} \\ \vdots & \vdots & \ddots & \vdots \\ X_{nu_s}\dfrac{\lambda_0}{\lambda_n}-X_{0u_s} & -X_{nu_s}\dfrac{\lambda_1}{\lambda_n}+X_{1u_s} & \cdots & -X_{nu_s}\dfrac{\lambda_{n-1}}{\lambda_n}+X_{n-1u_s} \\ n_n^T\dfrac{\lambda_0}{\lambda_n}-n_0^T & -n_n^T\dfrac{\lambda_1}{\lambda_n}+n_1^T & \cdots & -n_n^T\dfrac{\lambda_{n-1}}{\lambda_n}+n_{n-1}^T \\ n_n^1\dfrac{\lambda_0}{\lambda_n}-n_0^1 & -n_n^1\dfrac{\lambda_1}{\lambda_n}+n_1^1 & \cdots & -n_n^1\dfrac{\lambda_{n-1}}{\lambda_n}+n_{n-1}^1 \\ \vdots & \vdots & \ddots & \vdots \\ n_n^{k-2}\dfrac{\lambda_0}{\lambda_n}-n_0^{k-2} & -n_n^{k-2}\dfrac{\lambda_1}{\lambda_n}+n_1^{k-2} & \cdots & -n_{-1}^{k-2}\dfrac{\lambda_{n-1}}{\lambda_n}+n_{n-1}^{k-2} \end{pmatrix}$$

is n at $(u,\boldsymbol{\lambda})\in\Sigma_*(H)$. We write

$$\boldsymbol{a}_i={}^t(x_i(u),x_{iu_1}(u),\dots x_{iu_s}(u),n_i^T(u),n_i^1(u),\dots,n_i^{k-2}(u)).$$

Then we have

$$\mathrm{C}=\left(\boldsymbol{a}_n\frac{\lambda_0}{\lambda_n}-\boldsymbol{a}_0,-\boldsymbol{a}_n\frac{\lambda_1}{\lambda_n}+\boldsymbol{a}_1,\dots,-\boldsymbol{a}_n\frac{\lambda_{n-1}}{\lambda_n}+\boldsymbol{a}_{n-1}\right).$$

It follows that

$$\begin{aligned}\det\mathrm{C}\ &=\frac{\lambda_0}{\lambda_n}(-1)^{n-1}\det(\boldsymbol{a}_1,\dots,\boldsymbol{a}_n)+\frac{\lambda_1}{\lambda_n}(-1)^{n-2}\det(\boldsymbol{a}_0,\boldsymbol{a}_2,\dots,\boldsymbol{a}_n)\\ &\quad+\cdots+(-1)^0\frac{\lambda_{n-1}}{\lambda_n}\det(\boldsymbol{a}_0,\boldsymbol{a}_1,\dots,\boldsymbol{a}_{n-2},\boldsymbol{a}_n)+(-1)^1\frac{\lambda_n}{\lambda_n}\det(\boldsymbol{a}_0,\boldsymbol{a}_1,\dots,\boldsymbol{a}_{n-1}).\end{aligned}$$

Moreover, we define $\delta_i=\det(\boldsymbol{a}_0,\boldsymbol{a}_1,\dots,\boldsymbol{a}_{i-1},\boldsymbol{a}_{i+1},\dots,\boldsymbol{a}_n)$ for $i=0,1,\dots,n$ and

$$\boldsymbol{a}=(-(-1)^{n-1}\delta_0,(-1)^{n-2}\delta_1,\dots,(-1)^0\delta_{n-1},(-1)^1\delta_n).$$

Then we have

$$\boldsymbol{a}=(-1)^{n-1}\boldsymbol{X}\wedge\boldsymbol{X}_{u_1}\wedge\cdots\wedge\boldsymbol{X}_{u_s}\wedge\boldsymbol{n}^T\wedge\boldsymbol{n}_1\wedge\cdots\wedge\boldsymbol{n}_{k-2}.$$

We remark that $\boldsymbol{a} \neq 0$ and $\boldsymbol{a} = \pm\|\boldsymbol{a}\|\boldsymbol{n}_{k-1}$. By the above calculation, we have

$$\begin{aligned}\det \mathrm{C} &= \left\langle \left(\frac{\lambda_0}{\lambda_n}, \frac{\lambda_1}{\lambda_n}, \dots, \frac{\lambda_n}{\lambda_n}\right), \boldsymbol{a} \right\rangle = \frac{1}{\lambda_n}\left\langle \boldsymbol{X}(u) + \mu\left(\boldsymbol{n}^T(u) + \sum_{i=1}^{k-1}\xi_i \boldsymbol{n}_i(u)\right), \boldsymbol{a}\right\rangle \\ &= \frac{1}{\lambda_n} \times \pm\mu\xi_{k-1}\|\boldsymbol{a}\| = \pm\frac{\mu\xi_{k-1}\|\boldsymbol{a}\|}{\lambda_n} \neq 0.\end{aligned}$$

Therefore the Jacobi matrix of $\Delta^* H$ is non-singular at $(u, \boldsymbol{\lambda}) \in \Sigma_*(F)$.

For other local coordinates of S^n_1, we can apply the same method for the proof as the above case. This completes the proof. □

Here we consider the open set U_n^+ again. Since H is a Morse family of hypersurfaces, we have a Legendrian immersion

$$\mathscr{L}_H : \Sigma_*(H) \longrightarrow PT^*(S^n_1)|U_n^+$$

by the general theory of Legendrian singularities. By definition, we have

$$\frac{\partial H}{\partial \lambda_0}(u, \boldsymbol{\lambda}) = X_n(u)\frac{\lambda_0}{\lambda_n} - X_0(u),\ \frac{\partial H}{\partial \lambda_i}(u, \boldsymbol{\lambda}) = -X_n(u)\frac{\lambda_i}{\lambda_n} + X_i(u),\ (i = 1, \dots, n-1).$$

It follows that

$$\begin{aligned}&\left[\frac{\partial H}{\partial \lambda_0}(u, \boldsymbol{\lambda}) : \frac{\partial H}{\partial \lambda_1}(u, \boldsymbol{\lambda}) : \cdots : \frac{\partial H}{\partial \lambda_{n-1}}(u, \boldsymbol{\lambda})\right] \\ &\qquad = [X_n(u)\lambda_0 - X_0(u)\lambda_n : X_1(u)\lambda_n - X_n(u)\lambda_1 : \cdots : X_{n-1}(u)\lambda_n - X_n(u)\lambda_{n-1}].\end{aligned}$$

Therefore, we have

$$\mathscr{L}_H(u, \boldsymbol{\lambda}) = (\boldsymbol{\lambda}, [X_n(u)\lambda_0 - X_0(u)\lambda_n : X_1(u)\lambda_n - X_n(u)\lambda_1 : \cdots : X_{n-1}(u)\lambda_n - X_n(u)\lambda_{n-1}]),$$

where

$$\Sigma_*(H) = \{(u, \boldsymbol{\lambda}) \mid \boldsymbol{\lambda} = \mathbb{LH}_M(\boldsymbol{n}^T)(p, \boldsymbol{\xi}, t)\ ((p, \boldsymbol{\xi}), t) \in N_1(M)[\boldsymbol{n}^T] \times \mathbb{R}\}.$$

We observe that H is a generating family of the Legendrian immersion $\mathscr{L}_H$ whose wave front is $\mathbb{LH}_M(\boldsymbol{n}^T)(N_1(M)[\boldsymbol{n}^T] \times \mathbb{R})$. For other local coordinates of S^n_1, we have the similar results to the above case.

5. Contact with de Sitter lightcones

In this section, we consider the geometric meaning of the singularities of lightlike hypersurfaces in de Sitter space from the viewpoint of the theory of contact of submanifolds with model hypersurfaces in the view of Montaldi's theory. We review the theory of contact for submanifolds in [21]. Let X_i and Y_i, $i = 1, 2$, be submanifolds of $\mathbb{R}^n$ with $\dim X_1 = \dim X_2$ and $\dim Y_1 = \dim Y_2$. We say that *the contact of* X_1 and Y_1 at y_1 is the same type as *the contact of* X_2 and Y_2 at y_2 if there is a diffeomorphism germ $\Phi : (\mathbb{R}^n, y_1) \longrightarrow (\mathbb{R}^n, y_2)$ such that $\Phi(X_1) = X_2$ and $\Phi(Y_1) = Y_2$. In this case we write $K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$. Since this definition of contact is local, we can replace $\mathbb{R}^n$ by an arbitrary n-manifold. Montaldi gives in [21] the following characterization of contact by using $\mathcal{K}$-equivalence. We say that two function germs $h_i : (\mathbb{R}^s, \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$ $(i = 1, 2)$ are *$\mathcal{K}$-equivalent* if there exist a diffeomorphism germ $\psi : (\mathbb{R}^s, \mathbf{0}) \longrightarrow (\mathbb{R}^s, \mathbf{0})$ and a function germ $\lambda : (\mathbb{R}^s, \mathbf{0}) \longrightarrow \mathbb{R}$ with $\lambda(\mathbf{0}) \neq 0$ such that $\lambda(x)h_1 \circ \psi(x) = h_2(x)$ for $x \in (\mathbb{R}^s, \mathbf{0})$.

Theorem 5.1. *Let X_i and Y_i, $i = 1, 2$, be submanifolds of $\mathbb{R}^n$ for which $\dim X_1 = \dim X_2$ and $\dim Y_1 = \dim Y_2 = n - 1$. Let $g_i : (X_i, x_i) \longrightarrow (\mathbb{R}^n, y_i)$ be immersion germs and let $f_i : (\mathbb{R}^n, y_i) \longrightarrow (\mathbb{R}^p, 0)$ be submersion germs with $(Y_i, y_i) = (f_i^{-1}(0), y_i)$.*

Then $K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$ if and only if $f_1 \circ g_1$ and $f_2 \circ g_2$ are $\mathcal{K}$-equivalent.

We remark that the assertion of the above theorem holds for submanifolds Y_i with general codimension (cf., [21]).

Now, we return to the review of the theory of Legendrian singularities. We introduce a natural equivalence relation among Legendrian submanifold germs. Let

$$F, G : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$$

be Morse families of hypersurfaces. Then we say that $\mathscr{L}_F(\Sigma_*(F))$ and $\mathscr{L}_G(\Sigma_*(G))$ are *Legendrian equivalent* if there exists a contact diffeomorphism germ $H : (PT^*\mathbb{R}^n, z) \longrightarrow (PT^*\mathbb{R}^n, z')$ such that H preserves fibers of π and that $H(\mathscr{L}_F(\Sigma_*(F))) = \mathscr{L}_G(\Sigma_*(G))$, where $z = \mathscr{L}_F(\mathbf{0})$, $z' = \mathscr{L}_G(\mathbf{0})$. By using Legendrian equivalence, we can define the notion of Legendrian stability for Legendrian submanifold germs in the ordinary way (see, [1, Part III]). We can interpret Legendrian equivalence by using the notion of generating families. We denote by $\mathcal{E}_k$ the local ring of function germs $(\mathbb{R}^k, \mathbf{0}) \longrightarrow \mathbb{R}$ with the unique maximal ideal $\mathfrak{M}_k = \{h \in \mathcal{E}_k \mid h(\mathbf{0}) = 0\ \}$. Let $F, G : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$ be function germs. We say that F and G are *P-$\mathcal{K}$-equivalent* if there exists a diffeomorphism germ $\Psi : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$ of the form $\Psi(x, u) = (\psi_1(q, x), \psi_2(x))$ for $(q, x) \in (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$ such that $\Psi^*(\langle F\rangle_{\mathcal{E}_{k+n}}) = \langle G\rangle_{\mathcal{E}_{k+n}}$. Here $\Psi^* : \mathcal{E}_{k+n} \longrightarrow \mathcal{E}_{k+n}$ is the pull-back $\mathbb{R}$-algebra isomorphism defined by $\Psi^*(h) = h \circ \Psi$. We say that F is an *infinitesimally $\mathcal{K}$-versal deformation of $f = F|\mathbb{R}^k \times \{\mathbf{0}\}$* if

$$\mathcal{E}_k = T_e(\mathcal{K})(f) + \left\langle \frac{\partial F}{\partial x_1}|\mathbb{R}^k \times \{\mathbf{0}\}, \dots, \frac{\partial F}{\partial x_n}|\mathbb{R}^k \times \{\mathbf{0}\} \right\rangle_{\mathbb{R}},$$

where $T_e(\mathcal{K})(f) = \left\langle \frac{\partial f}{\partial q_1}, \dots, \frac{\partial f}{\partial q_k}, f \right\rangle_{\mathcal{E}_k}$, (see [19].) The main result in the theory of Legendrian singularities ([1], §20.8 and [24], THEOREM 2) is the following:

Theorem 5.2. *Let $F, G : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$ be Morse families of hypersurfaces. Then we have the following assertions:*
(1) *$\mathscr{L}_F(\Sigma_*(F))$ and $\mathscr{L}_G(\Sigma_*(G))$ are Legendrian equivalent if and only if F and G are P-$\mathcal{K}$-equivalent,*
(2) *$\mathscr{L}_F(\Sigma_*(F))$ is Legendrian stable if and only if F is an infinitesimally $\mathcal{K}$-versal deformation of $f = F|\mathbb{R}^k \times \{\mathbf{0}\}$.*

Since F and G are function germs on the common space germ $(\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$, we do not need the notion of stably P-$\mathcal{K}$-equivalence under this situation [24, page 27]. For any map germ $f : (\mathbb{R}^k, \mathbf{0}) \longrightarrow (\mathbb{R}^p, \mathbf{0})$, we define *the local ring of f* by $Q_r(f) = \mathcal{E}_k/(f^*(\mathfrak{M}_p)\mathcal{E}_k + \mathfrak{M}_k^{r+1})$. We have the following classification result of Legendrian stable germs (cf. [10, Proposition A.4]) which is the key for the purpose in this section.

Proposition 5.3. *Let $F, G : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$ be Morse families of hypersurfaces and $f = F|\mathbb{R}^k \times \{\mathbf{0}\}, g = G|\mathbb{R}^k \times \{\mathbf{0}\}$. Suppose that $\mathscr{L}_F(\Sigma_*(F))$ and $\mathscr{L}_G(\Sigma_*(G))$ are Legendrian stable. The the following conditions are equivalent:*
(1) *$(W(\mathscr{L}_F), \mathbf{0})$ and $(W(\mathscr{L}_G), \mathbf{0})$ are diffeomorphic as set germs,*
(2) *$(\mathscr{L}_F(\Sigma_*(F)), z)$ and $(\mathscr{L}_G(\Sigma_*(G)), z')$ are Legendrian equivalent,*
(3) *$Q_{n+1}(f)$ and $Q_{n+1}(g)$ are isomorphic as $\mathbb{R}$-algebras.*

We have the following basic observations.

Proposition 5.4. *Let $M = \boldsymbol{X}(U)$ be a spacelike submanifold with*

$$\kappa_\ell(\boldsymbol{n}^T)_i(p, \boldsymbol{\xi}) \neq 0 \quad \text{for} \quad i = 1, \dots s.$$

We consider $\boldsymbol{\lambda}_0 \in S^n_1$. Then $M \subset LC_{\boldsymbol{\lambda}_0} \cap S^n_1$ if and only if $\boldsymbol{\lambda}_0 = \mathbb{LF}_M(\boldsymbol{n}^T)$. In this case we have $\mathbb{LH}_M(\boldsymbol{n}^T)(N_1(M)[\boldsymbol{n}^T]) \subset LC_{\boldsymbol{\lambda}_0} \cap S^n_1$ and $M = \boldsymbol{X}(U)$ is totally lightcone umbilical.

Proof. By Proposition 3.1, $\kappa_\ell(\boldsymbol{n}^T)_i(p,\boldsymbol{\xi}) \neq 0$ for $i = 1, \dots s$ if and only if

$$\{(\boldsymbol{n}^T + \boldsymbol{n}^S), (\boldsymbol{n}^T + \boldsymbol{n}^S)_{u_1}, \dots, (\boldsymbol{n}^T + \boldsymbol{n}^S)_{u_s}\}$$

is linearly independent for $p_0 = \boldsymbol{X}(u_0) \in M$ and $\boldsymbol{\xi}_0 = \boldsymbol{n}^S(u_0)$, where $\boldsymbol{n}^S : U \longrightarrow N_1^{dS}(M)[\boldsymbol{n}^T]$ is a local section. By the proof of assertion (1) of Proposition 4.1, $M \subset LC_{\lambda_0} \cap S_1^n$ if and only if $h_{\boldsymbol{\lambda}_0}(u) = 0$ for any $u \in U$, where $h_{\boldsymbol{\lambda}_0}(u) = H(u, \boldsymbol{\lambda}_0)$ is the dS-height function on M. It also follows from Proposition 4.1 that there exists a smooth function $\eta : U \times N_1^{dS}(M)[\boldsymbol{n}^T] \longrightarrow \mathbb{R}$ and section $\boldsymbol{n}^S : U \longrightarrow N_1^{dS}(M)[\boldsymbol{n}^T]$ such that

$$\boldsymbol{X}(u) = \boldsymbol{\lambda}_0 + \eta(u, \boldsymbol{n}^S(u))(\boldsymbol{n}^T(u) \pm \boldsymbol{n}^S(u)).$$

In fact, we have $\eta(u, \boldsymbol{n}^S(u)) = -1/\kappa_\ell(\boldsymbol{n}^T)_i(p,\boldsymbol{\xi})$ $i = 1, \dots, s$, where $p = \boldsymbol{X}(u)$ and $\boldsymbol{\xi} = \boldsymbol{n}^S(u)$. It follows that $\kappa_\ell(\boldsymbol{n}^T)_i(p,\boldsymbol{\xi}) = \kappa_\ell(\boldsymbol{n}^T)_j(p,\boldsymbol{\xi})$, so that $M = \boldsymbol{X}(U)$ is totally lightcone umbilical. Therefore we have

$$\mathbb{LH}_M(\boldsymbol{n}^T)(u, \boldsymbol{n}^S(u), \mu) = \boldsymbol{\lambda}_0 + (\mu + \eta(u, \boldsymbol{n}^S(u))(\boldsymbol{n}^T(u) \pm \boldsymbol{n}^S(u)).$$

Hence we have $\mathbb{LH}_M(\boldsymbol{n}^T)(N_1(M)[\boldsymbol{n}^T] \times \mathbb{R}) \subset LC_{\lambda_0}$. By Corollary 4.2, the critical value set of $\mathbb{LH}_M(\boldsymbol{n}^T)(N_1(M)[\boldsymbol{n}^T] \times \mathbb{R})$ is the de Sitter lightlike focal set $\mathbb{LF}_M(\boldsymbol{n}^T)$. However, it is equal to λ_0 by the previous arguments.

For the converse assertion, suppose that $\boldsymbol{\lambda}_0 = \mathbb{LF}_M(\boldsymbol{n}^T)$. Then we have

$$\boldsymbol{\lambda}_0 = \boldsymbol{X}(u) + \frac{1}{\kappa_\ell(\boldsymbol{n}^T)_i(\boldsymbol{X}(u),\boldsymbol{\xi})}\mathbb{LG}(\boldsymbol{n}^T)(u,\boldsymbol{\xi}),$$

for any $i = 1, \dots, s$ and $(p,\boldsymbol{\xi}) \in N_1^{dS}(M)[\boldsymbol{n}^T]$, where $p = \boldsymbol{X}(u)$. Thus, we have

$$\kappa_\ell(\boldsymbol{n}^T)_i(\boldsymbol{X}(u),\boldsymbol{\xi}) = \kappa_\ell(\boldsymbol{n}^T)_j(\boldsymbol{X}(u),\boldsymbol{\xi})$$

for any $i, j = 1, \dots, s$, so that M is totally lightcone umbilical. Since $\mathbb{LG}(\boldsymbol{n}^T)(u,\boldsymbol{\xi})$ is null, we have $\boldsymbol{X}(u) \in LC_{\boldsymbol{\lambda}_0}$. This completes the proof. □

According to the above proposition, $LC_{\boldsymbol{\lambda}_0} \cap S_1^n$ is regarded as a model lightlike hypersurface in S_1^n. We define

$$T(S_1^n)_{\boldsymbol{\lambda}_0} = \{\boldsymbol{x} \in \mathbb{R}_1^{n+1} \mid \ \boldsymbol{x} - \boldsymbol{\lambda}_0 \in T_{\boldsymbol{\lambda}_0} S_1^n \ \},$$

where $T_{\boldsymbol{\lambda}_0} S_1^n$ is the tangent space of S_1^n at $\boldsymbol{\lambda}_0 \in S_1^n$. We call $T(S_1^n)_{\boldsymbol{\lambda}_0}$ a *tangent affine space* of S_1^n at $\boldsymbol{\lambda}_0 \in S_1^n$. It is easy to show that

$$LC_{\boldsymbol{\lambda}_0} \cap S_1^n = T(S_1^n)_{\boldsymbol{\lambda}_0} \cap S_1^n.$$

We write $LC_{\boldsymbol{\lambda}_0}(S_1^n) = LC_{\boldsymbol{\lambda}_0} \cap S_1^n = T(S_1^n)_{\boldsymbol{\lambda}_0} \cap S_1^n$, which is called a *dS-lightcone* with the vertex $\boldsymbol{\lambda}_0 \in S_1^n$. Therefore, the model lightlike hypersurface is a dS-lightcone.

We consider the contact of spacelike submanifolds with dS-lightcones. Let

$$\mathcal{H} : S_1^n \times S_1^n \longrightarrow \mathbb{R}$$

be a function defined by $\mathcal{H}(\boldsymbol{x}, \boldsymbol{\lambda}) = \langle \boldsymbol{x}, \boldsymbol{\lambda} \rangle - 1$. Given $\boldsymbol{\lambda}_0 \in S_1^n$, we write $\mathfrak{h}_{\lambda_0}(\boldsymbol{x}) = \mathcal{H}(\boldsymbol{x}, \boldsymbol{\lambda}_0)$, so that we have $\mathfrak{h}_{\lambda_0}^{-1}(0) = LC_{\boldsymbol{\lambda}_0}(S_1^n)$. For any $p_0 = \boldsymbol{X}(u_0) \in M$, $\mu_0 \in \mathbb{R}$ and $\boldsymbol{\xi}_0 \in N_1^{dS}(M)_p[\boldsymbol{n}^T]$, we consider the point $\boldsymbol{\lambda}_0 = \boldsymbol{X}(u_0) + \mu_0(\boldsymbol{n}^T(u_0) + \boldsymbol{\xi}_0)$. Then we have

$$\mathfrak{h}_{\lambda_0} \circ \boldsymbol{X}(u_0)) = \mathcal{H} \circ (\boldsymbol{X} \times 1_{AdS^{n+1}})(u_0, \boldsymbol{\lambda}_0) = H(p_0, \boldsymbol{\lambda}_0) = 0,$$

where $\mu_0 = 1/\kappa_\ell(\boldsymbol{n}^T)_i(p_0, \boldsymbol{\xi}_0)$, $i = 1, \dots, s$. We also have relations

$$\frac{\partial \mathfrak{h}_{\lambda_0} \circ \boldsymbol{X}}{\partial u_i}(u_0) = \frac{\partial H}{\partial u_i}(p_0, \boldsymbol{\lambda}_0) = 0, \ i = 1, \dots, s.$$

These imply that the dS-lightcone $\mathfrak{h}_{\lambda_0}^{-1}(0) = LC_{\boldsymbol{\lambda}_0}(S_1^n)$ is tangent to $M = \boldsymbol{X}(U)$ at $p_0 = \boldsymbol{X}(u_0)$. In this case, we call $LC_{\boldsymbol{\lambda}_0}(S_1^n)$ a *tangent dS-lightcone* of $M = \boldsymbol{X}(U)$ at $p_0 = \boldsymbol{X}(u_0)$, which is denoted by $TLC_{\boldsymbol{\lambda}_0}(M)_{p_0}$. Moreover, the tangent dS-lightcone $TLC_{\boldsymbol{\lambda}_0}(M)_{p_0}$ is called an *osculating dS-lightcone* if $\boldsymbol{\lambda}_0 = \mathbb{LF}_{\kappa_\ell(\boldsymbol{n}^T)_i(p_0,\boldsymbol{\xi}_0)}(u_0) \in \mathbb{LF}_M$, for one lightcone principal curvature $\kappa_\ell(\boldsymbol{n}^T)_i(p_0,\boldsymbol{\xi}_0)$. In this case, we call $\boldsymbol{\lambda}_0$ the *center of the lightcone principal curvature* $\kappa_\ell(\boldsymbol{n}^T)_i(p_0,\boldsymbol{\xi}_0)$. Therefore, we can interpret the lightlike focal set as the locus of the centers of the lightcone principal curvatures. This fact is analogous to the notion of the focal sets of submanifolds in Euclidean space.

We now describe the contacts of spacelike submanifolds in S^n_1 with dS-lightcones. We denote by $Q(\boldsymbol{X}, u_0)$ the local ring of the function germ $\widetilde{h}_{\boldsymbol{\lambda}_0} : (U, u_0) \longrightarrow \mathbb{R}$, where $\boldsymbol{\lambda}_0 = \mathbb{LC}_M(u_0, \boldsymbol{\xi}_0, \mu_0)$. We remark that we can explicitly write the local ring as follows:

$$Q_{n+1}(\boldsymbol{X}, u_0) = \frac{C^\infty_{u_0}(U)}{\langle \langle \boldsymbol{X}(u), \boldsymbol{\lambda}_0\rangle - 1\rangle_{C^\infty_{u_0}(U)} + \mathfrak{M}_{u_0}(U)^{n+2}},$$

where $C^\infty_{u_0}(U)$ is the local ring of function germs at u_0.

Let $\mathbb{LH}_{M_i}(\boldsymbol{n}^T_i) : (N_1(M_i)[\boldsymbol{n}^T_i] \times \mathbb{R}, (p_i, \boldsymbol{\xi}_i, \mu_i)) \longrightarrow (S^n_1, \boldsymbol{\lambda}_i)$, $(i = 1, 2)$ be two lightlike hypersurface germs of spacelike submanifold germs $\boldsymbol{X}_i : (U, u^i) \longrightarrow (S^n_1, p_i)$. Let

$$H_i : (U \times S^n_1, (u^i, \boldsymbol{\lambda}_i)) \longrightarrow \mathbb{R}$$

be the dS-height function germ of $\boldsymbol{X}_i$. Then we have the following theorem:

Theorem 5.5. *Let $\boldsymbol{X}_i : (U, u^i) \longrightarrow (S^n_1, p_i)$, $i = 1, 2$, be spacelike submanifold germs such that the corresponding Legendrian submanifold germs $\mathscr{L}_{H_i}(\Sigma_*(H_i))$ are Legendrian stable. We write $\boldsymbol{X}_i(U) = M_i$. Then the following conditions are equivalent*:

(1) $(\mathbb{LH}_{M_1}(N_1(M_1)[\boldsymbol{n}^T_1] \times \mathbb{R}), \boldsymbol{\lambda}_1)$ *and* $(\mathbb{LH}_{M_2}(N_1(M_2)[\boldsymbol{n}^T_2] \times \mathbb{R}), \boldsymbol{\lambda}_2)$ *are diffeomorphic,*
(2) $(\mathscr{L}_{H_1}(\Sigma_*(H_1)), z_1)$ *and* $(\mathscr{L}_{H_2}(\Sigma_*(H_2)), z_2)$ *are Legendrian equivalent,*
(3) H_1 *and* H_2 *are P-$\mathcal{K}$-equivalent,*
(4) h_{1,λ_1} *and* h_{2,λ_2} *are $\mathcal{K}$-equivalent,*
(5) $K(M_1, TLC_{\boldsymbol{\lambda}_1}(M_1)_{p_1}, p_1) = K(M_2, TLC_{\boldsymbol{\lambda}_2}(M_2)_{p_2}, p_2)$.
(6) $Q_{n+1}(\boldsymbol{X}_1, u^1)$ *and* $Q_{n+1}(\boldsymbol{X}_2, u^2)$ *are isomorphic as $\mathbb{R}$-algebras.*

Proof. By Proposition 5.3, conditions (1), (2) and (6) are equivalent. These conditions are also equivalent to the condition that two generating families H_1 and H_2 are P-$\mathcal{K}$-equivalent by Theorem 5.2. If we denote $h_{i,\boldsymbol{\lambda}_i}(u) = H_i(u, \boldsymbol{\lambda}_i)$, then we have $h_{i,\boldsymbol{\lambda}_i}(u) = \mathfrak{h}_{\boldsymbol{\lambda}_i} \circ \boldsymbol{X}_i(u)$. By Theorem 5.1, $K(\boldsymbol{X}_1(U), LC_{\boldsymbol{\lambda}_1}, p_1) = K(\boldsymbol{x}_2(U), LC\lambda_2, p_2)$ if and only if $\widetilde{h}_{1,\boldsymbol{\lambda}_1}$ and $\widetilde{h}_{2,\boldsymbol{\lambda}_2}$ are $\mathcal{K}$-equivalent. This means that (4) and (5) are equivalent. By definition, (3) implies (4). The uniqueness of the infinitesimally $\mathcal{K}$-versal deformation of $h_{i,\boldsymbol{\lambda}_i}$ (cf., [19]) leads that the condition (4) implies (3). This completes the proof. □

6. Spacelike submanifolds with codimension two

In [4], we previously investigated the singularities of lightlike surfaces along spacelike curves in S^3_1. As a consequence, we discovered a new invariant for spacelike curves which estimates the order of contact with de Sitter lightcones in S^3_1. After that, Kaseou [17] investigated the singularities of de Sitter lightlike hypersurfaces of spacelike submanifolds of codimension two in S^n_1. We remark that $N^{dS}(M)[\boldsymbol{n}^T]$ is a double covering of M for codimension two spacelike submanifold M in S^n_1. Then the de Sitter lightlike hypersurface is the image of the mapping $\mathbb{LH}^\pm_M(u, \mu) = \boldsymbol{X}(u) + \mu(\boldsymbol{n}^T \pm \boldsymbol{n}^S)(u)$, which coincides with the lightlike hypersurface along M in [17]. Therefore, all results in the previous sections for de Sitter space are generalizations of the results in [17]. We now consider spacelike surfaces in S^4_1 here. Let $\boldsymbol{X} : U \longrightarrow S^4_1$ be a spacelike embedding from an open subset $U \subset \mathbb{R}^2$. In [17], it was shown that there is the following generic classification theorem. We say that two map germs $f, g : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^p, 0)$ are *$\mathcal{A}$-equivalent* if there exists diffeomorphism germs $\phi : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^n, 0)$ and $\psi : (\mathbb{R}^p, 0) \longrightarrow (\mathbb{R}^p, 0)$ such that $f \circ \phi = \psi \circ g$. Let $\mathrm{Emb}_{\mathrm{sp}}(U, S^4_1)$ be a space of spacelike embeddings from U to S^4_1 with the Whitney C^∞-topology.

Theorem 6.1 ([17]). *There exists an open dense subset $\mathcal{O} \subset \mathrm{Emb}_{\mathrm{sp}}(U, S^4_1)$ such that for any $\boldsymbol{X} \in \mathcal{O}$, the germ of the corresponding lightlike hypersurfaces $\mathbb{LH}^{\pm}_M$ at any point $(u_0, \mu_0) \in U \times \mathbb{R}$ is $\mathcal{A}$-equivalent to one of the map germs A_k $(1 \le k \le 4)$ or $D_4^{\pm}$: where, A_k, $D_4^{\pm}$-map germs $f : (\mathbb{R}^3, 0) \longrightarrow (\mathbb{R}^4, 0)$ are given by*

$A_1;\ f(u_1, u_2, u_3) = (u_1, u_2, u_3, 0),$
$A_2;\ f(u_1, u_2, u_3) = (3u_1^2, 2u_1^3, u_2, u_3),$
$A_3;\ f(u_1, u_2, u_3) = (4u_1^3 + 2u_1u_2, 3u_1^4 + u_2u_1^2, u_2, u_3),$
$A_4;\ f(u_1, u_2, u_3) = (5u_1^4 + 3u_2u_1^2 + 2u_1u_3, 4u_1^5 + 2u_2u_1^3 + u_3u_1^2, u_2, u_3),$
$D_4^+;\ f(u_1, u_2, u_3) = (2(u_1^3 + u_2^3) + u_1u_2u_3, 3u_1^2 + u_2u_3, 3u_2^2 + u_1u_3, u_3),$
$D_4^-;\ f(u_1, u_2, u_3) = \left(\left(\dfrac{u_1^3}{3} - u_1u_2^2\right) + (u_1^2 + u_2^2)u_3, u_2^2 - u_1^2 - 2u_1u_3, 2(u_1u_2 - u_2u_3), u_3\right).$

As a corollary of the above theorem, we have the following generic local classification of AdS-lightlike focal sets along spacelike surfaces. We define $C(2,3,4) = \{(u_1^2, u_1^3, u_1^4) \mid u_1 \in \mathbb{R}\}$, which is called a $(2,3,4)$-*cusp*. We also define

$$C(BF) = \{(10u_1^3 + 3u_2u_1, 5u_1^4 + u_2u_1^2, 6u_1^5 + u_2u_1^3, u_2) \mid (u_1, u_2) \in \mathbb{R}^2\}.$$

We call $C(BF)$ a *C-butterfly* (i.e., the critical value set of the butterfly). Finally we define $C(2,3,4,5) = \{(u_1^2, u_1^3, u_1^4, u_1^5) \mid u_1 \in \mathbb{R}\}$, which is called a $(2,3,4,5)$-*cusp*.

Corollary 6.2. *There exists an open dense subset $\mathcal{O} \subset \mathrm{Emb}_{\mathrm{sp}}(U, S^4_1)$ such that for any $\boldsymbol{X} \in \mathcal{O}$, the germ of the corresponding dS-lightlike focal set $\mathbb{LF}^{\pm}_M$ at any point $(u_0, \mu_0) \in U \times \mathbb{R}$ is diffeomorphic to one of the following set germs at the origin in $\mathbb{R}^4$:*

$A_2;\ \{(0,0)\} \times \mathbb{R}^2,$
$A_3;\ C(2,3,4) \times \mathbb{R},$
$A_4;\ C(BF),$
$D_4^+;\ \{(2(u_1^3 + u_2^3) + u_1u_2u_3, 3u_1^2 + u_2u_3, 3u_2^2 + u_1u_3, u_3) \mid u_3^2 = 36u_1u_2\},$
$D_4^-;\ \left\{\left(\left(\dfrac{u_1^3}{3} - u_1u_2^2\right) + (u_1^2 + u_2^2)u_3, u_2^2 - u_1^2 - 2u_1u_3, 2(u_1u_2 - u_2u_3), u_3\right) \;\middle|\; u_3^2 = u_1^2 + u_2^2\right\}.$

Proof. For A_3, we can calculate the Jacobi matrix of the normal form f in Theorem 5.9:

$$J_f = \begin{pmatrix} 12u_1^2 + 2u_1 & 2u_1 & 0 \\ 12u_1^3 + 2u_1u_2 & u_1^2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

so that rank $J_f < 3$ if and only if $6u_1^2 + u_2 = 0$. Thus, the critical value set of f is

$$C(f) = \{(-8u_1^3, -3u_1^4, -6u_1^2, u_3) \mid (u_1, u_3) \in \mathbb{R}^2\}.$$

It is $C(2,3,4) \times \mathbb{R}$. By a similar calculation, we can show that the germ of A_4 is diffeomorphic to $C(BF)$. For D_4^+, we can calculate the Jacobi matrix o the normal form f:

$$J_f = \begin{pmatrix} 6u_1^2 + u_2u_3 & 6u_2^2 + u_1u_3, u_1u_2 & 0 \\ 6u_1 & u_3 & u_2 \\ u_3 & 6u_2 & u_1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Therefore, rank $J_f < 3$ if and only if

$$\begin{vmatrix} 6u_1^2 + u_2u_3 & 6u_2^2 + u_1u_3, u_1u_2 \\ 6u_1 & u_3 \end{vmatrix} = \begin{vmatrix} 6u_1^2 + u_2u_3 & 6u_2^2 + u_1u_3, u_1u_2 \\ u_3 & 6u_2 \end{vmatrix} = \begin{vmatrix} 6u_1 & u_3 \\ u_3 & 6u_2 \end{vmatrix} = 0,$$

which is equivalent to the condition that $u_3^2 = 36u_1u_2$. For D_4^-, by a calculation similar to the above, we have the condition that $u_3^2 = u_1^2 + u_2^2$. This completes the proof. □

By using the above normal forms, we can investigate the detailed geometric properties of spacelike surface in S_1^4 corresponding to the singularities of dS-lightlike focal sets . However, we have limited space, so that we omit these discussions here.

References

[1] V. I. Arnol'd, S. M. Gusein-Zade and A. N. Varchenko, *Singularities of Differentiable Maps vol. I*. Birkhäuser, 1986.

[2] R. Bousso, *The holographic principle*, REVIEWS OF MODERN PHYSICS 74 (2002), 825–874. DOI: 10.1103/RevModPhys.74.825

[3] S. Chandrasekhar, *The Mathematical Theory of Black Holes*, International Series of Monographs on Physics. 69 Oxford University press, 1983.

[4] T. Fusho and S. Izumiya Lightlike surfaces of spacelike curves in de Sitter 3-space, Journal of Geometry 88 (2008), 19-29 DOI: 10.1007/s00022-007-1944-5

[5] S. Frittelli, E. T. Newman and G. Silva-Ortigoza, *The Eikonal equation in flat space: Null surfaces and their singularities I*, J. Math. Phys. 40 (1999), 383–407. DOI: 10.1063/1.532777

[6] S. Izumiya, D. Pei and T. Sano, *The lightcone Gauss map and the lightcone developable of a spacelike curve in Minkowski 3-space* , Glasgow. Math. J. 42 (2000), 75–89. DOI: 10.1017/S0017089500010107

[7] S.Izumiya, D-H. Pei and T. Sano, *Singularities of hyperbolic Gauss maps*. Proc. London Math. Soc. 86 (2003), 485–512.

[8] S. Izumiya, D. Pei and M.C. Romero Fuster, The lightcone Gauss map of a spacelike surface in Minkowski 4-space, Asian Journal of Mathematics, 8 (2004), 511–530 DOI: 10.4310/AJM.2004.v8.n3.a7

[9] S. izumiya, D-H. Pei, M.C. Romero Fuster and M. Takahashi, The horospherical geometry of submanifolds in hyperbolic space, J. London Math. Soc. 71 (2005), 779–800. DOI: 10.1112/S0024610705006447

[10] S. Izumiya, M. Kossowski, D-H. Pei and M. C. Romero Fuster, *Singularities of lightlike hypersurfaces in Minkowski four-space*, Tohoku Math. J. 58 (2006), 71–88 DOI: 10.2748/tmj/1145390206

[11] S. Izumiya and M. C. Romero Fuster, *The lightlike flat geometry on spacelike submanifolds of codimension two in Minkowski space*, Sel. math., New ser. 13 (2007), 23–55

[12] S. Izumiya, *Lightlike hypersurfaces in Lorentz-Minkowski space*, The second Japanese-Australian Workshop on Real and Complex Singularities, RIMS Kokyuroku 1610 (2008), 51–65

[13] S. Izumiya, *Total lightcone curvatures of spacelike submanifolds in Lorentz-Minkowski space*, Diff. Geom. and Its Appl., 34 (2014) 103–127 DOI: 10.1016/j.difgeo.2014.03.012

[14] S. Izumiya and T. Sato, *Lightlike hypersurfaces along spacelike submanifolds in Minkowski spacetime*, Journal of Geometry and Physics, 71 (2013), 30–52 DOI: 10.1016/j.geomphys.2013.03.005

[15] S. Izumiya and T. Sato, *Lightlike hypersurfaces along spacelike submanifolds in anti-de Sitter space*, preprint (2012)

[16] M. Kasedou, *Spacelike submanifolds in de Sitter space*, Demonstratio Mathematica XLIII (2010),401–418

[17] M. Kasedou, *Spacelike submanifolds of codimension two in de Sitter space*, Journal of Geometry and Physics, 60, (2010), 31–42 DOI: 10.1016/j.geomphys.2009.08.005

[18] J. A. Little, *On singularities of submanifolds of high dimensional Euclidean space*, Ann. Mat. Pura Appl.(4) 83 (1969), 261–335. DOI: 10.1007/BF02411172

[19] J. Martinet, *Singularities of Smooth Functions and Maps*, London Math. Soc. Lecture Note Ser. 58, Cambridge Univ. Press, Cambridge-New York, 1982.
[20] C. W. Misner, K. S. Thorpe and J. W. Wheeler, *Gravitation*, W. H. Freeman and Co., San Francisco, CA, 1973.
[21] J. A. Montaldi, *On contact between submanifolds*, Michigan Math. J. 33, (1986), 81–85.
[22] B. O'Neill, *Semi-Riemannian Geometry*, Academic Press, New York, 1983.
[23] M. Siino and T. Koike, *Topological classification of black holes: generic Maxwell set and crease set if a horizon*, International Journal of Physics D, 20 (2011), 1095–1122 DOI: 10.1142/S0218271811019220
[24] V. M. Zakalyukin, *Lagrangian and Legendrian singularities*, Funct. Anal. Appl. (1976), 23–31.

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LINKS OF SINGULARITIES UP TO REGULAR HOMOTOPY

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ABSTRACT. We classify links of the singularities $x^2 + y^2 + z^2 + v^{2d} = 0$ in $(\mathbb{C}^4, 0)$ up to regular homotopies precomposed with diffeomorphisms of $S^3 \times S^2$. Let us denote the link of this singularity by L_d and denote by i_d the inclusion $L_d \subset S^7$. We show that for arbitrary diffeomorphisms $\varphi_d : S^3 \times S^2 \longrightarrow L_d$ the compositions $i_d \circ \varphi_d$ are image regularly homotopic for two different values of d, $d = d_1$ and $d = d_2$, if and only if $d_1 \equiv d_2 \mod 2$.

1. INTRODUCTION

It is well-known that the infinite number of Brieskorn equations in $\mathbb{C}^5$

$$z_1^{6k-1} + z_2^3 + z_3^2 + z_4^2 + z_5^2 = 0, \quad \text{(intersected with } S^9 = \{\Sigma|z_i|^2 = 1\})$$

describe the finite number of homotopy spheres. Why do we have infinitely many equations for a finite number of homotopy spheres? The answer was given in [E-Sz]: These equations give all the embeddings of these homotopy spheres in S^9 up to regular homotopy.

The present paper grew out from an attempt to investigate the analogous question for the equations

$$(*) \qquad x^2 + y^2 + z^2 + v^k = 0.$$

It was proved in [K-N] that the links of the singularities $(*)$ are S^5 or $S^3 \times S^2$ depending on the parity of k. Again we have infinite number of equations for both diffeomorphism types of links. So it seems natural to pose the analogous

Question: What are the differences between the links for different values of k of the same parity? Do they represent different immersions up to regular homotopy?

For k odd, when the link is S^5, the question about the regular homotopy turns out to be trivial, since any two immersions of S^5 to S^7 are regularly homotopic. (By Smale's result, see [S1], the set of regular homotopy classes of immersions $S^5 \longrightarrow S^7$ can be identified with $\pi_5(SO_7)$. The later group is trivial by Bott's result [B].)

The situation is quite different for k even. Put $k = 2d$ and let us denote by X_d the algebraic variety defined by the equation $(*)$, by L_d its link, and by i_d the inclusion $L_d \hookrightarrow S^7$. In this case the question on regular homotopy classes of i_d turns out to be not well-posed.

It is true that L_d is diffeomorphic to $S^3 \times S^2$ for any d, but the question about the regular homotopy makes sense only after having given a concrete diffeomorphism $\varphi_d : S^3 \times S^2 \longrightarrow L_d$, and only then we can ask about the regular homotopy classes

$$i_d \circ \varphi_d : S^3 \times S^2 \longrightarrow S^7.$$

(In the case of Brieskorn equations precomposing an immersion $f : \Sigma^7 \longrightarrow S^9$ with an orientation preserving self-diffeomorphism of the homotopy sphere Σ^7 does not change the regular homotopy class of the immersion f. This is not so for the manifold $S^3 \times S^2$.)

Definition (see [P]). Given manifolds M, N, and two immersions f_0 and f_1 from M to N, we say that f_0 and f_1 are *image-regular homotopic* if there is a self-diffeomorphism φ of M such that f_1 is regularly homotopic to $f_0 \circ \varphi$.

Notation:

1) $I(M,N)$ will denote the image-regular homotopy classes of immersions of M to N. The image regular homotopy class of an immersion f will be denoted by $\mathrm{im}\,[f]$.

2) Recall that an immersion is called framed if its normal bundle is trivialized. Fr-Imm (M,N) will denote the framed regular homotopy classes of framed immersions of M to N.

In the case when the immersion f is framed $\mathrm{reg}\,[f]$ will denote its framed regular homotopy class.

Remark. Note that for the inclusions $i_d : L_d \subset S^7$ their regular homotopy classes $\mathrm{reg}\,[i_d]$ are not well-defined, but their image regular homotopy classes $\mathrm{im}\,[i_d]$ are well-defined.

Formulation of the results

Theorem 1. *For any simply connected, stably parallelizable, 5-dimensional manifold M^5 the framed regular homotopy classes of framed immersions in S^7 can be identified with $H^3(M;\mathbb{Z})$, i.e.*

$$\text{Fr-Imm}\,(M^5, S^7) = H^3(M;\mathbb{Z}).$$

Corollary. *In particular,*

$$\text{Fr-Imm}\,(S^3\times S^2, S^7) = \mathbb{Z}.$$

Theorem 2. *The set $I(S^3\times S^2, S^7)$ of image-regular homotopy classes of framed immersions $S^3\times S^2 \longrightarrow S^7$ can be identified with $\mathbb{Z}_2$.*

Theorem 3. *The inclusions $i_d : \ L_d \hookrightarrow S^7$ for $d = d_1$ and d_2 represent the same element in $I(S^3\times S^2, S^7) = \mathbb{Z}_2$ (i.e. $\mathrm{im}\,[i_{d_1}] = \mathrm{im}\,[i_{d_2}]$) if and only if $d_1 \equiv d_2 \mod 2$.*

Remark. The identifications in the above Theorems arise only after we have fixed a parallelization of the manifolds (or a stable parallelization). (Different parallelizations provide different identifications. For the Corollary these identifications differ by an affine shift $x \mapsto x + a$, where $a \in \pi_3(SO) = \mathbb{Z}$ is the difference of the two parallelizations. Similarly, in Theorem 2, a is replaced by a mod 2 in $\mathbb{Z}_2 = \pi_3(SO)/\mathrm{im}\, j_*(\pi_3(SO_3))$, where j is the inclusion $j : SO_3 \subset SO$. Now we describe a concrete stable parallelization of $S^3\times S^2$ we shall use.

Hence, we want to choose a trivialization of the stable tangent bundle

$$T(S^3\times S^2)\oplus \mathcal{E}^1 \longrightarrow S^3\times S^2,$$

where $\mathcal{E}^1$ is the trivial real line bundle. This 6-dimensional vector bundle is the same as the restriction $T(S^3\times\mathbb{R}^3)\Big|_{S^3\times S^2} = (p_1^*TS^3 \oplus p_2^*T\mathbb{R}^3)\Big|_{S^3\times S^2}$, where p_1 and p_2 are the projections of $S^3\times\mathbb{R}^3$ onto the factors. The quaternionic multiplication on S^3 gives a trivialization of TS^3, i.e. an identification with $S^3\times\mathbb{R}^3$. We need a trivialization of $T(TS^3)$. The standard spherical metric on S^3 gives a connection on the bundle $TS^3 \longrightarrow S^3$, that is a "horizontal" $\mathbb{R}^3 \subset T(TS^3)$ at any point. The trivialization of TS^3 gives a trivialization of both the horizontal and the vertical (tangent to the fibers) components in $T(TS^3)$. Restricting this to the sphere bundle $S(TS^3) = S^3\times S^2$ we obtain the required trivialization of

$$T(TS^3)\Big|_{S^3\times S^2} = T(S^3\times\mathbb{R}^3)\Big|_{S^3\times S^2} = T(S^3\times S^2)\oplus\mathcal{E}^1.$$

Proof of Theorem 1. Having fixed a stable parallelization of M^5, any framed immersion $f :\ M^5 \longrightarrow \mathbb{R}^q$ gives a map $M^5 \longrightarrow SO_q$ that – by a slight abuse of notation – we will denote by df.

By the Smale–Hirsch immersion theory [S1, H] the map

$$\begin{array}{ccc} \text{Fr-Imm}\,(M,\mathbb{R}^q) & \longrightarrow & [M, SO_q] \\ \text{reg}\,[f] & \longrightarrow & [df] \end{array}$$

induces a bijection, where $[M, SO_q]$ denotes the homotopy classes of maps $M \longrightarrow SO_q$.

Since M^5 is simply connected there is a cell-decomposition having a single 0-cell, a single 5-cell, and no 1-dimensional, neither 4-dimensional cells.

Let $\overset{\circ}{M}$ be the punctured M^5: $\overset{\circ}{M} = M^5 \setminus D^5$. From the Puppe sequence of the pair $(\overset{\circ}{M}, \partial \overset{\circ}{M})$ (see [Hu]),

$$S^4 = \partial \overset{\circ}{M} \subset \overset{\circ}{M} \subset M \longrightarrow S^5,$$

it follows that the restriction map $[M^5, SO_q] \to [\overset{\circ}{M}, SO_q]$ is a bijection, since $\pi_4(SO) = 0$ and $\pi_5(SO_q) = 0$.

Now consider the Puppe sequence of the pair $(\overset{\circ}{M}, sk_2\, M)$. Note that $sk_2\, M$ is a bouquete of 2-spheres, while the quotient $\overset{\circ}{M}/sk_2\, M$ is homotopically equivalent to a bouquette of 3-spheres. Hence, a part of the Puppe sequence looks like this:

$$sk_2\, M \subset \overset{\circ}{M} \longrightarrow \vee S^3 \longrightarrow S(sk_2\, M) = \vee S^3$$

where $S(\ )$ means the suspension. Mapping the spaces of this Puppe sequence to SO_q, $q \geq 5$, we obtain the following exact sequence of groups (we omit q):

$$\big[sk_2\, \overset{\circ}{M},\, SO\big] \longleftarrow \big[\overset{\circ}{M},\, SO\big] \longleftarrow \big[\vee S^3,\, SO\big] \overset{\alpha}{\longleftarrow} \big[S(sk_2\, M), SO\big].$$

Here $\big[sk_2\, \overset{\circ}{M},\, SO\big] = 0$, because $\pi_2(SO) = 0$.

Since $\pi_3(SO) = \mathbb{Z}$ the group $[\vee S^3,\, SO]$ can be identified with the group of 3-dimensional cochains of M with integer coefficients, i.e. $[\vee S^3,\, SO] = C^3(M;\mathbb{Z})$.

Since there are no 4-dimensional cells this is also the group of 3-dimensional cocycles. The group $\big[S(sk_2\, M),\, SO\big]$ can be identified with the group of 2-dimensional cochains $C^2(M;\mathbb{Z})$.

Lemma. *The map α can be identified with the coboundary map*

$$\delta\colon C^2(M;\mathbb{Z}) \longrightarrow C^3(M;\mathbb{Z}).$$

Proof of this Lemma will be given in the Appendix.

Hence the cokernel of α, i.e. $\big[\overset{\circ}{M}, SO\big] = \text{Fr-Imm}\,(M, \mathbb{R}^q)$ can be identified with the cokernel of δ, i.e. with $H^3(M;\mathbb{Z})$. □

Remark 1. In the case when $M = S^3 \times S^2$ and $N \in S^2$ is a fixed point in S^2, for example the North pole, the inclusion $S^3 \hookrightarrow M$, $x \longrightarrow (x, N)$ gives an isomorphism

$$[M, SO] \longrightarrow [S^3, SO].$$

Hence, for $M = S^3 \times S^2$ two framed immersions $M^5 \longrightarrow \mathbb{R}^7$ (or $M^5 \longrightarrow S^7$) are regularly homotopic if their restrictions to $S^3 \times N$ are framed regularly homotopic (adding the two normal vectors of S^3 in M^5 to the framing).

Lemma 1. *The inclusion $j\,:\ SO_3 \hookrightarrow SO_q$ $(q \geq 5)$ induces in π_3 the multiplication by 2 (if we choose the generators in $\pi_3(SO_3) = \mathbb{Z}$ and in $\pi_3(SO_q) = \mathbb{Z}$ properly), i.e., for any $x \in \pi_3(SO_3) = \mathbb{Z}$ the image $j_*(x) \in \pi_3(SO) = \mathbb{Z}$ is $2x$.*

Proof. It is well-known that $\pi_3(SO_5) \approx \pi_3(SO_6) \approx \cdots \approx \pi_3(SO)$ and by Bott's result [B] $\pi_3(SO) \approx \mathbb{Z}$. Let us consider $V_2(\mathbb{R}^5) = SO_5/SO_3$. It is well-known that $\pi_3\big(V_2(\mathbb{R}^5)\big) = \mathbb{Z}_2$ (see for example [M-S]). It is also well-known that $\pi_3(SO_3) = \mathbb{Z}$.

Now the exact sequence of the fibration $SO_5 \longrightarrow V_2(\mathbb{R}^5)$ gives that the homomorphism $\pi_3(SO_3) \longrightarrow \pi_3(SO_5)$ induced by the inclusion is a multiplication by $+2$ (or -2, but choosing the generators properly it can be supposed that it is multiplication by $+2$). ☐

Remark 2. It is well-known that $\pi_3(SO_4) = \pi_3(S^3) \oplus \pi_3(SO_3)$ and the map $j_{4*}:\ \pi_3(SO_4) \longrightarrow \pi_3(SO_5)$ induced by the inclusion

$$j_4:\ SO_4 \hookrightarrow SO_5$$

is epimorphic.

It follows that j_{4*} maps $\pi_3(S^3) = \mathbb{Z}$ to the group $\mathbb{Z}_2 = \pi_3(SO_5)/j_{4*}\big(\pi_3(SO_3)\big)$ epimorphically.

From now on we shall denote by M the manifold $S^3 \times S^2$ (except in the Appendix). We shall write simply S^3 for the subset $S^3 \times N \subset S^3 \times S^2$, where $N \in S^2$.

Lemma 2. *For any class $2m \in 2\mathbb{Z} = \operatorname{im} j_{4*} \subset \mathbb{Z} = \pi_3(SO)$, there is a diffeomorphism $\alpha_m:\ M \longrightarrow M$ such that for any framed immersion $f:\ M \longrightarrow \mathbb{R}^7$ the difference of the regular homotopy classes of f and $f \circ \alpha_m$ is $2m$, i.e.*

$$\operatorname{reg}[f \circ \alpha_m] - \operatorname{reg}[f] \in \pi_3(SO) = \mathbb{Z}$$

is $2m$.

Proof. Let $\mu_m:\ S^3 \longrightarrow SO_3$ be a map representing the class $m \in \pi_3(SO_3)$ and define the diffeomorphism

$$\alpha_m:\ S^3 \times S^2 \longrightarrow S^3 \times S^2$$

by the formula

$$(x, y) \longmapsto (x, \mu_m(x)y).$$

We have the following diagram:

$$\begin{array}{ccc} \operatorname{reg}[f] \in \text{Fr-Imm}\,(M, \mathbb{R}^q) & \longrightarrow & \text{Fr-Imm}\,(S^3, \mathbb{R}^q) \ni \operatorname{reg}\left[f\Big|_{S^3}\right] \\ \downarrow\approx & & \downarrow\approx \\ [M, SO_q] & \longrightarrow & [S^3, SO_q] \\ df & \longmapsto & df\Big|_{S^3} \\ d(f \circ \alpha_m) & \longmapsto & d(f \circ \alpha_m)\Big|_{S^3} \end{array}$$

It shows that the regular homotopy class of the (framed) immersion f is detected by the homotopy class of $df\Big|_{S^3}$ in $\pi_3(SO)$, while the regular homotopy class of $f \circ \alpha_m$ is detected by the homotopy class of $d(f \circ \alpha_m)\Big|_{S^3}$.

So we have to compare the homotopy classes of maps

$$df\Big|_{S^3}:\ S^3 \longrightarrow SO_q \quad \text{and} \quad d(f \circ \alpha_m)\Big|_{S^3}:\ S^3 \longrightarrow SO_q.$$

By the chain rule one has:

$$d(f \circ \alpha_m)\Big|_{S^3} = df\Big|_{\alpha_m(S^3)} \cdot d\alpha_m\Big|_{S^3}.$$

The restriction map $\alpha_m\Big|_{S^3}:\ S^3 \longrightarrow S^3 \times S^2$ is homotopic to a map into $S^3 \vee S^2$, representing in the third homotopy group $\pi_3(S^3 \vee S^2) = \mathbb{Z} \oplus \mathbb{Z}$ the element $(1, *)$, where $*$ is an integer,

$* \in \pi_3(S^2) = \mathbb{Z}$ (at this point its value is not important, but later we shall show that it is m, see Lemma A). Since the map df maps $S^3 \times S^2$ into SO and $\pi_2(SO) = 0$, the map $df\Big|_{S^3 \vee S^2}$ can be extended to $S^3 \vee D^3 \cong S^3$.

Finally we have that $d(f \circ \alpha_m)\Big|_{S^3}$ is homotopic to the pointwise product of the maps $df\Big|_{S^3}$ and $d\alpha_m\Big|_{S^3}$.

But it is well-known that this gives the sum of the homotopy classes $\left[df\Big|_{S^3}\right] \in \pi_3(SO)$ and $\left[d\alpha_m\Big|_{S^3}\right] \in \pi_3(SO)$.

It remained to show the following

Sublemma. $\left[d\alpha_m\Big|_{S^3}\right] = 2m \in \pi_3(SO_q) = \mathbb{Z}$.

Proof. The differential $d\alpha_m$ acts on $T(S^3 \times \mathbb{R}^3)\Big|_{S^3 \times S^2} = p_1^* TS^3 \oplus p_2^* T\mathbb{R}^3\Big|_{S^3 \times S^2}$ as follows: by identity on $p_1^* TS^3$ and by $\mu_m(x)$ on $(x, y) \times \mathbb{R}^3$ for any $x \in S^3$, $y \in S^2$.

Hence, $d\alpha_m\Big|_{S^3}$ is $j \circ \mu_m$, where $j: SO_3 \hookrightarrow SO_q$ is the inclusion. Recall that the map $\mu_m: S^3 \longrightarrow SO_3$ was chosen so that its homotopy class $[\mu_m] \in \pi_3(SO_3)$ is $m \in \mathbb{Z} = \pi_3(SO_3)$. Since j_* is "the multiplication by 2" map it follows that $\left[d\alpha_m\Big|_{S^3}\right] = 2m$. □

This ends the proof of Lemma 2 too. □

Proposition. *Any self-diffeomorphism of $S^3 \times S^2$ changes the regular homotopy class of any immersion by adding an element of the subgroup in $\operatorname{im} j_* = 2\mathbb{Z} \subset \mathbb{Z} = \pi_3(SO)$. That is for any framed immersion $f: M \longrightarrow \mathbb{R}^q$ with (framed) regular homotopy class*

$$\operatorname{reg}[f] \in [M, SO] = \pi_3(SO)$$

and any diffeomorphism $\varphi: M \longrightarrow M$ the difference of regular homotopy classes

$$\operatorname{reg}[f] - \operatorname{reg}[f \circ \varphi]$$

belongs to the subgroup $\operatorname{im} j_* = 2\mathbb{Z}$ in $\mathbb{Z} = \pi_3(SO)$.

The proof will rely on the following two lemmas (Lemma A and Lemma B).

Definition. A self-diffeomorphism $\varphi: S^3 \times S^2 \longrightarrow S^3 \times S^2$ will be called *positive* if it induces on $H_3(S^3 \times S^2) = \mathbb{Z}$ the identity.

Lemma A. *For any positive self-diffeomorphism φ there exists a natural number $m \in \mathbb{Z}$ such that for $N \in S^2$ the restrictions $\varphi\Big|_{(S^3 \times N)}$ and $\alpha_m\Big|_{(S^3 \times N)}$ represent the same homotopy class in $\pi_3(M)$.*

Lemma B. *Let φ and ψ be self-diffeomorphisms of M such that the images of $S^3 \times N$ at φ and ψ represent the same element in $\pi_3(M)$. Then for any framed-immersion $f: M \longrightarrow \mathbb{R}^7$ the regular homotopy classes of $f \circ \varphi$ and $f \circ \psi$ coincide.*

Proof of Lemma B. Let us extend the self-diffeomorphisms φ and ψ to those of $M \times D^q$ by taking the product with the identity map of D^q, for some large q, and denote these self-diffeomorphisms of $M \times D^q$ by $\hat{\varphi}$ and $\hat{\psi}$. Similarly we shall denote by $\hat{f}$ the product of f with the standard inclusion $D^q \subset \mathbb{R}^q$.

By the Smale–Hirsch theory [S1, H] (or by the so-called Compression Theorem of Rourke–Sanderson [R-S]) the restriction induces a bijection

$$\text{Fr-Imm}\,(M,\mathbb{R}^7) \longleftarrow \text{Fr-Imm}\,(M\times D^q,\mathbb{R}^{7+q}).$$

Again the regular homotopy class of a framed immersion in

$$\text{Fr-Imm}\,(M,\mathbb{R}^{7+q}) = \text{Fr-Imm}\,(M\times D^q,\mathbb{R}^{7+q})$$

is uniquely defined by the restriction to $S^3 (= S^3\times N)$.

The maps $\hat\varphi$ and $\hat\psi$ restricted to the sphere $S^3\times N$ are framed isotopic. By Thom's isotopy lemma [T] there is an isotopy $\Psi_t:\ M\times D^q \longrightarrow M\times D^q$ such that $\Psi_0=\hat\varphi$ and $\Psi_1=\hat\psi$.

It follows that the induced maps $d\hat\varphi:\ M\longrightarrow SO$ and $d\hat\psi:\ M\longrightarrow SO$ are homotopic. Hence, the framed-regular homotopy classes of $\hat f\circ\hat\varphi$ and $\hat f\circ\hat\psi$ coincide. Then the compositions $f\circ\varphi$ and $f\circ\psi$ are also regularly homotopic. □

Proof of Lemma A. Let m be the homotopy class of the composition

$$S^3 \overset{i_\varphi}{\hookrightarrow} S^3\times S^2 \overset{p}{\longrightarrow} S^2,$$

where i_φ is the inclusion $x\mapsto\varphi(x,N)$ and p is the projection $S^3\times S^2\longrightarrow S^2$. We claim that the maps $\varphi' = p\circ\varphi\Big|_{(S^3\times N)}$ and $\alpha'_m = p\circ\alpha_m\Big|_{(S^3\times N)}$ are homotopic maps from S^3 to S^2. To show this it is enough to compute the Hopf invariants of these maps.

Let us consider first the case $m=1$. We need to show that the Hopf invariant of α'_1 is equal to 1.

The map $\mu_1:\ S^3\longrightarrow SO_3$ representing the generator in $\pi_3(SO_3)$ can be provided by the standard double covering $S^3\longrightarrow SO_3$. Then α_1 is the self-diffeomorphism of $S^3\times S^2$

$$\alpha_1(x,y) = \big(x,\mu_1(x)y\big)$$

and α'_1 is the composition of the following three maps: the inclusion

$$S^3\hookrightarrow S^3\times S^2,\quad x\longmapsto (x,N);$$

the map α_1 and the projection $p:\ S^3\times S^2\longrightarrow S^2$.

In order to compute the Hopf invariant of $\alpha'_1:\ S^3\longrightarrow S^2$ first we need to compute the preimage of a regular value. Let us compute first the preimage of N in S^3, i.e., $(\alpha'_1)^{-1}(N)$. The map α'_1 can be further decomposed as the composition of $\mu_1:\ S^3\longrightarrow SO_3$ with the evaluation map $e:\ SO_3\longrightarrow S^2$, $g\mapsto g(N)$, for $g\in SO_3$. The set $e^{-1}(N)$ is the subgroup $SO_2\subset SO_3$, which consists of the rotations around the line $\overrightarrow{(N,-N)}$ (the stabilizer subgroup of N).

When we identify SO_3 with the ball D^3_π of radius π with identified antipodal points on the boundary S^2_π, then this subgroup SO_2 corresponds to the diameter $\overline{N,-N}$ with identified endpoints N and $-N$. The preimage of this diameter at $\mu_1:\ S^3\longrightarrow SO_3$ is a great circle. If we take any other point V in S^2, then $e^{-1}(V)$ is a coset of the previous subgroup SO_2. Then its preimage at μ_1 is also a great circle. Therefore the linking number of two such preimages is 1.

The map α'_m can be obtained from α'_1 by precomposing it with a degree m map $S^3\longrightarrow S^3$. Hence the Hopf invariant of α'_m is m. □

Parametrizations of the links L_d (or, equivalently, of the singularities X_d)

Let us denote by ζ the complex $\mathbb{C}^2$-bundle $T\mathbb{C}P^1\oplus\varepsilon^1_C$ over $\mathbb{C}P^1=S^2$, where $T\mathbb{C}P^1$ is the tangent bundle of $\mathbb{C}P^1$, and ε^1_C is the trivial complex line bundle. Note that the bundle ζ considered as a real $\mathbb{R}^4$-bundle is isomorphic to the trivial bundle. Hence its total space is diffeomorphic to $S^2\times\mathbb{R}^4$. Let us denote by $E_0(\zeta)$ the complement of the zero section in the total space of the bundle ζ. We shall give below a *diffeomorphism* of this space $E_0(\zeta)$ onto $X_d\setminus 0$.

The existence of such a diffeomorphism will give a new proof of the result of [K-N] about the diffeomorphism type of L_d.

Proposition. *L_d is diffeomorphic to $S^3 \times S^2$.*

Proof. $X_d \setminus 0$ is diffeomorphic to $L_d \times \mathbb{R}^1$, and the space $E_0(\zeta)$ is diffeomorphic to $S^3 \times S^2 \times \mathbb{R}^1$. For simply connected 5-manifolds it is well-known, that two such manifolds are diffeomorphic if their products with the real line are diffeomorphic (see [Ba], Theorem 2.2). Hence L_d and $S^3 \times S^2$ are diffeomorphic. □

Next we give a concrete parametrization:

$$\varphi_d : \ E_0(\zeta) \longrightarrow X_d \setminus 0 = \big\{x, y, z, v \ \big| \ x^2 + y^2 + z^2 + v^{2d} = 0, \ |x| + |y| + |z| + |v| \neq 0\big\}.$$

The composition $i_d \circ \varphi_d$ (or its restriction to $\varphi_d^{-1}(S^7)$) will give a framed-immersion

$$S^3 \times S^2 \longrightarrow S^7,$$

and its regular homotopy class reg $[i_d \circ \varphi_d]$ will turn out to be the number

$$d \in \mathbb{Z} = \text{Fr-Imm}\,(S^3 \times S^2, S^7).$$

This will imply that the image-regular homotopy class of the link L_d in S^7 is $d \mod 2$ in $\mathbb{Z}_2 = I(S^3 \times S^2, S^7)$.

Proof of Theorem 3. For arbitrary manifolds N and Q the natural map

$$\text{Fr-Imm}\,(N, Q) \longrightarrow \text{Fr-Imm}\,(N, Q \times \mathbb{R}^1)$$

induces a bijection — by the Smale–Hirsch immersion theory (or by the Compression Theorem of Rourke–Sanderson). Hence Fr-Imm $(X_d \setminus 0 \subset \mathbb{C}^4 \setminus 0) =$ Fr-Imm $(S^3 \times S^2 \subset S^7)$. By a coordinate transformation of $\mathbb{C}^4$ we obtain the following equivalent equation defining X_d

$$X_d = \big\{x, y, z, v \ \big| \ xy - z(z + v^d) = 0\big\}.$$

The parametrization of $X_d \setminus 0$ is the following.

The inclusion

$$E_0(\zeta) \xrightarrow{\Psi} \mathbb{C}^4 = \{(x, y, z, v) \mid x, y, z, v \in \mathbb{C}\} \text{ with image } \ \text{im}\,\Psi = X_d \setminus 0$$

will be described on two charts:

1) $\big((a : b), x, v\big)$, where $a, b, x, v \in \mathbb{C}$, $b \neq 0$, $(a : b) \in \mathbb{C}P^1$, and $\|x\| + \|v\| \neq 0$. Put $t = \frac{a}{b} \in \mathbb{C}$. The map Ψ on this chart will be given by the formula

$$\Psi : \ (t, x, v) \longrightarrow (x, t^2x + tv^d, tx, v).$$

2) For $a \neq 0$ denote the quotient $\frac{b}{a}$ by t'. On the part of $E_0(\zeta)$ that projects to $\mathbb{C}P^1 \setminus (1 : 0)$ (that is diffeomorphic to $\mathbb{C}P^1 \setminus (1 : 0) \times (\mathbb{C}^2 \setminus 0)$) consider the coordinates (t', y, v) and define Ψ by the formula

$$\Psi : \ (t', y, v) \longrightarrow (t'^2y - t'v^d, y, t'y - v^d, v).$$

The change of coordinates between the two coordinate charts of $E_0(\zeta)$ is

$$t' = t^{-1}, \ v = v, \ x = t'^2y - t'v^d \ \text{ or equivalently}$$

$$y = t^2x + tv^d.$$

In order to see that these local coordinates give indeed the bundle ζ over $\mathbb{C}P^1$ we can precompose the first local system with the map $(t, x, v) \mapsto (t, x - tv^d, v)$. (Note that this map can be connected to the identity by the diffeotopy $(t, x, v) \mapsto (t, x - stv^d, v)$, $0 \leq s \leq 1$.) Then the change from the first coordinate system to the second one for $t \in S^1$ on the equator of $S^2 = \mathbb{C}P^1$ will be given by the map $(t, x, v) \mapsto (t, t^2x, v)$, where $x, v \in \mathbb{C}$. Now it is clear that the obtained bundle is $\zeta = T\mathbb{C}P^1 \oplus \varepsilon^1_C$. (The map of the equator to $U(2)$ defining the bundle ζ gives in $\pi_1(U(2))$ the double of the generator, and its image in $\pi_1(SO_4) = Z_2$ is trivial. That is why the bundle ζ is trivial as a real bundle although it has first Chern class equal 2 as a complex bundle.) Note that Ψ maps the part of the first chart corresponding to the points $t = 0$, (i.e., the space $\mathbb{C}^2 = \{(0:1), x, v\}$) identically onto the coordinate space $\mathbb{C}^2_{x,v} = \{x, 0, 0, v\}$ of $\mathbb{C}^4$. The restriction of Ψ to this part determines the framed immersion of $X_d \setminus 0$ to $\mathbb{C}^4$. Hence, the immersion itself is very simple: just the inclusion of $\mathbb{C}^2 \setminus 0 \to \mathbb{C}^4$. But we need to consider also the framing. It is coming a) from the paramatrization Ψ and b) from the defining equation of X_d.

a) The parametrization gives the complex vector field

$$\frac{\partial \Psi}{\partial t}\Big|_{t=0} = (0, v^d, x, 0).$$

b) The defining equation $g(x, y, z, v) = xy - z(z + v^d) = 0$ at the points $(x, 0, 0, v)$ gives the complex vector field

$$\operatorname{grad} g(x, 0, 0, v) = (0, x, -v^d, 0).$$

These two complex vector fields have zero first and last complex coordinates (on the coordinate subspace $\mathbb{C}^2_{x,v} = \{x, 0, 0, v\}$). Hence, we shall write only their second and third coordinates: those are (v^d, x) and $(-x, v^d)$ respectively. These two complex vectors give four real vector fields if we add their i-images as well. Let us denote by a_1 and a_2 the real and imaginary coordinates of v^d: $v^d = a_1 + ia_2$. Similarly x_1 and x_2 are those of x, i.e., $x = x_1 + ix_2$. Then the four real vectors in $\mathbb{R}^4 = \mathbb{C}^2 = (0, y, z, 0)$ are:

$$\begin{aligned} \mathbf{u}_1 &= (a_1, a_2, x_1, x_2) \\ \mathbf{u}_2 &= (a_2, -a_1, x_2, -x_1) \\ \mathbf{u}_3 &= (x_1, x_2, -a_1, -a_2) \\ \mathbf{u}_4 &= (-x_2, x_1, a_2, -a_1). \end{aligned}$$

The map $(x, v) \in \mathbb{R}^4 \setminus 0 \longrightarrow (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4)$ can be decomposed as a degree d branched covering $(x, v) \mapsto (x, v^d)$ and a map representing an element in $\pi_3(SO_4) = \pi_3(S^3) \oplus \pi_3(SO_3)$ of the form $(1, *)$ for some unknown element $*$ in $\pi_3(SO_3)$. (This is because the map

$$(x, v^d) = (x_1, x_2, a_1, a_2) \mapsto \mathbf{u}_1 = (a_1, a_2, x_1, x_2)$$

is almost the identity, it differs only by an even permutation of the coordinates.) Hence the composition represents an element of the form $(d, ?) \in \pi_3(S^3) \oplus \pi_3(SO_3)$, and its image in $\pi_3(SO)/j_{4*}(\pi_3(SO_3)) = \mathbb{Z}_2$ is $d \bmod 2$, see Remark 2. That finishes the proof of Theorem 3. □

Appendix

For any space Y let us denote by CY the cone over Y. Here we show that the map provided by the Puppe sequence

$$\alpha: \ \left[C(\overset{\circ}{M}) \cup C(sk_2 M), SO\right] \longrightarrow \left[\overset{\circ}{M} \cup C(sk_2 M), SO\right]$$

can be identified with the coboundary map in the cochain complex:

$$\delta: \ C^2(M; \mathbb{Z}) \longrightarrow C^3(M; \mathbb{Z}).$$

We have seen that the sources and targets of δ and α can be identified.

For simplicity let us consider the situation when $sk_2 M = S^2$ and M has a single 3-cell D^3, attached to this S^2 by a map θ of degree k. Then $\overset{\circ}{M} = S^2 \cup_\theta D^3$.

Let us denote the sets

$$\overset{\circ}{M} \cup C(sk_2 M) \quad \text{and} \quad C\overset{\circ}{M} \cup C(sk_2 M)$$

by A and B respectively.

Clearly we can choose *any* degree k map for θ in order to study the induced map α. Take for θ a branched k-fold cover of S^2 along S^0. Then the inclusion $A \subset B$ can be described homotopically as follows:

In $S^3 \times [0,1]$ contract an interval $* \times [0,1]$ for some $* \in S^3$ to a point. A will be identified with $S^3 \times \{0\}$. Further on $S^3 \times \{1\}$ identify the points that are mapped into the same point by the suspension of θ. The part of B coming from $S^3 \times \{1\}$ will be denoted by B_1. The space B_1 is a deformation retract of B.

Let us denote by r the retraction $B \longrightarrow B_1$. Clearly, its restriction $r|_A : A \longrightarrow B_1$ is a degree k map (it is actually the suspension of the branched covering θ). So the inclusion $A \subset B$ induces in the 3-dimensional homology group H_3 (or in π_3) a multiplication by k.

The proof of the special case (when in M there is a single 2-cell and a single 3-cell) is finished. The general case follows easily taking first the quotient of $sk_2 M$ by all but one 2-cell and considering any single 3-cell.

References

[Ba] D. Barden: Simply connected five-manifolds. Annals of Math. **82** , (1965), 365–385.

[B] R. Bott: The stable homotopy of the classical groups, *Proceedings of the National Academy of Sciences* **43** (1957), 933–935.

[E-Sz] T. Ekholm, A. Szűcs: The group of immersions of homotopy $(4k-1)$-spheres, *Bull. London Math. Soc* **38** (2006), 163–176.

[H] M. W. Hirsch: Immersions of manifolds, *Trans. Amer. Math. Soc.* **93** (1959), 242–276. DOI: 10.1090/S0002-9947-1959-0119214-4

[Hu] D. Husemoller: *Fibre bundles*, McGraw-Hill Book Co., New York–London–Sydney, 1966.

[K-N] A. Katanaga, K. Nakamoto: The links of 3-dimensional singularities defined by Brieskorn polynomials, *Math. Nachr.* **281** (2008), no. 12, 1777–1790.

[M-S] J. W. Milnor, J. D. Stasheff: *Characteristic classes*, Princeton University Press, Princeton, 1974.

[R-S] C. Rourke, B. Sanderson: The compression theorem I, *Geometry and Topology* **5** (2001), 399–429. DOI: 10.2140/gt.2001.5.399

[P] U. Pinkall: Regular homotopy classes of immersed surfaces, *Topology* **24** (1985), no. 4, 421–434.

[S1] S. Smale: Classification of immersion of spheres in Euclidean space, *Ann. of Math.* **69** (1959), 327–344.

[S2] S. Smale: On the structure of 5-manifolds, *Ann. of Math. (2)* **75** (1962), 38–46.

[T] R. Thom: La classification des immersions, *Séminaire Bourbaki*, 1957.

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PERIODIC SOLUTIONS OF DISCONTINUOUS SECOND ORDER DIFFERENTIAL SYSTEMS

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Abstract. We provide sufficient conditions for the existence of periodic solutions of some classes of autonomous and non-autonomous second order differential equations with discontinuous right-hand sides. In the plane the discontinuities considered are given by the straight lines either $x = 0$, or $xy = 0$. Two applications of these results are made, one to control systems with variable structure, and the other to small external periodic excitation of a discontinuous nonlinear oscillator.

1. Introduction and statement of the main result

In these last tens the study of discontinuous differential systems became relevant in the boundary between Mathematics, Physics and Engineering. In the book [2] and in the survey [10] there are different models coming from the impacting motion in mechanical systems, or from switchings in electronic systems, or from hybrid dynamics in control systems, and so on. All of these models are formulated with differential equations with discontinuous right–hand sides. Also, many studies have been done in the qualitative aspects of the phase space of discontinuous differential systems, see for instance the hundreds of references quoted in [2] and [10].

In this paper we are mainly interested in the study of the periodic solutions of autonomous and non–autonomous second order differential equations with discontinuous right–hand sides. Recently discontinuous second order differential equations have been studied for several authors, mainly non–autonomous ones. Thus, discontinuous differential equations of the form

$$u'' + u + \alpha \operatorname{sign}(y) = F(\theta),$$

where F is a periodic function has been studied in [7]. In [5] periodic solutions of discontinuous differential equations of the form $u'' + G(u) = F(\theta)$ are analyzed, where F is periodic and continuous, and G is continuous except at $u = 0$. In [6] the authors studied the periodic solutions of the discontinuous differential equations $u'' + \eta\operatorname{sign}(u) = \alpha \sin(\beta t)$.

Our main results will provide sufficient conditions for the existence of periodic solutions of the following two classes of autonomous second order differential equations with discontinuous right–hand sides:

$$u'' + u + \varepsilon\alpha \operatorname{sign}(u) G(u, u') = \varepsilon H(u, u'), \tag{1}$$

$$u'' + u + \varepsilon\alpha \operatorname{sign}(uu') G(u, u') = \varepsilon H(u, u'). \tag{2}$$

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Here $u = u(t)$, $\alpha \in \mathbb{R}$ is a parameter, ε is a small parameter, G and H are C^1 functions, and the prime denotes derivative with respect to the variable t. Note that the differential equation (1) is discontinuous at $u = 0$, and that the differential equation (2) is discontinuous at $uu' = 0$.

We also shall provide sufficient conditions for the existence of periodic solutions of the following two classes of non–autonomous second order differential equations with discontinuous right–hand sides:

$$r'' + \varepsilon^2 \alpha \operatorname{sign}(\cos\theta) G(\theta, r, r') = \varepsilon^2 H(\theta, r, r'), \tag{3}$$

$$r'' + \varepsilon^2 \alpha \operatorname{sign}(\sin(2\theta)) G(\theta, r, r') = \varepsilon^2 H(\theta, r, r'). \tag{4}$$

Here (r, θ) are the polar coordinates of the plane, i.e. $x = r\cos\theta$ and $y = r\sin\theta$, $\alpha \in \mathbb{R}$ is a parameter, ε is a small parameter, G and H are C^1 functions in the variables r and r', the functions G and H are continuous and periodic in the variable θ of period 2π, and the prime denotes derivative with respect to the variable θ. Note that the differential equation (3) is discontinuous at the straight line $x = 0$ of the plane in cartesian coordinates, and that the differential equation (4) is discontinuous at the straight lines $xy = 0$.

Denoting $x = u$ and $y = u'$ the autonomous differential equations of second order (1) and (2), respectively can be written as the following differential systems of first order in the plane

$$\begin{aligned} \frac{dx}{dt} &= x' = y, \\ \frac{dy}{dt} &= y' = -x - \varepsilon\alpha \operatorname{sign}(x) G(x, y) + \varepsilon H(x, y); \end{aligned} \tag{5}$$

with the discontinuity set $x = 0$, and

$$\begin{aligned} \frac{dx}{dt} &= x' = y, \\ \frac{dy}{dt} &= y' = -x - \varepsilon\alpha \operatorname{sign}(xy) G(x, y) + \varepsilon H(x, y); \end{aligned} \tag{6}$$

with the discontinuity set $xy = 0$.

Denoting $x = r$ and $y = r'/\varepsilon$ the non–autonomous differential equations of second order (3) and (4), respectively can be written as the following differential systems of first order in the plane

$$\begin{aligned} \frac{dx}{d\theta} &= x' = \varepsilon y, \\ \frac{dy}{d\theta} &= y' = -\varepsilon\alpha \operatorname{sign}(x) G(\theta, x, y) + \varepsilon H(\theta, x, y); \end{aligned} \tag{7}$$

with the discontinuity set $x = 0$, and

$$\begin{aligned} \frac{dx}{d\theta} &= x' = \varepsilon y, \\ \frac{dy}{d\theta} &= y' = -\varepsilon\alpha \operatorname{sign}(xy) G(\theta, x, y) + \varepsilon H(\theta, x, y); \end{aligned} \tag{8}$$

with the discontinuity set $xy = 0$.

The following propositions provide sufficient conditions for the existence of periodic solutions for the discontinuous differential systems (5), (6), (7) and (8), respectively.

Proposition 1. *For $\varepsilon \neq 0$ sufficiently small the discontinuous differential system (5) has a periodic solution $(x(t,\varepsilon), y(t,\varepsilon))$ for each simple zero r^* of the function*

$$
\begin{aligned}
f_1(r) = & \int_0^{2\pi} H(r\cos\theta, r\sin\theta)\sin\theta\, d\theta \\
& + \alpha\left(\int_{\pi/2}^{3\pi/2} G(r\cos\theta, r\sin\theta)\sin\theta\, d\theta - \int_{-\pi/2}^{\pi/2} G(r\cos\theta, r\sin\theta)\sin\theta\, d\theta\right),
\end{aligned}
$$

such that $(x(0,\varepsilon), y(0,\varepsilon)) \to (r^*, 0)$ when $\varepsilon \to 0$.

Proposition 2. *For $\varepsilon \neq 0$ sufficiently small the discontinuous differential system (6) has a periodic solution $(x(t,\varepsilon), y(t,\varepsilon))$ for each simple zero r^* of the function*

$$
\begin{aligned}
f_2(r) = & \int_0^{2\pi} H(r\cos\theta, r\sin\theta)\sin\theta\, d\theta \\
& - \alpha\left(\int_0^{\pi/2} G(r\cos\theta, r\sin\theta)\sin\theta\, d\theta + \int_{\pi}^{3\pi/2} G(r\cos\theta, r\sin\theta)\sin\theta\, d\theta\right) \\
& + \alpha\left(\int_{\pi/2}^{\pi} G(r\cos\theta, r\sin\theta)\sin\theta\, d\theta + \int_{3\pi/2}^{2\pi} G(r\cos\theta, r\sin\theta)\sin\theta\, d\theta\right),
\end{aligned}
$$

such that $(x(0,\varepsilon), y(0,\varepsilon)) \to (r^*, 0)$ when $\varepsilon \to 0$.

Proposition 3. *For $\varepsilon \neq 0$ sufficiently small the discontinuous differential system (7) has a periodic solution $(x(\theta,\varepsilon), y(\theta,\varepsilon))$ for each simple zero x^* of the function*

$$
f_3(x) = \int_0^{2\pi} H(\theta, x, 0)\, d\theta + \alpha\left(\int_{\pi/2}^{3\pi/2} G(\theta, x, 0)\, d\theta - \int_{-\pi/2}^{\pi/2} G(\theta, x, 0)\, d\theta\right),
$$

such that $(x(0,\varepsilon), y(0,\varepsilon)) \to (x^*, 0)$ when $\varepsilon \to 0$.

Proposition 4. *For $\varepsilon \neq 0$ sufficiently small the discontinuous differential system (8) has a periodic solution $(x(t,\varepsilon), y(t,\varepsilon))$ for each simple zero x^* of the function*

$$
\begin{aligned}
f_4(x) = & \int_0^{2\pi} H(\theta, x, 0)\, d\theta - \alpha\left(\int_0^{\pi/2} G(\theta, x, 0)\, d\theta + \int_{\pi}^{3\pi/2} G(\theta, x, 0)\, d\theta\right) \\
& + \alpha\left(\int_{\pi/2}^{\pi} G(\theta, x, 0)\, d\theta + \int_{3\pi/2}^{2\pi} G(\theta, x, 0)\, d\theta\right),
\end{aligned}
$$

such that $(x(0,\varepsilon), y(0,\varepsilon)) \to (x^*, 0)$ when $\varepsilon \to 0$.

The proof of these four propositions is given in section 2. The proofs are based in a recent result on the averaging theory applied to discontinuous differential systems obtained by the authors and also by Douglas Novaes, see the appendix.

In the study of control systems with variable structure appear the autonomous discontinuous second order differential equations similar to

$$
u'' + u + \varepsilon\alpha\,\mathrm{sign}(u) u u' = \varepsilon\frac{\alpha}{\pi}u', \tag{9}
$$

see for instance the book [1].

Corollary 5. *For $\varepsilon \neq 0$ sufficiently small the control system with variable structure (9) has one periodic solution $u(t,\varepsilon)$, such that $\sqrt{u(0,\varepsilon)^2 + u'(0,\varepsilon)^2} \to 3/4$ when $\varepsilon \to 0$.*

In the next corollary we apply Proposition 3 for studying the periodic solutions of the following small external periodic excitation of a discontinuous nonlinear oscillator

$$r'' + \varepsilon^2 \alpha \,\mathrm{sign}(\cos\theta) \left((2-3r)\cos\frac{\theta}{2} \right) = -\varepsilon^2 \frac{\sqrt{2}\,\alpha}{\pi} r^2. \tag{10}$$

Such kind of differential equations are considered in the book [11]. Note that equation (10) is a non–autonomous discontinuous second order differential equation.

Corollary 6. *For $\varepsilon \neq 0$ sufficiently small the small external periodic excitation of the discontinuous nonlinear oscillator (10) has two periodic solutions $r_k(\theta,\varepsilon)$ for $k = 1, 2$, such that $r_1(0,\varepsilon) \to \cos\theta$ and $r_2(0,\varepsilon) \to 2\cos\theta$ when $\varepsilon \to 0$.*

The proof of the two corollaries are given in section 3.

2. Proof of the propositions

In this section we prove the four propositions using the averaging theory for discontinuous differential systems described in the appendix.

Proof of Proposition 1. We write the discontinuous differential system (5) in polar coordinates (r,θ) where $x = r\cos\theta$ and $y = r\sin\theta$, and we obtain

$$\frac{dr}{dt} = \varepsilon\big(H(r\cos\theta, r\sin\theta) - \alpha\,\mathrm{sgn}(\cos\theta)G(r\cos\theta, r\sin\theta)\big)\sin\theta,$$
$$\frac{d\theta}{dt} = -1 + \frac{\varepsilon}{r}\Big(\big(H(r\cos\theta, r\sin\theta) - \alpha\,\mathrm{sgn}(\cos\theta)G(r\cos\theta, r\sin\theta)\big)\cos\theta\Big).$$

Now taking as new independent variable the angle θ this previous discontinuous differential system becomes

$$\begin{aligned}\frac{dr}{d\theta} &= \varepsilon\big(\alpha\,\mathrm{sgn}(\cos\theta)G(r\cos\theta, r\sin\theta) - H(r\cos\theta, r\sin\theta)\big)\sin\theta + O(\varepsilon^2)\\ &= \varepsilon F(\theta, r) + O(\varepsilon^2).\end{aligned} \tag{11}$$

This system is under the assumptions of Theorem 7, where the variables of this theorem are in our case $t = \theta$, $T = 2\pi$, $\mathbf{x} = r$, $\mathcal{M} = h^{-1}(0) = \{x = 0\}$. So we apply this theorem to our previous discontinuous differential equation and we compute

$$f(r) = \int_0^{2\pi} F(\theta, r)d\theta = f_1(r),$$

where $f_1(r)$ is the function defined in the statement of Proposition 1. Since by assumptions G and H are C^1 functions in their two variables, it follows that $f_1(r)$ is C^1. Consequently, if r^* is a simple zero of $f_1(r)$, i.e. $f_1(r^*) = 0$ and

$$\left.\frac{df_1}{dr}\right|_{r=r^*} \neq 0,$$

then the Brouwer degree $d_B(f_1, V, r^*) \neq 0$ being V a convenient open neighborhood of r^*, see for more details on the Brouwer degree [3] and [9]. Hence, by Theorem 7 it follows that for $\varepsilon \neq 0$ sufficiently small the discontinuous differential system (11) has a periodic solution $r(\theta,\varepsilon)$ such that $r(0,\varepsilon) \to r^*$ when $\varepsilon \to 0$. Going back through the polar change of variables we get that for $\varepsilon \neq 0$ sufficiently small the discontinuous differential system (5) has a periodic solution $(x(t,\varepsilon), y(t,\varepsilon))$ such that $(x(0,\varepsilon), y(0,\varepsilon)) \to (r^*, 0)$ when $\varepsilon \to 0$. So, the proposition is proved. □

Proof of Proposition 2. The discontinuous differential system (6) in polar coordinates (r,θ) becomes

$$\frac{dr}{dt} = \varepsilon\big(H(r\cos\theta, r\sin\theta) - \alpha\,\mathrm{sgn}(\sin(2\theta))G(r\cos\theta, r\sin\theta)\big)\sin\theta,$$
$$\frac{d\theta}{dt} = -1 + \frac{\varepsilon}{r}\Big(\big(H(r\cos\theta, r\sin\theta) - \alpha\,\mathrm{sgn}(\sin(2\theta))G(r\cos\theta, r\sin\theta)\big)\cos\theta\Big).$$

Taking as new independent variable the angle θ this discontinuous differential system becomes

$$\begin{aligned}\frac{dr}{d\theta} &= \varepsilon\big(\alpha\,\mathrm{sgn}(\sin(2\theta))G(r\cos\theta, r\sin\theta) - H(r\cos\theta, r\sin\theta)\big)\sin\theta + O(\varepsilon^2)\\ &= \varepsilon F(\theta,r) + O(\varepsilon^2).\end{aligned} \tag{12}$$

Applying Theorem 7 to this discontinuous differential equation, where the variables of this theorem are in our case $t=\theta$, $T=2\pi$, $\mathbf{x}=r$, $\mathcal{M}=h^{-1}(0)=\{xy=0\}$, we compute

$$f(r) = \int_0^{2\pi} F(\theta,r)d\theta = f_2(r),$$

where $f_2(r)$ is the function defined in the statement of Proposition 2. Since $f_2(r)$ is C^1, if r^* is a simple zero of $f_2(r)$, then the Brouwer degree $d_B(f_2,V,r^*)\neq 0$ being V a convenient open neighborhood of r^*. Therefore, by Theorem 7 it follows that for $\varepsilon\neq 0$ sufficiently small the discontinuous differential system (12) has a periodic solution $r(\theta,\varepsilon)$ such that $r(0,\varepsilon)\to r^*$ when $\varepsilon\to 0$. Going back through the polar change of variables we obtain that for $\varepsilon\neq 0$ sufficiently small the discontinuous differential system (6) has a periodic solution $(x(t,\varepsilon), y(t,\varepsilon))$ such that $(x(0,\varepsilon), y(0,\varepsilon))\to(r^*,0)$ when $\varepsilon\to 0$. This completes the proof of the proposition. □

Proof of Proposition 3. The discontinuous differential system (7) is already in the form (13) for applying the averaging theory described in Theorem 7, where now the variables of Theorem 7 are $t=\theta$, $T=2\pi$, $\mathbf{x}=(x,y)$, $\mathcal{M}=h^{-1}(0)=\{x=0\}$, $F(t,\mathbf{x})=F(\theta,x,y)=(F_1(\theta,x,y),F_2(\theta,x,y))$ where

$$F_1(\theta,x,y)=y,$$
$$F_2(\theta,x,y)=\alpha\,\mathrm{sign}(x)G(\theta,x,y)+H(\theta,x,y).$$

Therefore we apply Theorem 7 to the discontinuous differential system (7) and we obtain

$$f(x,y)=\int_0^{2\pi}F(\theta,x,y)d\theta,$$

where $f(x,y)=(g_1(x,y),g_2(x,y))$ with

$$g_1(x,y)=y,$$
$$g_2(x,y)=\int_0^{2\pi}H(\theta,x,y)\,d\theta+\alpha\left(\int_{\pi/2}^{3\pi/2}G(\theta,x,y)\,d\theta-\int_{-\pi/2}^{\pi/2}G(\theta,x,y)\,d\theta\right).$$

A solution (x^*,y^*) of the system $g_1(x,y)=g_2(x,y)=0$ satisfies $y^*=0$ and x^* is a solution of $f_3(x)=0$ where this function is the one defined in the statement of Proposition 3. Since G and H are C^1 functions in their two variables, it follows that $g_1(x,y)$, $g_2(x,y)$ and $f_3(x)$ are C^1. Consequently, if $(x^*,0)$ is a zero of the system $g_1(x,y)=g_2(x,y)=0$, and the Jacobian

$$\det\begin{pmatrix}\dfrac{\partial g_1}{\partial x} & \dfrac{\partial g_1}{\partial y}\\[2mm] \dfrac{\partial g_2}{\partial x} & \dfrac{\partial g_2}{\partial y}\end{pmatrix}\Bigg|_{(x,y)=(x^*,0)} = \frac{df_3}{dx}\Big|_{x=x^*}\neq 0,$$

then the Brouwer degree $d_B(f,V,(x^*,0)) \neq 0$ being V a convenient open neighborhood of $(x^*,0)$, see again for more details on the Brouwer degree [3] and [9]. Hence, by Theorem 7 it follows that for $\varepsilon \neq 0$ sufficiently small the discontinuous differential system (7) has a periodic solution $(x(\theta,\varepsilon), y(\theta,\varepsilon))$ such that $(x(0,\varepsilon), y(0,\varepsilon)) \to (x^*,0)$ when $\varepsilon \to 0$. So, the proposition follows. □

Proof of Proposition 4. The discontinuous differential system (8) is in the form (13) for applying the averaging theory described in Theorem 7, where the variables of Theorem 7 now are $t=\theta$, $T=2\pi$, $\mathbf{x}=(x,y)$, $\mathcal{M}=h^{-1}(0)=\{xy=0\}$, $F(t,\mathbf{x})=F(\theta,x,y)=(F_1(\theta,x,y),F_2(\theta,x,y))$ where

$$F_1(\theta,x,y)=y,$$
$$F_2(\theta,x,y)=\alpha\,\text{sign}(xy)G(\theta,x,y)+H(\theta,x,y).$$

By applying Theorem 7 to the discontinuous differential system (8) and we obtain

$$f(x,y)=\int_0^{2\pi} F(\theta,x,y)d\theta,$$

where $f(x,y)=(g_1(x,y),g_2(x,y))$ with

$$\begin{aligned} g_1(x,y) &= y,\\ g_2(x,y) &= \int_0^{2\pi} H(\theta,x,y)\,d\theta-\alpha\left(\int_0^{\pi/2} G(\theta,x,y)\,d\theta+\int_{\pi}^{3\pi/2} G(\theta,x,y)\,d\theta\right)\\ &\quad +\alpha\left(\int_{\pi/2}^{\pi} G(\theta,x,y)\,d\theta+\int_{3\pi/2}^{2\pi} G(\theta,x,y)\,d\theta\right). \end{aligned}$$

A solution (x^*,y^*) of the system $g_1(x,y)=g_2(x,y)=0$ satisfies $y^*=0$ and x^* is a solution of $f_4(x)=0$ where this function is the one defined in the statement of Proposition 4. Since $g_1(x,y)$, $g_2(x,y)$ and $f_4(x)$ are C^1, and if $(x^*,0)$ is a zero of the system $g_1(x,y)=g_2(x,y)=0$, then the Jacobian

$$\det\left.\begin{pmatrix} \dfrac{\partial g_1}{\partial x} & \dfrac{\partial g_1}{\partial y}\\ \dfrac{\partial g_2}{\partial x} & \dfrac{\partial g_2}{\partial y}\end{pmatrix}\right|_{(x,y)=(x^*,0)} = \left.\frac{df_4}{dx}\right|_{x=x^*}\neq 0,$$

then the Brouwer degree $d_B(f,V,(x^*,0)) \neq 0$ being V a convenient open neighborhood of $(x^*,0)$. Therefore, by Theorem 7 it follows that for $\varepsilon \neq 0$ sufficiently small the discontinuous differential system (8) has a periodic solution $(x(\theta,\varepsilon), y(\theta,\varepsilon))$ such that $(x(0,\varepsilon), y(0,\varepsilon)) \to (x^*,0)$ when $\varepsilon \to 0$. In short, the proposition is proved. □

3. Proof of the applications

Here we prove the two corollaries.

Proof of Corollary 5. The autonomous discontinuous differential equation of second order (9) is a particular case of equation (1) with

$$G(\theta,u,u')=uu' \quad \text{and} \quad H(\theta,u,u')=\frac{\alpha}{\pi}\,u'.$$

Then computing for equation (9) the function $f_1(r)$ given in the statement of Proposition 1 we get

$$f_1(r)=-\frac{\alpha}{3}r(4r-3).$$

Hence, $f_1(r) = 0$ has a unique positive simple root $r = 3/4$. Going back through the changes of variables described in the proof of Proposition 1, we obtain the result stated in the corollary. □

Proof of Corollary 6. The non–autonomous discontinuous differential equation of second order (10) is a particular case of equation (3) with

$$G(\theta, r, r') = (2 - 3r)\cos\frac{\theta}{2} \quad \text{and} \quad H(\theta, r, r') = -\frac{\sqrt{2}\,\alpha}{\pi}\, r^2.$$

Then computing for equation (10) the function $f_3(x)$ given in the statement of Proposition 3 we get

$$f_3(x) = -2\sqrt{2}\,\alpha(x-2)(x-1).$$

Therefore, $f_3(x) = 0$ has two simple roots $x = 1$ and $x = 2$. Going back through the changes of variables described in the proof of Proposition 3, it follows the result stated in the corollary. □

Appendix: Averaging theory of first order for discontinuous differential systems

We need the following recent result of [8] on averaging theory for computing periodic orbits of discontinuous differential systems. Its proof uses the theory on the Brouwer degree $d_B(f, V, 0)$ for finite dimensional spaces (see the appendix A of [8] for a definition of the Brouwer degree), and it is based on the averaging theory for continuous non–smooth differential system stated in [4].

Theorem 7. *We consider the following discontinuous differential system*

$$\mathbf{x}'(t) = \varepsilon F(t, \mathbf{x}) + \varepsilon^2 R(t, \mathbf{x}, \varepsilon), \tag{13}$$

with

$$F(t, \mathbf{x}) = F_1(t, \mathbf{x}) + \operatorname{sign}(h(t, \mathbf{x}))F_2(t, \mathbf{x}),$$
$$R(t, \mathbf{x}, \varepsilon) = R_1(t, \mathbf{x}, \varepsilon) + \operatorname{sign}(h(t, \mathbf{x}))R_2(t, \mathbf{x}, \varepsilon),$$

where $F_1, F_2 : \mathbb{R} \times D \to \mathbb{R}^n$, $R_1, R_2 : \mathbb{R} \times D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ and $h : \mathbb{R} \times D \to \mathbb{R}$ are continuous functions, T–periodic in the variable t and D is an open subset of $\mathbb{R}^n$. We also suppose that h is a C^1 function having 0 as a regular value. Denote by $\mathcal{M} = h^{-1}(0)$, by $\Sigma = \{0\} \times D \nsubseteq \mathcal{M}$, by $\Sigma_0 = \Sigma \backslash \mathcal{M} \neq \varnothing$, and its elements by $z \equiv (0, z) \notin \mathcal{M}$.

Define the averaged function $f : D \to \mathbb{R}^n$ as

$$f(\mathbf{x}) = \int_0^T F(t, \mathbf{x})dt. \tag{14}$$

We assume the following three conditions.

(i) *F_1, F_2, R_1, R_2 and h are locally L–Lipschitz with respect to $\mathbf{x}$;*
(ii) *for $a \in \Sigma_0$ with $f(a) = 0$, there exist a neighbourhood V of a such that $f(z) \neq 0$ for all $z \in \overline{V} \backslash \{a\}$ and $d_B(f, V, 0) \neq 0$.*
(iii) *If $\partial h/\partial t(t_0, z_0) = 0$ for some $(t_0, z_0) \in \mathcal{M}$, then $\big(\langle \nabla_{\mathbf{x}} h, F_1\rangle^2 - \langle \nabla_{\mathbf{x}} h, F_2\rangle^2\big)(t_0, z_0) > 0$.*

Then, for $|\varepsilon| > 0$ sufficiently small, there exists a T–periodic solution $\mathbf{x}(\cdot, \varepsilon)$ of system (13) such that $\mathbf{x}(0, \varepsilon) \to a$ as $\varepsilon \to 0$.

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References

[1] E.A. Barbashin, *Introduction to the theory of stability*, Translated from the Russian by Transcripta Service, London. Edited by T. Lukes Wolters-Noordhoff Publishing, Groningen, 1970.

[2] M. di Bernardo et al., *Bifurcations in non–smooth dynamical systems*, Publications of the Bristol Centre for Applied Nonlinear Mathematics, vol. **4**, 2005.

[3] F. Browder, *Fixed point theory and nonlinear problems*, Bull. Amer. Math. Soc. **9** (1983), 1–39. DOI: 10.1090/S0273-0979-1983-15153-4

[4] A. Buica and J. Llibre, *Averaging methods for finding periodic orbits via Brouwer degree*, Bulletin des Sciences Mathemàtiques **128** (2004),7–22. DOI: 10.1016/j.bulsci.2003.09.002

[5] J.A. Cid and L. Sanchez, *Periodic solutions for second order differential equations with discontinuous restoring forces*, J. Math. Anal. Appl. **288** (2003), 349–364. DOI: 10.1016/j.jmaa.2003.08.005

[6] A. Jacquemard and M.A. Teixeira, *Periodic solutions of a class of non–autonomous second order differential equations with discontinuous right–hand side*, Physica D **241** (2012), 2003–2009. DOI: 110.1016/j.physd.2011.05.011

[7] M. Kunze, T. Küpper and J. You, *On the application of KAM theory to discontinuous dynamical systems*, J. Differential Equations **139** (1997), 1–21. DOI: 10.1006/jdeq.1997.3286

[8] J. Llibre, D.D. Novaes and M.A. Teixeira, *On the birth of limit cycles for non–smooth dynamical systems*, to appear in Bull. Sci. Math. DOI: 10.1016/j.bulsci.2014.08.011

[9] N.G. Lloyd, *Degree Theory*, Cambridge University Press, 1978.

[10] O. Makarenkov and J.S.W. Lamb, *Dynamics and bifurcations of nonsmooth systems: a survey*, Physica D **241** (2012), 1826–1844. DOI: 10.1016/j.physd.2012.08.002

[11] N.Minorsky, *Nonlinear oscillations*, D. Van Nostrand Company, Inc., Princeton, 1962.

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ABELIAN SINGULARITIES OF HOLOMORPHIC LIE-FOLIATIONS

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ABSTRACT. We study holomorphic foliations with generic singularities and Lie group transverse structure outside of some invariant codimension one analytic subset. We introduce the concept of abelian singularity and prove that, for this type of singularities, the foliation is logarithmic. The Lie transverse structure is then used to extend the local (logarithmic) normal form from a neighborhood of the singularity, to the whole manifold.

1. Introduction

Foliations with Lie transverse structure are among the simplest constructive examples of foliations. They are however a natural object when one considers the possible applications of the theory of foliations in the classification of manifolds and dynamical systems. By a foliation with a Lie group transverse structure we mean a foliation that is given by an atlas of submersions taking values on a given Lie group G and with transition maps given by restrictions of left-translations on the group G. Such a foliation will be called a *G-foliation*. The theory of G-foliations is a well-developed subject and follows the original work of Blumenthal [2].

In the present work we study the possible Lie transverse structures associated to holomorphic foliations with singularities. This study initiated in [6] where we prove that a one-dimensional holomorphic foliation with generic singularities in dimension 3 and having a Lie transverse structure, outside of some analytic invariant subset of codimension one, is logarithmic.

As a consequence of our results, we conclude that, in dimension two, *the presence of generic singularities forces the transverse structure to be abelian.* The exact sense of the term generic is given below. We stress that our results are first steps in the comprehension of the possible Lie group for holomorphic foliations with singularities.

Abelian singularities. Let $\mathcal{F}$ be a germ of a one-dimensional foliation at the origin $0 \in \mathbb{C}^m$. We recall that $\mathcal{F}$ is *linearizable without resonances* if it is given in some neighborhood U of $0 \in \mathbb{C}^m$ by a holomorphic vector field X which is linearizable as

$$X = \sum_{j=1}^{m} \lambda_j z_j \frac{\partial}{\partial z_j}, \tag{1}$$

with eigenvalues $\lambda_1, \cdots, \lambda_m$ satisfying the following non-resonance hypothesis:
If $n_1, \cdots, n_m \in \mathbb{Z}$ are such that

$$\sum_{j=1}^{m} n_j \lambda_j = 0,$$

then $n_1 = n_2 = \cdots = n_m = 0$.

Now we consider a $(m-r)$-dimensional holomorphic foliation with singularities $\mathcal{F}$ in a connected open subset $V \subset \mathbb{C}^m$. Denote by $\operatorname{sing}(\mathcal{F}) \subset V$ the singular set of $\mathcal{F}$. The following definition is motivated by the two dimensional case (cf. Proposition 1):

Definition 1 (abelian singularity)**.** A $(m-r)$-dimensional singularity $p \in \text{sing}(\mathcal{F}) \subset \mathbb{C}^m$ is said to be *abelian* if $\mathcal{F}$ is given by a system of *commuting* vector fields $X_1, \cdots, X_{m-r}$ defined in a neighborhood U of p such that $X_1, \cdots, X_{m-r}$ vanish at p and are linearly independent off $\text{sing}(\mathcal{F}) \cap U$. The singularity $p \in \text{sing}(\mathcal{F})$ is *generic* if we can choose the system above such that:

(i) Each vector field is of the form $X_k = \sum\limits_{j=1}^{m} \lambda_j^k z_j \frac{\partial}{\partial z_j} + \text{h.o.t.}$.
(ii) The $m \times (m-r)$ matrix $A = (\lambda_j^k)$, where $j = 1, ..., m$ and $k = 1, ..., m-r$, is *nonresonant* in the following sense: the set of its $(m-r) \times (m-r)$ minor determinants is linearly independent over the integer numbers.
(iii) Some vector field X_j is *nonresonant* and analytically linearizable at the origin.

Remark 1. Regarding the notions above we have:

(1) A germ of a singular holomorphic vector field X at the origin $0 \in \mathbb{C}^m$ is in the *Poincaré domain* if the convex hull of its eigenvalues does not the origin $0 \in \mathbb{C}$. Otherwise it is in the *Siegel domain.* The so called Poincaré-Dulac theorem states that a Poincaré type singularity is analytically linearizable in the nonresonant case ([1]). In the generic case, a nonresonant Siegel type singularity is also linearizable ([5]).
(2) If $\mathcal{F}$ has dimension one then the singularity is generic if and only if it is generated by a generic vector field.

In this paper we consider the case where $\mathcal{F}$ has a G-transverse structure outside of some analytic codimension one subset Λ such that each irreducible component of Λ contains the origin $0 \in \mathbb{C}^m$. In this case, thanks to the linearization hypothesis, it is natural to assume that the germ of such a subset Λ at the origin is the germ of a union of coordinate hyperplanes.

A codimension r holomorphic foliation with singularities in a complex manifold V is *logarithmic* if it is given by a system of closed meromorphic one-forms with simple poles $\{\omega_1, ..., \omega_r\}$ in V. In this paper we prove:

Theorem 1. *Let $\mathcal{F}$ be a holomorphic foliation defined in an open connected neighborhood V of the origin $0 \in \mathbb{C}^m$, such that $\mathcal{F}$ has an abelian generic singularity at the origin. Assume that $\mathcal{F}$ has a G-transverse structure outside of some invariant codimension one analytic subset $\Lambda \subset V$, such that each irreducible component of Λ contains the origin. Then $\mathcal{F}$ is a logarithmic foliation.*

Remark 2. Theorem 1 contains the case of dimension two foliations (cf. Proposition 1) and of codimension one foliations (cf. [3]). We highlight the fact that the conclusion of Theorem 1 states that the foliation is logarithmic in the whole manifold V. From Lemma 1 we will see that the germ of singularity induced by the foliation at the origin, is already a germ of a logarithmic foliation. Thus, the main role of the Lie transverse structure is to extend this local (logarithmic) normal form from a neighborhood of the origin, to the manifold V.

2. Generic abelian singularities

In what follows we motivate and prove some results about the notion of abelian singularity. The next proposition motivates our approach.

Proposition 1. *Let $\{A_1, A_2\}$ be an integrable system of linear vector fields on $\mathbb{C}^m$. Assume that A_1 and A_2 are nonresonant. Then A_1 and A_2 commute. Indeed, A_1 and A_2 are simultaneously diagonalizable.*

Proof. Write $A = A_1 = (f_{ij})_{i,j=1}^m$. By hypothesis A_2 is nonresonant and therefore diagonalizable. Thus we may assume that A_2 is in the diagonal form D with eigenvalues $d_1, ..., d_m$. Also by hypothesis $[A, D] = c_1 A + c_2 D$, for some holomorphic functions c_1, c_2 defined in a neighborhood of the origin $0 \in \mathbb{C}^m$.

$$AD = \begin{pmatrix} f_{11}d_1 & f_{12}d_2 & \dots & f_{1n}d_m \\ f_{12}d_1 & f_{22}d_2 & \dots & f_{2n}d_m \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1}d_1 & f_{n2}d_2 & \dots & f_{nn}d_m \end{pmatrix}$$

and

$$DA = \begin{pmatrix} f_{11}d_1 & f_{12}d_1 & \dots & f_{1n}d_1 \\ f_{21}d_2 & f_{22}d_2 & \dots & f_{2n}d_2 \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1}d_m & f_{n2}d_m & \dots & f_{nn}d_m \end{pmatrix}$$

and

$$AD - DA = \begin{pmatrix} 0 & f_{12}(d_2 - d_1) & \dots & f_{1n}(d_m - d_1) \\ f_{21}(d_1 - d_2) & 0 & \dots & f_{2n}(d_m - d_2) \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1}(d_1 - d_m) & f_{n2}(d_2 - d_m) & \dots & 0 \end{pmatrix}.$$

On the other hand

$$c_1 A + c_2 D = \begin{pmatrix} c_1 f_{11} + c_2 d_1 & c_1 f_{12} & \dots & c_1 f_{1n} \\ c_1 f_{21} & c_1 f_{22} + c_2 d_2 & \dots & c_1 f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_1 f_{n1} & c_1 f_{n2} & \dots & c_1 f_{nn} + c_2 d_m \end{pmatrix}.$$

From $AD - DA = c_1 A + c_2 D$ we obtain:

$$c_1 f_{ij} = f_{ij}(d_j - d_i), \ c_1 f_{ji} = f_{ji}(d_i - d_j)$$

Assume $f_{ij} \neq 0$ for some i, j. Then $c_1 = d_j - d_i$. Notice that if also $f_{ji} \neq 0$ then $c_1 = d_i - d_j$ and therefore $d_i = d_j$, contradiction. Therefore, $f_{ij} \neq 0 \implies f_{ji} = 0$.

Given now an index $k \in \{1, ..., n\}$, as before we have $f_{ik} = 0$ or $f_{ki} = 0$. If $f_{ik} \neq 0$ we get $c_1 = d_k - d_i$ and therefore $d_k - d_i = d_i - d_j$ and thus $d_k - 2d_i + d_j = 0$, contradiction. If $f_{ki} \neq 0$ then $c_1 = d_i - d_k$ and thus $d_i - d_j = d_i - d_k$, that implies $d_j = d_k$, again a contradiction provided that $k \neq j$. We conclude that $f_{ik} = 0$ for all $k \neq i$ and $f_{ki} = 0$, $\forall k \neq j$. This means that, except for the elements f_{ii} on the diagonal of A, at most one element f_{ij} is different from zero. Since by hypothesis $A_1 = A$ is also nonresonant and diagonalizable, we conclude that A is also in the diagonal form and therefore A and D commute. □

A germ of a codimension one holomorphic foliation singularity at the origin is given in a neighborhood V of the origin $0 \in \mathbb{C}^m$ by an integrable holomorphic one-form ω. We can write $\omega = \omega_\nu + \omega_{\nu+1} + \cdots$ as a sum of homogeneous one-forms, where ω_ν is the first nonzero jet of ω. According to Cerveau-Mattei [3], under *generic* conditions on the coefficients of ω_ν, the foliation is given by an integrable system of $n - 1$ commuting vector fields, all of them with non-degenerate linear part at the origin. By generic we mean, for an open dense Zariski subset of the affine space of coefficients of ω_ν (see [3] in a more precise description).

In general, abelian singularities are *linearizable*, i.e., defined by simultaneously linearizable commuting vector fields, as the following proposition shows:

Proposition 2. *An abelian singularity is analytically linearizable provided that it is defined by commuting vector fields one of which has an analytically linearizable nonresonant singularity.*

Proof. It is enough to prove that given two commuting vector fields X and Y in a neighborhood U of $0 \in \mathbb{C}^m$, and such that X has an analytically linearizable nonresonant singularity at $0 \in \mathbb{C}^m$ then X and Y are simultaneously linearizable in a neighborhood of the origin. In fact, in a suitable local chart $x = (x_1, \cdots x_m)$ we have

$$X = \sum_{j=1}^{m} \lambda_j x_j \frac{\partial}{\partial x_j}, \qquad Y = \sum_{i=1}^{m} b_i(x) \frac{\partial}{\partial x_i}, \qquad [X,Y] = \sum_{i=1}^{m} \left(\sum_{j=1}^{m} \lambda_j x_j \frac{\partial b_i(x)}{\partial x_j} - \lambda_i b_i(x) \right) \frac{\partial}{\partial x_i}.$$

Since $[X,Y] = 0$ we get

$$\sum_{j=1}^{m} \lambda_j x_j \frac{\partial b_i(x)}{\partial x_j} - \lambda_i b_i(x) = 0,$$

for $i = 1, 2, \cdots, m$.

We write b_i in its Laurent series expansion in the variable x

$$b_i = \sum_{|(l_1,\ldots,l_m)|\neq 0} b^i_{l_1\cdots l_m} x_1^{l_1} \cdots x_m^{l_m}$$

$$x_j \frac{\partial b_i}{\partial x_j} = \sum_{|(l_1,\ldots,l_m)|\neq 0} l_j b^i_{l_1\cdots l_m} x^{l_1} \cdots x^{l_m}.$$

By hypothesis X is nonresonant. Therefore $\sum_{j=1}^{m} l_j \lambda_j - \lambda_i \neq 0$ and $Y = \displaystyle\sum_{j=1}^{m} \mu_j x_j \frac{\partial}{\partial x_j}$. □

Let now X be a linearizable vector field in neighborhood U of the origin where X can be written as in (1). We may introduce closed meromorphic one-forms $\omega_1, \cdots, \omega_{m-1}$ on U, linearly independent and holomorphic on $U \setminus \Lambda$, and such that $\omega_l(X) = 0, l = 1, ..., m-1$ by

$$\omega_l = \sum_{j=1}^{m} \alpha_j^l \frac{dz_j}{z_j} \tag{2}$$

where $l = 1, \cdots, m-1$ and the vectors $\vec{\alpha_l} := (\alpha_1^l, ..., \alpha_m^l) \in \mathbb{C}^m$ are suitably chosen in the hyperplane $z_1\lambda_1 + ... + z_m\lambda_m = 0$ in $\mathbb{C}^m$. We extend this fact by defining a *nonresonant linearizable abelian singularity* as an abelian singularity which is defined by $m-r$ simultaneously analytically linearizable nonresonant vector fields. Using this we prove:

Lemma 1. *A nonresonant linearizable abelian singularity is a germ of a logarithmic singularity.*

Proof. In fact, the singularity is given by a system of vector fields $X_k(y) = A_k y$, where $A_k \in \mathrm{GL}(m, \mathbb{C})$ is a diagonal matrix for each $k = 1, 2, \cdots, m-r$. If

$$A_k = \begin{pmatrix} \lambda_1^k & 0 & 0 \\ 0 & \dots & 0 \\ 0 & 0 & \lambda_m^k \end{pmatrix}$$

we define r one-forms $\omega_1, \cdots, \omega_r$ on $U \setminus \Lambda$ as in (2). Condition $\omega_l(X_k) = 0$ is equivalent to the following system of equations

$$\sum_{j=1}^{m} \alpha_j^l \lambda_j^k = 0, \quad l = 1, \cdots, r. \tag{3}$$

Set $\vec{\lambda}_k = \left(\lambda_1^k, \cdots, \lambda_m^k\right) \in \mathbb{C}^m$ and let $P_k \subset \mathbb{C}^m$ be the hyperplane given by

$$P_k = \left\{ (z_1, \cdots, z_m) \in \mathbb{C}^m \quad : \quad \sum_{j=1}^{m} \lambda_j^k z_j = 0 \right\}.$$

Then (3) is equivalent to $\vec{\lambda}_k \in P_k$. Because the vector fields X_k are linearly independent off the singular set of the foliation, which is of codimension ≥ 2, the vectors $\vec{\lambda}_1 \cdots, \vec{\lambda}_{m-r}$ are linearly independent in $\mathbb{C}^m$ and therefore the hyperplanes $P_1, \cdots, P_{m-r}$ intersect transversely at a linear subspace $Q = P_1 \cap \cdots \cap P_r \subset \mathbb{C}^m$ of dimension $m - r$. Since $\dim(Q) = m - r$, we can choose linearly independent vectors $\vec{\alpha_l} := (\alpha_1^l, ..., \alpha_m^l) \in \mathbb{C}^m$, $l = 1, ..., r$ so that the corresponding one-forms $\omega_1, ..., \omega_r$ defined by $\omega_l = \sum_{j=1}^{m} \alpha_j^l \frac{dz_j}{z_j}$ satisfy $\omega_l(X_k) = 0$ and the system $\{\omega_1, \cdots, \omega_r\}$ has maximal rank outside the set $\{\omega_1 \wedge \cdots \wedge \omega_r = 0\}$. Therefore $\mathcal{F}$ is logarithmic. □

3. Proof of Theorem 1

In this section we prove Theorem 1. The starting point in our study is the following characterization of G-foliations given by the classical theorem of Darboux-Lie ([2, 4]):

Darboux-Lie theorem. *Let $\mathcal{F}$ be a G-foliation on V. Then there are one-forms $\theta_1, ..., \theta_r$ in V such that: $\{\theta_1, ..., \theta_r\}$ is a rank r integrable system which defines $\mathcal{F}$ and the forms satisfy the Maurer-Cartan equation:*

$$d\theta_i = \sum_{j,k} c_{jk}^i \theta_j \wedge \theta_k. \tag{4}$$

The numbers $\{c_{ij}^k\}$ are the structure constants of a Lie algebra basis of G.
If $\mathcal{F}, V$ and G are complex (holomorphic) then the θ_j can be taken holomorphic.

The proof of Theorem 1 is also based on the following two lemmas:

Lemma 2. *Let $\{\omega_1, ..., \omega_r\}$ be a maximal rank system of logarithmic one-forms, say*

$$\omega_l = \sum_{j=1}^{m} \alpha_j^l \frac{dz_j}{z_j},$$

defined in an open connected neighborhood U of the origin $0 \in \mathbb{C}^m$. Assume that the coefficients matrix $B = (\alpha_j^l)_{j,l}$ is nonresonant in the following sense: the set of its $(m-r) \times (m-r)$ minor determinants is linearly independent over the integer numbers. Let $f : U \to \mathbb{C}$ be a holomorphic function such that $df \wedge \omega_1 \wedge ... \wedge \omega_r = 0$ in U. Then f is constant in U.

Proof. We have $\omega_1 \wedge ... \wedge \omega_r = \sum\limits_{j_1, ..., j_r} \alpha_{j_1}^1 ... \alpha_{j_r}^r \frac{dz_{j_1} \wedge ... \wedge dz_{j_r}}{z_{j_1} ... z_{j_r}} = \sum\limits_{j_1 < ... < j_r} \Delta(j_1, ..., j_r) \frac{dz_{j_1} \wedge ... \wedge dz_{j_r}}{z_{j_1} ... z_{j_r}}$ where $\Delta(j_1, ..., j_r)$ is the $r \times r$ minor determinant of the matriz $A = (\alpha_j^l)_{j,l}$ obtained by considering the lines $j_1 < ... < j_r$.

Write $f(z_1,...,z_m) = \sum\limits_{i_1,...,i_m} f_{i_1,...,i_m} z_1^{i_1}...z_m^{i_m}$. Then

$$df = \sum_{\ell=1}^{m} \sum_{i_1,...,i_m} i_\ell f_{i_1,...,i_m} z_1^{i_1}..z_\ell^{i_\ell - 1}....z_m^{i_m} dz_\ell.$$

Therefore

$$df \wedge \omega_1 \wedge ... \wedge \omega_r = \sum_{\ell=1}^{m} \sum_{i_1,...,i_m} i_\ell f_{i_1,...,i_m} z_1^{i_1}..z_\ell^{i_\ell - 1}....z_m^{i_m} dz_\ell \wedge \sum_{j_1<...<j_r} \Delta(j_1,...,j_r) \frac{dz_{j_1} \wedge ... \wedge dz_{j_r}}{z_{j_1}...z_{j_r}}$$

and then

$$df \wedge \omega_1 \wedge ... \wedge \omega_r = \sum_{\ell=1}^{m} \sum_{i_1,...,i_m} \sum_{j_1<...<j_r} i_\ell f_{i_1,...,i_m} \Delta(j_1,...,j_r)\, z_1^{i_1}..z_\ell^{i_\ell}....z_m^{i_m} \frac{dz_\ell}{z_\ell} \wedge \frac{dz_{j_1} \wedge ... \wedge dz_{j_r}}{z_{j_1}...z_{j_r}}$$

$$df \wedge \omega_1 \wedge ... \wedge \omega_r = \sum_{i_1,...,i_m} \sum_{j_1<...<j_r,\, \ell=1}^{\ell=m} (-1)^\ell i_\ell \Delta(j_1,...,j_r)\, f_{i_1,...,i_m}\, z_1^{i_1}..z_\ell^{i_\ell}....z_m^{i_m} \frac{dz_{j_1} \wedge ... \wedge dz_{j_r} \wedge dz_\ell}{z_{j_1}...z_{j_r}\, z_\ell}$$

Then $df \wedge \wedge\omega_1 \wedge ... \wedge \omega_r = 0$ implies

$$f_{i_1,...,i_m} \Big(\sum_{\ell \in \{1,...,m\} \setminus \{j_1,...,j_r\}} [(-1)^\ell i_\ell \Delta(j_1,...,j_r)] \Big) = 0$$

for all $j_1 < ... < j_r$ and for all $i_1,...,i_m$. Therefore, if $f_{i_1,...,i_m} \neq 0$ then we have

$$\sum_{\ell \in \{1,...,m\} \setminus \{j_1,...,j_r\}} (-1)^\ell i_\ell \Delta(j_1,...,j_r) = 0.$$

By the nonresonance hypothesis this occurs only for $(i_1,...,i_m) = (0,...,0)$.

□

Lemma 3. *Let $B = (\alpha_j^k)_{j,k}$ be a $r \times m$ matrix and let $A = (\lambda_j^k)$ a $m \times (m-r)$ matrix, such that $BA = 0$. Denote by $\Delta(B;\{k_1,...,k_r\})$ the $r \times r$ minor determinant obtained by choosing the columns $(k_1,...,k_r)$ in the matrix B and by $\Delta(A;\{k_1,...,k_r\}^c)$ the $(m-r) \times (m-r)$ minor determinant obtained by deleting in A the lines $k_1,...,k_r$. Then for any pair of choices $(k_1,...,k_r)$ and $(\tilde{k}_1,..,\tilde{k}_r)$ we have*

$$\frac{\operatorname{sign}(\sigma(k_1,...,k_r))}{\operatorname{sign}(\sigma(\tilde{k}_1,...,\tilde{k}_r))} \Delta(B;\{k_1,...,k_r\}\}\Delta(A;\{\tilde{k}_1,...,\tilde{k}_r\}^c) = \Delta(B;\{\tilde{k}_1,...,\tilde{k}_r\})\Delta(A;\{k_1,...,k_r\}^c)$$

where $\operatorname{sign}(\sigma(k_1,...,k_r))$ is the sign of the permutation $(k_1,...,k_r,j_1,...,j_{m-r})$, where

$$\{j_1 < ... < j_r\} = \{1,...,m\} \setminus \{k_1,...,k_r\}.$$

Assume that each such minor determinant is nonzero. Then we have

$$\operatorname{sign}(\sigma(k_1,...,k_r)) \frac{\Delta(B;\{k_1,...,k_r\}\}}{\Delta(A;\{k_1,...,k_r\}^c)} = \operatorname{sign}(\sigma(\tilde{k}_1,...,\tilde{k}_r)) \frac{\Delta(B;\{\tilde{k}_1,...,\tilde{k}_r\})}{\Delta(A;\{\tilde{k}_1,...,\tilde{k}_r\}^c)}$$

In particular, if A is nonresonant then B is also nonresonant.

Proof. The proof is standard Linear Algebra. Indeed, we first write $BA = 0$ as above in the following linear homogeneous system of equations

$$\sum_{j=1}^{m} \alpha_j^l \lambda_j^k = 0, \quad l \in \{1, \cdots, r\},\, k \in \{1, \cdots, m-r\}. \tag{5}$$

From now on it is just Gaussian elimination process. We give a sketch for the case $r = 2$ and $m = 4$. The general case is proved in the same way.

Write

$$A = \begin{pmatrix} \lambda_1 & \mu_1 \\ \lambda_2 & \mu_2 \\ \lambda_3 & \mu_3 \\ \lambda_4 & \mu_4 \end{pmatrix}$$

and

$$B = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{pmatrix}.$$

From $BA = 0$ we get

$$a_1\lambda_1 + a_2\lambda_2 + a_3\lambda_3 + a_4\lambda_4 = 0 \tag{6}$$

$$a_1\mu_1 + a_2\mu_2 + a_3\mu_3 + a_4\mu_4 = 0 \tag{7}$$

$$b_1\lambda_1 + b_2\lambda_2 + b_3\lambda_3 + b_4\lambda_4 = 0 \tag{8}$$

$$b_1\mu_1 + b_2\mu_2 + b_3\mu_3 + b_4\mu_4 = 0 \tag{9}$$

Multiplying equation (5) by b_2 and equation (7) by $-a_2$ and then summing up these resulting equations we eliminate λ_2 in the first and the third equations obtaining:

$$(b_2a_1 - a_2b_1)\lambda_1 + (b_2a_3 - a_2b_3)\lambda_3 + (b_2a_4 - a_2b_4)\lambda_4 = 0$$

Eliminating in a similar way μ_2 in the second and fourth equations we obtain

$$(b_2a_1 - a_2b_1)\mu_1 + (b_2a_3 - a_2b_3)\mu_3 + (b_2a_4 - a_2b_4)\mu_4 = 0$$

Using these two equations and eliminating the term $b_2a_3 - a_2b_3$ we obtain

$$\begin{vmatrix} \lambda_1 & \mu_1 \\ \lambda_3 & \mu_3 \end{vmatrix} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = - \begin{vmatrix} \lambda_3 & \lambda_4 \\ \mu_3 & \mu_4 \end{vmatrix} \begin{vmatrix} a_2 & a_4 \\ b_2 & b_4 \end{vmatrix}$$

Notice that, during the Gaussian elimination process, no division is performed. Thus, we do not need to make considerations regarding whether the coefficients are zero or not. □

Proof of Theorem 1. By Proposition 2 and Lemma 1, in a small neighborhood $U \subset V$ of the origin the foliation is defined by a system of logarithmic one-forms $\omega_1, ..., \omega_r$ where $\omega_l = \sum\limits_{j=1}^{m} \alpha_j^l \frac{dz_j}{z_j}$ and simultaneously by linear vector fields $X_1, ..., X_{m-r}$ of the form

$$X_k(z_1, .., z_m) = \sum_{i=1}^{m} \lambda_i^k \, z_i \frac{\partial}{\partial z_i}.$$

Since $\omega_l(X_k) = 0$ we have

$$\sum_{j=1}^{m} \alpha_j^l \lambda_j^k = 0, \quad l \in \{1, \cdots, r\},\, k \in \{1, \cdots, m-r\}. \tag{10}$$

Let $B = (\alpha_j^l)_{j,l}$ be the matrix of coefficients of the forms ω_l and $A = (\lambda_j^k)$ the matrix of coefficients of the vector fields X_k. From equation (10) we have $BA = 0$. Since A is nonresonant, by Lemma 3, B is also nonresonant.

On the other hand, by hypothesis the foliation is a Lie-foliation in $V \setminus \Lambda$. Let therefore $\{\theta_1, \cdots, \theta_r\}$ be a system of holomorphic one-forms in $V \setminus \Lambda$ defining $\mathcal{F}$ and satisfying the Maurer-Cartan equation as stated in Darboux-Lie theorem. Since $\{\omega_l\}_{l=1,\dots,r}$ and $\{\theta_l\}_{l=1,\dots,r}$ define the same foliation outside a codimension 1 analytical subset, given by the union of Λ with the singular locus of $\mathcal{F}$ (which has codimension ≥ 2), it is clear that there is a holomorphic map $F \colon U \setminus \Lambda \to \mathrm{GL}(r, \mathbb{C})$ given by $F(z) = (f_{ij})_{i,j=1}^r$ such that

$$\theta_i = \sum_{l=1}^r f_{il}\omega_l. \tag{11}$$

Since each ω_l is closed we have from the above equation

$$d\theta_i = \sum_{l=1}^r df_{il} \wedge \omega_l. \tag{12}$$

From equations (4) and (11) we have

$$d\theta_i = \sum_{j,k} c_{jk}^i \theta_j \wedge \theta_k = \sum_{l<t} \Big(\sum_{j,k} c_{jk}^i \left(f_{jl} f_{kt} - f_{jt} f_{kl}\right) \Big) \omega_l \wedge \omega_t. \tag{13}$$

Claim 1. *We have $df_{i1} \wedge \omega_1 \wedge \cdots \wedge \omega_{m-1} = 0$.*

Proof. Indeed, from equation (13) above we have

$$d\theta_i \wedge \omega_2 \wedge \cdots \wedge \omega_r = 0.$$

From this last equation and equation (12) we obtain

$$df_{i1} \wedge \omega_1 \wedge \cdots \wedge \omega_{m-1} = 0.$$

□

Similarly we prove that

$$df_{ij} \wedge \omega_1 \wedge \cdots \wedge \omega_{m-1} = 0, \, \forall i, j. \tag{14}$$

Since the matrix B of the coefficients of the forms ω_l is nonresonant, by Lemma 2 each f_{ij} is constant in a neighborhood of the origin in U. On the other hand, each one-form θ_j is defined in $V \setminus \Lambda$, and each irreducible component of Λ contains the origin. Therefore, by classical Levi-Hartogs' extension theorem (applied to each irreducible component of Λ) each one-form θ_i extends to Λ as a meromorphic one-form Θ_i in V. We claim:

Claim 2. *Each extension Θ_i is a closed meromorphic one-form with simple poles in V. Moreover the polar set $(\Theta_i)_\infty$ is contained in Λ.*

Proof. First we observe that the extension Θ_i is closed by the Identity Principle (also note that since Λ is a thin set, $V \setminus \Lambda$ is connected). In order to see that the poles of Θ_i are contained in Λ it is enough to observe that Θ_i ans θ_i coincide in $V \setminus \Lambda$, where θ_i is holomorphic. Finally, to see that each irreducible component of Λ is also contained in the polar set of each Θ_i it is enough to use the fact that this is true in a neighborhood of the origin and, by hypothesis, each irreducible component of Λ contains the origin. □

Since each Θ_i is a simple poles closed meromorphic one-form in V, the foliation $\mathcal{F}$ is logarithmic in V. This ends the proof of Theorem 1. □

References

[1] V. I. Arnold, *Geometrical methods in the theory of ordinary differential equations.* Springer, 1983.

[2] Blumenthal, R., 'Transversely homogeneous foliations', Ann. Inst. Fourier t. 29, no.4 (1979), 143–158.

[3] Cerveau, D. and Mattei, J.-F., 'Formes intégrables holomorphes singulières´. Astérisque, vol.97 (1982).

[4] Godbillon, C., Feuilletages. Études géométriques. With a preface by G. Reeb. Progress in Mathematics, 98. Birkhäuser Verlag, Basel, 1991.

[5] Yu. Ilyashenko & S. Yakovenko. *Lectures on analytic differential equations.* Graduate Studies in Mathematics. American Mathematical Society (December 27, 2007).

[6] Mafra, A. and Scardua, B., 'One-dimensional Lie foliations with generic singularities in complex dimension three´, J. Aust. Math. Soc. 91 (2011), no. 1, 13-28.

[7] Siu, Y., Techniques of Extension of Analytic Objects. Marcel Dekker, New York, 1974.

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SOME REMARKS ABOUT THE TOPOLOGY OF CORANK 2 MAP GERMS FROM $\mathbb{R}^2$ TO $\mathbb{R}^2$

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Abstract. Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a finitely determined map germ. The link of f is obtained by taking a small enough representative $f : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ and the intersection of its image with a small enough sphere S^1_ϵ centered at the origin in $\mathbb{R}^2$. We will use Gauss words to classify topologically corank 2 map germs. In particular, we will center our attention in map germs that belong to the Thom-Boardman class $\Sigma^{2,0}$.

1. Introduction

In a previous paper [9] we defined the Gauss word, which is a complete topological invariant for a finitely determined map germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ and we used it to classify corank 1 map germs. The following logical step is to try to extend this classification to germs of corank 2. This classification is also motivated by the fact that, as we will see in proposition 3.7, some examples of links are not realizable by corank 1 map germs, even if $|\deg(f)| \leq 1$ (see figure 1).

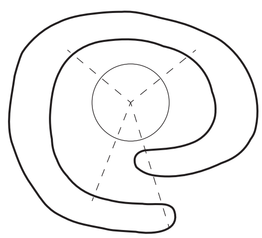

Figure 1.

This classification was completed for the $\Sigma^{2,0}$ Thom-Boardman class in the case of $\mathcal{K}$ - equivalence following Mather's techniques of classification (see for example [5]) and Nishimura proved in [10] that, dealing with $\mathcal{K}$ - $\mathcal{C}^0$ - classes, the absolute value of $\deg(f)$ becomes a complete topological invariant. In the complex case, we can find related results in [7] and a full clasification for weighted homogeneous map germs from $\mathbb{C}^2$ to $\mathbb{C}^2$ in an article of T.Gaffney and D.Mond in [4].

The fact that we are not able to consider our germs as 1-parameter unfoldings of functions, as we did in the corank 1 case, makes things to become much more complex. The absolute value of the topological degree does not have to be necessarily less or equal than 1 and although our Gauss words continue being a complete topological invariant, since their links are not constituted as the union of 2 curves (as we did in [9]) the simplifications of letters are not allowed anymore.

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In this work, we will classify corank 2 map germs with some additional convenient restrictions. Firstly we will suppose that f is of type $\Sigma^{2,0}$ (that is, f has corank 2, but the pair (f, df) has corank 0 at the origin). Departing from this point, we will establish a prenormal form of this kind of germs by using their $\mathcal{A}^2$-classes and Eisenbud-Levine formula([2]) will let us to compute their topological degree. As final step we will consider particular cases and under some restrictions on the number of monomials which appear in the second coordinate germ, we will obtain the different topological classes that we have in each case.

2. The link of a finitely determined map germ

We say that two smooth maps $f : M \to N$ and $g : M' \to N'$ between smooth manifolds are $\mathcal{A}$-equivalent if there exist diffeomorphisms $\phi : M \to M'$ and $\psi : N \to N'$ such that $g = \psi \circ f \circ \phi^{-1}$. If ϕ, ψ are homeomorphisms instead of diffeomorphisms, then we say that f, g are topologically equivalent.

In the same way, two smooth map germs $f, g : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ are $\mathcal{A}$-equivalent if there exist diffeomorphism germs $\phi, \psi : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ such that $g = \psi \circ f \circ \phi^{-1}$. If ϕ, ψ are homeomorphisms instead of diffeomorphisms, then we say that f, g are topologically equivalent.

We say that $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ is k-determined if for any map germ g with the same k-jet, we have that g is $\mathcal{A}$-equivalent to f. We say that f is finitely determined if it is k-determined for some k.

Let $f : U \to V$ be a smooth proper map, where $U, V \subset \mathbb{R}^2$ are open subsets. We denote by $S(f) = \{p \in U : Jf_p = 0\}$ the singular set of f, where Jf is the Jacobian determinant. It is a consequence of the Whitney's work [12] that f is *stable* if and only if the following two conditions hold:

(1) 0 is a regular value of Jf, so that $S(f)$ is a smooth curve in U.
(2) The restriction $f|_{S(f)} : S(f) \to V$ is an immersion with only transverse double points, except at isolated points, where it has simple cusps.

We denote $\Delta(f) = f(S(f))$ and we define $X(f)$ as the closure of $f^{-1}(\Delta(f)) \setminus S(f)$. If f is stable, then $S(f)$ is a smooth plane curve and $\Delta(f), X(f)$ are plane curves whose only singularities are simple cusps or transverse double points.

Given a finitely determined map germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$, if it is real analytic, we can consider its complexification $\hat{f} : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$. It is well known that $\hat{f}$ is also finitely determined as a complex analytic map germ. Then, by the Mather-Gaffney geometric criterion [11], it has an isolated instability. In other words, we can find a small enough representative $\hat{f} : U \to V$, where U, V are open sets, such that

(1) $\hat{f}^{-1}(0) = \{0\}$,
(2) the restriction $\hat{f}|_{U \setminus \{0\}}$ is stable.

From the condition (2), both the cusps and the double folds are isolated points in $U \setminus \{0\}$. By the curve selection lemma [6], we deduce that they are also isolated in U. Thus, we can shrink the neighbourhood U if necessary and get a representative such that $\hat{f}|_{U \setminus \{0\}}$ is stable with only simple folds. Coming back to the real map f, we have the following immediate consequence.

Corollary 2.1. *Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a finitely determined map germ. Then there is a representative $f : U \to V$, where $U, V \subset \mathbb{R}^2$ are open sets, such that*

(1) $f^{-1}(0) = \{0\}$,
(2) $f : U \to V$ *is proper,*
(3) *the restriction $f|_{U \setminus \{0\}}$ is stable with only simple folds.*

We finish this section with an important result due to Fukuda [3], which tell us that any finitely determined map germ, $f:(\mathbb{R}^n,0)\to(\mathbb{R}^p,0)$, with $n\leq p$, has a conic structure over its link. In order to simplify the notation, we only state the result in our case $n=p=2$.

Given $\epsilon>0$, we denote:

$$S^1_\epsilon=\{x\in\mathbb{R}^2:\|x\|^2=\epsilon\},\quad D^2_\epsilon=\{x\in\mathbb{R}^2:\|x\|^2\leq\epsilon\}.$$

and given a map germ $f:(\mathbb{R}^2,0)\to(\mathbb{R}^2,0)$ we consider a representative $f:U\to V$ and put:

$$\widetilde{S}^1_\epsilon=f^{-1}(S^1_\epsilon),\quad \widetilde{D}^2_\epsilon=f^{-1}(D^2_\epsilon).$$

Theorem 2.2. *Let $f:(\mathbb{R}^2,0)\to(\mathbb{R}^2,0)$ be a finitely determined map germ. Then, up to $\mathcal{A}$-equivalence, there is a representative $f:U\to V$ and $\epsilon_0>0$, such that, for any ϵ with $0<\epsilon\leq\epsilon_0$ we have:*

(1) *$\widetilde{S}^1_\epsilon$ is diffeomorphic to S^1.*
(2) *The map $f|_{\widetilde{S}^1_\epsilon}:\widetilde{S}^1_\epsilon\to S^1_\epsilon$ is stable, in other words, it is a Morse function all of whose critical values are distinct.*
(3) *$f|_{\widetilde{D}^2_\epsilon}$ is topologically equivalent to the cone of $f|_{\widetilde{S}^1_\epsilon}$.*

Definition 2.3. Let $f:(\mathbb{R}^2,0)\to(\mathbb{R}^2,0)$ be a finitely determined map germ. We say that the stable map $f|_{\widetilde{S}^1_\epsilon}:\widetilde{S}^1_\epsilon\to S^1_\epsilon$ is the *link* of f, where f is a representative such that (1), (2) and (3) of theorem 2.2 hold for any ϵ with $0<\epsilon\leq\epsilon_0$. This link is well defined, up to $\mathcal{A}$-equivalence.

Since any finitely determined map germ is topologically equivalent to the cone of its link, we have the following immediate consequence.

Corollary 2.4. *Two finitely determined map germs $f,g:(\mathbb{R}^2,0)\to(\mathbb{R}^2,0)$ are topologically equivalent if their associated links are topologically equivalent.*

3. Gauss words

In this section we recall briefly (for more information and examples see [9]) how we define an adapted version of the Gauss word in our particular case of study and some consequences of such definition.

Definition 3.1. Let $\gamma:S^1\to S^1$ be a stable map, that is, such that all its singularities are of Morse type and its critical values are distinct. We fix orientations in each S^1 and we also choose base points $z_0\in S^1$ in the source and $a_0\in S^1$ in the target.

Suppose that γ has r critical values labeled by r letters $a_1,\dots,a_r\in S^1$ and let us denote their inverse images by $z_1,\dots,z_k\in S^1$. We assume they are ordered such that $a_0\leq a_1<\dots<a_r$ and $z_0\leq z_1<\dots<z_k$ and following the orientation of each S^1.

We define a map $\sigma:\{1,\dots,k\}\to\{a_1,\dots,a_r,\overline{a}_1,\dots,\overline{a}_r\}$ in the following way: given $i\in\{1,\dots,k\}$, then $\gamma(z_i)=a_j$ for some $j\in\{1,\dots,r\}$; we define $\sigma(i)=a_j$, if z_i is a regular point and $\sigma(i)=\overline{a}_j$, if z_i is a singular point (i.e., the bar $\overline{a}_j$ is used to distinguish whether the inverse image of the critical value is regular or singular). We call *Gauss word* to the sequence $\sigma(1)\dots\sigma(k)$.

For instance, the link of the cusp $f(x,y)=(x,xy+y^3)$ has two critical values with four inverse images and the associated Gauss word is $a\overline{b}\overline{a}b$ (see figure 2).

It is obvious that the Gauss word is not uniquely determined, since it depends on the chosen orientations and base points in each S^1. Different choices will produce the following changes in the Gauss word:

(1) a cyclic permutation in the letters $a_1,\dots,a_r$;
(2) a cyclic permutation in the sequence $\sigma(1)\dots\sigma(k)$;

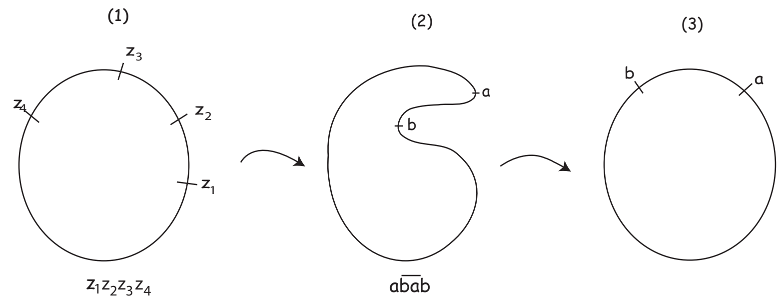


FIGURE 2.

(3) a reversion in the set of the letters $a_1, \ldots, a_r$;
(4) a reversion in the sequence $\sigma(1) \ldots \sigma(k)$.

We say that two Gauss words are equivalent if they are related through these four operations. Under this equivalence, the Gauss word is now well defined.

In order to simplify the notation, given a stable map $\gamma : S^1 \to S^1$, we denote by $w(\gamma)$ the associated Gauss word and by $\simeq$ the equivalence relation between Gauss words. We also denote by $\deg(\gamma)$ the topological degree. Then, we can state the main result of this section (see [9]).

Theorem 3.2. *Let $\gamma, \delta : S^1 \to S^1$ be two stable maps. Then γ, δ are topologically equivalent if and only if*

$$\begin{cases} w(\gamma) \simeq w(\delta), & \text{if } \gamma, \delta \text{ are singular,} \\ \\ |\deg(\gamma)| = |\deg(\delta)|, & \text{if } \gamma, \delta \text{ are regular.} \end{cases}$$

Given a finitely determined map germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$, we denote by $w(f)$ the Gauss word of its link and by $\deg(f)$ the local topological degree.

If $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ is a finitely determined map germ, then we can compute Gauss word of the link of f just by looking at the relative position of the branches of the three curves $S(f)$, $\Delta(f)$ and $X(f)$.

Example 3.3. Let us consider the finitely determined map germ $f(x, y) = (x, y^3 - x^2 y)$. The discriminant $\Delta(f)$ has a tree structure with one vertex at the origin and 4 adjacent edges labeled by 4 letters $a_1, \ldots, a_4$. Analogously, $S(f) \cup X(f)$ has also a tree structure with one vertex at the origin and 8 adjacent edges labeled by $Z_1, \ldots, Z_8$. We assume that the edges are well ordered $a_1 < \cdots < a_4$ and $Z_1 < \cdots < Z_8$ with respect to the chosen base points and orientations in the source and the target. We define the map $\sigma : \{1, \ldots, 8\} \to \{a_1, \ldots, a_4, \overline{a}_1, \ldots, \overline{a}_4\}$ in the following way: given $i \in \{1, \ldots, 8\}$, then $\gamma(Z_i) = a_j$ for some $j \in \{1, \ldots, 4\}$; we define $\sigma(i) = a_j$, if $Z_i \subset X(f)$ and $\sigma(i) = \overline{a}_j$, if $Z_i \subset S(f)$. Then, $\sigma(1) \ldots \sigma(8)$ is equal to the Gauss word of the link of f, obtaining in this case the word $a_1 \overline{a}_2 \overline{a}_1 a_2 a_3 \overline{a}_4 \overline{a}_3 a_4$ (see figure 3).

As a direct consequence, we have the following corollary.

Corollary 3.4. *Let $f, g : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be two finitely determined map germs. Then, if f and g are topologically equivalent, their links are topologically equivalent.*

Now,by using theorem 3.2 and corollaries 2.4 and 3.4 we can state the following result:

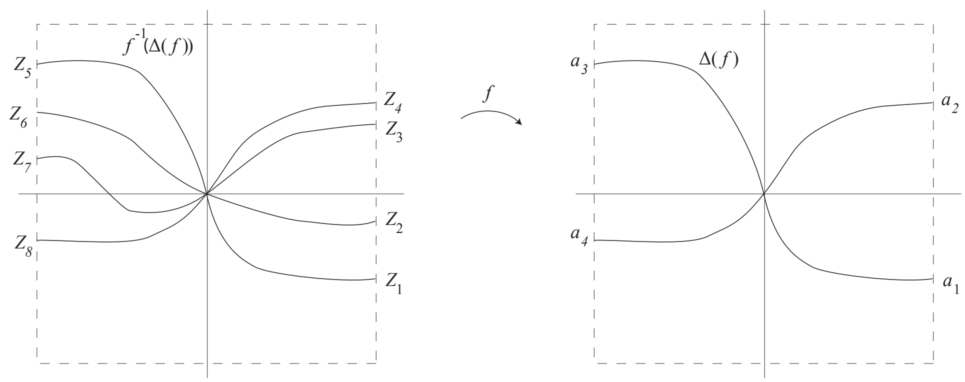


FIGURE 3.

Corollary 3.5. *Let $f, g : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be two finitely determined map germs. Then f, g are topologically equivalent if and only if*

$$\begin{cases} w(f) \simeq w(g), & \text{if } f, g \text{ are singular outside the origin,} \\ \\ |\deg(f)| = |\deg(g)|, & \text{if } f, g \text{ are regular outside the origin.} \end{cases}$$

Remark 3.6. If f is regular outside the origin and $|\deg(f)| = r$, then f is topologically equivalent to the germ $z \to z^r$, with $z = x + iy$.

Before finishing this section, let us state a result that will give us a necessary condition that a stable map $\gamma : S^1 \longrightarrow S^1$ should verify to be the link of a corank 1 map germ.

Proposition 3.7. *Any finitely determined map germ $f : (\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}^2, 0)$ of corank 1, with link γ, verifies that*

$$\text{mult}(\gamma) = \begin{cases} 0, & \text{if } \deg(f) = 0, \\ 1, & \text{if } \deg(f) = \pm 1. \end{cases}$$

We will define the multiplicity of a stable map $\gamma : S^1 \longrightarrow S^1$ as $\text{mult}(\gamma) = min_{p \in S^1} \text{mult}(p)$, with $\text{mult}(p) = \#\gamma^{-1}(p)$.

Proof. The three possible values of the topological degree of f are a consequence of a known result (see for example [9]). Let us suppose that $f(x, y) = (x, g_x(y))$, with

$$g_x(y) = y^n + a_{n-2}(x)y^{n-2} + \cdots + a_1(x)y.$$

If $\deg(f) = 0$, n is even, $n - 1$ is odd and, as a consequence, the both curves g_x^+, g_x^-, that will form the link of f will have both an odd number of folds. Thus, γ will not be surjective and $\text{mult}(\gamma) = 0$. If $\deg(f) = \pm 1$, n is odd, $n - 1$ is even and the union of both partial curves will completely fill S^1, so $\text{mult}(\gamma) = 1$. □

4. TOPOLOGICAL CLASSIFICATION OF MAP GERMS OF TYPE $\Sigma^{2,0}$

In this section of the chapter we will classify corank 2 map germs, $f : (\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}^2, 0)$ which are of type $\Sigma^{2,0}$.

First of all we will state a result that will give us two prenormal forms of map germs of this type.

Theorem 4.1. *Let $f : (\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}^2, 0)$ a corank 2 map germ of type $\Sigma^{2,0}$. Then, f can be written in one of the following prenormal forms:*

(1) $(xy, g(x, y))$
(2) $(x^2 + y^2, h(x, y))$,

where $g, h \in \mathcal{M}_2^2$

Proof. Firstly, we know that if we consider a map germ f of type $\Sigma^{2,0}$, its 2-jet $j^2 f(0)$ is situated in one of the following $\mathcal{A}^2$-classes (see for example [5]):

$$(xy, x^2 + y^2), \quad (xy, x^2), \quad (xy, 0), \quad (x^2 + y^2, 0).$$

Therefore, f will present one of the following forms:

(1) $(xy + a(x, y), b(x, y))$, with $a(x, y) \in \mathcal{M}_2^3$, $b(x, y) \in \mathcal{M}_2^2$
(2) $(x^2 + y^2 + c(x, y), d(x, y))$, with $c(x, y) \in \mathcal{M}_2^3$, $d(x, y) \in \mathcal{M}_2^2$

By applying Morse's lemma we know that if we consider a function germ $f : (\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}, 0)$ of the form $f(x, y) = u(x, y) + v(x, y)$, with $u(x, y)$ being a non degenerate quadratic form and with $v(x, y) \in \mathcal{M}_2^3$, we can choose a suitable change of coordinates

$$\begin{array}{rccc} \alpha : & (\mathbb{R}^2, 0) & \longrightarrow & (\mathbb{R}^2, 0) \\ & (x, y) & \to & (X, Y) \end{array}$$

such that $u = f \circ \alpha^{-1}$.

As we have a non degenerate quadratic form in the first component, if we apply this change of coordinates in (1) and (2), we arrive to the desired result. □

The first step to classify topologically this kind of germs will be to compute their topological degree. Taking it into account, we state and prove the following result.

Proposition 4.2. *Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a finitely determined map germ of type $\Sigma^{2,0}$.*

(1) *If $f(x, y) = (xy, g(x, y))$, f can have degree $0, \pm 1$ or ± 2.*
(2) *If $f(x, y) = (x^2 + y^2, h(x, y))$, f has degree 0.*

Proof. let us prove first (2). If our germ f has as first component $x^2 + y^2$ it is not surjective. Then, $\deg(f) = 0$.

For (1), we can suppose, without loss of generality, that

$$g(x, y) = ax^p + by^q + k(x, y)$$

where

$$p, q \geq 2, \qquad a, b > 0, \qquad \text{and} \qquad k(x, y) \in \langle xy \rangle.$$

As we know that $(xy, g(x, y))$ is $\mathcal{K}$-equivalent to $(xy, ax^p + by^q)$ and that the topological degree is a $\mathcal{K}$-invariant we only need to compute the topological degree of $(xy, ax^p + by^q)$. We will do it by applying Eisenbud-Levine's formula ([2]), given by

$$\deg(f) = \text{sign}\langle , \rangle_\varphi,$$

the signature of the quadratic form associated to a linear function $\varphi : Q(f) \to \mathbb{R}$ defined conveniently, with

$$Q(f) = \frac{\mathcal{E}_2}{\langle f_1, f_2 \rangle}.$$

Thus, we have that

$$Q(f) = \frac{\mathcal{E}_2}{\langle xy, ax^p + by^q \rangle}$$

and a basis of this space will be given by

$$\{1, x, x^2, \ldots, x^{p-1}, y, y^2, \ldots, y^{q-1}, J(f)\}$$

with $J(f) = qby^q - pax^p$.

We define the map

$$\begin{array}{rccc} \varphi : & Q(f) & \longrightarrow & \mathbb{R} \\ & J(f) & \to & 1 \\ & [1] & \to & 0 \\ & [x] & \to & 0 \\ & [y] & \to & 0 \\ & \vdots & \vdots & \vdots \\ & [x^{p-1}] & \to & 0 \\ & [y^{q-1}] & \to & 0 \end{array}$$

We will suppose that $a = b = \pm 1$, generalizing the result later.

The matrix of

$$\begin{array}{rccc} \langle , \rangle_\varphi : & Q(f) \times Q(f) & \longrightarrow & \mathbb{R} \\ & (p, q) & \to & \varphi(pq) \end{array}$$

with respect to the basis of $Q(f)$ is

$$A = \begin{array}{c} \\ 1 \\ x \\ x^2 \\ \vdots \\ x^{p-1} \\ y \\ y^2 \\ \vdots \\ y^{q-1} \\ J \end{array} \begin{array}{c} \begin{array}{cccccccccc} 1 & x & x^2 & \cdots & x^{p-1} & y & y^2 & \cdots & y^{q-1} & J \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & \mp\frac{1}{p+q} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & \mp\frac{1}{p+q} & 0 & \cdots & 0 & 0 & 0\cdots & 0 & 0 & \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \pm\frac{1}{p+q} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \pm\frac{1}{p+q} & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \end{array}$$

taking into account the following facts:

- Each element of the form $x^i y^j \in \langle xy \rangle$ and as a consequence is 0 in $Q(f)$
- Each element of the form $Jx^i, Jy^j, x^{p+i}, y^{q+j}$ can be written as linear combination of the components of f, that is, they are 0 in $Q(f)$
- The elements x^p and y^q can be written in the following form:
  - $x^p = \frac{\mp qy^q \pm px^p \pm q(\pm x^p \pm y^q)}{\pm(p+q)} = \frac{\mp 1}{p+q} J \pm \frac{q}{p+q}(\pm x^p \pm y^q)$, with $\varphi(x^p) = \mp\frac{1}{p+q}$
  - $y^q = \frac{\pm qy^q \mp px^p \pm p(\pm x^p \pm y^q)}{\pm(p+q)} = \frac{\pm 1}{p+q} J \pm \frac{p}{p+q}(\pm x^p \pm y^q)$, with $\varphi(y^q) = \pm\frac{1}{p+q}$

Therefore, by computing the determinant of the matrix $(xI - A)$ we obtain the following characteristic polynomials, depending on the parity of p and q:

- If p and q are odd, $\det(xI - A) = (x^2 - 1)(x^2 - \frac{1}{(p+q)^2})^{\frac{p+q-2}{2}}$ and, as a consequence, $\deg(f) = \operatorname{sign}\langle , \rangle_\varphi = 0$.
- If p and q are even, $\det(xI - A) = (x^2 - 1)(x^2 - \frac{1}{(p+q)^2})^{\frac{p+q-4}{2}}(x \mp \frac{1}{p+q})(x \pm \frac{1}{p+q})$ and, as a consequence, $\deg(f) = \operatorname{sign}\langle , \rangle_\varphi = \begin{cases} 0, & \text{if } ab > 0, \\ \pm 2, & \text{if } ab < 0. \end{cases}$

- If p and q have different parity, $\det(xI-A) = (x^2-1)(x^2-\frac{1}{(p+q)^2})^{\frac{p+q-3}{2}}(x \pm \frac{1}{p+q})$ and, as a consequence, $\deg(f) = \mathrm{sign}\langle,\rangle_\varphi = \pm 1$.

Let us see now that we are able to generalize this result for any $a, b \in \mathbb{R}$, with $a, b \neq 0$. We will prove this by constructing a homotopy.

Let $f_0(x,y) = (xy, ax^p + by^q)$, with $a > 0$ (analogous for $a < 0$), $f_1(x,y) = (xy, x^p + by^q)$ and we consider the family

$$f_t(x,y) = (xy, ((1-t)a + t)x^p + by^q),$$

with $t \in [0,1]$.

If we prove that for any t, $f_t^{-1}(0) = \{0\}$ and that if $t = 0$, $f_t = f_0$ and if $t = 1$, $f_t = f_1$, we will have that f_0 and f_1 are homotopic and, as a consequence, $\deg(f_0) = \deg(f_1)$.

As $(1-t)a + t \neq 0$ for any t, we will have that if we want that both terms of f_t vanish, x and y must be 0. Then, for any t, $f_t^{-1}(0) = \{0\}$. On the other hand, by substituting, if $t = 0$, $f_t(x,y) = (xy, ax^p + by^q) = f_0(x,y)$ and if $t = 1$, $f_t(x,y) = (xy, x^p + by^q) = f_1(x,y)$. Then, f_0 and f_1 are homotopic and $\deg(f_0) = \deg(f_1)$.

Analogously, we will have that $\deg(xy, x^p + by^q) = \deg(xy, x^p + y^q)$ if $b > 0$. Then,

$$\deg(xy, ax^p + by^q) = \deg(xy, x^p + y^q).$$

□

Now, putting together theorem 4.1 and proposition 4.2, we have the following corollary.

Corollary 4.3. *Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a finitely determined map germ of type $\Sigma^{2,0}$. Then, $|\deg(f)| \leq 2$.*

Proof. If f is of type $\Sigma^{2,0}$, by theorem 4.1 it can be written in the form $(xy, g(x,y))$ or in the form $(x^2+y^2, h(x,y))$, and we have just seen that the absolute value of their topological degree is less or equal than 2. □

Before starting to compute the different topological classes of this kind of germs, we should remember the concepts of admissible weights and weighted degrees of a weighted homogeneous map germ which were introduced by Gaffney and Mond in [4] and will be very helpful for us in our classification.

Definition 4.4. Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a weighted homogeneous map germ. We will say that its weights w_1, w_2 and its weighted degrees d_1, d_2 are *admissible* if they verify the two following conditions:

(1) $(w_1, w_2) = (d_1, d_2) = 1$
(2) $w_1 = w_2 = 1$ (homogeneous case) or $d_1 = k_1 w_1 w_2,\ d_2 = k_2 w_1 w_2 + w_1 + w_2$ (type 1) or $d_1 = k_1 w_1 w_2 + w_1,\ d_2 = k_2 w_1 w_2 + w_2$ (*type* 2).

Once we have introduced this concept, let us see its relation with finitely determined map germs.

Proposition 4.5. *Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a weighted homogeneous finitely determined map germ. Then, w_1, w_2, d_1, d_2 must be admissible.*

Proof. Given a weighted homogeneous finitely determined map germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$, since it is real analytic, we can consider its complexification $\hat{f} : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$. It is well known that $\hat{f}$ is also weighted homogeneous and finitely determined as a complex analytic map germ.Then, by applying [Proposition 5.3, 4], its weights w_1, w_2 and weighted degrees d_1, d_2 must be admissible for $\hat{f}$, and, as a consequence, for f. □

Remark 4.6. Let us see how we apply this result to a finitely determined map germ in our particular case of study.

- $f(x,y) = (xy, g(x,y))$
  If f is weighted homogeneous, that is,
  $$g(x,y) = \sum_{i=0}^{p} a_i (x^{w_2})^i (y^{w_1})^{p-i}$$
  we must have that $(w_1, w_2) = (w_1 + w_2, pw_1w_2) = 1$. Departing from the basis that w_1, w_2 must be relatively primes, we have the following consequences, according to the value of p.
  - If $p = 1$, f is generically finitely determined.
  - If $p = 2$, f won't be finitely determined if w_1 and w_2 are odd.
  - If $p = 3$, f won't be finitely determined if $w_1 + w_2 = 3k$, with $k \in \mathbb{N}$.
  - In general, if $p = t_1^{\alpha_1} \dots t_m^{\alpha_m}$, f won't be finitely determined if there exists i such that $w_1 + w_2 = kt_i$, with $1 \leq i \leq m$ and $k \in \mathbb{N}$.
- $f(x,y) = (x^2 + y^2, h(x,y))$
  Because of the first component, we are only able to study this kind of germ in the homogeneous case $w_1 = w_2 = 1$, with
  $$h(x,y) = \sum_{i=0}^{p} a_i x^i y^{p-i}$$
  and $(2,p) = 1$. We arrive quickly to the conclusion that if $p = 2k$, f won't be finitely determined.

4.1. **Germs with prenormal form** $(xy, g(x,y))$**.** We consider the special case of weighted homogeneous map germs, that is,
$$g(x,y) = \sum_{i=0}^{p} a_i (x^{w_2})^i (y^{w_1})^{p-i},$$
with (w_1+w_2, pw_1w_2) being the weighted degrees of our germ and $(w_1, w_2) = 1$. We also suppose that $p \leq 3$. Then, the following results will give us a complete topological classification of these particular cases.

Theorem 4.7. *($p = 1$) Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a finitely determined map germ of corank 2 of the form $f(x,y) = (xy, ax^{w_2} + by^{w_1})$. Then,*

(1) *if w_1, w_2 are odd, f is topologically equivalent to the fold (x, y^2),*
(2) *if w_1, w_2 have different parity, f is topologically equivalent to the cusp $(x, xy + y^3)$.*

Proof. Let us prove first (1).

If w_1, w_2 are odd, we know by the proof of theorem 4.2 that $\deg(f) = 0$. In addition to this, if we compute its singular set, we get the equation $w_1 b y^{w_1} - w_2 a x^{w_2} = 0$. Since this equation is irreducible, we can conclude that $S(f)$, and, as a consequence $\Delta(f)$, only present a single branch.

Let us see that we are going to have a single topological class which is the class of the fold. To prove this is enough to see that for any $a, b \in \mathbb{R} \setminus \{0\}$ there are points where f does not have any inverse image.

Let us consider the point $(1,0)$. We get the equations $xy = 1$ and $ax^{w_2} + by^{w_1} = 0$, obtaining that
$$y = \left(\frac{-a}{b} \right)^{1/(w_1+w_2)}.$$

Thus, if $ab > 0$ f does not have any inverse image and the result is proved (see figure 4).

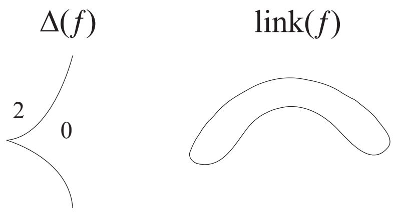


FIGURE 4.

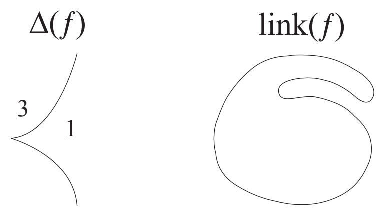


FIGURE 5.

Analogously, if we take now the point $(-1, 0)$ we have that every map germ f with $ab < 0$ does not have any inverse image either and we arrive to the conclusion again that we have a single configuration of inverse images in the discriminant curve, which is the one of the fold.

If w_1 and w_2 are of distinct parity, applying an analogous procedure as in (1) to prove the existence of points with a single inverse image, we obtain the desired result (see figure 5).

□

Theorem 4.8. *($p = 2$) Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a finitely determined map germ of corank 2 of the form $f(x, y) = (xy, ax^{2w_2} + bx^{w_2}y^{w_1} + cy^{2w_1})$. Then,*

1 *if w_1, w_2 have the same parity, f is not finitely determined,*

2 *if w_1, w_2 have distinct parity, we have three cases,*
- *if $(w_1 - w_2)^2b^2 + 16w_1w_2ac > 0$,*
  - *f is topologically equivalent to the map germ $(xy, x^2 + xy^2 + y^4)$ if $ac > 0$*
  - *f is topologically equivalent to the map germ $(xy, x^2 + 20xy^2 - y^4)$ if $ac < 0$*
- *if $(w_1 - w_2)^2b^2 + 16w_1w_2ac < 0$, f is topologically equivalent to the map germ $(xy, x^2 + xy^2 - y^4)$*
- *if $(w_1 - w_2)^2b^2 + 16w_1w_2ac = 0$, f is not finitely determined.*

Proof. If w_1, w_2 are both even or odd the result follows from remark 4.6. Let us suppose that w_1 and w_2 have different parity. The Jacobian determinant is given by

$$J(f) = -2w_2ax^{2w_2} + b(w_1 - w_2)x^{w_2}y^{w_1} + 2w_1cy^{2w_1},$$

that can be factorized in the form

$$-2w_2(x^{w_2} - \lambda_1y^{w_1})(x^{w_2} - \lambda_2y^{w_1}),$$

with $\lambda_i = \lambda_i(a, b, c, w_1, w_2) \in \mathbb{C}$, $i = 1, 2$. These λ_i are obtained by solving the quadratic equation given by the Jacobian determinant, whose discriminant is

$$(w_1 - w_2)^2b^2 + 16w_1w_2ac = 0.$$

Then, if this discriminant is positive we have two different real solutions for λ_i and as a consequence two branches in our singular set $S(f)$, if it is negative our singular set is empty outside of the origin and in the case that the discriminant vanishes, $\lambda_1 = \lambda_2$ and f won't be finitely determined. If the discriminant is negative, by remark 3.6 and proposition 4.2, taking into account that ac must be necessarily negative, we have that f will be topologically equivalent to the germ $(xy, x^2 - y^2)$. Since this germ is not finitely determined we can choose another member of this topological class that is finitely determined. let us take, for example, $(xy, x^2 + xy^2 - y^4)$.

Thus, we center our attention in the case $(w_1 - w_2)^2b^2 + 16w_1w_2ac > 0$. If we call

$$C_i \equiv x^{w_2} - \lambda_iy^{w_1} = 0$$

for $i = 1, 2$, and apply the coordinate changes

$$\begin{cases} x = \alpha t^{w_1} \\ y = \beta t^{w_2} \end{cases}$$

we have that

$$f|_{C_i}(t) = (\alpha\beta t^{w_1+w_2}, (a\lambda_i^2 + b\lambda_i + c)t^{2w_1w_2}),$$

whose derivative never vanishes out of 0 and it will present double folds if and only if $\alpha\beta = 0$, which is impossible. Let us observe that these curves are going to be symmetrical with respect to the y-axis (figure 6). From this point, we must consider two different cases:

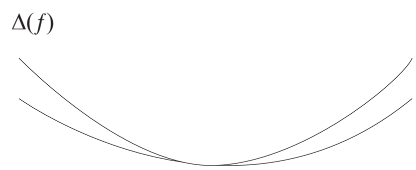


FIGURE 6.

- If $ac > 0$, by proposition 4.2 we know that $\deg(f) = 0$. Taking into account that our discriminant set has 2 branches and the link of f can't have more than one connected component, if we are able to prove that for any b we have points with no inverse images, we finish.

  If we consider the point $(0, -1)$ we obtain the equations

  $$xy = 0 \quad \text{and} \quad ax^{2w_2} + bx^{w_2}y^{w_1} + cy^{2w_1} = -1,$$

  getting the equality $y = \left(\frac{-1}{c}\right)^{1/(2w_1)}$ if $x = 0$ and $x = \left(\frac{-1}{a}\right)^{1/(2w_2)}$ if $y = 0$. In both cases if a and c are positive the equalities don't have any real solution. Thus, f does not present any inverse image (see figure 7).

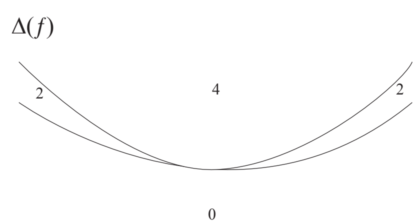


FIGURE 7.

Considering the point $(0, 1)$ and applying a totally analogous procedure we arrive to the conclusion that if $a, c < 0$ f does not present any inverse image either (see figure 8). Then, we have in both cases a single configuration of inverse images in the discriminant, obtaining the associated link and Gauss word that appear in figure 9. Thus, f is topologically equivalent to the known corank 1 normal form $(x, y^4 - xy^2 - x^2y)$. If we want to take a normal form of corank 2 we can choose, for example, $(xy, x^2 + xy^2 + y^4)$.

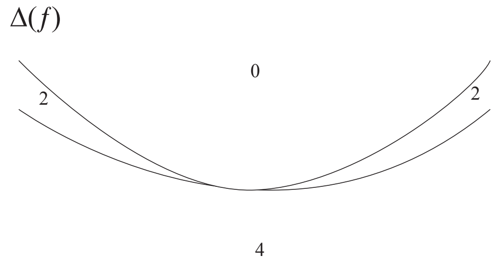


FIGURE 8.

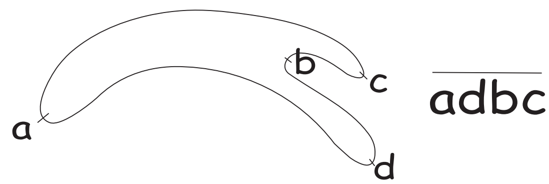


FIGURE 9.

- If $ac < 0$, using again proposition 4.2, we know that $\deg(f) = \pm 2$. Taking into account that we are dealing with a map germ whose discriminant only has two branches if we are able to prove that the maximum number of inverse images of f is 4 we finish.

  Let us consider the equations
  $$xy = d \quad \text{and} \quad ax^{2w_2} + bx^{w_2}y^{w_1} + cy^{2w_1} = e,$$
  with $(d, e) \in \mathbb{R}^2$. From here, we get the equality
  $$cy^{2(w_1+w_2)} + bd^{w_2}y^{w_1+w_2} - ey^{2w_2} + ad^{2w_2} = 0.$$
  Applying Descartes method and using the hypothesis $ac < 0$ we arrive to the conclusion that we can have three sign changes for $y > 0$ in the best of the cases and since all the exponents are even except $w_1 + w_2$, this is the only term whose sign is going to change when we consider $y < 0$. Then, we will have in this last case a single inverse image and a total of 4 inverse images, as we wanted to prove.

  Thus, the only possible configuration of the inverse images in the discriminant of a map germ of this type will be the one that appears in figure 10, having its correspondent associated link and Gauss word (figure 11).

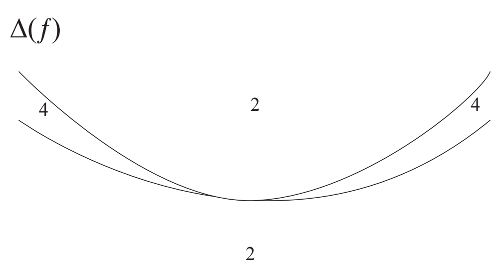


FIGURE 10.

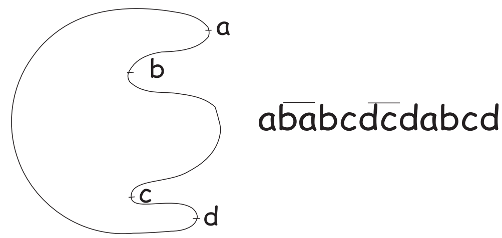


FIGURE 11.

To finish, let us choose a representative of this topological class, for example,

$$(xy, x^2 + 20xy^2 - y^4).$$

□

Theorem 4.9. *($p = 3$) Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a finitely determined map germ of corank 2 of the form $f(x, y) = (xy, ax^{3w_2} + bx^{2w_2}y^{w_1} + cx^{w_2}y^{2w_1} + dy^{3w_1})$. let us denote by*

$$A = -3w_2a\frac{(2w_1 - w_2)c}{3} - \left(\frac{(w_1 - 2w_2)b}{3}\right)^2,$$

$$B = -3w_2a3w_1d - \frac{(w_1 - 2w_2)b}{3}\frac{(2w_1 - w_2)c}{3},$$

$$C = \frac{(w_1 - 2w_2)b}{3}3w_1d - \left(\frac{(2w_1 - w_2)c}{3}\right)^2.$$

Then:

(1) *Let us suppose that w_1, w_2 have different parity,*
- *if $B^2 - 4AC > 0$, f is topologically equivalent to the simple cusp $(x, xy + y^3)$,*
- *if $B^2 - 4AC < 0$, f is topologically equivalent to one of the map germs that appear in table 1,*
- *if $B^2 - 4AC = 0$, f is not finitely determined.*

(2) *In the case that w_1, w_2 are both odd,*
- *if $B^2 - 4AC > 0$, f is topologically equivalent to one of the map germs that appear in the table 2,*
- *if $B^2 - 4AC < 0$, f is topologically equivalent to one of the map germs that appear in table 3,*
- *if $B^2 - 4AC = 0$, f is not finitely determined.*

Proof. If we compute the Jacobian determinant of f we get

$$Jf(x, y) = -3w_2ax^{3w_2} + (w_1 - 2w_2)bx^{2w_2}y^{w_1} + (2w_1 - w_2)cx^{w_2}y^{2w_1} + 3w_1dy^{3w_1}.$$

Let us realize that if we make the coordinate changes

$$\begin{cases} \overline{x} = x^{w_2} \\ \\ \overline{y} = y^{w_1} \end{cases}$$

we get the cubic form

$$Jf(\overline{x}, \overline{y}) = -3w_2a\overline{x}^3 + (w_1 - 2w_2)b\overline{x}^2\overline{y} + (2w_1 - w_2)c\overline{x}\overline{y}^2 + 3w_1d\overline{y}^3.$$

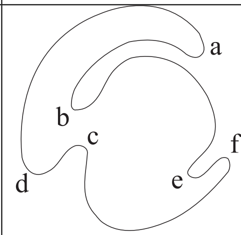
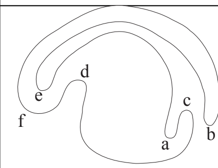
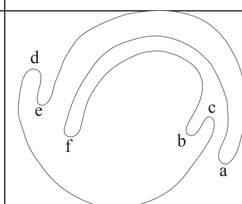
Degree	*Germ*	*Associated link*
1	$(xy, x^6 + 7x^4y^3 + 8x^2y^6 + y^9)$	 $a\overline{ba}bc\overline{dc}de\overline{fe}f$
	$(xy, x^6 + 2x^4y^3 + 9x^2y^6 + y^9)$	 $ab\bar{c}b\bar{a}bcd\bar{e}dc\bar{b}cde\bar{f}e\bar{d}ef$
	$(xy, x^6 - x^4y^3 + 7x^2y^6 + y^9)$	 $ab\overline{cb}cde\bar{f}edcb\bar{a}bcd\overline{ed}ef$

TABLE 1.

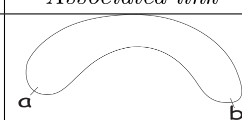
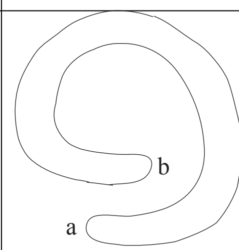
Degree	*Germ*	*Associated link*
0	(x, y^2)	 $\overline{ab}$
	$(xy, x^3 - x^2y^3 - xy^6 + y^9)$	 $\bar{a}ba\bar{b}ab$

TABLE 2.

From this point we apply a known result (see for example [5]) which tell us that a cubic form will be of symbolic, hyperbolic, parabolic or elliptic type if and only if its associated quadratic

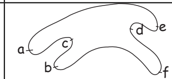
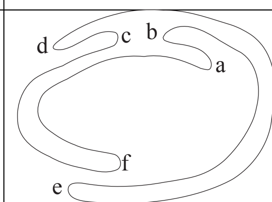
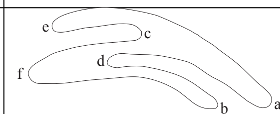
Degree	*Germ*	*Associated link*
0	$(xy, x^3 - x^2y^3 + 3xy^6 + y^9)$	 $c\overline{acb}cd\overline{fde}d$
	$(xy, x^3 - 6x^2y^3 + 4xy^6 + y^9)$	 $abc\overline{dc}de\overline{f}edcb\overline{ab}af\overline{e}f$
	$(xy, x^3 + 6x^2y^3 + 6xy^6 + y^9)$	 $\overline{a}bcd\overline{e}d\overline{c}de\overline{f}edc\overline{b}c\overline{d}cb$

TABLE 3.

form obtained by computing the Hessian determinant is of symbolic, hyperbolic, parabolic or elliptic type respectively. Thus, if we compute the Hessian determinant of $Jf(\overline{x}, \overline{y})$ we get the quadratic form $A\overline{x}^2 + B\overline{xy} + C\overline{y}^2$ with A, B, C depending on the values of the initial coefficients a, b, c, d and of the weights w_1, w_2 and undoing the coordinate changes we made earlier we get the function $Ax^{2w_2} + Bx^{w_2}y^{w_1} + Cy^{2w_1}$ which we will use to determine the different cases of study. Therefore, we have the following possibilities:

(1) Let us suppose that w_1 and w_2 have different parity. Firstly, if we consider as we did in the case $p = 2$ the coordinate changes

$$\begin{cases} x = \alpha t^{w_1} \\ y = \beta t^{w_2} \end{cases}$$

together with the image of the restriction of f to each one of the curves of the singular set, C_i, we get

$$f_{|C_i(t)}(t) = (\alpha\beta t^{w_1+w_2}, (a\lambda_i^3 + b\lambda_i^2 + c\lambda_i + d)t^{3w_1w_2}),$$

realizing that each one of these branches is symmetric with respect to the y-axis. Now, let us see the different configurations of inverse images that we can have in the discriminant, in order to obtain the distinct topological classes. As first step we will prove that $\#f^{-1}(z) \leq 5, \forall z \in \mathbb{R}^2$.

Let us take a point $(e, f) \in \mathbb{R}^2$ and let us consider the equations

$$\begin{cases} xy = e \\ ax^{3w_2} + bx^{2w_2}y^{w_1} + cx^{w_2}y^{2w_1} + dy^{3w_1} = f. \end{cases}$$

Taking in the first equation $x = \frac{e}{y}$, with $y \neq 0$ and substituting we get

$$a(\frac{e}{y})^{3w_2} + b(\frac{e}{y})^{2w_2}y^{w_1} + c(\frac{e}{y})^{w_2}y^{2w_1} + dy^{3w_1} = f.$$

As last step we multiply both sides of the equation by y^{3w_2}, obtaining the final equation

$$dy^{3(w_1+w_2)} + ce^{w_2}y^{2(w_1+w_2)} - fy^{3w_2} + be^{2w_2}y^{w_1+w_2} + ae^{3w_2} = 0.$$

Now, putting in order the monomials according to their weighted degree and taking into account that the order of appearance of $(c, -f, b)$ can suffer variations due to the different values of (w_1, w_2), we apply Descartes rule of signs. Since we are working with a a polynomial consisting of 5 monomials, the worst configuration (with a biggest number of inverse images) will be given by $+-+-+$. Then, we will have at most 4 inverse images for $y > 0$ or $y < 0$ indistinctly(let us take $y > 0$). If $y < 0$, taking into account the parity of the weighted degrees of the monomials, we have the configuration $---++$ (or $--+++$, depending on the parity of w_2), obtaining a single inverse image and a total of 5 inverse images as we wanted to prove. If $(c, -f, b)$ would appear in a distinct order, by applying an analogous procedure we would arrive to the same result.

Secondly, we are going to prove that our germ f is always going to have points with a single inverse image and points with 3 inverse images. To do this we take a point $(0, f) \in \mathbb{R}^2$ and consider the equations

$$\begin{cases} xy = 0 \\ ax^{3w_2} + bx^{2w_2}y^{w_1} + cx^{w_2}y^{2w_1} + dy^{3w_1} = f. \end{cases}$$

Since xy vanishes, x or y must be 0 and using the second equation we get in the first case $y = \left(\frac{f}{d}\right)^{1/(3w_1)}$ and in the second case $x = \left(\frac{f}{a}\right)^{1/(3w_2)}$. Therefore, if w_1 is even and w_2 is odd we will have 3 inverse images if $fd > 0$ and a single one if $fd < 0$; analogously, if w_1 is odd and w_2 is even we will have 3 inverse images if $fa > 0$ and a single one if $fa < 0$. Then, from this point, what we know for sure is that the sectors of our bifurcation set in the image of f created by the discriminant curves that contain the y-axis are going to have one of them 3 inverse images and the other, a single one.

With all these previous calculations we are now in conditions to obtain the different topological classes.

- If $B^2 - 4AC > 0$, we have a single branch in our singular set and as a consequence, the only possible configuration of inverse images in its single discriminant curve is the one that appear in figure 12, which is clearly identified with the Gauss word and the link of the simple cusp (figure 13). Then, f is topologically equivalent to the simple cusp.

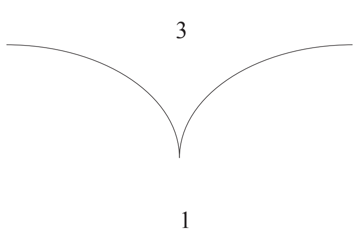


FIGURE 12.

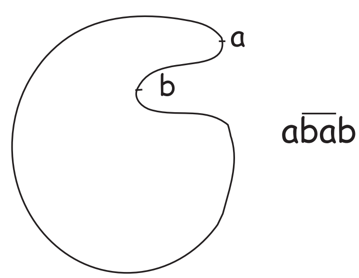


FIGURE 13.

- If $B^2 - 4AC < 0$ we have three branches and two possible configurations of inverse images in the discriminant curves (figure 14), obtaining in the first case the associated link and Gauss word that appears in figure 15 , with normal form

$(xy, x^6 + 7x^4y^3 + 8x^2y^6 + y^9)$ and in the second case the two different topological classes that appear in figure 16, having as normal forms $(xy, x^6+2x^4y^3+9x^2y^6+y^9)$ and $(xy, x^6 - x^4y^3 + 7x^2y^6 + y^9)$ respectively.

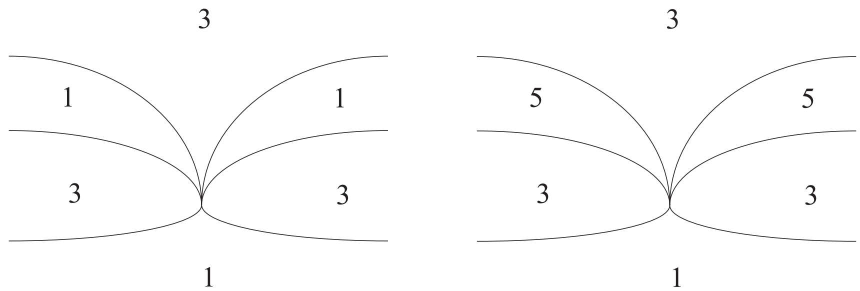


Figure 14.

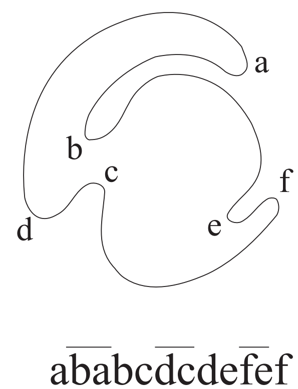


Figure 15.

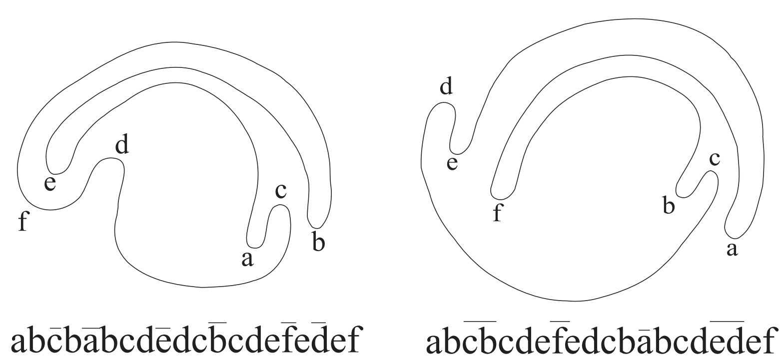


Figure 16.

- If $B^2 - 4AC = 0$ we will have a non reduced component in our singular set and f won't be finitely determined.

(2) If w_1 and w_2 are odd, we consider again the coordinate changes

$$\begin{cases} x = \alpha t^{w_1} \\ y = \beta t^{w_2} \end{cases}$$

together with the image of the restriction of f to each one of the curves of the singular set, C_i. In this case, these images are symmetric with respect to the x-axis. Now, let us see the different configurations of inverse images that we can have in the discriminant, in order to obtain the distinct topological classes. Firstly, we will prove that $\# f^{-1}(z) \leq 6$, $\forall z \in \mathbb{R}^2$.

Following a totally analogous procedure to the case of weights with different parity, taking a point $(e, f) \in \mathbb{R}^2$ we arrive to the equation

$$dy^{3(w_1+w_2)} + ce^{w_2}y^{2(w_1+w_2)} - fy^{3w_2} + be^{2w_1}y^{w_1+w_2} + ae^{3w_2} = 0$$

and applying Descartes method we conclude that points situated in the image of f are going to present 6 inverse images at most.

Let us see now that f is always going to have points with 2 inverse images in the y-axis. To prove this, we consider a point $(0, f) \in \mathbb{R}^2$. Since the first component of f must vanish we get the equalities $y = \left(\frac{f}{d}\right)^{1/(3w_1)}$, with a single inverse image $\left(0, \left(\frac{f}{d}\right)^{1/(3w_1)}\right)$ and $x = \left(\frac{f}{a}\right)^{1/(3w_2)}$, with a single inverse image $\left(\left(\frac{f}{a}\right)^{1/(3w_2)}, 0\right)$, getting a total of 2 inverse images, as we wanted to prove.

With all these previous remarks we are in conditions of giving a restricted list of the possible distribution of inverse images that we can have in the discriminant curves.

- If $B^2 - 4AC > 0$ our singular set and as a consequence the discriminant has a single real branch. Therefore, we only have two possible distributions of inverse images (figure 17), getting in the first case the link and Gauss word that appear in the left hand side of figure 18, with the associated normal form of the fold (x, y^2), and in the last case the one that appear in the right hand side of figure 18, with the associated normal form $(xy, x^3 - x^2y^3 - xy^6 + y^9)$.

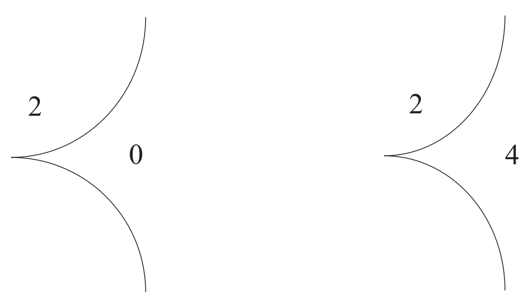


FIGURE 17.

- If $B^2 - 4AC < 0$ our singular set, and as a consequence the discriminant, has 3 distinct real branches and the initial number of possible configurations of inverse images in the discriminant is much bigger (see figure 19). Let us see that (d) and (e) can't occur.

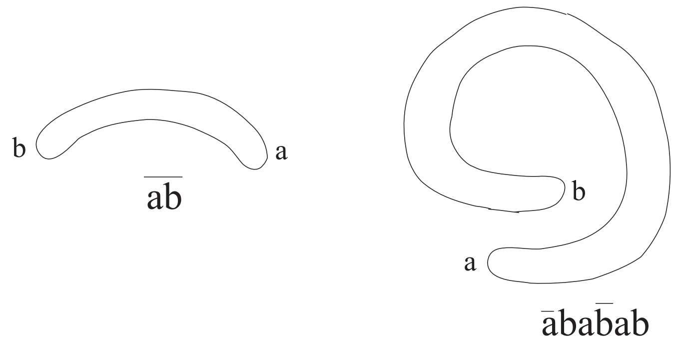


FIGURE 18.

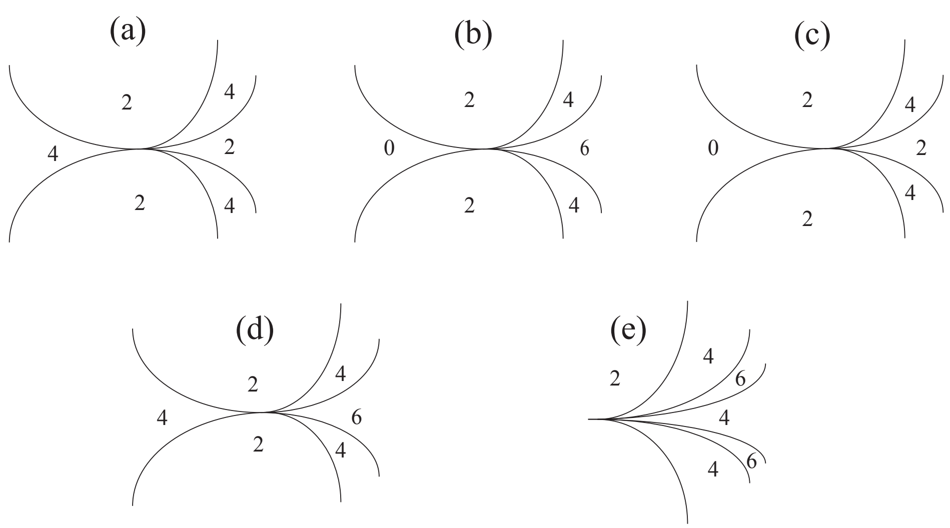


FIGURE 19.

If we had the configuration of (d), we would have points of the form $(e, 0) \in \mathbb{R}^2$ with 6 inverse images. let us suppose that $e > 0$. We obtain the equation

$$dy^{3(w_1+w_2)} + ce^{w_2}y^{2(w_1+w_2)} + be^{2w_1}y^{w_1+w_2} + ae^{3w_2} = 0.$$

If we apply Descartes method to this polynomial, the only possible signs configuration to get 6 inverse images is $+-+-$ for $y > 0$, obtaining $+-+-$ for $y < 0$. If (d) was possible, taking a point of the form $(e, 0)$ with $e < 0$ we should have 4 inverse images. But this is impossible because applying again Descartes method and using the sign of coefficients (a, b, c, d) we have had to choose to obtain 6 inverse images when $e > 0$ we obtain a signs configuration of the form $++++$ for any y. Therefore, we have just arrived to a contradiction and the configuration (d) is not possible.
To prove that (e) is not possible either we will choose a point of the form

$$(e^{w_1+w_2}, te^{3w_1w_2}) \in \mathbb{R}^2,$$

that is, a point of a generic cusp and we consider the equations

$$\begin{cases} xy = e^{w_1+w_2} \\ ax^{3w_2} + bx^{2w_2}y^{w_1} + cx^{w_2}y^{2w_1} + dy^{3w_1} = te^{3w_1w_2}. \end{cases}$$

If we suppose that $y \neq 0$, we can take $x = \frac{e^{w_1+w_2}}{y}$ and by substituting in the second equation and multiplying both terms by y^{3w_2} we have

$$a(e^{w_1+w_2})^{3w_2} + b(e^{w_1+w_2})^{2w_2}y^{w_1+w_2} + c(e^{w_1+w_2})^{w_2}y^{2(w_1+w_2)}$$

$$-te^{w_1w_2}y^{3w_2} + dy^{3(w_1+w_2)} = 0,$$

that is, a polynomial constituted by 5 monomials and where, applying Descartes method, we are going to have in the worst of the cases 4 sign changes, and as a consequence, 4 inverse images for $e > 0$ and $e < 0$. Then, (e) is not possible.
Thus, we only have 3 possible configurations ($(a), (b)$ and (c)) obtaining for each one a single topological class given by its correspondent associated link and Gauss word (see figure 20).

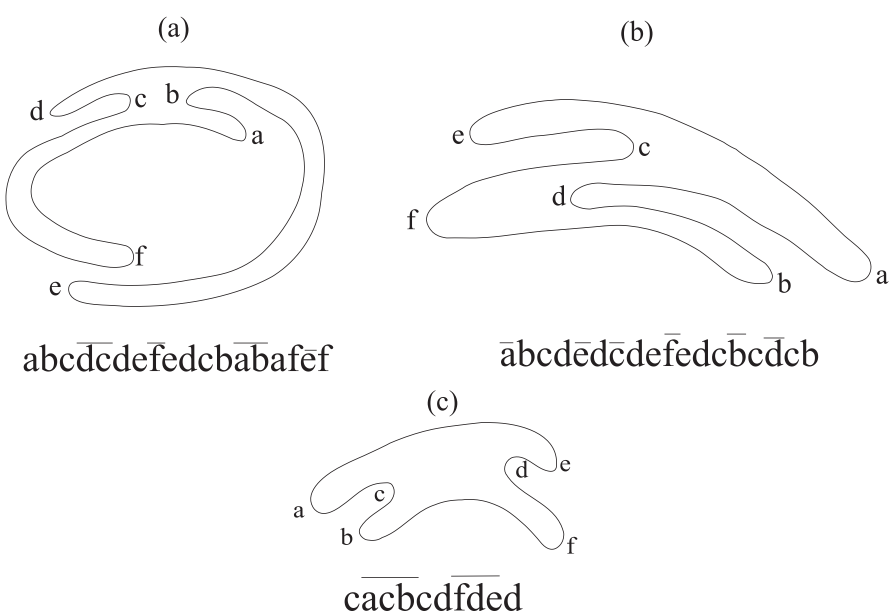


FIGURE 20.

To finish we associate to (a) the normal form $(xy, x^3 - 6x^2y^3 + 4xy^6 + y^9)$, to (b) $(xy, x^3 + 6x^2y^3 + 6xy^6 + y^9)$ and to (c) $(xy, x^3 - x^2y^3 + 3xy^6 + y^9)$.

- If we consider the remaining case, $B^2 - 4AC = 0$, following and analogous argument to the case of weights with different parity we conclude that f won't be finitely determined.

$\square$

4.2. **Germs with prenormal form** $(x^2+y^2, h(x,y))$**.** As we did with germs with prenormal form $(xy, g(x,y))$, we will suppose that $h(x,y)$ is a weighted homogeneous polynomial, that is,

$$h(x,y)=\sum_{i=0}^{p} b_i(x^{w_2})^i(y^{w_1})^{p-i},$$

although, in general, f won't be weighted homogeneous. We distinguish two different cases, according to the parity of p.

4.3. $\underline{p=2k}$**.** The following theorem will give us the classification of all germs of this type.

Theorem 4.10. *Let f be of type $\Sigma^{2,0}$,*

$$f(x,y)=(x^2+y^2,\sum_{i=0}^{p} b_i(x^{w_2})^i(y^{w_1})^{p-i}),$$

with $p=2k$. Then, f is not finitely determined.

Proof. We will prove it for $p=2$, being analogous for the remaining cases.

If w_1 or w_2 are greater than 1, when we compute the Jacobian determinant of f we obtain a expression of the form $2yA$ or $2xB$ with A, B depending on w_1, w_2, x, y. In the first case, we have the curve $y=0$ in the singular set, getting an image (x^2, ax^{2w_2}) that clearly presents double points. If we have $x=0$ by an analogous procedure we arrive to the same conclusion.

If $w_1=w_2=1$ we have branches of the form $x=\lambda y$ in the singular set, and as a consequence, each one of the discriminant curves will have the form $((\lambda y)^2+y^2, a(\lambda y)^2+b(\lambda y)y+cy^2)$ that present double points of the form $y_1=-y_2$. Then, f is not finitely determined either. □

4.4. **General case.** Firstly, we will see that if one of the weights is even and the other is different from 1, f won't be finitely determined.

Theorem 4.11. *Let f be of type $\Sigma^{2,0}$,*

$$f(x,y)=(x^2+y^2,\sum_{i=0}^{p} b_i(x^{w_2})^i(y^{w_1})^{p-i}).$$

Then, if w_1 or w_2 is even, with the other weight being greater than 1, f is not finitely determined.

Proof. Let us suppose that w_1 is even and $w_2>1$. If we compute the Jacobian determinant of f we get

$$Jf(x,y)=2x\sum_{i=0}^{p-1} w_1(p-i)b_i(x^{w_2})^i(y^{w_1})^{p-i-1}-2y\sum_{i=1}^{p} w_2 i b_i(x^{w_2})^{i-1}(y^{w_1})^{p-i}.$$

Since $w_2>1$ we can get one x out of the second summation, obtaining

$$Jf(x,y)=2x(\sum_{i=0}^{p-1} w_1(p-i)b_i(x^{w_2})^i(y^{w_1})^{p-i-1}-y\sum_{i=1}^{p} w_2 i b_i(x^{w_2})^{i-2}(y^{w_1})^{p-i}.$$

Therefore, one of the branches of $S(f)$ will always be given by the equation $x=0$ and

$$f_{|x=0}(y)=(y^2, y^{pw_1}),$$

that will always present double points of the form $y_1=-y_2$. Thus, f is not finitely determined. □

Let us see now what happen when both weights are odd. We will give some particular results about it.

Theorem 4.12. *($p = 1$) Let f be of type $\Sigma^{2,0}$, $f(x,y) = (x^2 + y^2, ax^{w_2} + by^{w_1})$, with w_1, w_2 both odd. Then, f is topologically equivalent to the germ $(x^2 + y^2, x^3 + y^5)$.*

Proof. Let us suppose that w_1, w_2 are both odd and greater than 1 (if one of them was 1, f wouldn't be of type $\Sigma^{2,0}$ anymore). In this case

$$Jf(x,y) = 2xy(w_1 b y^{w_1-2} - w_2 a x^{w_2-2}),$$

obtaining that our singular set $S(f)$ will have 3 branches, $x = 0$, $y = 0$ and $y^{w_1-2} = \frac{w_2 a x^{w_2-2}}{w_1 b}$. In the first two f does not present any problem. Let us see that it does not present any problem in the third one either. To see this we make the coordinates change $\begin{cases} x = \alpha t^{w_1-2} \\ y = \beta t^{w_2-2} \end{cases}$ with $\beta = (w_2 a)^{1/(w_1-2)} \in \mathbb{C}$ and $\alpha = (w_1 b)^{1/(w_2-2)} \in \mathbb{C}$. We have that

$$f|_{y^{w_1-2} = \frac{w_2 a x^{w_2-2}}{w_1 b}}(t) = (A(t), B(t)),$$

with $A(t) = \alpha^2 t^{2(w_1-2)} + \beta^2 t^{2(w_2-2)}$ and $B(t) = a\alpha^{w_2} t^{w_2(w_1-2)} + b\beta^{w_1} t^{w_1(w_2-2)}$. It is clear that although $A(t)$ is going to present double points of the form $t_1 = -t_2$, it is not going to happen with $B(t)$. Then, f is finitely determined.

Thus, $\Delta(f)$ will have three branches and we can only have two possible configurations (see figure 21):

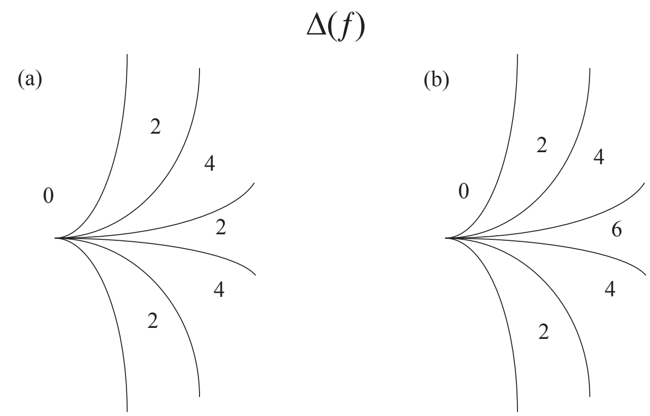


FIGURE 21.

Let us see that (b) is not possible. To prove this we consider a point $(e, 0) \in \mathbb{R}^2$ and we will prove by Descartes method that it will present at most 2 inverse images. We have the equations

$$\begin{cases} x^2 + y^2 = e \\ ax^{w_2} + by^{w_1} = 0 \end{cases}$$

obtaining a single equation of the form $A(y^{\frac{w_1}{w_2}})^2 + y^2 = e$ with $A > 0$. We consider the coordinate change $y = z^{w_2}$ in order to be able to work with integer exponents and we get $Az^{2w_1} + z^{2w_2} - e = 0$ that, applying Descartes method will always present at most 1 root if $z > 0$ and 1 root if $z < 0$, having a total of 2 roots z_1 and z_2 and as a consequence y_1 and y_2. Therefore, the only possible configuration is given by (a) and, since the 3 branches of the singular set are symmetric with respect to the origin of coordinates the only possible topological class is the associated to the link and Gauss word of figure 22.

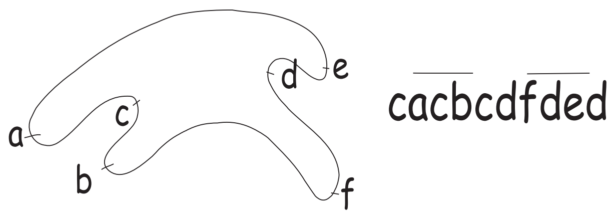


FIGURE 22.

Then, f is topologically equivalent to $(x, y^4 - x^2y^2 - \frac{1}{4}x^3y)$ and to the corank 2 normal form $(x^2 + y^2, x^3 + y^5)$.

□

Theorem 4.13. *($p = 3$, homogeneous case) Let f be of type of $\Sigma^{2,0}$, $f(x, y) = (x^2 + y^2, ax^3 + bx^2y + cxy^2 + dy^3)$. Then, if we denote by*

$$A = b(\frac{3d - 2b}{3}) - (\frac{2c - 3a}{3})^2,$$

$$B = -bc - \frac{(2c - 3a)(3d - 2b)}{9},$$

$$C = \frac{c(3a - 2c)}{3} - (\frac{3d - 2b}{3})^2$$

we have that

(1) *if $B^2 - 4AC > 0$, f is topologically equivalent to the fold,*
(2) *if $B^2 - 4AC < 0$, f is topologically equivalent to one of the germs that appear in table 4,*

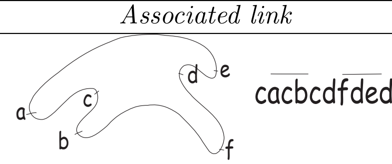
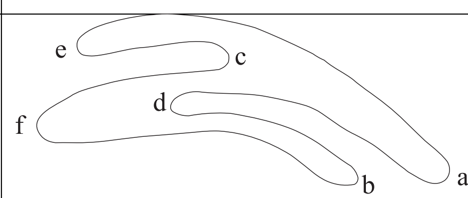
Degree	*Germ*	*Associated link*
0	$(x^2 + y^2, x^3 + y^5)$	
	$(x^2 + y^2, x^3 + x^2y - 3xy^2 + y^3)$	 $\text{bcd}\bar{\text{e}}\text{d}\bar{\text{c}}\text{de}\bar{\text{f}}\text{edc}\bar{\text{b}}\text{cdc}\bar{\text{a}}$

TABLE 4.

(3) *if $B^2 - 4AC = 0$, f is not finitely determined.*

Proof. Applying the result used earlier for map germs of the form $(xy, g(x,y))$ in the case $p = 3$ we obtain coefficients $A = A(a,,b,c,d)$, $B = B(a,b,c,d)$ and $C = C(a,b,c,d)$ such that $Jf(x,y)$ will present a symbolic, elliptical, hyperbolic or parabolic quadratic form if and only if $Ax^2 + Bxy + Cy^2$ presents a symbolic, elliptical, hyperbolic or parabolic quadratic form. Therefore, we have several cases:

(1) If $B^2 - 4AC > 0$, $S(f)$ presents a single branch $x = \lambda y$ whose image will be, as happen with all the germs of this form, symmetric with respect to the x-axis. Since the only possible configuration of inverse images is the one that appears in figure 23, f will be topologically equivalent to the fold.

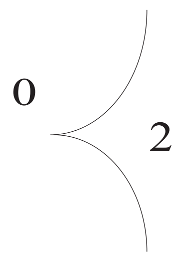


FIGURE 23.

(2) If $B^2 - 4AC < 0$, $S(f)$ will present three distinct real branches, obtaining in the discriminant the possible configurations of figure 24 and from each one of them a single topological class, symmetric with respect to the origin of coordinates. In case (a) we have the associated link and Gauss word of figure 25, taking as normal form $(x^2 + y^2, x^3 + y^5)$ and in case (b) we obtain the link of figure 26, taking as normal form $(x^2 + y^2, x^3 + x^2y - 3xy^2 + y^3)$.

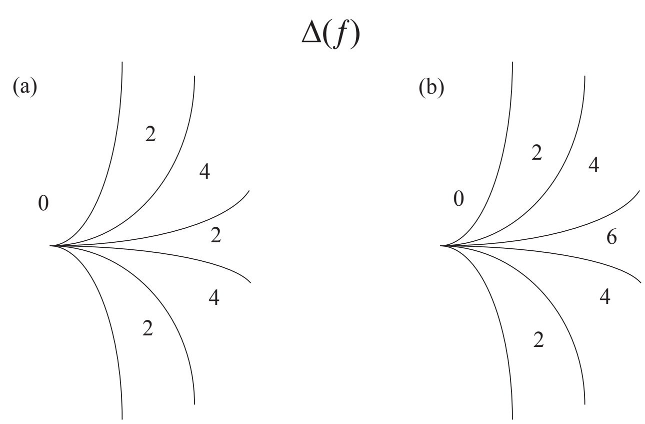


FIGURE 24.

(3) If $B^2 - 4AC = 0$ we obtain a non reduced component in $S(f)$. Then, f is not finitely determined.

□

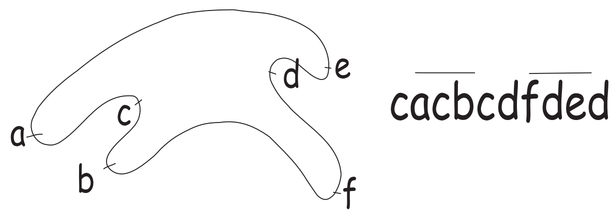


FIGURE 25.

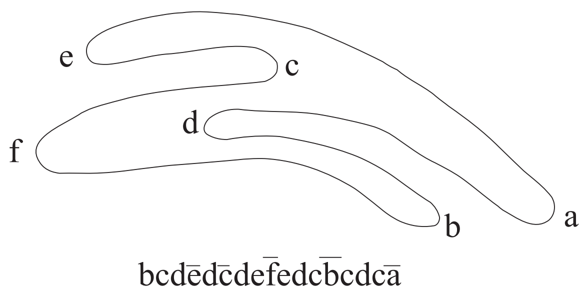


FIGURE 26.

REFERENCES

[1] C. Ehresmann, Les connexions infinitésimales dans un espace fibré différentiable *Colloque de topologie (espaces fibrés)*, Bruxelles, 1950, pp. 29–55.

[2] D. Eisenbud and H.I. Levine, An algebraic formula for the degree of a C^∞ map germ, *Annals of Mathematics*, 106 (1977), 19-37. DOI: 10.2307/1971156

[3] T. Fukuda, Local topological properties of differentiable mappings I, *Invent. Math.*, 65 (1981/82), 227–250. DOI: 10.1007/BF01389013

[4] T. Gaffney and D. Mond, Weighted homogeneous maps from the plane to the plane, *Math. Proc. Camb. Phil. Soc.*, 109 (1991), 451–470. DOI: 10.1017/S0305004100069905

[5] C.G. Gibson, Singular points of smooth mappings. *Research Notes in Mathematics*, 25. Pitman (Advanced Publishing Program), Boston, Mass.-London, 1979.

[6] J. Milnor, *Singular points of complex hypersurfaces.* Annals of Mathematics Studies, No. 61 Princeton University Press, 1968.

[7] A.J.Miranda, M.J.Saia and L.M.F.Soares, Topological classification of corank 2 map germs from the plane to the plane, *preprint*.

[8] A. Montesinos-Amilibia, *singR2R2*. software available here.

[9] J.A.Moya-Pérez and J.J.Nuño-Ballesteros, The link of a finitely determined map germ from $\mathbb{R}^2$ to $\mathbb{R}^2$, *J. Math. Soc. Japan* 62 (2010), No. 4, 1069–1092.

[10] T. Nishimura, Topological $\mathcal{K}$-equivalence of smooth map germs, in “Stratifications, Singularities and Differential Equations I”, *Travaux en Cours* 54, Hermann, Paris, 1997, 83–93.

[11] C.T.C. Wall, Finite determinacy of smooth map germs, *Bull. London. Math. Soc.*, 13 (1981), 481–539. DOI: 10.1112/blms/13.6.481

[12] H. Whitney, On singularities of mappings of Euclidean spaces. I. Mappings of the plane into the plane, *Ann. Math. (2)*, 62 (1955), 374–410. DOI: 10.2307/1970070

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LIPSCHITZ GEOMETRY OF COMPLEX CURVES

WALTER D NEUMANN AND ANNE PICHON

Abstract. We describe the Lipschitz geometry of complex curves. To a large part this is well known material, but we give a stronger version even of known results. In particular, we give a quick proof, without any analytic restrictions, that the outer Lipschitz geometry of a germ of a complex plane curve determines and is determined by its embedded topology. This was first proved by Pham and Teissier, but in an analytic category. We also show the embedded topology of a plane curve determines its ambient Lipschitz geometry.

1. Introduction

The germ of a complex set $(X, 0) \subset (\mathbb{C}^N, 0)$ has two metrics induced from the standard hermitian metric on $\mathbb{C}^N$: the *outer metric* given by distance in $\mathbb{C}^N$ and the *inner metric* given by arc-length of curves on X. Both are well defined up to bilipschitz equivalence, *i.e.*, they only depend on the analytic type of the germ $(X, 0)$ and not on the embedding $(X, 0) \subset (\mathbb{C}^N, 0)$. Studies of what information can be extracted from this metric structure have generally worked under analytic restrictions, e.g., that equivalences be restricted to be analytic or semi-algebraic or similar. In this note we prove the metric classification of germs of complex plane curves, but without any analytic restrictions (equivalence of item (1) of the following theorem with the other items):

Theorem 1.1. *Let $(C_1, 0) \subset (\mathbb{C}^2, 0)$ and $(C_2, 0) \subset (\mathbb{C}^2, 0)$ be two germs of complex curves. The following are equivalent:*

1. *$(C_1, 0)$ and $(C_2, 0)$ have same Lipschitz geometry, i.e., there is a homeomorphism of germs $\phi\colon (C_1, 0) \to (C_2, 0)$ which is bilipschitz for the outer metric;*
2. *there is a homeomorphism of germs $\phi\colon (C_1, 0) \to (C_2, 0)$, holomorphic except at 0, which is bilipschitz for the outer metric;*
3. *$(C_1, 0)$ and $(C_2, 0)$ have the same embedded topology, i.e., there is a homeomorphism of germs $h\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ such that $h(C_1) = C_2$;*
4. *there is a bilipschitz homeomorphism of germs $h\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ with $h(C_1) = C_2$.*

The equivalence of (1), (3) and (4) is our new contribution. The equivalence of (2) and (3) was first proved by Pham and Teissier [7]. By Teissier [8, Remarque, p.354] (see also Fernandes [5]) it then also follows that the outer bilipschitz geometry of any curve germ $(X, 0) \subset (\mathbb{C}^N, 0)$ determines the embedded topology of its general plane projection (Corollary 5.2).

For completeness we give quick proofs of all the equivalences. We start with the result for inner geometry, which will be used in examining outer geometry.

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2. Inner geometry

An algebraic germ $(X,0) \subset (\mathbb{C}^N,0)$ is homeomorphic to the cone on its link $X \cap S_\epsilon$, where S_ϵ is the sphere of radius ϵ about the origin with ϵ sufficiently small. If it is endowed with a metric, it is *metrically conical* if it is bilipschitz equivalent to the metric cone on its link. This basically means that the metric tells one no more than the topology (and is therefore uninteresting).

Proposition 2.1. *Any space curve germ $(C,0) \subset (\mathbb{C}^N,0)$ is metrically conical for the inner geometry.*

Proof. Take a linear projection $p\colon \mathbb{C}^N \to \mathbb{C}$ which is generic for the curve $(C,0)$ (*i.e.*, its kernel contains no tangent line of C at 0) and let $\pi := p|_C$, which is a branched cover of germs. Let $D_\epsilon = \{z \in \mathbb{C} : |z| \le \epsilon\}$ with ϵ small, and let C_ϵ be the part of C which branched covers D_ϵ. Since π is holomorphic away from 0 we have a local Lipschitz constant $K(x)$ at each point $x \in C \setminus \{0\}$ given by absolute value of the derivative map of π at x. On each branch of C this $K(x)$ extends continuously over 0, so the infimum and supremum K^- and K^+ of $K(x)$ on $C_\epsilon \setminus \{0\}$ are defined and positive. For any arc γ in C_ϵ which is smooth except where it passes through 0 we have $K^-\ell(\gamma) \le \ell'(\gamma) \le K^+\ell(\gamma)$, where ℓ respectively ℓ' represent arc length using inner metric on $\mathbb{C}_\epsilon$ respectively the metric lifted from B_ϵ. Since C_ϵ with the latter metric is strictly conical, we are done. ☐

3. Outer geometry determines embedded topological type

In this section, we prove (1) ⇒ (3) of Theorem 1.1, *i.e.*, that the embedded topological type of a plane curve germ $(C,0) \subset (\mathbb{C}^2,0)$ is determined by the outer Lipschitz geometry of $(C,0)$.

We first prove this using the analytic structure and the outer metric on $(C,0)$. The proof is close to Fernandes' approach in [5]. We then modify the proof to make it purely topological and to allow a bilipschitz change of the metric.

The tangent space to C at 0 is a union of lines $L^{(j)}$, $j = 1,\dots,m$, and by choosing our coordinates we can assume they are all transverse to the y-axis.

There is $\epsilon_0 > 0$ such that for any $\epsilon \le \epsilon_0$ the curve C meets transversely the set

$$T_\epsilon := \left\{(x,y) \in \mathbb{C}^2 : |x| = \epsilon\right\}.$$

Let μ be the multiplicity of C. The lines $x = t$ for $t \in (0,\epsilon_0]$ intersect C in μ points $p_1(t),\dots,p_\mu(t)$ which depend continuously on t. Denote by $[\mu]$ the set $\{1,2,\dots,\mu\}$. For each $j,k \in [\mu]$ with $j < k$, the distance $d(p_j(t),p_k(t))$ has the form $O(t^{q(j,k)})$, where $q(j,k) = q(k,j)$ is either a characteristic Puiseux exponent for a branch of the plane curve C or a coincidence exponent between two branches of C in the sense of e.g., [1, Chapitre 1, p. 12]. We call such exponents *essential*. For $j \in [\mu]$ define $q(j,j) = \infty$.

Lemma 3.1. *The map $q\colon [\mu] \times [\mu] \to \mathbb{Q} \cup \{\infty\}$, $(j,k) \mapsto q(j,k)$, determines the embedded topology of C.*

Proof. There are many combinatorial objects that encode the embedded topology of C, for example the Eisenbud-Neumann splice diagram [4] of the curve or the Eggers tree [3] (both are described, with the relationship between them, in C.T.C. Wall's book [9]). The "carrousel tree" described below is closely related (first described in [6]). All three are rooted trees with edges or vertices decorated with numeric labels.

To prove the lemma we will construct the carrousel tree from q. We also describe how one derives the splice diagram from it.

The $q(j,k)$ have the property that $q(j,l) \ge min(q(j,k),q(k,l))$ for any triple j,k,l. So for any $q \in \mathbb{Q} \cup \{\infty\}, q > 0$, the relation on the set $[\mu]$ given by $j \sim_q k \Leftrightarrow q(j,k) \ge q$ is an equivalence relation.

Name the elements of the set $q([\mu]\times[\mu])\cup\{1\}$ in decreasing order of size:

$$\infty = q_0 > q_1 > q_2 > \cdots > q_s = 1.$$

For each $i = 0,\dots,s$ let $G_{i,1},\dots,G_{i,\mu_i}$ be the equivalence classes for the relation $\sim_{q_i}$. So $\mu_0 = \mu$ and the sets $G_{0,j}$ are singletons while $\mu_s = 1$ and $G_{s,1} = [\mu]$. We form a tree with these equivalence classes $G_{i,j}$ as vertices, and edges given by inclusion relations: the singleton sets $G_{0,j}$ are the leaves and there is an edge between $G_{i,j}$ and $G_{i+1,k}$ if $G_{i,j}\subseteq G_{i+1,k}$. The vertex $G_{s,1}$ is the root of this tree. We weight each vertex with its corresponding q_i.

The *carrousel tree* is the tree obtained from this tree by suppressing valence 2 vertices: we remove each such vertex and amalgamate its two adjacent edges into one edge. We will describe how one gets from this to the splice diagram, but we first give an illustrative example.

We will use the plane curve C with two branches given by

$$y = x^{3/2} + x^{13/6}, \quad y = x^{7/3}\,.$$

Fig. 1 gives pictures of sections of C with complex lines $x = 0.1,\ 0.05,\ 0.025$ and 0. The central three-points set corresponds to the branch $y = x^{7/3}$ while the two lateral three-points sets correspond to the other branch.

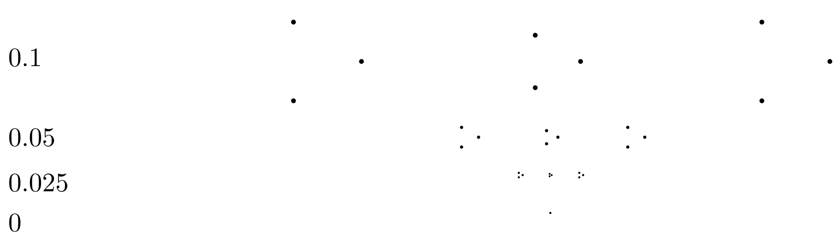

FIGURE 1. Sections of C

The carrousel tree for this example is the tree on the left in Fig. 2 and the procedure we will describe for getting from it to the splice diagram is then illustrated in the middle and right trees. We will follow the computer science convention of drawing the tree with its root vertex at the top, descending to its leaves at the bottom. At any non-leaf vertex v of the carrousel tree we

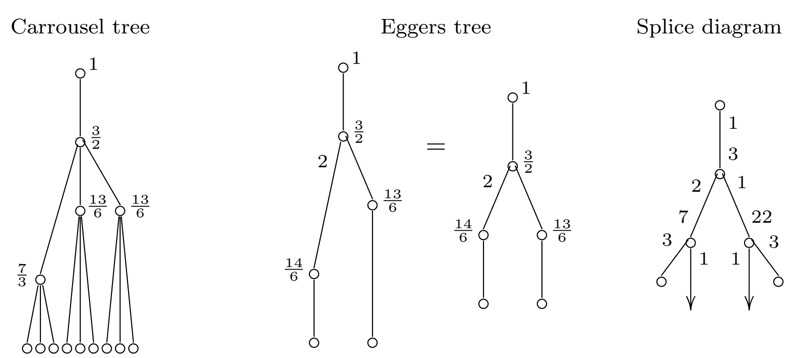

FIGURE 2. Carrousel tree to splice diagram

have a weight q_v, $1\le q_v\le q_1$, which is one of the q_i's. We write it as m_v/n_v, where n_v is the lcm of the denominators of the q-weights at the vertices on the path from v up to the root vertex. If

v' is the adjacent vertex above v along this path, we put $r_v = n_v/n_{v'}$ and $s_v = n_v(q_v - q_{v'})$. At each vertex v the subtrees cut off below v consist of groups of r_v isomorphic trees, with possibly one additional tree. We label the top of the edge connecting to this additional tree at v, if it exists, with the number r_v, and then delete all but one from each group of r_v isomorphic trees below v. We do this for each non-leaf vertex of the carrousel tree. The resulting tree, with the q_v labels at vertices and the extra label on a downward edge at some vertices is easily recognized as a mild modification of the Eggers tree.

We construct the splice diagram starting from this tree. We first replace every leaf by an arrowhead. Then at each vertex v which did not have a downward edge with an r_v label we add such an edge (ending in a new leaf which is not an arrowhead). Each still unlabeled top end of an edge is then given the label 1. Finally, starting from the top of the tree we move down the tree adding a label to the bottom end of each edge ending in a vertex v which is not a leaf as follows. If v is directly below the root the label is $m'_v := m_v$. For a vertex v directly below a vertex v' other than the root the label is $m'_v := s_v + r_v r_{v'} m'_{v'}$ if $r_{v'}$ does not label the edge $v'v$ and $m'_v := (s_v + r_v m'_{v'})/r_{v'}$ if it does (see [4, Prop. 1A.1]). □

As already noted, this discovery of the embedded topology involved the complex structure and outer metric. We must show we can discover it without use of the complex structure, even after applying a bilipschitz change to the outer metric.

Recall that the tangent space of C is a union of lines $L^{(j)}$. We denote by $C^{(j)}$ the part of C tangent to the line $L^{(j)}$. It suffices to discover the topology of each $C^{(j)}$ independently, since the $C^{(j)}$'s are distinguished by the fact that the distance between any two of them outside a ball of radius ϵ around 0 is $O(\epsilon)$, even after bilipschitz change to the metric. We therefore assume from now on that the tangent to C is a single complex line.

The points $p_1(t), \dots, p_\mu(t)$ we used to find the numbers $q(j,k)$ were obtained by intersecting C with the line $x = t$. The arc $p_1(t)$, $t \in [0, \epsilon_0]$ satisfies $d(0, p_1(t)) = O(t)$. Moreover, the other points $p_2(t), \dots, p_\mu(t)$ are in the transverse disk of radius rt centered at $p_1(t)$ in the plane $x = t$. Here r can be as small as we like, so long as ϵ_0 is then chosen sufficiently small.

Instead of a transverse disk of radius rt, we can use a ball $B(p_1(t), rt)$ of radius rt centered at $p_1(t)$. This $B(p_1(t), rt)$ intersects C in μ disks $D_1(t), \dots, D_\mu(t)$, and we have $d(D_j(t), D_k(t)) = O(t^{q(j,k)})$, so we still recover the numbers $q(j,k)$. In fact, the ball in the outer metric on C of radius rt around $p_1(t)$ is $B_C(p_1(t), rt) := C \cap B(p_1(t), rt)$, which consists of these μ disks $D_1(t), \dots, D_\mu(t)$.

We now replace the arc $p_1(t)$ by any continuous arc $p'_1(t)$ on C with the property that $d(0, p'_1(t)) = O(t)$, and if r is sufficiently small it is still true that $B_C(p'_1(t), rt)$ consists of μ disks $D'_1(t), \dots, D'_\mu(t)$ with $d\big(D'_j(t), D'_k(t)\big) = O(t^{q(j,k)})$. So at this point, we have gotten rid of the dependence on analytic structure in discovering the topology, but not yet dependence on the outer geometry.

A K-bilipschitz change to the metric may make the components of $B_C(p'_1(t), rt)$ disintegrate into many pieces, so we can no longer simply use distance between pieces. To resolve this, we consider both $B'_C(p'_1(t), rt)$ and $B'_C(p'_1(t), \frac{r}{K^4}t)$ where B' means we are using the modified metric. Then only μ components of $B'_C(p_1(t), rt)$ will intersect $B'_C(p_1(t), \frac{r}{K^4}t)$. Naming these components $D'_1(t), \dots, D'_\mu(t)$ again, we still have $d(D'_j(t), D'_k(t)) = O(t^{q(j,k)})$ so the $q(j,k)$ are determined as before. □

4. Embedded topological type determines outer geometry

In this section, we prove (3) ⇒ (2) of Theorem 1.1. The implication (2) ⇒ (1) is trivial, so we then have the equivalence of the first three items of Theorem 1.1.

We will use the following lemma:

Lemma 4.1. *Let $(C,0) \subset (\mathbb{C}^N,0)$ be a germ of complex plane curve and let $p\colon \mathbb{C}^N \to \mathbb{C}$ be a linear projection whose kernel does not contain any tangent line to C. Then there exists a neighborhood U of 0 in C and a constant $M>1$ such that for each $u,u' \in U \smallsetminus \{0\}$, there is an arc $\tilde\alpha$ in C joining u to a point u'' with $p(u'')=p(u')$ and*

$$d(u,u') \le L(\tilde\alpha) + d(u'',u') \le M d(u,u')$$

where $L(\tilde\alpha)$ denotes the length of $\tilde\alpha$.

Proof. There exists a neighbourhood U of 0 in C such that the restriction $p|_C$ is a bilipschitz local homeomorphism for the inner metric on $U \smallsetminus \{0\}$ (see proof of Proposition 2.1). Choose any $\delta > 1$. If 0 is not in the segment $[p(u),p(u')]$, we set $\alpha = [p(u),p(u')]$. If $0 \in [p(u),p(u')]$, we modify this segment to a curve α avoiding 0 which has length at most δ times the length of $[p(u),p(u')]$. Consider the lifting $\tilde\alpha$ of α by $p|_C$ with origin u and let u'' be its extremity. We obviously have:

$$d(u,u') \le L(\tilde\alpha) + d(u',u'')\,.$$

On the other hand, $L(\tilde\alpha) \le K_0 L(\alpha) \le \delta K_0 d(p(u),p(u'))$, where K_0 is a bound for the local inner bilipschitz constant of p on $U \smallsetminus \{0\}$. As $d(p(u),p(u')) \le d(u,u')$, we then obtain:

$$L(\tilde\alpha) \le \delta K_0 d(u,u').$$

If we join the segment $[u,u']$ to $\tilde\alpha$ at u we get a curve from u' to u'', so

$$d(u',u'') \le (1+\delta K_0) d(u,u').$$

We then obtain:

$$L(\tilde\alpha) + d(u',u'') \le (1+2\delta K_0) d(u,u'),$$

and $M = 1+2\delta K_0$ is the desired constant. □

Proof of (3) $\Rightarrow$ (2) *of Theorem 1.1.* Let $(C_1,0) \subset (\mathbb{C}^2,0)$ be an irreducible plane curve which is not tangent to the y-axis. Then there exists a minimal integer $n>0$ such that $(C_1,0)$ has Puiseux parametrization

$$\gamma_1(w) = \Big(w^n, \sum_{i\ge n} a_i w^i\Big)\,.$$

Denote $A := \{i : a_i \ne 0\}$. Recall that the embedded topology of C_1 is determined by n and the essential integer exponents in the sum $\sum_{i\ge n} a_i w^i$, where an $i \in A \smallsetminus \{n\}$ is an *essential integer exponent* if and only if $\gcd\{j \in \{n\}\cup A : j \le i\} < \gcd\{j \in \{n\}\cup A : j < i\}$ (equivalently $\frac{i}{n}$ is a characteristic exponent). Denote by A_e the subset of A consisting of the essential integer exponents.

Now let $(C_2,0) \subset (\mathbb{C}^2,0)$, given by

$$\gamma_2(w) = \Big(w^n, \sum_{i\ge n} b_i w^i\Big)\,,$$

be a second plane curve with the same embedded topology as C_1, so that the set of essential integer exponents $B_e \subset B := \{i : b_i \ne 0\}$ is equal to A_e.

We will prove that the homeomorphism $\Phi\colon C_1 \to C_2$ defined by $\Phi(\gamma_1(w)) = \gamma_2(w)$ is bilipschitz on small neighborhoods of the origin.

We first prove that there exists $K>0$ and a neighborhood U of 0 in $\mathbb{C}$ such that for each pair (w,w') with $w \in U$, $w \ne w'$ and $w^n = (w')^n$, we have

$$d\big(\gamma_1(w),\gamma_1(w')\big) \le K d\big(\gamma_2(w),\gamma_2(w')\big)$$

For (w, w') as above, consider the two real arcs $s \in [0,1] \mapsto \gamma_1(sw)$ and $s \mapsto \gamma_1(sw')$ and their images by Φ. Then we have

$$d\big(\gamma_1(ws), \gamma_1(w's)\big) = s^n \Big| \sum_{i>n} a_i s^{i-n}\big(w^i - (w')^i\big) \Big|$$

and

$$d\Big(\Phi\big(\gamma_1(ws)\big), \Phi\big(\gamma_1(w's)\big)\Big) = s^n \Big| \sum_{i>n} b_j s^{i-n}\big(w^i - (w')^i\big) \Big|$$

Let i_0 be the minimal element of $\{i \in A; w^i \neq (w')^i\}$. Then i_0 is an essential integer exponent, so a_{i_0} and b_{i_0} are non-zero. Moreover, as s tends to 0 we have

$$d\big(\gamma_1(ws), \gamma_1(w's)\big) \sim s^{i_0} |w^{i_0} - (w')^{i_0}||a_{i_0}|$$

and $d\big(\Phi\big(\gamma_1(ws)\big), \Phi\big(\gamma_1(w's)\big)\big) \sim s^{i_0}|w^{i_0} - (w')^{i_0}||b_{i_0}|$ and hence the ratio

$$d\Big(\gamma_1(ws), \gamma_1(w's)\Big) \Big/ d\Big(\Phi\big(\gamma_1(ws)\big), \Phi\big(\gamma_1(w's)\big)\Big) \qquad (*)$$

tends to the non zero constant $c_{i_0} = \frac{|a_{i_0}|}{|b_{i_0}|}$.

Notice that the integer i_0 depends on the pair of points (w, w'). But i_0 is either n or an essential integer exponent for γ_1. Therefore there are a finite number of values for i_0 and c_{i_0}. Moreover, the set of pairs (w, w') such that $w^n = (w')^n$ consists of a disjoint union of n lines. So there exists $s_0 > 0$ such that for each such (w, w') with $|w| = 1$ and each $s \leq s_0$, the quotient $(*)$ belongs to $[1/K, K]$ where $K > 0$. Then $U = \{w : |w| \leq s_0\}$ is the desired neighbourhood of 0.

We now prove that Φ is bilipschitz on $\gamma_1(U)$. Consider the projection $p\colon \mathbb{C}^2 \to \mathbb{C}$ given by $p(x, y) = x$. Let w and w' be any two complex numbers in U. Let α be the segment in $\mathbb{C}$ joining w^n to $(w')^n$ and let $\tilde\alpha_1$ (resp. $\tilde\alpha_2$) be the lifting of α by the restriction $p|_{C_1}$ (resp. $p|_{C_2}$) with origin $\gamma_1(w)$ (resp. $\gamma_2(w)$). Consider the unique $w'' \in \mathbb{C}$ such that $\gamma_1(w'')$ is the extremity of $\tilde\alpha_1$. Notice that $\gamma_2(w'')$ is the extremity of $\tilde\alpha_2$. We have

$$d\big(\gamma_1(w), \gamma_1(w')\big) \leq L(\tilde\alpha_1) + d\big(\gamma_1(w''), \gamma_1(w')\big).$$

According to Section 2, $p|_{C_1}$ (resp. $p|_{C_2}$) is an inner bilipschitz homeomorphism with bilipschitz constant say K_1 (resp. K_2). We then have $L(\tilde\alpha_1) \leq K_1K_2L(\tilde\alpha_2)$. Therefore setting $C = \max(K_1K_2, K)$, we obtain:

$$d\big(\gamma_1(w), \gamma_1(w')\big) \leq C\Big(L(\tilde\alpha_2) + d\big(\gamma_2(w''), \gamma_2(w')\big)\Big) \qquad (**)$$

Applying Lemma 4.1 to the restriction $p|_{C_2}$ with $u = \gamma_2(w)$ and $u' = \gamma_2(w')$, we then obtain:

$$d\big(\gamma_1(w), \gamma_1(w')\big) \leq CMd\big(\gamma_2(w), \gamma_2(w')\big)$$

This proves Φ is Lipschitz. It is then bilipschitz by symmetry of the roles.

In the general case where C_1 and C_2 are not necessarily irreducible, the same arguments work taking into account a Puiseux parametrization for each branch and the fact that the sets of characteristic exponents and coincidence exponents between branches coincide. □

5. Outer geometry of space curves

Before proving the final equivalence of Theorem 1.1 we give a quick proof, based on the preceding proof, of the following result of Teissier [8, pp. 352–354].

Theorem 5.1. *For a complex curve germ $(C, 0) \subset (\mathbb{C}^N, 0)$ the restriction to C of a generic linear projection $\ell\colon \mathbb{C}^N \to \mathbb{C}^2$ is bilipschitz for the outer geometry.*

Our notion of *generic linear projection* to $\mathbb{C}^2$, defined in the proof below, is equivalent to Teissier's, which says that the kernel of the projection should contain no limit of secant lines to the curve.

Proof of Theorem 5.1. We have to prove that the restriction $\ell|_C\colon C \to \ell(C)$ is bilipschitz for the outer metric. We choose coordinates (x, y) in $\mathbb{C}^2$ so $\ell(C)$ is transverse to the y-axis at 0 and coordinates $(z_1, \dots, z_n)$ in $\mathbb{C}^n$ with $z_1 = x \circ \ell$. So ℓ has the form $(z_1, \dots, z_N) \mapsto (z_1, \sum_1^N b_j z_j)$ and any component of C has a Puiseux expansion of the form (n is the multiplicity of the component):

$$\gamma(w) = \Big(w^n, \sum_{i\geq n} a_{2i}w^i, \dots, \sum_{i\geq n} a_{Ni}w^i\Big).$$

We first assume $(C, 0)$ is irreducible. We again denote $A := \{i : \exists j, a_{ji} \neq 0\}$ and call an exponent $i \in A \smallsetminus \{n\}$ an *essential integer exponent* if and only if

$$\gcd\{j \in \{n\} \cup A : j \leq i\} < \gcd\{j \in \{n\} \cup A : j < i\}.$$

Define $a_{1n} = 1$ and $a_{1i} = 0$ for $i > n$. We say ℓ is *generic* if $\sum_{j=1}^N b_j a_{ji} \neq 0$ for each essential integer exponent i. We now assume ℓ is generic.

As in the proof of the second part of Theorem 1.1 there then exists $K > 0$ and a neighborhood U of 0 in $\mathbb{C}$ such that for each pair (w, w') with $w \in U$ and $w^n = (w')^n$ we have

$$\frac{1}{K} d\big(\ell\gamma(w), \ell\gamma(w')\big) \leq d\big(\gamma(w), \gamma(w')\big) \leq K d\big(\ell\gamma(w), \ell\gamma(w')\big).$$

Lemma 4.1 then completes the proof, as before.

The proof when C is reducible is essentially the same, but the genericity condition must take both characteristic and coincidence exponents into consideration. Namely, ℓ should be generic as above for each individual branch of C; and for any two branches, given by (with n now the lcm of their multiplicities)

$$\gamma(w) = \Big(w^n, \sum_{i\geq n} a_{2i}w^i, \dots, \sum_{i\geq n} a_{Ni}w^i\Big), \quad \gamma'(w) = \Big(w^n, \sum_{i\geq n} a'_{2i}w^i, \dots, \sum_{i\geq n} a'_{Ni}w^i\Big),$$

we require $\sum_{j=1}^N b_j(a_{ji} - \lambda^i a'_{ji}) \neq 0$ for each n-th root of unity λ, where i is the smallest exponent for which some $a_{ji} - a'_{ji}$ is non-zero. □

Corollary 5.2. *Let $(C_1, 0) \subset (\mathbb{C}^{N_1}, 0)$ and $(C_2, 0) \subset (\mathbb{C}^{N_2}, 0)$ be two germs of complex curves. The following are equivalent:*

1. *$(C_1, 0)$ and $(C_2, 0)$ have same Lipschitz geometry i.e., there is a homeomorphism of germs $\phi\colon (C_1, 0) \to (C_2, 0)$ which is bilipschitz for the outer metric;*
2. *there is a homeomorphism of germs $\phi\colon (C_1, 0) \to (C_2, 0)$, holomorphic except at 0, which is bilipschitz for the outer metric;*
3. *the generic plane projections of $(C_1, 0)$ and $(C_2, 0)$ have the same embedded topology.* □

6. Ambient geometry of plane curves

To complete the proof of Theorem 1.1 we must show the implication (3) ⇒ (4) of that theorem, since (4) ⇒ (3) is trivial. We will use a *carrousel decomposition* of $(\mathbb{C}^2, 0)$ with respect to a plane curve, so we first describe this (it is essentially the one described in [2]).

The tangent space to C at 0 is a union $\bigcup_{j=1}^m L^{(j)}$ of lines. For each j we denote the union of components of C which are tangent to $L^{(j)}$ by $C^{(j)}$. We can assume our coordinates (x, y) in $\mathbb{C}^2$ are chosen so that no $L^{(j)}$ is tangent to an axis. Then $L^{(j)}$ is given by an equation $y = a_1^{(j)} x$ with $a_1^{(j)} \neq 0$.

We choose $\epsilon_0 > 0$ sufficiently small that the set $\{(x, y) : |x| = \epsilon\}$ is transverse to C for all $\epsilon \le \epsilon_0$. We define conical sets $V^{(j)}$ of the form

$$V^{(j)} := \big\{(x, y) : |y - a_1^{(j)} x| \le \eta |x|, |x| \le \epsilon_0\big\} \subset \mathbb{C}^2\,,$$

where the equation of the line $L^{(j)}$ is $y = a_1^{(j)} x$ and $\eta > 0$ is small enough that the cones are disjoint except at 0. If ϵ_0 is small enough $C^{(j)} \cap \{|x| \le \epsilon_0\}$ will lie completely in $V^{(j)}$.

There is then an $R > 0$ such that for any $\epsilon \le \epsilon_0$ the sets $V^{(j)}$ meet the boundary of the "square ball"

$$B_\epsilon := \big\{(x, y) \in \mathbb{C}^2 : |x| \le \epsilon, |y| \le R\epsilon\big\}$$

only in the part $|x| = \epsilon$ of the boundary. We will use these balls as a system of Milnor balls.

We now describe our carrousel decomposition for each $V^{(j)}$, so we will fix j for the moment.

We first truncate the Puiseux series for each component of $C^{(j)}$ at a point where truncation does not affect the topology of $C^{(j)}$. Then for each pair $\kappa = (f, p_k)$ consisting of a Puiseux polynomial $f = \sum_{i=1}^{k-1} a_i^{(j)} x^{p_i^{(j)}}$ and an exponent $p_k^{(j)}$ for which there is a Puiseux series

$$y = \sum_{i=1}^{k} a_i^{(j)} x^{p_i^{(j)}} + \dots$$

describing some component of $C^{(j)}$, we consider all components of $C^{(j)}$ which fit this data. If $a_{k1}^{(j)}, \dots, a_{km_\kappa}^{(j)}$ are the coefficients of $x^{p_k^{(j)}}$ which occur in these Puiseux polynomials we define

$$B_\kappa := \Big\{(x, y) : \alpha_\kappa |x^{p_k^{(j)}}| \le \Big|y - \sum_{i=1}^{k-1} a_i^{(j)} x^{p_i^{(j)}}\Big| \le \beta_\kappa |x^{p_k^{(j)}}|$$
$$\Big|y - (\sum_{i=1}^{k-1} a_i^{(j)} x^{p_i^{(j)}} + a_{kj}^{(j)} x^{p_k^{(j)}})\Big| \ge \gamma_\kappa |x^{p_k^{(j)}}| \text{ for } j = 1, \dots, m_\kappa\Big\}\,.$$

Here $\alpha_\kappa, \beta_\kappa, \gamma_\kappa$ are chosen so that $\alpha_\kappa < |a_{k\nu}^{(j)}| - \gamma_\kappa < |a_{k\nu}^{(j)}| + \gamma_\kappa < \beta_\kappa$ for each $\nu = 1, \dots, m_\kappa$. If ϵ is small enough, the sets B_κ will be disjoint for different κ.

The intersection $B_\kappa \cap \{x = t\}$ is a finite collection of disks with smaller disks removed. We call B_κ a *B-piece*. The closure of the complement in $V^{(j)}$ of the union of the B_κ's is a union of pieces, each of which has link either a solid torus or a "toral annulus" (annulus $\times\, \mathbb{S}^1$). We call the latter *annular pieces* or *A-pieces* and the ones with solid torus link *D-pieces* (a B-piece corresponding to an inessential exponent has the same topology as an A-piece, but we do not call it annular).

This is our carrousel decomposition of $V = V^{(j)}$. We call $\overline{B_\epsilon \smallsetminus \bigcup V^{(j)}}$ a $B(1)$ piece (even though it may have A- or D-topology). It is metrically conical, and together with the carrousel decompositions of the $V^{(j)}$'s we get a carrousel decomposition of the whole of B_ϵ.

Proof of (3) $\Rightarrow$ (4) *of Theorem 1.1.* Let $(C_1, 0) \subset (\mathbb{C}^2, 0)$ and $(C_2, 0) \subset (\mathbb{C}^2, 0)$ have the same embedded topological type. Consider two carrousel decompositions of $(\mathbb{C}^2, 0)$: one with respect to C_1 and the other with respect to C_2, constructed as above. The proof consists of constructing a bilipschitz map of germs $h\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ which sends the carrousel decomposition for C_1 to the one for C_2 (being careful to include matching pieces for inessential exponents which occur in just one of C_1 and C_2). We first construct it to respect the carrousels, but not necessarily map C_1 to C_2. Once this is done, we adjust it so that C_1 is mapped to C_2.

Let $L_1^{(j)}$ and $L_2^{(j)}$, $j = 1, \dots, m$, be the tangent lines to C_1 and C_2 and $C_1^{(j)}$ resp. $C_2^{(j)}$ the union of components of C_1 resp. C_2 which are tangent to $L_1^{(j)}$ resp. $L_2^{(j)}$. We may assume we have numbered them so $C_1^{(j)}$ and $C_2^{(j)}$ have matching embedded topology. Let $V_1^{(j)}$ and $V_2^{(j)}$, $j = 1, \dots, m$, be the conical sets around the tangent lines as defined earlier.

The $B(1)$ pieces of the carrousel decompositions for C_1 and C_2 are metrically conical with the same topology, so there is a conical bilipschitz diffeomorphism between them. We can arrange that it is a translation on each $x = t$ section of each $\partial V_1^{(j)}$. We will extend it over the cones $V_1^{(j)}$ and $V_2^{(j)}$ using the carrousels.

Consider the Puiseux series $y = \sum_{i=1}^k a_i^{(j)} x^{p_i^{(j)}} + \dots$ describing some component of $C_1^{(j)}$ and the Puiseux series $y = \sum_{i=1}^k b_i^{(j)} x^{p_i^{(j)}} + \dots$ describing the corresponding component of $C_2^{(j)}$. If a term with inessential exponent appears in one of the series, we include it also in the other, even if its coefficient there is zero. This way, when we construct the carrousel as above we have corresponding B-pieces for the two carrousels. Moreover, we can choose the constants $\alpha_\kappa, \beta_\kappa, \gamma_\kappa$ used to construct these corresponding B-pieces to be the same for both. The $\{x = t\}$ sections of a pair of corresponding A-pieces will then be congruent, so we can map the one A-piece to the other by preserving x coordinate and using translation on each $x = t$ section. The same holds for D-pieces. It then remains to extend to the B-pieces.

A B-piece B_{κ_1} in the decomposition for C_1 is determined by some $\kappa_1 = (f_1, p_k)$ with

$$f_1 = \sum_{i=1}^{k-1} a_i x^{p_i},$$

and is foliated by curves of the form $y = f_1 + \xi x^{p_k}$ for varying ξ (we call p_k the *rate* of B_κ). The corresponding piece B_{κ_2} for C_2 is similarly determined by some $\kappa_2 = (f_2, p_k)$ with

$$f_2 = \sum_{i=1}^{k-1} b_i x^{p_i}$$

and is foliated by curves $y = f_2 + \xi x^{p_k}$. The $x = \epsilon_0$ section of B_{κ_1} has a free cyclic group action generated by the first return map of the foliation, and the same is true for B_{κ_2}. We choose a smooth map $(B_{\kappa_1} \cap \{x = \epsilon_0\}) \to (B_{\kappa_2} \cap \{x = \epsilon_0\})$ which is equivariant for this action and on the boundary matches the maps, coming from A- and D-pieces, already chosen. This map extends to the whole of B_{κ_1} by requiring it to preserve the foliation and x-coordinate.

By construction, the resulting map of germs $\phi\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ is an isometry on the A- and D-pieces and bilipschitz on the $B(1)$ piece. We must check that it is bilipschitz on the B-pieces of type B_κ. Pick such a B and suppose the rate of B is r. The Lipschitz constant of ϕ is bounded in a neighborhood of the link $B^{(\epsilon)} := B \cap \{|x| = \epsilon\}$ of B by compactness. For $0 < \epsilon' < \epsilon$, if we move points inwards x-distance $\epsilon - \epsilon'$ along the leaves of the foliation of B, each section at $x = t$ with $|t| = \epsilon$ moves to the section at $x = \frac{\epsilon'}{\epsilon} t$ while scaling by a factor of $(\epsilon'/\epsilon)^r$. The same holds for the images of these sections in the carrousel for C_2. So to high order the Lipschitz constant of ϕ at a point of the $x = t$ section equals the Lipschitz constant at the corresponding point of the $x = \frac{\epsilon'}{\epsilon} t$ section. It follows that the local Lipschitz constant is bounded on the whole of B, so ϕ is bilipschitz.

However, ϕ maps C_1 not to C_2, but to a small deformation of it, since we constructed the carrousels by first truncating our Puiseux series beyond any terms which contributed to the topology. But it is not hard to see that, by a small change of the constructed map inside the D-pieces which intersect C_1, one can change ϕ so it maps C_1 to C_2 while changing the bilipschitz coefficient by an amount which approaches zero as one approaches the origin. Namely, let D_1 be such a piece and $D_2 = \phi(D_1)$ the corresponding piece for the curve C_2. In each $x = t$ slice $D_1(t) := D_1 \cap \{x = t\}$ we take the map $D_1(t) \to D_1(t)$ which moves the point $p_1(t) := D_1(t) \cap C_1$ to $p_2(t) := \phi^{-1}(D_2(t) \cap C_2)$ and maps each ray from $p_1(t)$ to a point $p \in \partial D_1(t)$ linearly to the ray from $p_2(t)$ to p. This gives a map $\psi\colon D_1 \to D_1$ whose bilipschitz constant rapidly approaches 1 as $t \to 0$ and $\phi \circ \psi$ does what is required on this piece. □

References

[1] Evelia García Barroso, Invariants des singularités de courbes planes et courbure des fibres de Milnor, PhD thesis, Paris 7, 2000.

[2] Lev Birbrair, Walter D Neumann and Anne Pichon, The thick-thin decomposition and the bilipschitz classification of normal surface singularities, Acta Math **212** (2014), 199-256. DOI: 10.1007/s11511-014-0111-8

[3] Harald Eggers, Polarinvarianten und die Topologie von Kurvensingularitäten, Bonn. Math. Schriften **147** (1983).

[4] David Eisenbud and Walter D Neumann, *Three dimensional link theory and invariants of plane curves singularities*, Annals of Mathematics Studies 110, Princeton University Press, 1985.

[5] Alexandre Fernandes. Topological equivalence of complex curves and bi-Lipschitz maps, Michigan Math. J. **51** (2003), 593–606. DOI: 10.1307/mmj/1070919562

[6] Walter D Neumann and Anne Pichon, Lipschitz geometry of complex surfaces: analytic invariants and equisingularity. arXiv:1211.4897v1

[7] Frédéric Pham and Bernard Teissier, Fractions Lipschitziennes d'une algèbre analytique complexe et saturation de Zariski. Prépublications Ecole Polytechnique No. M17.0669 (1969). Available at http://hal.archives-ouvertes.fr/hal-00384928/fr/

[8] Bernard Teissier, Variétés polaires, II. Multiplicités polaires, sections planes, et conditions de Whitney, *Algebraic geometry (La Ràbida, 1981)* 314–491, Lecture Notes in Math. **961** (Springer, Berlin, 1982).

[9] C.T.C. Wall, *Singular points of plane curves*, Cambridge University Press (2004). DOI: 10.1017/CBO9780511617560

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THE RIGHT CLASSIFICATION OF UNIVARIATE POWER SERIES IN POSITIVE CHARACTERISTIC

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Abstract. While the classification of univariate power series up to coordinate change is trivial in characteristic 0, this classification is very different in positive characteristic. In this note we give a complete classification of univariate power series $f \in K[[x]]$, where K is an algebraically closed field of characteristic $p > 0$ by explicit normal forms. We show that the right determinacy of f is completely determined by its support. Moreover we prove that the right modality of f is equal to the integer part of μ/p, where μ is the Milnor number of f. As a consequence we prove in this case that the modality is equal to the proper modality, which is the dimension of the μ-constant stratum in an algebraic representative of the semiuniversal deformation with trivial section.

1. Introduction

In [Arn72] V.I. Arnol'd introduced the "modality", or the number of moduli, for real and complex hypersurface singularities and he classified singularities with modality smaller than or equal to 2. In oder to generalize the notion of modality to the algebraic setting, the author and Greuel in [GN13] introduced the modality for algebraic group actions and applied it to high jet spaces.

Let the algebraic group G act on the variety X. Then there exists a *Rosenlicht stratification* $\{(X_i, p_i), i = 1, \dots, s\}$ of X w.r.t. G, i.e. the X_i is a locally closed G-invariant subset of X, $X = \cup_{i=1}^{s} X_i$ and the $p_i : X_i \to X_i/G$ a geometric quotient. For each open subset $U \subset X$ we define

$$G\text{-mod}(U) := \max_{1 \le i \le s} \{\dim\left(p_i(U \cap X_i)\right)\},$$

and for $x \in X$ we call

$$G\text{-mod}(x) := \min\{G\text{-mod}(U) \mid U \text{ a neighbourhood of x}\}$$

the *G-modality* of x.

Let K be an algebraically closed field of characteristic $p \ge 0$, let $K[[\mathbf{x}]] = K[[x_1, \dots, x_n]]$ be the formal power series ring and let the right group, $\mathcal{R} := Aut(K[[\mathbf{x}]])$, act on $K[[\mathbf{x}]]$ by $(\Phi, f) \mapsto \Phi(f)$. Two elements $f, g \in K[[\mathbf{x}]]$ are called *right equivalent*, $f \sim_r g$, if they belong to the same $\mathcal{R}$-orbit, or equivalently, there exists a coordinate change $\Phi \in Aut(K[[\mathbf{x}]])$ such that $g = \Phi(f)$.

Let $f \in \langle \mathbf{x} \rangle \subset K[[\mathbf{x}]]$ and let $\mu(f) := \dim K[[\mathbf{x}]]/\langle f_{x_1}, \dots, f_{x_n} \rangle$ be its Milnor number. We call f *isolated* if $\mu(f) < \infty$. By [BGM12, Thm. 5], f is isolated if and only if it is finitely right determined, i.e. f is right k-determined for some k. Here f is *right k-determined* if each $g \in K[[\mathbf{x}]]$ s.t. $j^k g = j^k f$, is right equivalent to f, where $j^k f$ denotes the *k-jet* of f in the *k-th jet space* $J_k := \langle \mathbf{x} \rangle / \langle \mathbf{x} \rangle^{k+1}$. The minimum of such k is called the *right determinacy* of f. For each isolated f, the *right modality* of f, $\mathcal{R}$-mod(f), is defined to be the $\mathcal{R}_k$-modality of $j^k f$ in J_k with $k \ge 2\mu(f)$ and $\mathcal{R}_k$ the k-jet of $\mathcal{R}$. Notice that if f is right equivalent to g then $\mathcal{R}$-mod$(f) = \mathcal{R}$-mod(g) (cf. [GN13, Prop. A.4]).

In Section 2, we show that the right determinacy of an isolated univariate formal power series f is equal to $d(f)$, which is defined by a concrete formula determined by the support of f (Definition 2.1, Proposition 2.8). Moreover we give an explicit normal form for any (not necessary isolated) univariate power series f w.r.t. right equivalence (Theorem 2.11). We prove in Section 3 that the right modality of an isolated series f is equal to the integer part of $\mu(f)/p$ (Theorem 3.1). As a consequence we show that the right modality is equal to the dimension of the μ-constant stratum in an algebraic representative of the semiuniversal deformation with trivial section (Corollary 3.6).

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2. Normal forms of univariate power series

Let $f = \sum_{n\geq 0} c_n x^n \in K[[x]]$ be a univariate power series, let $\operatorname{supp}(f) := \{n \geq 0 \mid c_n \neq 0\}$ be the *support* of f and $\operatorname{mt}(f) := \min\{n \mid n \in \operatorname{supp}(f)\}$ the *multiplicity* of f. If $\operatorname{char}(K) = 0$ and if $\varphi(x) = a_1 x + a_2 x^2 + \ldots, a_1 \neq 0$, is a coordinate change, then the coefficients a_i of φ can be determined inductively from the equation $f(x) = c_0 + (\varphi(x))^{\operatorname{mt}(g)}$ with $g(x) := f - c_0$. Hence f is right equivalent to $c_0 + x^{\operatorname{mt}(g)}$.

In the following we investigate $f \in K[[x]]$ with $\operatorname{char}(K) = p > 0$. The aim of this section is to give a normal form of f. It turns out that it depends in a complicated way on the divisibility relation between p and the support of f. To describe this relation we make the following definition, where later on Δ will be $\operatorname{supp}(f)$.

Definition 2.1. For each $n \in \mathbb{N}$ and each non-empty subset $\Delta \subset \mathbb{N} \setminus \{0\}$, we define

(a) $m := m(\Delta) := \min\{n \mid n \in \Delta\}$.
(b) $e := e(\Delta) := \min\{e(n) \mid n \in \Delta\}$, where $e(n) := \max\{i \mid p^i \text{ divides } n\}$.
(c) $q := q(\Delta) := \min\{n \in \Delta \mid e(n) = e\}$.
(d) $k := k(\Delta) := 1$ and $e_0(\Delta) := e + 1$ if $e(m) = e$ (i.e. $m = q$), otherwise,

$$k := k(\Delta) := \max\{k_\Delta(n) \mid m \leq n < q, n \in \Delta\},$$

where

$$k_\Delta(n) := \left\lceil \frac{q-n}{p^{e(n)} - p^e} \right\rceil \text{ denotes the ceiling of } \frac{q-n}{p^{e(n)} - p^e}$$

and

$$e_0 := e_0(\Delta) := \min\{e(n) \mid m \leq n < q, n \in \Delta\}.$$

(e) $d := d(\Delta) := q + p^e(k-1)$.
(f) $\bar{\Lambda}(\Delta) = \emptyset$ if $e(m) = e$, otherwise,

$$\bar{\Lambda}(\Delta) := \{n \in \mathbb{N} \mid m < n \leq d, e_0 \leq e(n)\} \cup \{q\}.$$

(g) If $e(m) > e$ (i.e. $m < q$) we define
$\Delta_0 := \{n \in \Delta \mid n < q\},\ q_0 := q(\Delta_0),\ d_0 := d(\Delta_0),\ \bar{d}_0 := \min\{d, d_0\}$,
$\Lambda_0(\Delta) := \emptyset$ if $e(m) = e_0$,
$\Lambda_0(\Delta) := \big\{n \in \mathbb{N} \mid m < n \leq \bar{d}_0, e_0 < e(n)\big\} \cup \{q_0\}$ if $e(m) > e_0$, and
$\Lambda_1(\Delta) := \{n \in \mathbb{N} \mid q \leq n \leq d, e \leq e(n) < e_0\}$.

(h) If $e(m) = e$ then $\Lambda(\Delta) := \emptyset$, otherwise,

$$\Lambda(\Delta) := \Lambda_0(\Delta) \cup \Lambda_1(\Delta).$$

Remark 2.2. If $f \in K[[x]]$ with $\mu(f) < \infty$ and $\Delta = \mathrm{supp}(f)$ then

(a) $m(\Delta) = \mathrm{mt}(f)$, the multiplicity (or, the order) of f.
(b) $q(\Delta) = \mu(f) + 1$, the first exponent in the expansion of f which is not divisible by p.
(c) $k_\Delta(n)$ is the minimum of l for which

$$\mathrm{mt}\left(\varphi(x^n) - x^n\right) \geq \mathrm{mt}\left(\varphi(x^q) - x^q\right) = q + l$$

with $q := q(\Delta)$ and $\varphi = x + u_{l+1}x^{l+1}$ + terms of higher order, $u_{l+1} \neq 0$, a coordinate change.

Indeed,

$$\begin{aligned} \varphi(x^n) &= \left(x + u_{l+1}x^{l+1} + \dots\right)^n \\ &= \left[\left(x + u_{l+1}x^{l+1} + \dots\right)^{n/p^{e(n)}}\right]^{p^{e(n)}} \\ &= \left[x^{n/p^{e(n)}} + (n/p^{e(n)}) \cdot u_{l+1}x^{n/p^{e(n)}+l} + \dots\right]^{p^{e(n)}} \\ &= x^n + (n/p^{e(n)})^{p^{e(n)}} u_{l+1}^{p^{e(n)}} x^{n+lp^{e(n)}} + \dots . \end{aligned}$$

It yields that

$$\mathrm{mt}\left(\varphi(x^n) - x^n\right) \geq q + l \Leftrightarrow l \geq \frac{q-n}{p^{e(n)} - 1}.$$

This proves the claim.

(d) $k(\Delta)$ is then the minimum of l for which

$$\varphi(f) = f \mod x^{q+l}$$

with $q = q(\Delta)$ and a coordinate change φ as above. This is used to show that:

(e) $d(\Delta)$ is the right determinacy of f, cf. Proposition 2.8.

Remark 2.3. The following facts (a)-(e) are immediate consequences of the definition. Property (f) follows from elementary calculations.

(a) $e(\Delta) < e_0(\Delta)$, $k(\Delta) > 0$.
(b) If $q(\Delta) = q(\Delta') =: q$ and $\Delta \cap \mathbb{N}_{<q} = \Delta' \cap \mathbb{N}_{<q}$, then $d(\Delta) = d(\Delta')$ and $\Lambda(\Delta) \equiv \Lambda(\Delta')$. That is, $q(\Delta)$ is the "determinacy" of $\Lambda(\Delta)$.
(c) If p does not divide $m(\Delta)$, then
  1. $e(\Delta) = e(m(\Delta)) = 0$ and $q(\Delta) = m(\Delta)$.
  2. $k(\Delta) = 1$ and $d(\Delta) = m(\Delta)$.
(d) If $e(m(\Delta)) = e(\Delta)$, then
  1. $q(\Delta) = m(\Delta)$.
  2. $k(\Delta) = 1$ and $d(\Delta) = m(\Delta)$.
(e) If $n + lp^{e(n)} \leq d(\Delta)$ for some l and some $n \in \Delta$, then $l \leq k(\Delta)$.
(f) If $k(\Delta) = k_\Delta(n)$, then

$$k(\Delta) - 1 + \frac{n}{p^{e(n)}} = \left\lfloor \frac{d(\Delta)}{p^{e(n)}} \right\rfloor,$$

where $\left\lfloor \frac{d(\Delta)}{p^{e(n)}} \right\rfloor$ denotes the floor (or, integer part) of $\frac{d(\Delta)}{p^{e(n)}}$.

In fact, one has, by denoting $e := e(\Delta)$, $q := q(\Delta)$, $k := k(\Delta)$, $d := d(\Delta)$, that

$$\begin{aligned}\frac{d}{p^{e(n)}} - \left(k - 1 + \frac{n}{p^{e(n)}}\right) &= \frac{q + p^e(k-1)}{p^{e(n)}} - \left(k - 1 + \frac{n}{p^{e(n)}}\right)\\ &= \frac{p^{e(n)} - p^e}{p^{e(n)}} \cdot \left(\frac{q-n}{p^{e(n)} - p^e} - k + 1\right).\end{aligned}$$

Then

$$0 < \frac{d}{p^{e(n)}} - \left(k + \frac{n}{p^{e(n)}} - 1\right) < 1$$

since $k = \left\lceil \frac{q-n}{p^{e(n)}-p^e} \right\rceil$. This gives us the formula.

Example 2.4. Let $p = \mathrm{char}(K) = 2$, let

$$f = x^8 + x^{36} + x^{37} + \text{terms of higher order in } K[[x]],$$

and let

$$\Delta := \mathrm{supp}(f) = \{8, 36, 37, \ldots\}.$$

Then

$$e = 0, q = 37, k = k_\Delta(8) = 5, d = 41,$$
$$e_0 = 2, q_0 = 36, d_0 = 60, \bar{d}_0 = d = 41.$$

and

$$\Lambda(f) = \{16, 24, 32, 36, 37, 38, 39, 40, 41\},$$
$$\sharp\Lambda(f) = 9 = \left\lfloor \frac{q}{p^{e_0}} \right\rfloor - \left\lfloor \frac{m}{p^{e_0}} \right\rfloor + 2.$$

The following proposition is the first key step in the classification.

Proposition 2.5. *With the notions as in Definition 2.1, assume that* $e(\Delta) = 0$. *Then*

$$\sharp\Lambda(\Delta) \le \left\lfloor \frac{q}{p} \right\rfloor - \frac{m}{p} + 1.$$

More precisely,

(i) *If* $e(m) < e_0$ *then* $\sharp\Lambda(\Delta) = 0$.

(ii) *If* $e(m) = e_0$ *then* $\sharp\Lambda(\Delta) = \left\lfloor \frac{q}{p^{e_0}} \right\rfloor - \frac{m}{p^{e_0}} + 1$.

(iii) *If* $e(m) > e_0$ *and*

(1) *if* $p > 2$ *then* $\sharp\Lambda(\Delta) \le \left\lfloor \frac{q}{p^{e_0}} \right\rfloor - \frac{m}{p^{e_0}} + 1$;

(2) *if* $p = 2$ *then* $\sharp\Lambda(\Delta) \le \left\lfloor \frac{q}{p^{e_0}} \right\rfloor - \frac{m}{p^{e_0}} + 2$.

Proof. (i) It is easy to see that, $e(m) < e_0$ if and only if $e(m) = e$ and then $\Lambda(\Delta) = \emptyset$.

(ii) Since $e(m) = e_0$, $\Lambda_0(\Delta) = \emptyset$ and $k_\Delta(m) = k$. Then

$$\Lambda(\Delta) = \Lambda_1(\Delta) = \{n \in \mathbb{N} \mid q \le n \le d, e(n) < e_0\}$$

and hence

$$\sharp\Lambda(\Delta) = k - \left(\left\lfloor \frac{d}{p^{e_0}} \right\rfloor - \left\lfloor \frac{q}{p^{e_0}} \right\rfloor\right) = \left\lfloor \frac{q}{p^{e_0}} \right\rfloor - \frac{m}{p^{e_0}} + 1$$

since $k - 1 + \frac{m}{p^{e(m)}} = \left\lfloor \frac{d}{p^{e(m)}} \right\rfloor$ due to Remark 2.3(f).

(iii) Since $e(m) > e_0$ one has

$$k(\Delta_0) - 1 = \left\lceil \frac{q_0 - n}{p^{e(n)} - p^{e_0}} \right\rceil - 1 < \frac{q_0 - m}{p^{e_0+1} - p^{e_0}}$$

for some $n \in \Delta_0,\ e(n) > e_0$, and

$$\Lambda_0(\Delta) = \left\{n' \in \mathbb{N} \mid m < n' \leq \bar{d}_0, e(n') > e_0\right\} \cup \{q_0\},$$

$$\Lambda_1(\Delta) = \{n' \in \mathbb{N} \mid q \leq n' \leq d, e(n') < e_0\}\,.$$

This implies that

$$\sharp\Lambda_0(\Delta) = \left\lfloor \frac{\bar{d}_0}{p^{e_0+1}} \right\rfloor - \frac{m}{p^{e_0+1}} + 1$$

and

$$\begin{aligned}\sharp\Lambda_1(\Delta) &= (d-q+1) - \left(\left\lfloor \frac{d}{p^{e_0}} \right\rfloor - \left\lfloor \frac{q}{p^{e_0}} \right\rfloor\right)\\ &= k - \left(\left\lfloor \frac{d}{p^{e_0}} \right\rfloor - \left\lfloor \frac{q}{p^{e_0}} \right\rfloor\right).\end{aligned}$$

We consider the following cases:

Case 1: $k_\Delta(q_0) = k$.

Then $k - 1 + \frac{q_0}{p^{e_0}} = \left\lfloor \frac{d}{p^{e_0}} \right\rfloor$ by Remark 2.3(f). We obtain

$$\begin{aligned}\sharp\Lambda(\Delta) &= \sharp\Lambda_0(\Delta) + \sharp\Lambda_1(\Delta) = \left\lfloor \frac{q}{p^{e_0}} \right\rfloor - \left(\frac{q_0}{p^{e_0}} - \left\lfloor \frac{\bar{d}_0}{p^{e_0+1}} \right\rfloor + \frac{m}{p^{e_0+1}} - 2\right)\\ &\leq \left\lfloor \frac{q}{p^{e_0}} \right\rfloor - \left(\frac{q_0}{p^{e_0}} - \left\lfloor \frac{d_0}{p^{e_0+1}} \right\rfloor + \frac{m}{p^{e_0+1}} - 2\right)\\ &\leq \left\lfloor \frac{q}{p^{e_0}} \right\rfloor - \left(\frac{q_0}{p^{e_0}} - \frac{q_0 + (k(\Delta_0) - 1)\,p^{e_0}}{p^{e_0+1}} + \frac{m}{p^{e_0+1}} - 2\right)\\ &< \left\lfloor \frac{q}{p^{e_0}} \right\rfloor - \left(\frac{(p^2-2p)q_0 + m}{p^{e_0+2} - p^{e_0+1}} + \frac{m}{p^{e_0+1}} - 2\right)\\ &\leq \left\lfloor \frac{q}{p^{e_0}} \right\rfloor - \left(\frac{m}{p^{e_0}} - 2\right),\end{aligned}$$

due to $k(\Delta_0) - 1 < \frac{q_0 - m}{p^{e_0+1} - p^{e_0}}$, respectively $q_0 > m$. Hence

$$\sharp\Lambda(\Delta) \leq \left\lfloor \frac{q}{p^{e_0}} \right\rfloor - \frac{m}{p^{e_0}} + 1.$$

Case 2: $k_\Delta(q_0) < k$.

Then

$$k = \left\lceil \frac{q-n}{p^{e(n)} - 1} \right\rceil < \frac{q-m}{p^{e_0+1} - 1} + 1$$

for some $n \in \Delta_0,\ e(n) > e_0$. It yields that

$$d = q + k - 1 > (k-1)p^{e_0+1} + m$$

and hence

$$\begin{aligned}
\sharp\Lambda(\Delta) &= \left\lfloor \frac{q}{p^{e_0}} \right\rfloor - \left(\left\lfloor \frac{d}{p^{e_0}} \right\rfloor - \left\lfloor \frac{\bar{d}_0}{p^{e_0+1}} \right\rfloor + \frac{m}{p^{e_0+1}} - k - 1 \right) \\
&\leq \left\lfloor \frac{q}{p^{e_0}} \right\rfloor - \left(\left\lfloor \frac{d}{p^{e_0}} \right\rfloor - \left\lfloor \frac{d}{p^{e_0+1}} \right\rfloor + \frac{m}{p^{e_0+1}} - k - 1 \right) \\
&\leq \left\lfloor \frac{q}{p^{e_0}} \right\rfloor - \left(\left\lfloor \frac{(p-1)d}{p^{e_0+1}} \right\rfloor + \frac{m}{p^{e_0+1}} - k - 1 \right) \\
&\leq \left\lfloor \frac{q}{p^{e_0}} \right\rfloor - \left((p-1)(k-1) + \frac{m}{p^{e_0}} - k - 1 \right) \\
&= \left\lfloor \frac{q}{p^{e_0}} \right\rfloor - \frac{m}{p^{e_0}} + 2 - (p-2)(k-1).
\end{aligned}$$

This completes the proposition. □

Note that if $f \in K[[x]]$ and $\mathrm{mt}(f) = 0$ then $\mathrm{mt}(f - f(0)) > 0$. Applying the results from $\mathrm{mt}(f) > 0$ to $f - f(0)$ we obtain that $f \sim_r f(0) + g$, where g is a normal form of $f - f(0)$ (cf. Theorem 2.11). From now on we assume that $\mathrm{mt}(f) > 0$. We denote, by using notations as in Definition 2.1 for $\Delta = \mathrm{supp}(f)$,

$$e(f) := e(\Delta),\ q(f) := q(\Delta),\ k(f) := k(\Delta),\ d(f) := d(\Delta)$$

and

$$\bar{\Lambda}(f) := \bar{\Lambda}(\Delta),\ \Lambda(f) := \Lambda(\Delta).$$

Remark 2.6. (a) The above numbers mt, e, q, k, d and the sets Λ and $\bar{\Lambda}$ are invariant w.r.t. right equivalence.

(b) Let $f = \sum_{n\geq 1} c_n x^n \in K[[x]]$ and let

$$\bar{f}(x) = \sum_{n \geq m(f)} c_n x^{n/p^{e(f)}}.$$

Then $\bar{f} \in K[[x]]$, $f(x) = \bar{f}(x^{p^{e(f)}})$ and $e(\bar{f}) = 0$. Moreover,

$$k(f) = k(\bar{f}), \sharp\Lambda(f) = \sharp\Lambda(\bar{f}),\ \sharp\bar{\Lambda}(f) = \sharp\bar{\Lambda}(\bar{f})$$

and if $\zeta(f)$ denotes one of $\mathrm{mt}(f), e(f), q(f), d(f)$ then

$$\zeta(f) = p^{e(f)}\zeta(\bar{f}).$$

(c) Note that $\mu(f) < \infty$ if and only if $e(f) = 0$ and then $q(f) = \mu(f) + 1$. By [BGM12, Thm. 2.1] f is then right $(2q(f) - \mathrm{mt}(f))$-determined. In Proposition 2.8 we will show that $d(f)$ is the right determinacy of f.

Lemma 2.7. *If $e(\mathrm{mt}(f)) = e(f)$ then $f \sim_r x^{\mathrm{mt}(f)}$.*

Proof. By Remark 2.6, there exists $\bar{f} \in K[[x]]$ such that $f(x) = \bar{f}(x^{p^{e(f)}})$ and $e(\bar{f}) = 0$. This implies that $\mu(\bar{f}) = q(\bar{f}) - 1$ and then $\mu(\bar{f}) = \mathrm{mt}(\bar{f}) - 1$ since $e(\mathrm{mt}(f)) = e(f)$. It follows from [BGM12, Thm. 2.1] that $\bar{f}$ is right $(\mathrm{mt}(\bar{f}) + 1)$-determined. That is,

$$\bar{f} \sim_r c_m x^{\mathrm{mt}(\bar{f})} \sim_r x^{\mathrm{mt}(\bar{f})}$$

and hence $f \sim_r x^{\mathrm{mt}(f)}$ with the same coordinate change.

In fact, in this case an inductive proof as in the case of characteristic 0 works. □

The next proposition is the second key step in the classification.

Proposition 2.8. *With f and $d(f)$ as above, assume that $\mu(f) < \infty$ then $d(f)$ is exactly the right determinacy of f.*

Proof. We may assume that $e(\mathrm{mt}(f)) > e(f)$ since the case $e(\mathrm{mt}(f)) = e(f)$ follows from Lemma 2.7. Let us denote $\Delta := \mathrm{supp}(f)$ and use the notions as in Definition 2.1.

Step 1: Let us show that if $g \in K[[x]]$ with $j^d(f) = j^d(g)$ and $d := d(f)$ then $f \sim_r g$.

By Remark 2.3(b), $d(g) = d(f) = d$ since

$$\mathrm{supp}(f) \cap \{n \in \mathbb{N} \mid n \leq q\} = \mathrm{supp}(g) \cap \{n \in \mathbb{N} \mid n \leq q\}.$$

It suffices to show that

$$f \sim_r f_0 := j^d(f).$$

Indeed, we write

$$f = f_0 + f_1 \text{ with } \mathrm{mt}(f_1) \geq d + 1.$$

and assume without loss of generality, that

$$f_1 = b_{q+l}x^{q+l} + \text{ terms of higher order, with } b_{q+l} \neq 0.$$

Then the coordinate change $\varphi_1(x) = x + u_{l+1}x^{l+1}$ with u_{l+1} a root of the following non-constant polynomial:

$$qc_qX + \sum_{\frac{q-n}{p^{e(n)}-1}=l} (n/p^{e(n)})^{p^{e(n)}} c_n X^{p^{e(n)}} + b_{q+l} = 0$$

is sufficient to increase the multiplicity of f_1 and does not change f_0 by Remark 2.2(d). We thus finish by induction.

Step 2: We now show that f is not right $(d-1)$–determined.

For this we need the following

Claim: $f \sim_r g$ if and only if $j^d g \in \mathcal{R}_k \cdot j^d f$, where

$$\mathcal{R}_k := \{\psi = u_0x + u_1x^2 + \ldots + u_{k-1}x^k \mid u_0 \neq 0\} \subset \mathcal{R}$$

and it acts on the jet space J_d by $(\psi, j^d h) \mapsto j^d(\psi(j^d h))$.

Proof of the claim. The "if"-statement follows easily from the first step. We assume that $f \sim_r g$, i.e. $g = \varphi(f)$ with

$$\varphi = u_0x + u_1x^2 + \ldots, u_0 \neq 0.$$

Setting

$$\psi := u_0x + u_1x^2 + \ldots + u_{k-1}x^k$$

and $\varphi_1 := \varphi \circ \psi^{-1}$ we obtain that $\varphi = \varphi_1 \circ \psi$ and that

$$\varphi_1 = x + a_kx^{k+1} + \text{ terms of higher order.}$$

Note that $k = k(f) = k(\psi(f))$ due to Remark 2.6(a). It follows from Remark 2.2(d) that

$$j^d(\varphi_1(\psi(f))) = j^d(\psi(f)).$$

Hence

$$j^d g = j^d\varphi(f) = j^d(\varphi_1(\psi(f))) = j^d(\psi(f)) = j^d(\psi(j^d f)).$$

This completes the claim.

We write, for new indeterminates $u_0, \ldots, u_{k-1}, t$,

$$f + tx^d - \psi(j^d f) = \sum_{i=m}^{d} b_i(u_0, \ldots, u_{k-1}, t)x^i$$

with $\psi := u_0x + u_1x^2 + \ldots + u_{k-1}x^k$ and $b_i \in K[u_0, \ldots, u_{k-1}, t]$, and define

$$V := Z(b_m, \ldots, b_d) := \{(u_1, \ldots, u_{k-1}, t) \in \mathbb{A}^k \mid b_i(u_0, \ldots, u_{k-1}, t) = 0\}$$

with the structure sheaf $\mathcal{O}_V$ and its algebra of global section

$$\mathcal{O}_V(V) = K[u_0, \ldots, u_{k-1}, t]/\langle b_m, \ldots, b_d\rangle.$$

We prove the second step by contradiction. Suppose the assertion were false. Then for all $t \in K$, f would be right equivalent to $f + tx^d$, equivalently, $j^d f + tx^d \in \mathcal{R}_k \cdot j^d f$ for all t due to the above claim. This implies that the map p defined by

$$\begin{array}{cccc} p & : & V & \to & \mathbb{A}^1 \\ & & (u_0, \ldots, u_{k-1}, t) & \mapsto & t \end{array}$$

is surjective. It yields that $\dim V \geq 1$. We may assume without loss of generality that $\dim_O V \geq 1$, where $O = (1, 0, \ldots, 0) \in V$ and $\dim_O V$ denotes the maximal dimension of irreducible components of V containing O. Since $\mathcal{O}_{V,O} \subset R := K[[u_0', u_1, \ldots, u_{k-1}, t]]/\langle b_m, \ldots, b_d\rangle$ with $u_0' = u_0 - 1$,

$$\dim R \geq \dim \mathcal{O}_{V,O} = \dim_O V \geq 1.$$

By the Curve Selection Lemma, there exists a non-constant K−algebra homomorphism

$$\begin{array}{rcl} \phi : K[[u_0', u_1, \ldots, u_{k-1}, t]] & \to & K[[\tau]] \\ u_0' & \mapsto & u_0'(\tau) \\ u_i & \mapsto & u_i(\tau) \\ t & \mapsto & t(\tau) \end{array}$$

such that

$$b_i\left(1 + u_0'(\tau), u_1(\tau), \ldots, u_{k-1}(\tau), t(\tau)\right) = 0 \quad \text{for all } i = m, \ldots, d.$$

Since $b_m = c_m(u_0^m - 1)$, it follows that

$$\left(1 + u_0'(\tau)\right)^m - 1 = 0$$

and therefore $u_0'(\tau) = 0$. Notice that, the series $u_i(\tau), i = 1, \ldots, k-1$ could not be all equal to zero since $\phi \neq 0$ and since

$$b_d(1, u_1, \ldots, u_{k-1}, t) = qc_q u_{k-1} + t + b_d'(u_1, \ldots, u_{k-1}), \text{ with } \operatorname{mt}(b_d') \geq 2.$$

We set

$$l := \min\{j \mid u_j(\tau) \neq 0\},$$
$$L := \min\{n + lp^{e(n)} \mid n \in \Delta\}$$

and

$$I := \{n \in \Delta \mid L = n + lp^{e(n)}\}.$$

By Remark 2.2 we can conclude that $m < L < d$ and that

$$\psi(f) - f = \sum_{n \in I} \left(n/p^{e(n)}\right)^{p^{e(n)}} c_n u_l(\tau)^{p^{e(n)}} x^L + \text{ terms of higher order}$$

where

$$\psi = x + u_l(\tau)x^{l+1} + \ldots + u_{k-1}(\tau)x^k.$$

It follows that

$$b_L\left(1, u_1(\tau), \ldots, u_{k-1}(\tau), t(\tau)\right) = \sum_{n \in I} \left(n/p^{e(n)}\right)^{p^{e(n)}} c_n u_l(\tau)^{p^{e(n)}} \neq 0,$$

which is a contradiction. This proves the second step. □

In Corollary 2.9, Lemma 2.10 and Theorem 2.11 below we do not assume that f is an isolated singularity, i.e. $\mu(f)$ may be infinite or, equivalently, $e(f)$ may be bigger than 0.

Corollary 2.9. *Let $f \in K[[x]]$ and $d = d(f)$. Let $g \in K[[x]]$ be such that $e(f) = e(g)$ and $j^d(f) = j^d(g)$. Then $f \sim_r g$.*

We have in particular that $f \sim_r j^d(f)$.

Proof. By Proposition 2.8, it suffices to prove the corollary for the case that $e := e(f) = e(g) > 0$. Taking $\bar{f} \in K[[x]]$ and $\bar{g} \in K[[x]]$ such that $f(x) = \bar{f}(x^{p^e}),\ g(x) = \bar{g}(x^{p^e})$ as in Remark 2.6 we have

$$e(\bar{f}) = e(\bar{g}) = 0,\ \bar{d} := d(\bar{f}) = d/p^e.$$

Since $j^d(f) = j^d(g)$, $j^{\bar{d}}(\bar{f}) = j^{\bar{d}}(\bar{g})$ and hence $\bar{f} \sim_r \bar{g}$ according to Proposition 2.8. This implies $f \sim_r g$ with the same coordinate change. □

Lemma 2.10. *With f, $\mathrm{mt}(f)$ and $\bar{\Lambda}(f)$ as above, we have*

$$f \sim_r x^{\mathrm{mt}(f)} + \sum_{n \in \bar{\Lambda}(f)} \lambda_n x^n,$$

for suitable $\lambda_n \in K$.

Proof. We decompose $f = f_0 + f_1$ with

$$f_0 := \sum_{e(f) \leq e(i) < e_0} c_i x^i \text{ and } f_1 := \sum_{e(n) \geq e_0} c_n x^n.$$

Then $\mathrm{mt}(f_0) = q(f)$ and $e(\mathrm{mt}(f_0)) = e(f_0) = 0$ and hence $f_0 \sim_r x^{q(f)}$ by Lemma 2.7. That is, $\varphi(f_0) = x^{q(f)}$ for some coordinate change $\varphi \in Aut(K[[x]])$. It yields that

$$g := \varphi(f) = \varphi(f_0) + \varphi(f_1) = x^{q(f)} + \varphi(f_1).$$

By Remark 2.6, $d(g) = d(f)$ and

$$\varphi(f_1) = \sum_{e(n) \geq e_0} \lambda_n x^n$$

for some $\lambda_n \in K$. Hence

$$f \sim_r g \sim_r j^{d(g)}(g) = x^{\mathrm{mt}(f)} + \sum_{n \in \bar{\Lambda}(f)} \lambda_n x^n$$

due to Corollary 2.9. □

From Proposition 2.5 and Remark 2.6(b), replacing f by $\bar{f}$ if $e(f) > 0$, and denoting $\Delta := \mathrm{supp}(f)$ we can conclude that

$$\sharp\Lambda(f) \leq \left\lfloor \frac{q}{p^{e_0}} \right\rfloor - \frac{m}{p^{e_0}} + 2 \leq \left\lfloor \frac{d}{p^{e_0}} \right\rfloor - \frac{m}{p^{e_0}} + 2 = \sharp\bar{\Lambda}(f).$$

The following theorem is therefore stronger than Lemma 2.10 because it reduces the number of parameters.

Theorem 2.11 (Normal form of univariate power series)**.** *With f, $\mathrm{mt}(f)$ and $\Lambda(f)$ as above, we have*

$$f \sim_r x^{\mathrm{mt}(f)} + \sum_{n \in \Lambda(f)} \lambda_n x^n$$

for suitable $\lambda_n \in K$.

Proof. We set $\Delta := \operatorname{supp}(f)$ and use the notations as in Definition 2.1. It is sufficient to prove the theorem for the case that $e(m) > e$, because the case $e(m) = e$ follows from Lemma 2.7. Then

$$\Lambda_0(\Delta) = \big\{n \in \mathbb{N} \mid m < n \leq \bar{d}_0, e_0 < e(n)\big\} \cup \{q_0\},$$
$$\Lambda_1(\Delta) = \{n \in \mathbb{N} \mid q \leq n \leq d, e \leq e(n) < e_0\}.$$

We decompose $f = f_0 + f_1$ with

$$f_0 := \sum_{i<q} c_i x^i \text{ and } f_1 := \sum_{n \geq q} c_n x^n.$$

Applying Lemma 2.10 to f_0 we obtain, by denoting $\Lambda_0'(\Delta) := \Lambda(\Delta) \cap \{n \in \mathbb{N} \mid n < q\}$ that

$$f_0 \sim_r x^m + \sum_{n \in \bar{\Lambda}(\Delta_0)} b_n x^n = x^m + \sum_{n \in \Lambda_0'(\Delta)} b_n x^n \mod x^q,$$

for suitable $\lambda_n \in K$, since

$$\bar{\Lambda}(\Delta_0) \cap \{n \in \mathbb{N} \mid n < q\} \subset \Lambda_0'(\Delta).$$

This means that there exists a coordinate change φ such that

$$\varphi(f_0) = x^m + \sum_{n \in \Lambda_0'(\Delta)} b_n x^n \mod x^q.$$

We denote $g := \varphi(f)$,

$$g_0 := x^m + \sum_{n \in \Lambda_0'(\Delta)} b_n x^n,$$

and

$$g_1 := g - g_0 := \sum_{n \geq q} b_n x^n, \; b_q \neq 0.$$

We will construct a series h such that $f \sim_r h$ and

$$h = x^m + \sum_{n \in \Lambda(\Delta)} \lambda_n x^n \mod x^d$$

by eliminating inductively all terms of exponent in

$$I := \{i \in \mathbb{N} \mid q \leq i \leq d, e \leq e(i)\} \setminus \Lambda(\Delta).$$

If we succeed then by Corollary 2.9

$$f \sim_r h \sim_r j^d h \sim_r x^m + \sum_{n \in \Lambda(\Delta)} \lambda_n x^n.$$

Let i_1 be the minimum exponent in I for which $b_{i_1} \neq 0$. According to Remark 2.3 the coordinate change

$$\varphi_1(x) = x + u_{l+1} x^{l+1}$$

with $l := \dfrac{i_1 - q_0}{p^{e_0}}$ and u_{l+1} a root of the non-constant polynomial:

$$\sum_{n + lp^{e(n)} = i_1} b_n \big(n/p^{e(n)}\big)^{p^{e(n)}} X^{p^{e(n)}} + b_{i_1} = 0,$$

makes the coefficient of x^{i_1} vanish, and no term of exponent i in I with $i < i_1$ occurs. We prove the last claim by contradiction. Suppose the claim were false, then we could find $j \in I, j < i_1$

such that the coefficient of x^j in $\varphi_1(g)$ differs from zero. That is, j is an exponent of a term in $(x+u_{l+1}x^{l+1})^n$ for some $n \in \Lambda(\Delta)$ with $b_n \neq 0$. Then there exists an $i \in \mathbb{N}$ such that

$$j = n + ilp^{e(n)}.$$

Note that $i > 0$ by the definition of i_1. This implies that

$$n + ilp^{e(n)} \geq n + lp^{e(n)} > j \text{ for all } n \in \Lambda(\Delta) \text{ with } b_n \neq 0,$$

because

- if $e(n) \leq e_0$ then n is either q or q_0, and hence
$$q_0 + lp^{e_0} = i_1 > j$$
and
$$q + lp^e \geq q_0 + lp^{e_0} = i_1 > j$$
since $l \leq k$ due to Remark 2.3(e).
- If $e(n) > e_0$ then $e(j) \geq e(n) > e_0$ and therefore $j > \bar{d}_0$. This implies that
$$\bar{d}_0 = d_0 < j < i_1 < d$$
and therefore
$$l = \frac{i_1 - q_0}{p^{e_0}} \geq k(\Delta_0).$$
It follows that
$$n + ilp^{e(n)} \geq n + lp^{e(n)} \geq q_0 + lp^{e_0} = i_1 > j.$$

This contradiction shows that there is no term of exponent i in I with $i < i_1$ in $\varphi_1(g)$. Hence we obtain by induction a series h as required. □

Note that the families over $\Lambda(f)$ resp. $\bar{\Lambda}(f)$ in Theorem 2.11 resp. Lemma 2.10 contain all possible normal forms having the same set Λ resp. $\bar{\Lambda}$ (and hence having the same m, q, k and d). The number of parameters of normal forms in the μ–constant stratum (proof of Theorem 3.1) could be bigger.

The following example shows that this normal form is in general not the best one we can get. This means that, we can sometimes reduce the number of parameters even more.

Example 2.12. We consider

$$f = x^8 + x^{36} + x^{37} + \text{terms of higher order}$$

in characteristic 2, as in Example 2.4. Then $d(f) = 41$ and

$$\Lambda(f) = \{16, 24, 32, 36, 37, 38, 39, 40, 41\}.$$

It follows from Theorem 2.11 that

$$f \sim_r x^8 + \lambda_1 x^{16} + \lambda_2 x^{24} + \lambda_3 x^{32} + \lambda_4 x^{36} + \lambda_5 x^{37} + \lambda_6 x^{38} + \lambda_7 x^{39} + \lambda_8 x^{40} + \lambda_9 x^{41}$$

for suitable $\lambda_i \in K$.

On the other hand, applying Lemma 2.7 to $f_1 := f - (x^8 + x^{36})$ we get $f_1 \sim_r x^{37}$. That is, $\varphi(f_1) = x^{37}$ for some coordinate change φ. It yields

$$\varphi(f) = a_0x^8 + a_1x^{16} + a_2x^{24} + a_3x^{32} + a_4x^{36} + x^{37} \mod x^{41}.$$

By Proposition 2.8,

$$f \sim_r \varphi(f) \sim_r a_0x^8 + a_1x^{16} + a_2x^{24} + a_3x^{32} + a_4x^{36} + x^{37} + a_5x^{40}$$

and hence

$$f \sim_r x^8 + b_1x^{16} + b_2x^{24} + b_3x^{32} + b_4x^{36} + b_5x^{37} + b_6x^{40}.$$

This shows that, we can find a "better normal form" for f. Moreover by the coordinate change

$$x + b_6/b_5 x^4,$$

we can even get rid of the term b_6x^{40} and obtain that

$$f \sim_r x^8 + c_1x^{16} + c_2x^{24} + c_3x^{32} + c_4x^{36} + c_5x^{37}.$$

In the following, we will give a set of terms of f which can not be removed by coordinate changes and then we conjecture the "best normal form" for f.

Remark 2.13. Let $f \in K[[x]]$ be such that $\mu(f) < \infty$. Let $\Delta := \mathrm{supp}(f)$ and let

$$q_i := \min\{n \in \Delta \mid e(n) \leq i\}.$$

Then

$$q(f) = q_0 \geq q_1 \geq \ldots \geq q_{e(m)} = m = q_i, \text{ for all } i \geq e(m).$$

We can see easily that the set $\{q_0, \ldots, q_{e(m)}\}$ is the set of exponents of terms which can not be removed by coordinate changes. However it is not true in general that

$$f \sim_r \sum_{i=1}^{e(m)} \lambda_i x^{q_i}$$

for suitable $\lambda_i \in K$ as the following example shows:

$$f = x^8 + x^{36} + x^{37} + x^{38} \in K[[x]] \text{ with } \mathrm{char}(K) = 2.$$

Then

$$q_0 = q_1 = q = 37, q_2 = 36, q_3 = m = 8.$$

It is not difficult to see that

$$f \not\sim_r \lambda_0 x^8 + \lambda_1 x^{36} + \lambda_2 x^{37}$$

for any $\lambda_0, \lambda_1, \lambda_2 \in K$.

We like to pose the following conjecture.

Conjecture 2.14. *With notations as in Remark 2.13, let $\Lambda^*(f) := \emptyset$ if $e(m) = 0$, otherwise*

$$\Lambda^*(f) := \{n \in \mathbb{N} \mid m < n \leq q, e(n) \geq i \text{ if } q_i \leq n < q_{i-1}\}.$$

Then f is right equivalent to

$$x^{\mathrm{mt}(f)} + \sum_{n \in \Lambda^*(f)} \lambda_n x^n$$

for suitable $\lambda_n \in K$, and moreover this is a modular family. That is, for each $\lambda = (\lambda_n)_{n \in \Lambda^*(f)}$, there are only finitely many $\lambda' = (\lambda'_n)_{n \in \Lambda^*(f)}$ such that

$$x^{\mathrm{mt}(f)} + \sum_{n \in \Lambda^*(f)} \lambda_n x^n \sim_r x^{\mathrm{mt}(f)} + \sum_{n \in \Lambda^*(f)} \lambda'_n x^n.$$

3. Right modality

Theorem 3.1. *Let* char$K = p > 0$. *Let* $f \in \langle x\rangle \subset K[[x]]$ *be a univariate power series such that its Milnor number* $\mu := \mu(f)$ *is finite. Then*

$$\mathcal{R}\text{-mod}(f) = \lfloor \mu/p \rfloor .$$

For the proof we need the following lemmas which are proven in [GN13] for unfoldings but the proof works in general (for algebraic families of power series).

Let us recall the notion of unfoldings (see, [GN13]). Let T be an affine variety over K with the structure sheaf $\mathcal{O}$ and its algebra of global section $\mathcal{O}(T)$. An element $f_t(x) := F(x,t) \in \mathcal{O}(T)[[x]]$ is called an *algebraic family of power series* over T. A family $f_t(x)$ is said to be *modular* if for each $t \in T$ there are only finitely many $t' \in T$ such that $f_{t'}$ is right equivalent to f_t. An *unfolding*, or *deformation with trivial section* of a power series f at $t_0 \in T$ over T is a family $f_t(x)$ satisfying $f_{t_0} = f$ and $f_t \in \langle x\rangle$ for all $t \in T$.

Remark 3.2. Let $f \in \langle x\rangle \subset K[[x]]$ be a univariate power series with Milnor number $\mu < \infty$. Then the system $\{x, x^2, \ldots, x^\mu\}$ is a basis of the algebra $\langle x\rangle/\langle x \cdot \frac{\partial f}{\partial x}\rangle$. By [GN13, Prop. 2.14] the unfolding over $\mathbb{A}^\mu$,

$$f_t(x) := f + \sum_{i=1}^{\mu} t_i \cdot x^i$$

with $t := (t_1, \ldots, t_\mu)$ the coordinates of $t \in \mathbb{A}^\mu$, is an algebraic representative of the semiuniversal deformation with trivial section of f.

Lemma 3.3. *With* f *and* $f_t(x)$ *as in Remark 3.2, assume that there exists a finite number of algebraic families of power series* $h_t^{(i)}(x)$ *over varieties* $T^{(i)}, i \in I$ *and an open subset* $U \subset \mathbb{A}^\mu$ *satisfying: for all* $t \in U$ *there exists an* $i \in I$ *and* $t_i \in T^{(i)}$ *such that* $f_t(x)$ *is right equivalent to* $h_{t_i}^{(i)}(x)$. *Then*

$$\mathcal{R}\text{-mod}(f) \le \max_{i=1,\ldots,l} \dim T^{(i)}.$$

Proof. cf. [GN13, Proposition 2.15(i)]. □

Lemma 3.4. *If* $f_t(x)$ *is a modular unfolding of* f *over* T *then*

$$\mathcal{R}\text{-mod}(f) \ge \dim T.$$

Proof. It follows from [GN13, Propositions 2.12(ii) and 2.15(ii)]. □

Proof of Theorem 3.1. We first prove the inequality $\mathcal{R}\text{-mod}(f) \le \lfloor \mu/p \rfloor$. Indeed, let

$$I := \{\Delta \subset \{1, \ldots, q(f)\} | \; q(f) \in \Delta\},$$

and let

$$h_{s_\Delta}(x) := x^{m(\Delta)} + \sum_{n \in \Lambda(\Delta)} s_\Delta^{(n)} x^n, \; \Delta \in I$$

the finite set of families over $A_\Delta \equiv \mathbb{A}^{l_\Delta}$ with $l_\Delta = \sharp\Lambda(\Delta)$ and $s_\Delta^{(n)}, \; n \in \Lambda(\Delta)$ the coordinates of s_Δ in A_Δ.

Notice that if $\Delta \in I$, then $e(\Delta) = 0, q(\Delta) \le q(f)$ and therefore, by Proposition 2.5,

$$\dim A_\Delta = \sharp\Lambda(\Delta) \le \lfloor q(\Delta)/p \rfloor \le \lfloor q(f)/p \rfloor = \lfloor \mu/p \rfloor .$$

With f_t as in Remark 3.2, setting

$$\Delta_t := \{n \in \text{supp}(f_t) \mid n \le q(f)\}$$

for each $t \in \mathbb{A}^\mu$, we conclude that $\Delta_t \in I$ and $\Lambda(\Delta_t) = \Lambda(\mathrm{supp}(f_t))$ according to Remark 2.3(b). By Theorem 2.11, $f_t \sim_r h_{s_{\Delta_t}}$ for some s_{Δ_t}.

This implies that the finite set of families $h_{s_\Delta}(x), \Delta \in I$ satisfies the assumption of Lemma 3.3. Hence

$$\mathcal{R}\text{-mod}(f) \le \max_{\Delta \in I} \dim A_\Delta \le \lfloor \mu/p \rfloor .$$

In order to prove the other inequality we consider the two following cases.

Case 1: $m(f) = p$.

Then $q := q(f) = \mu(f) + 1, k := k(f) = \left\lfloor \frac{q-p}{p-1} \right\rfloor, d := d(f) = q + k - 1$ and

$$\Lambda(f) = \{n \in \mathbb{N} \mid q \le n \le d, e(n) = 0\}$$

and $\sharp\Lambda(f) = \lfloor q/p \rfloor$ due to Proposition 2.5. It follows from Theorem 2.11 that

$$f \sim_r g := x^p + \sum_{n \in \Lambda(f)} c_n x^n$$

for suitable $c_n \in K$ with $c_q \neq 0$. Consider the unfolding

$$g_\lambda := g + \sum_{n \in \Lambda(f)} \lambda_n x^n$$

of g over $S := \left\{\lambda = (\lambda_n)_{n \in \Lambda(f)} \in \mathbb{A}^{\sharp\Lambda(f)} \mid \lambda_q + c_q \neq 0\right\}$, where $\lambda_n, n \in \Lambda(f)$ are the coordinates of λ. Let us show that g_λ is a modular unfolding. In fact, if $\lambda' = (\lambda'_n)_{n \in \Lambda(f)} \in S$ for which $g_\lambda \sim_r g_{\lambda'}$, then there exists a coordinate change

$$\varphi := ax + a_l x^{l+1} + \dots$$

such that

$$\varphi(g_\lambda) = g_{\lambda'}.$$

Looking at the coefficient of x^p we deduce that $a^p = 1$ and therefore $a = 1$. We have moreover that $l \ge k$, because if $l < k$, equivalently, $q + l > p(l+1)$ then $p(l+1) \in \mathrm{supp}(\varphi(g_\lambda))$ but $p(l+1) \notin \mathrm{supp}(g_{\lambda'})$, that is $\varphi(g_\lambda) \neq g_{\lambda'}$, a contradiction. It then follows from Remark 2.2(d) that

$$j^d(g_\lambda) = j^d(\varphi(g_\lambda)) = j^d(g_{\lambda'}),$$

i.e. $\lambda = \lambda'$. This implies that g_λ is a modular unfolding and hence

$$\mathcal{R}\text{-mod}(f) = \mathcal{R}\text{-mod}(g) \ge \sharp\Lambda(f) = \lfloor q/p \rfloor = \lfloor \mu/p \rfloor$$

due to Lemma 3.4

Case 2: $m(f) > p$.

By the upper semicontinuity of the right modality (cf. [GN13, Prop. 2.7]) one has

$$\mathcal{R}\text{-mod}(f) \ge \mathcal{R}\text{-mod}(f_s)$$

with $f_s = f + s \cdot x^p$, for all s in some neighbourhood W of 0 in $\mathbb{A}^1$. Take a $s_0 \in W \setminus \{0\}$ then $\mathcal{R}\text{-mod}(f_{s_0}) = \lfloor \mu/p \rfloor$ by the first case and hence

$$\mathcal{R}\text{-mod}(f) \ge \mathcal{R}\text{-mod}(f_{s_0}) = \lfloor \mu/p \rfloor .$$

□

Remark 3.5. We have $\mathcal{R}\text{-mod}(f) \ge \sharp\Lambda(f)$ by Theorem 3.1 and Proposition 2.5 with equality if $m(f) \le p$. Moreover, if $m(f) = p$, then $f_\lambda \sim_r f_{\lambda'}$ for $\lambda, \lambda' \in \Lambda(f)$ implies $\lambda = \lambda'$, which follows from the proof of Theorem 3.1.

The example $f = x^{p+1}$ with $\mathcal{R}\text{-mod}(f) = 1$ but $\Lambda(f) = \emptyset$ shows that a strict inequality $\mathcal{R}\text{-mod}(f) > \sharp\Lambda(f)$ can happen.

With f and the semiuniversal unfolding $f_t(x)$ as in Remark 3.2 we define

$$\Delta_\mu := \{t \in \mathbb{A}^\mu \mid \mu(f_t) = \mu\}$$

the *μ-constant stratum* of the unfolding f_t.

Corollary 3.6. *Let $f \in \langle x \rangle \subset K[[x]]$ with the Milnor number $\mu < \infty$. Then*

$$\mathcal{R}\text{-mod}(f) = \dim \Delta_\mu.$$

Proof. For each $t = (t_1, \ldots, t_\mu) \in \mathbb{A}^\mu$, if the set $N_t := \{i = 1, \ldots, \mu \mid t_i \neq 0, e(i) = 0\}$ is not empty, then $\mu(f_t) = n - 1 < \mu$ with $n := \min\{i \mid i \in N_t\}$. This implies that

$$\Delta_\mu = \{t = (t_1, \ldots, t_\mu) \in \mathbb{A}^\mu \mid t_i = 0 \text{ if } e(i) = 0\}.$$

It yields that

$$\dim \Delta_\mu = \sharp \{1 \leq n \leq \mu \mid e(n) > 0\} = \lfloor \mu/p \rfloor$$

and hence $\mathcal{R}$-mod$(f) = \dim \Delta_\mu$ by Theorem 3.1. □

References

[Arn72] Arnol'd V. I., *Normal forms for functions near degenerate critical points, the Weyl groups of A_k, D_k, E_k and Lagrangian singularities,* Functional Anal. Appl. 6 (1972) 254-272. DOI: 10.1007/BF01077644

[BGM12] Boubakri Y., Greuel G.-M., and Markwig T., *Invariants of hypersurface singularities in positive characteristic,* Rev. Mat. Complut. 25 (2012), 61-85. DOI: 10.1007/s13163-010-0056-1

[GN13] Greuel G.-M., Nguyen H. D., *Right simple singularities in positive characteristic,* to appear in the Journal für die Reine und Angewandte Mathematik (2013). arXiv:1206.3742

[Ng13] Nguyen H. D., *Classification of singularities in positive characteristic,* Ph.D. thesis, TU Kaiserslautern, Dr. Hut-Verlag (2013). www.dr.hut-verlag.de/9783843911030.html

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THE GEOMETRY OF DOUBLE FOLD MAPS

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Abstract. We study the geometry of a family of singular map germs $(\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ called *double folds*. As an analogy to David Mond's *fold map germs* of the form

$$f(x, y) = (x, y^2, f_3(x, y)), f_3 \in \mathcal{O}_2,$$

double folds are of the form

$$f(x, y) = (x^2, y^2, f_3(x, y)).$$

This family provides lots of interesting germs, such as finitely determined homogeneous corank 2 germs. We also introduce analytic invariants adapted to this family.

1. Introduction

A classification of complex analytic map germs from the plane to 3-space under $\mathcal{A}$-equivalence, that is, changes of coordinates in the source and target, was carried out by David Mond [8]. Like in the work of a taxonomist, Mond's list starts with the simplest singular map germs, the so called *fold maps*. We say that a map germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ is a fold map if its first two coordinate functions form a *Whitney fold*, $T : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$, $T(x, y) = (x, y^2)$. The image of a fold map $f(x, y) = (x, y^2, f_3)$ looks like the graph of the function f_3 'folded' along the OX axis. The third coordinate function of a fold map can be any but, under $\mathcal{A}$-equivalence, we can assume that it is of the form yp, where $p = T^*P$ for some germ P in the ring of germs of functions in two variables $\mathcal{O}_2$. Hence, the normal form of a fold map is

$$f(x, y) = (x, y^2, yp).$$

Fold maps are easy to study because they are germs of corank 1 and because they behave well under the action of the group $G = \{1, i\}$, generated by the reflection $i(x, y) = (x, -y)$. One can see that all lifted double points of a double fold f (that is, pairs $(z, z') \in \mathbb{C}^2 \times \mathbb{C}^2$ such that $f(z) = f(z')$ and, if $z = z'$, then f is singular at z) are of the form $(z, i(z))$.

In this work we explore a family which is also related to a group, while it contains lots of interesting corank 2 maps. In general, corank 2 maps are much harder to study than corank 1 ones, but the group action and some ideas lent by the fold case are going to help us. To generate the simplest corank 2 maps for our studies, we can not allow linear terms in f. Thus, we are going to 'fold' twice, once through OX and once through OY axis. We denote $\alpha : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ the *folded hankerchief*

$$\alpha(x, y) = (x^2, y^2).$$

Take the reflections $i_1(x, y) = (-x, y)$ and $i_2(x, y) = (x, -y)$ and the rotation $i_3(x, y) = (-x, -y)$. We write G for the group $\{1, i_1, i_2, i_3\}$. The orbit of any $z \in \mathbb{C}^2$ is $Gz = \alpha^{-1}(\alpha(z))$ and z is a singular point of α if and only if z belongs to $Fix(i_1) \cup Fix(i_2) = OX \cup OY$. Now, related to the group G, we have a family of maps of the form

$$f(x, y) = (x^2, y^2, f_3(x, y)),$$

which we call *double folds*.

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Section 2 covers the basics about double folds. First we compute their multiple point schemes (this was first done by Marar and Nuño-Ballesteros, who introduced double folds in [5]). Then we introduce a decomposition of the multiple point spaces related to the group G. In Section 3 we restrict ourselves to the double fold family and define the notion of DF-stability (and that of SDF-stability). DF-stable singularities are the ones preserved by small perturbations inside the double fold world. We show that the DF-stable singularities are the stable singularities, plus another kind of singularities, namely the standard self tangencies (and also the standard quadruple points in the special double fold case). We introduce an equivalent notion, DF-genericity, to characterize DF-stability in terms of transversality conditions on the facets of the Coxeter complex of the group G. Section 4 deals with DF-stabilizations, where only DF-stable singularities appear. We use these deformations and the decomposition of the multiple point spaces given in 2 to relate certain numbers to double folds. These numbers are candidates for $\mathcal{A}$-invariants (up to a permutation of indices induced by an isomorphism of G). In Section 5 we consider general families of map germs $(\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$, constructed in the same manner as the folds and double folds: choosing a finite map germ $\alpha : \mathbb{C}^n \to \mathbb{C}^n$ and attaching any $(n+1)$-th coordinate function to obtain a map germ of the form (α, f_{n+1}). We find results relating the $\mathcal{A}$-equivalence of this kind of germs to some subgroup of $\mathcal{K}$-equivalence adapted to each α. These results imply that the numbers introduced in section 4 are $\mathcal{A}$-invariant among the finitely determined quasihomogeneous double folds.

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2. MULTIPLE POINT SCHEMES

Definition 2.1. We call *double fold* (abbreviated as *DF*) any map germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ of the form $f(x, y) = (x^2, y^2, f_3(x, y))$. The function germ $f_3 \in \mathcal{O}_2$ can be written in the form $f_3(x, y) = P_0(x^2, y^2) + xP_1(x^2, y^2) + yP_2(x^2, y^2) + xyP_3(x^2, y^2)$, for some $P_i \in \mathcal{O}_2$. Under $\mathcal{A}$-equivalence, we can eliminate P_0. Then we obtain a *double fold in normal form*

$$f(x, y) = (x^2, y^2, xp_1 + yp_2 + xyp_3),$$

with $p_i = \alpha^* P_i$, for some $P_i \in \mathcal{O}_2$. We call *special double folds* (abbreviated as *SDF*) the double folds in normal form such that $p_3 = 0$.

Example 2.2. Fold and double fold families are not exclusive. The cross-cap is usually parameterized as a fold in normal form $(x, y) \mapsto (x, y^2, xy)$, but it can also be regarded as double fold with parameterization $(x, y) \mapsto (x^2, y^2, x + y)$ (see figure 1).

Multiple point spaces were introduced by Mond [9] as a key tool to study map germs

$$(\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0),\ n < p.$$

Initial papers about map germs $(\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ (like [7], [8] and [9]) focussed mainly on the case of corank 1, but some recent ones (for instance [5], [6] and the present paper) deal with corank 2 germs. Altough this was done first by Marar and Nuño-Ballesteros, who introduced double folds in [5], we shall sumarize here the computations of some of their multiple point spaces for a better understanding.

Multiple point spaces in the target are computed as described in [10]. Let $f : X \to (\mathbb{C}^{n+1}, 0)$ be a finite map germ, where X is a n-dimensional Cohen-Macaulay space. Let $f_*\mathcal{O}_X$ denote $\mathcal{O}_X$ as $\mathcal{O}_{n+1}$-module via f. The k-multiple point space in the target is given by the $(k-1)$-th

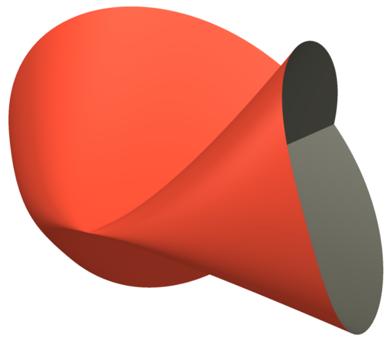

FIGURE 1. The cross-cap is a double fold.

Fitting ideal of the module $f_*\mathcal{O}_X$ defined next: Take a presentation of $f_*\mathcal{O}_X$, that is, an exact sequence

$$\mathcal{O}_{n+1}^p \xrightarrow{\lambda} \mathcal{O}_{n+1}^q \xrightarrow{\varphi} f_*\mathcal{O}_X \longrightarrow 0.$$

The matrix $M(f)$ which represents λ is called a *presentation matrix* for $f_*\mathcal{O}_X$. The k-th Fitting ideal of $f_*\mathcal{O}_X$ is the ideal $F_k(f)$ generated by the minors of size $\min(p,q)-k$ of $M(f)$ if $k<\min(p,q)$, and $F_k(f)=\mathcal{O}_{n+1}$ otherwise. The following method to compute certain presentation matrices can be found in [10, Section 2.2]: Assume $f=(f_1,\dots,f_{n+1}):X\to\mathbb{C}^{n+1}$ is such that $\tilde{f}=(f_1,\dots f_n):X\to(\mathbb{C}^n,0)$ is finite. If $g_1,\dots g_r$ are generators of $\tilde{f}_*\mathcal{O}_X$, then they are generators of $f_*\mathcal{O}_X$ too. Therefore, we obtain an epimorphism $\varphi:\mathcal{O}_{n+1}^r\to\mathcal{O}_X$ which sends the canonical vector e_i to the generator g_i. For any $1\le i\le r$, there exist germs $a_{ij}\in\mathcal{O}_n$, $1\le j\le r$ such that $f_{n+1}g_i=\sum_{j=1}^r\tilde{f}^*a_{ij}g_j$. If $X_1,\dots,X_{n+1}$ denote the variables in $\mathbb{C}^{n+1}$ and δ_{ij} is the Kronecker's delta function, then the matrix $M(f)$ with entries $a_{ij}(X_1,\dots,X_n)-\delta_{ij}X_{n+1}$ is a presentation matrix for $f_*\mathcal{O}_X$.

Given a double fold $f(x,y)=(x^2,y^2,xp_1+yp_2+xyp_3)$, we use the method explained above to find $M(f)$. Take $g_1=1,g_2=x,g_3=y,g_4=xy$ as generators of $\alpha_*\mathcal{O}_2$. For $i=1$, we have $f_3g_1=xp_1+yp_2+xyp_3=0\cdot g_1+\alpha^*P_1g_2+\alpha^*P_2g_3+\alpha^*P_3g_4$. Therefore, the elements of the first column of the matrix are $-Z,P_1,P_2,P_3$. After computing f_3g_i for $i=2,3,4$, we get the matrix

$$M(f)=\begin{pmatrix} -Z & XP_1 & YP_2 & XYP_3\\ P_1 & -Z & YP_3 & YP_2\\ P_2 & XP_3 & -Z & XP_1\\ P_3 & P_2 & P_1 & -Z\end{pmatrix},$$

where P_i represents $P_i(X,Y)$. Since $M(f)$ has size 4×4, f has no points with multiplicity greater than 4. For special double folds, the space of quadruple points in the image is given by the ideal $F_3(f)=\langle P_1(X,Y),P_2(X,Y),Z\rangle$ and $F_2(f)=(F_3(f))^2$. Hence, triple points of special double folds appear concentrated at quadruple points.

We define the source double point space $D(f)$ as the zero locus of the pull back $f^*(F_1(f))$. In the double fold case we have $D(f)=V\big((p_1+yp_3)(p_2+xp_3)(xp_1+yp_2)\big)$. Its defining ideal factorizes as the product of the ideals $I_1:=\langle p_1+yp_3\rangle$, $I_2:=\langle p_2+xp_3\rangle$ and $I_3:=\langle xp_1+yp_2\rangle$. Analogously, the source triple point space, defined as $V(f^*(F_2(f))$, is given by the product of the ideals $I_{1,2}:=\langle p_1+yp_3,p_2+xp_3\rangle$, $I_{1,3}:=\langle p_1+yp_3,p_2-xp_3\rangle$ and $I_{2,3}:=\langle p_2+xp_3,p_1-yp_3\rangle$. Quadruple points (again with the structure induced by the target) are given by the zeros of

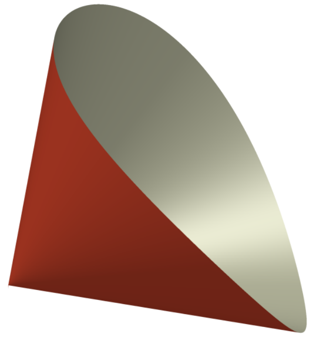

FIGURE 2. The image of a double cone.

$I := \langle p_1, p_2, p_3 \rangle$. We observe the collapse of triple points in the special double fold case: If p_3 equals zero, then the radical of $I_{1,2}I_{1,3}I_{2,3}$ is $\langle p_1, p_2 \rangle$, which is the ideal defining the quadruple point locus.

Definition 2.3. Given a double fold $f = (\alpha, xp_1 + yp_2 + xyp_3)$, we decompose the double point locus as the union of $D_i(f), 1 \leq i \leq 3$, with $D_i(f) := V(I_i)$ and the triple point space as the union of $D_{i,j}(f), 1 \leq i < j \leq 3$, with $D_{i,j} := V(I_{i,j})$. Finally, we denote $D_{1,2,3}(f) = V(I_{1,2,3})$ the quadruple point locus.

Remark 2.4. It's immediate that:

- w belongs to $D_l(f)$ if and only if $i_l(w)$ does so. Moreover $f(w) = f(i_l(w))$.
- w belongs to $D_{l,k}(f)$ if and only if $i_l(w)$ and $i_k(w)$ do so. Moreover

$$f(w) = f(i_l(w)) = f(i_k(w)).$$

- w belongs to $D_{1,2,3}(f)$ if and only if $i_1(w), i_2(w)$ and $i_3(w)$ do so. Moreover

$$f(w) = f(i_1(w)) = f(i_2(w)) = f(i_3(w)).$$

Example 2.5. Take the family $(x, y) \mapsto (x^2, y^2, \lambda_1 x + \lambda_2 y + \lambda_3 xy), \lambda_i \in \mathbb{C}$. Assume $\lambda_3 \neq 0$, then its double points are the following: $D_1(f) = V(\lambda_1 + y\lambda_3)$ is the line $y = -\lambda_1/\lambda_3$, which is obviously i_1-invariant, $D_2(f)$ is the i_2-invariant line $x = -\lambda_2/\lambda_3$ and, if $\lambda_2 \neq 0$, then $D_3(f)$ is the i_3-invariant line $y = -\lambda_1 x/\lambda_2$. We find the triple points where these lines meet:

$$D_{1,2}(f) = \{(-\lambda_2/\lambda_3, -\lambda_1/\lambda_3)\}, \quad D_{1,3}(f) = \{(\lambda_2/\lambda_3, -\lambda_1/\lambda_3)\}$$

and

$$D_{2,3}(f) = \{(-\lambda_2/\lambda_3, \lambda_1/\lambda_3)\}$$

(see figure 3). In the case $\lambda_3 = 0$ we have a special double fold. Thus, its triple points should appear collapsed at quadruple points, with equations $p_1 = p_2 = 0$. Since $p_1 = \lambda_1$ and $p_2 = \lambda_2$, the appearance of quadruple point forces $\lambda_1 = \lambda_2 = 0$ and hence, the map is the folded hankerchief. Another map that fits into this family is the so called *double cone* $(x, y) \mapsto (x^2, y^2, xy)$ (Figure 2). It parameterizes the cone $Z^2 = XY$, but does so in a two-to-one way. Indeed, its double point branch $D_3(f) = V(xp_1 + yp_2) = V(0)$ equals $\mathbb{C}^2$.

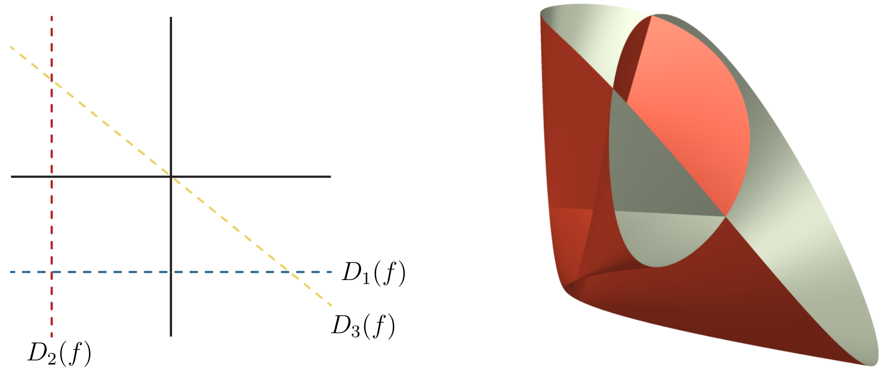

FIGURE 3. Image and double points of a double fold (see Example 2.5).

3. DOUBLE FOLD STABILITY

In this section we study the singularity types which are characteristic of the double folds. By a singularity type we mean an $\mathcal{A}$-equivalence class of multigerms $f : (\mathbb{C}^2, S) \to (\mathbb{C}^3, y)$. A singularity type, represented by f_0, is stable if it appears in any section $f_s, s \in \mathbb{C}$, of any deformation of f_0. It is well known that in the case $\mathbb{C}^2 \to \mathbb{C}^3$ the stable types are transverse double points, triple points and cross-caps. Our goal is to make a version of the concept of stability adapted specifically for double folds. Some types, despite not being stable, are preserved by deformations which occur inside the double fold world. We call them DF-stable types and these deformations DF-deformations. This concept can be adapted to the special double fold case and we shall use the notation (S)DF to refer respectively to both, the double fold and the special double fold case.

Definition 3.1. We call *(S)DF-deformation* of f_0 any germ $F : (\mathbb{C}^2 \times \mathbb{C}, 0) \to (\mathbb{C}^3, 0)$ of the form $F(x,t) = f_t(x)$, such that the germ $f_t : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ is a (special) double fold for all t. We call (S)DF-unfolding any map germ $F : (\mathbb{C}^2 \times \mathbb{C}, 0) \to (\mathbb{C}^3 \times \mathbb{C}, 0)$ of the form $F(x,t) = (f_t(x), t)$ such that $f_t(x)$ is a (S)DF-deformation.

Definition 3.2. We say a multigerm ξ is *(S)DF-stable* if any (S)DF-unfolding F of a multigerm f of type ξ is trivial. That is, if there exist some unfoldings of the identity Ψ, Φ such that $f \times id = \Psi \circ F \circ \Phi$. A (special) double fold $f : U \to \mathbb{C}^3$ is (S)DF-stable if all its multigerms at $f^{-1}(f(w)), w \in U$ are (S)DF-stable.

Remark 3.3. Every stable type is (S)DF-stable.

A priori, it might seem difficult to identify all possible (S)DF-stable maps, but a better understanding of the map α will help us to do so. The map α is the invariant map associated to the Coxeter group G (see [3] for Coxeter group theory). For any Coxeter Group there is a Coxeter complex, in this case $\mathcal{C} := \{\mathbb{C}^2 \setminus (OX \cup OY), OX \setminus \{0\}, OY \setminus \{0\}, \{0\}\}$. The Coxeter complex stratifies the space in a way such that the behavior of the group, and thus that of α, changes whenever we go from a facet to another. Consequently, much information about a double fold is contained in the way its multiple point spaces meet the Coxeter complex. The following proposition is an example of this.

Lemma 3.4. *The germ of a fold $f(x,y) = (x^2, y^2, xp_1 + yp_2 + xyp_3)$ centered at a point $w \in \mathbb{C}^2$ is a cross-cap if and only if one of the three conditions is verified:*

i) $w \in OX \setminus \{0\}$ *and the restricted function* $(p_2 + xp_3)|_{OX}$ *has a simple zero at* w.
ii) $w \in OY \setminus \{0\}$ *and the restricted function* $(p_1 + yp_3)|_{OY}$ *has a simple zero at* w.
iii) $w = 0$ *and* $p_1(w) \neq 0 \neq p_2(w)$.

Proof. A monogerm of map from $\mathbb{C}^2$ to $\mathbb{C}^3$ is a cross-cap if and only if its source double point space is smooth (this follows immediately from [6, Theorem 3.3]). Since cross-caps are singular monogerms, they lie on $OX \cup OY$. Assume first that $w \in OX \setminus \{0\}$. Looking at the 2×2 minors of the differential of f at w it follows that f is singular at w if and only if $p_2 + xp_3$ vanishes at w. Now the source double point space of the germ of f at w is $D_2(f)$, given by the zeros of $p_2 + xp_3$ (notice that, by Remark 2.4, the branches of double points $D_1(f)$ and $D_3(f)$ at $OX \setminus \{0\}$ produce multigerms, not monogerms). Therefore, the double point space of the germ of f at w is smooth if and only if the Milnor number of the germ of function $p_2 + yp_3$ at w equals 0. This happens if and only if at least one of the partial derivatives $\frac{\partial p_2 + xp_3}{\partial x}$ and $\frac{\partial p_2 + xp_3}{\partial y}$ does not vanish at w. Since p_2 and p_3 are functions of x^2 and y^2, we deduce that $\frac{\partial p_2 + xp_3}{\partial y}$ vanishes at OX. Hence, f has a cross-cap at $w \in OX \setminus \{0\}$ if and only $p_2 + xp_3$ vanishes at w and $\frac{\partial p_2 + xp_3}{\partial x}$ does not, that is, if and only if the restriction $(p_2 + xp_3)|_{OX}$ has a simple zero at w. The case $w \in OY \setminus \{0\}$ is analogous. Assume now $w = 0$. The source double point of f is the germ of complex space given by the zeros of $(p_1 + xp_3)(p_2 + p_3)(xp_1 + yp_2)$. The non vanishing of p_1 and p_2 at 0 is a necessary and sufficient condition for this germ of complex space to be smooth. □

Points where the source double point space meets the facets of the Coxeter complex in a generic way are called (S)DF-generic. We shall determine the different possible (S)DF-generic singularities and then show that they are exactly the (S)DF-stable singularities. Let us state the (S)DF-genericity conditions rigorously:

Definition 3.5. Let $f = (\alpha, xp_1 + yp_2 + xyp_3) : U \to \mathbb{C}^3$ be a double fold. We say that a point $w \in \mathbb{C}^2$, that belongs to a facet $C \in \mathcal{C}$, is *DF-generic* if:

1) $(p_1 + yp_3)|_C$, $(p_2 + xp_3)|_C$ and $(xp_1 + yp_2)|_C$ are transverse to $\{0\}$ at w, with the exception $(xp_1 + yp_2)|_{\{0\}}$ (notice that no double fold in canonical form could verify this transversality condition).
2) $(p_1 + yp_3, p_2 + xp_3)|_C$, $(p_1 + yp_3, p_2 - xp_3)|_C$ and $(p_2 + xp_3, p_1 - yp_3)|_C$ are transverse to $\{(0, 0)\}$ at w.
3) w is not a quadruple point of f.

A double fold $f : U \to \mathbb{C}^3$ is DF-generic if all points $w \in U$ are DF-generic

Conditions 1) and 2) adapt to the special double fold case just taking $p_3 = 0$ but, since quadruple points are more likely to appear at special double folds (they are the zeros of just two equations in $\mathbb{C}^2$), the SDF genericity conditions don't include condition 3).

Definition 3.6. Let $f = (\alpha, xp_1 + yp_2) : U \to \mathbb{C}^3$ be a special double fold, we say that a point $w \in \mathbb{C}^2$, that belongs to a facet $C \in \mathcal{C}$, is *SDF-generic* if:

1) $p_1|_C$, $p_2|_C$ and $(xp_1+yp_2)|_C$ are transverse to $\{0\}$ at w, with the exception $(xp_1+yp_2)|_{\{0\}}$.
2) $(p_1, p_2)|_C$ is transverse to $\{(0, 0)\}$ at w.

A special double fold $f : U \to \mathbb{C}^3$ is SDF-generic if all points $w \in U$ are SDF-generic

Remark 3.7. It is immediate from its defining ideals that every point belonging to $D_1(f) \cap OX$ or to $D_2(f) \cap OY$ must belong to $D_3(f)$ too. It is also immediate that $D_3(f)$ always crosses the facet $\{0\}$. Apart from these exceptions, which are inherent to the double fold family, the genericity conditions imply the following more geometric assertion: Given a regular stratification of $D(f)$, the strata have their expected dimension (double points have dimension 1 and triple (quadruple) points have dimension 0) and are transverse to the strata of the Coxeter complex $\mathcal{C}$.

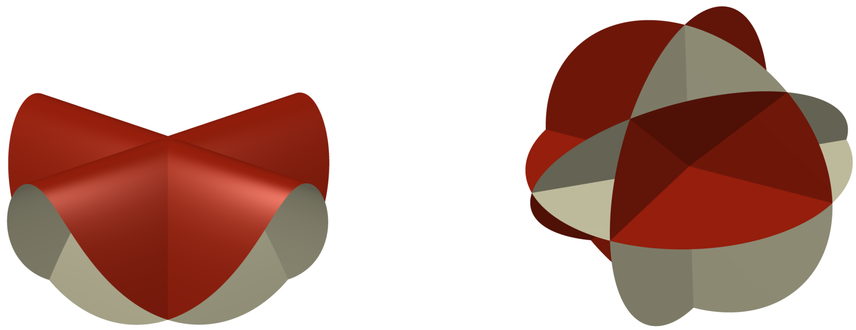

FIGURE 4. Images of a standard self tangency and a standard quadruple point.

Let us introduce our new candidates to be (S)DF-generic multigerms.

Definition 3.8. We call a *standard self tangency* the multigerm formed by two smooth branches with Morse contact. We call a *standard quadruple point* the multigerm formed by four smooth branches such that every three of them meet transversally. These singularities are depicted in Figure 4.

Proposition 3.9. *All standard self tangencies are $\mathcal{A}$-equivalent. All standard quadruple points are $\mathcal{A}$-equivalent.*

Proof. In [12] it is shown that the $\mathcal{A}$-class of a bigerm with smooth branches is determined by the contact type of its branches. Since there is only one contact class of Morse type, all standard self tangencies are equivalent. Let f be a multigerm of standard quadruple point. Any three of its branches form a triple point and there is only one $\mathcal{A}$-class of triple points. Therefore, there exists a change of coordinates that takes f to a multigerm whose branches send (x, y) respectively to $(x, y, 0), (x, 0, y), (0, x, y)$ and $g(x, y)$ for some regular monogerm g with $\operatorname{Im} g = \{U_1X + U_2Y + U_3Z = 0\}$, $U_i \in \mathcal{O}_3$. The plane tangent to $\operatorname{Im} g$ is determined by the equation $t_1X + t_2Y + t_3Z = 0$, with $t_i = U_i(0, 0)$. If we assume $t_1 = 0$, then the intersection of the tangent plane with the branches $\{Y = 0\}$ and $\{Z = 0\}$ is the line $\{Y = Z = 0\}$. This contradicts the transversality of these three branches. We deduce $t_1 \neq 0$ and, analogously, $t_2 \neq 0 \neq t_3$. The change $(X, Y, Z) \mapsto (U_1X, U_2Y, U_3Z)$ defines a germ of diffeomorphism that takes our multigerm to the one with image $\{XYZ(X + Y + Z) = 0\}$. Now the four branches of our multigerm send (x, y) to $(u_1x, u_2y, 0), (u_1x, 0, u_3y), (0, u_2x, u_3y)$ and

$$(a_1x + b_1y, a_2x + b_2y, -(a_1 + b_1)x - (a_2 + b_2)y),$$

where $u_i = U_i \circ f$, and a_1, a_2, b_1, b_2 are some function germs in $\mathcal{O}_2$. We take germs of diffeomorphisms at the source, at the four different points where our multigerm is centered. The first three diffeomorphisms send (x, y) respectively to $(x/u_1, y/u_2)$, $(x/u_1, y/u_3)$ and $(x/u_2, y/u_3)$. The fourth diffeomorphism is the inverse of the germ

$$(x, y) \mapsto (a_1x + b_1y, a_2x + b_2y).$$

These four source coordinate changes take the multigerm to one multigerm defined by four branches sending (x, y) respectively to $(x, y, 0), (x, 0, y), (0, x, y)$ and $(x, y, -x - y)$. Hence, all germs of standard quadruple point are equivalent. □

Lemma 3.10. *The (S)DF-generic points are regular points, transverse double points, cross-caps, standard self tangencies and triple points (resp. standard quadruple points).*

Proof. Given a (special) double fold f and a point $w = (x_0, y_0) \in \mathbb{C}^2$ satisfying the (S)DF-genericity conditions, we shall determine the type of singularity of the multigerm of f at $f^{-1}(f(w))$. First of all, notice that singular points lie in $OX \cup OY$ and the genericity condition 2) implies that all triple points belong to the facet $\mathbb{C}^2 \setminus (OX \cup OY)$. Hence, from genericity condition 1), together with Lemma 3.4, it follows that all points where f is singular are cross-caps.

Now suppose that f is regular at w and the point w belongs to $D_l(f)$, $1 \le l \le 3$. Take the vector fields along f defined by the cross product $\eta := \frac{\partial f}{\partial x} \times \frac{\partial f}{\partial y}$ and $\eta_l = \eta \circ i_l$, for $1 \le l \le 3$. The branches of the multigerm of f at w and $i_l w$ are transverse unless $\eta \times \eta_l$ or, equivalently, $\xi_l := (\eta - \eta_l) \times (\eta + \eta_l)$ vanish at w. We study the different cases a), b) and c), where w belongs to $D_1(f), D_2(f)$ and $D_3(f)$ respectively.

Case a) Let w belong to $D_1(f)$, then we have:
$\xi_1|_w = 4x_0y_0(4x_0\frac{\partial(xp_1+xyp_3)}{\partial y}|_w, 4y_0\frac{\partial(xp_1+xyp_3)}{\partial x}|_w, (\frac{\partial(xp_1+xyp_3)}{\partial y}|_w\frac{\partial yp_2}{\partial x}|_w - \frac{\partial(xp_1+xyp_3)}{\partial x}|_w\frac{\partial yp_2}{\partial y}|_w))$.
Suppose first $w \notin OX \cup OY$, then $\xi_1|_w$ vanishes if and only if $\frac{\partial p_1+yp_3}{\partial x}|_w = \frac{\partial p_1+yp_3}{\partial y}|_w = 0$, that is, if and only if $p_1 + yp_3$ is not transverse to $\{0\}$ at w. This is in contradiction with the first genericity condition. Suppose now $w \in OX \cup OY$ and notice $w \notin OY$ because it would be a singular point. Thus, we have $w \in OX \setminus \{0\}$. We claim that the bigerm of f at $(\pm x_0, 0)$ forms a standard self tangency at $(X_0, 0, 0)$, where $X_0 = x_0^2$. The genericity conditions imply that P_1 has a simple zero at $(X_0, 0)$ and P_2 does not vanish at $(X_0, 0)$. Let the germ of $f : \mathbb{C}^2 \to \mathbb{C}^3$ at x_0 parameterize one of the branches and let $\phi : \mathbb{C}^3 \to \mathbb{C}$ be the germ at $(X_0, 0, 0)$ which defines the other branch implicitly. Then, following Montaldi [11], the contact between the branches is given by the $\mathcal{K}$-class of the composition $\phi \circ f$. The branches are given by $(Z^2 \pm \sqrt{X}P_1)^2 - YP_2^2 \pm 2Y\sqrt{X}P_2P_3 - XYP_3^2 = 0$. After choosing the preimage $(x_0, 0)$ and composing we get the function $4x(p_1 + yp_3)(xp_1 + yp_2)$, which is of Morse type in $(x_0, 0)$. Therefore, the multigerm of f at $(\pm x_0, 0)$ is a standard self tangency.

Case b) is symmetric interchanging indices 1 and 2, and OX and OY.

Case c) If $w \in D_3(f)$, then we can assume $w \in D_3(f) \setminus (OX \cup OY)$ because otherwise $w \in D_1(f) \cup D_2(f)$. We have

$$\xi_3|_w = 4x_0y_0\left(4x_0\frac{\partial(xp_1+yp_2)}{\partial y}|_w,\ -4y_0\frac{\partial(xp_1+yp_2)}{\partial x}|_w,\right.$$
$$\left.\Big(\frac{\partial(xp_1+yp_2)}{\partial y}|_w\frac{\partial xyp_3}{\partial x}|_w - \frac{\partial(xp_1+yp_2)}{\partial x}|_w\frac{\partial xyp_3}{\partial y}|_w\Big)\right),$$

which vanishes if and only if $\frac{\partial xp_1+yp_2}{\partial x}$ and $\frac{\partial xp_1+yp_2}{\partial y}$ vanish in w, if and only if $xp_1 + yp_2$ is not transverse to $\{0\}$ at w.

As we have seen before, all triple points (and therefore all quadruple points) belong to the facet $\mathbb{C}^2 \setminus (OX \cap OY)$, where the second genericity condition implies that the branches are transverse. Therefore, all triple points are transverse (respectively all quadruple points are standard quadruple points). □

Lemma 3.11. *Every (special) double fold admits a (S)DF-deformation f_t defined in a neighborhood $U \times V$ of $(0,0) \in \mathbb{C}^2 \times \mathbb{C}$ such that, for every $t \in V$, f_t is (S)DF-generic.*

Proof. Let $f = (\alpha, xp_1 + yp_2 + xyp_3)$ be a representative defined at some neighborhood U of the origin. we consider DF-deformations of the form $f_{a,b,c} = (\alpha, x(p_1 + a) + y(p_2 + b) + xy(p_3 + c))$. Denote Δ the analytic space of the points $(a, b, c) \in \mathbb{C}^3$, such that for some point w in U the map $f_{a,b,c}$ does not satisfy all genericity conditions. We claim that Δ is a proper subspace of $\mathbb{C}^3$. Take the first function, p_1+yp_3, of the first condition and any facet of the Coxeter complex $C \in \mathcal{C}$. We consider the map $\psi : C \times \mathbb{C}^3 \to \mathbb{C}$, given by $\psi(w, a, b, c) = p_1(w)+a+y(p_3(w)+c)$. This is clearly a submersion. Therefore, the Basic Transversality Lemma [2, Lemma 4.6] tells us that, for almost every $(a,b,c) \in \mathbb{C}^3$, the map $f_{a,b,c}$ is transverse to 0. We can proceed analogously for all the maps given by the DF-genericity conditions to finally show that, for almost every $(a,b,c) \in \mathbb{C}^3$, all the genericity conditions hold at every point in U. Thus, Δ is a proper subspace. Hence, we can find some particular $(a,b,c) \in \mathbb{C}^3$ and some neighborhood V of 0, such that $t(a,b,c) \notin \mathbb{C}^3$ for any $t \in V$. If we take the DF-deformation

$$f_t(x,y) = (x^2, y^2, x(p_1+ta) + y(p_2+tb) + xy(p_3+tc)$$

defined at $U \times V$, then for any $t \in V$, the map f_t has only DF-generic points at U. The special double fold case is analogous. □

Theorem 3.12. *(S)DF-stable and (S)DF-generic points are the same. As a consequence:*
The DF-stable singularities are

- *Transverse double points, cross-caps and triple points.*
- *Standard self tangencies.*

The SDF-stable singularities are

- *Transverse double points and cross-caps.*
- *Standard self tangencies.*
- *Standard quadruple points.*

Proof. By Lemma 3.11, the DF-stable singularities must be DF-generic. Now take a DF-generic point w of a double fold f. If w is a transverse double point, a cross-cap or a triple point, then it is stable and, hence, DF-stable. Suppose w is a standard self tangency and Let $F = (f_t, t)$ be a DF-unfolding of f. Assume $w \in D_1(f)$. Then, as we have seen in the proof of Lemma 3.10, the point belongs to $OX \setminus \{0\}$, $(p_1 + yp_3)|_{OX}$ has a simple zero at w and the functions $p_2 + xp_3$ and $xp_1 + yp_2$ don't vanish at w. Therefore, there exist a neighborhood $U \times V$ of $(w, 0)$ and a curve of points $w_t \in U \cap OX \setminus \{0\}$, with $t \in V$ and $w_0 = w$, such that $(p_1 + yp_3)|_{OX}$ has a simple zero and the functions $p_2 + xp_3$ and $xp_1 + yp_2$ don't vanish at w_t. All this points are also standard self tangencies and, since they are all $\mathcal{A}$-equivalent by 3.9, they are DF-stable. The proof holds in the special case and is analogous for standard quadruple points. □

4. Counting (S)DF-stable points

A usual way to study germs is to count the number of stable 0-dimensional points of each type which appear in a stabilization of the original germ. One can show that these numbers can be obtained as the dimension (as $\mathbb{C}$-vector space) of certain local algebras related to the different stable 0-dimensional types. We adapt these techniques specifically to (S)DF-deformations and to (S)DF-stable points.

Definition 4.1. We call *(S)DF-stabilization* any (S)DF-deformation F such that there exists a neighborhood $U \times V$ of $(0,0) \in \mathbb{C}^2 \times \mathbb{C}$ such that, for every $t \in V$, f_t is (S)DF-stable.

Remark 4.2. By Lemma 3.11 and Theorem 3.12, every (special) double fold admits a (S)DF-stabilization.

Definition 4.3. For any (special) double fold f we define:

$ST_i(f) = \frac{1}{2}\dim_{\mathbb{C}} \mathcal{O}_1/j_i^* I_i(f)$, for $i = 1,2$,
$C_i(f) = \dim_{\mathbb{C}} \mathcal{O}_1/j_k^* I_i(f)$, for $(i,k) = (1,2),(2,1)$,
$T(f) = \dim_{\mathbb{C}} \mathcal{O}_2/I_{1,2}(f)$ (in the special DF case: $QD(f) = \frac{1}{4}\dim_{\mathbb{C}} \mathcal{O}_2/\langle p_1, p_2\rangle$),

where j_1 and j_2 denote the inclusions of OX and OY into $\mathbb{C}^2$ respectively.

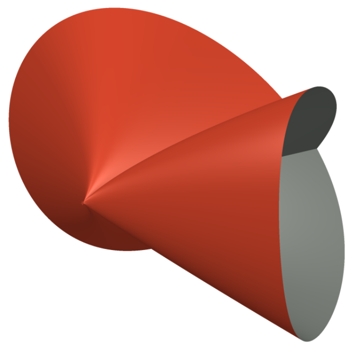

FIGURE 5. A non SDF-stable special double fold (see Example 4.6).

Remark 4.4. We don't include indices for the triple points in different branches because the complex spaces $D_{i,j}(f)$ are all isomorphic, since $\mathcal{O}_2/I_{1,2}(f) \cong \mathcal{O}_2/I_{1,3}(f) \cong \mathcal{O}_2/I_{2,3}(f)$ via the isomorphisms induced by i_1 and i_2.

Proposition 4.5. *Let $ST_i(f), C_i(f)$ and $T(f)$ (respectively $QD(f)$) be finite. Let f_s be a (S)DF-stabilization of f. Then, for a small enough $s \neq 0$, the following equalities hold:*

$ST_i(f) = \#$ *standard self tangencies* $f(D_i(f_s))$,
$C_i(f) = \#$ *cross-caps in* $D_i(f_s) \setminus \{0\}$,
$T(f) = \#$ *triple points of* f_s ($QD(f) = \#$ *standard quadruple points of* f_s).

Proof. Take the zero locus of the different ideals which appear in 4.3. If $ST_i(f), C_i(f)$ and $T(f)$ (respectively $QD(f)$) are finite, then the spaces are 0-dimensional. In this case, the codimension of any of these spaces equals the number of generators of its defining ideal. Hence, the spaces are complete intersection and the Principle of Conservation of Number (see for example [4, Theorem 6.4.7]) applies to them. We only need to check that, if the multigerm of f_s at $f_s^{-1}(f_s(w))$ is (S)DF-generic, then the numbers are 1 if it is the considered singularity and 0 otherwise. □

Example 4.6. Take the family of special double folds

$$(x, y) \mapsto (x^2, y^2, x(a_1x^2 + b_1y^2 - c_1) + y(a_2x^2 + b_2y^2 - c_2)).$$

The double points $D_1(f)$ and $D_2(f)$ are given by $a_1x^2 + b_1y^2 = c_1$ and $a_2x^2 + b_2y^2 = c_2$. In the real case, these two spaces collapse to the point 0 if $c_1 = c_2 = 0$. For the germ

$$f(x, y) = (x^2, y^2, x(x^2 + 2y^2) + y(2x^2 + y^2))$$

(Figure 5), we can easily compute $ST_1 = 1/2 \dim_{\mathbb{C}}(\mathcal{O}_1/\langle x^2\rangle) = 1$ and similarly $ST_2 = 1$ and $C_1 = C_2 = 2$. We also have $QD = 1/4 \dim_{\mathbb{C}}(\mathcal{O}_2/\langle 2x^2 + y^2, 2y^2 + x^2\rangle) = 1$. Now take the 2-parameter deformation $f_t = (x^2, y^2, x(x^2 + 2y^2 - t_1) + y(2x^2 + y^2 - t_2))$, where $t = (t_1, t_2)$. We see that, for almost every fixed t with $t_1 \neq 0 \neq t_2$, f_t is a SDF-stable map where we can find (Figure 6) a standard self tangency and two cross-caps along $D_1(f_t) \setminus \{0\}$ and the same on $D_2(f_t) \setminus \{0\}$. We also see the cross-cap at $f_t(0)$ and a standard quadruple point. For these good values of t we can also see that, apart from the restrictions on $D_i(f) \cap D_3(f)$ and $D_3(f) \cap \{0\}$ (see Remark 3.7), the regular stratification of $D(f_t)$ is transverse to every facet of the Coxeter complex.

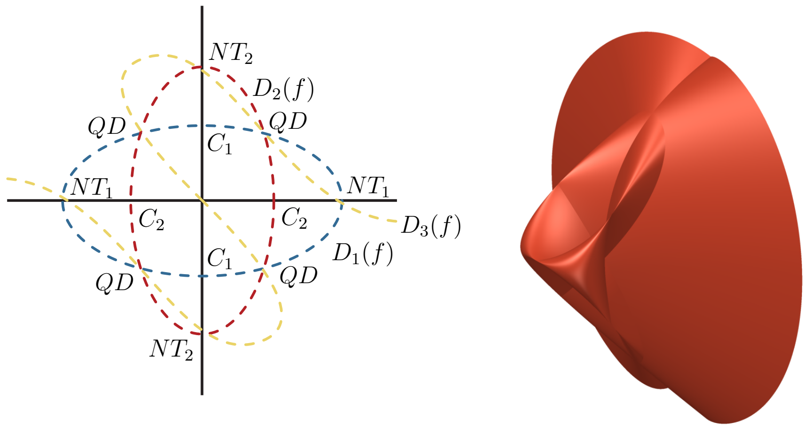

FIGURE 6. A SDF-stable deformation of the surface shown in figure 5.

Example 4.7. If we take the double cone $(x, y) \mapsto (x^2, y^2, xy)$ of Example 2.5, we see easily that $ST_i = 0$, $C_i = \dim_{\mathbb{C}} \mathcal{O}_1/\mathfrak{m}_1 = 1$ for $i = 1, 2$ and $T = \dim_{\mathbb{C}} \mathcal{O}_2/\mathfrak{m}_2$. In fact

$$f_t(x, y) = (x^2, y^2, tx + ty + xy)$$

is a DF-stabilization of the double cone where each section $t \neq 0$ has, as in figure 3, three cross-caps (one in $D_1(f) \setminus \{0\}$, one in $D_2(f) \setminus \{0\}$ and the other at 0) and one triple point.

Remark 4.8. Let $ST(f)$, $C(f)$, $T(f)$ (and respectively $QD(f)$ in the special case) denote the number of standard self tangencies, cross-caps, triple points (and standard quadruple points) respectively that appear taking a (S)DF-stabilization of f. It is known that $C(f)$ and $T(f)$ are well defined $\mathcal{A}$-invariants of f. It is immediate that $Q(f)$ is also invariant, because any map showing a quadruple point can be deformed (outside the special double fold world) into another that shows 4 triple points. It is not clear whether ST is $\mathcal{A}$-invariant or not, but it is easy to see that the numbers with indices $ST_i(f)$ and $C_i(f)$ are not. Given a double fold f, we can interchange x and y at the source and then permute the first two coordinates at the target to obtain a new double fold, say g, such that $ST_1(f) = ST_2(g)$, $ST_2(f) = N_1(g)$, $C_1(f) = C_2(g)$ and $C_2(f) = C_1(g)$. Apart from the permutation of indices 1 and 2 that this change of coordinates produces, examples suggest that changes of coordinates don't make the singularities jump from one space $D_i(f)$ to another one. Therefore, the numbers $ST_i(f)$ and $C_i(f)$ seem to be $\mathcal{A}$-invariant, modulo a simultaneous permutation of all indices 1 and 2 (and that would make ST $\mathcal{A}$-invariant). However, we have only succeeded in showing it for finitely determined quasi homogeneous double folds (Corollary 5.6).

5. $\mathcal{A}$-EQUIVALENCE AND $\mathcal{K}^{\alpha}$-EQUIVALENCE

The aim of this section is to mimic a result of David Mond [8, Theorem 4.1:1], which shows the coincidence between the $\mathcal{A}$-equivalence of folds $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$, $f(x, y) = (x, y^2, f_3)$ and some easier to use equivalence of the third coordinate function, f_3, defined ad hoc. This equivalence is given by a subgroup of $\mathcal{K}$ called $\mathcal{K}^T$ which behaves well with respect to the Whitney Fold $T(x, y) = (x, y^2)$. We take, instead of the Whitney Fold, any finite mapping $\alpha : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ and consider mappings $(\alpha, f_{n+1}) : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$. We define the group $\mathcal{K}^\alpha$ and the generalization of one direction of Mond's results comes easily: $\mathcal{K}^\alpha$-equivalence for f_{n+1} implies $\mathcal{A}$-equivalence for (α, f_{n+1}).

As usual, we denote $\mathcal{R}_n$ the group of germs of biholomorphism $\varphi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$.

Definition 5.1. Let $\alpha : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ be a finite germ. We define $\mathcal{R}^\alpha$ as the subgroup consisting of the germs $\varphi \in \mathcal{R}_n$ such that there exists a germ $\hat{\varphi} \in \mathcal{R}_n$ such that

$$\hat{\varphi} \circ \alpha = \alpha \circ \varphi.$$

We say that two germs $g, h \in \mathcal{O}_n$ are $\mathcal{K}^\alpha$*-equivalent* if there exist a function $\kappa \in \alpha^*\mathcal{O}_2, \kappa(0) \neq 0$ and a germ of diffeomorphism $\varphi \in \mathcal{R}^\alpha$, such that

$$g = \kappa \cdot h \circ \varphi.$$

Example 5.2. Let $\alpha(x, y) = (x^2, y^2)$, then any diffeomorphism $\varphi \in \mathcal{R}^\alpha$ is of the form

$$\varphi(x, y) = (x\varphi_1, y\varphi_2) \quad \text{or} \quad \varphi(x, y) = (y\varphi_1, x\varphi_2)$$

for some functions $\varphi_1, \varphi_2 \in \alpha^*\mathcal{O}_2$, $\varphi_i(0, 0) \neq 0$. In particular, if $g, h \in \mathbb{C}[x, y]$ are homogeneous $\mathcal{K}^\alpha$-equivalent polynomials, the factors κ and $h \circ \varphi$ are homogeneous. Hence, on one hand, κ is a constant in $\mathbb{C}^*$. On the other hand, since φ is a diffeomorphism, both h and $h \circ \varphi$ are homogeneous of the same degree. We can replace φ by its linear part without changing the composition. Thus, we can assume that φ is of the form $(x, y) \mapsto (ax, by)$ or $(x, y) \mapsto (by, ax)$.

Lemma 5.3. *A diffeomorphism $\varphi \in \mathcal{R}_n$ belongs to $\mathcal{R}^\alpha$ if and only if the algebras $\alpha^*\mathcal{O}_n$ and $(\alpha \circ \varphi)^*\mathcal{O}_n$ are equal.*

Proof. Let $\varphi \in \mathcal{R}^\alpha$ with $\hat{\varphi} \circ \alpha \circ \varphi = \alpha$. Any function $h \circ \alpha \in \alpha^*\mathcal{O}_n$ is equal to

$$(h \circ \hat{\varphi}) \circ \alpha \circ \varphi \in (\alpha \circ \varphi)^*\mathcal{O}_n.$$

Now take $h \circ \alpha \circ \varphi \in (\alpha \circ \varphi)^*\mathcal{O}_n$. This function is equal to $h \circ \hat{\varphi}^{-1} \circ \hat{\varphi} \circ \alpha \circ \varphi = (h \circ \hat{\varphi}^{-1}) \circ \alpha \in \alpha^*\mathcal{O}_n$.

Now suppose that the two sub-algebras above are equal, then there exist some functions $\hat{\varphi}_i$ such that $\alpha_i = \hat{\varphi}_i \circ \alpha \circ \varphi$. Take $\hat{\varphi} = (\hat{\varphi}_1, \ldots, \hat{\varphi}_n)$. Then we have $\alpha = \hat{\varphi} \circ \alpha \circ \varphi$. As α is finite and φ is a biholomorphism, α and $\alpha \circ \varphi$ have the same finite multiplicity. Therefore $\hat{\varphi}$ must have multiplicity 1, and hence is a biholomorphism. □

Theorem 5.4. *Let $\alpha : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ be a finite germ and f_{n+1}, g_{n+1} be two $\mathcal{K}^\alpha$-equivalent functions of $\mathcal{O}_n$, then the map germs $(\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ $f = (\alpha, f_{n+1})$ and $g = (\alpha, g_{n+1})$ are $\mathcal{A}$-equivalent.*

Proof. $f \sim_{\mathcal{K}^\alpha} g$ implies that there exists $\theta_\alpha : (\mathbb{C}^n \times \mathbb{C}, 0) \to (\mathbb{C}, 0)$ of the form

$$\theta_\alpha(X, Z) = \theta(\alpha(X), Z)$$

for some germ of function θ and such that $\theta_\alpha(0, \cdot)$ is a germ of biholomorphism, and there exists $\varphi \in \mathcal{R}_n^\alpha$ such that $g(X) = \theta_\alpha(X, f \circ \varphi(X))$. Since $\varphi \in \mathcal{R}_n^\alpha$, then there exists some germ of biholomorphism $\hat{\varphi}$ such that $\alpha = \hat{\varphi} \circ \alpha \circ \varphi$. We define $\psi_1 : \mathbb{C}^{n+1} \to \mathbb{C}^n$ by $\psi_1 = \hat{\varphi} \circ \pi_1$ and $\psi_2 = \theta \circ (\psi_1, \pi_2)$, where π_i represents the projection over the i-th component of $\mathbb{C}^n \times \mathbb{C}$. Now we define $\psi = (\psi_1, \psi_2) : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{n+1}, 0)$ and, for every $X \in \mathbb{C}^n$, we have

$$\psi \circ (\alpha, f) \circ \varphi(X) = \big(\hat{\varphi}(\alpha(\varphi(X))), \theta(\hat{\varphi}(\alpha(\varphi(X))), f(\varphi(X)))\big) =$$

$$\big(\alpha(X), \theta_\alpha(X, f(\varphi(X)))\big) = (\alpha, g)(X).$$

As a consequence of $\hat{\varphi}$ and $\theta_\alpha(X, \cdot)$ being biholomorphisms, we have that ψ is a biholomorphism. □

Again, examples suggest that the converse of Theorem 5.4 also holds: $\mathcal{A}$-equivalence of (α, f_{n+1}) and (α, g_{n+1}) implies $\mathcal{K}^\alpha$-equivalence of f_{n+1} and g_{n+1}. However we have not succeed in proving this in general. It was proved by Mond in [8] that it holds when α is the Whitney Fold. We have only succeeded in showing it for finitely determined quasihomogeneous double folds.

It is shown in [5] that any quasihomogeneous double fold must be a homogeneous one. There are only two ways to obtain a homogeneous double fold $f(x,y) = (\alpha, xp_1 + yp_2 + xyp_3)$. One is $p_3 = 0$ and the other $p_1 = p_2 = 0$. Every finitely determined double fold must have a reduced double point space, which is given by $(p_1+yp_3)(p_2+xp_3)(xp_1+yp_2) = 0$. We deduce immediately that every finitely determined quasihomogeneous double fold must be, in fact, a homogeneous special double fold.

Theorem 5.5. *Let $f = (\alpha, f_3)$ and $g = (\alpha, g_3)$ be $\mathcal{A}$-equivalent finitely determined quasihomogeneous double folds , then f_3 and g_3 are $\mathcal{K}^\alpha$-equivalent.*

Proof. Suppose there exist ψ and φ such that $g = \psi \circ f \circ \varphi$. Denote by φ_{i,x_j} the derivative of the i-th component with respect to the variable x_j. Taking into account that $p_1, p_2 \in \mathfrak{m}^2$, the 2-jet of the first two coordinate functions of the equality $g = \psi \circ f \circ \varphi$ gives us

$$x^2 = \psi_{1,X}(\varphi_{1,x}^2 x^2 + \varphi_{1,x}\varphi_{1,y}xy + \varphi_{2,y}^2 y^2) + \psi_{1,Y}(\varphi_{2,x}^2 x^2 + \varphi_{2,x}\varphi_{2,y}xy + \varphi_{2,y}^2 y^2),$$
$$y^2 = \psi_{2,X}(\varphi_{1,x}^2 x^2 + \varphi_{1,x}\varphi_{1,y}xy + \varphi_{2,y}^2 y^2) + \psi_{2,Y}(\varphi_{2,x}^2 x^2 + \varphi_{2,x}\varphi_{2,y}xy + \varphi_{2,y}^2 y^2).$$

Since $d\varphi$ is invertible, we have $\varphi_{1,x}\varphi_{2,y} \neq 0$ or $\varphi_{1,y}\varphi_{2,x} \neq 0$. In the first case from the equations we obtain $\varphi_{1,y} = \varphi_{2,x} = 0$ and, in the second case $\varphi_{1,x} = \varphi_{2,y} = 0$. Suppose we are in the first case (the second one is analogous). Then the differential of φ is of the form $d\varphi(u,v) = (au, bv)$ for some $a, b \in \mathbb{C}^*$.

Notice that w is a source double point of g if and only if it is so for $f \circ \varphi$, if and only if $\varphi(w)$ is a source double point of f. Since f and g are finitely determined, their double point spaces are reduced and thus $\varphi|_{D(g)} : D(g) \to D(f)$ is an isomorphism between complex space germs. We claim that $\varphi|_{D_3(g)}$ is an isomorphism between $D_3(g)$ and $D_3(f)$. We proceed by reduction to the absurd: suppose there is a irreducible component R of $D_3(g)$, such that $\varphi(R) \not\subset D_3(f)$. For example, suppose $\varphi(R) \subset D_1(f)$ (the other case, $\varphi(R) \subset D_2(f)$, is analogous). Since f and g are finitely determined, their diagonal double points are isolated and thus, since $R \subset D_3(g)$ and $\varphi(R) \subset D_1(f)$, we have $\varphi(i_3(R)) = i_1(\varphi(R))$. Let (u,v) be the tangent vector to the curve germ R, we have the equality $d\varphi(i_3(u,v)) = i_1(d\varphi(u,v))$, that is $(-au, -bv) = (-au, bv)$. The last equality implies (u,v) is a horizontal vector. Since g is homogeneous, the equation which defines R is also homogeneous and, thus, it is independent of x. This is implies that y divides $xq_1 + yq_2$, which in turn implies that y divides q_1. Then y^2 divides $q_1q_2(xq_1 + yq_2)$. This is a contradiction, because g is finitely determined and, thus, $D(g) = V(q_1q_2(xq_1 + yq_2))$ must be reduced.

Now we have the isomorphism of complex spaces $\varphi|_{D_3(g)} : D_3(g) \to D_3(f)$, that is, we have the equality $\langle g_3 \rangle = \varphi^* \langle f_3 \rangle$. This implies the existence of a function h, with $h(0,0) \neq 0$, such that $g_3 = h \cdot f_3 \circ \varphi$. Since g_3 y f_3 are homogeneous, we can take the diffeomorphism $\tilde{\varphi} = d\varphi$ and the constant $\kappa = h(0,0) \neq 0$ and get $g_3 = \kappa \cdot f_3 \circ \varphi$. Moreover, as we have seen before, $\tilde{\varphi}$ is a diagonal linear change and thus it belongs to $\mathcal{R}^\alpha$. □

Notice that the $\mathcal{K}^\alpha$-equivalence of f_3 and g_3 splits into two simultaneous equivalences between P_1, P_2 and Q_1, Q_2. In the diagonal case we get an expression

$$xQ_1(x^2, y^2) + yQ_2(x^2, y^2) = \kappa axP_1(a^2x^2, b^2y^2) + \kappa byP_2(a^2x^2, b^2y^2).$$

This is equivalent to $Q_1(x,y) = \kappa a P_1(a^2x, b^2y)$ and $Q_2(x,y) = \kappa b P_2(a^2x, b^2y)$. In the antidiagonal case we obtain the expression

$$xQ_1(x^2,y^2) + yQ_2(x^2,y^2) = \kappa ay P_1(a^2y^2, b^2x^2) + \kappa bx P_2(a^2y^2, b^2x^2),$$

which is equivalent to $Q_1(x,y) = \kappa b P_2(a^2y, b^2x)$ and $Q_2(x,y) = \kappa a P_1(a^2y, b^2x)$. Now the next corollary follows immediately.

Corollary 5.6. *Let f and g be two $\mathcal{A}$-equivalent quasihomogeneous finitely determined special double folds, then:*

$ST_i(f) = ST_j(g)$,
$C_i(f) = C_i(g)$,
$QD(f) = QD(g)$,
$\mu(D_i(f)) = \mu(D_j(g))$,

where $j = i$ in the diagonal case, and in the antidiagonal the pairs (i,j) are $(1,2)$, $(2,1)$, $(3,3)$.

References

[1] Gaffney, T.; *Polar Multiplicities and equisingularity of map germs.* Topology **32** (1993), no. 1, 185-223

[2] Golubitsky, M; Guillemin, V. *Stable Mappings and Their Singularities* Springer-Verlag.

[3] Humphreys, J.E; *Reflection Groups and Coxeter Groups.* Cambridge studies in advanced mathematics, no. 29 (1990).

[4] de Jong, Th; Pfister, G. *Local analytic geometry. Basic theory and applications.* Advanced Lectures in Mathematics. Friedr. Vieweg & Sohn, Braunschweig, 2000.

[5] Marar, W. L. and Nuño-Ballesteros, J. J.; *A note on finite determinacy for corank 2 map germs from surfaces to 3-space.* Math. Proc. Cambridge Philos. Soc. **145** (2008), no. 1, 153–163.

[6] Marar, W. L.; Nuño-Ballesteros, J. J.; Peñafort-Sanchis, G. *Double point curves for corank 2 map germs from C^2 to C^3.* Topology Appl. 159 (2012), no. 2, 526–536,

[7] Marar, W.L. and Mond, D.; *Multiple point schemes for corank* 1 *maps.* J. London Math. Soc. (2) **39** (1989), no. 3, 553–567.

[8] Mond, D.; *On the classification of germs of maps from R^2 to R^3.* Proc. London Math. Soc. (3) 50 (1985), no. 2, 333–369.

[9] Mond, D.; *Some remarks on the geometry and classification of germs of maps from surfaces to* 3*-space.* Topology **26** (1987), no. 3, 361–383.

[10] Mond, D.; Pellikaan, R.; *Fitting ideals and multiple points of analytic mappings.* Algebraic geometry and complex analysis (Pátzcuaro, 1987), 107–161, Lecture Notes in Math., 1414, Springer, Berlin, 1989.

[11] Montaldi, J. *On contact between submanifolds.* Michigan Math. J. 33 (1986), no. 2, 195–199.

[12] Wik Atique, R. *On the classification of multi-germs of maps from $\mathbb{C}^2$ to $\mathbb{C}^3$ under $\mathcal{A}$-equivalence.* Real and complex singularities. Chapman & Hall/CRC. Research Notes in Mathematics Series, 2000.

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KOENDERINK TYPE THEOREMS FOR FRONTS

KENTARO SAJI

Dedicated to Professor Shyuichi Izumiya on the occasion of his 60*th birthday*

Abstract. We prove Koenderink type theorems with the terminology of the singular curvatures of cuspidal edges of wave fronts.

1. Introduction

In 1984 and 1990, J. J. Koenderink showed theorems that relate to how one actually sees a surface. Let $f : U \to \boldsymbol{R}^3$ be a non-singular smooth surface in $\boldsymbol{R}^3$ and $M = f(U)$. Let $\pi : \boldsymbol{R}^3 \to P$ be the orthogonal projection onto a plane $P \subset \boldsymbol{R}^3$ and $\pi_{\mathbf{0}} : \boldsymbol{R}^3 \to S^2$ the central projection onto a unit sphere S^2 of $\boldsymbol{R}^3$ centered at $\mathbf{0} \in \boldsymbol{R}^3$. We denote the singular set of a map g by $S(g)$. Koenderink showed the following:

Theorem. ([11, Appendix], [12, page 433]) *Suppose $p \in S(\pi \circ f)$, and $\pi \circ f(S(\pi \circ f))$ is a regular curve near p. Let κ_1 be the curvature of the plane curve $\pi \circ f(S(\pi \circ f)) \subset P$, and κ_2 the curvature of the normal section of M at p by the plane that contains the kernel of π. Then*

$$K = \kappa_1 \kappa_2$$

holds at p, where K is the Gaussian curvature of M.

Suppose $p \in S(\pi_{\mathbf{0}} \circ f)$, and $\pi_{\mathbf{0}} \circ f(S(\pi_{\mathbf{0}} \circ f))$ is a regular curve near p. Let κ_g be the geodesic curvature of the curve $\pi_{\mathbf{0}} \circ f(S(\pi_{\mathbf{0}} \circ f))$ and d be the distance of p from $\mathbf{0}$. Then $K = \kappa_g \kappa_2 / d$ holds at p.

See [15, p223] for further considerations of this type problem. See also [3, 2, 14, 8, 9, 10]. If f has a singular point, generically the Gaussian curvature is unbounded. Thus this theorem does not hold at the singular points of f. In [16], it was shown that if f is a front, then the Gaussian curvature form $K d\hat{A}$ is bounded, and introduced the singular curvature function on the singular set which consists of cuspidal edges. The singular curvature has a certain geometric property. So it is natural to expect a Koenderink type theorem of fronts using the Gaussian curvature form and the singular curvature. In this paper, we give Koenderink type theorems for cuspidal edges with the terminology of the Gaussian curvature form and the singular curvature. We also give the same type theorems for the cuspidal edges in the hyperbolic space.

2. Singular curvature and statement of results

Let $(U; u, v) \subset \boldsymbol{R}^2$ be a domain, N a three dimensional manifold, and W a five dimensional contact manifold with a Legendrian fibration $\mathrm{pr} : W \to N$. A smooth map $f : U \to N$ is called a *front* if there exists a Legendrian immersion lift $L_f : U \to W$ of f; that is, L is an immersion, the pull-buck of the contact form vanishes on U, and $\mathrm{pr} \circ L_f = f$ holds. We remark that a front in a two dimensional manifold can be defined in a similar manner by replacing U with an interval, N with a two dimensional manifold, and W with a three dimensional contact manifold respectively. Let us consider the case W is the unit tangent bundle $T_1\boldsymbol{R}^3$ with the canonical contact form and pr is the Legendrian fibration $\mathrm{pr} : T_1\boldsymbol{R}^3 \to \boldsymbol{R}^3$. In this case, a smooth map $f : U \to \boldsymbol{R}^3$ is a front if there exists a unit vector field ν along f such that $L_f = (f,\nu) : U \to \boldsymbol{R}^3 \times S^2 = T_1\boldsymbol{R}^3$ is an immersion and the following orthogonality condition holds:

$$(df_p(X_p)\cdot\nu(p)) = 0 \quad (X \in TU, p \in U),$$

where $(\ \cdot\ )$ is the Euclidean inner product of $\boldsymbol{R}^3$. Let $f : U \to \boldsymbol{R}^3$ be a front. Set

$$\lambda(u,v) = \det\left(\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}, \nu\right)(u,v),$$

called the *signed area density function*. We also set

$$d\hat{A} = \lambda\, du \wedge dv, \tag{2.1}$$

called the *signed area form*. Suppose $p \in U$ is a singular point of f, then $\lambda(p) = 0$ holds. If $d\lambda(p) \neq 0$ holds, then there is a regular smooth curve $\gamma(t) : (-\varepsilon,\varepsilon) \to U$ $(\gamma(0) = p)$ such that the image of γ coincides with $S(f)$ near p. Furthermore, there exists a non-vanishing vector field η along γ satisfying

$$\langle \eta(t)\rangle_{\boldsymbol{R}} = \ker df_{\gamma(t)}.$$

We call γ the *singular curve* and η the *null vector field*.

It was shown in [13], if $\eta(0)$ transverse to $\gamma'(0)$, then the map germ f at p is $\mathcal{A}$-equivalent to a map germ $(u,v) \mapsto (u,v^2,v^3)$ at $\boldsymbol{0}$; that is, there exist diffeomorphic germs $\sigma : (\boldsymbol{R}^2,\boldsymbol{0}) \to (\boldsymbol{R}^2,p)$ and $\tau : (\boldsymbol{R}^3, f(p)) \to (\boldsymbol{R}^3,\boldsymbol{0})$ such that $\tau\circ f\circ\sigma(u,v) = (u,v^2,v^3)$ holds as map germs at $\boldsymbol{0}$. A singular point p of a front f is called a *cuspidal edge* if f at p is $\mathcal{A}$-equivalent to $(u,v) \mapsto (u,v^2,v^3)$.

Now we suppose that the singular curve γ of a front $f : U \to \boldsymbol{R}^3$ consists of cuspidal edges. Then we can choose the null vector field η such that $(\gamma'(t),\eta(t))$ is a positively oriented frame field along γ, where $' = d/dt$. We then define the *singular curvature* as follows ([16]):

$$\kappa_s(t) = \operatorname{sgn}(d\lambda(\eta))\frac{\det(\hat{\boldsymbol{\gamma}}'(t), \hat{\boldsymbol{\gamma}}''(t), \nu\circ\gamma(t))}{|\hat{\boldsymbol{\gamma}}'(t)|^3},$$

where $\hat{\boldsymbol{\gamma}} = f\circ\gamma$. For the geometric meanings of the singular curvature, and further details, see [16, 17].

Now we consider the Gaussian curvature form of fronts.

Proposition 2.1 ([16])**.** *Let $f : U \to \boldsymbol{R}^3$ be a front, and K the Gaussian curvature of f which is defined on the set of regular points of f. Then $K\,d\hat{A}$ can be continuously extended as a globally defined 2-form on U, where $d\hat{A}$ is the signed area form as in (2.1).*

A similar proposition as above also holds for plane curves. Let $\boldsymbol{c} : I \to \boldsymbol{R}^2$ be a front, and κ the curvature of $\boldsymbol{c}$, defined on the set of regular points. By the same method, one can show that $\kappa\, ds$ can be continuously extended as a globally defined 1-form on I, where s is the arclength parameter of $\boldsymbol{c}$.

Let $f : U \to \boldsymbol{R}^3$ be a front and $p \in U$ a cuspidal edge. Then one can see that a section of $M = f(U)$ near $f(p)$ by a plane through $f(p)$ which transverse to $df_p(M)$ is a 3/2-cusp, in particular a front (see [13, Proposition 2.9], for example). Several curvatures of fronts in the plane are investigated in [18].

Using the notions of the curvature forms above, we state the Koenderink type theorems for fronts.

Theorem 2.2. *Let $f : U \to \boldsymbol{R}^3$ be a front, $p \in U$ a cuspidal edge, and γ the singular curve with $\gamma(0) = p$. Set $\hat{\boldsymbol{\gamma}} = f\circ\gamma$, $\boldsymbol{\xi}_p = \nu(p)\times\hat{\boldsymbol{\gamma}}'(p)/|\hat{\boldsymbol{\gamma}}'(p)|$ and $\boldsymbol{v}_\theta = \cos\theta\boldsymbol{\xi}_p + \sin\theta\nu(p)$. Let P_θ be a plane normal to $\boldsymbol{v}_\theta$ and π_θ the orthogonal projection $\pi_\theta : \boldsymbol{R}^3 \to P_\theta$ with respect to $\boldsymbol{v}_\theta$. Let $\kappa_1(t)$ be the curvature of the plane curve $\boldsymbol{\gamma}_1(t) := \pi_\theta\circ\hat{\boldsymbol{\gamma}}(t)$, and $\kappa_2(s)$ the curvature of the intersection* *curve* $\boldsymbol{\gamma}_2$ *of* M *at* p *by the plane* $P := \langle \boldsymbol{\xi}_p, \nu(p) \rangle_{\boldsymbol{R}}$*, where* s *is the arclength parameter of* $\boldsymbol{\gamma}_2$*. If* $\theta \in (0, \pi/2)$ *then*

$$K d\hat{A} = \frac{1}{\cos\theta} \left(\sin\theta \kappa_s - \kappa_1\right) dt \wedge \kappa_2 \, ds \tag{2.2}$$

holds at p*, where* κ_s *is the singular curvature. Here, we give a orientation of* $\boldsymbol{\gamma}_2(s)$ *passing through* p *from the region* $\{\lambda < 0\}$ *to the region* $\{\lambda > 0\}$*. Also we give a orientation of* P_θ *such that* $\{-\sin\theta \boldsymbol{\xi}_p + \cos\theta \nu(p), \ \boldsymbol{\gamma}_1'(0)\}$ *forms a positive basis, and* P *such that* $\{\boldsymbol{\xi}_p, \nu(p)\}$ *forms a positive basis.*

3. Proof of theorem 2.2

Let $f : U \to \boldsymbol{R}^3$ be a front and $p \in U$ a cuspidal edge. Then by [16, Lemma 3.2], we can take a coordinate system (u, v) near p satisfying

- (u, v) is compatible with the orientation of U,
- $p = \mathbf{0}$ and the u-axis is the singular curve,
- the null vector field is ∂_v on U,
- $\lambda_v(\mathbf{0}) > 0$, and
- $|f_u(u, 0)| = 1$.

We call such a coordinate system (u, v) *adapted coordinate system* with respect to p. In an adapted coordinate system (u, v), since $\lambda_v > 0$, it holds that

$$\kappa_s(u) = \det(f_u, f_{uu}, \nu)(u, 0) = \left(f_{uu} \cdot \nu \times f_u\right)(u, 0), \tag{3.1}$$

where $f_{uu} = \partial^2 f / \partial u^2$, for example.

Proof of theorem 2.2. We take an adapted coordinate system (u, v). Since $f_v(u, 0) = \mathbf{0}$ and $f_{vv}(0, 0) \neq \mathbf{0}$, there exists a smooth function φ satisfying $\varphi(\mathbf{0}) \neq 0$ and

$$f_v(u, v) = v\varphi(u, v). \tag{3.2}$$

In this setting, the Gaussian curvature form has the following expression on U:

$$K \, d\hat{A} = \frac{-\left(f_{uu} \cdot \nu\right)\left(\varphi \cdot \nu_v\right) - v\left(\varphi \cdot \nu_u\right)^2}{\left(\varphi \cdot \varphi\right) - \left(f_u \cdot \varphi\right)^2} \sqrt{\left(\varphi \cdot \varphi\right) - \left(f_v \cdot \varphi\right)^2} \, du \wedge dv.$$

This is equal to

$$-\frac{\left(f_{uu} \cdot \nu\right)\left(f_{vv} \cdot \nu_v\right)}{\sqrt{\left(f_{vv} \cdot f_{vv}\right) - \left(f_u \cdot f_{vv}\right)^2}} \, du \wedge dv \tag{3.3}$$

at p. On the other hand, we calculate the curvatures κ_1 and κ_2. Let $\gamma_1(u)$ be the plane curve $\pi_\theta \circ f(u, 0)$. Then the curvature κ_1 of γ_1 is

$$\kappa_1 = \left(-\cos\theta \left(f_{uu} \cdot \nu(p)\right) + \sin\theta \left(f_{uu} \cdot \boldsymbol{\xi}_p\right)\right). \tag{3.4}$$

Let γ_2 be the plane curve of the intersection of $f(M)$ at p by P and κ_2 its curvature. Since $(f_u(u, v) \cdot f_u(p)) \neq 0$, by the implicit function theorem, there exists a function $u = u(v)$ such that

$$\left(f(u(v), v) \cdot f_u(p)\right) = 0.$$

Hence γ_2 is expressed by

$$\gamma_2(v) = \left(\left(f(u(v), v) \cdot \boldsymbol{\xi}_p\right), \left(f(u(v), v) \cdot \nu(p)\right) \right).$$

Using (3.2), since $\nu = f_u \times \varphi/|f_u \times \varphi|$, one can compute $\kappa_2\, ds$ as follows

$$\kappa_2\, ds = \frac{\det(f_u,\ \varphi,\ \varphi_v)}{(\varphi \cdot \varphi) - (f_u \cdot \varphi)^2}\, dv = -\frac{(\nu_v \cdot f_{vv})}{\sqrt{(f_{vv} \cdot f_{vv}) - (f_u \cdot f_{vv})^2}}\, dv, \tag{3.5}$$

at p, where s is the arclength parameter of γ_2. By (3.4) and (3.5), we have (2.2). □

To get the spherical projection version of the theorem, we need the following lemma.

Lemma 3.1. *Let $\boldsymbol{\gamma} : I \to \boldsymbol{R}^3$ be a smooth curve and κ its curvature as a space curve. Take a point $p \in I$ satisfying that $\boldsymbol{\gamma}(p)$ and $\boldsymbol{\gamma}'(p)$ are linearly independent. Let $\pi_{\mathbf{0}} : \boldsymbol{R}^3 \to S^2$ be the central projection onto a unit sphere S^2 centered at $\mathbf{0}$ and κ_g be the geodesic curvature of $\pi_{\mathbf{0}} \circ \gamma$ as a spherical curve. Then*

$$\kappa(p) = \frac{\kappa_g(p)}{d} \tag{3.6}$$

holds, where d is the distance of p from $\mathbf{0}$.

Proof. Direct computations. □

By Lemma 3.1 and Theorem 2.2, we have the following:

Corollary 3.2. *In the same setting as in Theorem 2.2, suppose that $\hat{\boldsymbol{\gamma}}(0)$ and $\hat{\boldsymbol{\gamma}}'(0)$ are linearly independent, and $\hat{\boldsymbol{\gamma}}(0)$ and $\boldsymbol{v}_\theta$ are parallel. Let $\pi_{\mathbf{0}} : \boldsymbol{R}^3 \to S^2$ be the central projection onto a unit sphere S^2 centered at $\mathbf{0}$ and κ_g the geodesic curvature of $\pi_{\mathbf{0}} \circ \hat{\boldsymbol{\gamma}}$ as a spherical curve. If $\theta \in (0, \pi/2)$, then*

$$K d\hat{A} = \frac{1}{\cos\theta}\left(\sin\theta \kappa_s - \frac{\kappa_g}{d}\right)\kappa_2\, du \wedge dv \tag{3.7}$$

holds at p, where d is the distance of $f(p)$ from $\mathbf{0}$.

4. Horospherical Koenderink type theorem

Recently an extrinsic geometry on submanifolds in the hyperbolic space is discovered by Shyuichi Izumiya and investigated [5, 7]. See also [4, 6]. It is called *horospherical geometry*. In this section, we show a horospherical geometric Koenderink type theorem for cuspidal edges. It should be noted that horospherical geometric Koenderink type theorems for regular surfaces in the hyperbolic space are shown in [9]. See also [8, 10].

To state a Koenderink type theorem, we prepare some notion. Let $\boldsymbol{R}^4_1$ be the Minkowski 4-space with the inner product $\langle\ ,\ \rangle = (-,+,+,+)$. We denote by $H^3_+(-1)$, LC^*_+ and $S^3_1(1) \subset \boldsymbol{R}^4_1$ the *hyperbolic space*, the *lightcone* and the *de Sitter space* defined by

$$\begin{aligned} H^3_+(-1) &= \{\boldsymbol{x} \in \boldsymbol{R}^4_1 \mid \langle \boldsymbol{x}, \boldsymbol{x}\rangle = -1, u_0 > 0\}, \\ LC^*_+ &= \{\boldsymbol{x} \in \boldsymbol{R}^4_1 \mid \langle \boldsymbol{x}, \boldsymbol{x}\rangle = 0, u_0 > 0\}, \\ S^3_1(1) &= \{\boldsymbol{x} \in \boldsymbol{R}^4_1 \mid \langle \boldsymbol{x}, \boldsymbol{x}\rangle = 1\}. \end{aligned}$$

Let $(U; u, v) \subset \boldsymbol{R}^2$ be a domain and $f : U \to H^3_+(-1)$ a smooth regular surface. Define a vector

$$\boldsymbol{e}(u, v) = \frac{f_u \wedge f_v \wedge f}{|f_u \wedge f_v \wedge f|}(u, v),$$

where $f_u = \partial f/\partial u$, for example. Here for any $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3 \in \boldsymbol{R}^4_1$, the vector $\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \boldsymbol{x}_3$ is defined as

$$\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \boldsymbol{x}_3 = -\det \begin{pmatrix} x_1^1 & x_2^1 & x_3^1 \\ x_1^2 & x_2^2 & x_3^2 \\ x_1^3 & x_2^3 & x_3^3 \end{pmatrix} \boldsymbol{e}_0 - \det \begin{pmatrix} x_0^1 & x_2^1 & x_3^1 \\ x_0^2 & x_2^2 & x_3^2 \\ x_0^3 & x_2^3 & x_3^3 \end{pmatrix} \boldsymbol{e}_1$$
$$+ \det \begin{pmatrix} x_0^1 & x_1^1 & x_3^1 \\ x_0^2 & x_1^2 & x_3^2 \\ x_0^3 & x_1^3 & x_3^3 \end{pmatrix} \boldsymbol{e}_2 - \det \begin{pmatrix} x_0^1 & x_1^1 & x_2^1 \\ x_0^2 & x_1^2 & x_2^2 \\ x_0^3 & x_1^3 & x_2^3 \end{pmatrix} \boldsymbol{e}_3$$

where $\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ is the canonical basis of $\boldsymbol{R}^4_1$ and $\boldsymbol{x}_i = (x_0^i, x_1^i, x_2^i, x_3^i)$ $(i = 1, 2, 3)$. We can easily show that $\langle \boldsymbol{x}, \boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \boldsymbol{x}_3 \rangle = \det(\boldsymbol{x}, \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3)$, so that $\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \boldsymbol{x}_3$ is orthogonal to any $\boldsymbol{x}_i$ $(i = 1, 2, 3)$. Thus we have $\langle \boldsymbol{e}, f_u \rangle = \langle \boldsymbol{e}, f_v \rangle = \langle \boldsymbol{e}, f \rangle = 0$ and $\langle \boldsymbol{e}, \boldsymbol{e} \rangle = 1$. This map $\boldsymbol{e} : U \to S^3_1(1)$ is called the *de Sitter Gauss image*. We also define a map

$$\boldsymbol{l}^{\pm}(u, v) = f(u, v) \pm \boldsymbol{e}(u, v) : U \to LC^*_+,$$

which is called the *lightcone Gauss image*. We consider the lightcone Gauss image as a Gauss map. See [5] for details. With this notion, we consider fronts in the hyperbolic space as follows. Consider the following double fibration:

- $H^3_+(-1) \times LC^*_+ \supset \Delta_2 = \{(\boldsymbol{x}, \boldsymbol{y}) \mid \langle \boldsymbol{x}, \boldsymbol{y} \rangle = -1\}$,
- $\pi_{21} : \Delta_2 \to H^3_+(-1)$, $\pi_{22} : \Delta_2 \to LC^*_+$,
- $\theta_{21} = \langle d\boldsymbol{x}, \boldsymbol{y} \rangle |_{\Delta_2}, \theta_{22} = \langle \boldsymbol{x}, d\boldsymbol{y} \rangle |_{\Delta_2}$.

Here,

$$\pi_{21}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{x},\ \pi_{22}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{y},\ \langle d\boldsymbol{x}, \boldsymbol{y} \rangle = -y_0\, dx_0 + \sum_{i=1}^{3} y_i\, dx_i, \text{ and } \langle \boldsymbol{x}, d\boldsymbol{y} \rangle = -x_0\, dy_0 + \sum_{i=1}^{3} x_i\, dy_i.$$

We remark that θ_{21} and θ_{22} define the same tangent hyperplane field over Δ_2 which is denoted by K_2. In [4], it has been shown that (Δ_2, K_2) is a contact manifold such that each fibration π_{2i} $(i = 1, 2)$ is a Legendrian fibration. See [4] for details.

As we have seen in Section 2, a smooth map $f : U \to H^3_+(-1)$ is a front if there exists a map $\boldsymbol{l} : U \to LC^*_+$ such that $(f, \boldsymbol{l}) : U \to \Delta_2$ is a Legendrian immersion with respect to K_2. The map $\boldsymbol{l}$ is called a Δ_2*-dual* of f. One can show that $-d_p\boldsymbol{l}$ is a linear transformation $-d_p\boldsymbol{l} : T_pU \to \big(\langle \boldsymbol{l}(p), f(p) \rangle_{\boldsymbol{R}}\big)^{\perp} \subset T_{f(p)}\boldsymbol{R}^4_1$, by an identification $T_{f(p)}\boldsymbol{R}^4_1 = \boldsymbol{R}^4_1$, where $\perp$ means the orthogonal complement. It is called the *hyperbolic shape operator*. The *hyperbolic Gaussian curvature* is defined as

$$K^h(p) = \det(-d_p\boldsymbol{l}),$$

and the *hyperbolic Gaussian curvature form* is defined as

$$K^h\, d\hat{A} = K^h \lambda^h\, du \wedge dv,$$

where λ^h is the the signed area density function $\lambda^h(u, v) = \det(f_u, f_v, \boldsymbol{l}, f)$. If K^h identically vanishes, then f is a one-parameter family of horocycles, more precisely, f is an envelope of a one-parameter family of horospheres and is a locus swept out by horocycles ([7]). It can be easily seen that if f is a front, then $K^h\, d\hat{A}$ can be continuously extended as a globally defined 2-form on U.

Let $f : U \to H^3_+(-1)$ be a front and $p \in U$ a cuspidal edge. We denote $\gamma(t) : I \to U$ by a parameterization of $S(f)$. Let $\boldsymbol{l}$ be a Δ_2-dual of f. We define the *hyperbolic singular curvature* κ^h_s as

$$\kappa^h_s(t) = \operatorname{sgn}(d\lambda(\eta)) \frac{\det(\hat{\gamma}', \hat{\gamma}'', \boldsymbol{l} \circ \gamma, \hat{\gamma})}{|\hat{\gamma}'|^3}(t),$$

where $\hat{\boldsymbol{\gamma}}(t) = f \circ \gamma(t)$ and $\eta(t)$ is a null vector field, namely, non-zero vector field along γ satisfying $\langle \eta(t) \rangle_{\boldsymbol{R}} = \ker df_{\gamma(t)}$ and (γ', η) is positively oriented. Here, $' = d/dt$ and $\hat{\boldsymbol{\gamma}}''(t) = D_t \hat{\boldsymbol{\gamma}}'(t)$, where D is the Levi-Civita connection of $H^3_+(-1)$. The hyperbolic singular curvature has the same type geometric meaning as the Euclidean case. See Section 2 and [16, 17].

4.1. **Curves in hyperbolic space.** For a vector $\boldsymbol{v} \in S^3_1(1)$, define the hyperplane normal to $\boldsymbol{v}$ as $HP(\boldsymbol{v}, 0) = \{\boldsymbol{x} \in \boldsymbol{R}^4_1 \mid \langle \boldsymbol{x}, \boldsymbol{v} \rangle = 0\}$. It is well known that the set $H^2(\boldsymbol{v}) = HP(\boldsymbol{v}, 0) \cap H^3_+(-1)$ is a totally geodesic hyperbolic plane. Let $\boldsymbol{c}(s) : I \to H^2(\boldsymbol{v})$ be a regular curve and s an arclength parameter. Then since $T_p H^2(\boldsymbol{v}) = \big(\langle \boldsymbol{v}, p \rangle_{\boldsymbol{R}}\big)^{\perp}$ holds for $p \in H^2(\boldsymbol{v})$, the geodesic curvature of $\boldsymbol{c}$ is $\det(\boldsymbol{c}', \boldsymbol{c}'', \boldsymbol{v}, \boldsymbol{c})$ modulo a sign. Thus we define the *curvature in* $H^2(\boldsymbol{v})$ *of* $\boldsymbol{c}$ by $\kappa^h(s) = \det(\boldsymbol{c}', \boldsymbol{c}'', \boldsymbol{v}, \boldsymbol{c})(s)$. It can be easily seen that if a curve germ $\boldsymbol{c} : (I, 0) \to H^2(\boldsymbol{v})$ is a *cusp* ($\mathcal{A}$-equivalent to $t \mapsto (t^2, t^3)$ at 0), then $\kappa^h(s)\, ds$ can be continuously extended as a globally defined 1-form on I, where s is the arclength parameter of $\boldsymbol{c}$.

4.2. **Projections to planes.** To state Koenderink type theorems, we need orthogonal projections in $H^3_+(-1)$ to hyperbolic planes. Let us consider a hyperplane

$$HP(\boldsymbol{v}, 0) = \{\boldsymbol{x} \in \boldsymbol{R}^4_1 \mid \langle \boldsymbol{x}, \boldsymbol{v} \rangle = 0\}$$

for a vector $\boldsymbol{v} \in S^3_1(1)$. Given a point $\boldsymbol{q} \in H^3_+(-1)$, there is a unique geodesic in $H^3_+(-1)$ which intersects orthogonally the hyperbolic plane $H^2(\boldsymbol{v}) = HP(\boldsymbol{v}, 0) \cap H^3_+(-1)$ at some point $r(\boldsymbol{q}, \boldsymbol{v})$. We call the point $r(\boldsymbol{q}, \boldsymbol{v})$ the orthogonal projection of $\boldsymbol{q}$ in the direction $\boldsymbol{v}$ to $H^2(\boldsymbol{v})$. The point $r(\boldsymbol{q}, \boldsymbol{v})$ is given by

$$r(\boldsymbol{q}, \boldsymbol{v}) = \frac{1}{\sqrt{1 + \langle \boldsymbol{q}, \boldsymbol{v} \rangle^2}} \big(\boldsymbol{q} - \langle \boldsymbol{q}, \boldsymbol{v} \rangle\, \boldsymbol{v}\big).$$

See [9] for details.

4.3. **Koenderink type theorem.** In this section, we prove the following theorem:

Theorem 4.1. *Let $f : U \to H^3_+(-1)$ be a front, $p \in U$ a cuspidal edge, $M = f(U)$ and γ a singular curve with $\gamma(0) = p$. Set $\hat{\boldsymbol{\gamma}} = f \circ \gamma$,*

$$\boldsymbol{\xi}_p = \hat{\boldsymbol{\gamma}}'(p)/|\hat{\boldsymbol{\gamma}}'(p)| \wedge \boldsymbol{l}(p) \wedge f(p),$$

and $\boldsymbol{v}_\theta = \cos\theta \boldsymbol{\xi}_p + \sin\theta \boldsymbol{l}(p)$. Let r_θ the orthogonal projection $r_\theta : H^3_+(-1) \to H^2(\boldsymbol{v}_\theta)$ in the direction $\boldsymbol{v}_\theta$. Let $\kappa^h_1(t)$ be the curvature in $H^2(\boldsymbol{v}_\theta)$ of the curve $\boldsymbol{\gamma}_1(t) = r_\theta \circ \hat{\gamma}(t)$, and $\kappa^h_2(s)$ the curvature in $H^2(\boldsymbol{l}(p) \wedge \boldsymbol{\xi}_p \wedge f(p))$ of the intersection curve $\boldsymbol{\gamma}_2$ of M at $f(p)$ by the hyperplane $HP(\boldsymbol{l}(p) \wedge \boldsymbol{\xi}_p \wedge f(p), 0)$, where s is the arclength parameter of $\boldsymbol{\gamma}_2$. If $\theta \in (0, \pi/2)$ then

$$K^h\, d\hat{A} = \frac{1}{\cos\theta}\big(-\cos\theta + \sin\theta \kappa^h_s - \kappa^h_1\big)\, dt \wedge \kappa^h_2\, ds$$

holds at p, where κ^h_s is the hyperbolic singular curvature. Here, we give a orientation of $\boldsymbol{\gamma}_2(s)$ passing through p from the region $\{\lambda^h < 0\}$ to the region $\{\lambda^h > 0\}$.

Proof. By changing coordinates on $(U; u, v)$, we may assume $p = \boldsymbol{0}$ and $S(f) = \{v = 0\}$. Also by isometries of $H^3_+(-1)$, we may assume

$$f(u, v) = \left(\sqrt{f_1(u, v)^2 + f_2(u, v)^2 + u^2 + 1}, f_1(u, v), f_2(u, v), u\right),$$

where $df_i = \boldsymbol{0}$ at $\boldsymbol{0}$ $(i = 1, 2)$. Then there exist functions $g_1(u)$, $g_2(u)$, $h_1(u, v)$, $h_2(u, v)$ such that $f_i(u, v) = u^2 g_i(u) + v h_i(u, v)$ $(i = 1, 2)$. Since $S(f) = \{v = 0\}$, it holds that $\partial h_i/\partial v(u, 0) = 0$ $(i = 1, 2)$. Thus there exist functions $\bar{h}_i(u, v)$ such that $h_i(u, v) = v \bar{h}_i(u, v)$ $(i = 1, 2)$. By a rotation of $H^3_+(-1)$, we may assume $\bar{h}_1(\boldsymbol{0}) = 0$. Thus we have $f_1(u, v) = u^2 a_1(u) + v^2 b_1(u, v)$

and $f_2(u,v) = u^2a_2(u) + uv^2a_3(u) + v^3b_2(u,v)$. where $a_1(u), a_2(u), a_3(u), b_1(u,v), b_2(u,v)$ are functions, and $b_1(\mathbf{0})b_2(\mathbf{0}) \neq 0$.

Then $\boldsymbol{l}(\mathbf{0}) = (0,0,1,0)$, $\boldsymbol{\xi_0} = (0,1,0,0)$ and $\boldsymbol{v}_\theta = (0, \cos\theta, \sin\theta, 0)$ holds. By a direct calculation, we have

$$\kappa_s^h = -2a_1(0),\ \kappa_1^h = 2a_2(0)\cos\theta - 2a_1(0)\sin\theta,\ \kappa_2^h\, ds = -\frac{3b_2(\mathbf{0})}{2b_1(\mathbf{0})}\, ds$$

at $\mathbf{0}$ since one can consider $\hat{\boldsymbol{\gamma}}(t) = f(t,0)$ and $\boldsymbol{\gamma}_2(t) = f(0,t)$. On the other hand,

$$K^h\, du \wedge dv = \frac{3(1+2a_2(0))b_2(\mathbf{0})}{2b_1(\mathbf{0})}\, du \wedge dv$$

holds at $\mathbf{0}$. By these computations, we have the result. □

References

[1] V. I. Arnol'd, S. M. Gusein-Zade and A. N. Varchenko, *Singularities of differentiable maps, Vol.* 1, Monogr. Math. **82**, Birkhäuser Boston, Inc., Boston, MA, 1985.

[2] J. W. Bruce and P. J. Giblin, *Outlines and their duals*, Proc. London Math. Soc. **50** (1985), 552–570. DOI: 10.1112/plms/s3-50.3.552

[3] T. Gaffney, *The structure of TA, classification and an application to differential geometry*, Proc. Sympos. in Pure Math. **40** (1983), 409–427. DOI: 10.1090/pspum/040.1/713081

[4] S. Izumiya, *Legendrian dualities and spacelike hypersurfaces in the lightcone*. Mosc. Math. J. **9** (2009), no. 2, 325–357.

[5] S. Izumiya, D. Pei and T. Sano, *Singularities of hyperbolic Gauss maps*. Proc. London Math. Soc. **86** (2003), no. 2, 485–512.

[6] S. Izumiya and K. Saji, *The mandala of Legendrian dualities for pseudo-spheres in Lorentz-Minkowski space and "flat" spacelike surfaces*. J. Singul. **2** (2010), 92–127. DOI: 10.5427/jsing.2010.2g

[7] S. Izumiya, K. Saji and M. Takahashi, *Horospherical flat surfaces in hyperbolic 3-space*. J. Math. Soc. Japan **62** (2010), no. 3, 789–849.

[8] S. Izumiya and F. Tari, *Projections of hypersurfaces in the hyperbolic space to hyperhorospheres and hyperplanes*. Rev. Mat. Iberoam. **24** (2008), no. 3, 895–920.

[9] S. Izumiya and F. Tari, *Projections of surfaces in the hyperbolic space along horocycles*. Proc. Roy. Soc. Edinburgh Sect. A **140** (2010), no. 2, 399–418.

[10] S. Izumiya and F. Tari, *Projections of timelike surfaces in the de Sitter space*. Real and complex singularities, 190–210, London Math. Soc. Lecture Note Ser., **380**, Cambridge Univ. Press, Cambridge, 2010.

[11] J. J. Koenderink, *What does the occluding contour tell us about solid shape?*, Perception **13** (1984), 321–330. DOI: 10.1068/p130321

[12] J. J. Koenderink, *Solid shape*, MIT Press Series in Artificial Intelligence. MIT Press, Cambridge, MA, 1990.

[13] M. Kokubu, W. Rossman, K. Saji, M. Umehara and K. Yamada, *Singularities of flat fronts in hyperbolic 3-space*, Pacific J. Math. **221** (2005), 303–351. DOI: 10.2140/pjm.2005.221.303

[14] E. E. Landis, *Tangential singularities*, Functional Anal. Appl. **15** (1981), no. 2, 103–114.

[15] I. R. Porteous, *Geometric differentiation. For the intelligence of curves and surfaces*, Second edition. Cambridge University Press, Cambridge, 2001.

[16] K. Saji, M. Umehara and K. Yamada, *The geometry of fronts*, Ann. of Math. **169** (2009), 491–529.

[17] K. Saji, M. Umehara and K. Yamada, *Coherent tangent bundles and Gauss-Bonnet formulas for wave fronts*, J. Geom. Anal. **22** (2012), no. 2, 383–409. DOI: 10.4007/annals.2009.169.491

[18] S. Shiba and M. Umehara, *The behavior of curvature functions at cusps and inflection points*, Differential Geom. Appl. **30** (2012), no. 3, 285–299.

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CLASSIFICATIONS OF COMPLETELY INTEGRABLE IMPLICIT SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Dedicated to Professor Shyuichi Izumiya on the occasion of his 60th birthday

Abstract. An implicit second order ordinary differential equation is said to be *completely integrable* if there exists at least locally an immersive two-parameter family of geometric solutions on the equation hypersurface like as in the case of explicit equations. An implicit equation may have an immersive one-parameter family of geometric solutions (or, singular solutions) and a geometric solution (or, an isolated singular solution). In this paper, we give a classification of types of completely integrable implicit second order ordinary differential equations and give existence conditions for such families of solutions.

1. Introduction

An implicit second order ordinary differential equation is given by the form

$$F(x, y, p, q) = 0,$$

where F is a smooth function of the independent variable x, the function y, its first and second derivatives $p = dy/dx$ and $q = d^2y/dx^2$ respectively.

It is natural to consider $F = 0$ as being defined on a subset in the space of 2-jets of smooth functions of one variable, $F : \mathcal{O} \to \mathbb{R}$ where $\mathcal{O}$ is an open subset in $J^2(\mathbb{R}, \mathbb{R})$. Throughout this paper, we assume that 0 is a regular value of F. It follows that the set $F^{-1}(0)$ is a hypersurface in $J^2(\mathbb{R}, \mathbb{R})$. We call $F^{-1}(0)$ the *equation hypersurface*. Let (x, y, p, q) be a local coordinate on $J^2(\mathbb{R}, \mathbb{R})$ and $\xi \subset TJ^2(\mathbb{R}, \mathbb{R})$ be the canonical contact system (the Engel structure) on $J^2(\mathbb{R}, \mathbb{R})$. It is well-known that locally the contact system is given by the vanishing of the two 1-forms $\alpha_1 = dy - pdx$ and $\alpha_2 = dp - qdx$.

We now define the notion of solutions. A *smooth solution* (or a *classical solution*) of $F = 0$ passing through a point z_0 is a smooth function germ $y = f(x)$ at a point t_0 such that

$$(t_0, f(t_0), f'(t_0), f''(t_0)) = z_0 \quad \text{and} \quad F(x, f(x), f'(x), f''(x)) = 0.$$

In other words, there exists a smooth function germ $f : (\mathbb{R}, t_0) \to \mathbb{R}$ such that the image of the 2-jet extension, $j^2f : (\mathbb{R}, t_0) \to (J^2(\mathbb{R}, \mathbb{R}), z_0)$, is contained in the equation hypersurface. It is easy to see that the map j^2f is an integral (Engel) immersion. More generally, a *geometric solution* of $F = 0$ passing through a point z_0 is an integral immersion $\gamma : (\mathbb{R}, t_0) \to (J^2(\mathbb{R}, \mathbb{R}), z_0)$ such that the image of γ is contained in the equation hypersurface, namely, $\gamma'(t) \neq 0$, $\gamma^*\alpha_1 = \gamma^*\alpha_2 = 0$ and $F(\gamma(t)) = 0$ for each $t \in (\mathbb{R}, t_0)$.

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In this paper, the following notions are basic (cf. [3, 6, 10, 11, 12, 20]):

A *smooth complete solution on* $F^{-1}(0)$ *at* z_0 is defined by a two-parameter family of smooth function germs $y = f(t,r,s)$ such that

$$F\left(t, f(t,r,s), \frac{\partial f}{\partial t}(t,r,s), \frac{\partial^2 f}{\partial t^2}(t,r,s)\right) = 0$$

and the map germ $j^2_* f : (\mathbb{R} \times \mathbb{R}^2, (t_0, r_0, s_0)) \to (F^{-1}(0), z_0)$ defined by

$$j^2_* f(t,r,s) = \left(t, f(t,r,s), \frac{\partial f}{\partial t}(t,r,s), \frac{\partial^2 f}{\partial t^2}(t,r,s)\right)$$

is an immersion. It follows that the equation hypersurface is foliated locally by a two-parameter family of smooth solutions.

On the other hand, consider the corresponding definition for geometric solutions. We call $\Gamma : (\mathbb{R} \times \mathbb{R}^2, (t_0, r_0, s_0)) \to (F^{-1}(0), z_0)$ a *complete solution on* $F^{-1}(0)$ *at* z_0 if Γ is a two-parameter family of geometric solutions of $F = 0$ and

$$\operatorname{rank} \begin{pmatrix} \partial x/\partial t & \partial y/\partial t & \partial p/\partial t & \partial q/\partial t \\ \partial x/\partial r & \partial y/\partial r & \partial p/\partial r & \partial q/\partial r \\ \partial x/\partial s & \partial y/\partial s & \partial p/\partial s & \partial q/\partial s \end{pmatrix} (t_0, r_0, s_0) = 3,$$

where $\Gamma(t,r,s) = (x(t,r,s), y(t,r,s), p(t,r,s), q(t,r,s))$. This condition means that Γ is an immersion germ, that is, the equation hypersurface is foliated locally by a two-parameter family of geometric solutions. We say that an equation $F = 0$ is *smoothly completely integrable* (respectively, *completely integrable*) at z_0 if there exists a smooth complete solution (respectively, a complete solution) on $F^{-1}(0)$ at z_0.

In the study of implicit ODEs from the view point of singularity theory, there is a lot of research. For example, generic singularities and properties were given in the case of first order in [1, 2, 4, 5, 7, 8, 10, 17, 19], in the case of second order in [14, 15] and in the case of any order in [9] etc. This paper is focused on the theory of completely integrable implicit ODEs (cf. [18, 20, 21]). Especially, we shall classify types of completely integrable implicit second order ODEs. In §2, we give previous results for completely integrable implicit second order ODEs, for more detail see [3, 19, 20]. In §3, we divide types of completely integrable implicit second order ODEs into ten and give an existence condition for families of geometric solutions for each type. In §4, we give examples which are useful to understand the notions of complete solutions and results. Moreover, as an application of the results, we consider the confluent hypergeometric equations (the degenerate hypergeometric equations) from the view point of complete integrability (Example 4.5). In Appendix, we give a corresponding result for completely integrable implicit first order ODEs. These results had been essentially given by Shyuichi Izumiya ([11]).

All map germs and manifolds considered here are differential of class C^∞.

2. Basic notions and previous results

Let $F(x,y,p,q) = 0$ be an implicit second order ODE. We denote the total derivative of F by $F_X = F_x + pF_y + qF_p$, where F_x (respectively, F_y, F_p, F_q) is the partial derivative with respect to x (respectively, y, p, q).

We say that $F = 0$ is of *(second order) Clairaut type* (for short, *type C*) at z_0 if there exists a function germ $\alpha : (F^{-1}(0), z_0) \to \mathbb{R}$ such that

$$F_X|_{F^{-1}(0)} = \alpha \cdot F_q|_{F^{-1}(0)},$$

and of *reduced type* (for short, *type R*) at z_0 if there exists a function germ $\beta : (F^{-1}(0), z_0) \to \mathbb{R}$ such that

$$F_q|_{F^{-1}(0)} = \beta \cdot F_X|_{F^{-1}(0)}.$$

Note that we call $F = 0$ is of reduced type as of first order type in [20]. Then we have shown the following result.

Theorem 2.1. ([20])

(1) *$F = 0$ is smoothly completely integrable at z_0 if and only if $F = 0$ is of type C at z_0.*

(2) *$F = 0$ is completely integrable at z_0 if and only if $F = 0$ is either of type C or of type R at z_0.*

We say that a geometric solution $\gamma : (\mathbb{R}, t_0) \to (F^{-1}(0), z_0)$ is a *singular solution* of $F = 0$ at z_0 if for any representative $\widetilde{\gamma} : I \to F^{-1}(0)$ of γ and any open subinterval $(a, b) \subset I$ at t_0, $\widetilde{\gamma}|_{(a,b)}$ is never contained in a leaf of a complete solution (cf. [3, 11, 13]).

Around $z \in F^{-1}(0)$ such that the contact plane ξ_z intersects $T_z F^{-1}(0)$ transversally, it is easy to see that a complete solution on $F^{-1}(0)$ exists by integrating the line field $\xi \cap TF^{-1}(0)$. We call points where transversality fails *contact singular points* and denote by $\Sigma_c = \Sigma_c(F)$ the set of contact singular points. It is easy to check that the contact singular set is given by

$$\Sigma_c = \{z \in J^2(\mathbb{R}, \mathbb{R}) |\ F(z) = 0, F_X(z) = 0, F_q(z) = 0\}.$$

From the definition of singular solutions, it is easy to see that a geometric solution

$$\gamma : (\mathbb{R}, t_0) \to (F^{-1}(0), z_0)$$

t is a singular solution only if it is contained in Σ_c (cf. [21]). We also consider the subset $\Delta = \Delta(F) \subset \Sigma_c$ which is defined to be the set of points $z \in \Sigma_c$ such that $T_z F^{-1}(0)$ coincides with the kernel of $\alpha_1(z)$. Explicitly, it is given by $\Delta = \{z \in \Sigma_c |\ F_p(z) = 0\}$.

Now suppose that $F = 0$ is completely integrable at z_0 and Σ_c is a 2-dimensional manifold around z_0. We say that a map germ

$$\Phi : (\mathbb{R} \times \mathbb{R}, (t_0, a_0)) \to (\Sigma_c, z_0)$$

is a *complete solution on Σ_c at z_0* if Φ is an immersion germ and $\Phi(\cdot, a)$ is a geometric solution for each $a \in (\mathbb{R}, a_0)$, that is, an immersive one-parameter family of geometric solutions of $F = 0$. Moreover, we call Φ a *complete singular solution on Σ_c at z_0* if $\Phi(\cdot, a)$ is a singular solution for each $a \in (\mathbb{R}, a_0)$.

If ξ_z intersects $T_z\Sigma_c$ transversally in $T_z F^{-1}(0)$, then integrating the line field $\xi \cap T\Sigma_c$ yields a complete solution on Σ_c. We call a point where transversality does not hold a *second order contact singular point* and denote the set of such points by $\Sigma_{cc} = \Sigma_{cc}(F)$ (cf. [3, 20, 21]).

Conditions for existence of a complete solution on $F^{-1}(0)$ and a complete (singular) solution on Σ_c for implicit second order ODEs were given under a regularity condition.

Theorem 2.2. ([3]) *Suppose that 0 is a regular value of $F_q|_{F^{-1}(0)}$.*

(1) *$F = 0$ is completely integrable at z_0 if and only if $z_0 \notin \Sigma_c$ or Σ_c is a 2-dimensional manifold around z_0.*

(2) *Let $F = 0$ be completely integrable.*

(i) *The leaves of the complete solution on $F^{-1}(0)$ which meet Σ_c away from Δ intersect Σ_c transversally.*

(ii) *The leaves of the complete solution on $F^{-1}(0)$ which meet Δ are tangent to Σ_c.*

(3) *Let $F = 0$ be completely integrable and $\Sigma_c \neq \emptyset$.*

(i) *There exists a complete singular solution on Σ_c at z_0 if and only if $z_0 \notin \Sigma_{cc}$ or Σ_{cc} is a 1-dimensional manifold around z_0.*

(ii) *Suppose that* $F = 0$ *admits a complete singular solution on* Σ_c. *Then each leaf of the complete singular solution on* Σ_c *intersects* Σ_{cc} *transversally.*
(4) *Let* $F = 0$ *be completely integrable at* $z_0 \in \Sigma_c$. *If* $z_0 \in \Delta$, *then* Δ *is a* 1*-dimensional manifold around* z_0.

Theorem 2.3. ([20]) *Suppose that* 0 *is a regular value of* $F_X|_{F^{-1}(0)}$.
(1) $F = 0$ *is completely integrable at* z_0 *if and only if* $z_0 \notin \Sigma_c$ *or* Σ_c *is a* 2*-dimensional manifold around* z_0.
(2) *Let* $F = 0$ *be completely integrable.*
(i) *The leaves of the complete solution on* $F^{-1}(0)$ *which meet* Σ_c *away from* Δ *intersect* Σ_c *transversally.*
(ii) *The leaves of the complete solution on* $F^{-1}(0)$ *which meet* Δ *are tangent to* Σ_c.
(3) *Let* $F = 0$ *be completely integrable and* $\Sigma_c \neq \emptyset$.
(i) *There exists a complete solution on* Σ_c *at* z_0 *if and only if* $z_0 \notin \Sigma_{cc}$ *or* Σ_{cc} *is a* 1*-dimensional manifold around* z_0.
(ii) *Suppose that* $F = 0$ *admits a complete solution on* Σ_c. *Then each leaf of the complete solution on* Σ_c *intersects* Σ_{cc} *transversally.*

Remark 2.4. The important differences between Theorems 2.2 and 2.3 are (3) and (4). One is an existence condition for a complete singular solution on Σ_c and the other is only for a complete solution on Σ_c. Moreover, if $F = 0$ is completely integrable at $z_0 \in \Delta$ and 0 is a regular value of $F_q|_{F^{-1}(0)}$, then Δ is a 1-dimensional manifold around z_0. However, Δ is not necessarily a 1-dimensional manifold around z_0 when 0 is a regular value of $F_X|_{F^{-1}(0)}$, see Examples 4.1 and 4.4.

Proposition 2.5. ([18, 20]) *Let* $F = 0$ *be completely integrable at* $z_0 \in \Sigma_c$.
(1) *If* 0 *is a regular value of* $F_q|_{F^{-1}(0)}$, *then* $F = 0$ *is of type* C *at* z_0.
(2) *If* 0 *is a regular value of* $F_X|_{F^{-1}(0)}$, *then* $F = 0$ *is of type* R *at* z_0.

Proposition 2.6. ([20]) *Let* $F = 0$ *be completely integrable at* z_0 *and* Σ_c *be a* 2*-dimensional manifold around* z_0. *Then the second order singular set* Σ_{cc} *is contained in* Δ.

3. Completely integrable implicit second order ODEs

In this section, we analyse completely integrable implicit second order ODEs in detail. Let $F(x, y, p, q) = 0$ be an implicit second order ODE at z_0. If $z_0 \notin \Sigma_c$, then $F = 0$ satisfies either $F_q(z_0) \neq 0$ or $F_X(z_0) \neq 0$.

First we assume that $F_q(z_0) \neq 0$. By the implicit function theorem, $F = 0$ can be represented by an explicit equation at least locally. In this case, $F = 0$ is of type C at z_0 and we call this type C_q. Next we assume that $F_X(z_0) \neq 0$. Then $F = 0$ is of type R at z_0 and we call this type R_X. In both cases, there is a unique geometric solution passing through each point of $F^{-1}(0)$. It follows that there is a complete solution on $F^{-1}(0)$ and no singular solution.

By Theorem 2.1, a completely integrable ODE at z_0 is either of type C or of type R at z_0. If $z_0 \in \Sigma_c$, then $F = 0$ satisfies either $F_p(z_0) \neq 0$ or $F_y(z_0) \neq 0$ by the assumption that $F = 0$ is regular at z_0 (see §1). The main purpose of this paper is to classify types of the completely integrable implicit second order ODEs at a point in detail, and to give existence conditions for a complete (singular) solution on Σ_c for each type respectively. It is concluded that there are ten kinds of types, see Table 1.

Conditions				Type	Name
$z_0 \notin \Sigma_c$	$F_q(z_0) \neq 0$			C	C_q
	$F_X(z_0) \neq 0$			R	R_X
$z_0 \in \Sigma_c$	$F_p(z_0) \neq 0$	z_0 is a regular point of $F_q\|_{F^{-1}(0)}$		C	RC_p
		z_0 is a regular point of $F_X\|_{F^{-1}(0)}$		R	RR_p
	$F_y(z_0) \neq 0$, $F_p(z_0) = 0$	z_0 is a regular point of $F_q\|_{F^{-1}(0)}$		C	RC_y
		z_0 is a regular point of $F_X\|_{F^{-1}(0)}$	$\Sigma_c = \Delta$	R	RR_y^1
			$\Sigma_c \supsetneq \Delta = \Sigma_{cc}$	R	RR_y^2
			$\Sigma_c \supsetneq \Delta \supsetneq \Sigma_{cc}$	R	RR_y^3
		z_0 is a singular point of $F_q\|_{F^{-1}(0)}$		C	SC_y
		and $F_X\|_{F^{-1}(0)}$		R	SR_y

Table 1. A classification of types of completely integrable implicit second order ODEs at z_0.

3.1. **On the types RC_p and RR_p.** If $z_0 \in \Sigma_c$ and $F_p(z_0) \neq 0$, by the implicit function theorem, there exists a smooth function $g : V \to \mathbb{R}$, where V is an open set in $\mathbb{R}^3$, such that in a neighbourhood of z_0, $(x, y, p, q) \in F^{-1}(0)$ if and only if $-p + g(x, y, q) = 0$. Thus we may assume without loss of generality that $F(x, y, p, q) = -p + g(x, y, q) = 0$. Under this notations, $F_q = g_q$ and $F_X = g_x + g \cdot g_y - q$. It follows that z_0 is a regular point of either $F_q|_{F^{-1}(0)}$ or $F_X|_{F^{-1}(0)}$.

If z_0 is a regular point of $F_q|_{F^{-1}(0)}$, then $F = 0$ is of type C at z_0 and Σ_c is a 2-dimensional manifold around z_0 by Proposition 2.5 and Theorem 2.2. We call this type RC_p. By $z_0 \notin \Delta$ and Proposition 2.6, we have $z_0 \notin \Sigma_{cc}$. Hence $F = 0$ has a complete singular solution on Σ_c at z_0.

On the other hand, suppose that z_0 is a regular point of $F_X|_{F^{-1}(0)}$. By Proposition 2.5 and Theorem 2.3, $F = 0$ is of type R at z_0 and Σ_c is a 2-dimensional manifold around z_0. We call this type RR_p. By $z_0 \notin \Delta$ and Proposition 2.6, we have $z_0 \notin \Sigma_{cc}$. Since the leaves of the complete solution which meet Σ_c away from Δ intersect Σ_c transversally, $F = 0$ has a complete singular solution on Σ_c at z_0.

3.2. **On the type RC_y.** If $z_0 \in \Sigma_c$ and $F_y(z_0) \neq 0$, again by the implicit function theorem, there exists a smooth function $f : U \to \mathbb{R}$, where U is an open set in $\mathbb{R}^3$, such that in a neighbourhood of z_0, $(x, y, p, q) \in F^{-1}(0)$ if and only if $-y + f(x, p, q) = 0$. Thus we may assume without loss of generality that $F(x, y, p, q) = -y + f(x, p, q) = 0$. Define the diffeomorphism $\phi : U \to F^{-1}(0), (x, p, q) \mapsto (x, f(x, p, q), p, q)$ and $u_0 = \phi^{-1}(z_0)$. Below, if $F_y(z_0) \neq 0$, we keep the notations of the above.

Suppose that z_0 is a regular point of $F_q|_{F^{-1}(0)}$. By Proposition 2.5 and Theorem 2.2, $F = 0$ is of type C at z_0 and Σ_c is a 2-dimensional manifold around z_0. We call this type RC_y. Moreover, $F = 0$ has a complete singular solution on Σ_c at z_0 if and only if $z_0 \notin \Sigma_{cc}$ or Σ_{cc} is a 1-dimensional manifold around z_0 by Theorem 2.2.

Remark 3.1. If Σ_{cc} is a 1-dimensional manifold around z_0, then $\Delta = \Sigma_{cc}$ and Σ_{cc} is an isolated singular solution passing through z_0 (see, [3, Proposition 1.4]). In this case, $F = 0$ have a two-parameter family of geometric solutions, a one-parameter family of singular solutions and an isolated singular solution passing through $z_0 \in \Sigma_{cc}$, see Example 4.2.

3.3. **On the type RR_y^1.** Let $z_0 \in \Sigma_c$ and $F_y(z_0) \neq 0$. Suppose that z_0 is a regular point of $F_X|_{F^{-1}(0)}$. By Proposition 2.5 and Theorem 2.3, $F = 0$ is of type R at z_0 and Σ_c is a 2-dimensional manifold around z_0. In this case, there are three types. First one is $\Sigma_c = \Delta$ around z_0 (type RR_y^1), second is $\Sigma_c \supsetneq \Delta = \Sigma_{cc}$ around z_0 (type RR_y^2), and the last is $\Sigma_c \supsetneq \Delta \supsetneq \Sigma_{cc}$ around z_0 (type RR_y^3). We may assume that $F_p(z_0) = 0$, namely, $z_0 \in \Delta$.

Let $F = 0$ be of the type RR^1_y at z_0. By Theorem 2.3, $F = 0$ has a complete solution of Σ_c at z_0 if and only if $z_0 \notin \Sigma_{cc}$ or Σ_{cc} is a 1-dimensional manifold around z_0. In this case, we have the following result, see Examples 4.1 and 4.4.

Theorem 3.2. *Let $F = 0$ be of type RR^1_y at $z_0 \in \Delta$. If $z_0 \notin \Sigma_{cc}$, then there exists a unique geometric solution passing through z_0.*

Proof. We denote $F(x, y, p, q) = -y + f(x, p, q) = 0$. Since $F = 0$ is of type R at z_0, there exists a smooth function germ $\alpha : (F^{-1}(0), z_0) \to \mathbb{R}$ such that

$$f_q = \alpha \cdot (f_x - p + qf_p). \tag{1}$$

A complete solution, $\Gamma : (\mathbb{R} \times \mathbb{R}^2, 0) \to (F^{-1}(0), z_0)$, is given by integrating the vector field $\phi_* X$, where $X : U \to TU$ is given by

$$X = (-\alpha, -\alpha \cdot q, 1)$$

(cf. [3, Lemma 3.1]). By (1), we have

$$(f_x - p + qf_p)_q = (\alpha_x + q\alpha_p) \cdot (f_x - p + qf_p) + \alpha \cdot ((f_x - p + qf_p)_x + q(f_x - p + qf_p)_p) + f_p.$$

It follows from the assumption $\Sigma_c = \Delta$ that

$$(f_x - p + qf_p)_q|_{\phi^{-1}(\Sigma_c)} = \alpha|_{\phi^{-1}(\Sigma_c)} \cdot ((f_x - p + qf_p)_x + q(f_x - p + qf_p)_p)|_{\phi^{-1}(\Sigma_c)}.$$

In this case, a complete solution on Σ_c, $\Phi : (\mathbb{R} \times \mathbb{R}, 0) \to (\Sigma_c, z_0)$, is given by integrating the vector field $\phi_* Y$, where $Y : \phi^{-1}(\Sigma_c) \to T\phi^{-1}(\Sigma_c)$ is given by

$$Y = (-\alpha|_{\phi^{-1}(\Sigma_c)}, (-\alpha \cdot q)|_{\phi^{-1}(\Sigma_c)}, 1)$$

(cf. [20, Lemma 3.5]). It follows that $\Gamma|_{\Gamma^{-1}(\Sigma_c)} = \Phi$ and hence there is a geometric solution on Σ_c. Let $\gamma : (\mathbb{R}, t_0) \to (\Sigma_c, z_0); \gamma(t) = (x(t), y(t), p(t), q(t))$ be a geometric solution passing through z_0. Since $z_0 \notin \Sigma_{cc}$, we have $x'(t) + \alpha \cdot q'(t) = 0$ at t_0. It follows that we can reparametrise $\gamma(t)$ as $(x(t), y(t), p(t), t)$. By the analogous way in the proof of Lemma 3.2 in [21], we can show uniqueness of the geometric solution passing through z_0. □

Proposition 3.3. *Let $F = 0$ be of type RR^1_y at $z_0 \in \Delta$. If Σ_{cc} is a 1-dimensional manifold around z_0, then Σ_{cc} is a singular solution passing through z_0.*

Proof. It is easy to see that Σ_{cc} is a geometric solution passing through z_0. By definition,

$$\phi^{-1}(\Sigma_c) = (f_x - p + qf_p)^{-1}(0)$$

and

$$\phi^{-1}(\Sigma_{cc}) = (f_x - p + qf_p)^{-1}(0) \cap ((f_x - p + qf_p)_x + q(f_x - p + qf_p)_p)^{-1}(0).$$

To show that Σ_{cc} is not a leaf of the complete solution on $F^{-1}(0)$ (and on Σ_c) at z_0, it is sufficient to check that the scalar product of $\mathrm{grad}((f_x - p + qf_p)_x + q(f_x - p + qf_p)_p)$ and the vector field X is non-zero at u_0. Now

$$\begin{aligned}
&\langle \mathrm{grad}((f_x - p + qf_p)_x + q(f_x - p + qf_p)_p), (-\alpha, -\alpha \cdot q, 1)\rangle \\
&= -\alpha \cdot ((f_x - p + qf_p)_x + q(f_x - p + qf_p)_p)_x - \alpha \cdot q((f_x - p + qf_p)_x + q(f_x - p + qf_p)_p)_p \\
&\quad + ((f_x - p + qf_p)_x + q(f_x - p + qf_p)_p)_q.
\end{aligned} \tag{2}$$

It follows from (1) that (2) is equal to $2(f_{xp} + qf_{pp}) - 1$ at u_0. By the assumption $\Sigma_c = \Delta$, there exists a smooth function germ β such that $f_p = \beta \cdot (f_x - p + qf_p)$ at least locally. Differentiating this equality with respect to x and p, we get

$$f_{xp} = \beta_x \cdot (f_x - p + qf_p) + \beta \cdot (f_x - p + qf_p)_x$$

and

$$f_{pp} = \beta_p \cdot (f_x - p + qf_p) + \beta \cdot (f_x - p + qf_p)_p.$$

It follows that (2) is non-zero at u_0. □

3.4. **On the type RR_y^2.** Suppose that $F = 0$ is of type RR_y^2 at z_0. See Example 4.2. Then $\Sigma_c \supsetneq \Delta = \Sigma_{cc}$ around z_0. By Theorem 2.3, $F = 0$ has a complete solution on Σ_c at z_0 if and only if Σ_{cc} is a 1-dimensional manifold around z_0. In this case, we have the following result.

Theorem 3.4. *Let $F = 0$ be of type RR_y^2 at $z_0 \in \Delta$. $F = 0$ has a complete singular solution on Σ_c at z_0 if and only if Σ_{cc} is a 1-dimensional manifold around z_0.*

Proof. By Theorem 2.3, each leaf of the complete solution on $F^{-1}(0)$ which meet Σ_c away from Σ_{cc} intersect Σ_c transversally, and each leaf of the complete solution on Σ_c intersects Σ_{cc} transversally. Therefore the complete solution on Σ_c is the complete singular solution on Σ_c. □

By the definition of Σ_{cc},

$$(f_x - p + qf_p)_x + q(f_x - p + qf_p)_p = 0, \ (f_x - p + qf_p)_q = 0$$

at $z_0 \in \Sigma_{cc}$. Since z_0 is a regular point of $F_X|_{F^{-1}(0)}$, $(f_x - p + qf_p)_p \neq 0$ at z_0. The equation $F = 0$ satisfies either

$$\text{(i) } ((f_x - p + qf_p)_x + q(f_x - p + qf_p)_p)_q \neq 0$$

or

$$\text{(ii) } ((f_x - p + qf_p)_x + q(f_x - p + qf_p)_p)_q = 0$$

at z_0. It follows that z_0 is a regular point of $(f_x-p+qf_p)_x+q(f_x-p+qf_p)_p$, or of $(f_x-p+qf_p)_q$.

Proposition 3.5. *Let $F = 0$ be of type RR_y^2 at $z_0 \in \Delta$. Suppose that Σ_{cc} is a 1-dimensional manifold around z_0.*

(1) *If $F = 0$ satisfies the condition* (i)*, then each leaf of the complete solution on $F^{-1}(0)$ is intersects Σ_{cc} transversally and hence Σ_{cc} is a singular solution passing through z_0.*

(2) *If $F = 0$ satisfy the conditions* (ii) *and $F_{pq}|_{\Sigma_{cc}} \equiv 0$ around z_0, then each leaf of the complete solution on $F^{-1}(0)$ is tangent to Σ_{cc}. If $\gamma(t) = (x(t), y(t), p(t), q(t)) \in \Sigma_{cc}$ is a geometric solution, $\gamma(t)$ is represented by the form (a, b, c, t), where $a, b, c \in \mathbb{R}$. Moreover, $\gamma(t)$ is a leaf of the complete solution on $F^{-1}(0)$.*

Proof. (1) Since $\phi^{-1}(\Sigma_{cc}) = (f_x - p + qf_p)^{-1}(0) \cap ((f_x - p + qf_p)_x + q(f_x - p + qf_p)_p)^{-1}(0)$, it is sufficient to check that the scalar product of $\mathrm{grad}((f_x - p + qf_p)_x + q(f_x - p + qf_p)_p)$ and the vector field X is non-zero at u_0. By the same calculations in Proposition 3.3,

$$\langle \mathrm{grad}((f_x - p + qf_p)_x + q(f_x - p + qf_p)_p), (-\alpha, -\alpha \cdot q, 1)\rangle = 2(f_{xp} + qf_{pp}) - 1$$

at u_0. The condition (i) guarantees that $2(f_{xp} + qf_{pp}) - 1 \neq 0$ at u_0. Therefore each leaf of the complete solution on $F^{-1}(0)$ intersects Σ_{cc} transversally and hence Σ_{cc} is a singular solution passing through z_0.

(2) Since $\phi^{-1}(\Sigma_{cc}) = (f_x - p + qf_p)^{-1}(0) \cap ((f_x - p + qf_p)_q)^{-1}(0)$, it is sufficient to check that the scalar product of $\mathrm{grad}(f_x - p + qf_p)_q$ and the vector field X is zero. By the direct calculations, the consequence follows from the condition $F_{pq}|_{\Sigma_{cc}} \equiv 0$ around z_0.

Let $\gamma(t) = (x(t), y(t), p(t), q(t)) \in \Sigma_{cc}$ be a geometric solution passing through z_0. By differentiating $f_p(x(t), p(t), q(t)) = 0$ with respect to t, we get

$$(f_{xp} + qf_{pp})(x(t), p(t), q(t)) \cdot x'(t) + f_{pq}(x(t), p(t), q(t)) \cdot q'(t) = 0.$$

By the condition (ii), we have $f_{xp}+qf_{pp}=1/2$ at u_0 and hence $x'(t)\equiv 0$. This means that $x(t)$ is constant on Σ_{cc} around z_0. Differentiating (1) with respect to p, we have

$$f_{pq}=\alpha_p\cdot(f_x-p+qf_p)+\alpha\cdot(f_x-p+qf_p)_p.$$

It follows that $\alpha|_{\Sigma_{cc}}\equiv 0$ around z_0. By the form of the vector field X (see, in the proof of Theorem 3.2), $\Gamma|_{\Gamma^{-1}(\Sigma_{cc})}=\gamma$. □

3.5. **On the type RR^3_y.** Suppose that $F=0$ is of type RR^3_y at z_0. See Example 4.3. Then $\Sigma_c\supsetneq\Delta\supsetneq\Sigma_{cc}$ around z_0. In this subsection, assume that Δ is a 1-dimensional manifold around z_0 and $z_0\notin\Sigma_{cc}$, since we consider complete solutions. By Theorem 2.3, $F=0$ has a complete solution on Σ_c at z_0. If Δ is not a geometric solution passing through z_0, the complete solution on Σ_c is the complete singular solution on Σ_c. On the other hand, if Δ is a geometric solution passing through z_0, we have the following result.

Proposition 3.6. *Let $F=0$ be of type RR^3_y at $z_0\in\Delta\setminus\Sigma_{cc}$. If $\gamma(t)=(x(t),y(t),p(t),q(t))\in\Delta$ is a geometric solution passing through z_0, then $\gamma(t)$ is represented by the form (a,b,c,t) where $a,b,c\in\mathbb{R}$. Moreover, $\gamma(t)$ is a leaf of both complete solutions on $F^{-1}(0)$ and Σ_c.*

Proof. Since $z_0\notin\Sigma_{cc}$, we have $(f_x-p+qf_p)_x+q(f_x-p+qf_p)_p\neq 0$ at u_0. Differentiating equalities $(f_x-p+qf_p)(x(t),p(t),q(t))=0$ and $f_p(x(t),p(t),q(t))=0$ with respect to t, we have

$$\begin{pmatrix}(f_x-p+qf_p)_x+q(f_x-p+qf_p)_p & (f_x-p+qf_p)_q\\ f_{xp}+qf_{pp} & f_{pq}\end{pmatrix}\begin{pmatrix}x'(t)\\ q'(t)\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}.$$

Since $\gamma(t)$ is a geometric solution, $(x'(t),q'(t))\neq(0,0)$ on Δ. Thus

$$\det\begin{pmatrix}(f_x-p+qf_p)_x+q(f_x-p+qf_p)_p & (f_x-p+qf_p)_q\\ f_{xp}+qf_{pp} & f_{pq}\end{pmatrix}=0$$

on Δ. It follows that $\alpha|_\Delta\equiv 0$ and hence $x'(t)\equiv 0$. This means that $x(t)$ is constant on Δ around z_0. By the forms of the vector field X for a complete solution on $F^{-1}(0)$ and of the vector field Y for a complete solution on Σ_c (which appeared in the proof of Theorem 3.2), it follows that $\Gamma|_{\Gamma^{-1}(\Delta)}=\Phi|_{\Phi^{-1}(\Delta)}=\gamma$. □

3.6. **On the type SC_y.** Suppose that $F=0$ is of type C at $z_0\in\Sigma_c$ and z_0 is a singular point of $F_q|_{F^{-1}(0)}$ and $F_X|_{F^{-1}(0)}$. We call this type SC_y. See Example 4.4.

Proposition 3.7. *Let $F=0$ be of type SC_y at z_0. If Σ_c is a 2-dimensional manifold around z_0, then $z_0\notin\Sigma_{cc}$.*

Proof. Let $F(x,y,p,q)=-y+f(x,p,q)=0$. Since $F=0$ is of type C at z_0, there is a function germ $\alpha:(F^{-1}(0),z_0)\to\mathbb{R}$ such that

$$f_x-p+qf_p=\alpha\cdot f_q. \tag{3}$$

By differentiating (3) with respect to p, we have $f_{xp}-1+qf_{pp}=\alpha_p\cdot f_q+\alpha\cdot f_{pq}$. Hence $f_{xp}+qf_{pp}=1$ at u_0. By a direct calculation,

$$(f_x-p+qf_p)_{xq}+q(f_x-p+qf_p)_{pq}=(f_{xq}+qf_{pq})_x+q(f_{xq}+qf_{pq})_p+f_{xp}+qf_{pp}. \tag{4}$$

On the other hand, by (3),

$$\begin{aligned}&(f_x-p+qf_p)_{xq}+q(f_x-p+qf_p)_{pq}\\ &=(\alpha_{xq}+q\alpha_{pq})\cdot f_q+\alpha_q\cdot(f_{qx}+qf_{pq})+(\alpha_x+q\alpha_p)\cdot f_{qq}+\alpha\cdot(f_{xqq}+qf_{pqq}).\end{aligned} \tag{5}$$

By definition, $\phi^{-1}(\Sigma_c)=f_q^{-1}(0)$. Since Σ_c is a 2-dimensional manifold around z_0, there is a regular function germ $g:(U,u_0)\to\mathbb{R}$ and a function germ $k:(U,u_0)\to(\mathbb{R},0)$ such that

$\phi^{-1}(\Sigma_c) = g^{-1}(0)$ and $f_q = k \cdot g$ at least locally. By a direct calculation, the right hand of (4) is given by

$$((k_x+qk_p)_x+q(k_x+k_p)_p)\cdot g+2(k_x+qk_p)\cdot(g_x+qg_p)+k\cdot((g_x+qg_p)_x+q(g_x+qg_p)_p)+f_{xp}+qf_{pp}.$$

Also the right hand of (5) is given by

$$\begin{aligned}&(\alpha_{xq}+q\alpha_{pq})\cdot k\cdot g+\alpha_q\cdot((k_x+qk_p)\cdot g+k\cdot(g_x+qg_p))+(\alpha_x+q\alpha_p)\cdot(k_q\cdot g+k\cdot g_q)\\&\quad+\alpha\cdot((k_{xq}+qk_{pq})\cdot g+k_q\cdot(g_x+qg_p)+(k_x+qk_p)\cdot g_q+k\cdot(g_{xq}+qg_{pq}))\,.\end{aligned}$$

If $z_0 \in \Sigma_{cc}$, then $g = g_x + qg_p = g_q = 0$ at u_0. This contradicts the fact that (4) = (5), namely 1=0 at u_0. □

Under the assumption of Proposition 3.7, it follows from $z_0 \notin \Sigma_{cc}$ that there is a complete solution on Σ_c at z_0. According to Theorem 3.11 in below, a geometric solution passing through z_0 on Σ_c is a singular solution for type C. Hence the complete solution on Σ_c is the complete singular solution on Σ_c at z_0.

3.7. **On the type SR_y.** Suppose that $F = 0$ is of type R at $z_0 \in \Sigma_c$ and z_0 is a singular point of $F_q|_{F^{-1}(0)}$ and $F_X|_{F^{-1}(0)}$. We call this type SR_y. We can also prove the following result by using the same arguments in the proof of Proposition 3.7, so we omit the proof.

Proposition 3.8. *Let $F = 0$ be of type SR_y at z_0. If Σ_c is a 2-dimensional manifold around z_0, then $z_0 \notin \Sigma_{cc}$.*

Moreover, we have the following result.

Proposition 3.9. *Let $F = 0$ be of type SR_y and not of type C at z_0. If Σ_c is a 2-dimensional manifold around z_0, then Δ is a 1-dimensional manifold around z_0. Moreover, Δ is not a geometric solution passing through z_0.*

Proof. By (1), $f_q = \alpha \cdot (f_x - p + qf_p)$ with $\alpha(z_0) = 0$. Since $\phi^{-1}(\Sigma_c) = (f_x - p + qf_p)^{-1}(0)$ is a 2-dimensional manifold around z_0, there exist a regular function germ $g : (U, u_0) \to (\mathbb{R}, 0)$ and a function germ $k : (U, u_0) \to (\mathbb{R}, 0)$ such that $f_x - p + qf_p = k \cdot g$ and $k^{-1}(0) \subset g^{-1}(0)$ at least locally. By a direct calculation, we have

$$(f_x - p + qf_p)_{xq} + q(f_x - p + qf_p)_{pq} = 1$$

at u_0. On the other hand,

$$(f_x - p + qf_p)_{xq} + q(f_x - p + qf_p)_{pq} = k_q \cdot (g_x + qg_p) + (k_x + qk_p) \cdot g_q$$

at u_0. Hence $k_q \cdot (g_x + qg_p) + (k_x + qk_p) \cdot g_q = 1$ at u_0. If $g_q(u_0) = 0$, then $k_q(u_0) \neq 0$. It follows that k is represented by $\lambda(x, p, q) \cdot (q - \mu(x, p))$ at least locally, where λ and μ are function germs with $\lambda(u_0) \neq 0$. Since $k^{-1}(0) \subset g^{-1}(0)$, $g(x, p, \mu(x, p)) = 0$. By differentiating this equality with respect to x and p, we have

$$g_x(x, p, \mu(x, p)) + \mu_x(x, p)g_q(x, p, \mu(x, p)) = 0$$

and

$$g_p(x, p, \mu(x, p)) + \mu_p(x, p)g_q(x, p, \mu(x, p)) = 0.$$

This contradicts the fact that g is regular at u_0. Therefore we have $g_q \neq 0$ at u_0.

By the definition of Δ, $\phi^{-1}(\Delta) = g^{-1}(0) \cap f_p^{-1}(0)$. To show that Δ is a 1-dimensional manifold around z_0, it is sufficient to show that the matrix

$$A = \begin{pmatrix} g_x & g_p & g_q \\ f_{xp} & f_{pp} & f_{pq} \end{pmatrix}$$

has rank 2 at u_0. Since $f_x - p + qf_p$ and f_q are singular at u_0, $f_{xp} + qf_{pp} = 1$ and $f_{pq} = 0$ at u_0. Therefore rank $A = 2$ at u_0.

Next suppose that $\gamma : (\mathbb{R}, t_0) \to (\Delta, z_0); \gamma(t) = (x(t), y(t), p(t), q(t))$ is a geometric solution passing through z_0. By differentiating equalities $g(x(t), p(t), q(t)) = 0$ and $f_p(x(t), p(t), q(t)) = 0$ with respect to t, we have

$$\begin{pmatrix} (g_x + qg_p)(x(t), p(t), q(t)) & g_q(x(t), p(t), q(t)) \\ (f_{xp} + qf_{pp})(x(t), p(t), q(t)) & f_{pq}(x(t), p(t), q(t)) \end{pmatrix} \begin{pmatrix} x'(t) \\ q'(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since the determinant of the matrix

$$\begin{pmatrix} g_x + qg_p & g_q \\ f_{xp} + qf_{pp} & f_{pq} \end{pmatrix}$$

does not vanish at t_0, $(x'(t), q'(t)) = (0, 0)$ at t_0. This contradicts the fact that $\gamma(t)$ is a geometric solution passing through z_0. □

As a conclusion, if $F = 0$ is of type SR_y, not of type C at z_0 and Σ_c is a 2-dimensional manifold around z_0, then there is a complete singular solution on Σ_c at z_0 by Propositions 3.8 and 3.9.

Finally, in this section, we give an important difference between type C and type R.

Lemma 3.10. *Let $F = 0$ be of type RC_y at z_0. If $z_0 \in \Delta \setminus \Sigma_{cc}$, then Δ is not a geometric solution passing through z_0.*

Proof. By Theorem 2.2, Δ is a 1-dimensional manifold around z_0. Suppose that

$$\gamma : (\mathbb{R}, t_0) \to (\Delta, z_0); \gamma(t) = (x(t), y(t), p(t), q(t))$$

is a geometric solution passing through z_0. Differentiating

$$f_p(x(t), p(t), q(t)) = 0 \quad \text{and} \quad f_q(x(t), p(t), q(t)) = 0$$

with respect to t, we have

$$\begin{pmatrix} (f_{xp} + qf_{pp})(x(t), p(t), q(t)) & f_{pq}(x(t), p(t), q(t)) \\ (f_{xq} + qf_{pq})(x(t), p(t), q(t)) & f_{qq}(x(t), p(t), q(t)) \end{pmatrix} \begin{pmatrix} x'(t) \\ q'(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Moreover, differentiating (3) with respect to p and q, $f_{xp} - 1 + qf_{pp} = \alpha_p \cdot f_q + \alpha \cdot f_{pq}$ and $f_{xq} + f_p + qf_{pq} = \alpha_q \cdot f_q + \alpha \cdot f_{qq}$ respectively. Then

$$\det \begin{pmatrix} (f_{xp} + qf_{pp})(x(t), p(t), q(t)) & f_{pq}(x(t), p(t), q(t)) \\ (f_{xq} + qf_{pq})(x(t), p(t), q(t)) & f_{qq}(x(t), p(t), q(t)) \end{pmatrix} = f_{qq}(x(t), p(t), q(t)).$$

The condition $z_0 \notin \Sigma_{cc}$ guarantees that $f_{qq} \neq 0$ at u_0. It follows that $(x'(t), q'(t)) = (0, 0)$ at t_0. This contradicts the fact that $\gamma(t)$ is a geometric solution passing through z_0. □

Theorem 3.11. *Let $F = 0$ be of type C at z_0. If $\gamma(t) = (x(t), y(t), p(t), q(t)) \in \Sigma_c$ is a geometric solution passing through z_0, then $\gamma(t)$ is the singular solution.*

Proof. First we assume that z_0 is a regular point of $F_q|_{F^{-1}(0)}$. If $z_0 \notin \Delta$, then $\gamma(t)$ is a singular solution passing through z_0 and hence we may regard that $\gamma(t) \subset \Delta$ by Theorem 2.2. Also if $z_0 \notin \Sigma_{cc}$, then $\gamma(t)$ is not a geometric solution passing through z_0 by Lemma 3.10. We may assume that $\gamma(t) \subset \Sigma_{cc}$. Then we can conclude that $\gamma(t)$ is a singular solution passing through z_0, see Remark 3.1.

Next we assume that z_0 is a singular point of $F_q|_{F^{-1}(0)}$. Also we may regard that $\gamma(t) \subset \Delta$. By differentiating $f_p(x(t), p(t), q(t)) = 0$ with respect to t,

$$(f_{xp} + qf_{pp})(x(t), p(t), q(t)) \cdot x'(t) + f_{pq}(x(t), p(t), q(t)) \cdot q'(t) = 0.$$

Since $f_{xp} - 1 + qf_{pp} = \alpha_p \cdot f_q + \alpha_p \cdot f_{pq}$, we have

$$(1 + \alpha \cdot f_{pq}(x(t), p(t), q(t))) \cdot x'(t) + f_{pq}(x(t), p(t), q(t)) \cdot q'(t) = 0.$$

By the assumption, $f_{pq}(u_0) = 0$. Hence $x'(t_0) = 0$ and $q'(t_0) \neq 0$. It follows from the form of smooth complete solution, $\gamma(t)$ is the singular solution passing through z_0. This completes the proof of Theorem 3.11. □

As a consequence, if $F = 0$ is of type C and there exists a geometric solution on the contact singular set, then uniqueness for geometric solutions does not hold.

4. Examples

We give examples of completely integrable second order ODEs. For more examples, refer to [3, Examples 5.1 and 5.2] etc.

Example 4.1. Let $F(x, y, p, q) = y + (1/2)p^2q^{2n+1} = 0$, where n is a natural number. In this case, $F_X = p(1+q^{2n+2})$ and $F_q = (1/2)(2n+1)p^2q^{2n}$. Hence $F = 0$ is of type R at $z_0 \in F^{-1}(0)$. Since 0 is a regular value of $F_X|_{F^{-1}(0)}$, and

$$\Sigma_c = \{(x, y, p, q) \mid y = p = 0\} = \Delta, \quad \Sigma_{cc} = \{(x, y, p, q) \mid y = p = q = 0\},$$

$F = 0$ is of type RR^1_y at $z_0 \in \Sigma_c$. By Theorems 2.3, 3.2 and Proposition 3.3, there exist a complete solutions on $F^{-1}(0)$ and Σ_c, and a singular solution. Indeed, the complete solutions $\Gamma : \mathbb{R} \times \mathbb{R}^2 \to F^{-1}(0), \Phi : \mathbb{R} \times \mathbb{R} \to \Sigma_c$ and the singular solution $\gamma : \mathbb{R} \to \Sigma_{cc}$ are given by

$$\Gamma(t, r, s) = \Big(-\frac{2n+1}{2} r \int (1 + t^{2n+2})^{-\frac{6n+5}{4(n+1)}} t^{2n} dt + s,$$
$$-\frac{1}{2} r^2 t^{2n+1} (1 + t^{2n+2})^{-\frac{2n+1}{2(n+1)}}, r(1 + t^{2n+2})^{-\frac{2n+1}{4(n+1)}}, t \Big),$$

$\Phi(t, a) = (a, 0, 0, t)$ and $\gamma(t) = (t, 0, 0, 0)$. We can observe that $\Gamma|_{\Gamma^{-1}(\Sigma_c)} = \Phi$.

Example 4.2. Let $F(x, y, p, q) = -y + pq^n - (n/(2n+1))q^{2n+1} = 0$, where n is a natural number. In this case, $F_X = -p + q^{n+1}$ and $F_q = -nq^{n-1}(-p + q^{n+1})$. Hence $F = 0$ is of type C and of type R for $n = 1$, and of type R for $n \geq 2$ at $z_0 \in F^{-1}(0)$. Since 0 is a regular value of $F_X|_{F^{-1}(0)}$ and

$$\Sigma_c = \left\{ (x, y, p, q) \mid y = \frac{n+1}{2n+1} q^{2n+1},\ p = q^{n+1} \right\},\ \Delta = \{(x, y, p, q) \mid y = p = q = 0\} = \Sigma_{cc},$$

$F = 0$ is of type RR^2_y at $z_0 \in \Delta$. Note that $F = 0$ is also of type RC_y at z_0 if $n = 1$. By Theorems 2.3 and 3.4, there exist a complete solution on $F^{-1}(0)$ and a complete singular solution on Σ_c. Moreover, $F = 0$ satisfies the condition (i) of Proposition 3.5 in §3.4, Σ_{cc} is an isolated singular solution. Indeed, the complete solution on $F^{-1}(0)$, the complete singular solution on Σ_c and the isolated singular solution are given by

$$\Gamma(t, r, s) = \left(t^n + r, \frac{n^2}{(n+1)(2n+1)} t^{2n+1} + st^n, \frac{n}{n+1} t^{n+1} + s, t \right),$$

$$\Phi(t, a) = \left(\frac{n+1}{n} t^n + a, \frac{n+1}{2n+1} t^{2n+1}, t^{n+1}, t \right) \text{ and } \gamma(t) = (t, 0, 0, 0).$$

If $n=1$, the complete solution on $F^{-1}(0)$ can be parametrised by

$$\Gamma(t,r,s)=\left(t,\frac{1}{6}t^3+\frac{1}{2}rt^2+st+rs-\frac{1}{3}r^3,\frac{1}{2}t^2+rt+s,t+r\right).$$

Example 4.3. Let

$$F(x,y,p,q)=-y+(1/2)x^2-(1/n)pq^n+(1/n)xq^n+(1/2n^2)q^{2n}-(1/n(2n+1))q^{2n+1}=0,$$

where n is a natural number. In this case, $F_X=x+(1/n)q^n-p-(1/n)q^{n+1}$ and $F_q=q^{n-1}F_X$. Since 0 is a regular value of $F_X|_{F^{-1}(0)}$ and

$$\Sigma_c=\left\{(x,y,p,q)\mid y=\frac{1}{2}x^2-\frac{1}{2n^2}q^{n+1}+\frac{n+1}{n^2(2n+1)}q^{2n+1}\right\},$$

$$\Delta=\left\{(x,y,p,q)\mid y=\frac{1}{2}x^2,p=x,q=0\right\},\ \Sigma_{cc}=\emptyset,$$

$F=0$ is of type RR_y^3 at $z_0\in\Delta$. Note that if $n=1$, then $F=0$ is also of type RC_y at z_0. By Theorem 2.3, there exist complete solutions on $F^{-1}(0)$ and Σ_c. Since Δ is not a geometric solution, the complete solution on Σ_c is the complete singular solution on Σ_c. The complete solution on $F^{-1}(0)$ and the complete singular solution on Σ_c at 0 are given by

$$\Gamma(t,r,s)=\left(-\frac{1}{n}t^n+r,\frac{1}{(n+1)(2n+1)}t^{2n+1}-\frac{1}{n}st^n+\frac{1}{2}r^2,-\frac{1}{n+1}t^{n+1}+s,t\right),$$

$$\Phi(t,a)=\left(x(t,a),\frac{1}{2}x(t,a)^2-\frac{1}{2n^2}t^{n+1}+\frac{n+1}{n^2(2n+1)}t^{2n+1},x(t,a)+\frac{1}{n}t^n-\frac{1}{n}t^{n+1},t\right),$$

where

$$x(t,a)=-\frac{1}{n}\left(\frac{n+1}{n}t^n+\frac{1}{n-1}t^{n-1}+\cdots+\frac{1}{2}t^2+t+\log|t-1|\right)+a.$$

Example 4.4. Let $F(x,y,p,q)=-y+xp-(1/2)x^2q+x^n=0$, where n is a natural number. In this case, $F_X=nx^{n-1}$ and $F_q=-(1/2)x^2$. Hence $F=0$ is of type R for $n=1$ and 2 at $z_0\in F^{-1}(0)$. Also $F=0$ is both types of C and R for $n=3$, and of type C for $n\geq 4$ at z_0.

First suppose that $n=1$. Since $F_X=1$, we have $\Sigma_c=\emptyset$. It follows that $F=0$ is of type R_X at z_0. The complete solution on $F^{-1}(0)$ at 0 is given by

$$\Gamma(t,r,s)=\left(\frac{2r}{1-rt},\frac{4r}{1-rt}\log|1-rt|+\frac{4r+2rs}{1-rt}+\frac{2r}{(1-rt)^2},2\log|1-rt|+\frac{2}{1-rt}+s,t\right).$$

Second suppose that $n=2$. Since 0 is a regular value of $F_X|_{F^{-1}(0)}$ and

$$\Sigma_c=\{(x,y,p,q)\mid x=y=0\}=\Delta,\ \Sigma_{cc}=\emptyset,$$

$F=0$ is of type RR_y^1 at $z_0\in\Delta$. The complete solutions on $F^{-1}(0)$ and Σ_c are given by

$$\Gamma(t,r,s)=\left(re^{\frac{t}{4}},\frac{r^2}{2}te^{\frac{t}{2}}-3r^2e^{\frac{t}{2}}+rse^{\frac{t}{4}},rte^{\frac{t}{4}}-4re^{\frac{t}{4}}+s,t\right),$$

$\Phi(t,a)=(0,0,a,t)$. We can observe that $\Gamma|_{\Gamma^{-1}(\Sigma_c)}=\Phi$.

Finally suppose that $n\geq 3$. Since 0 is a singular value of $F_q|_{F^{-1}(0)}$ and $F_X|_{F^{-1}(0)}$, $F=0$ is of type SC_y at $z_0\in\Delta$. We have

$$\Sigma_c=\{(x,y,p,q)\mid x=y=0\}=\Delta,\ \Sigma_{cc}=\emptyset.$$

The complete solution on $F^{-1}(0)$ and the complete singular solution on Σ_c are given by

$$\Gamma(t,r,s)=\left(t,\frac{2}{(n-2)(n-1)}t^n+\frac{1}{2}rt^2+st,\frac{2n}{(n-2)(n-1)}t^{n-1}+rt+s,\frac{2n}{n-2}t^{n-2}+r\right),$$

$\Phi(t, a) = (0, 0, a, t)$. Note that if $n = 3$, then $F = 0$ is also of type SR_y at z_0.

Example 4.5. Let $F(x, y, p, q) = xq + (a - x)p - by = 0$ be the confluent hypergeometric equations (the degenerate hypergeometric equations), where $a, b \in \mathbb{R}$, see in [16]. The equation have the confluent hypergeometric function as a solution. However, we can decide by using the results whether the equation have a complete solution or not. This is a new viewpoint for the equation as far as we know.

Since we consider the regular equation, we may assume that $b \neq 0$. By

$$F_X = q(1 + a - x) - p(1 + b) \quad \text{and} \quad F_q = x,$$

$$\Sigma_c = \{(x, y, p, q) \mid x = 0, ap - by = 0, q(1 + a) - p(1 + b) = 0\}.$$

If $z_0 \notin \Sigma_c$, then there exist a complete solution at z_0 and also a unique geometric solution passing through z_0. If $z_0 \in \Sigma_c$ and $a = -1, b = -1$, then $F_X = q \cdot F_q$, Σ_c is a 2-dimensional manifold and $\Sigma_{cc} = \emptyset$. It follows that $F = 0$ is of type RC_y at z_0. By Theorem 2.2, there exist a complete solution on $F^{-1}(0)$ and a complete singular solution on Σ_c. The complete solution on $F^{-1}(0)$ and the complete singular solution on Σ_c are given by

$$\Gamma(t, r, s) = \left(t, re^t + (1 + t)s, re^t + s, re^t\right), \ \Phi(t, a) = (0, a, a, t).$$

If $z_0 \in \Sigma_c$ and $a = -1, b \neq -1$ (respectively, $a \neq -1$), then Σ_c is a 1-dimensional manifold. Hence $F = 0$ is not completely integrable at z_0.

Appendix A. Completely integrable implicit first order ODEs

In this appendix, we quickly review known results for the theory of completely integrable implicit first order ODEs

$$F(x, y, p) = 0, \ p = dy/dx.$$

For more detail, see [10, 11, 12, 13, 19]. Assume that 0 is a regular value of F. We say that $F = 0$ is completely integrable at a point if there exists an immersive one-parameter family of geometric solutions on $F^{-1}(0)$ at the point. The contact singular set $\Sigma_c = \Sigma_c(F)$ is given by

$$\Sigma_c = \{z \in J^1(\mathbb{R}, \mathbb{R}) \mid F(z) = 0, F_X(z) = 0, F_p(z) = 0\}.$$

Here $F_X = F_x + pF_y$. We say that an equation $F = 0$ is of *(first order) Clairaut type* (for short, *type C*) at z_0 if there exists a function germ $\alpha : (F^{-1}(0), z_0) \to \mathbb{R}$ such that

$$F_X|_{F^{-1}(0)} = \alpha \cdot F_p|_{F^{-1}(0)},$$

and of *reduced type* (for short, *type R*) at z_0 if there exists a function germ $\beta : (F^{-1}(0), z_0) \to \mathbb{R}$ such that

$$F_p|_{F^{-1}(0)} = \beta \cdot F_X|_{F^{-1}(0)},$$

In [11], it has been shown the following results.

Theorem A.1. ([11]) *Let $F(x, y, p) = 0$ be an implicit first order ODE at z_0. The following are equivalent:*

(1) *$F = 0$ is completely integrable at z_0.*
(2) *$F = 0$ is either of type C or of type R at z_0.*
(3) *$z_0 \notin \Sigma_c$ or Σ_c is a 1-dimensional manifold around z_0.*

Moreover, if Σ_c is a 1-dimensional manifold around z_0, then Σ_c is a singular solution of $F = 0$ passing through z_0.

Now suppose that $z_0 \in \Sigma_c$. Since $F = 0$ is regular, $F_y(z_0) \neq 0$. By the implicit function theorem, there exists a smooth function $f : U \to \mathbb{R}$, where U is an open set in $\mathbb{R}^2$, such that in a neighbourhood of z_0, $(x, y, p) \in F^{-1}(0)$ if and only if $-y + f(x, p) = 0$. Thus we may assume without loss of generality that $F(x, y, p) = -y + f(x, p) = 0$. It follows that z_0 is a regular point of either $F_p|_{F^{-1}(0)}$ or $F_X|_{F^{-1}(0)}$. Therefore, completely integrable implicit first order ODEs have four kinds of types (cf. [19]), see Table 2.

Conditions			Type	Name
$z_0 \notin \Sigma_c$	$F_p(z_0) \neq 0$		C	C_p
	$F_X(z_0) \neq 0$		R	R_X
$z_0 \in \Sigma_c$	$F_y(z_0) \neq 0$	z_0 is a regular point of $F_p\|_{F^{-1}(0)}$	C	RC_y
		z_0 is a regular point of $F_X\|_{F^{-1}(0)}$	R	RR_y

Table 2. A classification of types of completely integrable implicit first order ODEs at z_0.

References

[1] V. I. Arnol'd, *Geometrical methods in the theory of ordinary differential equations*, Second edition, Springer-Verlag, New York (1988).
[2] V. I. Arnol'd, *Ordinary differential equations*, Springer Textbook, Springer-Verlag, Belin (1992). DOI: 10.14492/hokmj/1350659159
[3] M. Bhupal, On singular solutions of implicit second-order ordinary differential equations, *Hokkaido Math. J.* **32** (2003), 623–641.
[4] J. W. Bruce and F. Tari, On binary differential equations, *Nonlinearity.* **8** (1995), 255–271. DOI: 10.1088/0951-7715/8/2/008
[5] J. W. Bruce and F. Tari, Implicit differential equations from the singularity theory viewpoint, *Banach Center Publ.,* **33** (1996), 23–38.
[6] R. Courant and D. Hilbert, *Methods of Mathematical Physics II*, Wiley, New York (1962).
[7] L. Dara, Singularites generiques des equations differentielles multiformes, *Bol. Soc. Brasil. Mat.* **6** (1975), 95–128. DOI: 10.1007/BF02584779
[8] A. A. Davydov, The normal form of a differential equation, that is not solved with respect to the derivative, in the neighborhood of its singular point, *Funktsional. Anal. i Prilozhen.* **19** (1985), 1–10.
[9] M. Fukuda and T. Fukuda, Singular solutions of ordinary differential equations, *Yokohama Math. J.* **25** (1977), 41–58.
[10] A. Hayakawa, G. Ishikawa, S. Izumiya and K. Yamaguchi, Classification of generic integral diagrams and first order ordinary differential equations, *Int. J. Math.* **5** (1994), 447–489. DOI: 10.1142/S0129167X94000255
[11] S. Izumiya, Singular solutions of first-order differential equations, *Bull. London Math. Soc.* **26** (1994), 69–74. DOI: 10.1112/blms/26.1.69
[12] S. Izumiya, On Clairaut-type equations, *Publ. Math. Debrecen.* **45** (1995), 159–166.
[13] S. Izumiya and J. Yu, How to define singular solutions, *Kodai Math. J.* **16** (1993), 227–234. DOI: 10.2996/kmj/1138039786
[14] M. Lemasurier, Singularities of second-order implicit differential equations: a geometrical approach, *J. Dyn. Control Syst.* **7** (2001), 277–298. DOI: 10.1023/A:1013007405832
[15] Y. Machida and M. Takahashi, Classifications of implicit second-order ordinary differential equations of Clairaut type, *Proc. Roy. Soc. Edinburgh Sect. A.* **138** (2008), 821–842. DOI: 10.1017/S0308210506000977
[16] A. D. Polyanin and V. F. Zaitsev, *Handbook of exact solutions for ordinary differential equations*, Second edition, Chapman & Hall/CRC, Boca Raton, FL (2003).
[17] A. O. Remizov, The multidimensional Poincare construction and singularities of lifted fields for implicit differential equations, *Sovrem. Mat. Fundam. Napravl.* **19** (2006), 131–170.
[18] M. Takahashi, On implicit second order ordinary differential equations: Completely integrable and Clairaut type, *J. Dyn. Control Syst.* **13** (2007), 273–288. DOI: 10.1007/s10883-007-9013-9

[19] M. Takahashi, On completely integrable first order ordinary differential equations, *Proceedings of the Australian-Japanese Workshop on Real and Complex singularities.* (2007), 388–418.

[20] M. Takahashi, On complete solutions and complete singular solutions of second order ordinary differential equations, *Colloquium Math.* **109** (2007), 271–285. DOI: 10.4064/cm109-2-9

[21] M. Takahashi, Completely integrable implicit ordinary differential equations, *Yokohama Math. J.* **58** (2012), 17–29.

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LIPS AND SWALLOW-TAILS OF SINGULARITIES OF PRODUCT MAPS

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Abstract. Lips and swallow-tails are generic local moves of singularities of a smooth map to a 2-manifold. We prove that these moves of singularities of the product map of two functions on a 3-manifold can be realized by isotopies of the functions.

1. Introduction

We consider the relationship between a pair of maps and the product map. Let M, P, Q be smooth manifolds and let $C^\infty(M, *)$ denote the space of smooth maps from M to a smooth manifold $*$ endowed with the Whitney C^∞ topology. Two smooth maps $F \in C^\infty(M, P)$ and $G \in C^\infty(M, Q)$ determine the product map $(F, G) \in C^\infty(M, P \times Q)$ by $(F, G)(p) = (F(p), G(p))$. Conversely, a smooth map $\varphi \in C^\infty(M, P \times Q)$ can be decomposed into $\pi_P \circ \varphi \in C^\infty(M, P)$ and $\pi_Q \circ \varphi \in C^\infty(M, Q)$, where $\pi_P : P \times Q \to P$ and $\pi_Q : P \times Q \to Q$ are the projections. By Chapter II, Proposition 3.6 of [1], this correspondence gives the homeomorphism

$$(\sharp) \qquad C^\infty(M, P) \times C^\infty(M, Q) \cong C^\infty(M, P \times Q).$$

The homeomorphism ($\sharp$) however does not mean the singularity theoretic equivalence. More specifically, isotopies of F and G do not always induce an isotopy of (F, G), and an isotopy of φ does not always induce isotopies of $\pi_P \circ \varphi$ and $\pi_Q \circ \varphi$. Here, an isotopy of a map is a homotopy preserving the topological properties of the map. The partition of a mapping space into isotopy classes is of general interest in singularity theory, but few things are known about the relation between the partitions of the both sides of the homeomorphism ($\sharp$).

We focus on the case where M is closed and 3-dimensional, and P, Q are 1-dimensional. We do not assume the orientability of M. Suppose $F : M \to P$ and $G : M \to Q$ are smooth functions such that $\varphi = (F, G)$ is stable. A singular point of φ is then either a fold point or a cusp point. By Levine's [5] theorem, we can eliminate the cusp points by a homotopy of φ. It implies that we can eliminate the cusp points by homotopies of F and G. Note that we cannot reduce the number of cusp points by an isotopy of φ. We propose the following question.

Question 1. *Can we eliminate the cusp points of φ by (quasi-)isotopies of F and G?*

Our strategy to attack Question 1 is to deform φ by a sequence of global isotopies and local homotopies which can be realized by (quasi-)isotopies of F, G. Johnson [3, Section 6] showed what kind of global isotopy of φ can be realized by (quasi-)isotopies of F, G (see Corollary 7). In this paper, we prove the following.

Theorem 2. *The local moves of the discriminant set of φ as in Figure 1 can be realized by isotopies of F and G.*

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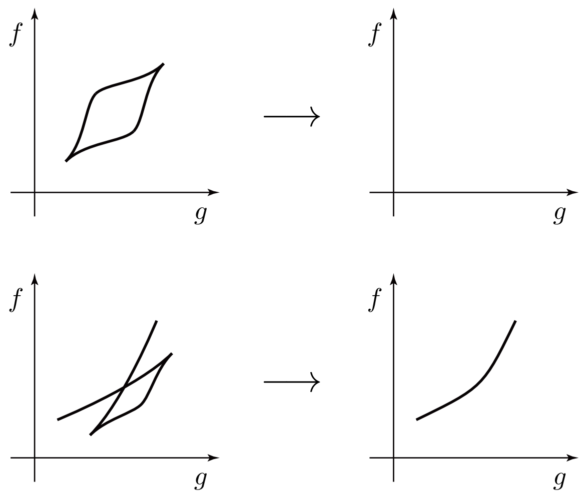

FIGURE 1. Local moves which reduce the number of cusp points of φ. Here, (f, g) is the coordinate system given by the product structure of $P \times Q$. We require that the local moves do not involve tangent lines of the discriminant set parallel to the axes.

Forgetting the axes, these moves are ones of generic local moves of singularities of φ known as a "lip" and a "swallow-tail", respectively. See [11] for the classification of generic local moves. We expect that we can use our method (Proposition 11) to work out other local moves, and we hope that we can use them to approach a global theory.

We would like to mention here the relation of this work to Heegaard theory of 3-manifolds. Rubinstein–Scharlemann [13] introduced *the graphic* for comparing two Heegaard splittings. Kobayashi–Saeki [4] interpreted the graphic as the discriminant set of the product map of two functions representing the splittings. Johnson [3] gave an upper bound for the *Reidemeister–Singer distance* between two Heegaard splittings in terms of the graphic. The author [14] developed Johnson's idea to show that the Reidemeister–Singer distance is at most the sum of the genera of the splittings plus the number of cusp points of the product map. If Question 1 is answered positively, it ensures that the Reidemeister–Singer distance is at most the sum of the genera, which is the best possible bound by Hass–Thompson–Thurston's [2] example.

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2. Morse functions and stable maps

In this section, we briefly review standard definitions and facts on singularities of smooth maps. We refer the reader to [10] for basic notions in Morse theory, and to [1] for detailed description of stable maps.

A *Morse function* on a compact smooth manifold M possibly with boundary is a smooth function F from M to either $\mathbb{R}$ or S^1 satisfying the following:

- All the critical points of F are non-degenerate and belong to $\text{int}M$.
- The function F is constant on each component of ∂M.

We regard $\mathbb{R}$ and S^1 as oriented. This gives the distinction between locally minimal components and locally maximal components of ∂M with respect to F. A smooth homotopy consisting of Morse functions is called a *quasi-isotopy*. Two Morse functions are said to be *quasi-isotopic* if there is a quasi-isotopy between them.

We consider generic homotopies of smooth functions from a compact connected surface Σ to $\mathbb{R}$. Let $\{\alpha_t : \Sigma \to \mathbb{R}\}_{t\in[-1,1]}$ be a smooth homotopy. A *birth* (resp. *death*) of $\{\alpha_t\}_{t\in[-1,1]}$ is the pair (o, σ) of $o \in (-1, 1)$ and $\sigma \in \mathrm{int}\Sigma$ such that $\alpha_\tau(\xi, \eta) = \tau\xi_\tau - \xi_\tau^3 + \eta_\tau^2$ for a local coordinate system (ξ, η) at σ, a local coordinate τ at o whose direction agrees (resp. disagrees) with that of t, and a local coordinate of the target $\mathbb{R}$. Here $\{(\xi, \eta) \mapsto (\xi_\tau, \eta_\tau)\}_\tau$ is a smooth family of coordinate transformations. A *passing* of $\{\alpha_t\}_{t\in[-1,1]}$ is the pair $(o, \{\sigma, \sigma'\})$ of $o \in (-1, 1)$ and $\{\sigma, \sigma'\} \subset \mathrm{int}\Sigma$ such that σ, σ' are non-degenerate critical points of α_o with the same value, and $\{(t, v) \in \mathbb{R}^2 \mid v \text{ is a critical value of } \alpha_t|_{U\cup U'}\}$ has a transverse crossing at $(o, \alpha_o(\sigma))$ for small neighborhoods U, U' of σ, σ', respectively. The homotopy $\{\alpha_t\}_{t\in[-1,1]}$ is said to be *generic* if it consists of Morse functions whose critical points have pairwise distinct values except that $\{\alpha_t\}_{t\in[-1,1]}$ has either a single birth, a single death or a single passing at each t in a finite subset of $(-1, 1)$. Note that a generic homotopy without births and deaths is a quasi-isotopy.

Theorem 3 (Maksymenko [7])**.** *Two Morse functions from Σ to $\mathbb{R}$ are quasi-isotopic if and only if they have the same number of critical points of each index, and the same sets of locally minimal and locally maximal components of $\partial\Sigma$.*

An *isotopy* of a smooth map $\varphi : M \to N$ between general smooth manifolds M, N is a homotopy $\{\varphi_t : M \to N\}_{t\in[0,1]}$ which is decomposed as $\varphi_t = H_t^N \circ \varphi \circ H_t^M$. Here $\{H_t^M\}_{t\in[0,1]}$, $\{H_t^N\}_{t\in[0,1]}$ are smooth ambient isotopies of M, N, respectively, such that $H_0^M = id_M$ and $H_0^N = id_N$. Two smooth maps are said to be *isotopic* if there is an isotopy between them.

A *stable map* from M to N is a smooth map $\varphi : M \to N$ such that there exists an open neighborhood U of φ in $C^\infty(M, N)$ such that every map in U is isotopic to φ. We remark that an equivalent definition of stable map is given by using "*right-left equivalent*" in place of "isotopic". We note that, in the case where M is closed and N is either $\mathbb{R}$ or S^1, the smooth map φ is stable if and only if φ is a Morse function whose critical points have pairwise distinct values.

Consider the case where M is a closed smooth 3-manifold and N is a smooth surface. Recall that $p \in M$ is a *regular point* of φ if the differential $(d\varphi)_p : T_pM \to T_{\varphi(p)}N$ is surjective, and otherwise a *singular point*. The set S_φ of singular points of φ is called the *singular set* and its image $\varphi(S_\varphi)$ is called the *discriminant set* of φ. At a regular point $p \in M \setminus S_\varphi$, the map φ has the standard form $\varphi(u, x, y) = (u, x)$ for some coordinate neighborhoods of p and $\varphi(p)$. Standard forms are also known for generic types of singular points as follows.

A *fold point* is a singular point p where φ has the form $\varphi(u, x, y) = (u, x^2 \pm y^2)$ for a coordinate neighborhood U of $p = (0, 0, 0)$ and a coordinate neighborhood of $\varphi(p) = (0, 0)$. The Jacobian matrix of $\varphi(u, x, y) = (u, x^2 \pm y^2)$ says that the singular set $S_\varphi \cap U$ is the arc $\{(u, 0, 0)\}$. It follows that each singular point on $\{(u, 0, 0)\}$ is also a fold point by a translation of the local coordinates. The arc $\{(u, 0, 0)\}$ is embedded to the arc $\{(u, 0)\} \subset N$ by φ.

A *cusp point* is a singular point p where φ has the local form $\varphi(u, x, y) = (u, ux - x^3 + y^2)$. One can check that the singular set $S_\varphi \cap U$ is the arc $\{(3x^2, x, 0)\}$, and consists of fold points except for the cusp point $p = (0, 0, 0)$. Note that the arc $\{(3x^2, x, 0)\}$ is a regular curve but its image $\{(3x^2, 2x^3)\} \subset N$ has a cusp at $\varphi(p) = (0, 0)$.

Assume that the singular set S_φ consists only of fold points and cusp points. By the above local observations and the compactness of M, we can see the outline of S_φ. It is a 1-dimensional submanifold of M, namely a collection of smooth circles. There are finitely many cusp points and the restriction $\varphi|_{S_\varphi}$ is an immersion except that each cusp point maps to a cusp. The next characterization of stable maps follows from Mather's theorems [8, Theorem A, Proposition 1.8] and [9, Theorem 4.1].

Theorem 4 (Mather)**.** *A smooth map φ from a closed smooth 3-manifold to a smooth surface is stable if and only if:*

- *The singular set S_φ consists only of fold points and cusp points.*
- *The restriction $\varphi|_{S_\varphi}$ has no double points at cusps, and the immersion $\varphi|_{S_\varphi \setminus (cusp\ points)}$ has only normal crossings.*

The *Stein factorization* W_φ of a general smooth map $\varphi : M \to N$ is the quotient space $M/\sim$, where $p_1 \sim p_2$ if p_1, p_2 belong to the same connected component of a level set of φ. Let q_φ denote the quotient map form M to W_φ. We can see that there is also a unique continuous map $\bar{\varphi} : W_\varphi \to N$ such that $\varphi = \bar{\varphi} \circ q_\varphi$. The Stein factorization of a stable map from a closed smooth 3-manifold to a smooth surface is, in fact, a 2-dimensional cell complex. See [6] for example.

3. Two functions and the product map

In this section, we review a local theory of singularities of two functions and the product map.

We use the following notation. Suppose M is a closed smooth 3-manifold and P, Q are either $\mathbb{R}$ or S^1. Let $F : M \to P$ and $G : M \to Q$ be smooth functions, and φ denote the product map (F, G). While we do not assume F, G to be Morse, we assume φ to be stable in this section.

The singular set S_φ includes the critical points of F and G, which can be seen as follows. For each point $p \in M$, there is a local coordinate system (f, g) at $\varphi(p)$ given by the product structure of $P \times Q$. The Jacobian matrix of φ with respect to this coordinate system is composed of the gradients of F and G. If p is a critical point of F or G, the Jacobian matrix has rank at most one, namely p is a singular point of φ.

We read information about F, G from the discriminant set of φ. Note that $\varphi|_{S_\varphi}$ is an immersion of circles with finitely many cusps. We can define the *slope* of the discriminant set $\varphi(S_\varphi)$ at $\varphi(p)$ for each $p \in S_\varphi$ with respect to the coordinate system (f, g). In particular, a point on the discriminant set with slope zero (resp. infinity) is called a *horizontal* (resp. *vertical*) *point*. We can also define the second derivative of $\varphi(S_\varphi)$ outside of vertical points and cusps, by regarding an arc of $\varphi(S_\varphi)$ as the graph of a function. In particular, a point with second derivative zero is called an *inflection point*. Since zero or non-zero of the second derivative is preserved by rotating the coordinate system, an inflection point can be defined also for vertical points.

Lemma 5. *A point $p \in M$ is a critical point of F (resp. G) if and only if $\varphi(p)$ is a vertical (resp. horizontal) point of the image of a small neighborhood of p in S_φ.*

Lemma 6. *A critical point p of F (resp. G) degenerates if and only if p is a fold point of φ and $\varphi(p)$ is a vertical (resp. horizontal) inflection point of the image of a small neighborhood of p in S_φ.*

The above lemmas were originally described by Johnson [3, Lemmas 10 and 11], and simple analytic proofs were given by the author [14, Lemmas 11 and 12]. Both Johnson and the author considered only the case of $P = Q = \mathbb{R}$, but the proofs are independent of whether P, Q are $\mathbb{R}$ or S^1.

By Lemmas 5 and 6, the function F (resp. G) is Morse if the discriminant set $\varphi(S_\varphi)$ does not have vertical (resp. horizontal) inflection points. Note that the f-coordinate (g-coordinate) of each vertical (resp. horizontal) point of $\varphi(S_\varphi)$ corresponds to the critical value of F (resp. G). It follows that the Morse function F (resp. G) is stable if $\varphi(S_\varphi)$ does not have vertical (resp. horizontal) double tangent lines.

Corollary 7. *A deformation of $\varphi(S_\varphi)$ by an ambient isotopy of $P\times Q$ can be realized by isotopies of F, G if it keeps $\varphi(S_\varphi)$ without horizontal or vertical, inflection points and double tangent lines.*

Proof. Let $\{H_t\}_{t\in[0,1]}$ be a smooth ambient isotopy of $P \times Q$ such that $H_0 = id_{P\times Q}$. By the definitions, $\{H_t \circ \varphi\}_{t\in[0,1]}$ is an isotopy of φ and consists of stable maps. It induces homotopies $\{F_t = \pi_P \circ H_t \circ \varphi\}_{t\in[0,1]}$ of F and $\{G_t = \pi_Q \circ H_t \circ \varphi\}_{t\in[0,1]}$ of G. The deformed discriminant set $H_t(\varphi(S_\varphi))$ is the discriminant set of $H_t \circ \varphi = (F_t, G_t)$ for each $t \in [0, 1]$. Since $H_t(\varphi(S_\varphi))$ does not have horizontal or vertical, inflection points and double tangent lines, F_t and G_t are stable for each $t \in [0, 1]$. The homotopies $\{F_t\}_{t\in[0,1]}$ and $\{G_t\}_{t\in[0,1]}$ are therefore isotopies. □

4. Restrictions to level surfaces

In this section, we consider the relation between a product map restricted to an appropriate domain and the family of the restrictions of one function to level surfaces of the other function.

We use the following notation again. Suppose M is a closed smooth 3-manifold and P, Q are either $\mathbb{R}$ or S^1. Let $F : M \to P$ and $G : M \to Q$ be smooth functions, and φ denote the product map (F, G). We do not assume F, G to be Morse nor φ to be stable at this stage.

We consider the restriction of F to a level surface of G. Suppose $p \in M$ is a regular point of G, and hence the level set $G^{-1}(G(p))$ is a regular surface near the point p. The point p is a critical point of $F|_{G^{-1}(G(p))}$ if and only if p is a singular point of φ, which can be seen as follows. The gradient of the restriction $F|_{G^{-1}(G(p))}$ is the projection of the gradient of F to the orthogonal complement of the gradient of G. It is zero if and only if the gradients of F and G are linearly dependent. They are linearly dependent if and only if the differential $(d\varphi)_p$ is not surjective.

Lemma 8. *A critical point p of $F|_{G^{-1}(G(p))}$ is non-degenerate if and only if p is a fold point of φ.*

Proof. Since p is a regular point of G, there is a local coordinate system (ξ, η, τ) of M at $p = (0, 0, 0)$ such that $G(\xi, \eta, \tau) = \tau + G(p)$ and (ξ, η) is a local coordinate system of $G^{-1}(G(p))$ at p. This gives us $\varphi(\xi, \eta, \tau) = (F(\xi, \eta, \tau), \tau + G(p))$ and $F|_{G^{-1}(G(p))}(\xi, \eta) = F(\xi, \eta, 0)$. By Morin's [12, Lemme 1] characterization, the point p is a fold point of φ if and only if the critical point p of $F(\xi, \eta, 0)$ is non-degenerate. □

We consider the restrictions of F to the level surfaces of G in a domain $V \subset M$ which is defined as follows. Note that there is a canonical covering $\mathbb{R}^2 \to P \times Q$ by identifying S^1 with $\mathbb{R}/\mathbb{Z}$. Let $\bar{R} \subset \mathbb{R}^2$ be the region $\{(f, g) \in \mathbb{R}^2 \mid h_-(g) \le f \le h_+(g),\ g_- \le g \le g_+\}$, where $g_-, g_+ \in \mathbb{R}$ are constants such that $g_- < g_+$ and $h_-, h_+ : \mathbb{R} \to \mathbb{R}$ are smooth functions such that $h_-(g) < h_+(g)$ for every $g \in [g_-, g_+]$. We assume that $\bar{R}$ is embedded to $R \subset P \times Q$ by the covering map. Let V be a connected component of the preimage $\varphi^{-1}(R)$. From now on, we consider φ, F, G restricted to V, which allows us to assume that $P = Q = \mathbb{R}$. We assume that G does not have critical points in V, and that the discriminant set of φ does not intersect the two edges $\{f = h_-(g), h_+(g),\ g_- \le g \le g_+\}$ of R. Each level set $G^{-1}(g) \cap V$ is then a regular surface whose boundaries are regular level curves of $F|_{G^{-1}(g)}$. The space V is therefore a Σ-bundle over $[g_-, g_+]$, namely the direct product $\Sigma \times [g_-, g_+]$. Here Σ is a compact connected surface and each $\Sigma \times \{g\}$ is the level surface $G^{-1}(g) \cap V$.

The restrictions of F to the level surfaces of G determine a homotopy $\{\alpha_t : \Sigma \to \mathbb{R}\}_{t\in[g_-,g_+]}$. That is to say, $\alpha_t(\sigma) = F(\sigma, t)$ for each point (σ, t) in $\Sigma \times [g_-, g_+] = V$. The range of each α_t is contained in $[h_-(t), h_+(t)]$, and each component of $\partial\Sigma$ is either at the minimal level $h_-(t)$ or at the maximal level $h_+(t)$. In particular, $\{\alpha_t\}_{t\in[g_-,g_+]}$ preserves the sets of locally minimal and locally maximal components of $\partial\Sigma$. By Lemma 8 and the definition of a quasi-isotopy, we have the following.

Corollary 9. *The homotopy* $\{\alpha_t\}_{t\in[g_-,g_+]}$ *is a quasi-isotopy if and only if the singular set* $S_\varphi \cap V$ *consists only of fold points.*

The "only if" direction of this corollary extends to the following.

Lemma 10. *If the homotopy* $\{\alpha_t\}_{t\in[g_-,g_+]}$ *is generic, the map* φ *is stable in* V*.*

Proof. At each birth or death, $\{\alpha_t\}_{t\in[g_-,g_+]}$ has the local form $\alpha_\tau(\xi,\eta) = \tau\xi_\tau - \xi_\tau^3 + \eta_\tau^2$ with the notation of Section 2. Choosing a local coordinate system (u, x, y) of M as $u = \tau$, $x = \xi_\tau$, $y = \eta_\tau$, the map φ has the local form $\varphi(u, x, y) = (u, ux - x^3 + y^2)$, which is of a cusp point. Taking this together with the "only if" direction of Corollary 9, the singular set $S_\varphi \cap V$ consists only of fold points and cusp points. The conditions of a generic homotopy about the critical values imply the second condition in Theorem 4. □

The discriminant set $\varphi(S_\varphi \cap V)$ is the so-called *Cerf graphic* of $\{\alpha_t\}_{t\in[g_-,g_+]}$. That is to say, the intersection of $\varphi(S_\varphi \cap V)$ with each line $l_t = \{(f, g) \in \mathbb{R}^2 \mid g = t\}$ corresponds to the critical values of α_t, and we can read from $\varphi(S_\varphi \cap V)$ how the critical values of α_t moves with t.

We can read more about the behavior of $\{\alpha_t\}_{t\in[g_-,g_+]}$ from the Stein factorization $q_\varphi(V)$. For a general homotopy $\{\beta_t : \Sigma \to \mathbb{R}\}_{t\in[g_-,g_+]}$, we call the Stein factorization of the map $(\sigma, t) \mapsto (\beta_t(\sigma), t)$ from $\Sigma \times [g_-, g_+]$ to $\mathbb{R}^2$ the *Cerf complex* of $\{\beta_t\}_{t\in[g_-,g_+]}$. Note that for each $t \in [g_-, g_+]$, the intersection of the Cerf complex $q_\varphi(V)$ of $\{\alpha_t\}_{t\in[g_-,g_+]}$ with the preimage $\bar{\varphi}^{-1}(l_t)$ is the Stein factorization W_{α_t} of α_t, and that the composition $\pi_P \circ \bar{\varphi}|_{q_\varphi(V)\cap\bar{\varphi}^{-1}(l_t)}$ is the map $\bar{\alpha}_t : W_{\alpha_t} \to \mathbb{R}$. Suppose φ is stable in V and l_t is disjoint from cusps and crossing points of the discriminant set $\varphi(S_\varphi \cap V)$. The function α_t is Morse by Lemma 8, and the critical points have pairwise distinct values. The Stein factorization $W_{\alpha_t} = q_\varphi(V) \cap \bar{\varphi}^{-1}(l_t)$ is then a graph whose vertex has valence 1, 2 or 3. Here, points in $q_\varphi(S_\varphi \cap V) \cap \bar{\varphi}^{-1}(l_t)$ are considered as vertices of the graph W_{α_t}. We remark that W_{α_t} has no valence 2 vertices if M is orientable. We can see that a vertex of valence 2 or 3 corresponds to an index 1 critical point of α_t. Regarding $\bar{\alpha}_t$ as a height function, a locally minimal (resp. locally maximal) valence 1 vertex corresponds to an index 0 (resp. 2) critical point of α_t except that those at the minimal level $h_-(t)$ (resp. the maximal level $h_+(t)$) correspond to minimal (resp. maximal) components of $\partial\Sigma$.

For example, consider the situation of the bottom left of Figure 1. We can choose a parallelogram R as in the top of Figure 2 after an appropriate isotopy of $\varphi(S_\varphi)$. There exists a component V of the preimage $\varphi^{-1}(R)$ containing the two cusp points. The left of the bottom four rows of Figure 2 shows the possible structures of $q_\varphi(V)$, and the right shows the corresponding structures of W_{α_t} for $t = g_-, g^-, g^+, g_+$. We remark that the structures as in the bottom row may not appear if M is orientable.

5. Moves of product maps

For the proof of Theorem 2, we make more general statements about moves of singularities of product maps.

We use the following notation. Suppose M is a closed smooth 3-manifold and P, Q are either $\mathbb{R}$ or S^1. Let $F : M \to P$ and $G : M \to Q$ be smooth functions, and φ denote the product map (F, G). Let $R \subset P \times Q$, $V \subset M$ and $\{\alpha_t : \Sigma \to \mathbb{R}\}_{t\in[g_-,g_+]}$ be as described in Section 4. We consider φ, F, G restricted to V and we can assume that $P = Q = \mathbb{R}$. We assume that φ is stable in V, and the following:

(1) The region R is a parallelogram

$$\{(f, g) \in \mathbb{R}^2 \mid f_- + a(g - g_-) \leq f \leq f_+ + a(g - g_-),\ g_- \leq g \leq g_+\},$$

where $f_- < f_+$, $g_- < g_+$, $a \in \mathbb{R}$ and $f_+ < f_- + a(g_+ - g_-)$.

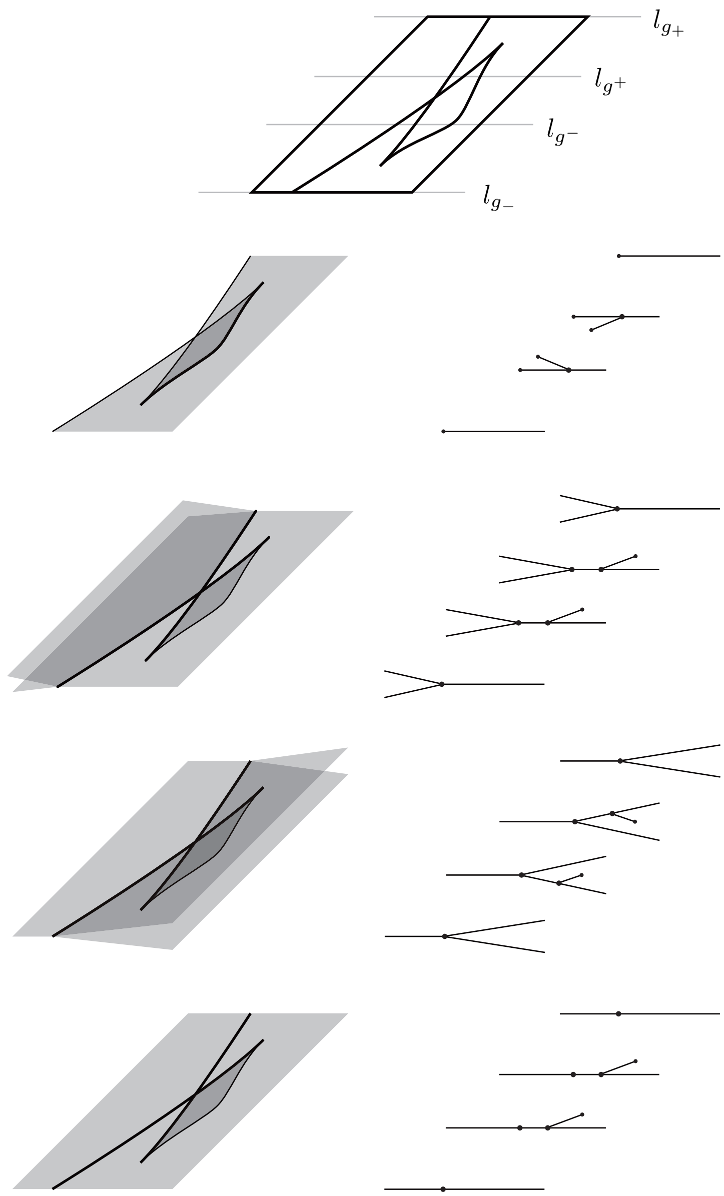

FIGURE 2. A choice of R and the four possible structures of $q_\varphi(V)$ and the corresponding structures of W_{α_t}.

(2) The discriminant set $\varphi(S_\varphi \cap V)$ has neither horizontal points nor vertical points.
(3) The Stein factorization $q_\varphi(V)$ has at least one edge which maps to one of the edges $\{f = f_\pm + a(g - g_-),\ g_- \leq g \leq g_+\}$ of R.

We can then regard a modification of the homotopy $\{\alpha_t\}_{t\in[g_-,g_+]}$ as an isotopy of the function F in the sense of the following proposition.

Proposition 11. *Let $\{\beta_t : \Sigma \to \mathbb{R}\}_{t\in[g_-,g_+]}$ be a generic homotopy from $\beta_{g_-} = \alpha_{g_-}$ to $\beta_{g_+} = \alpha_{g_+}$. Then there exists a smooth function $\tilde{F}$ isotopic to F such that $\tilde{F}|_{M\setminus V} = F|_{M\setminus V}$, and $\tilde{\varphi} = (\tilde{F}, G)$ is stable, and the Stein factorization $q_{\tilde{\varphi}}(V)$ is homeomorphic to the Cerf complex of $\{\beta_t\}_{t\in[g_-,g_+]}$.*

Proof. We can assume that $a = 1$, $f_- = 0$, $f_+ = \frac{1}{3}$, $g_- = 0$ and $g_+ = 1$ after an isotopy of $\varphi(S_\varphi)$ by the condition (1) and Corollary 7. Let $\bar{\beta}_t(\sigma) = \beta_t(\sigma) - t$ for $t \in [0,1]$, $\sigma \in \Sigma$, and let $c = \left|\inf\left\{\frac{\partial}{\partial t}\bar{\beta}_t(\sigma) \mid t \in [0,1],\ \sigma \in \Sigma\right\}\right|$. We define a continuous homotopy $\{\hat{\beta}_t\}_{t\in[0,1]}$ by

$$\hat{\beta}_t(\sigma) = \begin{cases} \left(1 - \frac{6c+2}{2c+1}t\right)\bar{\beta}_0(\sigma) + \frac{4}{3}t & \left(t \in \left[0, \frac{1}{3}\right]\right) \\ \frac{1}{6c+3}\bar{\beta}_{3t-1}(\sigma) + t + \frac{1}{9} & \left(t \in \left[\frac{1}{3}, \frac{2}{3}\right]\right) \\ \left(\frac{6c+2}{2c+1}t - \frac{4c+1}{2c+1}\right)\bar{\beta}_1(\sigma) + \frac{2}{3}t + \frac{1}{3} & \left(t \in \left[\frac{2}{3}, 1\right]\right) \end{cases}.$$

In the first interval $\left[0, \frac{1}{3}\right]$, the derivative $\frac{\partial}{\partial t}\hat{\beta}_t(\sigma)$ is positive since

$$\frac{\partial}{\partial t}\hat{\beta}_t(\sigma) = -\frac{6c+2}{2c+1}\bar{\beta}_0(\sigma) + \frac{4}{3} = -\frac{6c+2}{2c+1}\alpha_0(\sigma) + \frac{4}{3}$$

and $0 \le \alpha_0(\sigma) \le \frac{1}{3}$. It is positive also in the last interval $\left[\frac{2}{3}, 1\right]$ similarly. In the middle interval, since

$$\frac{\partial}{\partial t}\hat{\beta}_t(\sigma) = \frac{1}{6c+3}\frac{\partial}{\partial t}\bar{\beta}_{3t-1}(\sigma) + 1$$

and

$$-3c \le \frac{\partial}{\partial t}\bar{\beta}_{3t-1}(\sigma) \le 3c,$$

we have

$$\frac{1}{2} < \frac{-3c}{6c+3} + 1 \le \frac{\partial}{\partial t}\hat{\beta}_t(\sigma) \le \frac{3c}{6c+3} + 1 < \frac{3}{2}.$$

The derivative $\frac{\partial}{\partial t}\hat{\beta}_t(\sigma)$ is thus positive for $t \in [0,1]$ except that the right and left derivatives may disagree at $t = \frac{1}{3},\ \frac{2}{3}$.

Note that the ranges of $\hat{\beta}_0$, $\hat{\beta}_{\frac{1}{3}}$, $\hat{\beta}_{\frac{2}{3}}$, $\hat{\beta}_1$ are bounded as follows:

$$\hat{\beta}_0(\sigma) = \bar{\beta}_0(\sigma) = \beta_0(\sigma) = \alpha_0(\sigma) \in \left[0, \frac{1}{3}\right],$$

$$\hat{\beta}_{\frac{1}{3}}(\sigma) = \frac{1}{6c+3}\bar{\beta}_0(\sigma) + \frac{4}{9} \in \left[\frac{4}{9}, \frac{1}{18c+9} + \frac{4}{9}\right] \subset \left[\frac{4}{9}, \frac{5}{9}\right],$$

$$\hat{\beta}_{\frac{2}{3}}(\sigma) = \frac{1}{6c+3}\bar{\beta}_1(\sigma) + \frac{7}{9} \in \left[\frac{7}{9}, \frac{1}{18c+9} + \frac{7}{9}\right] \subset \left[\frac{7}{9}, \frac{8}{9}\right],$$

$$\hat{\beta}_1(\sigma) = \bar{\beta}_1(\sigma) + 1 = \beta_1(\sigma) = \alpha_1(\sigma) \in \left[1, \frac{4}{3}\right].$$

Since $\{\hat{\beta}_t\}_{t\in[0,\frac{1}{3}]}$ connects $\hat{\beta}_0$ and $\hat{\beta}_{\frac{1}{3}}$ linearly, we can see that $\hat{\beta}_t(\sigma) \in \left[t, \frac{1}{3} + t\right]$ for $t \in \left[0, \frac{1}{3}\right]$. The same holds for $t \in \left[\frac{2}{3}, 1\right]$. We can see that the same holds also for $t \in \left[\frac{1}{3}, \frac{2}{3}\right]$ since $\hat{\beta}_{\frac{1}{3}}(\sigma) \in \left[\frac{4}{9}, \frac{5}{9}\right]$, $\hat{\beta}_{\frac{2}{3}}(\sigma) \in \left[\frac{7}{9}, \frac{8}{9}\right]$ and $\frac{1}{2} < \frac{\partial}{\partial t}\hat{\beta}_t(\sigma) < \frac{3}{2}$. The range of $\hat{\beta}_t$ is thus contained in $\left[t, \frac{1}{3} + t\right]$ for $t \in [0,1]$ as well as that of α_t.

Note that the differential $(d\hat{\beta}_t)_\sigma$ is a scalar multiplication of $(d\beta_{t'})_\sigma$, where $t' = 0$ if $t \in \left[0, \frac{1}{3}\right]$, $t' = 3t - 1$ if $t \in \left[\frac{1}{3}, \frac{2}{3}\right]$ or $t' = 1$ if $t \in \left[\frac{2}{3}, 1\right]$. The homotopy $\{\hat{\beta}_t\}_{t\in[0,1]}$ therefore keeps $\partial\Sigma$ without critical points as well as $\{\beta_t\}_{t\in[0,1]}$. In particular, $\{\hat{\beta}_t\}_{t\in[0,1]}$ preserves the sets of locally minimal and locally maximal components of $\partial\Sigma$. By deforming $\{\hat{\beta}_t\}_{t\in[0,1]}$ in a collar neighborhood of $\partial\Sigma$, we can obtain a homotopy $\{\tilde{\beta}_t\}_{t\in[0,1]}$ such that the locally minimal (resp. the locally maximal) components of $\partial\Sigma$ are at the level t (resp. $\frac{1}{3}+t$) for each $t\in[0,1]$.

The homotopy $\{\tilde{\beta}_t\}_{t\in[0,1]}$ then determines a continuous function $\tilde{F}: M \to P$. That is to say, $\tilde{F}(\sigma,t)=\tilde{\beta}_t(\sigma)$ for each point (σ,t) in $V=\Sigma\times[0,1]$ and $\tilde{F}|_{M\setminus V}=F|_{M\setminus V}$. By arbitrarily small deformation in V, we can make $\{\tilde{\beta}_t\}_{t\in[0,1]}$ generic and $\tilde{F}$ smooth keeping the differential $\frac{\partial}{\partial t}\tilde{F}(\sigma,t)=\frac{\partial}{\partial t}\tilde{\beta}_t(\sigma)$ positive. The map $\tilde{\varphi}=(\tilde{F},G)$ is then stable by Lemma 10. The Stein factorization $q_{\tilde{\varphi}}(V)$ is homeomorphic to the Cerf complex of $\{\beta_t\}_{t\in[g_-,g_+]}$ by the constructions of $\{\bar{\beta}_t\}_{t\in[0,1]}$, $\{\hat{\beta}_t\}_{t\in[0,1]}$ and $\{\tilde{\beta}_t\}_{t\in[0,1]}$.

The condition (3) implies that Σ has non-empty boundary, and hence $V=\Sigma\times[0,1]$ is a handlebody. By the condition (2) and Lemma 5, the original function F has no critical points in the handlebody V. By $\frac{\partial}{\partial t}\tilde{F}(\sigma,t)>0$, the new function $\tilde{F}$ also has no critical points in V. The topologies of the level sets $F^{-1}(f)\cap V$ and $\tilde{F}^{-1}(f)\cap V$ change with f according to singularities of $F|_{\partial V}$ and $\tilde{F}|_{\partial V}$, respectively. Since $F|_{\partial V}$ and $\tilde{F}|_{\partial V}$ coincide, there is a homeomorphism of V which takes each $F^{-1}(f)\cap V$ to $\tilde{F}^{-1}(f)\cap V$. It is known that the canonical homomorphism from the mapping class group of a handlebody to the mapping class group of the boundary surface is injective. It follows that the homeomorphism is isotopic to the identity, and so $\tilde{F}$ is isotopic to F. □

Corollary 12. *Assume the following in addition to the above. Then there exists a smooth function $\tilde{F}$ isotopic to F such that $\tilde{F}|_{M\setminus V}=F|_{M\setminus V}$, and $\tilde{\varphi}=(\tilde{F},G)$ is a stable map without cusp points in V.*

(4) *The discriminant set $\varphi(S_\varphi\cap V)$ has no cusps and no crossing points on the two edges $\{f_-+a(g-g_-)\le f\le f_++a(g-g_-),\ g=g_-,g_+\}$ of R.*

(5) *The intersections of the Stein factorization $q_\varphi(V)$ with the preimages $\bar{\varphi}^{-1}(l_{g_-}),\bar{\varphi}^{-1}(l_{g_+})$ have the same numbers of locally minimal valence 1 vertices, valence 2 or 3 vertices and locally maximal valence 1 vertices.*

Proof. By the condition (4) and Lemma 8, $\alpha_{g_-},\alpha_{g_+}$ are Morse functions whose critical points have pairwise distinct values. By the condition (5) and the arguments in Section 4, $\alpha_{g_-},\alpha_{g_+}$ have the same number of critical points of each index. By Theorem 3, there exists a quasi-isotopy $\{\beta_t\}_{t\in[g_-,g_+]}$ from $\beta_{g_-}=\alpha_{g_-}$ to $\beta_{g_+}=\alpha_{g_+}$. Corollary 12 follows from the proposition and Corollary 9. □

The local moves in Theorem 2 are the simplest ones to which we can apply the above. We can see that the choice of R in Figure 2 satisfies the conditions (1), (2) and (4). We can also see that the structures of $q_\varphi(V)$ in Figure 2 satisfies the conditions (3) and (5). By Corollary 12, we can cancel the pair of cusp points, and the result is uniquely as in the bottom right of Figure 1. Similarly, we can obtain the local move of the top of Figure 1. This completes the proof of Theorem 2.

References

[1] M. Golubitsky and V. Guillemin, Stable mappings and their singularities, Graduate Texts in Mathematics, Vol. **14**, Springer-Verlag, New York-Heidelberg, 1973. DOI: 10.1007/978-1-4615-7904-5

[2] J. Hass, A. Thompson and W. Thurston, *Stabilization of Heegaard splittings*, Geom. Topol. **13** (2009), no. 4, 2029–2050.

[3] J. Johnson, *Stable functions and common stabilizations of Heegaard splittings*, Trans. Amer. Math. Soc. **361** (2009), no. 7, 3747–3765.

[4] T. Kobayashi and O. Saeki, *The Rubinstein-Scharlemann graphic of a 3-manifold as the discriminant set of a stable map*, Pacific J. Math. **195** (2000), no. 1, 101–156.

[5] H. Levine, *Elimination of cusps*, Topology **3** (1965), 263–296. DOI: 10.1016/0040-9383(65)90078-9
[6] H. Levine, Classifying immersions into R^4 over stable maps of 3-manifolds into R^2, Lecture Notes in Mathematics, **1157**, Springer-Verlag, Berlin, 1985.
[7] S. Maksymenko, *Path-components of Morse mappings spaces of surfaces*, Comment. Math. Helv. **80** (2005), no. 3, 655–690.
[8] J. Mather, *Stability of C^∞ mappings. IV. Classification of stable germs by R-algebras*, Inst. Hautes Études Sci. Publ. Math. No. 37 1969 223–248. DOI: 10.1007/BF02684889
[9] J. Mather, *Stability of C^∞ mappings. V. Transversality*, Advances in Math. **4** (1970), 301–336. DOI: 10.1016/0001-8708(70)90028-9
[10] J. Milnor, Morse theory, Annals of Mathematics Studies, No. **51**, Princeton University Press, 1963.
[11] L. E. Mata-Lorenzo, *Polyhedrons and pi-stable homotopies from* 3*-manifolds into the plane*, Bol. Soc. Brasil. Mat. (N.S.) **20** (1989), no. 1, 61–85.
[12] B. Morin, *Formes canoniques des singularités d'une application différentiable*, C. R. Acad. Sci. Paris **260** (1965), 6503–6506.
[13] H. Rubinstein and M. Scharlemann, *Comparing Heegaard splittings of non-Haken 3-manifolds*, Topology **35** (1996), no. 4, 1005–1026.
[14] K. Takao, *Heegaard splittings and singularities of the product map of Morse functions*, Trans. Amer. Math. Soc. **366** (2014), no. 4, 2209–2226.

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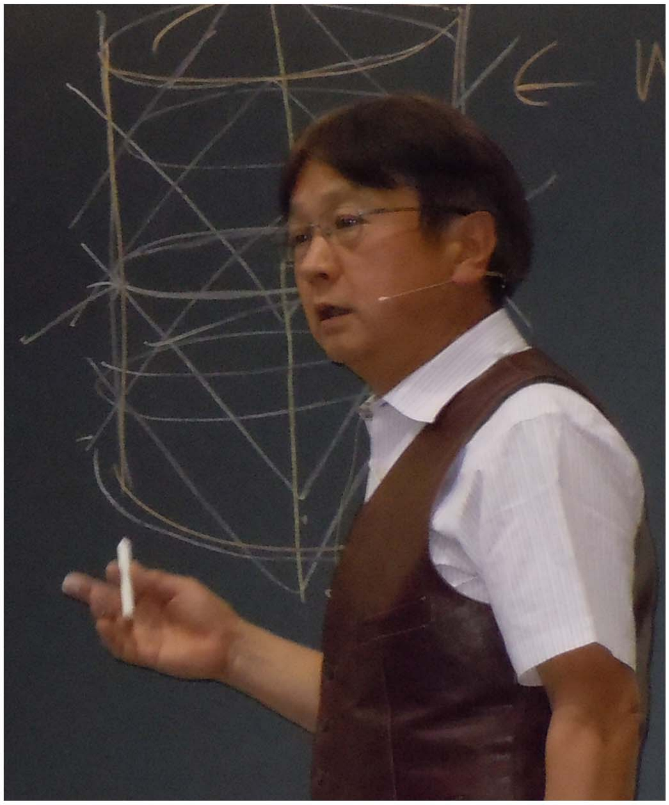

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